Proof Writing and Comprehension in Topology

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The pedagogy of proof has been a topic of research interest in mathematics education literature (e.g., Melhuish et al., 2022). While proof writing is a formal and abstract topic for many students, topology proofs typically demand a level of visual comprehension as well, which places topology in a spot of particular interest. Many argue that learning topology relies on spatial reasoning, yet others have noted that students' shortcomings are typically in the formal, proof-based aspects of problems (e.g., Narli, 2010). Various researchers have sought to understand how we can teach topology by studying how learning takes place (e.g., Cook et al., 2017; Wilkerson-Jerde & Wilensky, 2011), what methods of teaching can support this learning (e.g., Griffiths, 1971; Deogratias, 2022; Estabrooks & McArdle, 2022; Stewart, 2000), and what concepts students typically understand (e.g., Aksu, Gedik, & Konyalioglu, 2021; Cheshire, 2017; Gallager & Infante, 2019). However, there is a lack of research into how students perceive their own experiences in learning topology.

We seek to build a model based on Tall's (2008; 2013) framework of mathematical thinking and APOS theory in order to gain insights into the cognition of topology. In addition, we will examine how students communicate their experiences and think about topology. Tall's three worlds suggests that modes of thinking can be described as three different "worlds" of thought: the embodied, symbolic, and formal worlds (Tall, 2008; 2013). The embodied world refers to aspects of thought informed by perception, the symbolic world contains the thinkable concepts as generalized objects, and the formal, axiomatic world derives meaning from definitions and formal constructions. APOS theory refers to the depth of understanding that learners may demonstrate as they come to understand a topic (Dubinsky & McDonald, 2001). Using the three worlds of mathematical thinking and the action, process, and object levels from APOS, we will create a table synthesizing both ideas inside a grid (APOS on the first column, three worlds on the top row). Networking these theories gives language to what behaviors are exhibited in proof as the writer transitions across various types of thinking and levels of complexity. Changes throughout a proof would be characterized as transitions of thought, demonstrating how various ideas are handled, transformed, or ignored. We will demonstrate this model with a particular example from topology and elaborate on the mathematics behind it in order to develop it.

This three worlds-APOS model can be employed as novice graduate students complete work in topology. Students may understand concepts as actions, then generalized processes, then complete objects, and have a schema to allow them to transition between all three worlds of mathematical thinking. This research study is sought to determine students' struggles and experiences with topology using the model. We seek to answer the following questions: How can this model be used to examine the level of understanding of topology? What improvements can be made to this model? What topics in topology does this model reveal that are typically neglected on the formal level? What topics in topology are transitioned into spatial reasoning? What topics in topology are kept on the formal level of understanding?

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INTRODUCTION

The pedagogy of proof has been a topic of research interest in mathematics education literature (e.g., Melhuish et al., 2022). While proof writing is a formal and abstract topic for many students, topology proofs typically demand a level of visual comprehension as well, which places topology in a spot of particular interest. Many argue that learning topology relies on spatial reasoning, yet others have noted that students' shortcomings are typically in the formal, proof-based aspects of problems (e.g., Narli, 2010). A portion of the literature can be summarized with the following:

What do students struggle with?	What methods of teaching can help students?	How do students develop understanding?	How can we reinvent how topology is taught?
Countable unions/intersections Non-metrizable spaces Escaping the usual topologies Applying topology to geometry	Topology can be taught motivated from analysis General topologies are abstractions of metric spaces Physical objects can be helpful, though sometimes confusing	Students reinvent and adapt their own axiomatic systems Students reconcile misconceptions Experts learn well from proofs	Classes based around students discovering the concepts do incredibly well Motivations for topology exist in games/art
e.g., Deogratias 2022; Aksu et al. 2021; Cheshire 2017; Gallagher 2019; Narli 2010	e.g., Shipman & Stephenson 2022; Helmstutler et al. 2012; Griffiths 1971	e.g., Cheshire 2017; Wilkerson-Jerde, Michelle, & Wilensky 2011	e.g., Estabrooks & Mcardle 2022; Cook et al. 2017; Stewart 2000; Poggi 1985

Still, there is a lack of research into how students navigate transitions from the spatial concepts to formal logic and how they perceive their own actions in topology.

