A Framework for Characterizing and Identifying Playful Mathematical Experiences

Jeremy Bernier
Arizona State University

Play has been recognized as a component of mathematical practice; accordingly, this manuscript explores the potential role of play in undergraduate mathematics education. The manuscript introduces a novel framework for characterizing play, wherein a person is said to be playing in a scenario if they are (1) making non-routine, voluntary, and free choices/actions, (2) experiencing an appropriate level of challenge/uncertainty, and (3) experiencing amusement, satisfaction, or excitement. This framework is applied to provide a definition of playful mathematical experiences and consider how playful mathematical experiences relate to other concepts, like mathematical play and authentic mathematics.

Keywords: mathematical play, characteristics of play, rehumanizing mathematics

In the traditional mathematics classroom, students learn mathematics through some combination of teacher demonstration and repetitive practice of key computational skills While there is no consensus as to what mathematics *is*, articulations of the nature of mathematics tend to refer to concepts like problem solving, logical reasoning, modeling, or creative exercise (Halmos, 1980; Mura, 1993; Sfard, 1998; Strauss, 2011) – characterizations which stand in stark contrast to the operation of traditional mathematics classrooms (Boaler, 2002; Mann, 2006). Many approaches under the umbrella of Inquiry-Based Mathematics Education (IBME) have sought to incorporate more components of authentic (cf. Brown et al., 1989) mathematical practice – i.e., ways mathematicians do mathematics – into undergraduate mathematics classrooms as part of larger efforts to improve undergraduate mathematics instruction and learning (Laursen & Rasmussen, 2019).

One component of authentic mathematical practice – that is, something which mathematicians actually *do* – not included in Laursen and Rasmussen's (2019) synthesis of IBME approaches is the notion of *play*. Play is a near-universal human experience, and also one with much ambiguity (Sutton-Smith, 1997); much of the remainder of this manuscript will be dedicated to characterizing it more fully. When mathematicians like Su (2020) and Lockhart (2008) discuss play as part of mathematical practice, they refer to the pleasure and whimsy of freely exploring mathematical structures. Su (2022) argues that "contemplating patterns, playing with ideas, exploring what's true, and enjoying the surprises that arise along the way" (p. 53) is key to mathematical research, while Lockhart (2008) states that in mathematics, "we get to play and imagine whatever we want and make patterns and ask questions about them" (p. 4). Beyond mathematicians' descriptions of their own practice, Horne et al. (2023) recently examined the problem-solving practices of professional mathematicians and found evidence that playfulness is at least sometimes invoked within their mathematical practice.

Now, play being a part of authentic mathematical practice does not necessitate the inclusion of play at all levels of learning mathematics. Reviewing journal articles and attending faculty meetings are practices of mathematicians, but I suspect most would agree that first year undergraduate students need not participate in these practices. So, while I *assert* that play is legitimate for inclusion in mathematics classes due to its role in authentic mathematical practice (Holton et al., 2001; Horne et al., 2023; Lockhart, 2008; Su, 2020), I go further and *conjecture* that play may be a useful element to introduce to undergraduate mathematics classrooms for reasons beyond its validity.

My evidence for this conjecture is the current status of participation in STEM. Many more students enter college with interests in STEM than complete degrees in STEM fields (Bressoud, 2020; Seymour & Hunter, 2019). One reason for this is that undergraduate mathematics classrooms can be dehumanizing (cf. Gutiérrez, 2018) spaces where students are not able to interact with mathematics as their whole authentic selves; this is especially so for students from minoritized groups (Leyva et al., 2021, 2022). Recently, Caldwell et al. (2023) have argued that play can contribute to rehumanizing mathematics classrooms. According to their argument, play can challenge traditional mathematics classroom practices and create space for students to assert their own understandings of mathematics (Caldwell et al., 2023). While Caldwell et al. (2023) placed their work in the context of elementary schools, the fact that people of any age can and do engage in play means the same ideas may apply to the undergraduate mathematics classroom.

