How do calculus instructors frame tasks for introducing derivatives symbolically? Identifying Calculus-specific instructional situations

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In this study, I describe how seven U.S. college calculus instructors framed instructional tasks for introducing derivatives symbolically to students. During one-on-one interviews, the instructors were presented with before and after student conceptions and were asked to propose tasks for introducing derivatives symbolically using the limit definition of derivative, both at a point and as a function. The instructors' task framings reveal the types of mathematical problems students are expected to work on with respect to the content at stake: symbolic definitions of derivatives. Although the instructors heavily relied on calculating situations to introduce derivatives symbolically, some also used graphing, exploring, installing, and proving situations, which shows the high variability of task framing in calculus even when student conceptions before and after working on the tasks are predetermined.

Keywords: Calculus teaching, instructional tasks, derivatives, instructional situations

Despite the continuing research on students' understandings of calculus concepts, there is less research on the teaching of calculus (Larsen et al., 2017). The scarce research in this area focuses on the student outcomes of instructional treatments (e.g., Borji et al., 2018; Hähkiöniemi, 2006) or tasks that instructors use in their assignments, textbooks, and exams (e.g., Tallman et al., 2016; White & Mesa, 2014). I focus on teaching of calculus by investigating instructional tasks, or activities that are used during instruction "to focus students' attention on a particular mathematical idea" (Stein et al., 1996, p. 460). As basic units of instruction and "objects of students' activity" in mathematics classrooms (Ni et al., 2018; Sullivan et al., 2009, p. 859), instructional tasks are often seen as the conceptual bridge between teaching and learning (Christiansen & Walther, 1986; Stein & Lane, 1996).

I contribute to this line of research by focusing on the ways college Calculus I instructors shape, or *frame*, students' mathematical work during instruction of one specific piece of content: derivatives. By focusing on these subject-specific *framings* in tasks, we learn about the learning opportunities that are offered to students during instruction (Herbst et al., 2018). Here, I address the following research question: How do college Calculus I instructors frame instructional tasks to introduce derivatives symbolically?

Theoretical Frameworks

Framing

The concept of framing originates from the work of social scientists and sociologists Gregory Bateson (1904-1980) and Erving Goffman (1922-1982). Bateson (1974/2003) referred to framings as definitions of reality, determined by the culture, that allow people to interpret and respond to objects and events during social interactions. For Goffman, framing is simply one's answer to the question "What is it that's going on here?"; the answer, implicitly or explicitly, informs the person about acceptable and unacceptable (re)actions and behaviors during a social event (1974/1986, p. 8). To unpack college calculus instructors' framing of instructional tasks for teaching derivatives, I divide Goffman's question of 'what is it that is going on here' into two questions: 1) What is it that is going on here with respect to the content presented in the task? and 2) What is it that is going on with respect to interaction with other actors? I refer to the answers to these questions as *Framing for Interaction with Content* and *Framing for Social Interaction*. Although students make sense of what they should and should not do during a lesson by answering both questions, I only focus on the former here due to space reasons.

To answer the question about framing of a task for interaction with the content, I rely on Herbst and colleagues' (2020) notion of instructional situations (or situations in short) within the notion of didactical contract that govern the teacher-student relationship in the teaching and learning of mathematics (Brousseau, 1984). Instructional situations, as subject-specific types of mathematical problems in a course of studies, communicate teacher's and students' appropriate and customary units of work regarding the knowledge at stake (Herbst et al., 2020, p. 5). Given an instructional situation (e.g., finding the equation of a line in an algebra task), students know what kind of problem they are presented with and what kind of mathematical work and interactions they should prototype (Herbst & Chazan, 2012). The eight generic types of problems they have identified so far in geometry and algebra include: *graphing*; *calculation*; *exploration/conjecturing*; *doing proofs*; *generating a new definition or installing a new concept*; *installing a new theorem, property, or formula*; *solving equations with known methods*; and *solving word problems* (see Herbst et al., 2010).

Conceptions of Derivatives

To ground instructors in the students' conceptions of derivatives, I used Zandieh's (2000) framework for the concepts of derivative (Table 1). Zandieh (2000) organized students' conceptions by representation (graphical, verbal, physical, symbolic) and process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of* finding the limit of the ratio as an *object*. Here, I focus on instructors' tasks that were proposed to introduce derivatives symbolically at the limit layer and at the function layer.

