# U-Substitution through Quantitative Reasoning: A Conceptual Analysis

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*Research has shown how crucial quantities-based meanings are for calculus concepts. While past work has developed important quantitative approaches to integration, the major topic of u-substitution typically has not been fully detailed in these paradigms. This theoretical paper extends this past work to clearly define and elaborate u-substitution through a quantitative perspective. We use conceptual analysis based on quantities, starting with a concrete example of a solar panel producing energy. We abstract from this example to define u-substitution as a transformation from one quantitative relationship, via nested multivariation, to another quantitative relationship. We also detail a three-part structure within this transformation.*

*Keywords:* calculus, integrals, u-substitution, quantitative reasoning, conceptual analysis

A growing body of work has established that quantitatively-based meanings for calculus concepts such as derivatives and integrals are essential for robust student understanding and productive usage outside of math classes (Byerley, 2019; Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). For integrals, this means moving away from the purely "area under a curve" meaning in favor of a "sum of small bits" meaning (Ely, 2017; Jones, 2015; Sealey, 2006). Much work has been done in introducing integrals through this meaning and in helping students reason and model with integrals (Bajracharya et al., 2023; Blomhøj & Kjeldsen, 2007; Chhetri & Oehrtman, 2015; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). Yet, integration chapters typically conclude with the major "u-substitution" method, which is the first in a long line of substitution techniques, and which is also utilized in the sciences and engineering to convert between quantitative expressions (see Koretsky, 2012). Unfortunately, the current literature does not adequately depict how to incorporate u-substitution into a quantitative paradigm. It would be detrimental to work toward quantitative meanings for integrals only to have this major method exist outside them. Treating u-substitution quantitatively would not only allow students to effectively *use* it, but to understand the mechanisms for *why* it works. To build on the important prior work on quantitative reasoning in calculus, in this paper we induct u-substitution into it as well. As the first key step, this paper focuses on the *theoretical* side of a quantitatively-based approach to u-substitution. We use a quantities-based conceptual analysis to define the quantitative relationships within u-substitution and its three-part quantitative structure. In a separate paper, we build on this theoretical work to engage in empirical examinations of learning u-substitution through a quantitative approach.

## Brief Review of Closely Related Literature

There is a proposed quantitative structure for definite integrals called *adding up pieces* (AUP) (Jones, 2013; Jones & Ely, 2023). AUP is comprised of three parts: *partition*, *target quantity*, and *sum* (see also Dray & Manogue, 2023; Sealey, 2014; Von Korff & Rebello, 2012). *Partition* is taking a quantitative object (e.g., a length, a space, a time duration) and segmenting it into tiny pieces. If the pieces are essentially infinitesimal in size, they are called "differentials" and are denoted with a "*d*", as in $dx$ or $dt$ (Ely, 2020). In other words, a $dx$ or $dt$ can be thought of as an incredibly tiny $\Delta x$ or $\Delta t$. While differentials can be rigorously formalized through limits or hyperreals (see Jones & Ely, 2023), the more loosely-defined "essentially infinitesimal" is quite common across the sciences and is a crucial idea for quantitative reasoning, even if no specific threshold is defined for passing from "Δ" to "*d*" (Amos & Heckler, 2015; Hu & Rebello, 2013; Pina & Loverude, 2019; Thompson & Dreyfus, 2016; Von Korff & Rebello, 2014).

*Target quantity* refers to determining the sought-after quantity within each partition piece (Jones & Ely, 2023). For example, if one is determining the distance travelled under variable velocity, and time has been partitioned into infinitesimal *dt* pieces, one conceptualizes that each *dt* piece corresponds to a tiny bit of distance travelled, *dD*. In AUP, any quantitative relationship can be used to determine the target quantity. The distance example uses a simple product, $D = v \cdot t$, but other quantitative relationships can be used when appropriate (Simmons & Oehrtman, 2017). Oehrtman and Simmons (2023) called the quantitative relationship that holds for constant values, such as $D = v \cdot t$, the *basic model*. If some quantities vary, such as a varying velocity, the essentially infinitesimal nature of the partition pieces allows this quantity to be considered essentially constant over each piece. Thus, the basic model can be applied to the infinitesimal partition piece, as in $dD = v \cdot dt$, which Oehrtman and Simmons (2023) called the *local model*.

