Student Reasoning and Cooperative Learning while using a Dynamic Geometry Environment in Taxicab Geometry: An APOS Perspective

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The goal of this research is to investigate ways a dynamic geometry environment (DGE) and interactions during cooperative learning can leverage student understanding in geometry. Data from 18 students enrolled in a College Geometry course were collected by video recording in-class group work while students explored concepts in Taxicab geometry in a DGE. The textbook from this course and its activities are based on Action-Process-Object-Schema (APOS) Theory. As such, APOS Theory was used as a theoretical framework to analyze student reasoning during these activities. For this report, results are presented for one group of students and their discussion while working on an activity which encouraged the exploration of the mathematical definition of a circle in Taxicab geometry in a DGE. Trends emerged about how group structure while working in DGEs may influence interactions and outcomes for students. Some pedagogical suggestions are provided based on the results of this study.

Keywords: Geometry, Taxicab, Dynamic Geometry Software, Circle, Definitions

Introduction

By exploring concepts and definitions in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands et al., 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry, which is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegal et al. (1998) and Dreiling (2012) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it more accessible for students to reason and abstract concepts. Further, the use of a dynamic geometry environment (DGE) for the teaching and learning of geometry is encouraged (Liljedahl, P., 2020; Hollebrands 2003; Hollebrands et al., 2010; Glass & Deckert, 2001; Contreras, 2013; Kemp, 2018). Since properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition" (Fischbein, 1993, p. 141), rather than a geometric representation influencing the definition. As DGEs offer opportunities for students to interact with accurate constructions rather than drawings on paper, students can explore and focus on relationships between concepts and their definitions. This can guide students to differentiate between drawings and constructions, abstract properties and relationships, and develop a higher level of geometric reasoning and understanding. Cooperative learning, which involves students working in groups to complete a common goal, can help students to maximize their own and each other's learning (Johnson & Johnson, 1999). In cooperative learning settings, students are also more likely to reflect on the procedures they perform (Vidakovic, 1993). Researchers also emphasize the importance of cooperative learning on psychological health and social competence, as this can encourage students to value themselves and increase independence (Johnson & Johnson, 1999).

For this report, results are presented on the following research questions: (a) How do students in a College Geometry course develop their understanding of the definition of a circle in Taxicab geometry using a DGE? (b) How does the use of cooperative learning and DGEs help students to construct knowledge in geometry?

Theoretical Framework

Action-Process-Object-Schema (APOS) Theory is a constructivist framework based on Piaget's *reflective abstraction*. According to this framework, there are four different stages of cognitive development (Action, Process, Object, and Schema) and mechanisms to move between these levels (e.g., interiorization, encapsulation) (Arnon et al., 2014; Dubinsky, 2002; Asiala et al., 1996). In APOS Theory, an *Action* is an externally driven transformation of a mathematical object. For example, an action conception of **Circle** may be exhibited by a student being prompted to identify a point on a circle when given a specific center and radius. Once an individual performs an Action enough and reflects on this, they can *interiorize* this Action into a *Process*. That is, a student demonstrating a process conception of **Circle** can imagine how to draw a circle given any center and radius, understanding that the circle is made up of an infinite number of points that satisfy a property. A Process is *encapsulated* into an *Object* once the individual is aware of it as a totality on which other actions can be performed. As an example of this for **Circle**, a student may exhibit this level of cognitive development when they construct a perpendicular bisector using circles, generally understanding how properties of circles justify why this creates a set of points equidistant from the endpoints of a segment. Once a student constructs an object in APOS Theory, it may be necessary to *de-encapsulate* it. For example, Kemp and Vidakovic (2019) present results of students de-encapsulating their object conception of **Distance** to coordinate their **Euclidean distance** and **Taxicab distance** processes. The entire collection of Actions, Processes, Objects, and other Schemas that are connected to the original concept that form a coherent understanding in the mind of the individual is called a *Schema* and is uniquely formed based on experiences (Dubinsky, 2002). The concepts involved with the *circle schema* are identified to be **Distance, Radius, Center,** and **Locus of points** (Kemp, 2018; Kemp & Vidakovic, 2018, 2019, 2021b, 2023). As a part of these students' developing *circle schemas,* we analyze their conception of the definitions of these concepts as they emerged during a class activity. Detailed descriptions of the levels of cognitive development in APOS Theory associated with these and other concepts in Euclidean and Taxicab geometry are provided in Kemp (2018) and Kemp and Vidakovic (2018, 2019, 2021a, 2021b, 2023).