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical	Verbal	Physical	Symbolic
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

Methods

Using an in-take survey of 48 calculus instructors, I purposefully selected eight interviewees with different patterns of inquiry (see Gerami, 2023 for details): Adrian, Barry, Gopher, Justin, Matthew (He/Him pronouns), Monica (She/Her pronouns), Max (They/Them pronouns), and Alex (all pronouns). The participants were tenured and had more than four semesters of experience teaching Calculus I with inquiry. My focus on teaching experience and using inquiry was intentional; as my aim was to have instructors suggest diverse instructional tasks based on targeted students' conceptions of derivatives during the interviews, I assumed that experienced instructors who employ inquiry-based teaching methods for calculus are more likely to know of and consider student thinking when proposing tasks, and that they would propose a broader range of instructional tasks compared to those who teach calculus more traditionally (e.g., following conventional textbooks, lecturing). I conducted four semi-structured interviews (1-2 hour long each) with each instructor, as they proposed up to eight instructional tasks for introducing derivatives. I structured the interviews based on Zandieh's (2000) framework. During each interview, the instructors were asked to propose two tasks that targeted students' conception at the limit and the function layer within one representation. They could use their teaching materials to propose already used tasks or design new tasks for the future. Here, I report the findings based on the tasks proposed in response to Prompt 7 and 8:

Prompt 7: Assume that you have already taught about the difference quotient ($\frac{f(x)-f(x_0)}{x-x_0}$ or $\frac{f(x+h)-f(x)}{h}$) and want students to learn about the derivative of a function at a point ($f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ or $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$). Propose a task where students have to figure out the derivative of a function at a point.

Prompt 8: Assume that you have already taught about the derivative of a function at a point ($f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ or $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$) and want students to learn about the derivative function ($f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$). Propose a task where students have to represent the derivative function.

Because calculus-specific instructional situations have not been identified in the literature, I used inductive/deductive hybrid thematic analysis (Proudfoot, 2022) to identify them. The analysis entailed using pre-ordinate themes "through the application of an explicit theoretical framework developed through engagement with the literature" (the deductive element) to generate themes from the data (the inductive element; Proudfoot, 2022, p. 1). For the deductive portion, I used the generic types of problems: *graphing*; *calculation*; *exploration/conjecturing*; *doing proofs*; *generating a new definition or installing a new concept*; *installing a new theorem, property, or formula*; *solving equations with known methods*; and *solving word problems* (Herbst et al., 2010). For the inductive portion, I identified the emerging instructional situations by focusing on the mathematical work that instructors expected of their students would do within a generic type of problem. I also listened to the interviews and read the transcripts to find relevant information that instructors mentioned but did not include in the task description. After three rounds of coding, I identified 36 calculus-specific instructional situations across *all* eight tasks.

Findings

Figure 1 lists the 12 calculus-specific instructional situations that instructors used in their two tasks proposed for teaching derivatives symbolically at a point and as a function (Prompt 7 an 8).[1] All seven instructors who proposed tasks used one or two specific calculating situations to frame their tasks: *Calculating derivative at a point using a limit definition* (C6) and *Calculating derivative function $f'(x)$ using a limit definition* (C11). Four instructors utilized other types of situations to frame their tasks. Alex, Max, and Adrian used graphing situations at different layers

[1] Each instructional situation is identified by a letter representing the generic situation type (C for Calculating, E for Exploring/Conjecturing, G for Graphing, I for Installing, and P for Proving), and a number to differentiate it from other situations in the same category. Although Figure 1 only includes the situations that appeared in the tasks for introducing derivatives symbolically, I kept the original numbering system to keep my findings across research reports consistent.

(secant line [ratio, G1], tangent line [limit, G2], derivative function by plotting [function, G3]). Alex, Max, and Monica used the tasks to install ideas and formulas (e.g., power rule, I7). Finally, Max was the only instructor who used exploring and proving situations (E11 and P1).

