Is it Possible to Instruct College Students in the Role of Inventors? A Case Study of a Mathematics Instructor in India

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This qualitative case study attempts to understand instructional goals that a mathematics instructor in India sets and envisions for engaging students. The instructor participated in a professional development program provided by the authors on the active role of students in learning mathematics. The authors' interpretations from three interviews were guided by two theoretical lenses of Realistic Mathematics Education and Rehumanizing Mathematics. In return, the interpretations offer a context to imagine how the lenses might interact.

Keywords: Realistic Mathematics Education, Rehumanizing Mathematics, Undergraduate mathematics,

Today, a substantial number of students worldwide experience a disconnection from mathematics and adopt a passive role as mere recipients of knowledge in classrooms. What if our mathematics instruction was a journey of students' inventions built on situations experienced as "realistic" based on what they know, their intuition, and even their misconceptions (Van den Heuvel-Panhuizen & Drijvers, 2020)? What if we approached students' non-traditional conceptions as avenues to expand mathematical knowledge, as a form of rehumanizing mathematics (Gutiérrez, 2018)? Such an approach has the potential to enhance students' senses of connection with abstract mathematical concepts, foster active engagement and participation, and enable them to view themselves as "authors of mathematics" (Povey & Burton, 1999, p. 235). This study aims to investigate how a mathematics instructor in India, who has been provided professional development on the active role of students in learning mathematics, might conceive of engaging students in classrooms. As a qualitative case, this study seeks to shine light on the meaning that this one instructor made about his instructional aims while engaging in learning about ways to highlight the human nature of mathematics. The interpretations were guided by two theoretical lenses of Realistic Mathematics Education and Rehumanizing mathematics.

Literature Review and Theoretical Framework

Realistic Mathematics Education (RME), conceptualizes the use of "realistic" situations to teach mathematical knowledge (Artigue & Blomhøj, 2013; Gravemeijer, 2020; Van den Heuvel-Panhuizen & Drijvers, 2020). An underlying goal of RME is an instructional approach beyond just exposing the students to mathematics as ready-made, which makes it different from traditional instruction. Realistic Mathematics Education is theorized through six principles: activity, interactivity, reality, intertwinement, level, and guidance (Van den Heuvel-Panhuizen & Drijvers, 2020). Being focused on instructional practices, this study will center on the guidance principle or guided reinvention process, but not in isolation from other principles because other principles interact with and influence guided reinvention. Guided Reinvention (GR) theorizes how teachers can provide opportunities for students by selecting "realistic" contexts to facilitate them to reinvent "higher" level formal concepts individually and collaboratively. (Gravemeijer, 2008; Van den Heuvel-Panhuizen & Drijvers, 2020). Because GR appears to be a set of prescribed practices to teach predetermined concepts, is there room for instructors to be led more by "alternative conceptions" or student's conceptions that differ from or conflict with standard understanding of mathematics concepts (Fujii, 2020, p. 625)?

Rehumanizing Mathematics (RM) is a framework that centers queer BIPOC students, as they are most harmed by standard practices and forms of mathematics (Gutiérrez, 2018). RM seeks to correct for an overemphasis on abstraction, objectivity, and binary logic. The framework advocates for humane practices, such as the incorporation of the cultural and linguistic resources of the participants of school mathematics. RM involves eight dimensions: (a) participation/positioning, (b) cultures/histories, (c) windows/mirrors, (d) living practice/futures, (e) creation, (f) broadening mathematics, (g) body/emotions, and (h) ownership/stewardship. To understand how instructors might be led more by "alternative conceptions" or student's conceptions that differ from or conflict with standard understanding, we choose the Creation dimension of RM. The Creation dimension (Cr) acknowledges students' conceptions and diverse forms of expressing mathematics, which relates to possible students' conceptions in group discussions in GR (Gutiérrez, 2018). In this way, Cr along with GR can be considered as a more comprehensive way of being able to expand beyond traditional knowledge forms or representations and that can then feel humanizing for students who might possess other ways of knowing and viewing the world and help affirm their relationships with mathematics as authors/creators of mathematics. The overarching question – What are the perspectives and intentions of a mathematics instructor in India about his roles/goals in classroom instruction in terms of engaging in abstract concepts? More specifically, [GR] How does the instructor conceive of and facilitate students to learn abstract concepts of mathematics? [Cr] In what ways does the instructor respond to his students when their work and conceptions do not align traditional mathematics or when they could be perceived as "wrong"? (e.g., How does the instructor address students' responses that are "misconceptions"?)

