Connecting Proof Comprehension with the Student Perspective: The Case of Christy

Lino Guajardo
Texas State University

Kristen Lew
Texas State University

The current state of proof comprehension research has focused on the instructor or researcher perspective. Either through assessing student proof comprehension (e.g., Mejia-Ramos, 2017) or developing interventions to improve student proof comprehension (e.g., Hodds et al., 2014). In this report, we discuss findings from one student from an exploratory study. We consider the influence of prior experiences on what it means to them to understand a proof and the reasoning behind the proof comprehension strategies used to understand a given proof.

Keywords: Proof comprehension strategies, student experiences, reading proof

Comprehending proof is one important task that a mathematician must do during their career. A goal for instructors of proof-based courses is to have their students develop the skills needed for mathematics research (Weber & Mejia-Ramos, 2011). One way of doing this is to have students participate in tasks done by mathematicians, such as comprehending proof. The existing literature has either focused on developing ways to assess student proof comprehension (e.g., Davies et al., 2020; Mejia-Ramos et al., 2017) or interventions to help improve student proof comprehension (e.g., Alcock et al., 2015; Hodds et al., 2014; Roy et al., 2010; Samkoff & Weber, 2015). Each of these studies take the perspective of researchers or instructors of proof-based courses and the student perspective is largely absent from the literature on proof comprehension (e.g., Weber, 2015). In this preliminary report, we begin to investigate the following research questions: (a) How do students' prior experiences inform their understanding of proof? (b) What reasons do students provide for using strategies to understand proof?

Relevant Literature

Proof Comprehension Assessment

Mejia-Ramos et al.'s model (2012) and tests (2017) have been used by multiple researchers to assess the impact of interventions. Alcock et al. (2015) used the model to create proof comprehension assessments to assess if workbooks given to students in an abstract algebra course improved their independent study of proofs. Zazkis and Zazkis (2016) used the model to identify what aspects (with respect to the seven question types) of a proof on the Pythagorean theorem preservice teachers focus on during a proof script task. As creating a valid and reliable proof comprehension test can be a daunting task (Mejia-Ramos et al., 2017), Davies et al. (2020) attempted to find another method to assess student proof comprehension by using student proof summaries. Investigating how student proof summaries could be used to assess student's comprehension of a given proof, the authors also hoped this would provide instructors with an alternative to the long process of creating a proof comprehension test. The authors found this method of comparative judgement to be reliable and valid, but the judgements from the experts correlated less with the final course grades in comparison to the comprehension test. The findings from Davies et al.'s study suggest that the proof comprehension tests created by Mejía-Ramos and colleagues may be more effective in assessing students' understanding with respect to course grades than the comparative judgements of student proof summaries.

Improving Student Proof Comprehension

Researchers have attempted to develop ways to present proofs to students so that the proofs can be easier for students to understand. Leron (1983) suggested an alternative to the linear presentation of proof by describing structuring mathematical proofs to highlight the overarching ideas of the argument. Rowland (2001) presented generic proofs for number theory, which used specific examples to highlight the general approach of the argument. Using the specific example sought to improve clarity and to remove abstractness for the reader.

More recently, researchers have also developed interventions to help with student proof understanding. These interventions attempted to help students in different ways, such as providing proofs that had visual and audio aspects to focus student attention (Roy et al., 2017), helping students learn how to productively study independently (Alcock et al., 2015), or providing students with self-explanation training (Hodds et al., 2014). These interventions attempted to support students in various aspects of reading proof. Either by providing students the chance to read the proof at their own pace and revisit instructor-provided explanations as many times as they would like (Roy et al., 2017), providing students opportunities to engage deeper with the concepts and proofs (Alcock et al., 2015), or by providing students with insight into how to productively summarize arguments (Hodds et al., 2014).

Weber (2015) and Samkoff and Weber (2015) offer seven strategies that successful students implement as they read proof for understanding. These authors implemented two instructional interventions, finding mixed results, to help students improve their proof comprehension skills.

Conceptual Framework

Since reading proof is an important part of a mathematician's career, it is equally an important task for students to take part in within their proof-based courses (Weber & Mejia-Ramos, 2011). Since students must read the proofs they encounter in class and textbooks, they are consumers of the presented proofs. As consumers, they are tasked (implicitly or explicitly) to understand a given proof. While we acknowledge Mejia-Ramos et al.'s (2012) model for assessing students' proof comprehension and the impact of this model on our own conceptions of proof comprehension, this study focuses on what it means to students to understand proof. When a student is given a proof with the goal of understanding, students may act to better understand particular points or aspects of the proof. That is, a student will employ a proof comprehension strategy.