So far in this manuscript, I have provided a brief overview of *why* we might want to integrate play into undergraduate mathematics classrooms. Before we can proceed to fully considering *how* play might be integrated into undergraduate mathematics classrooms, further clarity is needed on *what* play actually *is* and how I suggest we should operationalize it in this context. The remainder of this manuscript serves to articulate a framework for play oriented towards identifying whether someone is playing or not in a given scenario, using that framework to define *playful mathematical experiences*, and considering how this definition relates to and differs from existing work on play in mathematics.

Characterizing and Contextualizing Play in Mathematics

Characterizing exactly what play *is* is a challenge. The conceptual ambiguity of play makes it difficult to define in a way which captures all applications of the word while being operationalizable. As a result, diverse definitions and frameworks exist which vary greatly depending on both the backgrounds of the authors developing the frameworks, as well as the audience they are speaking to (Sutton-Smith, 1997; Salen and Zimmerman, 2004; Bergen, 2015). Zoologists, philosophers, psychologists, and game designers all study play, each emphasizing different aspects of it which are particularly relevant to their domains and motivations. Some characterizations of play conflict somewhat with the idea of play in the classroom – for example, Huizinga (1949) articulates that play must be separate from "ordinary life" and involves no material gain. As a further complication, within mathematics, the concept of mathematical play exists and has been operationalized differently by different researchers (Holton et al., 2001; Horne et al., 2023; Williams-Pierce, 2019), but typically as a very specific type of play.

As a transdisciplinary scholar, I felt it was important to leverage the collective knowledge about play from diverse perspectives. Moreover, as I examined many frameworks of play, I found that they often were oriented to be describing the environment that a person was playing in as much as the actions of the player – and in some cases, seemed to imply that environments which meet certain conditions demand play. Under certain understandings of play, this is sensible. However, this orientation is not useful if one's goal – as mine is – is to clearly evaluate the extent to which the design and implementation of an activity intended to be playful results in playful experiences. In other words, in the environment of the undergraduate mathematics classroom (or indeed, any classroom), where both playful and non-playful behaviors are possible, how do we differentiate between those? So, to resolve this, I have pragmatically drawn on articulations of what play is from diverse authors and perspectives (Bergen, 2015; Callois, 2001/1961; Csikszentmihalyi & Bennett, 1971; Henricks, 2008; Holton et al., 2001; Huizinga, 1949; Salen & Zimmerman, 2004; Sutton-Smith, 1997; Sylva et al., 1976; Williams-Pierce & Thevenow-Harrison, 2021) to develop a framework of characteristics of play.

The Framework

Within any given scenario – whether designed or naturalistic – a person is likely engaging in play if, at some point, their interaction with the scenario meets each of the following criteria:

1. The person is making non-routine, voluntary, and free choices and/or actions.
2. The person is experiencing an appropriate level of challenge and/or uncertainty.
3. The person is experiencing amusement, satisfaction, or excitement.

In the following subsections, I describe in more detail how these characteristics draw on existing literature on play from multiple disciplines and authors. I also suggest (non-exhaustively) a few actions we may observe that would indicate that each characteristic is met in a scenario.

Making Non-Routine, Voluntary, and Free Choices and Actions. Across diverse authors and contexts, play is strongly associated with the idea of freedom (Henricks, 2008; Holton et al., 2001; Huizinga, 1949; Salen & Zimmerman, 2004; Williams-Pierce & Thevenow-Harrison, 2021). Although play is always bound by some sort of structure – be it an implicit societal structure or the more explicit rules of mathematics and games – play is defined by the player's ability to interact with that structure in ways that they choose. Moreover, for these choices to be truly free, they must be made voluntarily – they must not be imposed upon the potential player by an outside source (Huizinga, 1949; Williams-Pierce & Thevenow-Harrison, 2021). Of course, the degree to which a choice or action is taken freely and voluntarily may be challenging to determine as an observer. Observing kids at a playground at a public park, perhaps we could assume that their choices are all free and voluntary. In a classroom or even a playground attached to a school, this is a harder assumption to make – the rules of schools and classrooms make them spaces where students' freedom and agency are inherently reduced.

