Developing Personal Representations to Access a Set-Oriented Perspective on Counting

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In some classical semiotic perspectives, signs and sign systems can be viewed as either institutional or personal. This dichotomy can place personal representations as antecedent to institutional ones, sometimes with the explicit goal of transitioning away from the personal ones entirely. In this paper I draw from qualitative data to argue that personal representations are legitimate and sometimes necessary components of developing combinatorial facility, and they play an ongoing role in understanding that is not replaced by institutional representations. I frame this theoretical argument around a discussion of common institutional semiotic systems for combinatorics, which can insufficiently support a set-oriented perspective on counting. I discuss how to incorporate personal representations into analysis that uses a classical semiotic lens. Doing so includes attending to the motivations for creating (or refining) a sign system, the connections between those systems and others, and how the form and nature of the sign systems seem to impact or reflect student reasoning. I also discuss how some differences between combinatorics and other areas of mathematics might affect the personal/institutional dichotomy in semiotic analysis.

Keywords: combinatorics, semiotics, student reasoning, personal representations

Consider the counting problem *how many ways are there to flip a coin three times in a row?* This problem is an appropriate introduction to combinatorics because it invites solutions that draw from multiple modalities, including and certainly not limited to: flipping physical coins to exhaust outcomes (systematically or not), listing entire or partial sets of written outcomes such as HHT or 001, reasoning structurally about enumeration processes or representations of those processes such as <u>H/T</u> <u>H/T</u> <u>H/T</u>, using embodied and/or gestural reasoning with or without iconic imagery, and applying abstract counting principles formally or informally. Each of the possibilities mentioned here, and countless others, can find the same numerical solution—there are 8 possible ways to flip a coin three times in a row. And yet the approach might impact how the solution is expressed. While one approach might yield 2^3, others might yield $1 + 3 + 3 + 1$ or simply the integer 8. Institutional combinatorial texts (such as textbooks or lecture notes) might only present solutions within limited modalities, often confined to a natural language (e.g., English), symbolic mathematics, and figures (if the text is generous). I consider the limit of modalities to be a byproduct of the textual medium and not necessarily an indication that the solutions are more sophisticated than those drawing from other (often non-institutional) semiotic systems. Moreover, the limited presentations of solutions can perpetuate negative attitudes towards combinatorics when the singular way combinatorial reasoning is presented does not align with what is natural for learners. This is especially unfortunate considering that the abundance of solution approaches enrichens combinatorics.

The purpose of this paper is to legitimize personal representations in combinatorics by discussing why they have an ongoing role in the production and justification of solutions, and why institutional representations seem insufficient in satisfying this role. I apply a fairly narrow view on semiotics which examines the inscribed signs and sign systems students create and use as they solve counting problems. Although this view does not incorporate other semiotic resources such as rhythm and gesture, it is consistent with prior undergraduate combinatorics education literature while still offering additional contributions. My core argument is as follows. Semiotic systems are ubiquitous in mathematics in part because they facilitate mathematical production. Yet, common combinatorial notation is more conducive to expressing solutions than producing them. Students develop personal representations for combinatorics in order to produce mathematics externally because there are limited institutional means of doing so. Institutional and personal representations can cohabitate in combinatorics because they are used for different purposes. One is not transitional or auxiliary to the other, and personal representations are not antecedent to institutional ones. However, personal representations are not neutral mediators of combinatorial activity. Two different personal representations might lead to two different arguments and two different mathematical expressions of the same integer value, numerically equal but indicative of different structural reasoning. A semiotic lens on combinatorics must account for such differences.

Combinatorics and a Set-Oriented Perspective

Combinatorics is a broad area of mathematics that includes enumeration and existence. Most introductions are through enumerative combinatorics, more commonly called counting problems. From an expert's perspective counting problems task the counter (i.e., the person solving the counting problem) with determining the cardinality of a set of objects or determining the number of ways an event can be carried out. Lockwood (2013) characterizes combinatorial activity as occurring between three components: sets of outcomes, counting processes, and formulas/expressions. The set of outcomes is the collection of objects being counted in a problem, and the solution to the problem is typically the cardinality of that set. Counting processes are the real or imagined enumerative processes the counter engages in as they solve a problem. Formulas/expressions are the mathematical expressions that yield some numerical value, and solutions to counting problems are typically given as some expression. Lockwood (2014) describes a set-oriented perspective on counting as "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). At the core of the set-oriented perspective is the connection between mathematical expressions and the structure in a set of outcomes. A mathematical formula can be applied because it reflects structural elements of the set of outcomes. The set-oriented perspective contrasts other approaches to counting which involve identifying problems types and applying formulas associated with those problem types.

