Knowledge Resources for Multiplication and Challenges Reasoning About the Product Layer

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Research on students' construction and interpretation of integrals has grown significantly over the past decade. Some work has identified classes of interpretation for definite integral notation that include areas under curves, antiderivatives, and adding up pieces. The present study contributes to recent research on challenges students experience with the "product layer," $f(x) \cdot dx$*, when reasoning about adding up pieces. In particular, we present a case study in which 1 college student in a first-semester, calculus-based physics course drew on multiple knowledge resources associated with multiplication as she tried to construct a definite integral in a situation that for her was novel. Our results suggest that students' understandings of multiplication with quantities is understudied in the literature on integration and an important direction for further research.*

Keywords: Calculus, Integration, Product layer, Knowledge-in-Pieces

Introduction

An earlier generation of research on the teaching and learning of calculus focused primarily on limits and differentiation (e.g., Larsen et al., 2017; Rasmussen et al., 2014). Although research on integration goes back several decades (e.g., Orton, 1983), research on the topic has accelerated considerably in the last decade (e.g., Jones & Ely, 2023). We concur with Oehrtman and Simmons' (2023) recent observation that research to date has focused on "students' meanings associated to components and relationships within the standard definition of a limit of Riemann sums" (p. 36). Less research exists on students' moment-to-moment reasoning when constructing integrals to model problem situations. Furthermore, in our reading, existing literature has hinted at challenges students can experience coordinating multiplicative relationships between quantities with understandings of integration. This same literature, however, has stopped short of examining such challenges closely.

In the present study, we analyze in detail the reasoning of Larisa, one college student drawn from a larger study in which we examined how first-semester, calculus-based physics students coordinated reasoning about multiplication with quantities with reasoning about definite integrals. The analysis provides an existence proof that such coordination can be complex and, thus, suggests that inattention to students' reasoning about multiplication with quantities is understudied in the literature on integration and an important direction for further research. We asked the following research questions:

1. *What knowledge resources did Larisa evidence when reasoning about multiplication with quantities?*
2. *How did Larisa coordinate her multiplication resources with understandings of definite integrals to solve what was for her a novel problem?*

The first question asks about resources related to multiplication broadly and independently of integration. The second question avoids situations where Larisa might simply recall previously learned relationships—for instance, that velocity is the integral of acceleration.

Literature Review

Although definite integrals are defined as limits of Riemann sums, researchers have debated the role of these sums in students' reasoning. Jones (2013, 2015a) drew on Sherin's (2001) symbolic forms to classify how nine college students interpretated definite integral notation. He reported three primary forms. For the *perimeter and area* form students interpreted the integrand, the limits of integration, and the x-axis as boundaries of an enclosed region *not* partitioned further or approximated. For the *function matching* form, students interpreted the differential, *dx*, as indicating that the function to be integrated, *f*(*x*), is the derivative of some other function that is yet to be determined. For the *adding up pieces* form, students interpreted the $f(x)dx$ notation as indicating small amounts of a target quantity that are then added. Finally, if one interprets $f(x)dx$ as a product, then one can think of a multiplicatively-based summation (e.g., Jones 2015a). Thus, this last form is well-aligned with the definition of a Riemann sum. (Thompson & Silverman, 2008, developed an alternative approach to integration based on *accumulation from rate*.)

