Analogical Quotient Structure Sense in Abstract Algebra: An Expansion of University Structure Sense

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Despite existing research describing students' understanding of group concepts, little research in undergraduate mathematics education has attended to students' understanding of ring concepts. In this paper, we present an analysis of four students engaged in task-based interviews that provided insights into their understanding of quotient ring when constructed as an analogy to quotient group. Using university structure sense (Novotná & Hoch, 2008) as a foundation, we propose an expansion of structure sense to include attention to various structures (e.g., subgroups and quotient groups), and attention to analogous structure across different contexts (e.g., attending to quotient structure across group theory and ring theory.) Findings suggest that while students uniformly attended to similar structural components when creating the concept of quotient ring, there was variation in the depth of their reasoning about why certain structures exist.

Keywords: Abstract algebra, Analogical reasoning, Quotient groups, Structure sense

Attention to structure is a key component for understanding advanced mathematical concepts. Various studies have investigated students' attention to structure in high school algebra, linear algebra, and abstract algebra. Efforts have also been made to parse students' attention to structure for practical applications to teaching. One approach to parsing students' understanding of and attention to structure is structure sense (Linchevski & Livneh, 1999), which refers to the ability to recognize the same structure flexibly and creatively across various contexts. Although structure sense was originally defined in the context of algebraic equation solving, Novotná and Hoch (2008) extended structure sense to also include university (or abstract) algebra. Specifically, they identified two categories of structure sense: (1) structure sense of elements within a set and their behavior under a binary operation (e.g., understanding closure), and (2) properties of a binary operation (e.g., understanding the relationship between inverses and identities).

The current undergraduate mathematics education literature emphasizes attention to basic aspects of algebraic structure in advanced mathematics, most often with a focus on central algebraic objects (such as group), and the operations associated with them. For instance, Serbin (2023) identified three ways in which secondary pre-service teachers reason about identities and found that a unified understanding of identity could lead to productive reasoning about identities in more abstract contexts. Cook (2014) described students' emerging understanding of ring-theoretic concepts of unit and zero-divisor. Such research has provided a rich understanding of students' sense of basic structure. However, structure sense has not been explicated in the literature for attention to structure in abstract algebra beyond groups, although examples of implicit attention to a wider variety of structure senses exist. For example, Melhuish et al. (2020) presented a case study investigating students' unified understandings of the function and (group) homomorphism concepts and described a variety of understandings, ranging from students who could not recognize homomorphisms as functions at all, to students who could explicitly leverage homomorphisms as functions when approaching tasks.

Given that several important structures exist within abstract algebra (Dubinsky et al., 1994), there is a need to explicate different types of structure sense. However, being that similar structures are also present in many areas across the advanced mathematical curricula (e.g., sub-structures in group theory, ring theory, and topology), there is also a need to describe students' sense of structure *across different contexts.* In this paper, we expand even further upon the proposed extension by Novotná and Hoch of structure sense to abstract algebra to also include structure sense for a wider variety of structures, as well as introducing a mechanism for describing structure sense across contexts. To elucidate this expansion, we pose the following research question: *What are four students' sense of structure associated with quotient groups and quotient rings?*

Theoretical Framing

Expanding on Structure Sense

To describe students' structure sense, we adopt Novotná and Hoch's (2008) structure sense for university (i.e., abstract) algebra as being able to recognize sets of elements together with binary operations and their properties in a range of familiar and non-familiar structures. However, our expansion of structure sense also includes recognizing broader structures found across the advanced mathematics context. For instance, structure sense for 'subgroups' may include the ability to coordinate a subset of a group with the same operation as the parent group. We call the original structure sense in abstract algebra (as discussed by Novotná and Hoch) a *fundamental* structure sense in reference to it being the structure sense for a fundamental object of study in advanced mathematics, such as a group, ring, or topological space. Thus, structure sense need not be constrained to a set and a binary operation, but may be generalized to a set of elements together with some collection of properties that define an object (i.e., a topological space is a set together with a topology defined on the set). In this study, we attended specifically to *quotient structure sense*. Table 1 below summarizes the two types of standard structure senses to be discussed in this paper.

Table 1. Types of Structure Sense in Advanced Mathematics

Fundamental Structure Sense	Recognition of properties inherent to the main object of study (e.g., groups in group theory)	Example: Recognizing the identity element in the dihedral group of a regular polygon.
Quotient Structure Sense	Recognizing properties required to formulate a quotient of a structure, or properties that follow from the definition.	Example: Understanding that the role of a normal subgroup is to allow for well-defined binary operations defined on cosets that respects the original group operation.

