Conceptualizing Students' Ways of Understanding Bijections in Combinatorics

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Pairing and one-to-one correspondences are natural activities, which can be formalized using bijections. Despite bijections being central to mathematics, few studies have explicitly attended to how students construct, understand, and use them. Because bijections show up naturally in combinatorial arguments, I situate this work in combinatorics. In this theoretical paper, I propose three ways of understanding bijections in combinatorics—relabeling, representational, and relational bijections—and discuss possible affordances and constraints of each. I also provide combinatorial examples using binomial coefficients to exemplify each way of understanding.

Keywords: Combinatorics, bijection, ways of understanding

In elementary combinatorics there are two main proof techniques: double counting and constructing a bijection or correspondence (Benjamin & Quinn, 2003; Mazur, 2010). While there has been work done with counting arguments (Lockwood, 2013; Lockwood et al., 2015; Lockwood & Purdy, 2019; Lockwood & Reed, 2020) and double counting proofs (Engelke Infante & CadwalladerOlsker, 2011; Engelke & CadwalladerOlsker, 2010; Erickson & Lockwood, 2021; Lockwood et al., 2021), there has been little work done with bijections in combinatorics. In this theoretical paper, I define a bijective way of thinking and introduce three ways of understanding (Harel, 2008a) bijections in combinatorics: *relabeling bijections*, *representational bijections*, and *relational bijections*. For each of these, I provide one or more examples to illustrate the way of understanding and discuss the potential affordances and constraints.

Literature Review

In early childhood, one-to-one correspondences play an important role in children learning number concepts such as the cardinality principle and exact number. Mix (2002) found that children 12-38 months can experience and use one-to-one correspondences in a wide range of contexts and that certain types of correspondences may serve as bridging activities for students to develop numerical equivalence. For example, pairing objects that have some conceptual relationship, such as frogs and lily pads, serve as cues of equinumerosity for young children (Muldoon et al., 2005). Different types of correspondences may foster different skills. For example, item-to-item correspondences help with set pairing, whereas word-to-item correspondences help children with cardinality (Muldoon et al., 2009). However, just because children can understand cardinality does not mean they understand exact number, i.e., cardinality is an invariant for equinumerous sets (Muldoon et al., 2009). In fact, the relationship between exact number, cardinality, and one-to-one correspondences is complex (e.g., Izard et al., 2014; Sarnecka & Carey, 2008; Sarnecka & Gelman, 2004; Sarnecka & Wright, 2013).

As students progress in mathematics, one-to-one correspondences are formalized as the study of bijections, which are important for discussing cardinality of infinite sets, defining equivalent mathematical objects via isomorphism, and in making combinatorial arguments. In regards to infinite sets, research has found that students experience paradoxes related to arithmetic with infinite sets (Mamolo, 2014), which may arise because students rely on their knowledge of finite sets to reason about infinite sets (Fischbein et al., 1981; Tirosh, 1991). Moreover, students can often hold multiple conflicting criteria for comparing infinite sets, including bijection, single infinity, and subset inclusion (Tsamir, 1999). In fact, the presentation of two infinite sets can affect whether students reason using bijections or some other criterion (Tirosh & Tsamir, 1996; Tsamir & Tirosh, 1992). Even when students use bijections as a comparison method, they often rely on informal arguments rather can constructing explicit bijections (Hamza & O'Shea, 2011), which may be because students are limited in the bijections they can construct based on the functions they know (Tsamir & Dreyfus, 2005).

In combinatorial contexts, the research show mixed results related to students' bijective reasoning. Muter and Uptegrove (2011) found that a group of high school students could recognize and connect different combinatorial problems, such as pizza toppings and block towers, by constructing bijections, whereas Mamona-Downs and Downs (2004) found that students did not construct bijective solution unless supported in this approach, and even then "students did not make a clear distinction between the bijection itself and the result it was justifying (i.e., the number of elements of the two sets corresponded are equal)" (p. 244).