We highlight four themes related to the *adding up pieces* form. First, even after instruction, Riemann sums often remain peripheral to students' understanding of definite integrals (e.g., Jones, 2015a, 2015b; Jones et al., 2017; Rasslan & Tall, 2002). Second, students can be confused by using rectangular areas to model other quantities, such as distances (Thompson et al., 2013). Third, within the traditional $\int_a^b f(x)dx$ notation students can have trouble interpreting $f(x)dx$ as expressing multiplication (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). Sealey provided the most direct result: She decomposed reasoning about definite integrals into five layers—orientation, product, summation, limit, and function—and reported that college calculus students had the most difficulty reasoning about $f(x)dx$ in the product layer. She concluded that students had trouble not with calculations but with "**understanding what is being multiplied together and what quantity is produced from that multiplication**" (p. 240). Fourth, there is debate over the range of situations to which the multiplicatively-based summation and Riemann sum interpretations can apply. Meredith and Marrongelle (2008) claimed that (a) multiplicatively-based summation only makes sense when the quantity being integrated can be conceived of as a rate and (b) thinking about the electric field at a point, q_1, due to a bar charge is better thought of as summation of small effects. We question whether this really is a counter-example: The small effects in this linear situation can be thought of as the accumulation of the rate of field strength per unit of charge, $\frac{k}{r^2}$, multiplied by the charge in small segment of the bar, Δq, a distance *r* from q_1.Thus, conceiving of the quantity being integrated as a rate applies more broadly than previously acknowledged.

The present study builds most directly on Oehrtman and Simmons' (2023) recent discussion of *Emergent Quantitative Models* for definite integrals, which consists in turn of three primary models. A *basic* model is based on constant values for quantities. A *local* model is an adaptation of a basic model to a small portion of an object or event in which quantities may be approximated as constant. A *global model* is derived from the accumulation of local models. These researchers presented examples to demonstrate that student reasoning can involve varied interactions among basic, local, and global models. Examples include making different choices for (a) which factor in a product to treat as infinitesimal and (b) partitioning. At the same time, they stopped short of considering students' ecology of knowledge resources for multiplication.

Theoretical Framework

Our theoretical frame combines mathematical structure (as perceived by experts) with cognitive components evidenced by students. Vergnaud (1983, 1988) analyzed mathematical structures for multiplication with quantities and distinguished two subtypes of multiplication situations. In an *isomorphism-of-measure-spaces* situation (I-O-M) the multiplicative structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). An example is a measure space for time in seconds, a measure space for distances in meters, and the relationship $\frac{meters}{second} \cdot seconds$. In a *product-of-measure-spaces* situation (P-O-M) the multiplicative structure consists of the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). An example is a measure space for area in cm^2, a measure space for height in cm, and the relationship $area\ (cm^2) \cdot height\ (cm)$. The distinction between I-O-M and P-O-M situations highlights two different ways one might interpret $A \bullet B$ as a basic model.

For the cognitive component, we draw from the knowledge-in-pieces epistemological perspective. The perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006) and has been applied to various topics in mathematics, including multiplication (Izsák, 2005; Izsák et al., 2021), functions (e.g., Moschkovich, 1998), and integrals (Jones, 2013). From this perspective, reasoning is supported by diverse, fine-grained knowledge resources and novice knowledge evolves into expert knowledge through processes such as the construction of new knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some resources and loosing connections among others. In the present study, we examined knowledge resources Larisa cued and evidenced when reasoning about multiplication with quantities in the context of definite integrals.

Methods

We conducted one-to-one, semi-structured interviews (e.g., Bernard, 1994; Ginsburg, 1997) to assess how physics students reason about multiplication with quantities and the integral concept in both I-O-M and P-O-M situations. In Spring 2023, we recruited five students enrolled in first-semester, calculus-based physics at a selective university. Each participant completed three, 1-hour interviews spaced a few weeks apart.

The first interview asked students to reason about three situations—two I-O-M situations about estimating distance from varying velocity and one P-O-M situation about estimating volume from varying cross-sectional areas. The tasks were ones for which one could first multiply and then add to construct a total, but there was no mention of integration. We paid close attention to how students assigned units of measurement to quantities when multiplying. The second interview asked students to reason about two P-O-M problem situations (one estimating volume from cross-sectional areas and one about person-hours) and then to interpret the $\int_a^b f(x)dx$ notation in the context of each. We paid close attention to how students reasoned about units and graphical representations when interpreting integrals. The third interview asked students to reason about two I-O-M situations (one about water pressure on wall and one about acceleration from rest) and then to interpret the $\int_a^b f(x)dx$ notation in the context of each. Again, we paid close attention to how students reasoned about units and graphical representations when interpreting integrals.