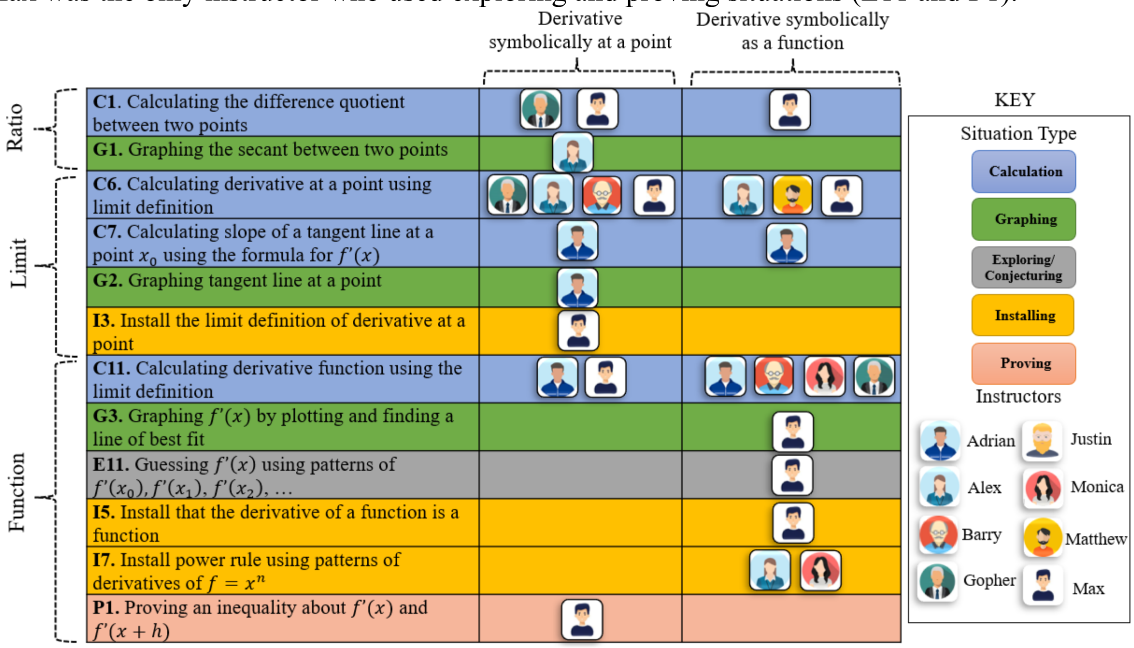
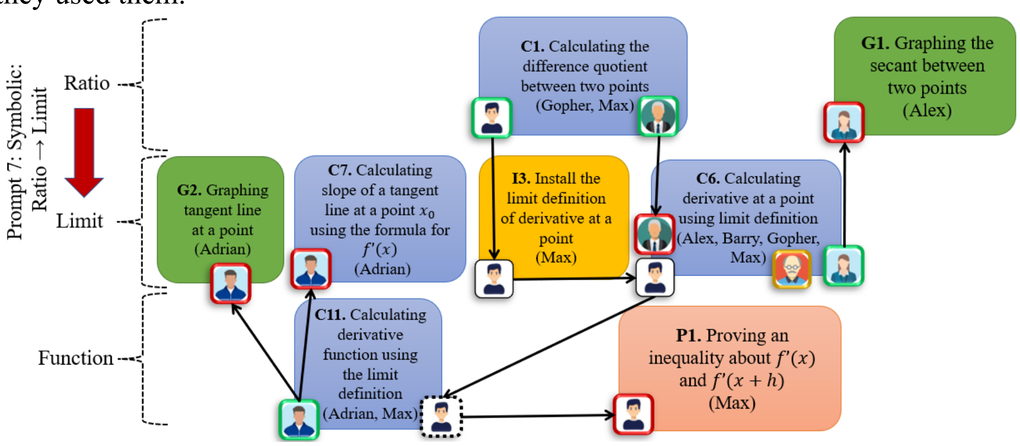


Figure 1. Identified instructional situations for introducing derivatives symbolically at a point and as a function and instructors who used them. Situations are organized by layer and generic type.

