Playful Math, Problem Posing, and Discovered Complexity

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Problem posing is an important part of mathematical inquiry, but students can struggle to pose problems that are mathematically relevant. One route for supporting meaningful problem posing is through playful math tasks, which can emphasize agency and exploration. In this paper we report findings from a small-group teaching experiment with five pre-service secondary teachers who explored a variety of tasks, including open tasks, problem-solving tasks, and tasks designed to foster mathematical play. In investigating students' problem posing, we found that developing playful challenges for one another supported instances of discovered complexity, the experience of new conceptual challenges. We discuss examples of discovered complexity and the mathematical ideas students developed when grappling with novel ideas.

Keywords: Problem posing, mathematical play, covariation

Instructional approaches that engage students in experiences that are authentic to disciplinary inquiry should offer opportunities for problem posing, exploring, and conjecturing (Bonotto, 2013). Problem posing can be a particularly powerful activity to support students' mathematical inquiry (Cai et al., 2015), and classrooms that do not include problem posing can curtail students' mathematical exploration (Ellerton, 2013). Collective problem posing activities are one way to build students' autonomy in the development of the mathematics they learn, and they can provide a space for students to develop their creativity and problem-solving skills (Cai, 2012). There are, however, challenges with meaningfully fostering problem-posing skills. Students and teachers can pose a variety of problems when presented with a given set of information (Stickles, 2011), but may struggle to pose problems that are mathematically relevant, appropriate, and solvable (e.g., Cai & Hwang, 2003; Silver & Cai, 1996; Silver et al., 1996). As Cai et al. (2015) noted, more research is needed to understand not only which strategies are needed to pose mathematically interesting problems, but also how to effectively teach for problem posing.

We conjecture that *playful math* could be one way to foster productive problem posing. We have been running a series of studies investigating how to meaningfully incorporate playful elements into task design and instruction, and have found that playful mathematics tasks can emphasize student agency, exploration, and goal selection (e.g., Bloodworth et al., 2023; Horne et al., 2023; Ellis et al., 2022). When crafting playful mathematics tasks, we shift the design role to the student, enabling students to construct challenges for one another. This provides an opportunity to engage in problem posing that may support discovered complexity (Williams, 2001; 2002; 2003), which occurs when problem solvers "perceive *intellectual* and *conceptual* complexities not evidence at the commencement of the task" (p. 378, emphasis original). In this paper we address the following research questions: When does students' problem posing lead to discovered complexity? In particular, how does discovered complexity emerge in different task types, and what mathematical ideas are developed through discovered complexity?

Background Literature and Theoretical Perspectives: Problem Posing and Play

Cai and Hwang (2020) defined Mathematical Problem Posing (MPP) as "the process of formulating and expressing a problem within the domain of mathematics" (p. 2). While some emphasize problem posing in terms of the generation of a new problem from a given set of conditions (e.g., Stoyanova, 1998), others also include in problem posing the activity of reformulating existing problems (e.g., Silver, 1994). Problem-posing tasks, then, are ones that require students (or teachers) to generate new problems based on given situations, expressions, or diagrams (Cai et al., 2020). As an example, Stoyanova (1998) offered the following task: "Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on the previous ring. Ask as many questions as you can" (p. 66). Note that this task, which is typical in the literature, offers a constrained context with specific values, and then asks students to develop problems with the given information. That said, a good problem-posing task should leave room for different interpretations (Kontorovich et al., 2012; Silver & Cai, 1996).

Posing problems can deepen students' understanding of mathematics, as well as their understanding of problem-solving (Brown & Walter, 2004). Indeed, problem posing is an important part of problem-solving competence, and to become successful problem solvers, students need to encounter tasks that require both posing and solving problems (Cai et al., 2015; Kilpatrick, 1987; Niss & Højgaard, 2019). Problem posing can also capture students' interest, foster reasoning and communication abilities, and can help students understand that there is no one right way in mathematics (Cai et al., 2015; Silver, 1994).

