A Teaching Experiment for U-Substitution Based on Quantitative Reasoning

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Prior work has created approaches to calculus based on crucial quantitative reasoning. For integration, however, the major topic of u-substitution has generally not been fully detailed in these paradigms. This paper presents a study where students were taught u-substitution from a quantitative perspective based on a three-part quantitative structure: differential quantity, integrand quantity, bounds quantity. The students reasoned about the quantitative conversions in flexible ways, and used various quantitative relationship types in their reasoning. However, reasoning about the differential quantity was difficult, and a new type of "collapse" metaphor was identified. By the end, the students had all developed a good quantitative basis for u-sub.

Keywords: calculus, integrals, u-substitution, quantitative reasoning, teaching experiment

Research has strongly established quantities-based meanings for calculus concepts as essential for robust student understanding and for productive usage outside of math classes (Byerley, 2019; Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). For integrals, this means moving away from the purely "area under a curve" meaning in favor of a "sum of small bits" meaning (Ely, 2017; Jones, 2015; Sealey, 2006). Several studies have examined teaching integrals and students' reasoning and modeling with integrals through this meaning (Bajracharya et al., 2023; Blomhøj & Kjeldsen, 2007; Chhetri & Oehrtman, 2015; Dray & Manogue, 2023; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). However, the major "u-substitution" method has largely been absent from explicit research examination in these paradigms, despite its importance as the first in a long line of substitution techniques, and as a means in the sciences and engineering for converting between quantitative expressions (e.g., see Koretsky, 2012). Treating u-substitution quantitatively can allow students both to effectively *use* it and to understand the mechanisms for *why* it works. Very recent theoretical work has examined u-substitution through a quantitative paradigm (Jones & Fonbuena, 2024), and here we contribute by extending this theoretical work to an empirical examination of students learning u-substitution in this way. Our research questions were: (1) In teaching u-substitution through this paradigm, what quantitative reasoning did students use? (2) What understandings did they construct for the individual parts of u-substitution and for the overall u-substitution process?

Brief Review of Closely Related Literature

Research work has put forward a quantitative structure for definite integrals called *adding up pieces* (AUP) (Jones, 2013; Jones & Ely, 2023). AUP is comprised of three parts: *partition*, *target quantity*, and *sum* (see also Dray & Manogue, 2023; Sealey, 2014; Von Korff & Rebello, 2012). To explain these, consider the example of a solar panel generating energy (in kJ), where power is defined as the rate of energy generation (kJ/hr) over time (hr): $E = P \cdot t$. If power is constant, Oehrtman and Simmons (2023) call this a *basic model*. If power varies over time, this basic product cannot be used to find the total energy. Rather, time must be *partitioned* into essentially infinitesimally small pieces, denoted by the differential *dt* (Ely, 2020). In other words, *dt* is a very tiny Δt (Ely & Ellis, 2018). While differentials can be rigorously formalized by limits or hyperreals (see Jones & Ely, 2023), the more loosely-defined "essentially infinitesimal" is quite common across the sciences and is the crucial idea for quantitative reasoning, even if no specific threshold is defined for passing from "Δ" to "*d*" (Amos & Heckler, 2015; Hu & Rebello, 2013; Pina & Loverude, 2019; Thompson & Dreyfus, 2016; Von Korff & Rebello, 2014).

With infinitesimal partition pieces, power can be considered essentially constant over each *dt* piece, so the basic model can be applied to find the tiny bit of energy produced within each *dt*: $dE = P \cdot dt$. Here, energy is called the *target quantity*, and the application of the basic model to an infinitesimal piece is called the *local model* (Oehrtman & Simmons, 2023). While integrals in AUP can utilize any quantitative relationship (Simmons & Oehrtman, 2017), in this paper we primarily focus only on quantitative relationships defined through a product: $Q3 = Q1 \cdot Q2$.

With a conceptualization of tiny bits of the target quantity in each infinitesimal partition piece, *sum* refers to the literal summation of these target quantity bits to obtain the total target quantity amount. AUP follows Leibniz's convention (Katz, 2009) in using the integral symbol, $\int$, as a literal "sum" symbol. For example, $\int_{t1}^{t2} P(t)dt$ is the sum of infinitesimal bits of energy (each produced by $P \cdot dt$) across the *dt* pieces between $t = t_1$ and $t = t_2$, yielding the total energy.

