# Are These Vectors Linearly Independent? Conceptions of Linear (In)Dependence in a Linear Algebra Textbook and Student Responses

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*Using Balacheff's (2013) model of conceptions we analyzed textbook examples in the linear independence section of an interactive linear algebra textbook and 61 student responses to a similar reading question. Reading questions seek to entice students to read the textbook before attending the lesson when the ideas will be discussed; the responses are immediately available to instructors. We found ten additional parts of conceptions (control structures) used independently or in combination with the ones promoted in the textbook. We discuss the implications of our findings and our plans for future research.*

*Keywords:* linear independence, conception, linear algebra, cK¢ model of conceptions

We contribute to investigations of students' understanding of fundamental notions in linear algebra by bringing a different theoretical approach to such investigation with the goal of exploring its usability in large-scale analysis. Linear algebra is a course that is nearly universally taught at four- and two-year institutions across the United States (Blair et al., 2018). Extensive research documents the difficulties students experience with the high abstract level of the notions, which also demands substantial computational dexterity. While much work has been done to address student difficulties (e.g., using curricula that emphasize building mathematical understanding through concrete experiences and visualizations that are progressively mathematized; Wawro et al., 2012), managing these difficulties in the classrooms is not straightforward. As they learn the material, independently of the curriculum that is being used, students face difficulties in transitioning from concrete to more abstract and from managing visual representations to working with spaces that can not be visualized. Our position is that independently of the textbooks or the mode of teaching, such difficulties will emerge; thus, finding ways in which these difficulties can be readily identified could promote better teaching and learning.

## Literature Background

The research on the learning of linear (in)dependence has focused on identifying students' conceptual difficulties with the notion and its formal/abstract definition (Dorier, 2017; Hannah et al., 2013; Stewart & Thomas, 2007, 2009) and characterizing students' ways of thinking about linear (in)dependence. Regarding conceptual difficulties, researchers have shown that, although students make mistakes when solving routine problems, their procedural understanding is sounder and less varied than their conceptual understanding of this notion (Celik, 2015; Parker, 2010). To investigate ways of thinking, researchers have relied on a handful theorizations, and although they are named differently, the categorizations they produce are similar: abstract, algebraic, geometric (Ertekin et al., 2010; using Hillel's [2000] descriptive modes); travel, geometric, vector algebraic, matrix algebraic (Plaxco & Wawro, 2015; using Tall and Vinner's [1981] notion of concept image); embodied, symbolic, formal (Stewart & Thomas, 2007; using

Tall's [2004] three worlds of mathematics); or synthetic-geometric, analytic-arithmetic, analytic-structural (Dogan-Dunlap, 2010; Sierpinska's [2000] three worlds of mathematics). Across these studies, there is consensus that visual, embodied, or geometric forms of thinking about linear (in)dependence can help students overcome their difficulties with its algebraic and formal modes.

We focus on research addressing the problem of determining whether a set of vectors is linearly independent. Celik (2015) organized 186 student responses to this problem into 18 categories. The most common student approach was finding scalars that satisfied the definition of linear (in)dependence. Bouhjar et al. (2021) categorized the reasoning of 255 about the problem into six strategies: comparing Reduced Row Echelon Form (RREF) to the identity matrix, identifying the number of pivots, writing one vector as a linear combination of others, solving $Ax = 0$, comparing the number of rows and columns, and establishing the existence of free variables. Because linear independence was defined in terms of the solution to the homogeneous vector equation corresponding to $Ax = 0$, solving $Ax=0$ and comparing the RREF were considered the more appropriate approaches. Analyzing student responses to the same problem, Dogan (2010, 2018) showed that when not using geometric ways of thinking, most students use one of the following: (1) find a linear combination among the vectors, (2) find that the only solution to a linear combination is the trivial solution, (3) find whether scalar multiple and/or adding the vectors gives the zero vector, or (4) find RREF.

We complement this work in three significant ways. First, in addition to identifying student reasoning when learning how to solve the problem of determining whether a set of vectors is linearly independent, we study how their linear algebra textbook addresses it, which helps frame students' work. Second, while researchers have investigated students' various modes of thinking or types of reasoning, we only focus on one semiotic register: the symbolic vector-matrix. This allows us to fully investigate the difficulties of this representation, which is preferred for formalization to higher-order spaces. Third, although it is common to draw a line between procedural and conceptual understanding, we assume that these approaches are intertwined and can significantly inform one another. Therefore, we use a theoretical framework that bypasses analysis of conceptual and procedural understanding and instead focuses on the actions and the decision-making processes as students determine whether a set of vectors is linearly independent.