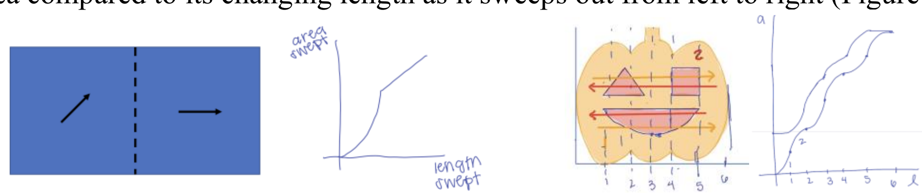
Research on people's abilities to pose problems is largely situated at the K-12 level, and the findings are mixed. Some studies show that both students and teachers can pose interesting and important problems (e.g., Silver & Cai, 1996; 2005; Cai & Hwang, 2020). However, more studies have identified challenges with successful problem posing (e.g., English, 1998). Stickles, for instance, found that secondary teachers could pose problems, but their success was partial and was related to experience and background (2011). Taken as a whole, the studies of students' and teachers' problem posing have identified issues such as a) posing nonmathematical problems, b) posing tasks that were not problems, and c) posing low-quality problems (Cai & Hwang, 2020; Leung & Silver, 1997; Silver et al., 1996).

In investigating strategies for supporting problem posing, researchers have noted the importance of providing opportunities for exploration (Koichu & Kontorovich, 2013). Crespo and Sinclair (2008), for instance, emphasized the need to allow students avenues to explore the limits of the mathematical situations they investigate. Cai and Cifarelli (2005) studied how undergraduate students posed problems, and identified two levels of reasoning strategies, hypothesis-driven and data-driven, that students incorporated into their problem-posing strategies. Brown and Walter (2004) proposed the "what-if-not" strategy, and others have suggested helping students learn to pose problems by extending or revising an existing problem (e.g., Abu-Elwan, 2002; Cai & Brook, 2006). However, as Cai et al. (2015) pointed out, "Much more research is needed to develop a broadly-applicable understanding of the fundamental processes and strategies of problem posing" (p. 17). As we discuss below, we hypothesize that some features of playful math could foster meaningful problem posing.

Mathematical Play and Playful Math. Drawing on five characteristics that recur throughout the literature, we define mathematical play in terms of a person's experiences, behaviors, and affective states during activity. First, mathematical play entails *freedom* and *agency*. It is player-centric, with the player in charge of the process (Holton et al., 2001; Gresalfi et al., 2018). Participation is voluntary (Williams-Pierce & Thevenow-Harrison, 2021) and ludic (Featherstone 2000). Second, mistakes have *low stakes*. When playing, a person is not afraid of failing (Williams-Pierce & Thevenow-Harrison, 2021) and may not even experience anything as "failure" (Barab et al., 2010; Su, 2017). Third, mathematical play entails *engagement and* *immersion* in one's activity (Gresalfi et al., 2018). Learners find pleasure in their experience (e.g., Burghardt, 2011; Sukstrienwong, 2018), and activity is imaginative and creative (e.g., Parks, 2015; Su, 2017). Fourth, mathematical play is *experimental and exploratory* (Mason, 2019). Play often leads to surprises (Su, 2017), and when one plays, one is in a state of openness to unexpected experiences and outcomes (Davis, 1996). Finally, mathematical play is *goal-driven* (Huizinga, 1955). Goals can emerge as play proceeds, and researchers have described instances of goal-directed behavior, such as learners solving challenges in a video game (Williams-Pierce, 2019). Consequently, we consider mathematical play to entail the following three traits: (a) exploration, (b) self-selection of goals, and (c) immersion, investment, and/or enjoyment.

Researchers have identified a number of learning benefits from mathematical play. It can support experimentation, reflection, and persistence (Barab et al., 2010; Gresalfi et al., 2018; Mason, 2019) and can provide a productive route for exploring and conjecturing (Jasien & Horn, 2018; Mason, 2019; Williams-Pierce, 2019). Beyond mathematics, classroom-based play offers a supportive environment for risk-taking and creativity (Barab et al., 2010; Brown, 2009; Radke & Ma, 2018; James & Nerantzi, 2019). It can help students connect theory and practice (Barnett, 2007), it can foster openness to new learning (Forbes, 2021), and it can support classroom engagement (James & Nerantzi, 2019; Whitton & Moseley, 2014). Many of these practices mirror mathematicians' activity, such as detecting patterns (Fernández-León et al., 2021; Melhuish et al., 2021), generalizing (Bass, 2008; Martín-Molina et al., 2018), conjecturing (Fernández-León et al.; 2021; Harel, 2008), and experimenting (Watson, 2008). These similarities have resulted in calls for instruction that encourages playful engagement. For instance, Gresalfi and colleagues (2018) argued that "much of what is considered to be sophisticated disciplinary engagement involves many of the same features as play" (p. 1335).

