Two Students' Meanings for Partial and Directional Derivatives when Constructing Linear Approximations: The Cases of Alonzo and John

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In this paper, I describe two vignettes of two students' meanings for partial and directional derivatives. The data was collected in the Spring 2023 academic semester from two STEM students who had completed multivariable calculus at least two semesters prior to participating in the study. The interviews were conducted using task-based, exploratory teaching interviews with the goal of creating explanatory models of student thinking to highlight the nuance in their meanings for differential multivariable calculus ideas. The two students, Alonzo and John, each exhibited novel meanings for partial derivatives that were constructed over several semesters of mathematics and physics instruction. These results add to the field's understanding of students' thinking about a relatively underdeveloped area of the research literature, specifically multivariable calculus education.

Keywords: Students' meanings, partial derivatives, directional derivatives, multivariable calculus education

Introduction and Literature Review

Investigations of students thinking about multivariable and vector calculus ideas have been of increasing interest to undergraduate mathematics education researchers over the past several years (Jones, 2020, 2022; Mhkatshwa, 2021; Moreno-Arotzena et al., 2021). Despite a relative lack of mathematics education research literature investigating students' conceptions of multivariable calculus topics (Rasmussen & Wawro, 2017), Multivariable and Vector Calculus (MVC) is positioned at an important point in the sequencing of the undergraduate mathematics curriculum. Specifically, MVC is typically offered as the third semester of university-level Calculus and is sometimes required as a prerequisite course for upper-division mathematics courses such as Linear Algebra or Differential Equations. Further, the differential calculus ideas normally discussed in a MVC course are often necessary to make sense of certain concepts and systems of equations encountered in upper-division physics and engineering courses (Bajracharya et al., 2019; Dray & Manogue, 2006; Roundy et al., 2015). To better understand how undergraduate STEM majors think about partial and directional derivatives, I conducted a set of exploratory interviews in the Spring 2023 semester. In this paper, I share two vignettes from this study that highlight nuance across two students' meanings for partial and directional derivatives.

Theoretical Perspective

Discussions about students' meanings for mathematical ideas necessarily elicit recursive discussions about what we, as mathematics education researchers, "mean" by meaning.

The Meaning of "Meaning."

Thompson (2013), described the recursive nature of meaning, and the development of meaning, in the context of mathematics education in the United States. Thompson provided an in-depth analysis of what several philosophers meant by "meaning." As a genetic epistemologist, Piaget appeared to understand "constructing a meaning" as "constructing an understanding," and appeared highly related in Piaget's Genetic Epistemology (Glasersfeld, 1995; Montangero & Maurice-Naville, 1997; Piaget, 2001). In the context of Piaget's theory, *assimilation* is a biological notion in which an individual grafts new sensorimotor experiences, or mental activity, onto previously constructed mental structures, or *schemes*. For Piaget, assimilation was the source of all schemes, and *accommodation* was the driving mechanism behind the creation of new schemes and differentiations in existing schemes to create new schemes through the process of Reflective Abstraction (Glasersfeld, 1991; Piaget, 2001). In the context of the learning theory that emerges from adopting Radical Constructivism as a background theory or an epistemological stance (Simon, 1995), learning is conceptualized as the process of iterative cycles of assimilation and accommodation driving individuals, conceptualized by Radical Constructivists as individual cognizers, towards a temporary state of *cognitive equilibrium* (Cobb & Steffe, 1983; Glasersfeld, 1991, 1995). If having an understanding is therefore the result of assimilating to a scheme, then the corresponding understanding accompanied by an inference, or a set of inferences in the moment, would be someone's meaning that resulted from the assimilation to their scheme(s) (Thompson, 2013; Thompson et al., 2014).

Students' Classroom Experiences and the Negotiation of "Shared Meaning."

