Pre-service Teachers Understanding of Commutativity, Associativity and the Challenge of Order

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Commutativity and associativity are recurring properties in school and university mathematics. In design experiments with pairs of secondary pre-service teachers (PSTs) we investigate what they are focusing when discussing commutativity and associativity, and how they deal with the notion of order. A qualitative discourse analysis shows for commutativity that the focus is on the order of two elements for most PSTs. Few focus on the order of several elements. Regarding associativity, bracketing is a main focus for many PSTs. The order in which operations are computed is sometimes discussed explicitly with also emphasizing the order of elements and sometimes more implicitly. The PSTs struggle with the notion of order so that we suggest coping strategies like "demanding precision in language" and "using alternative terms for order".

Keywords: pre-service teachers, commutativity, associativity, qualitative discourse analysis

Introduction

The arithmetic properties commutativity and associativity are recurring mathematical topics that can be found at various levels in school and university mathematics (Wasserman, 2016). One could assume that dealing with different binary operations and their properties in different settings should have eliminated any difficulties in dealing with the properties. However, according to empirical data, this does not appear to be the case (Melhuish & Fagan, 2018; Zaslavsky & Peled, 1996). When discussing commutativity and associativity, one of the biggest struggles for university students seems to be the notion of order (Larsen, 2010). Larsen (2010) suspects that a lack of precision in the informal language is one cause for this difficulty. Though it is not yet certain what language could be hindering or supportive, and what are possible coping strategies for dealing with the notion of order. This is why the reported study in this paper deals with the following research questions:

1. What do (German) secondary pre-service teachers near the end of their studies focus on when discussing commutativity and associativity?
2. How do secondary pre-service teachers deal with the notion of order concerning the properties commutativity and associativity?

Theoretical Background

In university mathematics, commutativity is defined for an operation $*: M \times M \to M$ on a set M referring to two elements $a * b = b * a, \forall\, a, b \in M$ and associativity referring to three elements $(a * b) * c = a * (b * c), \forall\, a, b \in M$ (Alcock, 2021). In (German) school textbooks, however, we sometimes find definitions of commutativity that are not based exclusively on two elements anymore, e.g. "The summands of a sum or the factors of a product can be changed in any way" (translated: Braun, 2019, p. 91). For commutativity, the school definition might evoke the idea that the order of several elements may be changed. Whereas the university definition has the idea that the order of two elements may be changed (Larsen, 2010). As a consequence, the understanding of commutativity in school and university mathematics may not match.

One understanding of associativity suggested by the formal definition is that, if the property is valid, brackets may be arbitrarily moved, inserted, or removed (Melhuish & Fagan, 2018). Behind this still rather superficial idea, if bracketing refers to two elements, is the understanding that elements can be grouped differently when computing the operations (Weber & Larsen, 2008). For this, the order in which the operations are computed does not matter (Larsen, 2010) One interesting aspect is, how Cayley formulated associativity as a condition for a group. In general, that commutativity does not hold would not actually need to be explicitly addressed. He postulated "symbols are not in general convertible [commutative], but are associative" (Kleiner, 1986, p. 208 cited in Larsen, 2010). Larsen (2010) observed exactly this phenomenon with his students. They operated with symmetry mappings and addressed (non)commutativity in their definitions for a group even though it is not necessary. To some students it seems inherently important to emphasize that the order of the elements from left to right is not allowed to be changed (Larsen, 2010). Nevertheless, there are also students who have difficulties with keeping the order of the elements when using associativity (Larsen, 2010; Zaslavsky & Peled, 1996). When checking operations for commutativity, one common difficulty is confusing the operation sign with the number sign when changing the order of the elements (Zaslavsky & Peled, 1996). For checking operations for associativity, difficulties can be caused if learners primarily have the understanding that associativity allows brackets to be moved (Melhuish & Fagan, 2018). Students are asked to verify that the operation of the mean $*$: $a * b = \frac{1}{2}(a + b)$ is associative. One given choice to justify that the operation is not associative is based on random shifting of brackets $\left(\frac{1}{2}a\right) + b \neq \frac{1}{2}(a + b)$. This option was chosen by 16.9% of PSTs (Melhuish & Fagan, 2018). More generally, it can be said that the understanding of and checking for commutativity seems to be easier for students than associativity (Findell, 2001).