To study how to foster mathematical play, we used the term "playful math" to describe the elements of classroom activities and environments that can facilitate mathematical play. Drawing on the literature, we have identified five design principles for playful math tasks (Ellis et al., 2022): (1) enable free exploration within constraints; (2) engender anticipation within the task; (3) provide a method for intrinsic feedback; (4) offer meaningful challenge while still being feasible; and (5) allow the student to act as both designer and player. As an example, we created the *Guess My Shape* game, which draws on a set of research-based tasks to support student understanding of function through examining covarying quantities (Ellis et al., 2020; Matthews & Ellis, 2018). In these tasks, students investigate dynamically growing shapes, graphing a shape's area compared to its changing length as it sweeps out from left to right (Figure 1a).



(a) *(b)*

Figure 1. An instructor generated (a) and student generated (b) shape and associated area-length graph.

To playify the tasks, the *Guess My Shape* game prompts students to create shapes of their choice (design principles 1 and 5), construct graphs comparing length and area (principles 2, 4, and 5), then challenge each other to determine the shape or the graph based on what is provided (principles 2, 3, and 4). Figure 1b shows an example of a particularly creative shape and its associated graph created by our participants, which entailed sweeping a pumpkin from left to right, and then sweeping backwards right to left to remove area to create a jack-o'-lantern.

The playful math design principles are consistent with many features of problem-posing tasks. Problem-posing tasks require students to generate new problems (principle 5), and they offer opportunities for exploration (principle 1). However, our tasks shift the design role to students in a manner that is more open-ended than what is seen in the problem-posing literature. For instance, in contrast to Stoyanova's (1998) doorbell task, in which both the situation and the relevant numbers or mathematical expressions are given, we provide fewer constraints and encourage students to design novel challenges.

Discovered Complexity. We draw on Williams' (2001) construct of discovered complexity to examine the consequences of students' problem posing, particularly the spontaneous problem posing that occurred in the course of both designing and solving tasks. Williams (2002) wrote that when students discover a complexity, they spontaneously formulate a question that leads to intellectual and conceptual challenges: "the mathematical ideas in a discovered complexity are new to all students in the group and the teacher does not contribute new mathematical ideas during the interaction" (p. 403). The process of discovered complexity is autonomous, spontaneous, and creative, similar to the activity described by a research mathematician (Williams, 2002). Discovered complexity is associated with high positive affect (Williams, 2003) and meets the conditions for flow (Williams, 2001), as students can work just above their present skill level to meet a challenge almost beyond their reach.

Methods

We conducted a videoed teaching experiment (TE; Steffe & Thompson, 2000) with five secondary pre-service teachers, Phyllis, Meredith, Kelly, Ryan, and Toby (all pseudonyms). The students had all completed their first semester in a secondary mathematics education program, and they were familiar with non-routine covariation and graphing tasks. The students enjoyed a positive rapport with one another and were comfortable questioning each other and themselves.

The TE met for a total of 6 hours. We began with the function growth activities described above, which we call standard tasks, and which addressed polynomial, trigonometric, and piecewise functions. We also incorporated playified tasks in the form of the *Guess My Shape* game, as well as more traditional problem-solving and open tasks (e.g., tasks that were open-ended with many possible solutions, such as "create two growing shapes that could be represented by the same graph"). The problem-solving tasks were ones such as the handshake task: "If there are *n* people in a room, and they all shake hands with each other exactly once, how many handshakes occurred?" All of the standard, problem-solving, and open tasks were challenging and creative, but only the *Guess My Shape* game adhered to all five playful math design principles. Over the course of the TE we implemented six standard tasks, three problem-solving tasks, two open tasks, and three playified tasks.

Analysis. As part of a larger study investigating the characteristics of students' mathematical play, we developed emergent codes describing students' experiences and behavior. One code that became relevant for this paper was "wonderment", which represented instances of students spontaneously communicating curiosity or interest about a new mathematical idea, problem, or relationship. Wonderment is an instance of spontaneous problem posing. We initially found 23 cases of wonderment, and examined each case to determine: a) did it occur during a standard, open, problem-solving, or playified task; b) did it result in discovered complexity? We identified instances of discovered complexity by seeing whether each wonderment yielded a novel or more complex mathematics challenge, as opposed to simply trying to understand the given question, establish a convention or definition, or wondering how to create a challenging task for someone else (but not actually creating such a task). During this process, we abandoned two of the 23 wonderments as not actual wonderments. In one case, a student was trying to understand how another student's shape was being swept out, and in the second case, a student was curious about the logic another student was using for her solution. We analyzed the remaining 21 wonderments collaboratively, meeting weekly to refine and adjust our meanings and resolve discrepancies.

