Calculus I Students' Understanding of Implicit Differentiation

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Implicit differentiation is an important topic in first-semester Calculus, yet only recently have researchers in the RUME community turned their attention to it. Our study seeks to contribute to this growing body of knowledge. We examined students' performance on an instructional activity that emphasized the graphical representation of implicit curves alongside the symbolic competence of implicit differentiation. The students worked in small groups during recitations and then completed a similar type of question on an assessment. A researcher-designed Hypothetical Learning Trajectory was used to structure the instructional task and to analyze student written data. The results suggest that students' attainment of symbolic learning goals is higher than that of graphical ones, with most difficulties experienced with coordination between symbolic and graphic modalities. Implications for further research are discussed.

Keywords: Implicit differentiation, Calculus learning, Hypothetical learning trajectory

Implicit differentiation is one of the core topics of Calculus, yet only recently researchers have turned their attention to studying it (Speer and Kung, 2016). Implicit differentiation is a technique for finding derivatives of equations where y cannot be explicitly expressed as a function of x. Many popular textbooks on single-variable Calculus in the United States (e.g., Stewart, 2016), present implicit differentiation as a computation technique for finding derivatives, with a strong emphasis on symbolic manipulation. Little attention is paid to building the conceptual basis of implicit functions, justifying the legitimacy of differentiating both sides of the implicit equation (Mirin & Zazkis, 2019), or developing the graphical meaning of implicit functions. All these contribute to students' conceptual and technical difficulties with the topic.

Within the existing research on implicit differentiation, we identified two strands: survey studies that characterize students' difficulties with implicit differentiation (e.g., Chu, 2019; Kandeel, 2021), and small-scale intervention studies that closely examine the development of students' understanding of this topic (e.g., Borji & Martinez-Planell, 2019, 2020; Jeppson, 2019).

Our study occupies a middle ground. It was conducted in the first-semester single-variable Calculus I course, in an authentic classroom setting. After attending a lecture on implicit differentiation, students worked in groups, during a recitation on an instructional centered on graphical representation and the connections between symbolic and graphical meaning of implicit differentiation. In this context, we examined students' performance on the implicit differentiation task and the follow-up performance on a similar exam question. Due to space constraints, here we focus on the class work only.

Literature review

Calculus topics such as derivatives, the chain rule, and related rates have been previously researched (e.g., Cottrill, 1999; Infante, 2007; Martin, 2000, Zandieh, 2000). However, implicit differentiation came to the researchers' attention only recently. Several studies focused on students' difficulties with this topic. Chu (2019) surveyed 136 first-semester calculus students and found that only about 50% of solutions were correct; with most students' errors being related to calculus concepts and procedures, rather than prerequisite algebra skills. Kandeel (2021) classified common errors in an implicit differentiation survey completed by 117 Calculus students. Among common algebraic errors were isolating a common factor, simplifying expressions, dealing with exponents and radicals, and isolating y', of responses. Common calculus errors occurred in the application of the chain rule, product rule, and differentiating functions. These types of errors appeared in 25% - 50% of responses.

Researchers also attempted to unpack what it means to "understand" implicit differentiation. Mirin and Zazkis (2019) point to the conceptual difficulty of understanding the legitimacy of differentiating both sides of implicit equations. Borji and Martinez-Planell (2019) assert that first-year Calculus students have neither the background nor "mathematical tools to consider the statement, and even less the proof, of the Implicit Function Theorem, upon which a rigorous study of implicit differentiation would be based" (p. 15). The authors utilized the APOS (action-object-process-schema) theory (Arnon et al., 2014) to develop a genetic decomposition for implicit differentiation –a hypothetical sequence of mental constructs involved in learning this topic and five technology-assisted activities supporting the development of these mental constructs. The genetic decomposition and the activities emphasized a graphical understanding of implicit functions and symbolic-graphic connections. The intervention was carried out with 14 students who have completed a lecture-based Calculus I course. Ten out of 14 students showed an increased understanding of implicit differentiation, but others had no evidence of progress.

In another intervention study, Jeppson (2019) developed a hypothetical learning trajectory (HTL) for implicit differentiation. HLT is a sequence of learning goals, activities, and conjectured students' understanding of a particular content (Simon, 1995). Jeppson's HLT integrates the chain rule, implicit differentiation, and related rates under the overarching concept of nested multivariation - a covariation of multiple variables related to one another through function composition. Jeppson conducted single-subject teaching experiments with four Calculus students to study how they develop an understanding of implicit differentiation through meaningful context and careful sequencing of mathematical. Yet, Jeppson's HTL did not include graphical representations of implicit functions nor symbolic-graphic connections.