As it is central to this paper, we briefly expound upon quotient structure sense. Quotient structures possess a rich and complex structure that involves multiple levels of coordination to grasp in full. For example, there must be a coordination of a parent structure with a particular sub-structure through which an equivalence relation is defined. In the context of abstract algebra, quotient groups and rings require the identification of well-defined binary operations on the quotient structure, thus allowing for the quotient structure to itself be described as an example of the central object of study (e.g., a quotient group is itself a group.)

Analogical Structure Sense

To conceptualize reasoning across different contexts, we view students' comparative activity across contexts as a form of analogical reasoning. We draw upon the Analogical Reasoning in Mathematics (ARM) framework for investigating students' analogical reasoning (Hicks, 2020) which adapts several components of Gentner's (1983) Structure-Mapping Theory to the context of mathematics while also respecting students' potentially idiosyncratic forms of analogical reasoning. Within this framework, analogies are determined by mapping content from a source domain to a target domain. Unlike typical frameworks for analogical reasoning, ARM parses students' reasoning into individual analogical activities, thus allowing for a close examination of one part of a student's analogy (known as an instance of analogical reasoning), followed by a reconstitution of the instances to analyze the student's analogy as a whole. In this way, even if the final result of two students' analogies appear to be the same, their analogies can be distinguished by investigating how the analogy was created.

We introduce *analogical structure sense* to parse students' attention to structure during analogical reasoning across different domains. We define analogical structure sense as the ability to either recognize or create similar structure across two distinct domains. In other words, it is the ability to either (1) reason about which aspects of a known pair of structures are relevant when comparing what is similar and different, or (2) reason about which aspects of a structure in the source domain are crucial to the creation and development of an analogous structure in the target and which aspects of the source are superfluous. Analogical structure sense can be explicated across widely different domains, suggesting that a certain level of abstraction may be required in some cases, such as comparing the subgroup concept and the concept of topological subspace (Hicks, Flanagan & Park, 2022; see also English & Sharry, 1996). In contrast, we refer to a sense of structure as *standard* when it refers to only one domain.

To a degree, standard structure sense already appeals to basic forms of analogical reasoning: a student comparing two examples of groups to identify how they are similar in terms of their structure is reasoning by analogy. When viewed in this way, we recognize that analogical structure sense can be embedded within the original definition. However, for the purposes of this paper, we consider analogical structure in the case where the fundamental structure senses are different between the source and target, such as between groups and rings. Thus, analogical structure sense captures a wider range of structure senses than are possible with the standard definition alone. Table 2 summarizes the two types of analogical structure sense explored in this study.

Table 2. Types of Analogical Structure Sense in Advanced Mathematics

Fundamental Analogical Structure Sense	Recognizing (or creating) similar/different properties inherent to the main objects of study across two domains (e.g., groups and rings).	Example: Articulating that inverses are not necessarily required for multiplication in rings in contrast to inverses always being required for groups operations.
Analogical Quotient Structure Sense	Recognizing (or creating) similar/different properties inherent to quotient structures across two domains.	Example: Considering the need for both additive and multiplicative cosets when defining quotient rings by analogy with quotient groups.

Methods

The data for this study was collected as part of a larger project investigating students' analogical reasoning in abstract algebra. In the Fall of 2019 and the Spring of 2020, emails were sent out to students requesting participation in the study. Four students responded to the request: three advanced undergraduate math majors named Ellen, Nathan, and Brandon (all pseudonyms), and one graduate student in mathematics education named Andrew. Each student was then asked to participate in a series of 5 clinical task-based interviews (Goldin, 2000), with each interview lasting between 60-90 minutes. Each of the interviews was conducted by the first author, who will be referred to throughout as the interviewer. In this paper, we focus our attention on the interview associated with quotient rings in which students were posed the following task: *Make a conjecture for a structure in ring theory that is analogous to quotient groups in group theory.* In addition to this task, students were also asked several questions inquiring into their construction, as well as provided several tasks in which they leveraged their constructed analogue to reason about relevant content, such as generating examples or conjecturing theorems related to the structure.

The analysis of the data occurred in two phases: (1) an analysis of the students' standard structure sense for quotient group, and (2) an analysis of the students' analogical structure sense of quotient structure. The goal was to analyze the students' evoked structure sense, meaning we do not claim that these interviews revealed the full range of the students' structure senses. The video and transcripts of the initial interview were reviewed multiple times, and notes were made whenever any evidence of standard structure sense was present. For example, if the student explicitly mentioned the need for a normal subgroup to describe a quotient group, then evidence of standard structure sense was present. Profiles for each student's senses of structure (totaling 4 profiles) were then written, thus providing a holistic summary of the key points that the student attended to when recalling (or reviewing) the definition of the structure.