Methods

India's new National Education Policy 2020 strongly emphasizes the achievement of "learning outcomes," which is evident from its repeated mention of the term in the document (Government of India, n.d., p. 3). The primary focus in traditional classrooms of India is often limited to demonstrating the prescribed curricula and preparing students for semester final exams (Venkataraman et al., 2012). As such, many instructors feel pressured to complete the curriculum, even though it is too much content to cover in a meaningful manner. Student-student and student-instructor interactions in these classrooms are usually minimal (Ramanujam, 2012). Thus, students tend to find themselves disconnected from the course content.

In order to build rapport and reciprocity, we organized an online professional development (PD) program consisting of seven interactive sessions (each 90-120 minutes long) for 10 college/university mathematics instructors in India. Inspired by RME and RM literature, we designed activities such as creating reinvention-style lesson plans and grading intuition-driven hypothetical student responses that conflict with standard ideas in order to facilitate participants to share their meanings around GR and Cr in their classroom practices. Five participants who volunteered to be part of the research study gave us permission to interview and record them as they reflected on their experiences in each workshop. The first author interviewed each research participant twice in two 60-minute semi-structured sessions conducted and recorded through Zoom. The first interview centered on Guided Reinvention included questions such as "How was the lesson plan activity with your group partner to create a lesson that can facilitate students to discover math concepts?" to probe how they make sense of reinvention-based instruction and their related instructional goals. The second interview, focusing on Creation, included questions such as, "In the workshop, you discussed with your group partner about 'wrong responses.' What were some important points that arose in the discussion?" to probe for participants' viewpoints for alternative conceptions."

Thus, the PD program served as the entry point for questions we developed for individuals based upon their work in groups and perceptions of the PD. The interview questions avoided evaluation and instead emphasized ways participants made meanings in the program sessions. We sought explanatory responses and offered interviewees a chance to share their perspectives and practices as a teacher and learner. Questions that asked participants to reflect on their experiences as learners offered them a chance to talk about "poor" or ineffective instructional practices without referring to themselves. For example, "In your career as a student and then as a teacher, can you recall an event when a learner's own ways of solving mathematics problems were supported or discouraged because of something? What was that?"

Data Analysis

After analyzing responses from the first two interviews, we selected one instructor, whom we call Aditya [pseudonym], to focus upon for a case study to fully engage and delve into more complexities of his identity and context in making pedagogical decisions (Yin, 2017). As a case study design, this study is not generalizable "from samples to universes" (Yin, 2017, p.20), but its findings might be "transferable" (Finfgeld‐Connett, 2010, p. 246) to other instructors' contexts. The final interview utilized a "Member Checking and Intervention Interview" protocol that was aimed (1) to determine the accuracy of interpretations of this instructor's identified practices in the first two interviews through member checking questions that consisted of summaries of our interpretations and (2) to extend/challenge the participants' practices toward GR and Cr through intervention questions or joint wonders that consisted of the interviewee's own perspectives about teaching and utilizing questions asked within the literature (Gambill, 2022). We wanted to honor the relational piece in research and promote collaborative wondering to make research more humane, empathetic, less extractive, and non-evaluative.

The data were analyzed in multiple phases. We first identified interesting segments and interpreted them with memos. When the authors had different interpretations, we attempted to reach a consensus or an "interrater reliability" (Belotto, 2018, p. 2625). We then labeled the segments and interpretations with short phrases of 3-4 words, which became sub-codes. These codes served as the basis of the findings we present below.

Findings

We found four themes: (a) Avoiding faulty foundations through correct proofs, (b) Offering exploratory opportunities through minimal-to-incremental guidance, (c) Encouraging student engagement by building upon their prior knowledge, and (d) Increasing motivation by providing the history of mathematical concepts. Due to space constrains, we present below the first two themes in this section and mainly the second in the discussion section later.