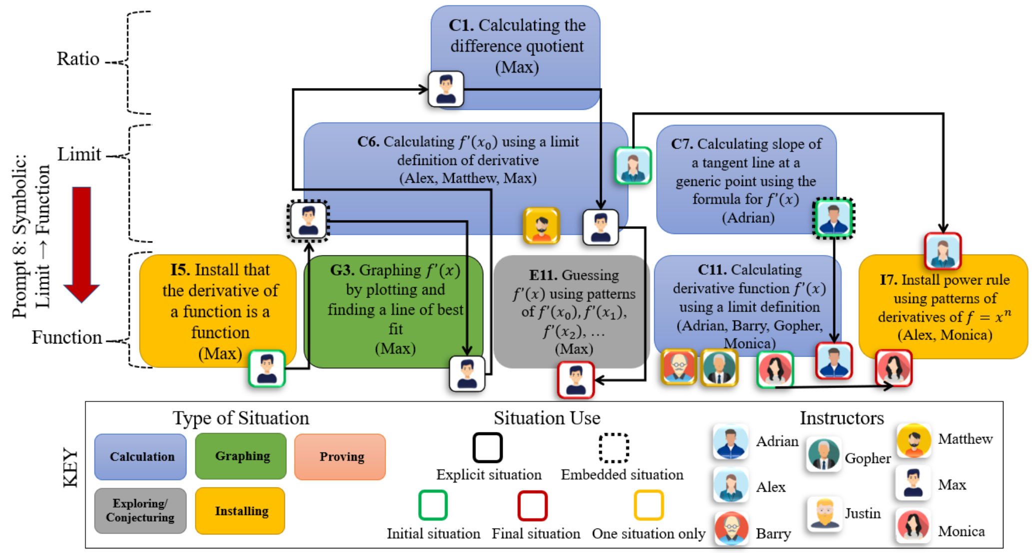
Justin did not propose any tasks for Prompt 7 and 8 (he stated that he does not explicitly talk about the difference quotient in the notation proposed by the prompts or the limit definition of derivative). Monica and Matthew did not propose tasks for Prompt 7, as they did not teach about derivative symbolically as a point, but they did propose tasks for Prompt 8. Next, I provide an overview of how the instructional situations were used in the tasks proposed for each prompt.

Tasks Proposed for Prompt 7: Ratio → Limit

Figure 2a shows the instructional situations that five instructors used to frame their tasks in the order they used them.



(a)



(b)

Figure 2. Calculus-specific instructional situations in the order they appeared in the tasks proposed for (a) Prompt 7 and (b) Prompt 8. The arrows show the order in which the situations were used by each instructor. A situation is explicit if students know what type of problem they are working on after being presented with a task, whereas a situation is embedded within an explicit (larger) situation if, while working on the explicit problem, students find out that they must solve another problem in order to solve the explicit situation.

Max and Gopher started their tasks at the ratio layer, before the limit/target layer, by asking students to: *calculate difference quotients between two points*, x_1 and x_2, or x_0 and $x_0 + h$ (C1). Gopher ended his with the situation of *calculating the derivative at a given point* x_0 at the limit layer (C6), which was the situation that Barry started and ended his task with. Barry, Gopher, and Max expected their students to use formal limit notation to calculate $f'(x_0)$ at a given point x_0 ($\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$). Max continued their task with four more situations after calculating difference quotients between two points (x_0 and $x_0 + h$, with decreasing h, the ratio layer; C1). First, Max *installed the limit definition of derivative at a point* ($\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$; I3). While other instructors installed the limit definition of derivative at a point *themselves* before the tasks (most probably via lecture), Max expected students to come up with the definition as part of the task by asking them: "What we can do to determine a value at $x = 0$? What concept from calculus helps us analyze questions about arbitrarily small sizes?". Next, they asked their students to: 1) use their newly defined derivatives to *calculate* $f'(x_0)$ *at some given points* (C6), and 2) to show an inequality $f'(x) < f'(x + h)$ for $h > 0$ (*P1. Proving an inequality is true for a specific function*). The proving situation has embedded calculating situations—*calculating* $f'(x)$ *and* $f'(x + h)$ *using the limit definition of derivative* (C11).

Alex started their task at the target/limit layer with the same situation that Barry, Gopher, and Max also used in their tasks: *calculating* $f'(x_0)$ *at some given points* (C6). However, unlike the other instructors, they did not use formal limit notation; instead, they expected their students to assume the denominator goes to zero when calculating the difference quotient $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$. Alex explained that, at this point in their course, the limits are replaced with arrows (→) instead of equal signs ($\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to c$ as $\Delta x \to 0$, instead of $\lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = c$). To assist students with confirming the value of the difference quotient with Δx going to zero, Alex then asked students to "Describe what happens to the secant lines as Δx gets closer to zero?", which is a situation at the ratio layer: *graphing secant lines Δx-units distanced to the right side of a fixed point x_0, assuming Δx gets closer to zero* (G1).

Adrian approached the task differently than the rest of the instructors by starting his task at the function layer (the layer after the target/limit layer): *calculating $f'(x)$ using the limit definition of derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$* (C11). He then asked his students to g*raph a tangent line at a given point $(x_0, f(x_0))$* (G2) and *find its slope* (i.e., the derivative at a point or $f'(x_0)$; C7), which are two situations at the limit/target layer. I capture the latter situation C7 separately than those used by Barry, Gopher, Max and Alex (*C6. Calculating $f'(x_0)$ at a given point x_0 using the limit definition of derivative at a point* or *Calculating the difference quotient assuming the denominator goes to zero*) because Adrian expected his students to substitute x_0 in the formula of $f'(x)$ found in his task's initial situation, instead of using the limit definition to calculate derivative at the point.