To analyze the students' analogical structure sense, the ARM framework was first used to code the students' instances of analogical reasoning while comparing and creating structures in each of the interviews 2-5. Each instance was then scrutinized for evidence of analogical structure sense, meaning whether or not there was evidence that they were reasoning about why certain features of the source should be mapped to the target when comparing or creating structures. In addition, these profiles included descriptions of the concepts that the students attended to while creating the analogous structures. Identifying these concepts often assisted with interpreting the students' analogical reasoning, and these differences revealed a variation in how the students were reasoning by analogy. Following this coding process, a profile was written for each student's sense of analogical structure associated with quotient ring (totaling 4 more profiles overall).

Findings

The four students displayed a range of analogical structure sense, from superficial grasp of the underlying analogous structures to a developing sense of the analogous structure for describing quotient rings. In the section that follows, we describe key holistic observations made about the students' attention to analogical structure for quotient structure. We present these findings in two sections: (1) describing what structures students considered to be relevant to their analogical construction of a quotient structure in ring theory, and (2) describing the variation of depth of the students' reasoning about quotient structure.

Attention to Existence of Analogical Structure: Normal Subrings and Cosets

All students acknowledged that a "normal subring" should be present when constructing the ring-theoretic analogy to quotient groups. Rather than immediately copy over the definition of normal subgroup to the ring context, each participant reflected upon what differences might be present when considering the definition of a normal subring. For example, Figure 1 below exhibits Andrew's constructed definition of a normal subring. Here, Andrew is attempting to reconcile the property of normality for subgroups as a property within the context of rings.

A Subring is Normal if:

$$\{g+x = g*x \mid x \in N\} = \{x+g = x*g \mid x \in N\}$$

$\forall g \in G$.

Figure 1. Andrew's definition of a "normal subring."

In the above example, Andrew is exhibiting two important features: (a) he is describing normality as a set of cosets being equivalent, and (b) he is appealing to the existence of two operations by expressing equivalence between additive and multiplicative cosets. This suggests that Andrew's sense of analogical structure included the existence of cosets and the manner in which binary operations were defined on those cosets. Such observations were present across each of the other students as well, meaning that all of the participants uniformly attended to the existence of normal subring, cosets, and adapting binary operations to the quotient ring context in some way.

Although analogical inferences related to the existence of structure across the contexts of group and ring theory may appear superficial on the surface, we contend that our participants' attention to the existence of these structures indicated a productive sense of analogical structure. Furthermore, as we discuss in the next section, the attention to the existence of analogous structure proved to be fertile ground for reasoning deeply about the meaning of those structures.

Variation in Students' Depth of Reasoning: Attending to Meaning of Structure

The students' sense of analogical structure not only entailed the existence of certain components related to quotient structure, but also included the underlying reasons for the existence of those components. However, the depth of reasoning varied from one student to another, thus distinguishing students' analogical structure senses in our analysis. Although all students identified the existence of a normal subring, only some students also investigated further by inquiring into why the normal subring concept was needed at all. In particular, Brandon's proposed description for the structure of a quotient ring was focused on describing a mapping from an element to the equivalence class of the element. Based upon evidence from the initial group interview, Brandon was presumably referring to an analogue for the canonical homomorphism defined for groups wherein every element in the domain is mapped to its coset in the quotient group. When asked what constituted normality in the context of his ring-theoretic analogue for quotient groups, Brandon responded: "I really only remember why we did it for groups." Although he was unable to expound further, he explained that if R' was normal, then the quotient R/R' would be a ring. Thus, Brandon's sense of analogical structure included attention to a quotient structure in fact being an example of the fundamental structure, and thus the normal subring would have to be carefully selected to make this happen.

While Brandon asserted that a normal subring was needed to form a quotient ring, he struggled to provide a description of normality. However, reasoning about the definition of normality for rings appeared in other ways across the students. In particular, each student attended to the need for adapting multiple binary operations to the ring-theoretic analogy for quotient groups. In general, the students made the following observations about binary operations and the quotient ring structure: (1) describing what coset operations look like, (2) determining how the existence of two binary operation affected the normal subring concept. Initially, Andrew focused his attention on just one operation at a time. He stated:

> What I would do, is I would literally just add the second operation to both of these. I'll do it in red what I have. I should probably make it clear though… This [digitally points to $x + H = y + \mathrm{H}$] is a plus operation, and… [writes "$x{*}H = y{*}H$"].