Lockwood's characterization of a set-oriented perspective is sufficiently broad so as to incorporate the manyfold ways of reasoning combinatorially, which inevitably raises the question of how the one engages in such a perspective. Antonides and Battista (2022a, b) captured certain cognitive processes for spatial-temporal-enactive structuring involved with geometric objects, which included recursive reasoning that ultimately led to closed-form solutions for permutation problems. Some work with younger students (e.g., English 1993; Speiser, 2010; Tarlow, 2010; Tillema, 2013, 2018) has examined how material resources, iconic imagery, and emergent representations are key components in developing and communicating combinatorial reasoning. Literature examining undergraduate students has often examined student inscriptions, such as lists of outcomes (Lockwood & Gibson, 2016), and how encoding outcomes impacts solutions and justifications (Lockwood et al., 2018; Wasserman & Galarza, 2019). This literature has all reported on various semiotic resources as important to student reasoning, even if the analysis was not focused on the mechanisms through which the semiotic resources mediated solutions. A further step towards understanding how students engage in a set-oriented perspective is to examine the role of the semiotic resources in solutions that use a set-oriented perspective.

Connecting Classical Semiotics to Combinatorics

Student use and development of personal semiotic representations in combinatorics are not new phenomena. They appear in literature spanning from early elementary to graduate school (e.g., Tillema, 2013; Wasserman & Galarza, 2019), and they are reported as indicative of or contributing to combinatorial facility. By using a semiotic lens to investigate personal representations I seek to better understand the representations themselves, why students create them, and how they contribute to solutions. I begin discussing semiotics by discussing semiotic systems, and how common combinatorial notation fits within broader semiotic systems. This will include a discussion of semiotic systems as communicative mediums as well as ways for producing new mathematics. Then, I will briefly discuss some insufficiencies of institutional combinatorial notation and why students create personal representations. In the following section I will discuss some personal representations in combinatorics, and use the examples to illustrate how the personal representations demonstrated sophisticated sign systems that contributed to mathematical production.

Semiotic Systems and the Combinatorics Register

In broad strokes, semiotic systems in mathematics are the systems of signs and symbols used to communicate and carry out mathematics. Common semiotic systems are fractions, decimals, set-builder notation, and function notation. Semiotic systems exist in many modalities (e.g., speech, manipulatives, imagery, pictures) but for the purpose of this paper I will focus solely on written (inscribed) and spoken signs and sign systems. Ernest (2008b) characterizes a semiotic system as being composed of three components: i) a set of signs, ii) a set of rules for sign use and production, and iii) an underlying meaning structure that incorporates the relationships between the signs and the rules for their use. For fractions we may all be aware that '0', '3', '8', and '/' are included in the set of signs, and that the set of rules for sign use says that '30/8' is a permissible sign whereas '038/' is not. The rules for sign use are a consequence of using signs as a communication vehicle for an underlying meaning structure. The rules ensure that there is a way of interpreting the signs, and that the interpretations of the signs are consistent with what they represent. The fraction 30/8 is permissible because it is interpreted as the ratio between the integers 30 and 8. The sign 038/ does not have such an interpretation, and so it is not permissible.

Sign systems are also used to produce new mathematics through rules that state how some signs can be transformed into others (Duval, 2006). That is, beginning with the expressions $\frac{d}{dx}3x^2$ we can carry out the derivative purely symbolically to find $\frac{d}{dx}3x^2 = 2 \cdot 3x^{2-1} = 6x$. Although we can relate this transformation back to the underlying meaning structure, in practice doing so is rare. In this instance the set of rules in the semiotic system allow for more efficient mathematical production because they allow for a temporary foregoing of the underlying meaning structure. Duval (1995, 2006) distinguishes between two types of transformations. Treatments are transformations that occur within the same semiotic system (e.g., a fraction to a fraction), and conversions are transformations that occur between different semiotic systems (e.g., a fraction to a decimal). Duval calls semiotic systems that permit both treatments and conversions *registers*. With these theoretical constructs in mind, we can characterize some mathematical activity as reasoning within or between various semiotic systems (registers), and we can also characterize some difficulties as difficulty with types of transformations or semiotic systems.