With a conceptualization of tiny bits of the target quantity in each essentially-infinitesimal partition piece, *sum* refers to the literal summation of these target quantity bits to obtain the total amount of the target quantity. We follow Leibniz's convention (Katz, 2009) in using the integral symbol, $\int$, as a literal "sum" symbol. As an example, $\int_a^b v(t)dt$ is the summation of tiny distances (produced by $v \cdot dt$) across the many *dt* pieces, yielding the total distance.

## Theoretical Lens: Quantitative Relationships and Multivariation

Thompson (Smith & Thompson, 2007; Thompson, 1990) defined a "quantitative relationship" as a system of three quantities where any two can determine the third. Thompson's three-quantity relationships can be represented as a triangle with a quantity on each vertex (Figure 1a). Jones (2022) elaborated on these quantitative relationships to create a set of distinct relationship types called *multivariation* (MV) relationships (though MV can also include relationships with more than three quantities). One MV relationship type is *independent MV*, where multiple inputs influence a single output, but where the inputs are independent from each other (Figure 1b). Another type is *dependent MV*, where all the quantities are co-dependent and a change in a single quantity implies changes in all others (Figure 1c). A third type is *nested MV*, where the quantities are structured in a chain of influence, congruent to the structure in function composition (Figure 1d). Our conceptual analysis of u-substitution in this paper relies heavily on Thompson's idea of a quantitative relationship and on the dependent MV and nested MV types.

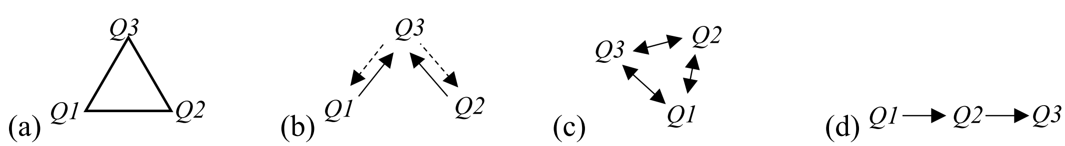
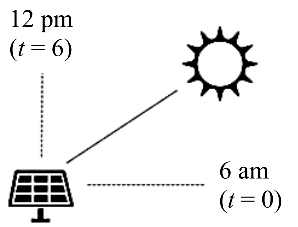



*Figure 1. (a) Thompson's quant. relationship; with Jones' (b) independent MV, (c) dependent MV, (d) nested MV*

## Conceptual Analysis of U-Substitution: Starting with an Example Context

We begin our conceptual analysis with an example, from which we can abstract the general relationships. For our example, consider Figure 2a of a solar panel, with the sun rising at 6 am and reaching its zenith at 12 pm. The main quantities in this context are time (in hours, hr), the energy produced by the solar panel over a period of time (in kilojoules, kJ), and the power, or the rate at which energy is generated over time (in kilojoules per hour, kJ/hr). While a "watt" is a standard power unit (joules per second), we use kJ/hr to match the unit of time (hr), and to make the rate structure more obvious. Time, power, and energy are related by $E = P \cdot t$ (Figure 2b).

Notice that early in the morning there is less power because the sun makes a shallow angle with the panel. The power increases as the sun approaches its zenith, meaning that power is a function of time. In fact, we call time the main *input quantity* because it is an input both for the power, $P(t)$, and for the energy, $E(P, t)$. For this context, if we let 6:00 am be $t = 0$, we can use $P(t) = 1500 \sin\left(\frac{\pi}{12} t\right)$ kJ/hr as a reasonable model. The sine function suits the sun's increasing angle over time: it is zero at $t = 0$, increases from $0 \leq t \leq 6$, and is maximized at $t = 6$.

(a) 

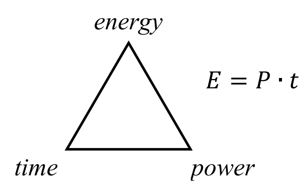

(b) 


*Figure 2. (a) A solar panel with varying power as the sun rises, and (b) the quantitative relationship in this context*

The following initial question leads to a regular definite integral: How much energy is produced by the solar panel in this six-hour window? If power were constant over the six hours, we could use the basic model to calculate it: $E = P \cdot t$. But because power varies, we use the quantitative AUP structure to set up an integral. If we partition the six hour interval into essentially infinitesimal time intervals, *dt*, then we can assume power to be essentially constant over each interval. We can then use the local model $dE = P \cdot dt$ to capture the tiny bit of energy generated over a *dt* interval. Power, $P(t)$, is determined by a *t* value associated with the *dt* interval, so we can write our local model as: $dE = 1500 \sin\left(\frac{\pi}{12} t\right) \cdot dt$. We can then add up these bits of energy between $t = 0$ and $t = 6$ to find the total energy: $E = \int_{t=0}^{t=6} 1500 \sin\left(\frac{\pi}{12} t\right) dt$.