There is some research that utilize APOS Theory in relation to DGEs (Hollebrands, 2003; Patsiomitou, 2019; Trigueros et al., 2022; Kemp, 2018), but a search in the literature implies a need for more. Further, the vast majority of this research focuses on Euclidean geometry. As exploration in non-Euclidean geometry can offer opportunities for students to refine their understanding of concepts, it is important to investigate how exploration in non-Euclidean geometries in DGEs can also contribute to this refinement. Results presented in Hollebrands (2003), where high schoolers' use of a DGE and how these students reasoned about transformations in geometry was investigated using APOS Theory, is used as a model in this report as the author provides a framework based in APOS Theory for student understanding as it evolves while using DGEs. The author indicated student understanding of the domain of a transformation may have been influenced by their interactions with the computer as the students interiorized the actions they performed on the computer, which contributed to their ability to form explanations of the transformations. Without the opportunity to use a DGE, students in the control group of this study were only able to examine and perform transformations on the limited number of samples provided to them and did not develop as dynamic of an understanding of these transformations. Adapted from Hollebrands (2003) for the context of this study, an action conception of *circle* in Taxicab geometry in a DGE may be exhibited by a student counting units to find a point on a circle given a specific center and radius. Once a student interiorizes this action, they can consider all points on the circle and operate with the theoretical definition of a point on a circle, rather than particular points on the screen. They can also imagine drawing a circle given any center and radius. An object conception of *circle* in Taxicab geometry in a DGE may be demonstrated by a student considering general properties and behaviors of circles in Taxicab geometry without relying on the specific examples in their DGE. As a note from Hollebrands (2003), students with a process conception of a transformation can anticipate the results of a transformation without having to perform the actual transformation on the computer.

Methodology

This study was conducted in a College Geometry course at a large university. There was a prerequisite for the class of an introduction to proofs course, and the enrollment in the class was comprised of seven undergraduate math majors and eleven graduate students who were pre-service or in-service secondary teachers. The textbook used for the course was College Geometry Using the Geometer's Sketchpad (Fenton & Reynolds, 2011) and is based on APOS Theory and the Activities, Classroom discussions, and Exercises (ACE) Teaching Cycle (Asiala et al., 1996). It is noted the course utilized Geometer's Sketchpad (GSP), which is a platform no longer supported for use. As GSP is a DGE and operates similarly to other DGEs, the methodology and results of this study can be transferrable to other DGEs. Concepts learned throughout the course in Euclidean geometry were introduced in Taxicab geometry during the last four 75-minute classes of the semester. All 18 students enrolled in the course volunteered to participate in this study, and audio and video recordings from in-class group work, work completed in GSP, and written work or notes during these class sessions for all 18 students were collected as data. The activity corresponding to the results presented in the current report was intended to guide students through the construction of a circle in Taxicab geometry in GSP and generalize this construction. Examples of a circle in Euclidean geometry and a circle in Taxicab geometry are illustrated in Figure 1. The activity is described below:

1. Plot points at $P(3,4)$, $A(2,2)$, $B(3,7)$, $C(2,5)$, and $D(5,5)$. By counting the number of blocks from P to A, we find that the *taxi-distance* PA is 3 units. Find the taxi-distances PB, PC, and PD. Two of these points are the same taxi-distance from P as A is. Which two?
2. The set of all points that are at the same taxi-distance from P form a *taxi-circle* centered at P. In part (a), three of the points lie on a taxi-circle of radius 3 centered at P. Find several additional points on this taxi-circle. Describe the set of all points that are at a taxi-distance of 3 units from a fixed point P. How is the shape of a taxi-circle different from (or similar to) the shape of an ordinary Euclidean circle?
3. If you are given a point $Q(x_q, y_q)$ and a radius r, how could you quickly sketch a taxi-circle of radius r centered at Q?

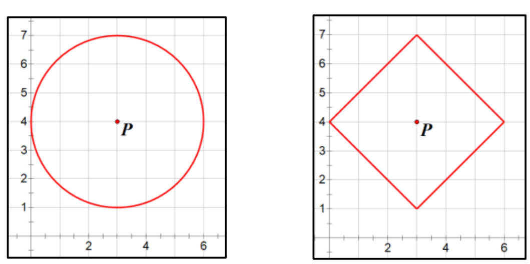

Figure 1. Geometric representations of a Euclidean and Taxicab circle each with center $P(3,4)$ and radius 3.