To allow for identification of playful behaviors under these environments of reduced freedom, I draw particularly on discussion by Sylva et al. (1976) about the relationship between routines and play. Sylva et al. (1976) articulate this by stating that play offers an "invitation to the possibilities inherent in things and events. It's the freedom to notice seemingly irrelevant detail" (p. 396). When operating under a routine, we do not give ourselves space to notice alternate possibilities (Sylva et al., 1976). In the undergraduate mathematics classroom, this is reflected in the difference between exercises and problems (cf. Schoenfeld, 1985): exercises, like completing a worksheet of u-substitution integration tasks after instruction on the technique, involve implementing routines and therefore have less space for freedom. Problems, like asking students who have not yet learned about integration to estimate the area under a curve, are non-routine and create more space for students to freely choose their strategy.

This is not to say that play never involves routine behaviors. Consider a pitcher's role in a baseball game – while the pitcher has practiced their routine for throwing fastballs and curveballs, how they implement these pitches in response to the state of the game and the actions of the offense is, at least occasionally, non-routine. In the undergraduate mathematics classroom, consider the prior example of asking students to estimate the area under a curve. As part of this task, students will likely compute the areas of figures they have well-established routines for – but the choices of how to deploy those routines is subject to students' choices. So, the notion of non-routine is useful to identify actions which are done freely and voluntarily in environments with reduced freedom, like the undergraduate mathematics classroom.

At this stage, I suggest three observable actions which would indicate to me that the person is making non-routine, voluntary, and free choices and/or actions. The first is when a person takes an action with an indication that the action is exploratory and that the outcome will be unknown. Routine actions inherently have at least expected outcomes, so an action taken with an explicit reference to uncertainty of outcome is indicative of a non-routine action. Second is when a person takes actions to test or push the bounds of a scenario. Testing and potentially adjusting the limitations of a scenario is reflective of asserting freedom over the scenario by specifically challenging it. An example of this is in the case of Calvin and the educational mathematical game *Boone's Meadow* (Wisittanawat & Gresalfi, 2021). While playing the game, Calvin tested the bounds of the scenario by seeing if he could hijack a vehicle and flying in different ways to see how the outcome of the game changed. Finally, the creation of individual goals would also indicate free action. These goals could be expressed explicitly as a goal or could be any desire to accomplish something that is not specifically implicated by the scenario.

Appropriate Level of Challenge or Uncertainty. In order to engage in playful behavior in a scenario, a person should find some challenge within that scenario (Csikszentmihalyi & Bennett, 1971; Salen & Zimmerman, 2004) or some uncertainty of how to act within that scenario (Holton et al., 2001; Salen & Zimmerman, 2004). Whether the play is provoked by challenge or by uncertainty roughly corresponds to Callois' concepts of ludus and paidia (2001/1961). When the play is provoked by challenge, the player is trying to accomplish some sort of specific goal within (and to some extent, perhaps against) the scenario – which is inclusive of the rule-based play of ludus. Conversely, if the play is provoked by uncertainty, the player may simply be trying to explore or understand the scenario and what the bounds of the scenario are – which would be more in alignment with the improvisational play of paidia. Of course, these are not mutually exclusive. How challenging a scenario is can fluctuate depending on a person's uncertainty, and the process of exploring uncertainty may lead to finding or creating challenges.

