Connecting Unique Factorization Domains with the Teaching of High School Algebra in Approximations of Practice

Sthefania Espinosa
The University of Texas, Rio Grande Valley

Kaitlyn Stephens Serbin
The University of Texas, Rio Grande Valley

Younggon Bae
The University of Texas, Rio Grande Valley

We designed approximations of practice (AoPs; Grossman et al., 2009) that pose hypothetical classroom scenarios of high school students learning about factoring polynomials. We investigated the ways in which prospective and in-service teachers leverage their understanding of unique factorization domains to inform their teaching practices of attending, interpreting, and deciding how to respond to students' thinking in the AoPs with the goal of making explicit connections between abstract algebra and the teaching of secondary mathematics.

Keywords: approximations of practice, noticing, unique factorization domain, abstract algebra

Connecting secondary mathematics and abstract algebra can support in-service and pre-service teachers (IPSTs) in their teaching. For IPSTs to recognize advanced mathematics courses, like Abstract Algebra, as beneficial for their teaching, they should be explicitly given opportunities in their mathematics courses to reflect on how the advanced content is connected to secondary mathematics and how those connections can inform their teaching. Some researchers and teacher educators (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Serbin & Bae, 2023) have provided such opportunities to IPSTs by using *Approximations of Practice* (AoPs) in advanced mathematics courses, which are activities situated in hypothetical classroom scenarios that require IPSTs to simulate certain teaching practices (Grossman et al., 2009). AoPs often include scripting tasks (Zazkis et al., 2013) and noticing tasks (Jacobs et al., 2010) that engage IPSTs in attending, interpreting, and deciding how to respond to students' thinking in samples of written student work or scripted class discussions.

Before teachers' understanding of connections between secondary and abstract algebra can become useful in their teaching or AoPs, those connections should first serve to fundamentally change the teachers' understandings of the content they teach (Wasserman, 2018). This principle guided our design of an instructional task sequence that guided IPSTs in reinventing (Gravemeijer, 1999) unique factorization domains (UFDs), connecting its properties to the factorization of integers and polynomials (commonly used in secondary algebra), and applying their understandings of those connections in AoPs. A *UFD* is an integral domain R in which every nonzero element $r \in R$ which is not a unit has the following two properties: (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1 p_2 \dots p_n$ and (ii) the decomposition in (i) is unique up to *associates*: namely, if $r = q_1 q_2 \dots q_m$ is another factorization of r into irreducibles, then $m = n$ and there is some renumbering of the factors so that p_i is associate to q_1 for $i = 1, 2, \dots, n$ (Dummit & Foote, 2003). After reinventing UFDs, the IPSTs were asked to complete a set of AoPs (Grossman et al., 2009) that were designed to lead them to use their understandings of UFDs and factorization while they noticed (attended to, interpreted, and decided how to respond to) student thinking or scripted class discussions in hypothetical classroom scenarios (Jacobs et al., 2010; Zazkis et al., 2013). We address this research question: *What aspects of IPSTs' understanding about UFDs do they leverage as they interpret and decide how to respond to students' mathematical thinking in AoPs?*

Literature Review

It is essential for teachers to have knowledge of the content they teach, as well as knowledge of advanced mathematics that is related to, but is still outside the scope of, secondary mathematics. Wasserman (2018) defines the mathematical knowledge outside of the scope that teachers teach as *nonlocal mathematics*. Nonlocal mathematics can influence the teachers' perceptions and interpretations of their students' thinking. Thus, it is important that preservice teachers are prepared beyond the local mathematics they plan to teach. However, it often occurs that IPSTs do not see any connections between the advanced mathematics courses they take with the secondary mathematics they eventually teach (Zazkis & Leikin, 2010).

Some researchers have tried to resolve this disconnect by studying how to support IPSTs in making connections between the content they learn and the content they teach on the secondary level (Alvarez et al., 2020; Wasserman, 2018; Wasserman et al., 2017; Larsen, 2013). However, there has not been much research done on how these connections actually influence the way graduate students teach mathematics (Wasserman et al., 2017). Abstract Algebra is one of the primary advanced mathematical courses required for IPSTs, where they can learn about algebraic structures (Pramasdyahsari, 2021; Ticknor, 2012). To make explicit connections between the abstract algebra and secondary mathematics, Álvarez et al. (2020) designed pedagogical tasks for use in abstract algebra courses that generated student-teacher interaction and helped students construct their own knowledge. Similarly, Zbiek & Heid (2018) proposed the Mathematical Understanding for Secondary Teaching framework where IPSTs connect abstract algebra and school algebra through the mathematical activities of mathematical noticing, mathematical reasoning and, mathematical creating, which leverage the practices of noticing structure and symbolic forms, proving and conjecturing, as well as representing and defining. Herbst et al. (2014) explored how AoPs using online learning can engage prospective secondary teachers as part of their practice-based teacher development. These studies exemplify efforts being made toward supporting IPSTs in connecting advanced mathematics to teaching.