However, the construction of meaning is incomplete without consideration of the conveyance of one's personal meanings to another individual, which will always impact the nature of the interaction between two or more people (Blumer, 1986; Thompson, 2013). For instance, in classroom contexts, one aspect of a teacher's role is to act as a brokering agent between the local classroom context and the larger community of mathematicians and mathematics students (Zandieh et al., 2017). It makes sense that a teacher's meanings will necessarily impact the meanings that students construct, and the negotiation of meaning is often a subtle and difficult-to-grasp process in settings as dynamic as classrooms (Thompson & Thompson, 1994). As such, I decided to first analyze the meanings that students *had* constructed as a start to building second-order models of students thinking about partial and directional derivatives (Cobb & Steffe, 1983; Steffe & Thompson, 2000).

Further, while first-order models of student thinking can inform the initial set of tasks or instructional materials, engaging with students' genuine mathematical realities will result in more viable models of student thinking. Therefore, a researcher who adopts Radical Constructivism as a background theory takes seriously the notion that knowledge resides in the mind of the individual and rejects the notion that the researcher has "direct" access to the student's internal cognitive process, or *students' mathematics*. Thus, adopting Radical Constructivism implies an acceptance of the building models of student thinking is the best we can do as qualitative researchers to build theories about student thinking (Cobb & Steffe, 1983; Steffe & Thompson, 2000). Since I chose to adopt Constructivism as a background theory, I decided to follow the recommendations of Steffe and Thompson (2000) to engage in exploratory teaching with students to get in touch with their mathematical realities, or *the mathematics of students*.

A Quantitative Meaning for Partial Derivative as a Directional Rate of Change Function

In the context of MVC instruction, Mhkatshwa (2021) highlighted students' struggles with conceptualizing partial derivatives in non-kinematics contexts using covariational reasoning. Mhkatshwa's results corroborated other research results that students will conceptualize derivatives differently depending on the problem context (Jones, 2017; Zandieh, 2000; Zandieh & Knapp, 2006). For example, conceptualizing a partial derivative as a directional rate of change function would entail (i) a meaning for constant rate of change as a constant of proportionality which multiplicatively compares an amount of variation in the value of a single independent quantity and the corresponding amount of variation in the value of the dependent quantity, while mentally holding the value of the other independent quantity fixed, and (ii) a meaning for rate of change function as a record of the relationship between the value of the independent quantity and how quickly the value of the dependent quantity changes due to infinitesimal amounts of variation in the value of the independent quantity. Such a meaning would lead to the anticipation that the symbolization $f_x(x,y) = x^2y$ where $z = f(x,y)$ would imply that $f_x(2,1)$ represents the rate of change of the z value for any change in x away from $x = 2$ where the y value is fixed at $y = 1$. The rate of change would represent an equivalence class of ratios leading to the inference that the corresponding change in z is $f_x(2,1)$ times as large as the corresponding change in the x value away from $x = 2$, e.g., $\Delta z \approx f_x(2,1)\Delta x$.

Conceptions of partial and directional derivatives align well with recent instructional recommendations of research groups investigating student thinking about partial and directional derivatives by explicitly lecturing on 3D slopes (Martínez-Planell et al., 2017; Martínez-Planell & Trigueros Gaisman, 2021; McGee & Moore-Russo, 2015; Moore-Russo et al., 2011) and tactile manipulatives from the *Raising Calculus to the Surface* workshops (Wangberg, & Dray, 2022). The issue is that students' conceptions of rates of change as slope, e.g., as a graphical measure of the slantiness of a line or plane, cannot be assumed to automatically "transfer" to non-graphical contexts (Lobato & Siebert, 2002). As such, further research is required to investigate students' quantitative conceptions of partial and directional derivatives as rate of change functions whose dependent quantities have an equivalent measure to the 3D slopes of tangent planes as described by previously mentioned research groups.

