Towards Systematically Examining Students' Epistemic Grounds of Developing Mathematical Generalization: A Preliminary Framework

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The practice of generalization is a fundamental aspect of mathematics that requires further investigation, particularly in advanced contexts. This paper has sought to address the research gap by presenting a preliminary framework that captures how and on what basis undergraduate students develop and evaluate mathematical generalization. I illustrate each aspect of the framework using examples from two focal students' generalization of the concept of a group in Abstract Algebra, uncovering important mechanisms and thinking behind their emergent generalizing ideas. The framework has potential to be used more broadly in other mathematical contexts and to inform instructional designs aimed at promoting students' generalization skills. Key takeaways and potential directions for future research are discussed.

Keywords: Student Thinking/Cognition, Abstract Algebra, Generalization

Generalization is a fundamental practice of the mathematics community (e.g., Aigner, 2003; National Council of Teachers of Mathematics, 2014). While a significant amount of research has investigated younger students' generalizing activities (Carraher, Martinez & Schliemann, 2008; Radford, 2006), much remains to be understood about how undergraduate students generalize formal, axiomatic mathematics concepts (Reed, 2018). In addition, despite the fact that frameworks exist for distinguishing different types of generalization (e.g., Radford, 2003) and for characterizing generalization processes (Ellis et al., 2017), many aspects on *how* and *on what basis* students develop and evaluate the logical validity of mathematical generalization are still incomplete. This paper seeks to contribute to the generalization literature by providing a preliminary framework on the epistemic grounds upon which students develop and evaluate mathematical generalization in advanced settings. For the purpose of this paper, I define *epistemic grounds* to be the foundation or basis upon which knowledge and beliefs are justified or supported. Some examples of epistemic grounds include evidence from empirical data, logical arguments, personal experience, and testimony from reliable sources.

Given that Abstract Algebra is a natural extension of arithmetic and generalization naturally manifests itself in its content (Fraleigh, 2003), I intentionally situate my work in this area. Widely acknowledged as challenging (Melhuish, 2015), Abstract Algebra demands a significant leap to formalism and axiomatization that many mathematics majors find intimidating (Huang, 2022). It is my hope that by investigating students' generalization of a fundamental Group-theory concept, this work can contribute to a more solid understanding of generalization and the ways in which students reason about axiomatic constructs in Abstract Algebra and beyond.

Theoretical Background

Studies have drawn attention to how generalization is vital in fundamental areas such as algebraic reasoning (Amit & Neria, 2008; Stacey, 1989), problem solving (Sriraman 2003), and functional thinking (Ellis 2011; Rivera & Becker 2007). Radford's (2003) categorization of factual, contextual, and symbolic generalization reflects the broader trend among researchers to classify forms of generalization. According to Radford (2003), factual generalization is abstracted from "actions undertaken on objects at the concrete level" (p. 47); contextual generalization occurs when students manage to objectify an "operational scheme that acts on abstract — although contextually situated — objects" (p. 54); and symbolic generalization capitalizes on symbols in the generalizing activity. At the undergraduate level, Harel and Tall (1991) categorized three types of generalization: expansive generalization involves expanding the scope of an existing schema without reconstructing it; reconstructive generalization entails reconstructing relevant schema to adapt to the expansion; disjunctive generalization occurs when students create a new, "disjoint" schema to cope with new problems (Harel & Tall, 1991, p. 39).

In contrast with the common approach of evaluating students' generalization with respect to certain predetermined benchmarks (e.g., see above), Ellis (2007) proposed to understand generalization from an actor-oriented perspective. This perspective foregrounds actions that students take in the generalization process and views these actions from the perspective of the students (Reed, 2018). According to Ellis (2007), there are three major categories of generalizing actions: relating, searching, and extending. Relating involves recognizing and creating mathematical relationships between objects or situations; searching is an active investigation of whether certain mathematical properties persist across different cases; and extending entails expanding a "pattern or relationship into a more general structure" (Ellis, 2007, p. 241). While Ellis' (2007) taxonomy originated from observations of middle-school students' generalizing activities, it has been applied to the undergraduate level (e.g., Reed & Lockwood, 2018) and it lays the foundation for the Relating, Forming, Extending (R-F-E) framework (Ellis et al., 2017).

