Learning in Pieces: Horizontal Décalage in Students' Thinking on Homework Tasks

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This report (a) documents the phenomenon that student learning from online homework may occur across a series of mathematical tasks and (b) provides a theoretical explanation for the phenomenon by characterizing some of the cognitive mechanisms that underpin it. Specifically, findings indicate one way students learn from online homework is by comparing similar tasks. The results add to knowledge about what and how students learn from homework. They also have implications both for practice and research in terms of how we define and measure learning. In particular, the results support the need to consider and design for learning across multiple mathematical tasks.

Keywords: assimilation, accommodation, online homework, equation solving

Rationale and Overview

Although many university students spend more time doing mathematics homework than they do in class (Ellis et al., 2015; Krause & Putnam, 2016; Lew & Zazkis, 2019), there is a paucity of research about student learning from homework. Research about what students learn from homework and the mechanisms by which they learn it can inform the creation of more effective homework assignments, which, given the amount of time students spend doing homework, could greatly improve student learning. This report, which focuses on student learning while doing online homework, makes several contributions. First, it documents what Piaget called horizontal décalage, or the phenomenon that there is often a temporal delay in learning. Specifically, the paper shows how some students' learning from online homework occurred across a series of mathematical tasks. Second, the paper characterizes some of the cognitive mechanisms that supported students' learning over the series of tasks.

Theoretical Perspective and Theoretical Literature

von Glasersfeld (1995) summarized Piaget's definition of learning as follows: learning "takes place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium" (p. 68). A *scheme* is a construct describing how people organize knowledge. It consists of "(1) recognition of a certain situation; (2) a specific activity associated with that specific situation; and (3) the expectation that the activity produces a certain previously experienced result" (von Glasersfeld, 1995, p.65). The framework was chosen for the study in part because of its clear definition of learning, which afforded identifying when learning happened and what supported that learning.

People employ, construct, and modify schemes via assimilation, perturbation, equilibration, and accommodation. Consider a person engaging with a math task. *Assimilation* occurs if the person "treats new material *as an instance of something known*… [the] cognizing organism fits an experience into a conceptual structure it already has" (emphasis original; von Glasersfeld, 1995, p. 62). If the person is unable to fit the experience into an existing conceptual structure, they may be *perturbed,* meaning they recognize assimilation has failed. von Glasersfeld (1995) gives disappointment or surprise as examples of indicators that a person is perturbed. *Accommodation* is the modification of a scheme to remove perturbation. *Equilibration* describes the “construction process and [a] mechanism of change” by which a person accommodates a scheme and resolves perturbation. One way accommodation could occur is through the person reviewing the situation and identifying characteristics missed in the assimilation attempt; the identified characteristics “may effect a change in the recognition pattern and this in the conditions that will trigger the activity in the future” (von Glasersfeld, 1995, p. 65). A second way accommodation could occur is if the person creates a new recognition pattern (and hence a new scheme).

An accommodation is a semipermanent state; Piaget notes “it can take a certain time for a particular way of operating to spread to other contexts (*horizontal décalage*)” (von Glasersfeld, 1995, p. 71-72). Montangero and Maurice-Naville (1997) note that horizontal décalage “illustrates the limits of the generalization of a mental structure” (p. 89). That is, horizontal décalage can be thought of as the temporal delay in an individual’s capacity to apply a scheme to a structurally-isomorphic task (i.e., engage in observer-oriented transfer). The idea that accommodation (learning) takes place over time and across contexts instead of as a single “lightbulb” moment, is the focus of this paper. The theory that learning occurs over time and across contexts is neither new nor unique to Piaget. di Sessa (1993) proposed the knowledge in pieces (KiP) framework, and Wagner (2006) characterized transfer as occurring in ‘pieces.’ KiP, developed in the context of physics learning, has been employed in empirical mathematics education work (e.g. Izsák, 2005).

Empirical Literature

The findings about student learning from homework indicate students’ affinity for learning from examples (Dorko, 2021, 2020; Dorko, Cook, & DeHoyos, 2023; Aichele et al., 2011; Bissel, 2012; Erickson, 2020; Kanwal, 2020; Krause & Putnam, 2016; Lithner, 2003; Weinberg et al., 2012). Students appear to like examples in part because they provide an opportunity to reason via viewing a similar problem. However, the extant literature also indicates the learning from these problems tends to be procedural in nature, with students missing conceptual insights the professor intended.

When learning from online homework, multiple researchers have found students employ immediate feedback to inform their work (Dorko, 2020; Kontorovich & Locke, 2022). That is, students tend to submit an answer to a part of a problem, see if it is correct, and then proceed to the next part of the problem.