Tasks Proposed for Prompt 8: Limit → Function

Figure 2b shows the instructional situations that the seven instructors used to frame their tasks in the order they used them. At the function/target layer, *calculating $f'(x)$ using the limit definition of derivative* (C11) was the most common situation, used by four instructors—Barry, Gopher, Adrian, and Monica—with Barry and Gopher solely using this situation to frame their tasks. In Adrian's task, the situation was embedded within another—*calculating slope of a tangent line at a point x_0* (C7), as students were expected to calculate and use the formula for $f'(x)$ to find the slope of the tangent line at x_0. Monica asked her students find $f'(x)$ by calculating the difference quotient, $\frac{f(x+h)-f(x)}{h}$, and assuming that *h* goes to zero. After finding $f'(x)$ for $f = x$, $f = x^2$, and $f = x^3$, she used an installing situation, expecting students to *install the power rule using patterns of derivatives of $f = x^n$ for $n \geq 1$* (I7): "What is the general formula for the derivative of x^n?"

Matthew and Alex started their tasks with a situation within the limit layer, where students were expected to *calculate $f'(a)$ for a generic variable (a) either using the limit definition of derivative at a point using the formal or informal definition of derivative with limit of difference quotient* ($\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ for Matthew, and $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to c$ *as* $\Delta x \to 0$ for Alex; C6). Although the instructors were prompted to introduce students to the derivative symbolically as a function, Matthew only used the situation of calculating $f'(a)$ for a generic value of a at the limit layer, not $f'(x)$. Because Matthew used a linear function in the task—$f(x) = 5x - 3$, with $f'(x) = 5$ for all values of x, it is possible that he equated finding $f'(a)$ to $f'(x)$, which, mathematically, is true. However, from the task's description, it is not obvious whether students would conclude that the "derivative 5" is in fact a function. Alex continued their task like Monica's by installing the power rule using patterns of derivatives of $f = x^n$ *for* $n \geq 1$ (I7).

To frame their task, Max used five explicit situations across all three layers. Max started the task by asking students to review their work for Task 7, in which students found the derivative at various points. They then encouraged students to *install that derivative of a function is a function* (I5) by asking: "Because we have a derivative value at every function input value, what do we have?". Next, Max asked students to *graph f' by plotting derivative values at different points and finding a curve of best fit* (G3). Although students used the derivative values they found at various points in Task 7, I captured *calculating* $f'(x)$ *using the limit definition of derivative* $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (C6) as an embedded situation before the graphing situation to count for the work students should have done to complete this part. Next, Max used a situation at the ratio layer by asking students to *calculate the difference quotient* $\frac{f(x_0+h)-f(x_0)}{h}$ (C1) for various values of x_0, which was immediately followed by asking students to *calculate* $f'(x_0)$ *using the limit definition of derivative* $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ (C6) for their chosen x_0 values. Max ended his task at the function/target layer with an exploring/conjecturing situation: *guessing* $f'(x)$ *using patterns of* $f'(x_0), f'(x_1), f'(x_2), \ldots$ (E11).

Discussion

In this report, I identify the ways seven college Calculus I instructors framed their instructional tasks to introduce the symbolic representations of the derivative at a point and as a function, that is the limit definitions of derivative. While all instructors used calculating derivative at a point and/or calculating $f'(x)$ using a limit definition, four instructors also relied on other types of situations: graphing, exploring/conjecturing, installing, and proving. Although I did not review the findings from the rest of their tasks in this report, the proving situation (P1), the installation of power rule (I7) and installation of the limit definition of derivative at a point (I3) were only used to frame tasks for Prompt 7 and 8. Moreover, the proving situation was the only proving situation used across all tasks, which suggests that the instructors avoided using proving situations when introducing derivatives. More specifically, regarding the installation of power rule (I7), it is noteworthy that two instructors (Alex and Monica) used the symbolic representations to install one specific derivative rule—the power rule. Lastly, although some instructors used similar calculus-specific instructional situations within the three generic problem types, only two instructors proposed the same task consisting of all the same instructional situations (Barry and Gopher for their second tasks). This illustrates instructors' varied ways of shaping students' work when introducing derivatives, which shows that inquiry methods are implemented considerably differently among instructors. Nonetheless, most instructors relied on explicit, rather than implicit situations, which shows their inclinations towards scaffolded inquiry, rather than open-problem or discovery-based inquiry.

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