Scholars have studied what students do when they are reading proof. For example, Weber (2015) investigated how successful mathematics majors attempt to understand a given proof through task-based interviews. Dawkins and Zazkis (2021) compared the reading behaviors between students who had little to no experience with reading proof to those who had experience. Additionally, Samkoff and Weber (2015) attempted to teach students seven productive strategies that can be employed to understand a given proof, such as dividing the given proof into independent sections.

Literacy education scholars have discussed student reading strategies and reading skills, with some scholars differentiating the constructs and others suggesting the two are the same. In both cases, scholars rarely (if at all) formally define them. Afflerbach et al. (2008) attempted to bring cohesion within the field by defining both constructs, offering the following definition of a reading strategy: "[r]eading strategies are deliberate, goal-directed attempts to control and modify the reader's efforts to decode text, understand words, and construct meanings of text" (pg. 368). With respect to reading proof, we define a strategy to be deliberate acts from the reader with the goal of understanding some subset of a given proof. To help in operationalizing this, we consider the ostensive strategies that a student enacts when attempting to understand a given proof. Thus, we define a proof comprehension strategy as any action of the reader that is done outside of the proof text with the goal to understand some portion of a given proof (inclusive of the whole proof). By outside of the proof, we mean the student engages in furthering their understanding through actions not done or recommended within the proof.

Meanwhile, the literature suggests that students' beliefs and prior experiences can have a significant impact on their actions within a classroom (Muis, 2004; Schoenfeld, 1989). Thus, as students are reading proofs to understand, they may rely on previous experiences with proofs or related topics. That is, they may reflect on past lived experiences, similar proofs they have seen, or on what proof comprehension strategies have previously been useful or productive. As such, we believe it is possible that students' beliefs and experiences may influence their own meanings of comprehension and the actions they take to understand a proof.

Theorem 1. *Suppose* $f : \mathbb{N} \to \mathbb{Z}$ *such that* $f(n) = \frac{(-1)^n(2n-1)+1}{4}$ *for* $n \in \mathbb{N}$. *Then* f *is a bijective function.*

Theorem 2. *Suppose* $f : A \to B$ *is a function. Then* f *is injective if and only if* $X = f^{-1}(f(X))$ *for all* $X \subseteq A$.

Theorem 3. *Suppose* A *is a set. Then,* $|A| < |\mathscr{P}(A)|$.

Figure 1: Three theorem statements

Methods

The data presented is from a basic qualitative study (Merriam & Tisdell, 2015) that investigated what students do as they attempt to understand given proofs on functions. Three participants were recruited from a large southern Hispanic-serving institution. Participants were interviewed during the winter break of 2022-2023 or during the Spring 2023 semester. Each participant had taken an introduction to proof course during the Fall 2023 semester. Participants were met with 1-3 times for hour-long clinical interviews (Ginsburg, 1997), based on their availability. For each interview, participants were given a theorem statement and its proof. Students were tasked to read the proof until they felt they understood the proof to the best of their ability. Each proof focused on functions, a topic covered in each of the participants' introduction to proof courses. Figure 1 shows the three theorem statements. Relevant definitions were available and provided if requested. After the participants indicated they understood the proof, interview protocol questions were asked. These questions focused on what it means to the participant to understand a proof, what the participants had done to understand the proof, and what they felt as they read the proof for understanding. All interviews were video and audio recorded. For this report, we focus on Christy[1], a computer science major at the time of the study (now a CS and mathematics double major). Christy successfully completed the course, continued in proof-based mathematics courses, and eventually engaged in undergraduate mathematics research. For the study, Christy met with the first author for three interviews. Data for this report was analyzed using open and axial coding by both authors (Corbin & Strauss, 2015). Both

[1] Pseudonym chosen by participant.

authors coded for student experiences and indications for how these experiences influenced what it meant for the student to understand a proof. Additionally, the authors coded for reasoning provided by the student for the use of identified proof comprehension strategies.

Preliminary Results

Christy's Definition of Understanding. In each interview, Christy was asked what it meant to them to understand a proof. Each time, Christy's response was similar: that understanding a proof means to understand the reasoning behind each action taken and decision made by the author throughout the proof. For instance, consider Christy's answer during the first interview:

> To understand, I think for me it's like understanding why people do things or like why certain actions we're taking. Like we are doing two parts cuz for proving it you need the onto and the one-[to]-one. ***But then like why do you have to assume - understanding why these assumptions are made. Because it's one thing to just do it and another thing to understand why it is*** […] ***[L]ike computer sciences, like CS. There's a difference between like having a program that works and actually knowing why it works. Because if you don't know why it works, then you could mess you up over later on when you're doing other things.*** […] Yeah, like why we make certain assumptions with things and like when we have to.