Theoretical Background

Building Up From Practice – Stepping Down to Practice

To help teachers make connections between advanced and secondary mathematics, Wasserman et al. (2017) proposed the *building up-stepping down* model for a real analysis course. The model consists of providing IPSTs with pedagogical situation tasks that can help them recognize the utility of advanced mathematics, in this case, real analysis, in their teaching. Thus, they start from a practical scenario in the context of teaching mathematics and build up the advanced mathematical content from that practice. Once the students learn advanced mathematics, they step down to teaching practice, where the ideas of real analysis are explicitly connected to secondary mathematics, and IPSTs can use their knowledge about real analysis to support their teaching practices such as attending and responding to hypothetical students' ideas.

Approximations of Practice

Teaching requires multifaceted skills, like interpreting and responding to students' mathematical thinking, explaining mathematical definitions, posing questions, and responding to questions that can advance the students' reasoning (Álvarez et al. 2020). Therefore, it is crucial that IPSTs have opportunities to practice these skills without the complexity of teaching in a real-life scenario. Grossman et al. (2009) defined *AoPs* as "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (p. 2058). AoPs, like scripting and noticing tasks, serve as practice for IPSTs to implement the pedagogical moves they learn (Crespo, 2018). For instance, Zaskis & Marmur (2018) argue that scripting tasks provide IPSTs the opportunity to "explore erroneous or incomplete approaches of a student, revisit and possibly enhance personal understanding of the mathematics involved and enrich the repertoire of potential responses to be used in future 'real' teaching" (p. 294). Scripting tasks also prompt prospective teachers to reflect and explain why they would continue the discussion in that manner (Campbell & Baldinger, 2021). By having these opportunities in advanced mathematics courses, IPSTs can develop teaching practices in connection to the advanced mathematical content they learn in the courses (Álvarez et al., 2020, Zaskis & Marmur, 2018).

Noticing Students' Mathematical Thinking

In our task design, we leveraged the three components of professional teacher noticing of mathematical thinking: attending students' mathematical thinking, interpreting students' understanding and deciding how to respond (Jacobs et al., 2010), as we consider them key pedagogical moves that teachers can implement to guide a class discussion and answer their students' questions. Zambak et al. (2023) recognize that "noticing provides pedagogical readiness for PSTs to correctly identify students' mathematical conceptions and misconceptions and guides their instructional strategies" (p. 4). In noticing tasks, prospective teachers are posed with a hypothetical classroom scenario where the mathematical activity and thinking of a student are described. The goal of such tasks is for IPSTs to attend, interpret, and decide how to respond to the student's ideas by reasoning about the strategies the student used to solve the given task. We investigate the aspects of IPSTs' understandings of UFDs that IPSTs leverage as they interpret and respond to students' mathematical thinking in AoPs.

Therefore, our study design is informed by Wasserman et al.'s (2017) *building up-stepping down* model for designing curricula that connects advanced mathematics to teaching, along with the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999), informed our creation of local instructional theory, in which we guided graduate student IPSTs' reinvention of UFDs. We connected the first tasks in the sequence to the teaching of factorization in high school algebra by using algebra tile manipulatives. We built up the advanced mathematical content by guiding the IPSTs to reinvent UFDs (see Bae et al., 2024). We then guided the IPSTs to step down to practice by using their understandings of UFDs in their responses to AoPs situated in hypothetical classroom situations. The design of the AoPs was informed by Grossman et al.'s (2009) AoPs and Jacobs et al.'s (2010) noticing framework.

Methods

We designed a sequence of tasks to guide IPSTs to reinvent the reducibles and irreducibles (Unit 1), reinvent the definition of UFD (Unit 2 & 3), and apply their knowledge to AoPs (Unit 4). In Unit 4, we created AoPs where IPSTs engaged in noticing students' mathematical reasoning about concepts related to factorizations of integers and polynomials. The tasks were designed to elicit IPSTs' use of their knowledge of UFD from the previous units as they attended to, interpreted, and decided how to respond to students' thinking in written work and class discussion scripts. In this paper, we focus on their work on the AoPs tasks in Unit 4.