Finally, to introduce some terminology, because time is an input for the varying power, $P(t)$, as well as for the target quantity, $E(P, t)$, we call it the main *input quantity* (Jones & Fonbuena, 2024). Because power becomes the integrand in the integral, we call it the *integrand quantity*.

Theoretical Perspective: Quantitative Reasoning Applied to U-Substitution

Thompson (Smith & Thompson, 2007; Thompson, 1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure" (1990, p. 12). Quantitative structures are based on quantitative relationships, defined as "the conception of three quantities, [any] two of which determine the third" (1990, p. 12) (Figure 1a, below). In particular, a product-based definite integral captures a quantitative relationship between an *input quantity*, an *integrand quantity*, and a *target quantity* (Figure 1b). Our previous work applied quantitative reasoning to u-substitution (Jones & Fonbuena, 2024), which we recap here through the solar panel example.

Suppose the sun rises at 6am ($t = 0$) and reaches its zenith at 12pm ($t = 6$). A horizontal solar panel would have less power in the early morning, but power would increase to a maximum at noon. We use $P(t) = 250 \sin\left(\frac{\pi}{12}t\right)$ kJ/hr as a reasonable model for this situation. Based on the explanations in the previous section, this means that for each *dt* interval, the local model would be $dE = 250 \sin\left(\frac{\pi}{12}t\right) \cdot dt$, and the total energy would be given by $E = \int_0^6 250 \sin\left(\frac{\pi}{12}t\right) dt$.

To now move to u-substitution, note that while energy can be tracked over time, the power is mostly due to the sun's *angle* with the panel, so it might make sense to switch from time as the input to the sun's *angle*. This is exactly the quantitative question we claim u-substitution is built on: What if we want to track the target quantity in terms of a new input quantity (e.g., θ), rather than the original input (e.g., t)? To start the conversion, note there is a relationship from $t \to \theta \to P$, or $P(\theta(t))$, which is called *nested multivariation (MV)* (Jones, 2022). In the "quantity triangle" (Figure 1c), this nested MV is located along the edge of the triangle between the original input and integrand quantity. Time and angle are related by $\theta = \frac{\pi}{12}t$, meaning we can

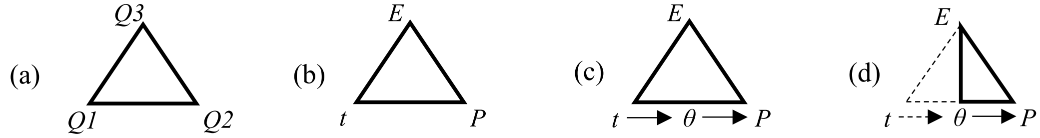


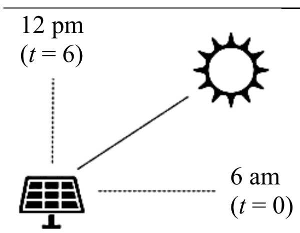
Figure 1. A basic quantitative relationship (a); and u-substitution as the transformation of the original integral relationship (b), through the nested MV between t, θ, and P (c), to a new integral relationship (d)

convert $P\big(\theta(t)\big) = 250\sin\big(\theta(t)\big)$ to simply $P(\theta) = 250\sin(\theta)$. This relation also means that a change in angle is always $\pi/12$ as big as the change in time, so $d\theta = \frac{\pi}{12}dt$, or $dt = \frac{12}{\pi}d\theta$. The local model can now be fully described in terms of θ: $dE = 250\sin(\theta)\cdot\frac{12}{\pi}d\theta$. Finally, the sum over dt pieces from $0 \le t \le 6$ hr is equivalent to a sum over $d\theta$ pieces from $0 \le \theta \le \frac{\pi}{2}$ rad. Taken together, $E = \int_{t=0}^{t=6} 250\sin\left(\frac{\pi}{12}t\right)dt$ transforms to $E = \int_{\theta=0}^{\theta=\pi/2} 250\sin(\theta)\frac{12}{\pi}d\theta$. Our previous theoretical work defined u-substitution quantitatively as *transforming the original relationship by shifting the vertex from t to θ, to form a new relationship* (Figure 1d).