The design of this activity guides students through performing actions by prompting them to calculate the distance between points in Taxicab geometry to find examples and non-examples of points lying on a circle centered at (3,4) with radius 3. The students are prompted to find additional points that are 3 units away from the center and describe the set of all points that make up this circle. This is intended to guide students to interiorize the actions performed previously in the DGE into a process by focusing on the theoretical definition of a circle rather than the points on their screen. Students are then asked to compare their geometric understanding of a circle in Taxicab geometry to a circle in Euclidean geometry to aid in generalizing properties of a circle (i.e., all points are equidistant from the center, a circle looks different in Taxicab and Euclidean geometry based on how distance is defined), indicative of an object conception of *circle*.

The 18 students enrolled in the course were divided into six groups of three prior to this activity and had already worked through several activities about distance in Taxicab geometry. Students were used to working in groups and in GSP as these were regular and integral parts of the course. The students had access to a class set of laptops that had GSP installed on them and were at liberty to either each use their own laptop or share a laptop to work on the DGE activities in their groups. The classroom was set up with six square tables with two seats on two of the sides opposite one another, and students were free to move chairs around to work together in different ways in their groups. Using the adapted framework from Hollebrands (2003), data was analyzed from each group to investigate their individual understanding in terms of APOS Theory, their use of the DGE, and how their interactions within the group influenced both of these things. This report focuses on data collected from one of the groups, comprised of one undergraduate student, Ally, and two graduate students who were pre-service secondary teachers, Amy and Brianna, during the activity described above. This group was chosen in this report as they were representative of groups that worked cooperatively towards a shared goal and demonstrated various levels of cognitive development of the concept of **Circle**.

Results

As the prompts for this activity were intended to guide students through various levels of cognitive development, the data collected during this activity provided a rich understanding of how these levels can be exhibited by students while working in a group in a DGE. Ally, Amy, and Brianna all worked on their own laptops while communicating regularly with one another and often talking out loud to themselves to try and articulate their thinking. Ally's work in GSP for this activity is shown in Figure 2 below as a representative of the group. It is noted that students had access to a tool in GSP that was designed to measure the distance between two points in Taxicab geometry. This tool would also display the algebraic representation of the Taxicab distance between two points, seen in the top left of Figure 2.

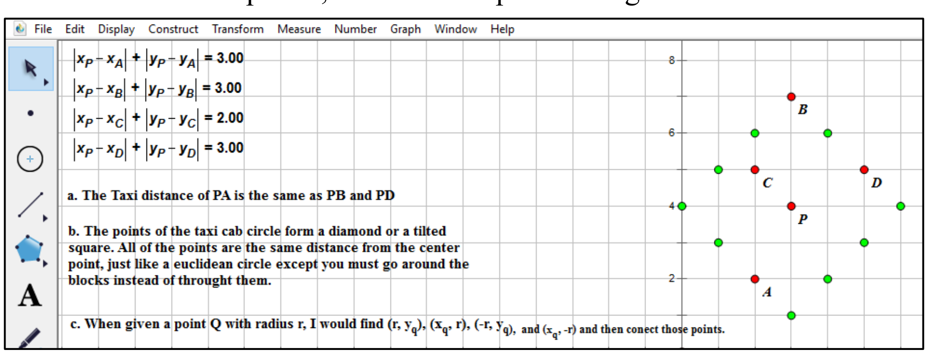

Figure 2. Work in GSP submitted by Ally.

After they read part (2) of the problem which asked students to plot several other points that are three units away from P, Brianna was looking at her graphical representation of the problem and said, "wait, why is PB the same?... Just cause it's three straight up?" to which Amy replied, "'Cause it's like the radius, yeah... of that circle." Shortly after this exchange, Amy began to investigate why the distance from P to A was three and the following interaction occurred:

Amy: 'Cause if that was a triangle, then the length of that hypotenuse be… wait, because PA is like… sides one and two.

Brianna: But we're just finding other like… how many points?

Ally: So, we just need to form like a taxi-circle.

Brianna: But I think we're trying to find other points that we think lie on the taxi-circle.

Ally: (4, 6), umm…

Brianna: Wouldn't $(3,1)$? 'Cause it's the same as PB, just the opposite direction? Like that's the radius, right?