With this definite integral in place, we can now ask a different question that leads to u-substitution: What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock? This is the fundamental quantitative question we claim u-substitution is based on: How do you convert from one main *input quantity* to a different main *input quantity*? This conversion affects each of the three parts of the AUP structure. For the first part (partition), we need to determine what each infinitesimal *dt* piece corresponds to in terms of infinitesimal angles, *dθ*. For this conversion, notice that six hours corresponds to $\pi/2$ radians, three hours to $\pi/6$ radians, one hour to $\pi/12$ radians, and so on. In fact, for any time interval, the corresponding angle will always be $\pi/12$ as big, even at very small scales. Thus, $d\theta = \frac{\pi}{12} dt$, or equivalently $dt = \frac{12}{\pi} d\theta$, where the conversion factor $\frac{12}{\pi}$ has units of $\frac{\text{hr}}{\text{rad}}$. Switching from time pieces to angle pieces gives: $E = \int_{t=0}^{t=6} 1500 \sin\left(\frac{\pi}{12} t\right) \cdot \frac{12}{\pi} d\theta$. Of course, for this simpler context, the relationship between time and angle is linear at all scales. Later in the paper we examine a context where the relationship is *not* linear at all scales.

In tracking the units up to this point, we have $\left[1500 \sin\left(\frac{\pi}{12} t\right)\right] \frac{\text{kJ}}{\text{hr}} \cdot \left[\frac{12}{\pi}\right] \frac{\text{hr}}{\text{rad}} \cdot [d\theta]$ rad, which still yields a measure of kJ. This is valid quantitatively, though there is now a mismatch between defining part of the integral (the integrand) in terms of time and another part (the differential) in terms of angles. We also want to convert from power defined by time to power defined by angles. To do so, we note the nested MV relationship from *time → angle → power*, given by the function composition $\theta(t) = \frac{\pi}{12}t$ and $P\big(\theta(t)\big) = 1500\sin\big(\theta(t)\big)$. However, if we no longer care about time, and only want to track the power in angles, this can become $P(\theta) = 1500\sin(\theta)$, leading to the integral expression: $E = \int_{t=0}^{t=6} 1500\sin(\theta)\frac{12}{\pi}\,d\theta$. Note that $1500\sin(\theta)$ still has units of kJ/hr, with the kJ/hr rate now determined by the angle. The factor 12/π still has units of hr/rad, so it is the entire "$1500\sin(\theta) \cdot 12/\pi$" that has units of kJ/rad.

While energy bits are now determined purely through the angle, $dE = 1500\sin(\theta) \cdot \frac{12}{\pi}\,d\theta$, the sum itself is still defined in terms of time. We want the sum to also be described in terms of the angle. Because $0 \le t \le 6$ hours corresponds to $0 \le \theta \le \frac{\pi}{2}$ radians, a sum running across *dt* pieces between $0 \le t \le 6$ hours covers the same ground as a sum running across $d\theta$ pieces between $0 \le \theta \le \frac{\pi}{2}$ radians. Thus, $E = \int_{\theta=0}^{\theta=\pi/2} 1500\sin(\theta)\frac{12}{\pi}\,d\theta$ and we have successfully described the total energy completely in terms of the *angle* as opposed to *time*.

**Abstracting from the Example to a Quantitative Definition of U-Substitution**

Before proceeding, we make some terminology clear. We have already used the term *input quantity* for the quantity that the other two depend on and that gets partitioned into infinitesimal pieces (*time*, in our example). We have also used *target quantity* to represent the quantity whose bits are added up by the integral and whose total amount is given by the value of the integral (*energy*, in our example). We introduce a new term, *integrand quantity*, for the quantity that functionally depends on the input quantity and that also combines with the input quantity to produce the target quantity. In our example, the integrand quantity was *power*.