We recorded the interviews using two cameras, one to capture the student (body movements, hand gestures, etc.) and the interviewer and one to capture close ups of the student's written work. We collected all written work for later analysis at the end of each interview. We used a computer to synchronize and combine the two video recordings file into a single file. In addition, we transcribed the interviews verbatim.

We watched videos for all five students side-by-side with the transcripts. After multiple viewings of the interview data, we narrowed our attention to Larisa and studied her as a particularly good case for our research questions. She provided evidence for several resources related to multiplication with quantities but, at the same time, coordinating those resources with her understandings of definite integrals was not straight forward, particularly when working on the Water Pressure task in Interview 3. We then traced Larisa's spoken language, hand gestures, and drawings in more detail to make sense of her reasoning.

Results

Larisa was majoring in engineering and self-reported good grades (B range) in her prior precalculus and calculus courses and on the calculus AB exam. We summarize a few key points from Interview 1 and Interview 2 before discussing in detail her performance in Interview 3.

In Interview 1, Larisa used the trapezoid method on a velocity graph to estimate distance traveled and focused on computations—for instance, when asked where she saw distance in her work, she canceled units in $\frac{feet}{\cancel{second}} \cdot \cancel{seconds}$. She appeared to understand such cancelation as a formal method for determining units for the product, but she did not connect her symbol manipulation to her graph. She also gave a fluent explanation for how to estimate the volume of liver when given cross-sectional areas in cm^2 and spaced 2 *cm* apart. In particular, she recalled multiplication from the formula "length times width times height gives you volume." In Interview 2, Larisa considered a similar liver problem when given cross-sectional areas in cm^2 and spaced 1/3 *inches* apart. This time, she discussed the integral as giving the area under a curve; but, her explanation for the role of multiplication was imprecise. At one point she commented "if you multiply the x axis by the y axis….you get the whole area under [the graph]." At another point, she commented "I just kind of know you have to write the dx. I kind of, like I haven't done integrals in a while, so I kind of forget why it's exactly that way, but it's like this means integral. So this means area under the curve." Finally, she mentioned "the integral is the opposing, I guess the inverse of the derivative." Thus, in Interview 1 and Interview 2, Larisa articulated two meanings for integrals (areas under curves and anti-derivatives) and recalled how multiplication related quantities in familiar situations; but, she did not provide evidence for interpreting $f(x)dx$ as a product.

For Interview 3, we adapted the Water Pressure task (Figure 1) from prior studies (e.g., Oehrtman & Simmons, 2023; Sealy, 2014). In so doing, we deliberately omitted direct references to multiplication and integration because we wanted to see how students would reason about these spontaneously in what for them was a novel task. We also omitted units for pressure and for force, because we did not want students to rely on formal cancelation of units. Using Oehrtman and Simmons' (2023) terms, the basic model would be *pressure* • *area* = *force*, or $P \bullet A = F$, and the corresponding local model would be $15x \bullet 4\Delta x$. Larisa did not produce a complete correct solution, but the multiplication knowledge resources she cued could play a central role in one or another of at least three different correct solutions. The first solution follows Oehrtman and Simmons' local model and computes $\int_0^3 15x \bullet 4dx = 270$. The second solution considers the average pressure on the wall. Because the pressure varies linearly with respect to depth, the total force on the wall is the pressure at the average depth multiplied by the area of the wall. In this case, 22.5 • 12 = 270. The third solution considers the force per 1-foot width of wall multiplied by the width of the wall in feet. In this case, $\int_0^3 15x dx$ • 4 = 67.5 • 4 = 270.