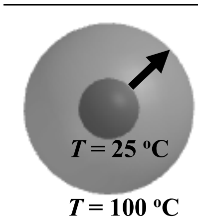
How is this transformation enacted? Our previous work also detailed a three-part process to do so: *differential, integrand,* and *bounds* (Jones & Fonbuena, 2024). These parts match the *partition, target quantity,* and *sum* structure of AUP. *Differential* means converting infinitesimal pieces of the original input to infinitesimal pieces of the new input (e.g., $dt \to [conversion]d\theta$). *Integrand* means using the nested MV structure to redefine the integrand quantity in terms of the new input (e.g., $P(t) \to P(\theta)$), allowing the target quantity to be determined by the new input (e.g., $E(P,t) \to E(P,\theta)$). *Bounds* means re-describing the sum over the original input pieces to a sum over the new input pieces (e.g., a sum over $0 \le t \le 6$ hrs to a sum over $0 \le \theta \le \pi/2$).

Methods

For this study, we designed an experimental lesson consisting of two u-substitution tasks using real-world contexts (Figure 2). The first author was the researcher-instructor for the lesson. In Task 1, the goal was for students to construct an integral and then enact the u-substitution conversions outlined in our theoretical perspective section. In Task 2, the goal was similarly for students to identify the three-part transformation of (a) *bounds*: converting between the sum over $25 \le T \le 100$ and the sum over $10 \le r \le 15$, (b) *integrand*: converting between $4\pi\left(\sqrt{T}+5\right)^2$ and $4\pi r^2$, and (c) *differential*: using the relation $dr = \frac{1}{2\sqrt{T}}dT$ to convert between dr and dT.



Task 1: A solar panel generates energy in joules, J. Power is the rate at which energy is produced (joules per hour, J/hr). Note that power depends on the location of the sun and is maximized when the sun is directly overhead (12 pm). We'll use $P(t) = 250\sin\left(\frac{\pi}{12}t\right)$ as a reasonable model, because it is zero at 6 am (t = 0), increases from $0 \le t \le 6$ hours, and is maximized at 12 pm (t = 6).
(a) Construct an integral (in time) for the total energy generated over these 6 hours.
(b) The sun's angle is what really matters for power. What if we tracked the total energy produced with respect to angle instead? How does that change the integral?



Task 2: The gas inside a spherical balloon is heated from the inside from T = 25 ºC to T = 100 ºC. As the gas expands, the sphere grows [a dynamic visual is provided]. Suppose the radius (in cm) is given by $r(T) = \sqrt{T} + 5$. Note that at a given moment, for a tiny change in temperature, the tiny bit of increased volume is essentially the product of the current surface area and the corresponding tiny change in the radius.
(a) Consider the following two integrals, and explain how they each find the total change in volume: $\int_{T=25}^{T=100} 4\pi\left(\sqrt{T}+5\right)^2 \frac{1}{2\sqrt{T}}dT$ and $\int_{r=10}^{r=15} 4\pi r^2\, dr$.
(b) Explain how these two integrals are equivalent to each other.

Figure 2. Abbreviated versions of the two interview lesson tasks

To ensure participating students had the needed background, we recruited from a first-semester calculus class (*not* taught by the authors) that incorporated quantitative meanings for calculus concepts. We recruited when the unit on integration had started, so they would have basic knowledge about definite integrals through an AUP perspective. We recruited six students and paired them into three groups of two students each: Luke and Liam, Matt and Mark, and Ruby and Rafael. We analyzed the data as follows. First, for quantitative reasoning, we coded excerpts according to whether they referred to (a) the quantities, (b) the units, (c) relationships between quantities, and (d) operations done on quantities. We also coded for contraindications of quantitative reasoning, such symbol manipulations without attention to quantities. Second, to connect their reasoning to u-substitution, we coded excerpts according to the three parts in our breakdown: (a) *differentials*, (b) *integrand*, and (c) *bounds*. We looked for connections between their quantitative reasoning and each of the parts. Finally, the researcher-instructor asked the students at times to recap what they understood about u-substitution and what it meant. We summarized these general understandings for each student about u-substitution.