Results: When Wonderment Leads to Discovered Complexity

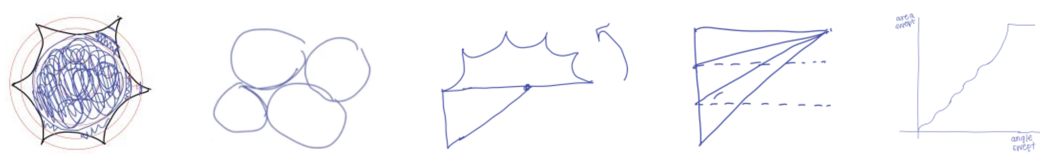
Table 1 shows the number of wonderments per task type and the number of wonderments within each type that led to discovered complexity. Five of the 21 wonderments occurred during the two open tasks, and the remaining 16 occurred during the three playified tasks. We did not find any instances of wonderment in the six standard tasks or the three problem-solving tasks.

Table 1. Number of wonderments and number of resulting discovered complexities by task type.

Task Type	Wonderments	Discovered Complexity
Standard	0	0
Problem Solving	0	0
Open	5	3
Playified	16	9

In examining how discovered complexity emerged, we first identified cases of wonderments that were not discovered complexities. As an example, Toby grappled with an open task in which he had to invent a shape with a monotonically increasing graph that changed concavity. He drew a trapezoid, and a corresponding piecewise area-length graph that had a concave up portion, a linear portion, and a concave down portion. Toby asked himself, "Does that count as a change in concavity? Like, what is the concavity of a straight constant line?" Toby's wonderment did not introduce a new mathematical idea or problem, and it did not count as problem posing. Instead, it was an attempt at a clarification of meaning.

In another example, during the *Guess My Shape* game, Phyllis and Meredith explored a novel way of sweeping. They created a figure that they called a police badge (Figure 2a) and imagined sweeping area from a point in the center moving outward in a circle. Phyllis wondered, "I'm just thinking about, like, now just the geometry stuff. When there's a bunch of circles connected (draws four circles, Figure 2b). You see what I mean, you see spikes here. Is there any relationship with that at all?" Later, she clarifies her wonderment: "How many circles could you fit around one circle of the same size?" Here Phyllis's wonderment posed a problem about circle packing, addressing mathematics that was novel to her. However, before she had an opportunity to attempt a solution, the students were interrupted and told to move back to the whole group, and thus she did not get a chance to engage with intellectual or conceptual complexity.



(a) (b) (c) (d) (e)

Figure 2. Phyllis and Meredith's shape (a), Phyllis's illustration of a circle packing problem (b), Meredith and Toby's shape (c), triangle divided into sub triangles (d), Meredith and Toby's graph (e)

In contrast, we present a case in which a wonderment did lead to discovered complexity. As seen in Table 1, discovered complexity only occurred a little over half the time during students' activity in the playified *Guess My Shape* game. It occurred more often when the students were creating a challenge for their classmates, a formal act of problem posing, than when solving a challenge (discovered complexity occurred during 5 out of 7 wonderments when creating a challenge, but only 4 out of 9 wonderments when solving a challenge). We found that discovered complexity emerged when students either developed curiosity about the nature of covariation, rates of change, or related graphs, or when students became curious about why a particular solution method or strategy was valid.

In the following example, Toby and Meredith posed a problem by creating a shape for their classmates, which they imagined sweeping radially from 0° to 360° (Figure 2c). The students wondered whether the associated graph comparing area and angle swept would be constant once the angle reached 180°:

Toby: So, what happens when we do that? I feel like you're getting more triangle each time you open up, right? Because this length gets longer (points to radial lines in Figure 2d; dotted lines are not yet present), but the base is the same and the height is the same, right? Wait, the base is definitely the same. I don't know about the height, it could also be increasing.

Meredith: I think it –

Toby: – Wait, wait, there's only base and height. So, the base is the same, the height's increasing, so it would get bigger.

Meredith: The base has to be the same.