One of the difficulties within combinatorics is that there are limited institutional semiotic systems students can access as they solve problems. Many counting problems are given in a natural language (e.g., English), and they task the counter to determine the cardinality of a set of outcomes, or the number of ways in which an event can occur. Solutions to counting problems are integers or parametric expressions that can be transformed into integers, which may include signs and symbols associated with combinatorics, including $P(8,3)$ and $\binom{n}{k}$. I refer to the semiotic system used for combinatorics as the *combinatorics register*. Seen purely as signs that can be transformed into integer values, the combinatorics register might be considered an extension of the fraction register or algebraic register. However, the underlying meaning structure of the combinatorics register is quite different, which leads to a duality of the notation (Wasserman, 2019). Wasserman claims that the combinatorics register (which he described as combinatorial notation) can be interpreted in two ways, first as integer-valued expressions, but also as representations of enumerative or structural processes. Underlying any combinatorial expression is the need for that expression to be tied to a set of outcomes. Not only does the expression need to permit a transformation into an integer, but the expression also needs to permit an interpretation as an enumerative expression for a set (Wasserman, 2019). This connection has repercussions for the set of rules for sign use. From a purely algebraic/arithmetic standpoint it is difficult to connect the two signs 2^n and $\sum_{k=0}^{n}\binom{n}{k}$, and substituting one for the other (although correct) might be considered lacking justification. Interpreted from a combinatorics standpoint this substitution is justified because the two expressions represent different enumerative/structural arguments for the power set of a set of n. The underlying meaning structure reorganizes and rewrites the rules for sign use and production because it focuses interpretation on cardinalities and enumeration of sets rather than abstract integers.

The combinatorics register provides a means of communicating cardinality and enumerative processes simultaneously, but it does not provide a means of producing the mathematics. The interpretations of the combinatorics register are purely internal. This contrasts with the prior calculus example where the semiotic system also permitted treatments that facilitated producing mathematics purely symbolically. The combinatorics register thus conflicts with a set-oriented perspective because it does not provide notation specifically for the objects being counted, requiring students to either form the connection mentally or develop entirely new notation for the objects being counted. In practice students do both, and in the following section I present some examples of personal representations to illustrate how personal representations inform and impact solutions to counting problems.

Examples of Personal Representations

In this section I discuss some examples of personal representations students created over the course of several interviews. These data come from a larger study that aimed to examine the roles of representations in solutions to counting problems. I have chosen these particular examples because they exemplify the potential symbiosis between personal and institutional representations. Although the students expressed a ‘final answer’ using an institutional representation, the bulk of their work was done using personal representations. Moreover, the students developed and refined the personal systems of representations over the course of multiple interviews, and the systems indicate sophisticated attention to structural aspects of the sets of outcomes which impacted the form of their solutions.

Lists and listing processes: Hallie and Leah

The students Hallie and Leah used lists and listing processes repeatedly over the course of six 90-minute interviews. During the first few interviews Hallie and Leah used the lists to completely enumerate a set of outcomes, solving a counting problem by tallying the total number of outcomes in their list. This use of lists became increasingly intractable as counting problems became more complex. Hallie and Leah observed symmetric properties in their lists, and their subsequent uses of lists exploited symmetry to decrease the number of outcomes they needed to write down. The exploitation of symmetry was accompanied by newly introduced signs (such as arrows) to indicate continuations of patterns. In early problems Hallie and Leah would list half of the total outcomes and indicate a continued pattern. In later problems the listing process was increasingly segmented, and symmetry was exploited throughout. Figure 1 provides an example of a listing process represented through segmentation of symmetric components of the list. This example solves a combination whose solution can be written as $\binom{13}{3}$.