Results

Research Question #1: Students' Use of Quantity and Quantitative Relationships

We first describe how our students used quantities and quantitative relationships. One important aspect was what Redish (2005) called loading meaning onto symbols, which means the students directly interpreted the symbols and expressions as referents to the quantities in a productive way. To illustrate, consider these excerpts from Matt and Ruby:

> *Matt*: This [points to sin(π/12 t)], is showing how much joules you're getting per hour. And this [points to dt] is showing small time, in hours.

> *Ruby*: That equation [i.e., expression, points to $\frac{1}{2\sqrt{T}}\,dT$] is essentially equal to bits of radius.

Next, recall that Thompson's (1990) definition of quantitative relationships is of three quantities, where any two determine the third. Our students certainly used this type of relationship across the interviews for time-power-energy, angle-power-energy, temperature-area-volume, and radius-area-volume. However, our students also exhibited different types of relationships outside of this classic definition. The first type was "two-quantity relationships."

> *Rafael* (Task 2, after converting to radius): Well, it's just a much more direct relationship between radius and volume. The relationship between temperature and volume isn't as direct.

> *Liam* (Task 1, in converting the differential): We also want the relationship between dt and $d\theta$.

In fact, these two-quantity covariational relationships (Carlson et al., 2002) were a key part of the students' work. While each context held four quantities (e.g., time, angle, power, energy), decomposing into two-quantity relationships helped them track the conversions (cf. Jones, 2022).

Also, some three-quantity relationships the students imagined did not have Thompson's triangle format (Figure 1). Rather, some relationships followed the nested MV structure instead.

> *Liam*: At that time… The angle of the sun at a certain time, and like the amount of energy produced.

> *Mark*: Our tiny change in temperature… how this temperature affects the radius at that specific temperature and all that stuff we're adding up… we're adding up different volumes.

Liam described a relationship of *time* → *angle* → *energy*. Mark described a relationship of *temperature* → *radius* → *volume*. Importantly, though, note that the students' descriptions go from *original input* → *new input* → ***target*** *quantity*, skipping the integrand quantity. This result conflicts with the study's intended *input* → *new input* → ***integrand*** *quantity*, suggesting the lesson did not properly scaffold this particular nested MV relationship entailed in u-substitution.

Research Question #2: Students' Quantitative Conversions

We now explain the results pertinent to each of the three parts in the u-substitution structure: *differential, integrand,* and *bounds*. First, the *integrand* and *bounds* were mostly unproblematic for the students. The students readily converted the summation bounds from one quantity to the other and demonstrated understanding the quantitative equivalence. For example Matt explained,

> *Matt*: Your degree Celsius is going to start at 25 degrees and going to 100 degrees… It's kind of the same thing as radius… We're both [i.e., temperature and radius] changing at the same time. It's just you calling this one in terms of temperature instead of radius, even though they both happen at the same time… It's the same thing as showing when your radius is changing from 10 to 15.

The students also readily converted the integrand quantity from a dependence on the original input to the new input. These conversions, though, tended to be based on symbolic appearance at first, and were only later justified through quantities. Yet, we view this symbolic work as fine, so long as the quantitative relationships were understood, as demonstrated in these excerpts:

> *Mark*: Basically, because this *is* [emphasis in original] the angle right here [points to $\frac{\pi}{12}t$]. Like, that's the sine of theta [$\sin(\theta)$], it's our power output and this [again points to $\frac{\pi}{12}t$] would be what θ would be equal to.

> *Luke*: [We have] the relationship between *r* and *T* [i.e., the function $r(T) = \sqrt{T} + 5$], so we know that, like, for whatever *T* we put in there [i.e., into "$4\pi\left(\sqrt{T} + 5\right)^2$"], it's gonna come up to the right *r* to get the same result as this one [points to the integral with "$4\pi r^2$"].