Theoretical Perspectives

Following Borji and Martinez-Planell (2019, 2020), we place a strong emphasis on symbolic-graphical connections in developing student understanding of implicit differentiation. However, the scale and methods of our study, detailed below, do not allow for examining mental constructs or cognitive processes underlying student understanding of implicit differentiation. Instead, we adopt Vygotsky's (1978) *sociocultural perspective* to examine what learners *say and do* while solving an implicit differentiation task in a collaborative group context, supported by more knowledgeable others – teaching and learning assistants (TAs and LAs). Within the sociocultural perspective, student activity is characterized by their discursive performances, which include vocabulary, visual mediators, actions, etc. (Lave & Wegner, 1990; Sfard, 2015)[1]. Student learning is cast in terms of increased, proficient participation in the discursive practices of a particular community and sifting from ritualistic to goal-oriented participation. In the Calculus course, students are enculturated into discursive practices of the mathematical community. We described these practices in a Hypothetical Learning Trajectory of implicit differentiation.

Hypothetical Learning Trajectory for Implicit Differentiation

An HLT is a theoretical model to design instruction supporting conceptual learning, which consists of learning goals, tasks, and hypothesized learning processes (Sion, 1995). Jeppson's

[1] See Sfard (2015) for the discussion of parallels between cognitive constructs and discursive perspective.

(2019) HTL was a main inspiration for our model. But our HTL (Table 1) had a strong focus on graphical representation and graphic-symbolic coordination (c.f., Borji & Martinez-Planell, 2019). The last column matches the HTL's goal to the item number in the activity (Figure 1).

Table 1. Hypothetical Learning Trajectory for Implicit Differentiation

Category	Description of Goal	Item #
Symbolic	**Develop symbolic meaning of implicit equations and correctly implement differentiation procedures**	
Symbolic Recognition (S_Rec)	Recognize an implicit equation cannot be written explicitly as $y = f(x)$.	1b
Symbolic implicit equation (S_Eq)	Given an equation with variables x and y, recognize y as an implicit function of x.	2
Symbolic Chain Rule (S_Ch)	Given implicit equations recognize the need for the chain rule in taking the derivative with respect to the implicit independent variable.	2
Symbolic differentiation (S_Diff)	Can correctly perform procedures for finding $\frac{dy}{dx}$ for an implicit equations.	2
Symbolic Evaluation (S_Eval)	Can correctly compute the derivative $\frac{dy}{dx}(x_0, y_0)$ at a point	3a, 3b
Symbolic Tangent Line (S_Tan)	Can correctly find the equation of a tangent line at a point.	3a
Graphic	**Interpret graphical properties of implicit functions**	
Graphic Recognition (G_Rec)	Recognize graphic representation of implicit equations as a curve in a cartesian place that does not pass the vertical line test.	1a
Graphic Tangent Slope (G_TS)	Interpret the meaning of $\frac{dy}{dx}(x_0, y_0)$ a slope of a tangent line at a point (x_0, y_0), and $\frac{dy}{dx}$ as slope of the tangent line or instantaneous rate of change at any point.	2a, 2b
Graphic Vertical Tangent (G_VT)	Recognize that tangent lines to a graph of implicit equations can be vertical, and at the points of tangency the derivative does not exist.	4
Graphic Constant Rate (G_CR)	Recognize in an explicit function, the lack of x in the derivative formula means the function has a constant rate of change, but a derivative formula of the implicit equation can contain no x, while the curve does not have a constant rate of change.	2b
Graphic Coordination (G_Cor)	Coordinates graphical and symbolic representations.	2b, 3a, 3b, 4

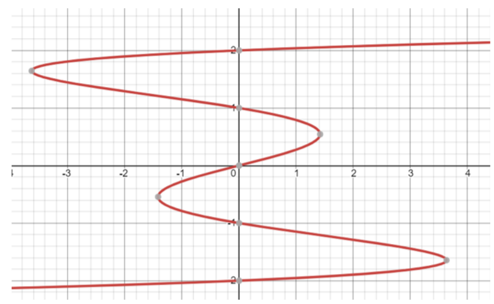
We are mindful that the construct of HTL belongs to the cognitive paradigm, hence our use of the term may not exactly align with the literature. We operationalize the HTL categories in Table 1 as *discursive competencies* that students can exhibit in their mathematical work. This allows us to characterize not just students' difficulties, but also their areas of strength. We examine the following research question: *What aspects of HTL can be observed in students' written responses to an implicit differentiation activity?*