## Theoretical Framework

Balacheff (2013; Balacheff & Gaudin, 2009) posits that meaning derives from the system that encompasses a milieu and a cognizant subject rather than being inscribed in either of them alone; meaning is created through the interactions between the milieu and the subject through the actions of the subject on the milieu and the feedback the milieu provides to the subject. His work responds to three needs related to investigating student thinking (1) explaining why particular held conceptions of a notion that appear contradictory to an observer are not to the holder of the conceptions; (2) describing the array of conceptions that can be held by learners, which are the result of historical and pedagogical process of concept development; and (3) understanding the connection between behaviors and knowing. Following this tradition, *savoir* (knowledge), the subject matter developed by a community, is differentiated from *connoisseur* (knowing, as a noun), the knowledge held by an individual, which can be incomplete or even mathematically invalid. For this reason, knowledge and knowings are tied to the problems in which they emerge. Thus, understanding the practices in which specific mathematical ideas are called for is the first step in understanding conceptions.

The cK¢ model is a heuristic of sorts for identifying conceptions. It has four components: problems, operators, semiotic and representation systems, and control structures. *Problems* correspond to the "class of the disequilibria the considered conception is able to recover from" (Balacheff & Gaudin, 2009, p. 190); they emerge from the sets of practices in which individual concepts are called for. *Operators* refer to "actions on the milieu" including those needed "to transform and manipulate linguistic, symbolic or graphical representations" (p. 190). The learner receives feedback from the milieu as a result of their actions (action-feedback loop). The *semiotic and representation systems* are defined as the "linguistic, graphical or symbolic means which support the interaction between the subject and the milieu" (p. 190). Finally, the *control structures* refer to the "components supporting the monitoring of the equilibrium of the [Subject - Milieu] system" (p . 190), or said differently, the strategies that the cognizant subject relies on to decide whether they had solved the problem and that they had done so correctly. This model has been used to identify potential conceptions that could emerge as students work with different components in textbooks, such as the problems, the examples, or the textbook presentation with different concepts (functions, Mesa, 2004; differential equations, Mesa, 2010; angles, Mesa & Goldstein, 2016). We use this theorization to identify: (1) What conceptions of linear independence are promoted by examples in an interactive undergraduate linear algebra textbook? and (2) What control structures do students use when answering reading questions about linear independence embedded in these textbooks?

## Methods

The data for this study come from a larger study that investigates undergraduate mathematics students' and instructors' use of interactive textbooks in calculus, linear algebra, and abstract algebra courses (Beezer et al., 2018). The linear algebra textbook, *A First Course of Linear Algebra* (Beezer, 2021), is designed to be a bridge-to-proof linear algebra course and it follows the definition-theorem-proof presentation style (Love & Pimm, 1996). The textbook is authored in PreText (https://pretextbook.org/), which allows the inclusion of interactive features. *Reading Questions* are one such feature designed to entice students into reading material before coming to class; students are supposed to answer a few of these short answer questions in each section directly in their textbooks before coming to class. Their responses are then collected and made available for their instructor in real-time so that instructors can learn about their students' thinking before a lesson and alter their plans as needed.

To investigate how the textbook content related to the chosen reading question in this section (Figure 1), we identified one example in the Linear Independence and Spanning Sets (LISS) section directly related to it (Figure 2). The example, named Linear Independence in $M_{32}$ (LIM32) is about the vector space of all 3x2 matrices and illustrates the work needed to decide whether two different sets of matrix vectors are linearly independent or dependent. We analyzed the example using Balachef's (2013) cK¢ model.

**LISS Reading Questions**

**1.** Is the set of matrices below linearly independent or linearly dependent in the vector space $M_{22}$? Why or why not?

$$\left\{\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}, \begin{bmatrix}-2 & 3\\ 3 & -5\end{bmatrix}, \begin{bmatrix}0 & 9\\ -1 & 3\end{bmatrix}\right\}$$

*Figure 1: The first reading question in LISS*

**Example LIM32** Linear independence in $M_{32}$
Consider the two sets of vectors $R$ and $S$ from the vector space of all $3 \times 2$ matrices, $M_{32}$ (Example VSM)

$$R = \left\{ \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix}, \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} \right\}$$

One set is linearly independent, the other is not. Which is which? Let us examine $R$ first. Build a generic relation of linear dependence (Definition RLD),

$$\alpha_1 \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in $M_{32}$ (Example VSM) we obtain,

$$\begin{bmatrix} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 & -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 & 4\alpha_1 - 3\alpha_2 + -5\alpha_4 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 & -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 &= 0 \\ -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 &= 0 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 &= 0 \\ 4\alpha_1 - 3\alpha_2 + -5\alpha_4 &= 0 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 &= 0 \\ -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this matrix we are led to conclude that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. This means there is *only* a trivial relation of linear dependence on the vectors of $R$ and so we call $R$ a linearly independent set (Definition LI).