While *differential* and *integrand* were fairly unproblematic, one of the more important results was that the differential quantitative conversion was particularly cognitively demanding. The students spent significant time there and needed to carefully reason about the relationships. The first big difficulty was assuming that the differentials, such as *dt* and *dθ*, could be directly substituted. In fact, all three groups initially did so in Task 1 (Figure 3). It seemed based on the idea that all "infinitesimals" must be equivalent, since they are all so small. This idea is similar to Oehrtman's (2009) "collapse" metaphor, though it is different in that instead of collapsing to "nothing", these infinitesimals seem to all collapse to a single "infinitesimal sameness."

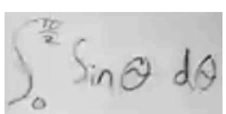
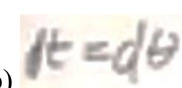
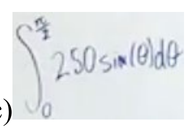
(a)  (b)  (c) 

Figure 3. Evidence from each group of initially equating dt and dθ

To help, the interviewer pointed out this implicit assumption and asked if the differentials were actually equal. The students reasoned about the quantities in order to realize they were not.

> *Luke*: I think they'd be proportional, but I don't know if they'd be exactly the same… One hour is equal to *π*/12, but in my mind they don't mean the same thing [*dt* and *dθ*]. Because *π*/12 is a ratio.

The interviewer followed up by asking if 1/100th of an hour was equal to 1/100th of a radian. This helped the students see and resolve the issue. This excerpt shows how one group began to do so:

> *Rafael*: *dt* is infinitesimal, getter smaller over a period of 6 hours. Where that's, where *dθ* is getting infinitely smaller over the range of *π*/2 [i.e., 0 to *π*/2].

Based on this reasoning, Rafael set up a proportion: $\frac{dt}{6} = \frac{d\theta}{\pi/2}$. He rearranged this relation to produce $d\theta = \pi/12\ \ dt$. We note that once this issue was resolved in Task 1, it did not reappear in Task 2. The students referred back to Task 1, now explaining that the differentials might be different sizes, and that some comparative relation between them needed to be identified.

A separate issue with differentials was interpreting "*d*" to just mean "derivative." This is sensible because the derivative was the other major concept they had experienced that used the differential notation "*d*", as in *dy/dx*. When interpreting *d* in this way, the students operated computationally and lost sight of the quantities. For example, in Task 2, Ruby interpreted "*dr*" as the derivative of the function $r(T) = \sqrt{T} + 5$, leading her to assert that $dr = \frac{1}{2\sqrt{T}}$. She then created the differential-less integral: $\int_{25}^{100} 4\pi(\sqrt{T}+5)^2\left(\frac{\sqrt{T}}{2T}\right)$. When asked about it, Rafael realized there was essentially no "local model" in the integral because of the absence of a *dT*.

> *Rafael*: This isn't, this is just for one temperature. It's not 'as we're getting smaller.'

This realization helped the students see the need to think of *dr* and *dT* as representing *quantities*.

Of course, using derivatives was not always a problem and was sometimes necessary and useful. The students needed to derive $r(T) = \sqrt{T} + 5$ to find the relationship $dr = 1/(2\sqrt{T})\,dT$. Even so, they still tended to lose sight of the quantities within that computation and had to reconceptualize the quantitative relationship in the differential equation after the fact.

Our last result to highlight here is that the students successfully enacted the three parts of u-substitution (*differential, integrand, bounds*) in different orders, suggesting that these do not need to be enacted in a specific order. Table 1 shows the orders the groups used for each task.

Table 1. The three-part u-substitution conversion enacted in different orders by the groups across the tasks

	Task 1: Solar panel	Task 2: Sphere volume
Group 1	*bounds → integrand → differential*	*integrand → bounds → differential*
Group 2	*integrand → differential → bounds*	*integrand → differential → bounds*
Group 3	*integrand → bounds → differential*	*integrand → differential → bounds*

Research Question #2: General Understanding of U-Substitution

By the end of the interview lesson, all of the students had developed an understanding of the three-part structure of u-substitution: *differential, integrand, bounds*. They expressed the need for the two integrals to be equivalent in all aspects, as seen in Mark's representative explanation.