Here, Amy had imagined a right triangle with PA as a hypotenuse and was expressing a comparison between the distance between P and A in Euclidean geometry ("hypotenuse") and in Taxicab geometry ("sides one and two"). This is evidence that Amy was exhibited a process conception of **Distance** by imagining this triangle and comparing the geometric representations of Euclidean and Taxicab distance between P and A, consistent with Kemp and Vidakovic (2023). Ally seemed to indicate she understood they were constructing a circle ("just need to form…a taxi-circle") and used this theoretical definition to anticipate all the points they were finding on the circle they were constructing, indicative of a process conception of **Circle** based on our framework adapted from Hollebrands (2003). Ally also seemed to recognize Brianna was not processing her comment that they are forming a circle and began to list specific points to help the group move into this reasoning. Brianna visualized the vertical distance PB as a radius and essentially reflected this distance ("the opposite direction") over the horizontal line through $(3,4)$ to obtain the point $(3,1)$. Aligning with Hollebrands (2003), since she anticipated the results and implications of a transformation on this radius without having to perform the actual transformation, we interpret this as her exhibiting evidence of a process conception of **Distance** and **Radius**.

As the group continued to explore in GSP while working on the activity, they began to anticipate the shape of the object they were constructing.

Ally: It forms a diamond.

Amy: So, it's not a circle.

Ally: This radius is the same…Well, an ordinary circle is round…I mean they're similar because… the radius is always the same.

Brianna: It's like the distance from…they're all the same distance from the center?

Amy: Well, this is the same distance too, but it's the same taxi-distance.

Brianna: But they're like a different type of distance cause you can go like up, you can move different ways, it's not like straight to it… like a clock.

Ally: They're similar cause its…the radius is all the way around [makes a circular motion with pointer finger].

Brianna: But it's like different.

Amy: They all are like… the same distance.

Brianna: The same distance, but it's like a different type of distance. I'm going to say it's a different type of distance than the radius of a circle.

In this cooperative learning exchange, the group seemed to be maximizing their own and each other's learning as described by Johnson and Johnson (1999) and were all reflecting on the procedures they had performed, consistent with Vidakovic (1993). All three students were working to articulate their thinking, bouncing phrasing and ideas off one another. At first, Amy implied that because this object was a "diamond" [instead of round as in Euclidean geometry], it was not a circle. As the conversation progressed, she heard what her groupmates said and indicated she processed their comments by referring to "the same taxi-distance" in her construction in the DGE. Brianna used a metaphor to explain that a circle in Euclidean geometry is round as a result of using a radius that is a straight segment, like the hand of a round clock. This is evidence Brianna was operating at a process conception of **Circle**, at least within Euclidean geometry, as she was able to generalize properties of the construction of a circle. Amy and Brianna ended up writing extremely similar responses in GSP for this prompt. Seen in Figure 3, Amy [and Brianna] wrote "they are all the same distance from the center, but it's a different type of distance than the radius of a circle." In other words, they seemed to agree that the way distance is measured is the cause of the difference in appearance of a circle in Euclidean geometry and in Taxicab geometry, but indicated they equated the radius of a circle with its representation as distance between two points in Euclidean geometry and that the distance between the same two points in Taxicab geometry is different than a radius.

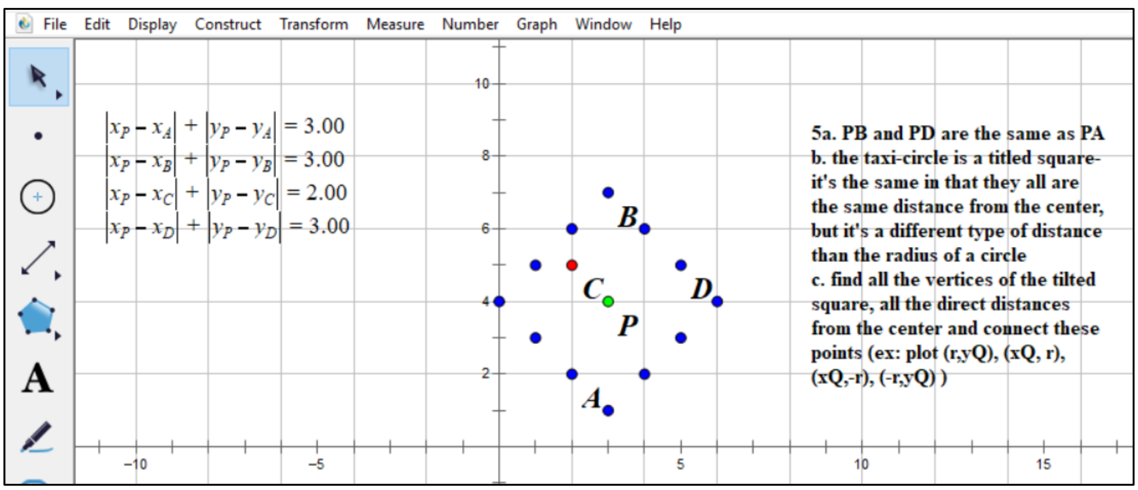

Figure 3. Work in GSP submitted by Amy.