The R-F-E framework draws from both cognitive and sociocultural research to create a more fine-grained model of students' generalizing activities (Reed, 2018). This framework distinguishes inter-contextual generalization from intra-contextual generalization, submitting that the former involves establishing similarities across situations and the latter entails forming and extending similarities within one task (Ellis et al., 2017, p. 680). As the primary form of inter-contextual generalization, *relating* centers on developing meaningful mathematical relations and it can be further divided into three subcategories: relating situations, relating ideas or strategies, and recursive embedding (Ellis et al., 2017). As a major form of intra-contextual generalization, *forming* is about perceiving regularities across one's own activity and it consists of four subcategories: associating objects, searching for similarities, isolating constancy, and identifying a regularity. *Extending* is another form of intra-contextual generalization, which involves applying established patterns and regularities (e.g., those from relating or forming) to novel contexts. According to Ellis et al. (2017), extending includes continuing an existing pattern beyond familiar contexts, operating on an identified relationship, transforming a generalization, and removing particular details so as to express an identified regularity more generally.

While the above perspectives and frameworks offer insights into characterizing generalization, they fall short in addressing the underlying mechanisms that drive students' generalization process and a majority of the existing frameworks are too broad to meaningfully inform instruction. Indeed, Radford's (2003) and Harel and Tall's (1991) works are valuable in distinguishing different forms of generalization, but their broad categorizations obscure students' actions and fail to shed light on the essential skills or practices required of developing robust mathematical generalization. Ellis' (2007) taxonomy foregrounds students' generalizing actions and the later R-F-E framework (Ellis et. al., 2017) provides fine-grained analysis; yet they do not capture the details or nuances in students' reasoning nor do they provide insights into why or on what basis students spontaneously think or behave in a certain way in the generalization process.

A Preliminary Framework on the Epistemic Grounds of Mathematical Generalization

Through extensive micro-level analysis of undergraduate students' generalization of the concept of a Group in Abstract Algebra, I developed a preliminary framework that aimed to capture how and on what basis students developed and evaluated their emergent generalizing ideas. The framework was created in the context of design-based research (Bakker, 2018), where I iteratively conducted four design-enactment-reflection-refinement cycles (Lewis et al., 2020) with volunteer participants recruited among undergraduates at UC Berkeley. I integrated Ellis' (2007) actor-oriented perspective with my observer-researcher lens in the data analysis process: I carefully analyzed video-recordings of the students' generalization, envisioned myself in the students' positions, and made informed hypotheses about how and why the students used certain strategies to develop or justify their emergent ideas. The resulting hypotheses became the initial draft of the framework shown in Figure 1, which was later revised according to results from a bottom-up approach in analyzing the video-recordings as well as students' written works. The fact that I have similar undergraduate mathematics experience as those of my research participants (as I earned a B.S. in Mathematics from a similar institution) and a relative expertise in the content (as demonstrated by my recent A in a graduate-level Abstract Algebra course) positions my data analysis and the creation of the framework as accounting for both students' and experts' perspectives. More information about the design and methods can be found in Huang (2022). For the sake of consistency and due to space constraints, I will illustrate the proposed framework using examples from two focal undergraduate participants only. Eden and Jane (pseudonym) participated in this study as a pair for two hours in 2022, and I had been a teaching assistant for their Calculus course for 13 weeks prior to their participation.