Avoiding faulty foundations through correct proofs

In this section, we will explore Aditya's role, which appeared in the data analysis, to instruct his students to refrain from faulty foundations through formal or correct proofs. Interestingly, with this goal in mind, he seemingly incorporates informality and motivates students to share diverse "guess[es]" or "wrong" conceptions. In his grading of a student's solution that included informal terms presented in our PD program, Aditya explained:

> [W]e [as students] **assume some results** are obvious or trivial. We simply **write** it down "clearly," "obviously." I forbid them [students] to **write** "obviously" or "clearly." I tell them that if it is so obvious or so clear to you, why don't you **produce**

a proof because it's not at all obvious to me? … if I could not do it, then either there's a problem in my understanding … or … what we are trying to solve is not true.

Like many mathematics instructors, Aditya seems to consider his instructional role to "forbid" or prevent students to always rely upon intuition/informality ("assume some results") so that they can understand the value of proofs. He might be recognizing that ideas that seem to be true informally might not be the case when they try to prove those ideas formally. And, in this case, he seems to be invested in supporting his students to learn how to prove something, as it will be an important skill in learning mathematics. Even if he stresses formal proofs, Aditya appears quite different from many instructors who consistently ignore intuition/informality in instruction to understand his goals toward both informality and formality. When he says, "I forbid them to write obviously, or clearly," he seems to value mathematical symbolic or formal proofs, especially when it comes to *writing* proofs. But contrasting with the traditional emphasis *only* on formal procedures, Aditya acknowledges, at another instance during an interview, informality's role to get a global bird's eye view or an "overall idea" of proofs. These views appear to underscore informalities for *thinking* processes in students' heads when they prove formally or even when they have acquired formal concepts. The way Aditya "forbids" his students to prevent errors is not necessarily instant rejection of their thoughts. At another moment, Aditya appears to value an open classroom space by motivating his students to propose "guess[es]" or conjectures and to ask questions without thinking of them as trivial and incorrect. This might be one way to accomplish the goal effectively because Aditya is able to correct more "wrong" points or "faulty" foundations. Thinking that an understanding based on a student's "guess" can differ from or conflict with an instructor's understanding, the first author asked him about contradictory understandings. He said:

> [If two understandings] are contradictory, one of the concepts must be wrong because **in maths the answer [to a question] is just one [unique]**. … if I realized that I was completely wrong, [or] I was on the wrong track, … [then it] would stay with me always, and I would never make that **mistake** [again]…. If it's not contradictory, then there are scopes [or chances] of learning. Even if it is contradictory …or one of us [with contradictory understandings] is completely wrong, [then] you can think of it as a pin [or reminder] or some kind of marking that would always stay with the student and would help him **not to get it wrong the next time**.

Saying "in maths the answer is just one [unique]" and "not to get it wrong the next time," Aditya appears to value correcting his students' "mistakes" or conceptions, which might otherwise be "contradictory" and ambiguous. Such an instruction can highlight a limited exploration of "wrong" things though because the instruction is only about identifying "mistakes" so that the students do not repeat them. The students' exploration through "guesses" or "questions" requires instructor's facilitation and guidance, which we understand in the next theme.

Offering exploratory opportunities through minimal-to-incremental guidance

Aditya expects his instructional role to offer students exploratory opportunities through his minimal-to-incremental guidance that encompasses exposing the least content possible in the beginning and providing students hints when believed necessary. When asked about students' exploration opportunities in the classroom, Aditya said:

> [W]hat I feel I would … follow in the class actually [offer the] **least bit of help**. So, as if there are several pages [of hints]. You flip the first page, you have [a hint on the first page] if you can solve [the given problem] ... If you can't, then move to the next

> page and see some **more hints or more suggestions** or some more help. So, step by step it's up to the students. … I get a feeling … how much **help they need**. So, depending on that I try to provide help, but at the very beginning. I would give them the minimum. … let them **struggle and learn**.

He appears to target his instruction by offering the "least bit of help" at first so that the students can "struggle and learn" and can get his support of "more hints or more suggestions" when they are perceived unable to complete the given tasks. Through such minimal-to-incremental guidance, as opposed to instructors who give them maximum guidance, his students are likely to first be given the opportunity to learn with guidance as per the extent of the "help they need." Aditya's exploratory and discovery-oriented goals for his students were also reflected in his epistemological perspective in the quote below:

> [Exploring] Proof first then [getting at] theorem statement should be a way [to teach] because that means students always would be **curious**. … How can we get this theorem? It is a **reverse process**; you don't get to eat your meal until, unless you cook it. … That's the way it should be natural.