Methods

The data presented here emerged from a study designed to highlight the nuance in STEM intending students' meanings for partial and directional derivatives across a set of symbolic, graphical, and dynamic geometry tasks (Bettersworth, 2023). The tasks were designed using a combination of literature on students' meanings for partial and directional derivatives (Jones, 2022; Martínez-Planell et al., 2017; Mhkatshwa, 2021) and my conceptual analysis for the two dynamic geometry tasks (Bettersworth, Year), to support students in engaging in the goal directed activity of using derivatives to create linear approximations for particular functions in graphical and symbolic settings.

Each student completed a 30-minute screening interview, which included questions about interpreting the limit definition for derivatives and partial derivatives in graphical and symbolic settings. Alonzo (he/him) and John (he/him) were invited to participate in up to five task-based, think-aloud style, exploratory teaching interviews with the goal of producing explanatory models of student thinking (Cobb & Steffe, 1983; Goldin, 1997; Steffe & Thompson, 2000). While the initial goal was to engage in *intuitive and responsive interaction* with each student, the interviewer would transition to making teacher moves if either student expressed discomfort or unfamiliarity with a particular subset of the ideas presented in my tasks or interview protocol. The teacher moves made by the teacher-interviewer were based on my a priori conceptual analysis for the previously mentioned geometry tasks (Bettersworth, 2021; Thompson, 2008).

I engaged in retrospective analysis of each interview recording by reviewing each video between interviews to determine any areas which required further clarification, or to inform the design of future tasks and teaching interventions. After the conclusion of the data collection, I engaged in ongoing analysis of the data through iterative rounds of open coding (Strauss & Corbin, 1998). Once an initial set of codes was established, the interview data was reanalyzed using quantitative reasoning (Thompson, 2022) as a lens for investigating the degree to which each student conceptualized variables (Carlson et al., 2002; Thompson & Carlson, 2017), function notation (Oehrtman et al., 2008; Thompson & Carlson, 2017), graphs (David Parr et al., 2018; Moore & Thompson, 2015), and rates of change (Thompson, 1994) as a record of their conception of the relevant quantities within each task.

Context of Student Data

Alonzo (he/him) and John (he/him) were selected from a group of five students who participated in a set of screening interviews in February 2023. Alonzo is a physics major at a large research university in the southwest United States. Alonzo completed the undergraduate calculus sequence at a local community college before transferring to the university. John is a double major in mathematics and physics at the same university. John completed the undergraduate mathematics sequence during high school and commented frequently about his positive STEM instructional experiences in high school. Both students were very vocal with their thinking and agreed to participate in the exploratory interviews after the completion of their individual screening interviews and after signing the approved study participation forms.

Results

Alonzo and John demonstrated varied and nuanced meanings for partial and directional derivatives that include interrelated conceptions of other MVC ideas (e.g., dot product, cross product, basis vectors, and projection) while constructing linear approximations. However, due to space limitation, I will share two vignettes related to Alonzo and John's meanings for partial and directional derivatives.

Vignette 1: Students' Meanings for Partial Derivatives

In the first vignette, I highlight Alonzo and John's meanings for partial derivatives which included: (i) curves on the surface in the x and y direction, (ii) tangent lines to each trace in the x and y direction, and (iii) slopes of the lines tangent to the curves on the surface in the x and y direction, or contained in a plane parallel to the xz or yz plane.

Meaning 1: Partial derivatives are the curves on the surface. Throughout the second and third day of Alonzo's interviews, it appeared that his meaning for partial derivative was evoked differently based on whether he anticipated mentally evaluating the partial derivative function for a particular set of values for both independent quantities. For instance, as demonstrated in the following transcript, when finding the partial derivatives of the function $f(x, y) = xy^2 - xy$ in task 2.1.2 (day 2, task 1, question 2), Alonzo interpreted $f_x(x, y)$ and $f_y(x, y)$ differently than he interpreted $f_x(2,1.5)$ and $f_y(2,1.5)$ in task 2.1.3. The task statements are pictured in Figure 1(b).

Alonzo: Okay, find the partial derivative of x of the function of f and the partial derivative of y with a function of f. What do these partial derivatives represent? So, find partial. I'll start with that, with partial of x. So, that would be f of x is equal to $y^2 - y$, f of y is equal to $2xy - x$.