We administered the AoPs to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in the Southern US. Six mathematics graduate student IPSTs participated in this study: Josie and Raul (Group A), Javier and Roberto (Group B), and Kim and Taylor (Group C) (pseudonyms). They all had taken an abstract algebra course but had not yet learned about UFDs. They had varied teaching experiences at different school and college levels ranging from student teaching to 20 years. Each group participated in five 90-minute sessions of teaching experiments with the research team. Groups A and B participated in person, and Group C participated in Zoom. We collected and transcribed the session recordings.

We implemented the AoPs in Figures 1, 2, and 3. The IPSTs responded to questions about their interpretation of what the students' approaches and understandings, and how they would respond to such student ideas if they were the students' teacher and how their responses were informed by their understanding of UFDs, providing us with evidence about how they used their knowledge about UFD to inform their pedagogical actions. We used inductive coding (Miles et al., 2013) to analyze how IPSTs use their knowledge about UFD as they worked on the AoPs. We coded the properties of UFDs the IPSTs referenced as they in noticed and responded to students' mathematical thinking, i.e., (Jacobs et al., 2010).

In Mr. Garcia's class, students were asked to factor a quadratic polynomial $-2x^2-4x+6$ for solving a quadratic equation $-2x^2-4x+6=0$. Mr. Garcia had four of his students, Aden, Brian, Cassy, and David come to the board and share how they factored the polynomial. The following is the solutions of the four students that involve different final forms of factorization of the polynomial $-2x^2-4x+6$.

Aden	Brian	Cassy	David
$-2x^2-4x+6$ $=-2(x^2+2x-3)$ $=-2(x+3)(x-1)$	$-2x^2-4x+6$ $=-(2x^2+4x-6)$ $=-(x-1)(2x+6)$	$-2x^2-4x+6$ $=x^2+2x-3$ $=(x+3)(x-1)$	$-2x^2-4x+6$ $=(-x+1)(2x+6)$ $=(-x+1)2(x+3)$

Jaime, one of the students in Mr. Garcia's class, said "So who's got it right? They all got it different, so someone must be wrong." Monica followed, "That doesn't matter, you have to check if they all give you the same roots".

Figure 1. Approximation of Practice Task 1 Scenario

1. In Ms. Gonzalez's last class, students went over problems that required factoring polynomials. Ms. Gonzalez wants her students to think about the factorization of polynomials. She said: "Today, we are going to explore how polynomials and integers are similar. What similarities do you think polynomials and integers have?"
2. A student Amy asked the following question to Ms. Gonzalez: That problem told us to factor a polynomial, and I was trying to remember where I saw factoring before. I remember that numbers have prime factorizations. Do polynomials have prime factorizations too? Or is that a different kind of factorization?

Figure 2. Approximation of Practice Task Scenarios from Task Set 2

The teacher asks students to find the factors of the polynomial x^2+5x+6.
When asking the students to share their answers, he receives the following answer:
Luis: The factors are 3 and 2.
Teacher: Can you explain your answer?
Luis: The factors of 6 are 3 and 2 because 3·2 is equal to 6 and 3 plus 2 gives me 5 which matches the polynomial.
Alex: I think he's right, he just needs to add x, so we have $(x+3)$ and $(x+2)$.
Mia: I have $(x+2)$ and $(x+3)$. Does the order for this matter?

Figure 3. Approximation of Practice Scenario from Task Set 3

Results

Episode 1: Reasoning About the Uniqueness Axiom of UFD to Inform their Interpretation of and Response to Hypothetical Students' Thinking

In the AoP task set 1, four hypothetical students, Aiden, Brian, Cassy, and David obtained different factorizations $-2x^2-4x+6$ (see Figure 1). In this hypothetical scenario, the students' work was on the classroom board, and the class was discussing the factorizations. Two students, Jaime and Monica, shared their ideas (see Figure 1). The participants were asked to attend to and interpret the student understanding evident in Jaime's and Monica's comments. Josie and Raul agreed that Jaime's comment was valid, arguing that his understanding of factoring is strong. For instance, Josie claimed: He knows that… *when you're factoring something to the irreducibles, it should be unique no matter what… The order doesn't matter, but it still should be unique factorization. So, they should still end up with the same solution if the task is to factor it out.* Josie recognized that Jaime thought only one of the distinct solutions from his classmates must be correct because he might know that the problem must have one unique solution. Raul interpreted Jaime's understanding of factorization as "*on point*" for this same reason. Both IPSTs connected Jaime's comment to the uniqueness axiom of the unique factorization domain.