In comparison of the two properties it becomes evident that the notion of order is apparent in the understanding of commutativity and associativity. However, in the case of commutativity, the order refers to the order in which the elements are arranged. While in the case of associativity, the order refers to the order in which the operations are computed. This similarity entails the risk that students could conflate the two properties when not paying attention to the differences (Larsen, 2010; Zaslavsky & Peled, 1996).

If both properties apply, they could be replaced by a single property (Pinto & Cooper, 2017). For example, if multiplication is considered "it is not clear why these two rules should not be replaced by a simpler multiply in any sequence rule" (ibid. p. 322). It would be sufficient to know that in a multiplication the order may be interchanged arbitrarily without differentiating what the order refers to. Empirically, it also appears that some students combine the two properties into one "order doesn't matter" (Findell, 2001, p. 148) property. This notion can lead students to make statements such as "because it's associative, you can move it all around" (Findell, 2001, p. 148) or "this property [meaning commutativity] allows us to switch around the elements in an expression so that it doesn't matter which elements will operate first" (Findell, 2001, p. 148). The difficulty with order and differentiating what the order refers to could also lead to the misconception that associativity also has something to do with swapping the order of elements (Ding et al., 2013). Furthermore, a common misconception that one property might imply the other property (Melhuish & Fagan, 2018; Tirosh et al., 1991).

Method

Data Collection

This study is part of a design-based research project at a German university with the aim of connecting university mathematics and teaching school mathematics for secondary pre-service teachers (PSTs) (Dellori & Wessel, in press). In this project, three teaching units following the sequencing instruction model (Wasserman et al., 2017) for PSTs algebra learning are developed and tested in laboratory design experiments. The second one addresses learning goals related to associativity and commutativity. At first (*building-up*), the PSTs work with school textbook excerpts about associativity and commutativity. In the second phase (*relearning*), the PSTs deal with commutativity and associativity in different algebraic structures. Finally (*stepping-down*), they reflect on the textbook excerpts with reference to the university mathematics. The data for this paper is situated at the beginning of the relearning phase of the teaching unit about commutativity and associativity. It concerns the following tasks:

1. Let M be a set with the binary operation $*: M \times M \to M$. Write down the definitions for commutativity and associativity.
2. In your own words, describe commutativity and associativity verbally.
3. What are differences and similarities between commutativity and associativity?

For the first two tasks, PSTs first discuss orally and then give a written answer. Task 3. is only discussed orally. Afterwards PSTs are prompted with the statement: “Commutativity implies associativity”. Although, some PSTs already mention this claim during task 3.

The design experiments were conducted with pairs of PSTs (N=8x2) by the first author. The PSTs were in their first to third semester of their master’s program at a German university and volunteered to participate in the study. At the time of data collection, 15 of the 16 PSTs were enrolled in an Abstract Algebra course. The design experiments were videographed and transcribed. The focus of the analysis is on tasks 2 and 3, as these encourage to reflect on the understanding of the properties.

Data Analysis

To answer the research question about what PSTs focus on when discussing commutativity and associativity, a qualitative content analysis is conducted (Kuckartz & Rädiker, 2022). The main categories are deductively formulated from previous research. For commutativity the two main categories are “order of two elements” and “order of elements” (Braun, 2019; Larsen, 2010). Inductively, the distinction is added as to whether a more general or example-based discussion of the order of the elements took place (examples see Figure 1).