Interviewer: Nice, and then how did you come up with those again?

Alonzo: So here [*points to f_x*], I held everything but the x variable is constant, so these y's are just constant in this case, and here [*points to f_y*] I held everything but the x variables everything with the y variable is constant. So, the x variables were constant.

Interviewer: Nice.

Alonzo: Okay, and then what do these represent? These represent I'd say curves on the 3D surface.

After I asked him to clarify his response, Alonzo sketched the diagram pictured in Figure 1(a). Alonzo explained that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ represented the curves on the surface in the x and y direction respectively. This meaning appeared highly associated with his attention to mentally fixing the value of one of the independent quantities by treating the variable as a constant when differentiating. Relatedly, when interpreting $f_x(2, 1.5)$ and $f_y(2, 1.5)$ in task 1.3.3, Alonzo explained that once he evaluated the partial derivative function, for example f_x, at $x = 2$ and $y = 1.5$, $\partial f/\partial x$ and $f_x(2, 1.5)$ represented the slope of the tangent line to the curve at the given point (as shown in Figure 1(b)) while mentally fixing the value of $y = 1.5$. This example of Alonzo's meaning for partial derivatives is interesting because it highlights how Alonzo evokes different aspects of his apparent scheme for partial derivative based on his current goal (either differentiating, evaluating the function at a point, or attending to graphical interpretations).

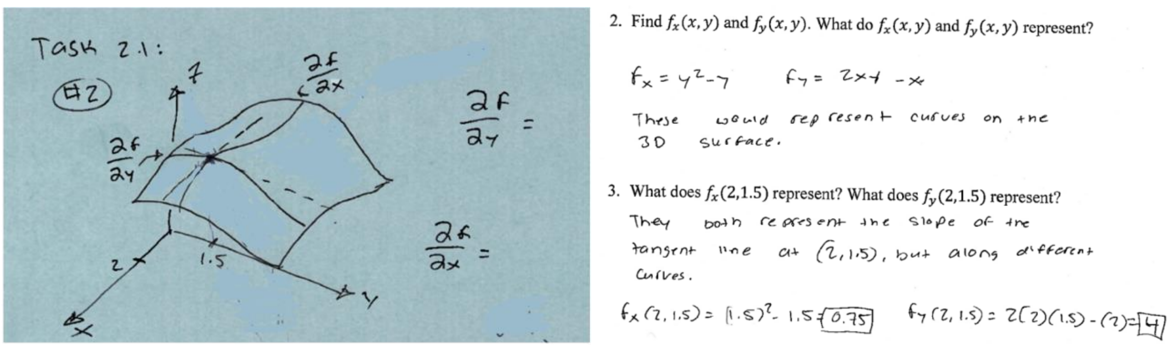

Figure 1: (a) Alonzo's sketch in task 2.1 accompanying his explanation for task 2.1.2 and 2.1.3. (b) Alonzo's response to task 2.1.2 and 2.1.3 demonstrating his dual meanings for partial derivative notation.

Meaning 2: Partial derivatives are the slopes of the tangent lines to each curve. John's meaning for partial derivative, on the other hand, consistently included a conception of partial derivatives as the slope of the tangent line contained within a plane parallel to either the xz or yz plane respectively. When responding to task 2.1.2, John's meaning for $f_x(x, y)$ and $f_y(x, y)$ included a graphical representation of the surface and slopes, as demonstrated by his gestures moving along an imagined surface graph. For John, mentally fixing the value of one of the independent quantities corresponded to isolating his attention on the graph of the function to a particular cross-section represented by intersecting the surface graph with a plane, as demonstrated in the following transcript.