Methods

This study is a part of the larger NSF-funded project for improving the teaching and learning of introductory STEM courses in a large public university in the northeast of the USA. Calculus I is taught in the format of large lectures (~120 students), taught by a faculty member three times a week; and recitations (~20 students), taught two times a week by graduate TAs. The reform efforts involved introducing active learning (CBMS, 2016) in recitations, the inclusion of LAs to support student work; and the use of conceptually rich activities that emphasize graphic representations of the calculus concepts. Figure 1 shows an activity on Implicit Differentiation.

Task 1: Consider the curve on the graph and the equation of the curve: $x = y^5 - 5y^3 + 4y$.

With your small group, answer the following questions using both the symbolic equation and the curve graph. Explain your reasoning.



1) Is it possible to express y as an explicit function of x?
   a) Explain your response using the graph.
   b) Explain using the symbolic equation for the curve: $x = y^5 - 5y^3 + 4y$.
2) Use implicit differentiation to find a formula for $\frac{dy}{dx}$.
   a) Explain the meaning of $\frac{dy}{dx}$. What does it represent?
   b) Sam noticed that his expression for $\frac{dy}{dx}$ does not contain x. Sam thinks that this means that the rate of change is constant for all points on the x-axes. Is Sam correct in his thinking or not? Explan.
3) a) Find the equation of the line tangent to the graph of $x = y^5 - 5y^3 + 4y$ at point (0,1). Add the tangent line to the sketch.
   b) Are there other points on the graph, where the slope of the tangent line is the same as you found in 3(a)? Respond using the graph, and then verify algebraically.
4) Determine all the points at which the graph of $x = y^5 - 5y^3 + 4y$ has a vertical tangent line. Explain how you know that. How many such points are there? Locate them on the graph.

Figure 1: Task 1 from the Implicit Differentiation Activity. Based on Boelkins et al., 2018 (p. 147).

Data Collection

Data collection occurred in Spring 2022. After a lecture on implicit differentiation, students worked on the Implicit Differentiation activity (ID-activity, hereafter) in the recitations. Students worked in groups of 3-4, but each person submitted their own worksheet, graded for completion. There was a standard textbook homework assignment, unrelated to graphical representation. The mid-term exam, about three weeks later, included an implicit differentiation close in style and content to the ID-activity. The data sources were 119 consenting students' written work on the ID-activity and the exam question. Here we only report on their ID-activity performance.

Data Analysis

The data were coded using the HLT (Table 1). Each response was coded as either exhibiting evidence of the student attaining a certain learning goal, not attaining it, or not enough evidence (Yes/No/NEE). For example, consider item 2a: "Explain the meaning of dy/dx. What does it represent?" A student's response "This represents the derivative, which is the slope of the tangent line. Rate of change" was coded as "Yes" for the evidence of attaining the goal *Graphic Tangent Slope* (G_TS). Only wrong or completely irrelevant answers were coded as "No" evidence of attaining an HLT goal. All other cases, including no response, were coded as NEE – not enough evidence. An example of NEE-coded response for question 2a is: "$\frac{dy}{dx} = y'$. Find derivative with respect to y." This response explains the notation rather than the meaning of dy/dx. Yet, it is not enough evidence to conclude the lack of attainment of the G_TS goal.

This method allowed us to identify the elements of HLT where students showed competence or lack thereof on a fine-grain scale. Figure 2 shows one student's work in response to question (2) "Use implicit differentiation to find the formula for dy/dx."

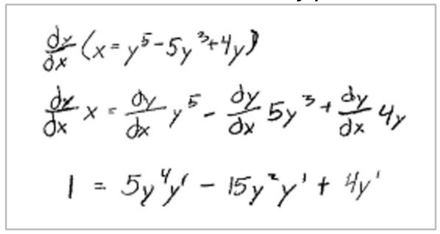


Figure 2: Coding Example

This work would score "Yes" for evidence of correctly using the chain rule to take the derivative with respect to the implicit independent variable (S_Ch) but would get NEE for symbolic differentiation (S_Diff), due to incomplete calculation of y'.