Category	Definition	Example
Commutativity		
Order of elements	There is a focus on the fact that commutativity allows the order of elements to be rearranged.	General: The set under the operation is commutative, if we can say we can change the order around. […] And that for all the elements. Example-based: So, we know that this is commutative, then we can rearrange it as we want. [points to the "a*b*c" on sheet 1]
Order of two elements	There is a focus on the fact that commutativity allows the order of two elements to be rearranged.	General: That it doesn't matter in which order I combine two elements? Example-based: Whether I now combine a with b or combine b with a doesn't matter.
Associativity		
Order of operation	There is a focus on the fact that associativity allows the order in which operations are computed to be interchanged arbitrarily.	First computation: So, it just doesn’t matter what we calculate first. Order: Here it is about the operation so to speak. It doesn't matter in which order we compute it. With regard to the operation. Example-based: And it doesn't matter for […] my result, if I, uh, first combine the elements a and b, the result from that with c. Or if I first combine b and c and then again combine what comes out of that with a.
Emphasis on order of elements	There is a focus on the fact that when the associative property is applied, the order of the elements is not changed	The order of the operation pair by pair doesn’t matter. […] Whereas maybe … The order of the operation doesn’t matter if the order of the elements is the same.
Bracketing	There is a focus on the fact that associativity allows brackets to be set arbitrarily.	When combining more than two elements of a set, brackets may be omitted or set as you like.

Figure 1: Categories for the focus on commutativity and associativity

For associativity the “order of operation”, “emphasis on order of elements” and “bracketing” are the three main categories (Larsen, 2010; Melhuish & Fagan, 2018) (see Figure 1). It has been observed that the order of operations could be addressed explicitly in using the term “order” or more implicitly in describing it on an “example-based” level or focusing on what is “computed first” (see Figure 1).

To answer the second research question (how do the PSTs deal with the notion of order), all passages in which the term “order” occurs were paraphrased and summarized for each pair. These summaries were contrasted and compared in order to identify typical difficulties and coping strategies for the notion of order.

Results

Pre-service teachers’ discussion of commutativity

To answer the first task and give a definition for commutativity and associativity, all PSTs give a symbolic based answer. The definition refers to *two elements* a and b for all pairs. Minor differences between the definitions are the notation of the operation ($*: M \times M \rightarrow M$) or the algebraic structure $(M,*)$. The pair #8 is the only pair that does not specify that $a * b = b * a$ must hold for all elements. The other pairs use the quantifier $\forall$ for this in their definitions. Furthermore, this pair also added $a * b * c = c * a * b$ later, when Sven (#8) talks about switching the elements back and forth. According to their understanding, commutativity is not just about the order of two elements but the *order of elements* in general.

Sven: If I can switch *a* and *b* and have a *c* there as well, then I can also, yes, … switch that back and forth if I want to.

In describing commutativity in their own words, six out of eight pairs clearly focus on the *order of two elements*. For example, pair #3 wrote: “The two combined elements of a set may be interchanged arbitrarily.”. The two pairs #1 and #5 start out with an example-based description of commutativity such as “It doesn't matter whether I combine *a* with *b* or combine *b* with *a*”. During their discussion, passages are also at a general level, for example like “It doesn't matter in which order I combine two elements”. Similarly, pair #8 has a mix of example-based and general passages in their discussion (see Figure 2). Pairs #2 and #4 tend to be almost exclusively at the general level while pairs #3 and #7 engage in discourse solely at this level (see Figure 2).

	#1	#2	#3	#4	#5	#6	#7	#8
example-based	3	1		1	2	4		2
general	4	7	2	7	5		3	4

Figure 2: Passages focusing on the order of two elements

Pair #6 was the exception to talk about the *order of two elements* merely on an *example-based* level. On a more general level, this pair only talked about the *order of elements* such as “An operation is commutative if the sequence of elements is freely selectable and may be changed.”. It cannot be said with certainty that they are talking about multiple elements, but as they never explicitly talk about only two elements, it is assumed. The two pairs #6 and #8 are the only ones where the focus is on the *order of elements* rather just two elements. In the discourse of the other pairs we find some passages in which it is not clear from the language whether two or more elements are considered. Since, however, these pairs previously have multiple passages in which they explicitly only refer to two elements, one can assume that the missing explication is only due to an abbreviated form of expression.

Pre-service teachers' discussion of associativity

Like in their definitions of commutativity, all PSTs first state a mainly symbolic definition of associativity. Again, the only differences are in the notation of the operation and the algebraic structure. Pair #8 is the only pair that doesn't express that $(a * b) * c = a * (b * c)$ must hold for all elements.