One-to-one correspondences are a natural object of study that are central to young children's development of number and play an important role in comparing infinite sets. Furthermore, bijections are central to discussing equivalent mathematical objects because an isomorphism (of graphs, groups, rings, topological spaces, etc.) is a structure-preserving bijection. I hypothesize that studying bijections in combinatorics, where the sets are typically discrete and easily visualized or listed, may provide students a space to understand and reason with bijections, injections, and surjections that could be drawn on and leveraged in other mathematical domains.

Theoretical Perspective

In this theoretical paper, I draw on Harel's dual constructs *ways of understanding* and *ways of thinking* (Harel, 2008a, 2008b). According to Harel, "a person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's *way of understanding* associated with that mental act" (Harel, 2008b, p. 490, emphasis original). For example, a solution is a product of the problem-solving mental act, and a particular proof of an assertion is a product of the proving mental act. Thus, a particular solution or proof is a way of understanding with respect to the corresponding mental act. Instead of focusing on the product of a single mental act, one can also look across ways of understanding to identify a way of thinking: "Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a *way of thinking* associated with that mental act" (Harel, 2008b, p.490, emphasis original). Sticking with the previous example, problem-solving strategies and proof schemes (Harel & Sowder, 1998) are examples of ways of thinking associated with the mental acts of problem solving and proving, respectively. These constructs are central to Harel's Duality-Necessity-Repeated Reasoning (DNR) instructional framework, and the Duality Principle notes that ways of understanding and ways of thinking are interdependent. As students engage in a mental act, they may use a way of understanding multiple times, which can contribute to the development of a way of thinking. Similarly, they may draw on a way of thinking to understand a particular idea or problem.

Researchers have used Harel's ways of thinking and ways of understanding to make sense of students' counting activity in enumerative combinatorics. Lockwood has repeatedly concluded that attending to sets of outcomes is an important way of thinking (Lockwood, 2014; Lockwood et al., 2017; Lockwood & Gibson, 2016). She calls this way of thinking a set-oriented perspective, which she has defined as "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). In her dissertation, Halani (2013) identified three broad categories of ways of thinking about combinatorial solution sets. Lockwood and Reed (2020) defined an equivalence way of thinking, which allows a student to frame "division as a way of accounting for duplicate outcomes, allowing a counter to identify such duplicates as belonging to the same equivalence class" (p. 4). This paper extends this work into bijective combinatorics.

Three Ways of Understanding Bijections in Combinatorics

Here, I will define and discuss three ways of understanding bijections in a combinatorial context: relabeling, representational, and relational. For each, I will present one or more combinatorial examples and discuss the potential affordances and constraints of thinking about a bijection from the perspective of each way of understanding. These three ways of understanding are associated with a bijective way of thinking, which is marked by a student conceptualizing two sets (possibly the same) and imagining how the elements of one set can be transformed or mapped to elements of the other set via a bijection.

Relabeling Bijections

Suppose two students are asked to solve a coin flipping problem such as the following: "If a coin is flipped five times in a row, what are the possible outcomes?". Student A solves the problem by denoting the possibilities of a single flip, using "H" for heads and "T" for tails. Student B, in contrast, decides to denote heads as "1" and tails as "0". Each student lists the 32 possible outcomes, and the question arises as to whether their answers are the same. The two solutions can be related to each other using a bijection that changes "H" to "1" and "T" to "0" and vice versa, and thus the students realize they have the same answer differing only by notation. This is an example of a *relabeling bijection*, which is a bijection that connects two sets by changing the surface features of those sets, typically by changing notation.

Relabeling bijections can serve as a starting point for students to recognize situations as similar, which can be leveraged to connect different representations. This way of understanding bijection may serve as a starting point into an intuitive understanding of injectivity, surjectivity, and invertibility, since the function is relatively simple. Additionally, relabeling bijections may be a way to begin to discuss permutations as a bijection from a set to itself. However, if a student only understands bijections as relabeling, it may difficult for them to construct a bijection between two sets whose elements differ in more substantive ways, such as subsets and lattice paths (discussed below).