THEORETICAL BACKGROUND

Tall's Three Worlds of Mathematical Thinking suggests that modes of thinking can be described as three different "worlds" of thought: the embodied, symbolic, and formal worlds (Tall, 2008; 2013). The embodied world refers to aspects of thought informed by perception, the symbolic world contains the thinkable concepts as generalized objects, and the formal, axiomatic world derives meaning from definitions and formal constructions.
APOS theory refers to the depth of understanding that learners may demonstrate as they come to understand a topic (Dubinsky & McDonald, 2001). First, there are actions taken. Processes are actions that are repeated and reflected upon, which then are encapsulated into objects that can be acted on. The schema is the collection of all of these.

DESIGN OF THE MODEL AND RESEARCH QUESTIONS

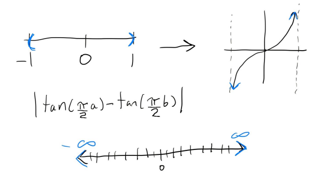
Seeking to gain insight into how students perceive their own actions in their topology proof writing, we network Tall's Three Worlds and APOS Theory together to develop a model. Putting these together gives a holistic perspective of the learner's thoughts and actions. We perform this networking by combining both theories into a single table with APOS columns and Tall's Three Worlds rows. With this model in mind, we seek to answer the following **research questions**:
How can this model be used to examine students' understanding of topology? What topics in topology does this model reveal that are typically neglected on the formal level? What topics in topology are transitioned into spatial reasoning? What topics in topology are kept on the formal level of understanding? What are the limitations of this model and what improvements can be made?

APPLYING THE MODEL: TASK AND SAMPLE SOLUTION

To help identify when transitions are made, the order of the items are numbered with superscripts.

Question 1: Give a metric on the set (-1,1) that induces the usual topology and is complete.

	Action	Process	Object
Embodied	-1→ -∞, 1→∞[6]	(-1,1)→R sends points to infinity[4]	Graph of Tan is infinite 1-manifold[1.5]
Symbolic	Compose distance with Tan[9]	d(a,b)=|a-b|[8]	R ≅ (a,b)[1] Metric has 2 inputs[7]
Formal		f(x)=Tan(x*pi/2)[5]	Reals are complete[2] (-1,1) doesn't have its limits[3]

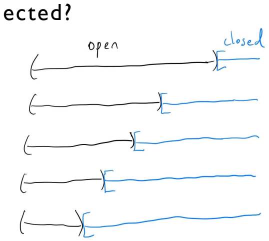


- Why are there no formal actions?
- The world with the most behavior is the symbolic, the stage most referenced is object. Does the performance on this question support this?
- When do transitions from worlds/stages happen?

What is the usual topology on this interval? It's the same as the reals, as is any basic open interval (via some variation on tangent). I must come up with a *metric* that induces the usual topology but is complete as a metric space, so I ask if the space is usually complete. The reals are complete with the usual metric, but clearly, as is the concept of the question, the interval given is not complete with the same metric. So, comparing the two spaces, how are they different from a metric standpoint: While the interval is finite in length, the reals are infinite, so we just need to identify how the interval is sent to the reals via tangent, then combine that with the usual metric on the reals.

Question 2: Is the topology given by T={ (0,a) ≤ R⁺ | 0 < a } path connected?

	Action	Process	Object
Embodied		Path-Connectedness is embedding lines throughout the space[2]	
Symbolic	Compliment of (0,a) is [a, ∞)[5]		Closed sets of R[6]
Formal	Courser topologies are not always subset topologies[1]		Definitions: Topology as a set-family[1.5], path-connected[3], continuous function[4]



- Why was this mostly a symbolic and object oriented proof?
- Does the lack of varying concepts reflect a lack of understanding or completion?

While being a subset of the reals, this is not given the subspace topology, so I need to be mindful of that distinction as I think of this space. My usual intuition for path-connectedness fails since this is not the usual topology. I fall back on the definition of path-connected, and, wanting to make a path between any two points, I realize I'm not sure what a continuous function into this space should look like. The definition of continuous is the preimage of closed sets are closed, so I need to know what sets are closed in this topology. Since these are just compliments of the open sets given to me, they all look the same as [a, ∞). These sets are already closed in the usual topology, so the usual paths are already continuous here, so any straight-line path is a path from any two points of the space, thus the space is path-connected.

REFLECTION

Although the model is adequate at showing how students think about concepts in topology in terms of complexity and form, as the examples above show a lack of formal actions, it needs more research directed towards questions demanding those behaviors.

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https://sites.google.com/view/proofwritingandcomp