The participants were prompted to decide how they would respond to the class following the comments by Jaime and Monica. Josie responded:

> Just check each student's solutions of like, you have student [work] on the board, and you might ask, who got it right? It's a two-part question, the factoring and solving the equation. So, to *check if you got the factoring right, distribute it, see if you got the original polynomial*. And then the second part of solving the equation, actually solve it. Set it equal to 0. Find the roots for each one. And that will give you the checkmark or the X for each one, whether they got both parts of the question right or wrong.

Josie and Raul decided that they would ask students to go back and check their classmates' answers for them to find whether the factorizations were correct.

Finally, the participants were asked how their understanding of UFD helped them interpret and decide how to respond to the students' thinking. Josie and Raul said:

> *Josie*: *It helps you address Jaime's comment, of the unique factorization domain tells us that the elements can be written as a unique product of irreducibles up to the order and associates of the elements*. So, it helps us address his comment right away of, *he understands that the factorizations should be unique*, so every student should have gotten the same factorization if the question is to factor it all the way down to irreducibles.
>
> *Raul*: [...]. So, I kind of push already for like, *you need to factorize it as much as possible, which is like this unique factorization idea, prime factorization idea*. Um, so, I guess my push is like, with this, with *this unique factorization domain idea in mind is that, we want it to be as small as possible, on what the product is that.* […]
>
> *Josie*: *Because we know… how the integral domain works, we can answer their question like well, but how? But why?* …I think that's the question. Which is why I thought that the question would be more like appropriately addressed to Jamie's comment, and maybe student Aiden for that matter too, that they understand that the factorization, you're going to get it down to like, *you want to break it down as much as possible, like the primes*. And that *factorization must be unique, which is the property of the UFD we're looking at***.**

Josie recognized that the way she could address the students' comments was by knowing how some mathematical structures such as rings and integral domains work. Specifically, she mentioned that the UFD axiom about uniqueness helped her respond to the students' mathematical thinking. Raul also referenced the UFD axiom regarding the existence of prime factorizations, as evident in his response of pushing students to "factorize it as much as possible" and "be as small as possible." Thus, Josie's and Raul's understandings of the axioms of UFDs informed their interpretation of and response to student thinking in this AoP.

Episode 2: Reasoning about the Reducibility and Irreducibility of Polynomials to Inform their Interpretation of and Response to Hypothetical Students' Thinking

In Task Set 2, the IPSTs were given the classroom scenario in Figure 2, where a teacher prompts her students to identify the similarities between integers and polynomials to help them develop an intuitive understanding of factoring polynomials. The IPSTs were asked to anticipate what similarities students might find between integers and polynomials. Taylor and Kim said that students would probably recognize that integers and polynomials have numbers, and they could do operations with them such as addition, subtraction, and multiplication. Both agreed that students would probably not think about factorization as a similarity between them. Kim and Taylor were then asked about how they would respond to these anticipated student ideas and how they could connect those ideas to the factorization of polynomials. Kim replied, "I think the big thing is… branching the ideas together. Right, okay, so…this is surface-level. Let's get a little bit deeper, to get the point of the factorization at least. Okay, yes, so you can factor polynomials." Next, they were asked about the characteristics of unique factorization domains that would help inform their response to students in this question. They answered as follows:

Kim: I think for that one, just the fact that you *had the unique factorization domain of like integers but also the ring of polynomials. Knowing that they are, that that is the same, inherently those structures, allows one I think to be more confident in saying, okay, they are similar, because they are the same structure*, right?

Taylor: *I still like the "breaking them down", that they can be broken down into irreducible things. […] Just knowing that you can break these things down*, I think that would be really beneficial, at least for that level. So that I guess the first point is the most important one for me, not so much that they're unique, but that you can do it to begin with.

Taylor related the reducibility of elements in UFDs as the main property that was connected to factoring polynomials, arguing that students could understand factoring as the action of "breaking things down" Therefore, Taylor's knowledge of the reducibility in UFDs helped her have an intuitive understanding of factoring in a high school setting and informed her way of sharing these ideas with the student.

The participants were next asked about how they would decide to respond Amy's question if they were her teacher (see task 2 in Figure 2). They replied as follows:

Taylor: Yeah, they do have prime factorizations, *and I don't think it's a different kind of factorization*. Because like, 2 and 3, you know, that's 6, but the factorization of 2 and 3 can be the same as a polynomial as long as you have the correct x value. So it's, I don't know if I would call it different kind of factorization. *I would say they're the same, but they're performed differently and for different reasons*.