For this, I assert three observable actions which may indicate that a person is experiencing an appropriate level of challenge or uncertainty. The first is self-explanatory: a direct statement or express that they are experiencing challenge or uncertainty, like saying "this is so hard" or "I'm not sure if this will work." The second regards the potential of frustration. Sylva et al. (1976) indicate that play comes with a "moratorium on frustration" (p. 245), but Holton et al. (2001) suggest that frustration is actually a common feature of play. Indeed, in my own experiences, it is typical that working on appropriately challenging tasks will involve some frustration, even as it gives way to excitement when a 'lightbulb moment' is achieved. So, I draw on Holton et al. (2001) to suggest that frustration may be experienced while engaging playfully in a scenario – but, if it is experienced, it is not an obstacle to the person's engagement. What this would manifest as, then, is that a person showing signs of frustration persists in the task. Finally, I suggest that if the person makes multiple attempts at interacting with the scenario – by changing an action midcourse, somehow resetting the scenario, or otherwise repeating their experience with the scenario – this indicates an appropriate level of challenge and/or uncertainty. In the world of games, this may be reflected in restarting a level or loading an earlier save file before a certain action was taken; in the world of mathematical problems, this could involve crumpling up a sheet of paper or erasing everything on a whiteboard up to a certain point.

Experiencing Amusement, Satisfaction, or Excitement. The affective component of play – that is, the idea that play is fun – is broadly acknowledged (Csikszentmihalyi & Bennett, 1971; Henricks, 2008; Sutton-Smith, 1997). In my view, it is minimized and under-addressed in importance in most frameworks of play. Indeed, many of the definitions and frameworks did not mention fun; some who did mention fun did so only to state that fun alone is insufficient to understand play (Csikszentmihalyi & Bennett, 1971). I would argue, however, that if a person is engaging in free action and experiencing a challenge or uncertainty, but is not "having fun," then they are not engaging in play. Therefore, if the goal of observing an experience is to determine if that experience was playful, then whether they are having fun ought to be considered. Of course, fun is too broad a concept, so I suggest that having fun can be operationalized as experiencing amusement, satisfaction, or excitement.

These do not represent the total of all feelings that can be experienced during play – I have already mentioned frustration, and other emotions like annoyance or even sadness can occur while playing. My exclusion of these experiences in this framework is not to say that they are never part of play. However, if the experience of frustration, annoyance, or sadness *never* gives way to amusement, satisfaction, or excitement, that it would be incorrect to characterize that activity as play. In other words, play need not *always* be fun, but *at some point*, it must.

Even just with the experiences of amusement, satisfaction, and excitement, there are limits to how I might observe these. People do not always outwardly express these emotions even as they are experiencing them. Moreover, the way emotions are expressed and recognized varies from culture to culture and person to person (Elfenbein & Ambady, 2003), although there is some evidence that expressions of joy may be broadly recognized (Sauter et al., 2010). Observing for this may be particularly challenging, and is the part of this framework most in need of further development. For now, I draw on things I am likely to do when I am enjoying my experience of playing a game and/or solving a mathematical problem: smiling, laughing, and engaging in celebratory moves (e.g. a fist pump, a 'woop', or self-applause).

Playful Mathematical Experiences, Mathematical Play, and Mathematical Problem Solving

Given my framework for play, I define a playful mathematical experience as an experience where a person or group of persons is (a) interacting with mathematical rules, ideas, and objects, and (b) where their interaction otherwise meets each of the three characteristics in the prior subsection. Playful mathematical experiences thus overlap with several different kinds of experiences discussed in this manuscript (Figure 1). Here, I wish to interrogate the relationship