Representational Bijections

In combinatorics, students often encounter or are introduced to a variety of different problems whose solutions are the same. As an example, let A be the set of k-element subsets of an n-element set, let B be the set of binary strings of length n with exactly k ones, and let C be the set of north-east lattice paths[1] from (0, 0) to (k, $n - k$) (see Figure 1). The cardinality of each set is $\binom{n}{k}$. This can be shown by starting with the standard combinatorial definition of $\binom{n}{k}$ as counting the number of k-element subsets of an n-element set. Thus, by definition, $|A| = \binom{n}{k}$. To also see that sets B and C have the same cardinality, I will construct two bijections.

The first bijection is between sets A and B. A k-element subset of A can be mapped to a binary string of B by labelling the n positions of the binary string 1 through n and placing a 1 in

[1] A north-east lattice path is a path on an integer lattice involving the steps (0, 1) and (1, 0).

the i^{th} position of the binary string if the i^{th} element is in the subset and a 0 otherwise. So, if n = 6 and k = 3, the subset $\{2, 4, 5\}$ will map to the binary string 010110 and the subset $\{1,2,3\}$ will map to 111000. Similarly, a given binary string can be uniquely mapped back to a subset by inverting the process, i.e., 110110 maps to $\{1, 2, 4, 5\}$ and 000100 maps to $\{4\}$. Since bijections preserve cardinality, $|B| = \binom{n}{k}$.

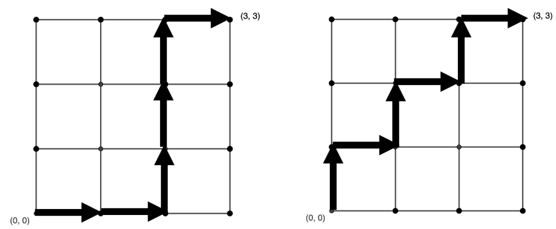

Figure 1. Two examples of a north-east lattice path from (0,0) to (3,3).

Finally, I will construct a bijection between sets *B* and *C*. Any binary string of *B* can be relabeled by changing each 1 to an "E" and each 0 to an "N," which can be turned into a lattice path by reading left to right and taking the step (0, 1) when there is an "E" and (1, 0) when there is an "N." For concreteness, I will continue to let n = 6 and k = 3. Consider the binary strings 110001 and 010101. These map to the binary words EENNNE and NENENE, which are exactly the two lattice paths shown in Figure 1. This mapping can be reversed by following the steps in reverse.

These two bijections are examples of what I call *representational bijections*. A representational bijection is a bijection between two sets represented in mathematically different ways that may afford different operations and ways of reasoning. In our example above, the elements of set *A* are subsets, which can be operated on naturally using intersections, unions, complements, and symmetric differences. In contrast, the elements of set *C* are lattice paths, which are geometric objects that can be reflected, rotated, and assigned an area (such as the area between the lattice path and the x-axis). It is not clear what the area of a subset would be, nor how to take the symmetric difference of a lattice path; however, it may be possible to interpret these through a representational bijection.

This way of understanding bijections enables students to connect different representations, which may allow them to better understand what a combinatorial expression such as $\binom{n}{k}$ counts. Additionally, having a range of contexts in which to interpret combinatorial expressions afford students choosing a representation with which they are comfortable working, which could support them in proving combinatorial identities (Lockwood et al., 2021). In the examples above, I have demonstrated bijections, and thus I know that every set can be counted by $\binom{n}{k}$. However, that does not answer the more fundamental question, which is, if I encounter a new context or representation, how do I recognize it as $\binom{n}{k}$? By examining the examples above, one can look for similarities and differences. In each case, there are n slots to be filled and two choices to be made:

1. **Set *A*:** To construct a subset of *A*, one decides whether to include the first element, the second element, and so forth. In order to get a subset with k elements, one must choose which k elements are included, at which point the $n - k$ excluded elements are determined.