Pressure, P, applied across a surface area, A, creates a total force, F. Consider a vertical side wall of a tank with a width of 4 feet and a maximum depth of 3 feet (see picture below). Assume that the tank is full of water. The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where x is the depth of the water (the deeper the water, the greater the pressure). Given this information, describe a method for approximating the total force from the water pressure on the wall.

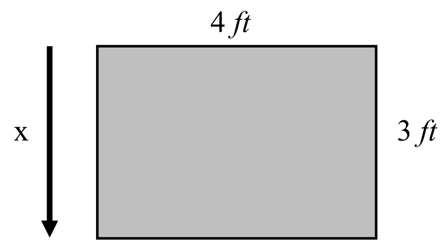

Figure 1. The Water Pressure task

Data

Upon reading the task aloud, Larisa reported that she did not recall working similar tasks in her prior experience. She graphed pressure as a function of depth, shaded the area under her graph, and stated that "taking like the integral of this bottom piece here would probably get you the total force kind of applied throughout" (Figure 2). She then computed the area of the triangle, $\frac{3•45}{2}$ = 45 + $\frac{45}{2}$ = 67.5, and stated she was not sure what units to attach to force. Finally, she computed the area of the wall and multiplied 67.5 • 12, stating that 67.5 was the "average pressure." She concluded with the following explanation for how she saw multiplication in the situation:

Larisa: Some pressure applied across a surface area creates a total force. That sounds like a very English way of saying like *P* times *A* equals *F*. Um, I think I would definitely like think you need to take the average in order to at least have it be an easier problem, because I don't know how you would do it if you were just kind of going like piece-by-piece as the pressure is increasing. That seems difficult and weird.

Analysis

Larisa relied on the text to cue multiplication in a novel situation and articulated an appropriate basic model, *P* • *A* = *F*. She also introduced both holistic and local approaches to the task. She initiated the holistic approach by cuing the notion of an average value, which she incorrectly associated with the integral or area under her graph of the pressure function. She then multiplied what she took to be the average pressure, 67.5, and multiplied by the area of the wall. This reasoning overlapped with the second solution discussed above when she multiplied by 12 and with the third solution discussed above when integrating the pressure function from 0 to 3.

When reasoning that integrating a pressure function would give an average pressure, Larisa did not seem aware that the units for the region under her graph would be different than those for pressure. This complemented the data from Interview 2 that multiplication with quantities was not well-coordinated with reasoning about integrals. At the same time, Larisa's discussion of "going like piece-by-piece" suggested that she gave initial consideration to a solution based on adding up pieces but predicted that this approach would be "difficult and weird."

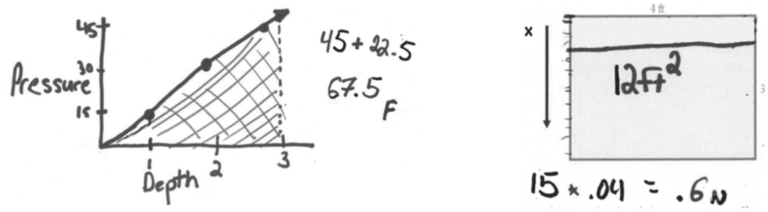

Figure 2. Larisa's work on Water Pressure task (converted to grayscale)

Data

Larisa continued to compare her approach based on averages to an approach based on adding up pieces. On the one hand, she recalled how the meaning of the integral was explained in her past calculus classes: "That's like how they explain integrals in class. They'll explain it as like a ton of really small sections, and then you just add them all up." At the same time, she remained skeptical: "I think if you were to do the force on like one of those really, really, really small sections, it would just be really small, because the surface area would be basically nothing. So you kind of need it to be a larger scale thing." At the interviewer's request, Larisa drew a horizontal line on the wall to indicate what she meant by a "small section" and wrote an appropriate multiplicative expression (i.e., the local model) for the force: $15 * .04 = .6\ N$ (Figure 2). She explained:

> *Larisa*: Like at 1 foot, and it's 0.01 part of a section. So it would be 15, which is the pressure at that depth, and then times .01 times 4, so like .04, and then that would equal .6 Newtons. And that's like just a really small amount. And like you can go smaller than that, and you like, it like just infinitely gets smaller to the point where like it doesn't matter, nobody's going to ask you that question. Because like you're asking about just like the pressure on like a crack in the glass, and that's not, that's not helpful."