John: Okay, so what do these represent? Well, if I had a sheet [*moves hands along an imagined bump which smooths out as he moves hands away from one another*] right? Then I can take a plane in the x direction [*chops hand down pointing to his left*] and if I fix the x value [*gestures with same hand to the right*] then I should be able to find the instantaneous slope [*moves pinched fingers up and to his right*] in the y direction. If I were to just look at that, that graph right there [*brings hands together flattened together at the fingertips*], then I would be able to figure out what the derivative would be with this, what the slope would be [*moves right hand along holding left hand fixed flat*].

While Alonzo's meaning for $f_x(x, y)$ and $f_x(2, 1.5)$ foregrounded different aspects of his more general meaning for partial derivatives, John's meaning for $f_x(x, y)$ and $f_x(2, 1.5)$ remained relatively consistent across all day 2 tasks.

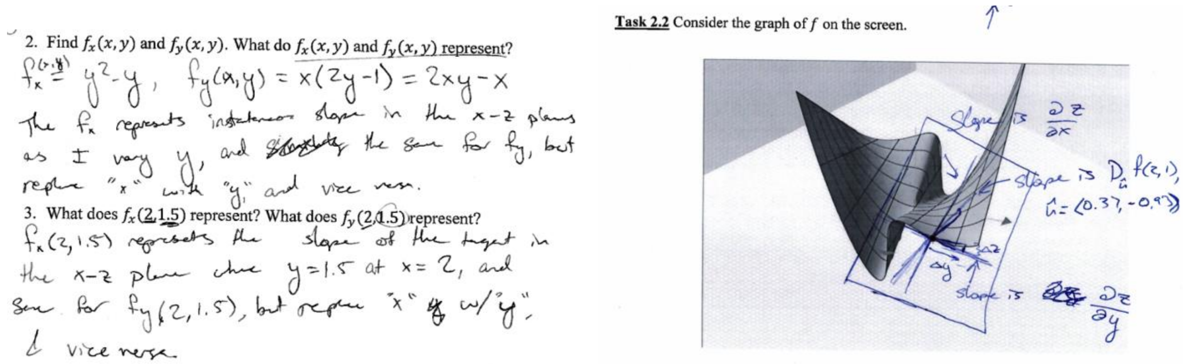

Figure 2: (a) John's response to tasks 2.1.2 and 2.1.3 demonstrating his consistent meaning for partial derivatives. (b) John's annotation of the surface graph in task 2.2 slope of the tangent line meaning.

Vignette 2: Student's Meanings for Directional Derivatives

Alonzo was not comfortable with the idea of directional derivatives, so we spent most of the remainder of his day 2 interview making teacher moves aligning with the meanings for directional derivative as outlined in my conceptual analysis. John, however, due to his recent Differential Geometry course, had revisited these ideas and exhibited numerous meanings for directional derivatives which included: (i) the rate of change of z in the $\hat{u}$ direction, (ii) the slope of the tangent line resulting from intersecting the graph of the surface with a new "shifted" uz plane, and (iii) as the amount of variation in z due to change in the $\hat{u}$ direction.

Meaning 1: The directional derivative is the slope of the tangent line to the curve in the new, shifted uz plane. As shown in Figure 2(b), John's meaning for directional derivative included a conception of slope in the $\hat{u}$ direction. While it may look like John conceived of partial and directional derivatives as the slopes of the tangent planes in task 2.2, this was not explicitly the case. In task 2.2, John was tasked with sketching the best local, linear approximation for the given function at a specific coordinate triple. He then discussed his meaning for $D_{\hat{u}}f(2, 1)$ using the relative size measurement of the horizontal and vertical segments he sketched in Figure 2(b). However, John's meaning for directional derivative is better demonstrated in the following excerpt from his work on task 2.1, where he describes his construction of the new "shifted" uz plane, which is also demonstrated in his response to task 2.1.5 in Figure 3(a).

John: Okay, this right here is a directional derivative. And so, if I were to want to find, turn my plane now [*rotates hands together*]. I'm going to turn my plane so that the u vector is pointing [*points thumb in a particular direction*] in the direction of my plane, in the xy plane, right? [***omitted***] And so, this u vector lies in here and so beforehand we were talking about this plane [*sketches xz plane*] we were talking about this plane right here [*sketches yz plane*] and now I want to talk about varying this plane right here [*sketches plane parallel to the z axis containing the u vector*].