So it must be that $S$ is linearly dependent. Let us see if we can find a nontrivial relation of linear dependence on $S$. We will begin as with $R$, by constructing a relation of linear dependence (Definition RLD) with unknown scalars,

$$\alpha_1 \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + \alpha_4 \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in $M_{32}$ (Example VSM) we obtain,

$$\begin{bmatrix} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 & \alpha_3 + 3\alpha_4 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 & -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 & 3\alpha_1 - 6\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 &= 0 \\ \alpha_3 + 3\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 &= 0 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &= 0 \\ 3\alpha_1 - 6\alpha_2 + 4\alpha_3 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & -2 & 0 & -4 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this we see that the system is consistent (we expected this since the system is homogeneous, Theorem HSC) and has $n - r = 4 - 2 = 2$ free variables, namely $\alpha_2$ and $\alpha_4$. This means there are infinitely many solutions, and in particular, we can find a nontrivial solution, so long as we do not pick all of our free variables to be zero. The mere presence of a nontrivial solution for these scalars is enough to conclude that $S$ is a linearly dependent set (Definition LI). But let us go ahead and explicitly construct a nontrivial relation of linear dependence.

Choose $\alpha_2 = 1$ and $\alpha_4 = -1$. There is nothing special about this choice, there are infinitely many possibilities, some "easier" than this one, just avoid picking both variables to be zero. (Why not?) Then we find the dependent variables to have values $\alpha_1 = -2$ and $\alpha_3 = 3$. So the relation of linear dependence,

$$(-2) \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + (1) \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + (3) \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + (-1) \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is an iron-clad demonstration that $S$ is linearly dependent. Can you construct another such demonstration? △

*Figure 2: Example LIM32 in LISS*

Next, we analyzed 61 responses to the reading question in LISS from 61 unique students from four different sections during Fall 19, Fall 20, Spring 21, and Fall 21. All four instructors used the textbook and assigned the reading questions to their students. During analysis, we noticed that the operators were implied in most of the student responses and the steps taken were not outlined. However, the justification for their conclusion was usually explicit. Therefore, we solely focused on the control structures stated in the student responses. After coding all the responses individually, we met to compare the coding and resolve disagreements. Six of the 61 students' responses to the reading question were uninterpretable or did not have a stated control structure. We also looked for patterns when students used more than one control structure.

## Findings

### Inferred conceptions from Example LIM32 in the textbook

Example LIM32 addressed the problem "where a given set of vectors is linearly independent" (see Figure 2). Relying on a 3x2 matrix, this example uses symbolic representations. We identified a total of five operators (OP) in the solution and two control structures (CS).

The author starts by constructing the homogeneous system by equating an arbitrary linear combination of all vector matrices with a zero matrix (OP1: *Construct homogeneous system*). He then creates an augmented matrix for the system (OP2: *Construct matrix*) and performs a row-reduced echelon transformation on the matrix (OP3: *Row reduction*). He proceeds to inspect the resulting RREF matrix. He first starts by asserting that "the system is consistent (we expected this since the system is homogeneous, Theorem HSC)", then he states the observation that the RREF "has $n - r = 4 - 2 = 2$ free variables, namely $\alpha_2$ and $\alpha_4$" (OP4.1: *Free variables*, feedback = “there are free variables”). This observation implies that “there are infinitely many solutions” (OP4.2: *Number of solutions*) which leads to the criterion “we can find a nontrivial solution.” This enables him to use the control structure *Nontrivial* (CS1) that alludes to the existence of nontrivial solutions to the homogeneous system to justify the claim that the given set is linearly dependent. Conversely, if the homogeneous system only has a trivial solution (which is implied by having one unique solution, as it can not have no solution because the system is consistent by default), then the given set is linearly independent. Even though “the mere presence of a nontrivial solution (…) is enough to conclude that *S* is a linearly dependent set,” the author also provides an optional operator, *Find nontrivial scalars* (OP5) by explicitly constructing a nontrivial relation of linear dependence. This suggests another control structure in solving this problem, namely *Scalars* (CS2) that justify the linear dependence of the set by providing a set of nonzero scalars that satisfy the homogeneous system.