In our HLT (Table 1) the two codes S_Eval and G_TS were matched with two items, and the code G_Cor with four items. To get a single score per code we considered where most evidence is leaning, e.g., two or more "Yes" with all other NEE was coded as "Yes" (evidence of G_Cor attainment). One "Yes" with all other NEE was coded as NEE. Two or more "No" with all other NEE was coded as "No". Getting NEE in all four G_Cor items was coded as "No," since the student used none of the four opportunities to exhibit any evidence of coordinating graphic-symbolic representations. There were no "Yes/No" combinations in our data. The rest of the data was analyzed similarly, resulting in a total of 1190 codes (119 students x 10 codes/ HLT goals[2]).

Results

Table 2 shows the distribution of Yes/No/Nee codes in each of the HLT categories.

Table 2: Distribution of Yes/No/NEE codes by category, N=119

	Symbolic HLT goals					Graphical HLT goals				
	S_Rec	S_Ch	S_Diff	S_Eval	S_Tan	G_Rec	G_TS	G_VT	G_CR	G_Cor
Yes	67	100	109	107	95	109	92	35	102	20
No	0	0	0	1	0	0	2	0	1	66
NEE	52	19	10	11	24	10	25	84	16	33

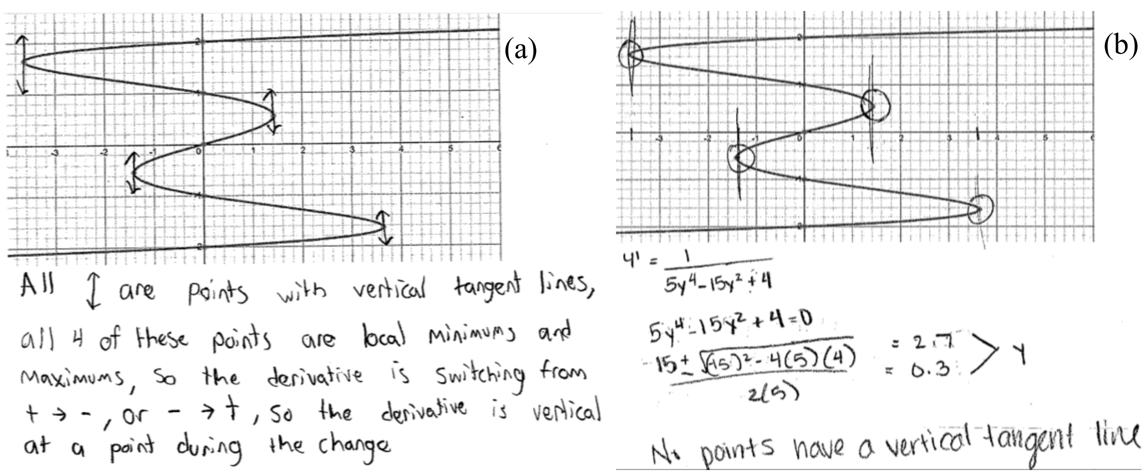
The modal categories with at least 90% show that students had the most success in calculating the derivative of the implicit function (S_Diff), evaluating the derivative at a given point (S_Eval), and recognizing that a graph of an implicit equation does not pass the vertical line test (G_Rec), hence cannot be considered as $y = f(x)$. We were encouraged by such a high percentage of "Yes" scores for the correct calculation of the implicit derivative (S_Diff), which requires correctly applying the chain rule (S_Ch) *and* correctly isolating y'. 100 students (84%) correctly applied the chain rule when taking the derivative with respect to an implicit variable.

The two goals posing the most difficulties to students were related to graphical representation: G_VT (vertical tangent) and G_Cor (coordination of symbolic and graphic modalities). The item related to G_VT was question 4 which called for finding all points where

[2] The codes S_Eq and S_Ch were merged since the worksheet explicitly asked for using implicit differentiation.

$x = y^5 - 5y^3 + 4y$ has vertical tangent line. Most students could correctly identify the four points on the graph, and state at these points y' is undefined. But only 35 students (29%) completed the calculation, and even fewer correctly interpreted the result.

We observed two types of difficulties in this item. One type is shown in Figure 3a. The student identified the points where the tangent line is vertical and marked them on the graph. But then proceeded to explain that “these points are local minima and maxima, so the derivative switches sign” from positive to negative and vice versa, hence the “derivative is vertical at a point during the change.” The use of “vertical derivative” instead of “vertical tangent line” can be a mere carelessness, but it seems alarming that the student refers to these points as “local minimum and maximum.” Moreover, although the derivative does change the sign at these points, it is unclear whether the student simply states known facts about local minima /maxima, or whether the student has some warrant for their claim, from the graph or from the derivative formula. This cannot be determined from a written text. Nevertheless, since this kind of response appeared repeatedly in our data, we assume that there is some underlying tendency at play, where students project their knowledge of critical points of explicit functions to implicit equations, but without recognizing how to adjust their prior knowledge to the new situation.