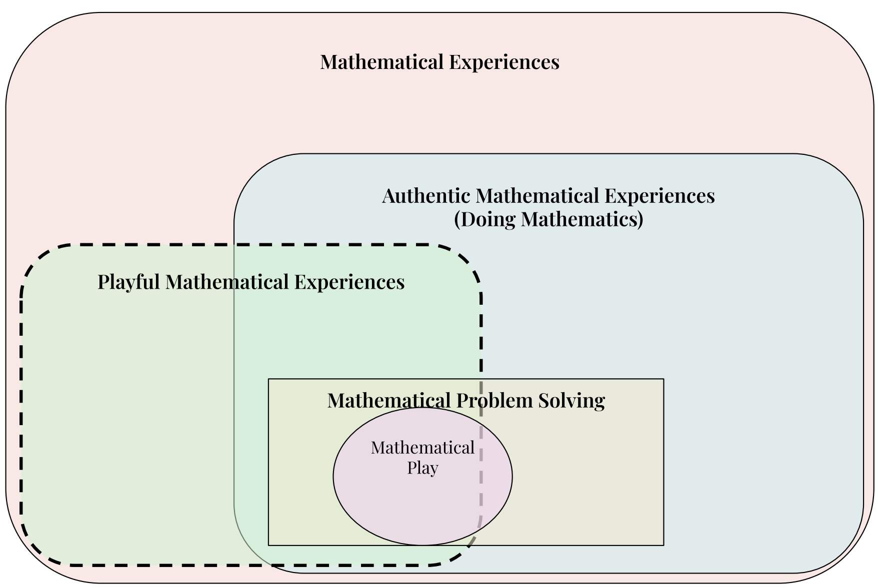

Figure 1. An Euler diagram showing the relationship between playful mathematical experiences (dashed lines) and other concepts discussed in this manuscript.

between playful mathematical experiences and mathematical play, and how these relate to authentic mathematical experiences.

Mathematical play is conceptualized differently by different scholars (Holton et al., 2001; Horne et al., 2023; Williams-Pierce, 2019). I start by considering those of Williams-Pierce (2019) and Holton et al. (2001), which I see as having the most differences from my definition of playful mathematical experiences. Williams-Pierce (2019) defines mathematical play as "voluntary engagement in cycles of mathematical hypotheses with occurrences of failure," (p. 591), where cycles of mathematical hypotheses entails "shift[ing] regularly between generalizing activity and generalizations…in which players learn to adjust their behavior and conceptualization" (p. 592). Holton et al. (2001) conceptualize it more generally:

> By mathematical play we mean that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion. Mathematical play involves pushing the limits of the situation and following thoughts and ideas wherever they may lead. (p. 403)

Both of these invoke ideas of freedom and exploration within mathematics, specifically with the solving of mathematical problems or development of mathematical concepts. I drew on both characterizations of mathematical play articulate in developing my own understanding of playful mathematical experiences.

Horne et al. (2023), meanwhile, operationalize mathematical play (reframed as playful math) in a way which is very well-aligned with my framework. In examining mathematicians' problem solving for evidence of play, they looked for evidence of three characteristics: "(a) agency in exploration or goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment" (Horne et al., 2023, p. 99). The first two of these characteristics align with my first characteristic of play, and their third characteristic lines up well with my third characteristic.

However, each of Williams-Pierce (2019), Holton et al. (2001), and Horne et al. (2023) specifically locate mathematical play within or very tightly adjacent to mathematical problem solving. While mathematical problem solving is a fertile ground for playful mathematical experiences, my definition for playful mathematical experiences is more broad. By way of example, consider a group of people who are telling a series of mathematical jokes to one another. Their experience may be playful in that (a) creating the jokes to tell is likely a non-routine process involving free exploration of different ideas, (b) the group members may view it as a "challenge" to try to get a bigger laugh with each joke, and (c) the shared laughter is an indication of their enjoyment of the situation. Since making jokes about mathematics involves interacting, in some manner, with mathematics, this would qualify as a playful mathematical experience. But, unless the jokes also represent a sequence of well-reasoned mathematical argumentation, it would likely not be construed as mathematical play.