**Control structures in student responses to LISS reading question**

*Nontrivial* (CS1) was observed in 14 responses (out of 55 total; e.g., “Linear independent, because it only contains the trivial solution”, #50). Despite OP7: *Find nontrivial scalars* being described as optional, *Scalars* (CS2) was observed in 10 responses (e.g., “This set is linearly dependent as there is the nontrivial dependence relation \`a_1=-2, a_2=-1, a_3=1\`.” #3). Here, because the student provided nontrivial scalars, we only coded the responses for *Scalars* and not *Nontrivial*.

We identified 10 additional control structures in student responses, four of which were identified in at least six responses: *FreeVar* (CS3), *NumSol* (CS4), *LinearCombo* (CS5), and *Pivots* (CS6). *FreeVar* (CS3, 10 responses) and *NumSol* (CS4, 6 responses) could derive from the solution path demonstrated in the textbook. *FreeVar* alludes to the number or existence of free variables in the justification (e.g., “Set is linearly dependent. These can be modeled by a system of 4 equations which can be represented in a matrix that, when row-reduced, yields a free variable.” #17) whereas *NumSol* alludes to the number of solutions to the homogeneous system in their justification. For example,

> Following Example 2 of this section, if we form a general relation of linear dependence, find the corresponding system of homogeneous equations of each of the matrix entries, create the coefficient matrix and row reduce, we see that the resulting RREF of the coefficient matrix gives us a 4x3 matrix that has one free variable. This says that there are infinitely many solutions for the scalars in the relation of linear dependence, which further means (by Def LI) that the given set of matrices is linearly dependent. (#15)[1]

The other two control structures—*LinearCombo* (CS5, 14 responses) and *Pivots* (CS6, 8 responses)—were not directly related to the textbook examples. *LinearCombo* (CS5) alludes to the existence of linear combinations within the set or to finding the exact linear combination by expressing one vector matrix in terms of others (e.g., “The set of matrices are linearly dependent because 2[[1,3],[-2,4]]+[[-2,3],[3,-5]] = [[0,9],[-1,3]] thus making one vector a linear combination of the others.” #1) whereas *Pivots* (CS6) is assigned when the students’ responses suggest a comparison of the number of pivots with the number of vectors in their justification (e.g., “The set of matrices are linearly independent because there are less pivots than vectors.”).

Other control structures were observed in at most three responses (“Others” in Figure 3). For example, students gave justifications: using the consistency of the homogeneous system (CS7:

[1] Note that this response also used *FreeVar* (CS3, “... has one free variable”). We could not find responses using only *NumSol* (CS4).

*Consistency*), using singularity of the coefficient matrix (CS8: *Singularity*), by reducing the vector matrix to identity (CS9: *VectorMatrix=I*), by reducing the coefficient matrix to identity (CS10: *CoefMatrix=I*), mentioning the shape of the coefficient matrix (e.g., number of rows or columns, number of vectors, number of equations; CS11: *MatrixShape*), and using a theorem, (Theorem Row-Equivalent Matrices represent Equivalent Systems, CS12: *REMES*).