There are three key points to this quote. First, we note that Christy expressed the need to prove that a function is both one-to-one and onto to prove that it is a bijection – meaning that she is engaging in some proof comprehension strategies addressed by the literature, namely splitting the proof into various cases. Second, Christy emphasizes that simply identifying the modular structure (or cases) of the proof is insufficient. Rather, she states that it is also necessary to understand *why.* Why those two cases are made, why any assumptions can be made, and why the author is able to take different actions. Finally, Christy connects her need to consider both the cases of the proof and the reasoning behind these cases to her experiences in Computer Science. Throughout each of her interviews, Christy's focus on the "why" was consistent, as was her connection to computer science.

Christy's Comprehension Strategies. Christy was given a goal of understanding a given proof to the best of their ability. With their definition of what it means to understand a proof, Christy focused on various aspects of the proof to meet this goal. Christy used various proof comprehension strategies, some of which were previously identified by other researchers as strategies used by mathematicians. For example, for each of the given proofs, Christy segmented the proof into parts. Specifically, she marked each proof to indicate independent portions of the proof (such as differentiating the onto and one-to-one portions of the first proof). Christy also frequently used examples or drew diagrams to illustrate portions of a given proof. However, the reasons Christy used these strategies do not necessarily match the reasons the literature has indicated that mathematicians use and promote the use of these same strategies.

Consider partitioning a proof into independent portions. Samkoff and Weber (2015) state that identifying the modular structure of a given proof can be used to see overall structure of the proof. Additionally, Mejia-Ramos et al. (2012) discuss mathematicians views on the benefits of using this strategy as connections can be made between the portions identified. Yet, for Christy, this strategy held a different purpose, "I just need something there so I can see it. […] Like, yeah, block out what I don't need like sometimes I'll drive with the visor down even if it's there's no sun. Just so I can like focus on what's important." For Christy, portioning the proof was to allow them to focus on the independent pieces. Thus, we see Christy using productive comprehension strategies, but the personal reasons for implementing them differ from the motivation suggested by the literature.

Christy also used strategies not identified by Weber (2012). For example, one way Christy tried to understand certain portions of a given proof was to continue reading beyond the section she did not initially understand. In the second proof, Christy read the line "Since $f(x') = y$ and $f(x) = y$, then $f^{-1}(y) = \{x, x'\}$." Christy expressed she "didn't think functions could equal sets." When asked how she made sense of the line, Christy responded "mine was just read the next line." Similarly, while reading the third proof and attempting to make sense of the definition of a set (an element of the powerset and the mechanism by which the contradiction arises), Christy read over the definition multiple times before jumping forward in the proof saying "trying to see if the conclusion helps." In both of these instances, Christy continued to read further lines in the proof in an attempt to increase their understanding. This strategy was not identified by Weber in his 2015 study.

Connections to Computer Science. Christy made connections to their experiences with computer science in each of their interviews. Moreover, the connections varied in terms of scale. For instance, as discussed earlier, her experiences appear to strongly influence her personal definition of understanding a proof. Meanwhile, she also made connections between specific content in each field. For instance, during the first interview, Christy connected the concept of hash maps in computer science to the concept of a bijection:

> Computer science, there's like a hash map. So like to get values, you have to like, know the key for it […] There's just an infinite number of keys […] that's the definition of bijection is that you can get every single value. […] we just need to be able to get every single integer. Yeah, that makes sense in my head.

Discussion

Scholars have mainly focused on the instructor or researcher perspectives with proof comprehension research, though few have begun to investigate the student perspective. This study expands on Weber's (2015) findings that identified six strategies that successful students use to comprehend proof. Christy was a successful student in their introduction to proof course, yet her reasoning for using strategies previously identified by Weber are different than the motivation mathematicians might have for using these same strategies. Additionally, Christy was seen using strategies not reported by Weber. This study contributes to the field's knowledge about how students attempt to understand proofs in two ways. First, there may be additional strategies that successful students implement when reading proof. Additionally, we gain insight into the reasons students use different strategies. In this study, Christy did not necessarily use strategies for the same reasons a mathematician may, yet she productively used strategies that helped her understand the given proofs.

Investigating student perspectives on proof comprehension can inform continued and improved support to students. Additionally, investigating more students may provide deeper insight into ways to support students in comprehending proofs they see in their courses. Researchers can better identify which proof comprehension strategies are productive and when they are most productive to further support students in comprehending proofs they see in their courses. We aim to address this need in current and future work. We are investigating how more students attempt to make sense of given proofs. Further, we will be connecting student beliefs about proof and prior experiences with proof to what proof comprehension strategies they use with a given proof.

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