> *Mark*: If we're going to go from one relationship to the other, like radius to temperature, or from, you know, time to degrees or I guess radians, we had to change our function, bounds, and our differential. We had to make sure that they were still equivalent statements.

The students also understood the need to find a relationship between the original input and the new input. As stated before, though, the nested MV relationship from these to the integrand quantity was somewhat blurred and pointed to a needed revision in the lessons.

We thus claim that this lesson was generally successful in helping the students (a) to reason about the quantitative nature of converting from one input quantity to another, and (b) to develop a personal understanding of the *differential-integrand-bounds* structure for u-substitution. However, we describe here one other unanticipated issue that cropped up during the lesson that points to a needed revision. While the students were reasoning quantitatively and understanding the u-substitution process, a couple of students (Ruby in particular) occasionally expressed confusion about what the point of doing these conversions would be.

> *Ruby*: Well, I feel like, I mean, there must be a reason that we're doing it [converting from time to angle], but it seems like a lot of work, um, for not really anything.

After the completed process, though, Ruby had some recognition why it might be useful.

> *Ruby*: I think that this [seeing the two completed integrals] solves my problem, that this [the integral in time] is harder than this [the integral in angle]… Over here [the integral in angle] we have, we have the numbers plugged in in a way that will solve that for us [i.e., an easier antiderivative].

While Ruby began to the see the computational benefits of u-substitution, the lesson still did not help the students see possible scientific motivation for converting between quantities. For example, in Task 2, radius information might be easier to come by than temperature information.

Discussion

The major contribution of this study is to help induct u-substitution within a quantitative reasoning paradigm for calculus. This study extended prior theoretical work (Jones & Fonbuena, 2024) to an empirical analysis of students learning u-substitution in this way, which provided several insights into a quantitative u-substitution. First, the students used several distinct quantitative relationships, including classic three-quantity relationships (Smith & Thompson, 2007; Thompson, 1990), two-quantity covariational relationships (Carlson et al., 2002), and nested MV relationships (Jones, 2022). However, interestingly, the identified nested relationship was not the intended *original input → new input → integrand quantity*. This result suggests the need to scaffold this particular MV relationship to make it more salient in the lesson.

In the three-part structure of u-substitution (*differential, integrand, bounds*), the quantitative treatment of the differential was the most cognitively demanding. Crucially, we observed a new type of "collapse" distinct from Oehrtman's (2009). Rather than collapsing to "nothing," the differentials seemed to collapse, in the students' minds, to a single infinitesimal "sameness." It took careful work to consider the relationship between changes in the original input versus the new input to identify that even at very small scales there is a conversion factor between an infinitesimal bit in one quantity versus an infinitesimal bit of another. This strengthens the case that gaining the ability to reason about increasingly small increments of a quantity in calculus is important for quantitative reasoning (Ellis et al., 2020; Ely & Ellis, 2018).

Another implication from our study for teaching u-substitution is the need to ensure a motivating "why" for doing it (Harel, 2013). A couple students were unsure of the benefit of converting from time to angle, or from temperature to radius. One motivation, of course is the computationally practical "easier antiderivative" used in typical pure-math approaches. But we believe it is important to also weave in the scientific reasons for converting between quantities at times (e.g., see Koretsky, 2012). For example, it is possible that certain data is more available, or easier to obtain, which would definitely be the case for radius versus temperature in Task 2.

Overall, we believe this study to be a proof-of-concept of teaching u-substitution through a quantitative paradigm and teaching it in a way that illuminates the quantitative structures inherent in it. We believe that doing so can help students construct the *differential-integrand-bounds* structure that would allow flexibility in using u-substitution. However, beyond simply enacting the substitutions, our students also came to understand the mechanisms for why the substitutions work, and could reason about the conversions in context. Further, we believe that understanding u-substitution in this way can provide a strong basis for understanding future substitution techniques, such as trigonometric substitution or multivariable change of variables. For example, trigonometric substitution uses a different nested MV relationship where the new input ($\sin(\theta)$) *precedes* the original input (x), as in $\sin(\theta) \to x \to f(x)$. Thus, we claim that understanding these relationships can help students in future math learning and science usage.

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