Methods

The data discussed are representative excerpts from video recorded sessions of five precalculus students doing online homework. I collected data at a large, public US university during fall 2022 and spring 2023. All five students had the same instructor. The entire data set consists of 27 online homework assignments between the six students, collected over a two-month period in each term. During the sessions, students were instructed to think aloud. Students received immediate feedback on their answers from the homework platform and had unlimited attempts for non-multiple choice problems.

During data collection, the researcher had observed that students might answer a problem correctly, only to struggle with a subsequent problem about the same mathematical idea. This was intriguing because it raised the question why the student did not employ the previously-successful scheme in the new context. Hence the first step in analysis was isolating all series of problems that dealt with the same mathematical idea and taking from that set the data in which a student struggled with some (but not all) of the problems. There were two goals in analyzing the data. The first was to evaluate the extent to which horizontal décalage described the students' struggles over a series of problems. Having determined that it did, the researcher then sought to characterize the mechanism of equilibration.

The criteria for evaluating if a student's work across the series of problems was an example of horizontal décalage was the student modifying a scheme over a series of problems, with evidence for those modifications coming from students' utterances and the answers they submitted. This follows from Piaget's characterization of accommodation as a semipermanent state and the definition of horizontal décalage (above). Evidence that students were perturbed came from their expressing confusion, disappointment, or surprise (von Glasersfeld, 1995). If students approached and solve a problem without difficulty (either explicitly expressed difficulty, or the researcher noticing activity like their writing/erasing in scratchwork), the students were considered to be in an unperturbed state and assimilating the problem to an extant scheme.

To characterize the mechanisms of equilibration, the researcher performed an analysis grounded in the data and informed by the theory. In terms of the theory, she sought evidence that the student had (upon an assimilation failing) explored the context to seek characteristics the assimilation had disregarded. This followed from von Glasersfeld's (1995) description of one way accommodation can occur. However, this explanation did not seem to fit the data under consideration. The researcher did notice that Dylan, the first student whose data she analyzed, made progress on a task in the middle of the series by re-examining his work on previous problems that he identified as similar. In accordance with grounded methods of analysis, the researcher evaluated data from other students to see if *comparison* appeared to be a cognitive mechanism that supported equilibration for other students (it did). This work is ongoing, and subsequent analysis will seek to identify other mechanisms of equilibration. The results shown below provide examples of (a) horizontal décalage as explaining students' difficulty on a series of tasks that employed the same mathematical idea and (b) comparison as a cognitive mechanism supporting equilibration. The illustrative excerpts correspond to the students' work on the following tasks[1], which are given in the order they appeared in the students' assignment. The course coordinator chose the problems because they all focused on the "solve for" idea but had variations in the formulas and task wording that might lead students to see them as different. For example, the GDP and gas mileage part A problems did not involve any constants, while the other tasks did. The gas mileage part B task differed from the other tasks because students had to set a variable as constant. The wordings also differed from "solve for E" to include "solve for t and express t as a function of V" to "to obtain a formula". That is, in some wording, students were told the answer was a function or a formula, while in others, they were not. Hence the tasks had enough variation that students might have to engage in cognitive work to see them as similar.

Gross domestic product (GDP) task. The gross domestic product, P, is calculated as the sum of personal consumption expenditures, C, gross private domestic investment, I, government gross investment G, and net exports E of goods and services. All are measured in billions of dollars. The formula is P = C + I + G + E. Solve the equation for E.

Boat task. The resale value, V, in thousands of dollars, of a boat is a function of the number of years *t* since the start of 2011, and the formula is V = 12.5 – 1.5t. Solve for t in the formula to obtain a formula expressing t as a function of V.

[1] The tasks are from Crauder et al. (2018). The presentation here paraphrases wording and omits parts of the tasks irrelevant to the report.

Gas mileage task: The distance, d, in mi., that you can travel without stopping depends on the number of gallons g of gas in your tank and the gas mileage, m, in mi/gal. The relationship is d = gm. (A) Solve the equation for m. (B) An engineer wants to ensure the car she is designing can go 415 miles on a full tank of gas. She does not yet know what gas mileage the car will get. Solve the equation d = gm for g using 415 for the distance.

Ant task: A scientist observed that the speed S (cm/sec) at which ants run was a function of T, the temperature (°C). He discovered the formula S = 0.2T – 2.7. Solve for T to obtain a formula expressing the temperature as a function of the speed.

Illustrative Excerpts

Dylan

Dylan's work on a series of three problems provides an example of horizontal décalage. The course content included solving an equation for a particular variable and having the answer be a formula instead of a number. That "solve for *x*" could result in an *x* = … expression in which the right side of the equation contained variables was a new idea for Dylan. By 'new idea', I mean he had been exposed to this meaning of "solve for" in lecture but was grappling with it on his own for the first time during the homework session. Dylan solved the GDP and boat tasks on the first try. On Gas Mileage part A, he solved d = gm for m correctly on his first try. In part B, however, Dylan was perturbed by the fact that a "solve for…" answers was a formula. Dylan's facility with the boat task and part A of the gas mileage task, but perturbation on part B of the gas mileage task, indicated to the researcher that Dylan was in the process of accommodating his "solve for…" scheme. Specifically, the perturbation on part B of the gas mileage task appeared to be an example of horizontal décalage because Dylan had just answered two similar tasks unperturbed. Dylan's thinking about part B of the gas mileage task is below. It provides evidence of how he equilibrated by *comparing* the problem at hand to the previous problems to remove his perturbation.

