Professional Mathematicians' Levels of Understanding and Pseudo-Objectification

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The notion of mathematical understanding has always been central to mathematics education research, including how students understand mathematical concepts (e.g., Wawro et al., 2019; Zandieh, 2000). More recently, researchers have been interested in the mathematical practices of mathematicians, including how they understand mathematical concepts (e.g., Flanagan, 2022; Oehrtman et al., 2019; Shepherd & van de Sande, 2014; Wilkerson-Jerde & Wilensky, 2011). Related to understanding is the notion of abstraction, which includes various process-object theories (e.g., Dubinsky & McDonald, 2001; Sfard, 1991). Likely the most prominent theory of abstraction is Piaget's reflective abstraction and his broader genetic epistemology (Piaget, 1970). In this broader theory, Piaget (1964) distinguished between two different ways to understand a mathematical concept, namely "act on it," and to being able to "understand the process of this transformation, and as a consequence to understand the way the object is constructed" (p. 176).

Using this distinction by Piaget, this study introduces three different theoretical levels of understanding: pseudo-object-level, process-level, and object-level. Pseudo-object-level understanding consists of being able to act on the concept like an object without understanding the underlying processes. Process-level understanding consists of being able to understand the underlying processes of the concept. Object-level understanding consists of being able to both act on the concept and understand the underlying processes. Pseudo-object-level understanding also closely aligns with the notion of a pseudostructural conception (Linchevsky & Sfard, 1991; Sfard, 1992; see also Zandieh, 2000).

Through using these levels of understanding, this study addresses the following two research questions:

1. In what ways do professional mathematicians operate with highly-abstract, advanced mathematical concepts at different levels of understanding?
2. What factors can influence a professional mathematician's level of understanding for a given mathematical concept?

To answer these research questions, three semi-structured Zoom interviews (Hammer & Wildavsky, 1993) were conducted with six professional mathematicians specializing in algebra research. The first interview was a semi-structured task-based interview, where the participants completed two tasks with category theory concepts unfamiliar to the participants. The second interview utilized the participants' own research journal publications to generate discussion on what influenced their understanding of the concepts they used in that journal article. The final interview was more reflective, utilizing stimulated recall to triangulate the other two interviews.

The primary forms of data analysis consisted of conceptual analysis (Thompson, 2008) and thematic analysis (Braun & Clarke, 2006). The results provided evidence that professional mathematicians often function at a pseudo-object-level understanding for various mathematical concepts, and that they tend to operate differently with the concepts depending on their level of understanding. Moreover, numerous factors were shown to influence mathematicians' level of understanding, including the particular way the concept is being utilized in their work, as well as other sociocultural factors like one's field of study or what their research community values.

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