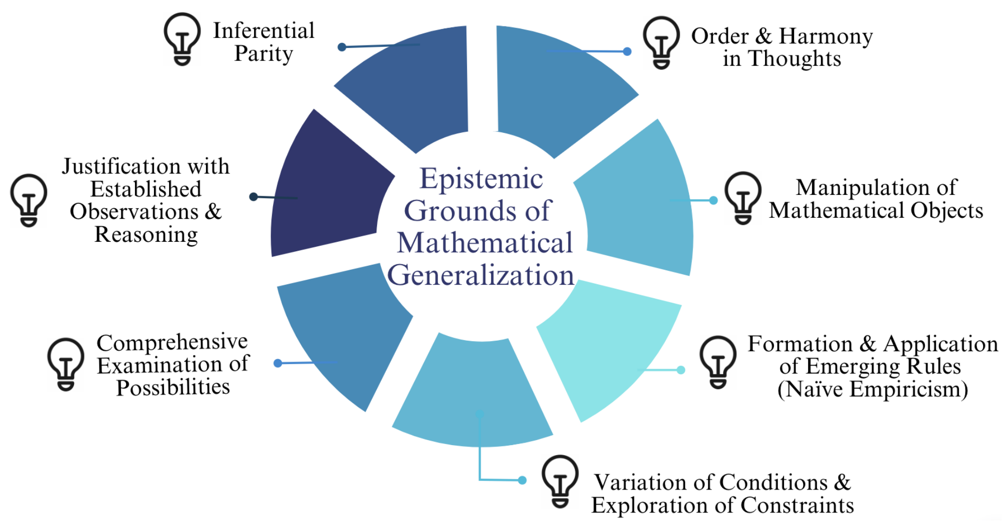

Figure 1. A preliminary framework on the epistemic grounds of developing mathematical generalization

Order and Harmony in Thoughts. This fundamental ground for developing generalization is closely related to the notion of equilibrium in Gestalt psychology (Koffka, 2013) and the common attribution of understanding to structured, harmonious mental organization (e.g., Sierpinska, 1994). According to Piaget (1976), the equilibrium of cognitive structures can be achieved through mental operations of assimilation and accommodation, which will then lead to order and harmony in people's minds. In a similar vein, Sierpinska (1994) proposes that understanding entails "finding a unifying principle, a relation that 'founds' what we want to understand" (p. 33), and that order and harmony as well as the feeling that "it fits" constitute the "most obvious criterion" of understanding (p. 32). In the process of zooming in and out of different algebraic systems and formulating similarities across different cases, Eden and Jane bounced ideas off of each other and made multiple utterances that conveyed similar meanings as

“Oh, it makes sense”. I believe these utterances signify moments when they felt a sense of order, harmony, and coherence in their thoughts, and thus accepted their emergent mathematical ideas.

Manipulation of Mathematical Objects. This epistemic ground of generalization centers on an important epistemological practice of generating and validating mathematical knowledge by means of *doing* mathematics. The (manual and mental) manipulation of mathematical objects can involve a variety of techniques and tools, including physically manipulating artifacts, performing algebraic operations, constructing geometric figures, visualizing abstract properties, applying analytic techniques, and many more. Overall, the manipulation of mathematical objects is a crucial epistemic ground for developing generalization, as it allows students to generate and justify new insights, techniques, and results, providing a solid foundation for students to explore mathematical structures and relationships. It is important to note that besides attending to students’ “mental math”, I intentionally leveraged the embodied aspect of mathematical manipulation (which is often overlooked in undergraduate math classrooms). More specifically, I provided physical artifacts to invite rotation and reflection of geometric figures as a means to explore example Groups (e.g., Dihedral Groups), and scaffolded with semiotic tools such as tables to support students’ reflexive abstraction (Vygotsky, 1978) of defining Group properties.