Analysis

Larisa clearly recalled from past instruction the adding up pieces interpretation of the definite integral, and she gave a clear explanation for a normatively correct local model where she assumed a constant pressure over a thin, rectangular area. At the same time, even though she knew such reasoning would be encouraged in class, she rejected it in favor of "a larger scale thing." Thus, Larisa's appropriate local model was in tension with her sense of what could contribute to a consequential effect. Part of the issue may have been lack of experience accumulating many small contributions into a larger effect.

Data

An exchange later, the interviewer recalled Larisa's comment about an English way of indicating multiplication and asked for more detail. Larisa responded:

> *Larisa*: Multiplication makes sense because I don't, because addition doesn't because, and I don't know the units, but like in order to get a new unit, it would need to be multiplication or division. Because addition would just be like, you can't add pressure and surface area. It's not like terms. You would need to multiply them to create a new term, which would be force.

She reiterated that "total" cued multiplication and explained that division would be for a "specific point." A few exchanges later, she restated that the area under her graph for the pressure function gave an average pressure and provided another example analogous to her 67.5 • 12 calculation—she considered 1 foot of water depth and multiplied the triangular area under the pressure graph by the area of a 1 *ft*-by-4 *ft* section of wall, 7.5 • 4.

Finally, after working for a while on another task, Larisa returned to the Water Pressure task, wrote "$\int of\ P$", computed $\frac{15x^2}{2}$, and explained that she was trying to see if this would also solve the problem. She evaluated her expression at $x = 3$ and got 67.5 once more.

Analysis

Similar to her work in Interview 1, Larisa demonstrated connections between multiplication and transformation of units when she said, "You would need to multiply them to create a new term, which would be force." We think it likely she was thinking about "new terms" when she performed her 67.5 • 12 and 7.5 • 4 calculations, even if she was unsure which units to use for pressure and for force. At the same time, she did not reference transformation of units when thinking about area under her graph of the pressure function. She characterized this area as the average pressure early on in her work and restated this interpretation after discussing multiplication to create a "new term." Somehow notions of area under a curve did not cue for Larisa a similar transformation of units.

Conclusion

To answer our first research question, Larisa demonstrated a variety of knowledge resources associated with multiplication, including some specific results (e.g., recalled *speed* • *time* = *distance* and *area* • *height* = *volume* relationships) and some general associations with finding totals, transforming units, and taking averages. She used these resources to distinguish multiplication situations from those calling for addition, subtraction, or division. Because each of these multiplication resources could support one or another of the three solutions we mapped for the Water Pressure task, none was inherently correct or incorrect. What mattered was how she employed subsets of the these resources when reasoning. To answer our second research question, Larisa mentioned antiderivatives in passing when working on the Water Pressure task, and discussed antiderivatives to a greater extent when working on further tasks about relationships among distance, velocity, and acceleration. More often, she attended to areas under curves, but did so in different ways. In Interview 1, she clearly broke up areas into trapezoidal pieces and added, but in Interview 3 she did not partition the area under her graph for the pressure function. Instead, motivated by what she perceived to be an efficient solution, she computed integrals as areas of triangles. From our perspective, when interpreting these areas as averages, Larisa combined using the total area of the wall from the second solution we mapped with the force per 1-foot width from the third solution we mapped. Again, using averages was not inherently correct or incorrect. Most striking was her use of multiplication to construct the normatively correct local model which she then rejected as less useful than a holistic approach. Larisa' use of relevant resources in combinations that do quite work and competition between a normatively correct local model with others of her resources underscore the complexity of coordinating reasoning about multiplication with reasoning about integration.

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