2. **Set *B*:** To construct a binary string, one decides where to put a 0 or a 1. To ensure the binary string is an element of *B*, one has to choose *k* places to put a 1, which forces where a 0 can be put.
3. **Set *C*:** At each step of the lattice path, a choice to go east or north is made. The given constrains of the lattice (starting at (0, 0) and ending at (*k*, n – k)) imposes the restriction that *k* norths must be chosen, which will determine where the east steps must be.

In each of these cases, one decides between two choices (include or not include, 1 or 0, and east or north), and once *k* choices of one type have been made, the remaining *n* – *k* choices must be of the other type. This affords a student being able to look for similar reasoning in other problems and contexts, which they connect to this counting process (Lockwood, 2013), and by extension, to $\binom{n}{k}$.

When a combinatorial expression, such as $\binom{n}{k}$, is examined across the multiple contexts and representations, one can gain insight into the underlying counting process. However, this does not immediately afford an understanding of how a set counted by $\binom{n}{k}$ can be structured: Is there some recursion that determines the set? Does the set exhibit some symmetry? Can the set be restructured and counted according to some property? Being able to understand how a set can be structured, partitioned, refined, and counted is important for proving combinatorial identities, a student with a representational way of understanding bijections may overlook these finer processes.

Relational bijections

One goal of combinatorics is to understand the structure of the sets counted by a combinatorial sequence. As mentioned above, $\binom{n}{k}$ counts subsets, binary strings, and lattice paths; however, what properties do these objects have and how are they related to the binomial coefficients that count them? To be more concrete, I will discuss two binomial identities and highlight how each identity captures some information about the structure of sets counted by $\binom{n}{k}$.

The first identity is $\binom{n}{k} = \binom{n}{n-k}$. If I take the left-hand and right-hand side to be the cardinality of two sets, then this identity suggests a symmetry among the sets. To illustrate this, let us interpret these coefficients as counting the *k*-elements subsets of an *n*-element set and as counting the (*n* – *k*)-element subsets of the same *n*-element set. Taking the set complement (with respect to the *n*-element set) defines a bijection between the two sets, because the complement of *k*-element subset is an (*n* – *k*)-element subset and vice versa. What is this bijection really saying about objects counted by $\binom{n}{k}$? It indicates that if any *k* elements are chosen, then the other *n* – *k* element are determined. Thus, I could either count which *k* elements are included in a subset or count which *n* – *k* elements are excluded from the subset, which indicates a type of symmetry of $\binom{n}{k}$.

As a second example, consider the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Once again, I interpret these in the contexts of subsets in order to describe a bijection. The left-hand side counts the number of *k*-element subsets of an *n*-element set while the right-hand side counts the number of subsets with either *k* or *k* – 1 elements chosen from an *n* – 1 element set. Given a *k*-element subset of *n*-elements, it either contains the n^{th} element, or it does not. If it does not contain the n^{th} element, then it is already a *k*-element subset of an (*n* – 1)-element set and can be mapped to itself. Otherwise, it contains the n^{th} element and can be mapped to a set counted by $\binom{n-1}{k-1}$ by removing the n^{th} element. This process is reversible and thus defines a bijection. In this case, this bijection illustrates a recursive structure of the sets counted by $\binom{n}{k}$, since each element of this set can be built from an element of the set counted by $\binom{n-1}{k}$ or by $\binom{n-1}{k-1}$ as described in the bijection.

Each of these examples illustrate a *relational bijection*. Understanding a relational bijection entails thinking not only about the sets being counted but also about a property of the sets being counted, which can be related to some other property of the sets. This provides additional information about how sets counted by combinatorial expressions, such as sets counted by $\binom{n}{k}$, are structured. In the first example, the bijection provided information about the symmetry between the subsets of an n-element set, i.e., one can either focus on which elements the set contains or which elements the set does not contain. In the second example, the bijection describes a systematic process for building new subsets out of old subsets; thereby, it defines a two-variable recurrence for binomial coefficients.