Formation & Application of Emerging Rules (Naïve Empiricism). This epistemic ground for developing generalization involves discerning an emerging mathematical rule from a limited number of empirical cases and utilizing it without rigorous justification. Historically, humans have made extensive use of this approach, due to various constraints such as limited access to data, insufficient analytical tools, and cost and time constraints (Gibson, & Papafragou, 2016). In the context of abstracting properties of multiplication in number systems (and later operations across example Groups), Eden and Jane concluded that “for three numbers in multiplication, order doesn’t matter [or more precisely, it is associative]” after randomly picking some set of integers and checking that $(3 \times 4) \times 5 = 3 \times (4 \times 5)$ and $(7 \times 9) \times 100 = 7 \times (9 \times 100)$. As another example, Eden used $2 \heartsuit 6 = 5$ (marked blue in Fig. 2) to challenge Jane’s result of $6 \heartsuit 2 = 4$ (marked red in Fig. 2) in the Symmetry Group of an equilateral triangle, based on their established observation that “order doesn’t matter [or more precisely, operation is commutative]” for previous example Groups. But making $6 \heartsuit 2 = 6 \heartsuit 2 = 5$ raised another problem, as Jane quickly noticed that there was already a 5 in the same row of the operation table (marked green in Fig. 2). Jane’s refutation was supported by their agreed-upon observation that each element appears exactly once in each row and each column of an operation table. Note that

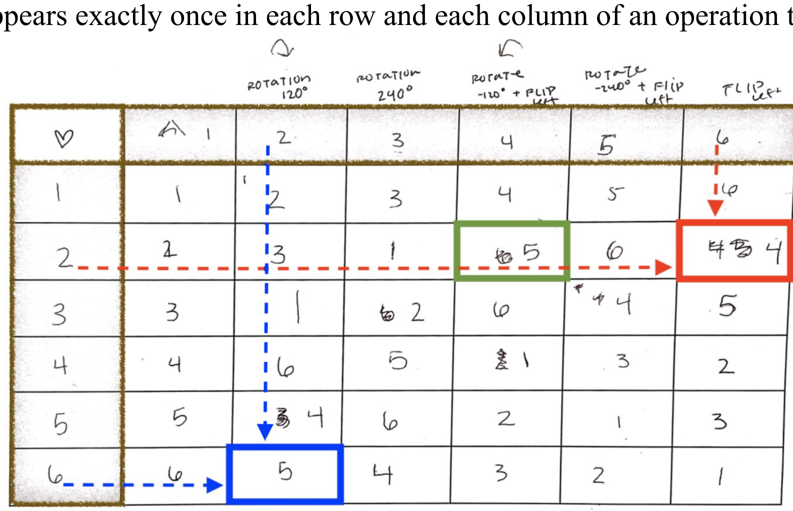

Figure 2. Formation and application of emerging rules: the case of 6♡2

the three emerging rules (i.e., multiplication is associative, commutativity is a defining similarity, operation table satisfies the "Sudoku property") played a crucial role in Eden and Jane's generalization of a Group, although they made no attempts to justify or refine these rules. Since students tend to search for similarities across their mathematical activities and generalize inductively from concrete cases, this epistemic ground is closely related to the idea of forming in the R-F-E framework (Ellis et al., 2017), Radford's (2003) notion of factual generalization, as well as Pólya's (1945) heuristics of "consider special cases" and "look for patterns". Despite its common usage, however, relying on naïve empiricism can lead to over-generalization or underestimation of the underlying mathematical structures, resulting in incorrect conclusions (Bollen et al., 2015). Thus, more sophisticated approaches (e.g., see below) are recommended to extend empirical findings to more generalizable and robust mathematical principles.

Variation of Conditions & Exploration of Constraints. This epistemic ground foregrounds an essential aspect of developing mathematical generalization, that of exploring conditions under which a generalization holds true and identifying any constraints that may apply. This is similar to Pólya's (1945) "variation of problems" heuristic in that both involve changing conditions; yet Pólya's (1945) heuristic focuses on generating new problems to help solve the original problem, while this epistemic ground aims to support the creation of more precise, powerful generalization by specifying its scope and constraints. During Eden and Jane's investigation of "order" (a student-generated vocabulary that bootstrapped their later differentiation between commutativity and associativity), I as their researcher-tutor probed: "We agree that 'order doesn't matter' for two elements in the first few examples, but what about three elements? Will adding parentheses in any way we want affect the result of operations?" In a swift response, Eden bounced back with an unanticipated question: "Does it have to be a set of three elements?" In retrospect, I realize that Eden's question was intended to vary the condition and facilitate an exploration of relevant constraints within their domain of investigation. After knowing that they could certainly investigate more than three elements at the same time (which broadened the scope of my original question), Eden and Jane went on to test operations with three and then four elements, creating a richer foundation for them to unpack underlying principles (e.g., associativity for three elements implies associativity for more elements) and generalize defining properties for Groups. Note that by varying conditions and exploring constraints, students not only generalize actions taken for specific numbers but also abstract the actions themselves. This suggests that they move beyond the realm of concrete examples and consider a broader set of possibilities; in the words of Radford (2003), students are engaged in contextual (and possibly symbolic) generalization.