Interviewer: Great.

John: And so, if I take that plane and I move it around however I need to [*moves vertically-held hands around in front of him*] and find the directional derivative that's the derivative in that direction [*points with pen*]. So, it represents the rate of change in the direction of u-hat. Yeah, I guess that's probably the easiest way to say that.

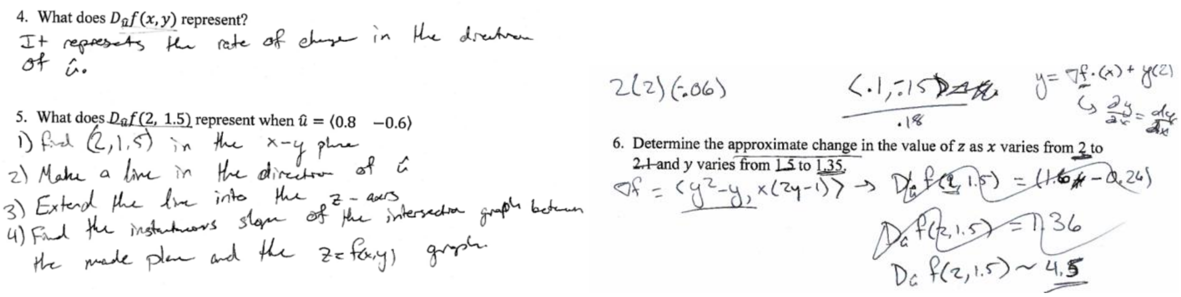

Figure 3: (a) John's response to task 2.1.4 and 2.1.5 demonstrating his slope in the uz plane meaning. (b) John's response to task 2.1.6 demonstrating his meaning for directional derivative as the amount of variation in z.

Meaning 2: The directional derivative is the amount of variation in z. While John appeared to have a robust meaning of linearization which was highly interrelated to his meaning for derivative, it appeared that, at times, John conceptualized rates of change as amounts of variation in the value of the dependent quantity, as shown in Figure 3(b). For instance, in task 2.1.5, John described the directional derivative as "the best approximation you can make if you're assuming that it's from here starting at this point [*underlines x=2 and y=1.5*] and goes to these points [*underlines x=2.1 and y=1.35*]." While it is not uncommon that students conceptualize rates of change as amounts of variation in the value of the dependent quantity of the function, it does highlight the importance of supporting students in differentiating between the concepts of rate of change and variation earlier in their mathematics instruction, as the complexity of attending to the values of more than two quantities as they change together, vectors and vector notation, and the calculus of MV functions is enough of a cognitive load for students without worrying about conflating rates of change with amounts of variation.

Discussion and Conclusion

In this paper, I shared two vignettes of two student's nuanced meanings for partial and directional derivatives. These vignettes highlight the importance of ensuring that students construct a strong foundational understanding of derivatives as rate of change functions in single variable calculus, to support the generalization of their meanings to MVC settings (Harel, 2021; Martínez-Planell & Trigueros Gaisman, 2021; Thompson, 2019). Second, these results highlight a need to support students in differentiating between amounts of variation and rates of change as distinct quantities (Thompson, 2022). This is an important, but difficult, conceptual distinction that students struggle to make in single variable calculus settings. Since students' meanings for derivatives of functions appear to influence the meanings they construct in MVC settings (Bettersworth, 2023), more research is required to support students in making this distinction.

Limitations

The mathematics of students presented here represent second-order models of student thinking emerging from the analysis of two students' responses within the context of exploratory teaching interviews. As such it is impossible to claim that these are representative of every possible meaning a student might possess or construct. Future studies could attempt to replicate or generalize the results presented here with a larger collection of students from a different or more varied cross-section of students majoring in STEM.

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