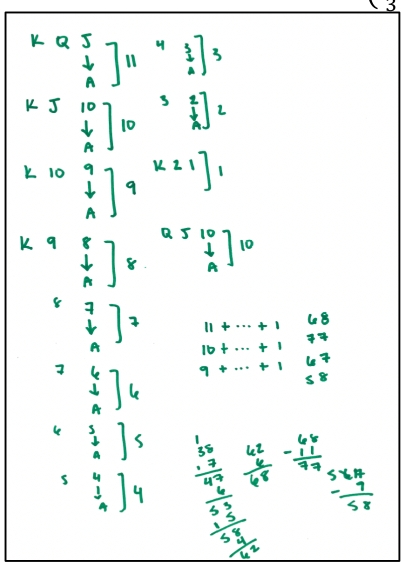

Figure 1: Representation of a listing process exploiting symmetry

In Figure 1 a listing process is represented using alpha-numerical characters, brackets, and arrows. The arrows are used to indicate a continuation of a pattern, and the brackets segment the list of outcomes into constituents. Hallie and Leah placed numbers next to the brackets to indicate the number of outcomes in each constituent piece. The list indicated in Figure 1 is only a partial list, but the sums in the middle of the figure illustrate how Hallie and Leah generalized the results of the partial listing to the remainder of the outcomes. The partial list in the example corresponds to the sum $11 + \cdots + 1$, and the remaining two sums $10 + \cdots + 1$ and $9 + \cdots + 1$ correspond to two other segments of the entire set of outcomes. Hallie and Leah used the partial sums to count the number of times each summand appeared in a partial sum. They found that 11 appears in one partial sum, 10 appears in two partial sums, and so forth, leading to their solution given in Figure 2. I present this example because it demonstrates that Hallie and Leah used personal representations (the lists and listing processes) to produce their solution, and their solution reflects the structure they observed in the outcomes.

$$11+2(10)+3(9)+...+10(1)$$

Figure 2: Solution to a combination problem

Sequences of characters: Pat and Damien

The students Pat and Damien repeatedly solved counting problems by representing outcomes as sequences of alphanumeric characters and leveraging know structural information about alphanumeric characters to solve their counting problems. This technique was challenged when they solved the road problem, which states "A certain map connecting locations A and B is given below. You are creating routes between A and B so that no roads are repeated. How many routes are there?"

Pat and Damien's solution to this problem consisted of iteratively refining the method of encoding paths in the diagram as sequences of characters. The first method of encoding paths consisted of mapping the route of an imagined agent in terms of left and right path segments. This method of encoding was difficult to count because it resulted in paths of variable sizes with constraints that were challenging to characterize. Through several refinements consisting of reformulating how information was encoded or selectively choosing information to leave out of the outcomes, Pat and Damien found that each outcome could be encoded as a length-6 sequence of Ts and Bs. A T was used to indicate movement along the top of the diagram, a B was used to indicate movement along the bottom of the diagram, and vertical movement was omitted because it could be deduced by movement along the top or bottom of the diagram. After finding this method of encoding Pat and Damien concluded that there were 2^6 total paths along the diagram. I include this example because it illustrates that most of Pat's and Damien's solution consisted of using personal representations to characterize the outcomes in ways that were conducive to counting. Their final solution, which connected sequences of Ts and Bs to the expression 2^6, was merely one step in a larger process of iteratively refining representations of the outcomes, and the bulk of the mathematical production occurred in personal representational systems.

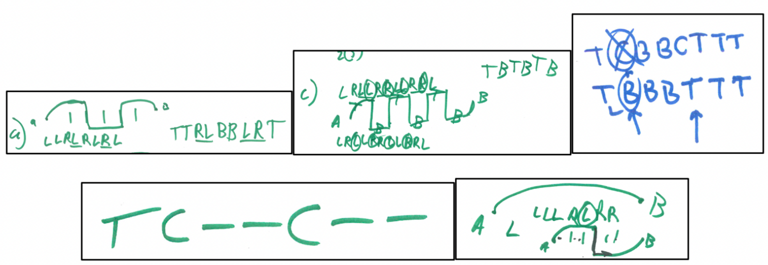

Figure 3: Multiple iterations of encoding paths as sequence of characters

Conclusions

In this paper I have argued that personal representations play a key role in developing combinatorial facility. In part, this role is necessitated by the combinatorial register (the common institutional semiotic system for combinatorics) insufficiently supporting mathematical production. Unlike semiotic systems common to calculus and algebra, the combinatorial register is most conducive to communication. The commonality of personal representations in combinatorics appears to be a result of this. Personal representations appear to facilitate mathematical production while reflecting student understanding.

Wasserman (2019) previously discussed the duality of combinatorial notation, and that the difficulty of learning to interpret combinatorial notation as representative of enumerative processes poses a pedagogical challenge. He proposed the development of institutional notation for sets of outcomes specifically in ways that align with the combinatorics register. In my view personal representations can fill the same role, and attending to personal representations can additionally increase student agency in solutions. This argument, of course, must weigh the possibility that students can still be challenged by creating personal representations, and that some personal representations might be more difficult to use in ways that are tractable. Further study is needed to determine how to leverage personal representations in a didactical setting, and potential tradeoffs between attending to personal representations and creating institutional representations that incorporate elements of observed personal representations.

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