Comprehensive Examination of Possibilities. In developing generalization, this epistemic ground concerns rigorously examining different situations as a means to assess the validity and soundness of an emerging generalizing idea, which is in mutual reinforcement with the previous epistemic ground. For example, when Eden and Jane tried to test whether "order matters" (or more precisely, whether associativity holds) for more than three numbers in multiplication, they attempted to exhaust all possible ways of adding parentheses in their randomly chosen expression $4 \times 2 \times 3 \times 5$. As a summary, they examined $(4 \times 2) \times 3 \times 5$, $4 \times (2 \times 3) \times 5$, and $4 \times 2 \times (3 \times 5)$, which all turned out to be 120. It is important to note that the first attempt $(4 \times 2) \times 3 \times 5$ was equivalent to $(4 \times 2 \times 3) \times 5$ and that those *were* all the possibilities they could thought of at the moment, despite the fact that advanced thinkers may find another possibility of $4 \times (2 \times 3 \times 5)$ left untested. Through comprehensive analysis of all possible cases, students can identify patterns and connections that may not be immediately apparent, contributing to their more robust understanding of the underlying principles. Since both *comprehensive examination of possibilities* and *variation of conditions & exploration of constraints* demand active investigation of whether certain emerging mathematical properties persist across different situations, the two epistemic grounds are inseparably linked to the idea of searching in Ellis' (2007) taxonomy. Moreover, as students often use logic to systematically exhaust all possibilities, it naturally leads to the next epistemic ground.

Justification with Established Observations & Reasoning. This epistemic ground highlights the importance of justification and proofs in developing mathematical generalization (or more generally in *any* mathematical activity). With the help of existing observations and logical reasoning, mathematics students learn to justify conjectures by showing that it is a logical consequence of certain principles and assumptions. As a standard practice in mathematics, justification involves higher-order thinking skills such as deductive reasoning, analyzing, and critical thinking, providing a rigorous and reliable way to establish the truth of mathematical statements. An example of this could be found in Eden and Jane's scrutiny of "order" in the modulo 5 system under addition. Knowing that "order doesn't matter" when they add integers, Jane argued that "order doesn't matter" in the modulo 5 system either because "You can mod 5 with like any number, and if you are gonna be adding something to it, order does not matter." Jane managed to extend her understanding of the set of integers to the new context of the modulo 5 system, and it is clear that her use of logic was mathematically correct. Note that Eden and Jane had not successfully distinguished commutativity and associativity at this point; the reasoning was valid regardless of the specific "order" they referred to. This example illustrates how justification is closely intertwined with the notion of extending in the R-F-E framework (Ellis et al., 2017), as it operates to establish a universal structure that transcends specific cases, personal viewpoints, and historical eras. Moreover, this epistemic ground bears resemblances to Pólya's (1945) heuristics of "using direct reasoning" as well as Radford's (2003) contextual and symbolic generalization in that all four prioritize logical reasoning in mathematical pursuits.