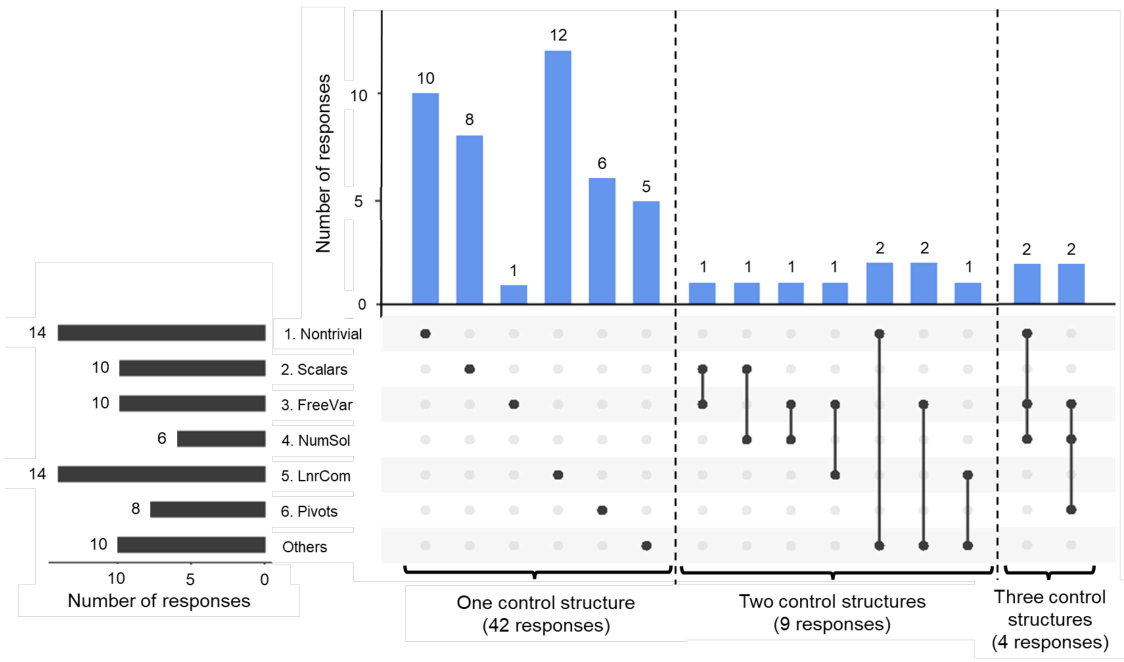


*Figure 3: Distribution of control structures used in LISS-RQ1 (n = 55)*

Figure 3 is an UpSet plot (Conway et al., 2017; Lex et al., 2012) that summarizes the distribution of control structures in student responses. Among the 55 interpretable responses, 42 used one control structure; nine used two control structures; and four used three control structures. *FreeVar* (CS3) and *NumSol* (CS4) was the most common combination, recorded in five of 13 responses with multiple control structures.

## Discussion and Conclusion

Analyzing the control structures in student responses to reading questions allowed us to learn more about how students interpret the content in the textbook and potentially rely on other resources to tackle the problem of deciding whether a set of vectors is linearly independent. In the textbook example, the connection between the existence of free variables, the number of solutions to the homogeneous system, and the existence of nontrivial solutions are demonstrated sequentially, as if one needs to go through all these steps to answer the question. It is possible that the author wants to stay consistent with the definition of linear independence in the textbook (Figure 4) by ending the action-feedback loop with the criteria "whether there are non trivial solutions to the homogeneous system." However, given that the conditions are equivalent mathematically (e.g., free variable(s) exist ⇔ infinitely many solutions to the homogeneous system ⇔ existence of a nontrivial solutions), students may have realized that is possible to justify linear dependence after carrying out either or both operations—*Free variables* (OP4.1) and *Number of solutions* (OP4.2)—and inspecting the feedback. This suggests that students may see these three conditions as equivalent, rather than as a uni-directional line of reasoning as described by the author (free variable(s) exist ⇒ infinitely many solutions to the homogeneous system ⇒ existence of non-trivial solutions), which is an important learning objective and must be emphasized throughout the course.

> **Definition LI. Linear Independence.** Suppose that $V$ is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ from $V$ is ***linearly dependent*** if there is a relation of linear dependence on $S$ that is not trivial. In the case where the ***only*** relation of linear dependence on $S$ is the trivial one, then $S$ is a ***linearly independent*** set of vectors.

*Figure 4: Definition of linear independence in section LISS.*

*LinearCombo* (CS5, 14 responses) and *Pivots* (CS6, 8 responses) were not mentioned in the textbook example. Independently, CS5 was more frequently observed than *Nontrivial* (CS1, 10 responses), which was demonstrated in the example. We suspect that students may be relying on the textbook's definition of the dependence relation (Figure 5) and not necessarily on the operator paths or the control structures presented in the textbook example. Likewise using *Pivots* might be justified by students making connections across sections as the connection between pivots and number of solutions is explained in section Types of Solution Sets (TSS), 17 sections before LISS. This implies a potential avenue for future research, which entails conducting a longitudinal examination of students' responses to reading questions across different sections in order to observe how their conceptions evolve over time.

> **Definition RLD. Relation of Linear Dependence.** Suppose that $V$ is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, an equation of the form
>
> $$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$
>
> is a ***relation of linear dependence*** on $S$. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is a ***trivial relation of linear dependence*** on $S$.

*Figure 5: Definition of linear dependence relation in section LISS.*

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