With this terminology in place, recall that a definite integral answers the following question: If a target quantity is some combination of an input quantity and an integrand quantity, how can we determine its total amount if the integrand quantity varies over the input quantity? Relatedly, we claim that u-substitution now answers this question: How can we determine the total amount of the target quantity if we want to switch from tracking it in terms of the original input quantity to a new, different input quantity? This conversion is based on an inherent nested MV structure going from *original input → new input → integrand quantity*. In our solar panel example, the nested MV was $time \rightarrow angle \rightarrow power$, described by the function composition $P\big(\theta(t)\big)$.

To provide a clear quantitative definition, we wish to use generic symbols for these quantities that are not specific to any context. We use "*a*" for the original input, "*b*" for the new input, "*Q*" for the integrand quantity, and "*T*" for the target quantity. In our solar panel example, "*a*" was time, "*b*" was angles, "*Q*" was power, and "*T*" was energy. Recall that for a basic integral, *a*, *Q*, and *T* are in a dependent MV quantitative relationship described by a Thompson triangle (Figure 3a). The nested MV relationship resides along the triangle's edge between the input quantity and the integrand quantity: *a* → *b* → *Q* (Figure 3b). Thus, *b* is always an intermediary between *a* and *Q*. We can now define u-substitution quantitatively as the act of shifting the vertex "*a*" of the

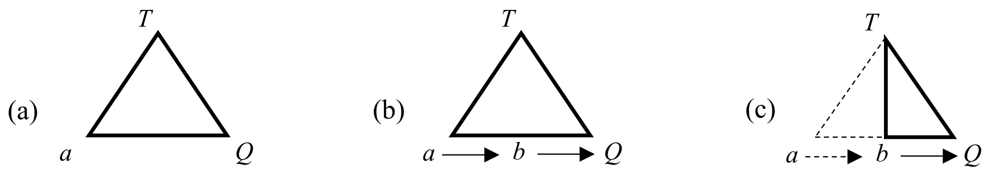



*Figure 3. U-substitution as the transformation of the original dependent MV relationship (a); through the nested MV relationship between a, b, and Q (b); to a new dependent MV relationship (c).*

original three-quantity triangle to the new input along this edge, "$b$," thereby creating a new three-quantity triangle between $b$, $Q$, and $T$ (Figure 3c).

### The Three-part Structure within this Quantitative Definition

Having defined u-substitution as a transformation from one quantitative relationship to another closely-related quantitative relationship, as in Figure 3, we now examine the detailed structure contained within this transformation. Recall that the quantitative AUP structure for definite integrals contains three parts: *partition*, *target quantity*, and *sum*. U-substitution similarly contains a three-part structure, with the three parts corresponding to those in AUP.

The first part corresponds to AUP's *partition* and involves converting the partition pieces from the original input quantity to *equivalent* partition pieces for the new input quantity. In the solar panel example, we saw that one cannot simply substitute $d\theta$ for $dt$, because it would change the size of the partition pieces. That is, a 0.001 second interval is not equivalent to a 0.001 radian interval, and a product between power and a 0.001 second interval does not yield the same energy as the product between power and a 0.001 radian interval. Rather, one must determine how big an infinitesimal piece of the new input quantity is in relation to an infinitesimal piece of the original input quantity. We call this part of the u-substitution structure: *differential* (Table 1).

The second part corresponds to AUP's *target quantity*. It involves taking the integrand quantity that functionally depends on the original input and converting it to a functional relation based on the *new* input. In the solar panel example, power was dependent on time, and we converted it to a functional dependency on angle. This conversion permits the infinitesimal bits of the target quantity in each partition piece to be completely determined by the new input quantity, since both differential and integrand are now in terms of the new input quantity. We call this second part of the u-substitution structure: *integrand* (Table 1).

The third part is to re-describe the *summation*. The original sum is in terms of infinitesimal pieces of the original input quantity, and we need to instead describe it in terms of the new input quantity. In the solar panel example, we originally summed across infinitesimal time pieces between $0 \le t \le 6$, and we needed to switch to an equivalent sum across infinitesimal angle segments between $0 \le \theta \le \frac{\pi}{2}$. We call this third part of the structure: *bounds* (Table 1).