I am motivated by the authenticity of play to mathematics. Based on the substantial overlap between these different characterizations, it seems to me that nearly any playful mathematical experience which involves doing mathematics (i.e., that is an authentic mathematical experience) would necessarily fit these definitions of mathematical play. Indeed, as I proceed in developing this framework, I am looking to mathematical problem solving experiences to better understand what play might look like in mathematics. However, I think it is important to include other ways of playing with mathematics, like my previous joke-making example, within the definition of playful mathematical experiences for two reasons.

First, students may hold beliefs about mathematics which preclude the possibility of playing in mathematics classrooms (cf. the case of Keegan in Wisittanawat & Gresalfi, 2021) or have anxiety about mathematics. Such students may not be 'ready' to engage playfully with mathematics while doing mathematics until they become more comfortable with doing mathematics itself. If playfulness is a goal, it may be useful to use activities which provoke playfulness but which are not necessarily authentic to mathematics as a bridge. In the classroom, this could involve an activity sequence which first prompts students to play more generally, then adding mathematics to their play, before finally inviting them to engage in mathematical play through mathematical problem solving.

Second, in a humanized mathematics classroom, students must be able to "draw upon all of their cultural and linguistic resources" and to "see aspects of themselves" (Gutiérrez, 2018, p. 1) in their mathematical work. To me, this means that students are not just invited to participate in existing practices of mathematicians, but are also invited to bring practices from other communities to bear on mathematics. Since one of my motivations for integrating play into undergraduate mathematics classrooms is to contribute to rehumanizing them, then it seems appropriate to include space for playful mathematical experiences beyond doing mathematics.

Discussion

The framework of play and definition of playful mathematical experiences in this manuscript started being developed when I, as a mathematics educator and gamer, first noticed that the traits of many of my favorite games were much the same as those of my favorite mathematical experiences. Since then, I have developed my understanding of play and playful mathematical experiences through the careful transdisciplinary examination of diverse sources of literature. While this has been a substantive endeavor, this is still an early draft of this framework, and further development is needed before it is used in the evaluation of designed learning activities. Even in between my first articulation of this framework and the writing of this manuscript, recently-published works (i.e., Caldwell et al., 2023; Horne et al., 2023; Su, 2020) led me to further develop my ideas.

The greatest weakness of my framework for play and my associated definition of playful mathematical experiences is that they are informed primarily by research literature and secondarily by my own experiences, and they are not yet informed by data. To address this, I am presently in the process of conducting a study which will use this framework to examine students' interactions with digital puzzle games and mathematical problems in a clinical interview environment. From this process, I expect this framework will evolve and deepen greatly. In particular, this process should allow me to better understand the relationship between playful experiences in general, playful mathematical experiences, and mathematical play, by examining how student interactions are similar and different in more- and less-explicitly mathematical situations. Even as a work in progress, I present this framework with the intent to prompt mathematicians, mathematics educators, and researchers of mathematics education to consider for themselves: what is, can be, and should be the role of play in undergraduate mathematics classrooms?

Acknowledgments

I must thank Elisabeth Gee, Michelle Zandieh, Brian Nelson, and Naneh Apkarian for mentoring the development of this framework as my dissertation committee. I also wish to thank Zac Bettersworth and Jason Guglielmo for providing valuable feedback on this manuscript.

References

Bergen, D. (2015). Psychological approaches to the study of play. *American Journal of Play*, *7*(3), 101–128.

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. *For the Learning of Mathematics*, *22*(1), 42–47.

Bressoud, D. (2020). Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education (Book review). *Notices of the American Mathematical Society*, *67*(09), 1. https://doi.org/10.1090/noti2145

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32–42. https://doi.org/10.3102/0013189X018001032

Caldwell, B., Gresalfi, M., & Vescio, J. (2023). What counts? Play as a mechanism for disrupting participation patterns in school mathematics. *Proceedings of the 17th International Conference of the Learning Sciences - ICLS 2023*.

Callois, R. (2001). *Man, play, and games*. https://doi.org/10.1080/21594937.2020.1778277

Csikszentmihalyi, M., & Bennett, S. (1971). An exploratory model of play. *American Anthropologist*, *73*(1), 45–58.