Kim: …I just immediately am like…what do you mean by prime, right? Like prime, just give the definition of prime... You can't reduce it, right? So, *talk about reducibility, make that connection, and then you can get the polynomial to a reducibility* as well, kind of depending on like that factorization and with the root finding.

Taylor agrees with Amy's reasoning about the similarities between the prime factorizations of integers and polynomials, and Kim relates such property of polynomials to the reducibility in the elements of UFDs given that both integers and polynomials with integer coefficients are UFDs. We asked the participants what aspects of UFD informed their responses to students. Kim said:

> *Just knowing that the UFD is both a definition for the integers and the polynomials ring is, is important for this response*, like with the last one. Because, okay, if you know they are the same, you can confidently say yes, of course, without having that doubt. Because if you have that… second of doubt of like, oh, are they the same or are they not? And you're discouraging the student from like, but they were right, and they had the right idea. *So setting a foundation of yes, I know for a fact, because it's the same structure, okay. And it instills confidence in the student as well.*

Kim's understanding of the structure of the integers and polynomials as UFDs gave her the confidence to assert that the factorizations the student Amy asked about were indeed similar because they have the same UFD structure. Having this understanding about the structure of factorization in $\mathbb{Z}$ and $\mathbb{Z}[x]$, informed Kim's response to the hypothetical student in the AoP.

Episode 3: Reasoning About the Commutativity of Polynomial Factors to Inform their Interpretation of and Response to Hypothetical Students' Thinking

The last set of tasks consisted of the hypothetical classroom scenario in Figure 3. When asked to interpret Mia's understanding, Javier replied*: "Mia's just trying to I guess clarify if commutativity is going to work here also."* We asked the participants how they would address and respond to the students' comments if they were their teacher. Javier and Roberto responded:

Javier: For Mia, *I'd probably just tell her to multiply those out,* $(x+2)(x+3)$*, and then multiply* $(x+3)(x+2)$*, to see if it really does matter if we multiply them in any order.*

Roberto: *…Mia to expand both terms and see if they give you the same answer*. I think the way the task is given is that I think she knows how to factor everything out. She knows how to get those two answers, but *she's just not sure if commutativity matters or not*. It can be solved just by *telling her to distribute*. *It doesn't matter in this case, it's the same*.

The IPSTs were asked about how their knowledge of unique factorization domains was informing their responses to students. Javier responded, "Seeing the commutativity in the tiles," and Roberto agreed. The IPSTs explained that their knowledge of the commutativity property of UFDs helped them explain with confidence why the order of the factors does not matter when factoring a quadratic. They both referred to the "uniqueness up to the order" of the factors, which results from the multiplicative commutativity that can be applied to the factors in a factorization.

Discussion and Conclusion

We exemplified how IPSTs' understandings of the connections between the abstract algebraic concept of UFD and the concepts of integer and polynomial factorization from secondary mathematics seemed to support them in their pedagogical practices. Specifically, we identified how the participating IPSTs seemed to leverage their reasoning about the existence and uniqueness axioms of UFDs to inform their responses to AoPs tasks that prompted them to interpret and decide how to respond to hypothetical students' thinking or classroom scenarios. Our work contributed to the literature on how IPSTs' knowledge of abstract algebraic concepts can inform their pedagogical practices. Our findings also contribute to the literature on the utility of AoPs in supporting IPSTs in connecting advanced and secondary mathematics. Álvarez et al. (2020) suggested that these AoPs can be "used both as a vehicle for bridging undergraduates' advanced mathematical knowledge to secondary school mathematics and for strengthening undergraduates' understanding of the advanced mathematics from an encounter with school mathematics" (p. 16). Additionally, AoPs can give IPSTs experience in using their knowledge of advanced mathematics in their teaching practices, which can contribute to improvements in their perceptions of advanced mathematics as being useful for their teaching of school mathematics (Serbin & Bae, 2023). Future research can address how IPSTs use their knowledge of other advanced mathematical concepts, as well as their pedagogical or mathematical practices (e.g., Wasserman, 2022), as they perform AoPs. Lastly, we propose that mathematics instructors who prepare IPSTs should implement AoPs in their assignments to support IPSTs in developing coherent understandings of the connections between abstract algebra and the teaching and learning of secondary mathematics.

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