Dylan: I'm thinking, thinking I don't have enough information but, because I'm missing, missing m.

Interviewer: What do you mean [by] you don't have enough information?

Dylan: I mean well right now I'm thinking I only have one, only have one variable and I need two.

Int.: What makes you think you need two?

Dylan: That's kind of how it works, right? You're given, like you're given a variable, you're given like an equation and then you put in all the numbers.

Int.: So you think the answer, you think it needs to be g equals some number.

Dylan: Yeah. Because it's solving for the equation. But I guess I solved for the equation up here [moves mouse to part A]. So I guess not… if it [Webassign] took this [m=d/g as correct in part A] and it's asking part B the same that it's asking [in] this [part] I mean it's asking it in the same way. [Submits g=415/m, correct]

I take Dylan's comments about not having enough information and needing two variables so he could "put in all the numbers" as evidence of perturbation. I focus on how, following those comments, Dylan reviewed his earlier, correct m=d/g. He noted that his present task was worded the same way as that task; that is, he compared the wording of the tasks and employed the answer from the previously-solved task to support his solution in the unsolved task. I use the word *comparison* to characterize the mechanism of change as Dylan equilibrated[2]. Dylan solved the subsequent Ant Task without perturbation, which we take as evidence that accommodation to his "solve for…" scheme had occurred:

Int.: You have a formula that's got another variable and it's not just like all numbers. You feel okay about that?

Dylan: Yeah, yeah I do this time. Because I remember from up [above] that it worked.

Amy

Amy's work on a series of 4 tasks provides an example of horizontal décalage. Amy struggled with GDP task. She opened the e-book, located the answer to the problem, and said she understood why the answer was correct. Amy copied the book answer, submitted it, and the platform marked it correct. On the boat task, Amy noted the problem was "another 'solve for'." She wrote t = V(t); note this is an equation in which t = … and the right side of the equation contains only variables[3]. After submitting this answer and seeing it was wrong, Amy located the problem's answer in the e-book and commented she had never seen an answer like that before. She said the solution "looked right", though she did not understand why.

Amy: I don't quite get what that's asking for [opens the e-book].

Int.: What are you hoping to find there?

Amy: Something that explains this specific problem. [finds exact problem with answer key, looks at answer] this [answer] is not what I would have thought of… like I would think that you'd solve this equation for what E is, not in terms of variables. In my mind whenever you say solve that means like get an answer for E.

Int.: Get an answer like a number?

Amy: Yes…. [looks over answer in e-book] I understand why that's the answer.

Amy: [Boat task] It's another "solve for"… like "solve for t in the formula, expressing t as a function of V." My mind goes to like this type of answer [types t = V(t)] but I don't know if that's correct because given the past problem it probably isn't. [opens e-book] That's what we've always used for like functions of, whenever you're expressing t as a function of V it's like anything is a function of something else it's always been like, A, like B is a function of A it's like A(B). [submits answer, incorrect] I don't quite understand why this is wrong. [looks at answer in book] I've never seen something like this before. And I don't know how I would get this answer.

Int.: Does anything in that [book answer] make sense?

Amy: I mean yeah because I mean given the whole picture like this [highlights formula answer] does make sense and it is correct… Like looking at it, it looks right. Like if someone showed me this and explained how to get to it I would understand more how it was right.

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[2] A difference in Gas Mileage Parts A and B is that A had three variables while B directed students to use d = 415. We do not have enough evidence to know if this difference caused Dylan's perturbation. Similarly, von Glasersfeld (1995) hypothesized a scheme could be modified to include new recognition criteria or the learner could create a new scheme. It is impossible to know which is the case here (in part because of the difficulty defining the boundaries of a scheme). These concerns are beyond the scope of this report; our goal is to (a) identify comparison as the mechanism of change and (b) show an example of horizontal décalage.

[3] As the interview excerpts indicate, Amy switched the independent and dependent variables here and in her generic A(B); this is immaterial to the point that what she wrote was an equation in which the right-hand side was an expression with variables, not a number.