These three parts are exactly what enable the transformation from the original quantitative relationship to the new quantitative relationship by shifting the triangle's vertex from $a$ to $b$. If only some of these parts are enacted, the original input still exists as a vertex in the triangle, making it a four-quantity relationship in the triangle. While this is not incorrect, it creates issues for computation. In the solar panel example, when the differential in time was converted to a differential in angle, the quantitative structure still worked, but it was odd to have parts of the integral defined in one input quantity with other parts defined in another input quantity. Once the

*Table 1. The three-part u-substitution structure, with original input a, new input b, and integrand quantity Q*

| **Initial AUP parts** | **Corresponding parts in the u-substitution structure** | |
|---|---|---|
| 1. Partition: $da$ | 1. Differential: Convert partition from *da* pieces to equivalently-sized pieces in *db*. | $da \rightarrow [conversion]db$ |
| 2. Target quantity: $Q \cdot da$ | 2. Integrand: Convert integrand quantity, $Q$, from depending on $a$ to depending on $b$. Target quantity now determined entirely by new input. | $Q(a) \rightarrow Q(b)$<br>$Q(a)da \rightarrow Q(b)[conv]db$ |
| 3. Sum: $\int_{a1}^{a2} Q(a)da$ | 3. Bounds: Convert sum from running across original input pieces to new input pieces. | $a_1 \le a \le a_2 \rightarrow$<br>$b_1 \le b \le b_2$ |

three parts of the u-substitution structure are enacted, the original input quantity's vertex is eliminated and the triangle transforms to the new vertex. Table 1 summarizes this three-part structure of u-substitution. Of course, we are quick to point out that in reasoning about u-substitution, one does *not* need to follow these three parts in a specific, prescribed order. Rather, it is possible to work through them in any order (see Oehrtman & Simmons, 2023).

**Applying this Definition and Structure to Another (Non-Linear) Example**

Having exposited the quantitative theory, we feel it beneficial to zoom back out to another contextual example to apply our quantitative definition and the three-part structure of the quantitative relationship transformation. We also use a context in which the relationship between the original input and the new input is *not* linear. Suppose a sphere is growing over time and we want to know the accumulated volume between $t = 5$ and $t = 10$ mins (Figure 4a, below). Note that at a given time the volume will grow at a rate proportional to its surface area, meaning there is a three-quantity relationship between *time*, *surface area*, and *volume* (Figure 4b). Here, time is the *input quantity*, volume is the *target quantity*, and surface area is the *integrand quantity*. Just to have something to work with, suppose the sphere's radius (in cm) is given by the function of time: $r(t) = t^2 + 5$ cm. Thus, surface area at any moment is $S(t) = 4\pi(t^2 + 5)^2$. Also, the instantaneous rate of radius growth is $dr/dt = 2t$ cm/min, so that for an essentially infinitesimal interval of time, $dt$, the radius grows by $dr = 2t \cdot dt$ cm. To more quickly get to u-substitution, we simply state that for an infinitesimal $dt$, volume will grow by the product of the surface area and the corresponding change in radius, leading to the integral: $V = \int_{t=5}^{t=10} 4\pi(t^2 + 5)^2\, 2t\, dt$.

We now ask the question specifically relevant to u-substitution: While volume grows over time, could we track its accumulation with respect to growth in the *radius* rather than *time*? This question is exactly what we claimed the domain of u-substitution to be. That is, we want to determine the total amount of the target quantity (volume), but we want to switch from one main *input quantity* (time) to another main *input quantity* (radius). Further, there is a nested MV relationship between *time* → *radius* → *surface area*, or $S\big(r(t)\big)$ (Figure 4c). U-substitution will transform the original quantitative relationship into the new desired relationship (Figure 4d).

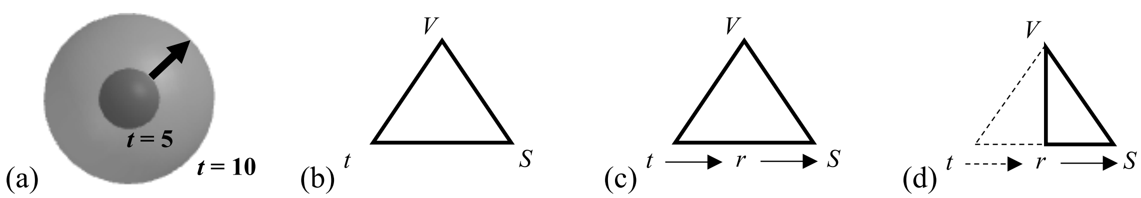



*Figure 4. (a) growing sphere context; (b) time, surface area, volume quantitative relationship, (c) nested MV relationship between time, radius, and surface area, and (d) transformation to the new quantitative relationship*

