Obstacles of Learning Proof by Cases

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Epistemological Obstacles (EOs) are defined as challenges that students face when exposed to concepts that contradict their pre-existing mathematical intuition (Norton & Arnold, 2023). The objective of our poster is to address EOs that emerge during the lesson on proving by cases, rather than circumventing them. This approach, as pursued by Norton and Arnold (2023), elicited EOs such as the Principal of Universal Generalization (PUG) and understanding the difference between the role versus the value of a quantified variable (Qrv). PUG entails proving "for all" statements with an arbitrary value. It was mainly encountered during mathematical induction by Norton and Arnold (2022), but its relation to proofs by cases was not studied. When proving statements with PUG, it may not occur to students that they need specific (x=0), or arbitrary (x is even) cases. Qrv teaches students to prioritize maintaining generality; however, loss of generality is allowed with proof by cases as it is preserved in the aggregate. The obstacle comes from students struggling to understand that generality is maintained differently depending on the situation. The two EOs are closely related; Qrv can be thought of as a direct product of PUG.

At the study institution, students generally take introductory proofs their Junior or Sophomore year and are required to have passed the second of a three-part calculus sequence and discrete math. As such, they are expected to have experience with symbolic logic and operations on conditional statements (converse, contrapositive, negation), set theory and proofs on sets, and mathematical induction. Many upper-level students transfer from the local community college, which introduces a deficiency in the information on what students have and have not covered.

Class size ranges from 12-20 students in a room with whiteboards on most walls. An undergraduate learning assistant was present to walk around during group discussions to aid students and note student reactions for post lesson reflections. The instructor has a degree in mathematics, with a specialization in undergraduate mathematics education. He relied on that background to focus the course goals on inculcating social practices, specifically the norms and practices of the mathematics community. He intentionally structured the class to get students to 'play mathematician' by questioning arguments, positing hypotheses, and proving or disproving those hypotheses through routine think-pair-share cycles.

Our course begins with a review of symbolic logic. We then cover translating written propositions into logical forms, performing operations on those propositions, and discovering how universal and existential quantifiers modify them. After covering our first proof method, the direct proof, we move on to proof by cases. This lesson was immediately after direct proof because the instructor was following the order of topics set out in the course textbook suggested by the department. Our poster will detail the lesson tasks, the rational behind the tasks, and student thinking that came up during the lesson. In particular, the instructor was aiming to introduce students to proof by cases, when to use cases, what cases to consider, and how to use the statement 'without loss of generality'. Through reflection on student reactions to lecture content, certain areas of the lesson could be improved. In particular, in-class examples can be chosen in a way that requires students to think more about how the constructs are used. Our intention with this poster is to get ideas on how we can improve the lesson and produce a lesson analysis manuscript (Corey & Jones, 2023) from this material for a future journal submission.

Acknowledgements

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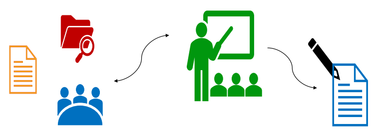
BACKGROUND: Epistemological obstacles (EO's) are challenges students face when exposed to concepts that contradict their pre-existing math intuition. A framework of eliciting and addressing EOs was developed by and practiced at a partnering institution. The particular purpose of this work was to use that EO framework in crafting a proof by cases lesson.

METHODS/CONTEXT

1. Lesson was taught at Eastern US University, class size about 15-20 students.
2. Student Population: had discrete math pre-requisite where a bit of proof was introduced
3. Structured lesson to elicit the EOs that arise in proof by cases- based on suggestions from collaborators



Like the Following:
Task 1) … For each 'for all' statement, discuss what might make proving the claim tricky. What additional info might you want to know about the arbitrary variables to prove the statement?
a. If $n\in$"Z", then n^2+n is an even integer
b. Given two integers a, b, if a and b have opposite parity (i.e. characteristic of being even or odd), then a+b is odd
c. $\forall x\in$R, $x\leq|x|$
d. $\forall(x,y)\in$R^2 , $y=mx$ for some $m\in$"R"
e. If x^2−5x+6>0, then x>3 or x<2