When asked, Aditya could not recall any particular example of such a "reverse process" at that time but it appears worthy to mention. This "reverse process" of getting a theorem statement after a proof differs from a widespread perception among mathematics professors and in mathematics textbooks to follow Definition – Theorem – Proof format. If directly taught, students might not feel supported to explore in the traditional format. On the other hand, with Aditya igniting "curious[ity]," students might come up with brand new conjectures because they are free to explore and build knowledge. However, the final interview's member checking revealed the strategy does not always play out in the ways he hopes and revealed that his students just wonder individually in their heads instead of contributing to proofs by sharing aloud with others. This is understandable because it might be difficult for students to understand what is expected of them. By practicing minimal-to-incremental guidance, he values neither complete absence of guidance by letting them explore without facilitation nor excessive guidance.

The analysis of the first two interviews highlights his emphasis on asking questions to the students, which can appear as "hints." Thus, in the final interview, we asked how he might model questions to students. Some questions such as "what-if" questions were found to be open-ended and promote exploration while others such as "this-and that" questions could guide students toward a particular line of thinking. He said:

> [I]magine that I have **changed one condition** from there or the other condition from there **[in a known result]**. [Then I ask them] do you still think the result is true? So, some questions of this form [I would ask]. Can we remove these conditions or can we add [some], can we **weak[en]** them? So, with [their present] knowledge they [students] can come up with … counterexample[s] … You said that both A & B would hold? Almost probably A would hold or B would hold. **Can you guess which of them [A and B] would hold?** … They are just **interpolating [guessing] from what they know**.

When he says, "Can you guess which of them [A and B] would hold?", he appears to prompt the students to consider specific and important ideas or concepts by "interpolating from what they know." Such a binary question is important so that students can understand the intended goals of their instructor. Moreover, Aditya's "what-if" questions like "chang[ing] one condition … in a known result" in particular can be seen as open-ended questions and beyond mere yes/no questions because the "students can come up with … counterexamples" to justify and there can be brainstorming on tweaking conditional statements, like "weaken[ing] them" in his classroom. Thus, the what-if questions in Aditya's classrooms are likely to provide support for exploration.

Discussion

Aditya had been introduced to the concepts of Guided Reinvention (GR) and Creation (Cr) through a series of PD sessions and was offered the opportunity to reflect on these concepts and how they (might) play out in his teaching. With the objective of how he makes sense of GR and Cr in mind, we identified four main themes through the teaching practices that he employs or conceives: (a) Avoiding faulty foundations through correct proofs, (b) Offering exploratory opportunities through minimal-to-incremental guidance, (c) Encouraging student engagement by building upon their prior knowledge, and (d) Increasing motivation by providing the history of mathematical concepts.

Due to space restraints, we mainly focus on the theme of *minimal-to-incremental guidance* in this section. Even so, we believe it is important to highlight some of the interactions between the themes and with the conceptual tools of GR and Cr. We found that Aditya envisions his responsibility to provide opportunities for exploration through an instructional strategy of giving minimum assistance in the beginning and increasing the assistance gradually through "hints" and questions as per the perceived need of the students. Through the instructor's hints, such an approach has potential to empower students to reinvent intended concepts that GR contends. Due to the lack of expository teaching, the "curious" students might also have opportunities to *explore* and develop brand new conjectures in such a minimal-to-incremental guided instruction; thus, the instruction can be seen consistent with Cr. Aditya's technique appears unique, given this type of guidance does not appear in Learning Outcomes based Curriculum Framework (LOCF) and the standard faculty induction program called GURU-DAKSHTA published by University Grants Commision (UGC), a statutory body in India. Aditya' questioning strategy in the classroom supports both GR and Cr. The binary question, observed in the findings section, is important so that the students can understand the intended and clear goals of their instructor, which provides support for GR because RME-instruction sequences appear to have a clear specific path to follow. Moreover, Aditya's open-ended questions like what-if questions appear consistent with King and Rosenshine (1993) who argue for "guided cooperative questioning strategy" in which teachers ask generic thought-provoking questions that can provide support for exploration and, thus, Creation. Such an approach is more process-oriented (as opposed to product-oriented), which makes it different from the learning outcomes approach highlighted in India's education policy and GR's "intended mathematics" goals (Gravemeijer, 1999).