Ally explained that a Euclidean and Taxicab circle were similar in the way they are constructed and compared these constructions across geometries ("they're similar because…the radius is the same…the radius is all the way around.") As she motioned with her pointer finger in a smooth, circular motion it is possible she was imagining a continuous, dynamic procedure defining the locus of points of a generalized circle by rotating a radius 360 degrees. Note she was speaking about an arbitrary circle in both Euclidean and Taxicab geometry at the same time. This is reiterated in her GSP work in Figure 2, where she writes, "All of the points are the same distance from the center point, just like a euclidean circle except you must go around the blocks instead of throught (*sic*) them." It is interpreted that by "go around the blocks," Ally was talking about distance in Taxicab geometry and by "through them," she was talking about distance in Euclidean geometry. As Hollebrands (2003) describes an object conception as a student considering general properties and behaviors rather than relying on the images on their screen, Ally demonstrated here she was exhibiting an object conception of **Circle**.

In the third prompt which asked students how they might quickly sketch an arbitrary circle in Taxicab geometry, the group began discussing geometrically how they would do so. While attempting to articulate this, they shifted over to attempting to write an algebraic explanation for this process. Specifically, for a circle in Taxicab geometry centered at $Q(x_q, y_q)$ with radius r, they determined the points (r, y_q), (x_q, r), $(-r, y_q)$ and $(x_q, -r)$ would be on the circle and by connecting these points, they will have constructed the entire circle. Brianna wrote in GSP, "I would find the 4 vertices (or points that are a straight/direct distance from the center) and then connect them to find the other points that lie on the taxi-circle." Brianna indicated here that she could imagine and understood all the points she would draw would be on the circle, indicating she was operating with at least a process conception of **Circle**. Amy wrote, "find all the vertices of the tiled square…and connect these points." This is notable because it may be the case that Amy was uncomfortable with calling the object a circle, perhaps an example of the tendency to neglect the definition of a concept under the pressure of [assumed] figural constraints, described by Fischbein (1993). Although the procedure determined by the group would only be true for a circle centered at the origin, this indicates a conceptual understanding in terms of identifying the intended points. Right after they wrote these coordinates, the class period ended. If they had more time, it is possible that they would have been able to test their conjecture of these coordinates and adjust them.

Discussion

As they worked to abstract properties, Amy, Brianna, and Ally bounced reasoning off one another. Ally demonstrated operating at a variety of levels of conception of **Circle**, moving between these levels during conversations to either explain a concept to a group member or try and abstract an idea. Brianna and Amy both provided evidence of operating at the action and process levels of conception of **Circle** and made connections through their conversation and explorations in the DGE. In this cooperative learning setting, all three students provided evidence that they were working toward a shared goal. They leveraged their exploration individually on their own laptops in their discussion and used comments from one another to help guide their exploration. When one group member was confused, they tended to try and clarify or help explain something. In this way, the group seemed to support the notion that the activity encouraged them to help one another as it would also benefit themselves, maximizing their own and each other's learning, while increasing their independence (Johnson & Johnson, 1999). As this group all explored on their own laptops, they often were not looking at the same screen while referencing the same objects. While this did not seem to hinder their conversation, it is possible this could deter some groups from conversing meaningfully about the activity. Some pedagogical suggestions based on this research include structuring DGE work in groups to encourage individual exploration while explicitly working toward a shared goal. By doing this, it is possible students will leverage their understanding to help explain concepts to one another and maximize the entire group's understanding, as the participants in this study did. Further, offering more opportunities for students in College Geometry courses to explore non-Euclidean geometry earlier in the course and in DGEs may be beneficial to their understanding of Euclidean concepts as they study them. Future research may investigate different types of structure in DGE group work and how this affects individual understanding. For example, what affordances or hindrances might arise if students all shared one laptop and were looking at the same screen rather than exploring individually? As this study utilized APOS Theory, DGEs, and cooperative learning, it may help fill a gap in the literature of undergraduate mathematics education.

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