How do we enact this transformation? By using the three-part structure we proposed for u-substitution based on AUP. To show these can be done in any order, we purposefully use a different order here than for the solar panel example. First, the summation currently runs over $dt$ pieces from $5 \le t \le 10$ min, but we could cover the exact same ground if we instead ran a summation over $dr$ pieces from $30 \le r \le 105$ cm. This is the "*bounds*" part, making the integral so far: $V = \int_{r=30}^{r=105} 4\pi(t^2 + 5)^2\, 2t\, dt$. But the sum now is in terms of the radius, while all else is still described in terms of time. Second, we could re-describe the *integrand quantity* in terms of the radius. From the nested MV relationship, we have $4\pi(t^2 + 5)^2 = 4\pi\big(r(t)\big)^2 = S\big(r(t)\big)$, but since we want to eliminate time, we can rewrite this dependence simply as $S(r) = 4\pi r^2$. This is the "*integrand*" part, and makes the integral: $V = \int_{r=30}^{r=105} 4\pi r^2 \ 2t \ dt$. Lastly, we need to convert the partition pieces in *dt* to partition pieces in *dr*. As explained before, a *dr* piece is not equivalent to a *dt* piece, so we need to know the conversion factor. While *t* and *r* do not have a linear relationship, we know that for any time *t*, $dr$ will be exactly 2*t* times as big as *dt*: $dr = 2t \cdot dt$. This is the *differential* part and we now have a fully transformed integral from the original input quantity to the new input quantity: $V = \int_{r=30}^{r=105} 4\pi r^2 \ dr$ (see Figure 4b-d).

### Comparison to "Pure Math" U-Substitution

In a traditional "pure math" sense, u-substitution is depicted as undoing the chain rule: $\int_{x_1}^{x_2} f'\big(g(x)\big)g'(x)dx$ . The procedure is to label the "inside" function, $g(x)$, as an arbitrary symbol "$u$", and to label $du = g'(x)dx$ in order to get $\int_{u_1}^{u_2} f'(u)\ du = f(u_1) - f(u_2)$. However, these symbols tend to be just notational conveniences and have little (if any) meaning, especially *du*. Here we show how our quantitative definition for u-substitution corresponds to this pure-math procedure and how it can provide conceptual meaning. If we think of *x*, $f'$, and $f$ as representing three quantities in a relationship, *u* is an intermediary quantity between *x* and $f'$. This is why *u* is always in a nested MV relationship with *x* and $f'$, as in $x \to u \to f'$. In the quantitative paradigm, instead of *du* being just a notational convenience, $du = g'(x)dx$ actually represents the conversion factor from *dx* partition pieces to *du* partition pieces. Lastly, since we are converting from one input quantity to another, the switching of the bounds indicates a re-describing of the sum as it now ranges across infinitesimal pieces of the *new* input quantity.

### Discussion

The contribution of this theoretical paper is to induct u-substitution into a quantitative paradigm, thereby extending the crucial prior work on quantitative reasoning in calculus (Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). In short, u-substitution deals with taking a quantitative relationship (Smith & Thompson, 2007; Thompson, 1990) in a definite integral and re-describing it through a *new* input quantity, where a nested MV exists between *original input → new input → integrand quantity* (Jones, 2022). We defined u-substitution as transforming this quantitative relationship by sliding the vertex from the original input to the new input. To fully accomplish this transformation, we described a three-part structure for u-substitution (*differential, integrand,* and *bounds*) that correspond to the three-part quantitative structure of AUP for definite integrals (Jones & Ely, 2023). These parts do not need to be executed in one specific order (Oehrtman & Simmons, 2023). In fact, a benefit to quantitative reasoning is actually reasoning about the context to flexibly carry out operations (Ely, 2017).

Our contribution allows for a quantitative reasoning paradigm in calculus based on informal infinitesimals (Ely, 2020) and AUP (Jones & Ely, 2023) to extend all the way through the introductory calculus curriculum, including u-substitution. This extension permits coherence across the entirety of the calculus course, founded on a reasoning type that has been shown to be extremely important for understanding and productive usage (Jones, 2015; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Pina & Loverude, 2019; Von Korff & Rebello, 2012). We are also building on this theoretical work through teaching experiments, teaching students u-substitution based on these quantitative meanings (which we report on in a separate paper). We believe that understanding u-substitution quantitatively in this way will enable students to actually understand the mechanisms behind it and to become more flexible users of it, as they can see what it means and the three-part structure for executing the quantitative transformation.

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