A minimal guidance might facilitate students' exploration, but there are chances that such explorations can lead to students' proofs that are not "correct" or to their "contradictory understandings" that are revealed in the theme on correct proofs. We note from Aditya's practice that if guidance is increased too quickly, it might restrict exploration or discovery. In other words, if his students solve mathematics problems using informal or intuitive language that does not align with standard proofs, he might feel prompted to make them aware of the inaccuracy and tell them or motivate them to use formal proofs.

More than simply showing how Cr and GR arise in Aditya's views and instruction, his case highlights how the two conceptual tools Cr and GR relate and support or expand the other theoretically. Both concepts acknowledge: (a) students as mathematicians, (b) students participating and connecting with each other, and (c) instruction as process-oriented. Let's consider how they expand each other. First, Cr expands GR in following ways: (a) The GR process aims to facilitate students to reinvent instructor's or curriculum's predetermined concepts. Cr acknowledges that the foundations of the predetermined concepts can be contextual, for example, following one of several appropriate sets of axioms or a collection of self-evident statements. In this way, Cr might go beyond just considering predetermined concepts in teaching and learning mathematics by taking into account the "misconceptions" as alternative, yet viable, ways to think about or develop a concept. (b) GR appears to be a linear sequence of levels starting from situation → referential → general → formal in "Model of - Model for: emergent modeling" (Gravemeijer et al., 2000; Streefland, 1985; Van den Heuvel-Panhuizen, 2003). Creation can inform us that instructional practices are not necessarily sequential due to the probable presence of an interaction between correct and "wrong" conceptions. Second, GR expands Cr in the following ways: (a) Goal-oriented: Cr does not speak about achieving specific goals that can indicate success. Considering GR with Cr can be helpful because GR appears goal-oriented by focusing on facilitating students to reinvent a predetermined concept. (b) Concrete practice: GR can offer instructional steps, for example the linear sequence of levels, should an instructor be confused or uncertain about how to approach Cr in the classroom. Thus, Cr and GR show the potential to expand each other.

There are several strengths of this study. The study was not extractive because we attempted to build through (a) the PD program before we generated data with Aditya and (b) the joint wonders during interviews that included the first author's teaching experience to identify the resistance in practicing GR and/or Cr in Aditya's instruction during the interview. Second, the interview protocols were tailored to the participant's responses and according to the examples and comments they provided in preceding interviews. Even so, there are some limitations to this study. For example, although several interviews were conducted with each participant, there is a single data source: interview responses from Aditya. Multiple data sources, such as classroom observations and comments from students during focal groups or interviews, or survey responses would have allowed for triangulation and increased credibility and a more comprehensive understanding.

Conclusion and Implications

This case study investigated how a mathematics instructor in India conceived of his practice in relation to facilitating abstraction and addressing students' conceptions. This study raises several issues for future researchers. In this study, Aditya described what he does and what he thinks about his teaching through interviews and, in that sense, his case offers researchers a glimpse of the potential approaches and challenges of incorporating GR and Cr into mathematics instruction at the college level. However, the data fail to capture how his students feel about his instructional practices. Future research might consider observing the classroom in an ethnographic study in order to get some evidence from students on how GR and Cr are experienced or might be supported in the classroom. This approach is especially important in the case of RM because instructors alone cannot determine if their pedagogy is being felt in rehumanizing ways (Gutiérrez, 2018). Potential research questions for the study might include: (a) How does a mathematics instructor practicing both GR and Cr know how much informality is appropriate during a course primarily based on abstract concepts or in courses that feature "pure" mathematics? (b) What instructional practices, such as debates in history, might offer Nepantla moments for students so they can appreciate conflicting conceptions in classrooms (Scott & Tuana, 2017)? and (c) How might seeing informality and formality as part of a larger cycle affect a mathematization process in GR during the instruction?

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