The understanding that associativity allows *brackets* to be set arbitrarily is coded for six pairs. Pairs #2 and #4 are the only ones that don't discuss brackets in connection with associativity at all. For four pairs (#3, #5, #7, #8), their initial main idea for the verbal description of the property is associated with brackets. For example, in the verbal definition of pair #8: "The result is independent of the bracket placement." Later in the design experiment, these pairs are asked to give an alternative definition without using the idea of bracketing. The other pairs (#1, #6) discuss the notion of bracketing, e.g., see Pia and Nora (#6), but it does not seem to be the focus of their understanding of associativity. For Nora, focusing on the order seems more meaningful than the idea of bracketing.

Pia: Then/ Mhh. Yes, first you are allowed to place the brackets differently somehow. Then, uh, it doesn't matter how the brackets are set.

Nora: Mhh yes but that/ I think that doesn't say much. Because I would just write that/ here the order doesn't matter, right? So, whether I do *a* and *b* first or *b* and c first.

Looking at the discussions about associativity, it can be noted that an *emphasis on the order of the elements* is apparent in six of the pairs while pairs #3 and #8 didn't talk about it all. In some cases, students point out that the order of *a*, *b* and *c* from left to right is not allowed to be changed and explicitly contrast this with the property of commutativity. For two pairs (#4, #7) it was such an integral part of their understanding of associativity that they included it in their verbal definition: "Considering 3 elements from a set, the order of the pairwise composition does not matter if the order of the elements remains the same."

	#1	#2	#3	#4	#5	#6	#7	#8
first computation	5	1	2	1	6	1	1	2
order	2	3	3	5	1	4	8	3
example-based	2		2	1	2	1	6	

Figure 3: Passages focusing on the order of operations

In some way, all PSTs discuss the *order of operations*. Initially, three pairs (#3, #5, #7) consider the order of operations at an *example based-level*. Like Nico and Marcel (#5, see three passages below) these pairs use the specific elements *a*, *b* and *c* to implicitly take the order into consideration (first part). In further discussion, they move away from the three concrete elements and start contemplating that it doesn't matter what is *computed first* (second part). Afterwards, they go further and explicitly interpret "what you do first" as the *order of operations* (third part).

Nico: Yes, well. Um. You could say something like: First *a* with *b* and then with *c* or/

Marcel: First/

Nico: First *b* with *c* and then with *a*.

Marcel: Kind of like … Yeah, you can do it the same way or? Associativity means that it doesn't matter/

Nico: Which operation you do first.

Nico: Mhh. So here we have/ Here it is about the operation. There it doesn't matter in which order we compute it. Referring to the operation.

The other five pairs don't begin their discussion about associativity on an *example-based* level. With two pairs not referring to associativity on this level at all. Some mention the *order of operations* right at the beginning while some start with discussing which operation is *computed first*. Essentially, it can be observed that two pairs (#1, #5) have their biggest focus on which operation is computed first, and two pairs (#4, #6) explicitly on the order of operations. For the other pairs, it's more broadly distributed (see Figure 4).

Pre-service teachers dealing with the notion of order

While for some pairs (#1, #5, #6, #8) the notion of order is only provoked by the third task about similarities and differences between/ of the properties, some pairs (#2, #3, #4, #7) already discuss the notion of order when describing the properties.

The two pairs who manage particularly well to elaborate the similarities and differences always specify what the order refers to when they use the term. Adrian (#3) emphasizes that the order regarding commutativity refers to the elements and the order regarding associativity refers to the computation of the operations. Adrian (#3) has already shown the precise use of the term "order" in the verbal definitions in task 2. Pair #5, on the other hand, was the only one of all pairs that did not use the term order in task 2 at all. Nevertheless, most of the other pairs struggle more to specify what the order refers to. They often simply talk about the "order of operations" and not the "order in which the operations are computed".

Adrian: In both cases, it's about a kind of variable order. Just in one case I may switch the elements and in the other case I may/ It's maybe a bit fare fetched. But there I am also allowed to choose the order, but just which operation I compute first. So that would be a similarity and a difference. So, the similarity is that it's about an order that I'm free to choose. And the difference that it's not about elements, it's about, um, the operation.