RESULTS

- Students found it difficult to determine whether to use cases and which cases to use
- Were some tasks too basic & didn't elicit obstacles? We need insights on how to improve the lesson if so
- Or is proof by cases too easy & there aren't many EOs?
- Or is there only 1 EO (identifying cases thinking is needed and what are the cases) but that itself is a huge challenge?
- Or are things more nuanced because of contextual differences: students at the study institution had experience with baby proofs in discrete mathematics

Teaching Materials:

Feedback:

Students find it difficult to determine **whether** to use cases and **which** cases to use early in the proof writing process.

How might you address such obstacles? What would be productive tasks to move students through this struggle?

Some tasks are too easy:

4) Look over the following proofs. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

b) Prove: $\forall\ a,b \in$ R, $|ab| = |a||b|$

Assume WLOG, that a>0 and b<0, then

$$|ab| = a(-b) = |a||b|$$

4) Look over the following proofs. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

a) Prove: $\forall\ a,b \in$ R, $|a+b| \ \leq |a| + |b|$

Case 1: assume a>0, b>0, then $|a+b| = a+b$, and $|a|+|b| = a+b$. Since $a+b = a+b$, then $|a+b| = a+b \leq a+b = |a|+|b|$ as well

Case 2: assume a<0, b<0, then $|a+b| = -(a+b)$, and $|a|+|b| = -a+(-b)$. Since $-(a+b) = -a-b$, then $|a+b| = -(a+b) \leq -a+(-b) = |a|+|b|$ as well

Case 3: assume a and b have opposite sign. Without loss of generality (wlog) assume a>0 and b<0 then

$$|a+b| = a-(-b) = a-|b| \leq |a|+|b|$$

How might **you** adjust these problems?

3b) Given two integers a, b, if a and b have opposite parity (i.e. characteristic of being even or odd), then a+b is odd

2) Find all cases that help prove the following. Time permitting, then prove/disprove the statements

c. $(\forall\ a,b \in$ R, ab=0$)\ (a = 0 \vee b = 0)$

d. $\forall\ n \in$ Z where n is odd, n divided by 4 leaves a remainder of 1 or 3

This task was added to the second implementation in the current spring semester. Part (d) specifically was added as a challenging problem for students to think about which cases are needed.

4) Look over the following proof. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

Prove: $\forall\ a,b \in$ R, $|a+b| \ \leq |a| + |b|$

Case 1: assume a>0, b>0, then $|a+b| = a+b$, and $|a|+|b| = a+b$. Since $a+b = a+b$, then $|a+b| = a+b \leq a+b = |a|+|b|$ as well

Case 2: assume a<0, b<0, then $|a+b| = -(a+b)$, and $|a|+|b| = -a+(-b)$. Since $-(a+b) = -a-b$, then $|a+b| = -(a+b) \leq -a+(-b) = |a|+|b|$ as well

Case 3: assume a and b have opposite sign. Without loss of generality (wlog) assume a>0 and b<0 then

$$|a+b| = a-(-b) = a-|b| \leq |a|+|b|$$

Compared to the first implementation, many students did not realize that we need the zero case

6) Determine if the follow statement is true or false. If true, prove your claim. If false, give a counter example: '$\forall\ n \in$ Z where n is odd, n divided by 4 leaves a remainder of 1 or 3'

This problem was skipped in the first implementation due to time constraints. But in the second implementation, the problem was tackled and asked students to identify cases.

The EO we most frequently noticed was the Principle of Universal Generalization (PUG), which is when students struggle with the concept of maintaining generality in a proof. In our case, they confuse the generality of the proof with the generality of cases. Another is Qrv which is when students mistake the role and the value of a quantified variable.

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