Inferential Parity. This epistemic ground is taken from Abrahamson (2014) to allow for a more detailed elaboration on how students reconcile their intuitive generalization with formal mathematics concepts. According to Abrahamson, inferential parity refers to the idea that students perceive a formal structure as "bearing the same information about the target property of the source phenomenon as their informal judgment" (2014, p. 13). Through the lens of inferential parity, researchers can capture moments when students recognize a connection between their informal knowledge and the established mathematical ideas, shedding light on how students utilize their intuitive generalization to navigate through formal mathematics. An illustrative example could be found when Eden and Jane compared their generalized concept with the formal definition of a Group. "A Group is an ordered pair where G is a set and ★ is a binary operation on G," read Eden, "that's kind of what we have here, right?" Jane nodded, pointing to the first row of their similarity table (marked blue in Fig. 3). Moments later, they skimmed through the defining properties of identity and inverse in the formal definition, prompting Eden to exclaimed: "Ohhh! That's cool! We are doing that!" They stared at their self-generated terminologies of "prime" (see Fig. 3, which bears the same meaning as identity or "e") and "undoing #s" (see Fig. 3, which bears the same meaning as inverse or " a^{-1} "), turned to each other, and then laughed. In the meantime, Jane copied "e" and " a^{-1} " from the formal definition to the left of their corresponding terminologies (marked red in Fig. 3). A later reflective interview revealed that the experience of struggling through unfamiliar mathematics (with support) and eventually uncovering how their intuitive generalization connected to (or more precisely, encapsulated) the formal, textbook definition was a valuable learning experience, leading to Eden and Jane's renewed sense of agency and ownership in their mathematical learning.

	Systems / Properties	$(\mathbb{Z}, +)$	$(\mathbb{Q}, \times)$	$(\mathbb{Z}/5\mathbb{Z}, +)$ "Set & Operation"	~~☆~~ (☆, ♡) # operation
'e' ☆	"PRIME" element that will not affect results	"Identity" 0	1	0	1
a^{-1} ☆	Undoing #'s	"Inverse" (−)	$\frac{1}{x}$	There's always a # that you add, you can use ~~#~~ to undo it	Sometimes undoing element is the # itself, sometimes it's a different #. (always has one!)
	Common order	order doesn't matter for these operations	→	→	yes, order matters (the order of #'s matter) But the order of parenthesis does not.

Figure 3: Inferential parity: connections between the formal and the informal

Concluding Remarks

While a fair amount of research has investigated younger students' generalizing activities, existing frameworks and perspectives often fall short in more advanced contexts and they fail to address the specific grounds upon which students develop mathematical generalization. This paper has sought to address the research gap by presenting a preliminary framework that captures *how* and *on what basis* students develop and evaluate their emergent generalizing ideas at the undergraduate level. The seven epistemic grounds highlighted by the framework are: Order & Harmony in Thoughts, Manipulation of Mathematical Objects, Formation & Application of Emerging Rules (Naïve Empiricism), Variation of Conditions & Exploration of Constraints, Comprehensive Examination of Possibilities, Justification with Established Observations & Reasoning, and Inferential Parity. I provided illustrations of each epistemic ground with examples from two undergraduate students' generalization of the concept of a Group in Abstract Algebra. It is important to note that this paper not only uncovers important mechanisms and thinking behind students' emergent generalizing ideas in Abstract Algebra, but also develops theoretical constructs that characterize students' generalization of axiomatic mathematics concepts and that hold potential to be applied more broadly to other mathematical contexts.

In addition to the framework presented in this paper, there are several key takeaways and potential directions for future research. First, while the proposed framework is grounded in literature and my observations in students' generalization of an Abstract Algebra concept, it is preliminary in nature and it requires refinement and empirical validation from other advanced content areas (e.g., Real Analysis). It is hoped that this paper can inspire further investigation into mathematical generalizations that are of a formal, axiomatic nature. Second, I do not claim that the identified epistemic grounds are mutually exclusive; further research is needed to better understand the interrelations among them for different students and different contexts. Third, the framework identifies important generalization-related mechanisms that could potentially inform or even guide instructional designs; future research could explore the framework's effectiveness in facilitating students' critical reflection upon and the development of their generalization-related skills and practices. Finally, it is worth emphasizing that mathematical generalization is a complex and multifaceted practice that demands students to engage in knowledge integration, problem solving, and critical thinking. By enhancing our understanding of the epistemic grounds that underpin this practice, we can better support students in developing the essential skills and knowledge needed for success in (both formal and informal) mathematics and beyond.

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