He then attempted to reconcile the meaning of normality when using the multiplicative operation: "Oh man, this is gonna be trickier. Because would they be the same? That's harder, let's think about this. I feel like this would be so hard to make happen [digitally points to $x{*}H = y{*}H$]." Returning to Andrew's construction of normal subring in Figure 1 above, we see this attention to multiple operations on display as Andrew eventually conjectured that the desired property for a normal subring was that $x + g = x * g$ and $g + x = g * x$

In contrast to Brandon and Andrew, Ellen did not attempt to describe the meaning of normality, nor did she ever formally define a coset in the context of rings. However, Ellen spent significant time reflecting upon how operations would be defined on cosets, thus attending to the need for a set of cosets to form a ring. She proposed the description seen in Figure 2 below. Of particular note was Ellen's attention to a "general" coset of the form aH as seen in the figure. Attention to this general coset included reasoning about a general coset operation, suggesting that Ellen was attending to more than just two operations for the specific context of rings. When asked to explain, Ellen stated: "I'm saying what operation specifically does this need to work for. For rings it's addition and multiplication. So, if you use addition, will that work, and if you use multiplication, will that work."

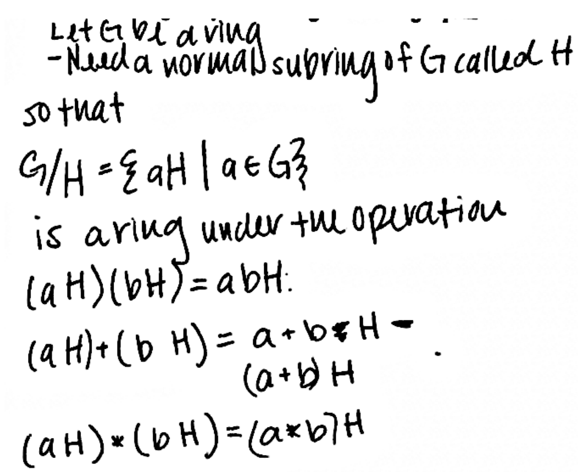

Figure 2. Ellen's proposed quotient ring definition and coset operations.

Summary of Findings

The findings presented above reveal that although all students in this study possessed a strong sense of analogical structure for the *existence* of structural analogies between quotient group and ring, their sense of analogical structure differed greatly in how they attended to the meaning of those structural analogies. Specifically, all students recognized ring-theoretic analogies to normal subgroups, cosets, and the role of binary operations within quotient groups. However, the students' sense of underlying meaning of those structures ranged from deep considerations about the purpose and definition of the structure, to a complete lack of attention to purpose or definition at all.

Discussion and Implications

Our proposed expansion of structure sense, to accommodate a wider variety of structures as well as structures across different contexts, allows for deeper explorations of students' attention to advanced mathematical structures. In this study, we elucidated this framework by investigating students' understanding of a particularly difficult concept: quotient ring. Findings indicate that students do indeed productively attend to several aspects of analogical structure when developing a ring-theoretic analogy to quotient groups, including attention to the need for a "normal subring," and attention to adapting multiple binary operations to the ring context. Overall, while all students were successfully able to make headway into describing a ring-theoretic analogy to quotient groups by spontaneously identifying the so-called "normal subring" concept, the notion of constructing a ring-theoretic analogy to normal subgroups proved to be a difficult task for students.

In contrast to structural similarity, semantic similarity (Holyoak & Thagard, 1989) refers to the degree of similarity between the meanings of objects within a domain rather than the global similarity of the structures as a whole. By investigating students' sense of analogical structure, we discovered that our participants uniformly attended to structural similarity, but varied in terms of their attention to semantic similarity. Although high semantic similarity exists between structures in group and ring theory due to their historical development (see Hausberger, 2018), only a subset of our participants spontaneously inquired into potential semantic similarities, focusing instead on structural similarity.

Our findings warrant suggestions for teaching abstract algebra concepts as well as considerations about the guided reinvention (Gravemeijer, 1999) of ring-theoretic concepts by analogy. The existence of a "normal subring" appeared to be a natural first step for students' construction of the quotient ring concept; however, their understanding of the analogous role that a normal subring should play appeared to be impeded by their understanding of the normal subgroup concept. Thus, our investigation of students' analogical structure sense suggests that, if analogies are to be used during instruction to compare quotient groups and rings, then there is a need to not only emphasize the coordination of structures on their own (e.g., describing quotient groups as a group together with a normal subgroup), but to also emphasize the conceptual underpinnings as to *why* normality is an important component of establishing quotient groups. We assert that this approach could establish an effective first step toward developing curricula for the guided reinvention of the concept of ideal. Furthermore, effectively leveraging of student analogical reasoning could promote productive instances of backward transfer (Hohensee, Willoughby & Gartland, 2022) in which students' understanding of quotient group and normal subgroup could be enriched by attending to the underlying relationship between quotient rings and ideals and making comparisons between the domains. Future research can investigate these matters in greater detail.

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