We take Amy's "that's what we've always used for like functions of, whenever you're expressing t as a function of V it's like… like B is a function of A it's like A(B)" as evidence that she assimilated the boat task to a scheme for "function of" instead of a "solve for" scheme, though she clearly knew (because she had stated) it was a "solve for" problem. We consider this an example of horizontal décalage because she had indicated that she understood why the answer to the previous GDP problem was correct, identified the boat problem as similar to the GDP problem because they were both "solve for" tasks, but did not use her "solve for" scheme. That is, the boat problem context was somehow different enough for there to be a time delay (decalage) in terms of Amy's scheme use. On the next task, Amy correctly solved the problem and stated that while she thought "solves" meant to get a number, she was seeing in the problems that these answers were not numbers. Amy described the answers in these tasks as "rearrang[ing]" the equations. We take those statements as evidence that her "solve for" scheme was being modified from her engagement in the tasks. Like Dylan, the mechanism in equilibration was comparing. Amy compared the gas mileage task to the previous problems:

> *Amy*: [Gas mileage tasks] [types m = d/g] Given like these previous problems I'm starting to pick up that what I think would be the answer, like my original first thought, is not the answer.
> *Int.*: You said this is like, so so far these haven't been what you thought it would be.
> *Amy*: Yes.
> *Int.*: So you still want that m to be some number, like that's your initial instinct?
> *Amy*: Yes.
> *Int.*: But, you said based on the past couple problems, you feel like that's not the way to go on these.
> *Amy*: Yes. I guess like looking at like "solve", like I said "solves" to me means get a number… but because of the previous problems and like the previous explanations via the textbooks, the answers the textbooks have given, and also this thing down here, it's very obvious that like you're not getting a number. This [highlights GDP problem] is technically the same format that this [highlights b in current problem] is asking this in [the "solve for…"], solve the given equation for a variable, I put a number in here, it was not correct, and then it gave me basically this original formula, rearranged, and then the same down here, this original formula, rearranged… that's how I'm getting this, it's just this original formula that's given to you, essentially rearranged… Like solving the given equation for a variable means, I guess in like the simplest form of like terms, like rearranging the original formula to put t on, like by itself, and then all the other ones would be rearranged to equal t.

In the ant task, Amy answered the question correctly on the first answer. When asked how she got the answer, Amy said, "rearranging this [given] formula." We took this as more evidence that she had accommodated her "solve for" scheme.

Discussion, Significance, and Implications for Instruction and Research

A limitation of this work is that it only analyzed data from five students completing somewhat procedural problems. However, these preliminary results support extant classroom practices (a legitimate way research can inform practice). For example, Dylan and Amy learned when "solving for" a variable, the answer can be a formula. This is an example of student learning via online homework, which supports other evidence of online homework as efficacious (e.g. Dorko, 2021, 2020; Ellis et al., 2015). Another implication is that instructors should consider providing students with explicit opportunities to compare problems to identify underlying mathematical structure, if they are not already doing so. This was the mental activity that supported Dylan and Amy's success on the task series in question. Specifically, it seems important to provide students with multiple tasks that explore the same mathematical idea, followed by an explicit reflection task engaging students in identifying the underlying similarity.

The results here extend prior work about student learning from similar examples in homework (Dorko, 2021). While that work identified one way of learning as students following steps of similar problems, this study found another way students learn from homework is by identifying a series of tasks as similar, then comparing across them to identify an underlying mathematical structure or meaning. That students here learned a new meaning for "solve for…" is also a novel finding because it gives empirical evidence of conceptual learning from homework, somewhat contrasting other findings that students' learning from online homework is primarily procedural in nature and that students often miss conceptual insights the professor intended they learn (Dorko, 2019; Dorko, Cook, & DeHoyos, 2023). A limitation of the paper is that the tasks were somewhat procedural practice problems. Future research could focus on homework problems that are intended to extend what students might have seen in class.

Various features within the online homework platform supported students' learning. For example, Amy opened the e-book and made use of the solutions it provided to homework problems. While printed textbooks tend to have solutions to some exercises in the back of the book, the e-book in this course had direct links from some problems to their solutions. The homework platform's immediate feedback supported Dylan's reasoning because he knew something had "worked" in a previous problem that the platform marked as correct, and this allowed him to compare subsequent tasks to correct solutions. This finding provides more support for findings that students employ immediate feedback from online homework platforms to guide their work (Dorko, 2020; Kontorovich & Locke, 2022).

The results have implications for how we define and measure learning in classroom and research settings. The evidence shows that learning occurs over time, implying a need for classroom practices that support this. This could include helping students form growth mindsets so they come to see struggling as a necessary and acceptable part of the learning process. Another way to acknowledge learning as happening over time could be extant practices such as mastery-based grading (e.g., allowing a final exam grade to replace a lower preliminary exam score). In terms of research, the horizontal décalage supports designs that examine student thinking about a mathematical idea in multiple contexts because a student might struggle with the idea in one context but not another. This is likely of particular importance in a study that employs a single task-based interview with multiple students, because one question about a given mathematical idea could yield an incomplete picture about the student's understanding of the idea.

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