Figure 2d divides the triangle into three smaller triangles, and Toby and Meredith referred to the vertical portion on the left of each smaller triangle as its base. They erroneously assumed that for each equal angle increment, the base of each small triangle would be the same. Toby and Meredith initially conceived of the height of each triangle differently. Toby assumed the heights would be the solid radial lines in Figure 2d, whereas Meredith realized that they would need to be the dotted horizontal lines, which she drew in: "I think the height has to be perpendicular to the base." The students agreed that the heights of each triangle must be equal, and therefore, they concluded that the amount of area added for each equal angle increment would be equal. Thus, the graph for that portion of the shape should be linear.

Toby remained, however, uncertain about this conclusion. As they worked on an initial graph, he returned to the issue of the triangle's growth, saying, "I still find that hard to believe, but I think it's true. How come this ends up being the same as if we had done that (draws another triangle with the base at a 45° angle, rather than vertical) instead?" Both students, however, believed that it must be linear, because they had justified it based on the belief that the base of each incremental triangle would be equal. Later, a teacher-researcher asked the students about their thinking, and Toby expresses doubt again, but this time with a different outcome:

TR1: How did that triangular part, what was your reasoning there?

Meredith: We were saying that for the bases, we're saying all have to be the same because that's how we're designing the –

Toby: Wait. Are they though? I'm questioning myself again. Just because the angle increases by the same amount, does that mean the base is the distance away, because…(pause). I'll draw a bigger one. There's this angle, and you open it up by the exact same amount. I feel like you're going to have *more* base over here (points to the bottom sector in Figure 2d) than you did up here (points to the top sector). You know what I mean?

Toby drew a much larger triangle, and relying on the grid feature of his notetaking app, then carefully subdivided it into triangles with equal increment angles. In this manner it became apparent that each new triangle had a larger base than the previous triangle. Consequently, the students revised their graph to reflect that the portion of the graph from 180° to 270° would not be constant, but rather would increase at an increasing rate (Figure 2e). Indeed, this can be determined by considering the smaller triangles as each having a swept angle θ, a height of 1, and a base *x*, in which *x* grows as area is swept. The area of each triangle will be $\left(\frac{1}{2}\right)x$. Because $\tan\theta = x$, the area function will be $A = \frac{1}{2}\tan\theta$, $0 \leq \theta \leq \frac{\pi}{4}$, which yields a graph that increases at an increasing rate.

In posing the radial sweeping problem, Toby and Meredith encountered a discovered complexity. Toby's wonderment about whether the relevant portion of the graph would be linear led to the students perceiving a conceptual complexity, that of how to determine the growth of the triangle. That this was an intellectual challenge for the students was apparent in their discomfort with their initial graph, as well as the need to revisit the question more than once. In resolving the complexity, Toby and Meredith shifted from reasoning only about increasing or decreasing rates, without attending to a way to quantify area amounts, to reasoning geometrically with the triangle area formula to justify why the area had to grow at an increasing rate for each equal-increment angle sweep. In this manner, the problem Meredith and Toby posed for their classmates required them to grapple with a novel conceptual complexity.

Discussion

The *Guess My Shape* game formalizes problem posing by asking students to design problems, but in a manner that is less constrained than what is typically seen in the literature (e.g., Stoyanova, 1998). The primary constraint on the students was that they had to determine how to graph any shape they developed. Because the students were motivated to make their tasks "fun", "tricky", and "unexpected", they pushed themselves to create shape-graph pairs that included novel and creative elements, some that were particularly challenging to solve. However, wonderments emerged not just during formal problem posing; they also occurred when students were solving one another's challenges and exploring open tasks. Wonderments are spontaneous instances of problem posing, which may or may not be pursued to fruition. When they were pursued and resulted in discovered complexity, mathematical ideas emerged that addressed topics such as constant versus nonconstant rates of change in novel sweeping situations, the effects of the center of rotation on the change in area for equal changes in angle measure, and determining the areas of segments of circles formed by two non-radii chords.

Given the small numbers of each task type, we cannot make definitive claims about which types of tasks will necessarily lead to discovered complexity. That said, it was compelling that the students did not demonstrate any wonderments when solving standard or problem-solving tasks. It may be that the motivation to develop creative tasks and solutions during the playified and open tasks, combined with an emphasis on agency and exploration, afforded students opportunities to embrace risk taking, openness, and experimentation, mirroring results from studies on mathematical play (e.g., Gresalfi et al., 2018; Mason, 2019; Williams-Pierce, 2019). We also found that it was often the students' need for causality (Harel, 2013) – the need to explain or determine the cause of a phenomenon – that resulted in a wonderment becoming a discovered complexity. These findings suggest that playful math could potentially be one avenue meriting further exploration for supporting productive problem posing.

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