Conceiving of bijections in this way pushes students to go beyond pairing two sets and arguing they are equinumerous. It pushes students to examine what properties the bijection may be preserving or what structure the bijection is hinting at, e.g., a partitioned structure, a refinement, or a recursive structure. All of these properties are related to size and cardinality, since that is what a bijection is guaranteed to preserve. However, this way of understanding may provide a chance for students to build intuition about preserving structure and connecting (counting) processes, which could be leveraged in future discussions about isomorphisms of rings, fields, and groups, homeomorphisms of topological spaces, and isometries in geometry.

One constraint with this way of understanding relates to the difficulty of constructing these types of bijections. A priori, it is not clear what property of the sets one should be exploring and trying to connect, and thus it takes time to explore and understand the sets.

Discussion

These three ways of understanding bijection are not mutually exclusive; it is possible for a student to simultaneously understand a bijection in two or more of these ways. To illustrate, I return to the identity $\binom{n}{k} = \binom{n}{n-k}$. Instead of interpreting the binomial coefficients as counting subsets, I could interpret them as counting binary strings. Then, $\binom{n}{k}$ counts the number of binary strings of length n with k ones, and $\binom{n}{n-k}$ counts the number of binary strings of length n with $n - k$ ones. One can form a bijection between these two sets by interchanging zeros and ones. Certainly, this bijection can still be understood as relational for the reasons noted above; however, a student may not attend to a counting process and instead see this bijection as relabeling. Alternatively, a student could interpret $\binom{n}{k}$ as counting binary strings and $\binom{n}{n-k}$ as counting k-element subsets. In this case, a bijection between the two sets may be interpreted as (1) relational if the student understands how the bijection is relating the structuring of the two sets; (2) representational if the student sees subsets and binary strings as affording different mathematical ideas and approaches; and (3) relabeling if the student has internalized the relationship between binary strings and subsets and views the bijection as changing notation from binary strings to a more familiar context of subsets (or vice versa). Thus, it is important to attend to how the student is reasoning about the bijection in order to understand what way of understanding the student may have.

I see these ways of understanding as hierarchical in the sense that there is a natural progression from relabeling to representational to relational. Relabeling bijections are conceptually easier for students to construct and reason about since they usually involve notational changes, which can be written down explicitly and properties like injectivity and surjectivity are easier to check. Representational bijections can be more challenging because different sets of objects have different properties, and it is not immediately obvious what the relationship between two sets of objects is, if any. Thus, an instructor may leverage relabeling bijections to begin a discussion of representational bijection. For example, a bijection between lattice paths and binary strings can be thought of as a relabeling bijection where each east step is relabeled as 1 and each north step is relabeled as 0. However, a discussion can be focused around how the lattice paths and binary strings are mathematically different by focusing on, for example, the geometric properties of lattice paths and whether there are corresponding properties in the binary strings. Discussing properties and operations of a set can progress to relational bijections. For example, a student may wonder what happens to a lattice path that is reflected across the line $y = x$. Using the representational bijection, a student could convert each lattice path and its reflection to binary strings and recognize that the two binary strings have their zeros and ones swapped, i.e., the ones in the first binary string are precisely the zeros in the second binary string and vice versa. With some further exploration or guidance, this could lead to a justification of the identity $\binom{n}{k} = \binom{n}{n-k}$ using binary strings and using lattice paths.

In this paper, I have provided a theoretical account of three ways of understanding bijections in combinatorics: relabeling, representational, and relational. I provided combinatorial examples to exemplify each way of understanding, and I discussed some potential affordances and constraints of reasoning with each type of bijection. My hope is that these ways of understanding can serve as a foundation for future empirical research and conceptual analyses, both to provide evidence and to suggest refinements and additions to these three categories. Moreover, conceiving of combinatorial bijections in terms of relabeling, representational, and relational provides one avenue for designing tasks to target a bijective way of thinking in combinatorics. By elaborating ways of understanding bijections in combinatorics, I hope to inspire thought and research into bijections and how this research can be connected to other areas of mathematics education research, such as research in combinatorics education, functions, equivalence and sameness, and topics such as isomorphism, homeomorphism, and isometry.

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