Figures 3 a & b: Sample student responses to item 4

Figure 3b shows another interesting response type. Here, the student marked the vertical tangent lines on the graph and correctly calculated the y-coordinate of the points where the derivative is undefined. But then the student made a seemingly contradicting conclusion, that there are “no points” with vertical tangent lines. Another student with a similar solution wrote that the answers “don’t work.” We conjecture the following explanation. The student solves the equation for y, obtaining the y-coordinates of the marked points on the graph. But then attempted to match these numerical values to the x-coordinates of these points. The x-coordinates being ± 3.63 and ± 1.91, they indeed do not match the calculations of 2.7 and 0.3, leading the student to a quite paradoxical conclusion that “no points have a vertical tangent line.”

Students who correctly identified the points where the tangent line is vertical and calculated the y-coordinates by finding where the derivative is undefined scored a “Yes” for attaining the G_VT goal. However, correctly interpreting the meaning of the calculations corresponds to another HLT goal: coordinating graphic and symbolic representations of implicit equations (G_Cor). Four items contributed to a single G_Cor score: item 4, discussed above, and items 2b, 3a, 3b (Table 1). So, there were four opportunities for students to show some attainment of the

G_Cor goal, yet it appeared to be the most difficult one for the students. It was the only one with 66 students (55%) scoring "No" for that goal, and another 33 students (28%) scoring NEE.

The items that contributed most to these low scores were 3a and 3b, where students were asked to sketch the tangent lines, whose equations they found algebraically, on the graph. In item 3a, 95 students (80%) correctly found the equation of the tangent line at a point (0,1), but only 49 students (41%) could correctly draw this tangent line on the graph. The rest of the students either did not attempt this item (getting the NEE score) or drew an intersecting line instead of a tangent. In item 3b, 66 students (55%) justified algebraically that the tangent parallel to $y = -\frac{1}{6}x + 1$ will pass through (0,-1), but only 17 students (14%) drew the correct tangent line.

Overall, aggregating across all items and HLT goals, we obtained that the percentage of "Yes" scores was 80% for all symbolic HLT goals and only 60% for all graphic goals.

Discussion and Scientific Significance of the Study

The goal of the study was to examine students' performance on the implicit differentiation activity as they worked in small groups during a recitation in an otherwise traditional first-semester single-variable Calculus 1 course. Our study contributes to the existing literature in two ways. First, rather than using surveys or small-scale teaching experiments, our study was conducted in an authentic classroom setting and reflects what students can do collaboratively while being supported by GTA and LA. Second, our implicit differentiation instructional activity emphasized the graphical representation of implicit equations and coordination between the symbolic and graphic meanings of implicit differentiation.

Our results show that students performed much better on the symbolic HLT goals compared to graphical ones. Moreover, the percent of correct performance on utilizing the chain rule (S_Ch) and calculating y' (S_Diff) were much higher than what has been described in the literature 84% and 92% respectively, compared to about 50% in the studies by Chu (2019) and Kandeel (2021). The outcome may be attributed to the collective nature of student work, as opposed to individual survey data reported by Chu and Kandeel. In fact, the percentage of correct performance for S_Ch and S_Diff dropped to 66% and 62% respectively on individual exams, but the full discussion of these outcomes is beyond the scope of this paper.

The graphical interpretation of implicit curves, including graphing tangent lines and, specifically, coordinating between graphic and symbolic meanings of implicit functions presented the biggest challenges for the students. This outcome concurs with Borji and Martínez-Planell (2019). Students' challenges seem to be related to applying procedures and types of reasoning borrowed from their experience with explicit functions, e.g., attempting to interpret points with vertical tangent lines as "local minima and maxima", possibly due to the visual similarity, while ignoring the graph's orientation. Similarly, being used to finding the x-coordinates of critical points first, when obtaining y-coordinates from the calculation, students got confused and couldn't match them to the point on the graph. These challenges suggest that students require more exposure to and experiences with implicit curves to develop a better understanding of implicit functions. More research is needed to understand how students coordinate graphic-symbolic representations of implicit functions to support their learning.

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