Elfenbein, H. A., & Ambady, N. (2003). Universals and cultural differences in recognizing emotions. *Current Directions in Psychological Science*, *12*(5), 159–164. https://doi.org/10.1111/1467-8721.01252

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Annual Perspectives in Mathematics 2018: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). National Council of Teachers of Mathematics.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly*, *87*(7), 519–524.

Henricks, T. S. (2008). The nature of play: An overview. *American Journal of Play*, 157–180.

Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology*, *32*(3), 401–415. https://doi.org/10.1080/00207390118654

Horne, D., Bloodworth, A., & Ellis, A. B. (2023). Modeling mathematicians' playful enagement in task-based clinical interviews. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 99–107.

Huizinga, J. (1949). *Homo ludens: A study of the play-element in culture*. https://doi.org/10.4324/9781315824161

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–149.

Leyva, L. A., McNeill, R. T., Balmer, B. R., Marshall, B. L., King, V. E., & Alley, Z. D. (2022). Black queer students' counter-stories of invisibility in undergraduate STEM as a white, cisheteropatriarchal space. *American Educational Research Journal*, *59*(5), 863–904. https://doi.org/10.3102/00028312221096455

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *Journal of Higher Education*, *92*(5), 784–814. https://doi.org/10.1080/00221546.2021.1879586

Lockhart, P. (2008). A mathematician's lament. *MAA Online*, *March 2008*, 25.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236–260. https://doi.org/10.4219/jeg-2006-264

Mura, R. (1993). Images of mathematics held by university teachers of mathematical sciences. *Educational Studies in Mathematics*, *25*(4), 375–385.

Salen, K., & Zimmerman, E. (2004). *Rules of play: game design fundamentals*. The MIT Press.

Sauter, D. A., Eisner, F., Ekman, P., & Scott, S. K. (2010). Cross-cultural recognition of basic emotions through nonverbal emotional vocalizations. *Proceedings of the National Academy of Sciences of the United States of America*, *107*(6), 2408–2412. https://doi.org/10.1073/pnas.0908239106

Schoenfeld, A. H. (1985). A framework for the analysis of mathematical behavior. In *Mathematical problem solving* (pp. 11–45). https://doi.org/10.1016/b978-0-12-628870-4.50007-4

Seymour, E., & Hunter, A. (2019). *Talking about Leaving Revisited*. https://doi.org/10.1007/978-3-030-25304-2

Sfard, A. (1998). The many faces of mathematics: Do mathematicians and researchers in mathematics education speak about the same thing? In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 491–511). https://doi.org/10.2307/749348

Strauss, D. (2011). Defining mathematics. *Acta Academica*, *43*(4), 1–28.

Su, F. (2020). play. In *Mathematics for human flourishing* (pp. 48–66). Yale University Press.

Sutton-Smith, B. (1997). *The ambiguity of play*. Harvard University Press.

Sylva, K., Bruner, J. S., & Genova, P. (1976). The role of play in the problem-solving of children 3-5 years old. In J. S. Bruner, A. Jolly, & K. Sylva (Eds.), *Play: Its role in development and evolution* (pp. 244–257). Penguin.

Williams-Pierce, C. (2019). Designing for mathematical play: failure and feedback. *Information and Learning Sciences*, *120*(9–10), 589–610. https://doi.org/10.1108/ILS-03-2019-0027

Williams-Pierce, C., & Thevenow-Harrison, J. T. (2021). Zones of mathematical play. *Journal of the Learning Sciences*, *30*(3), 509–527. https://doi.org/10.1080/10508406.2021.1913167

Wisittanawat, P., & Gresalfi, M. S. (2021). The “tricky business” of genre blending: Tensions between frames of school mathematics and video game play. *Journal of the Learning Sciences*, *30*(2), 240–278. https://doi.org/10.1080/10508406.2020.1817747