The pairs that discuss order at the beginning have in common that they use or wanted to use the term "order" in both their verbal definitions of commutativity and associativity. This causes students to struggle with the notion of order like Anton and Lina (#1, see below). They have a problem using the term "order" for both properties and their way of dealing with it is deciding not to use the term "order" for both properties after all. For commutativity, they prefer to use the alternative term of "direction", which "fits better" to commutativity, according to Lina.

Anton: Because somehow order fits both. That's why I was a bit confused by the term. But somehow… But I mean, it fits more to this [means associativity].

Lina: Maybe it's better to use direction, which is what we said first. So, because you can combine from the left or from the right.

Anton: Yes. I mean/ So I would say both of them are correct. But/

Lina: No, direction fits better. Because that's really, you can look at it from the left and from the right. And with order, there the direction stays the same and then you have *a* and then the bracket or you have first the bracket and then your element.

This approach of resorting to alternative terms - such as direction, placement and position - can be observed in several pairs (#2, #6, #7). In all of these occasions, the alternative terms are used to refer to commutativity. Using alternative terms can lead to rejecting the notion of order for a property, as it is the case for Lina. However, when trying to explain the similarities and differences in task 3, the alternative terms can also be supportive in disentangling the notion of order concerning commutativity and associativity. For example, Anton later used the term "placement order" to clarify what the order is for him in commutativity.

In those pairs (#1, #6, #8) that use the term "order" for only one property in task 2, many of the students tend to associate the term more strongly with that one property even later on. Eva and Mirco (#1) use the term "order" only for commutativity in task 2. In task 3, they quickly give the answer that both properties are in fact about some sort of order. However, Eva is not particularly firm in this view. In the subsequent discussion she is tempted to reject the notion of order for associativity because of her understanding of bracketing. Andreas (#8) behaves like Eva. A similar pattern can be observed for Pia (#6) and Sven (#8). Only that in their case the term "order" originally was used for associativity, and they are inclined to reject order for commutativity.

Eva: Yes, strictly speaking, we don't switch the order [means associativity]. We just rearrange the brackets.

Mirco: But that is an order in which you compute the operation.

Discussion

In summary, for the majority of the PSTs in this study, the focus of commutativity is on the order of two elements and consequently aligns with the formal definition in university mathematics. For almost all PSTs we found passages in which they just spoke about the "order of elements". It is assumed that for most students, this is just due to inaccuracy in their language. For associativity, PSTs seem to have a strong focus on the idea of bracketing. Their focus on bracketing leads some students to reject the notion of order for associativity. Just like Larsen (2010), we also observe that the majority of the PSTs emphasize the order of elements when speaking about associativity. The task about finding similarities and differences between commutativity and associativity seems to be very productive in initiating discussions about the notion of order. Those PSTs who lack precision in their language concerning what the order refers to have difficulties with the notion of order. Struggling with using the term "order" for both properties, some PSTs (momentarily) reject the notion of order for one of the properties. To cope, some PSTs develop alternative terms for the term "order" such as "direction".

The results indicate that a formal symbolic representation of the properties seem not to be particularly challenging for the PSTs. Difficulties arise when explicating the meaning of the properties is concerned. As shown in previous research, symbolic representations and formal vocabulary are not sufficient to explain meanings of mathematical concepts. Thus, meaning-related language is needed "especially for expressing and thinking about abstract relationships" (Post & Prediger, 2020, p. 112). Post and Prediger (2020) suggest providing meaning-related phrases in learning arrangement to support students in understanding the meaning of the part-whole relationship in probability (Post & Prediger, 2020). For understanding the meaning of order regarding commutativity and associativity, our findings suggest that phrases with alternative terms for order seem to be part of the meaning-related language. In line with the theory of scaffolding, we suggest providing students with phrases like that to support them in verbalizing the meaning of order. Besides using meaning-related language, demanding precision in language (in this case to clarify what the term "order" refers to) seems to be productive for explaining the meaning of order as Larsen (2010) already assumed.

The methodological limitations of this study, above all (a) the small sample size, (b) being tied to the specific teaching units, and (c) being situated in a German context (language and university), must be considered when interpreting the empirical findings. Concerning the latter, seeing how the identified language difficulties, coping strategies and proposed supportive means appear and may vary in different languages with other vocabulary could be an interesting aspect for further research.

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