Reinventing the Definition of Unique Factorization Domains

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In this study, we conducted a teaching experiment to test hypothetical learning trajectories of reinventing the definition of the unique factorization domain. Six graduate mathematics students who are preservice and in-service teachers were given experientially real tasks of examining factorizations of integers and quadratics using algebra tiles. The experiment showed that their intuitive and informal reasonings about the reducibility and unique factorization were leveraged to formalize two defining axioms of the unique factorization domain by the emergent model of algebra tiles and by the pedagogical moves of the teacher-researcher who led the sessions.

Keywords: Unique factorization domains, Mathematics for teachers, Defining, Conjecturing

The importance of making connections between advanced mathematics courses required in teaching preparation programs and the secondary mathematics that students may teach has been highlighted in recent research (Álvarez et al., 2020; Wasserman, 2018; Wasserman et al., 2017). Abstract Algebra is a course where students learn about algebraic structures, including the properties of number systems, sets, and operations, including those commonly found in secondary mathematics contexts. Abstract Algebra serves as a course where in-service and prospective teachers (IPSTs) can learn the structural and fundamental reasons why secondary mathematics works, e.g., why the factors of a polynomial $(a + b)(c + d)$ are equal to $(c + d)(a + b)$. One concept typically covered in a graduate algebra course is unique factorization domain (UFD), which is defined as an integral domain R such that for every nonzero non-unit element $r \in R$, (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1 p_2 \dots p_n$ and (ii) the decomposition in (i) is unique up to *associates*: namely, if $r = q_1 q_2 \dots q_m$ is another factorization of r into irreducibles, then $m = n$, and there is some renumbering of the factors such that p_i is associate to q_i for $i = 1, 2, \dots, n$ (Dummit & Foote, 2004). The structure of UFD is commonly used in secondary mathematics contexts related to the fundamental theorem of arithmetic and the fundamental theorem of algebra.

There exists a gap in the literature on students' understanding and their reinvention of UFDs. In our study, grounded in the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1973; Gravemeijer, 1999), we designed a local instructional theory (Gravemeijer, 2004) with the goal of students reinventing UFDs. This concept of UFD is particularly useful for IPSTs, as it can support them in understanding that polynomials and integers have similarities in their structure and properties, which can be useful knowledge in their teaching of high school and college algebra. In this paper, we describe two hypothetical learning trajectories that comprise this local instructional theory where graduate students, who are IPSTs, engage in the mathematical activities of conjecturing and defining UFD by intuitively reasoning about the side lengths of rectangles represented in arrays and quadratics represented with algebra tiles. The students construct the axiom about the factorization of an element into a product of irreducibles. This helps students recognize the structural similarities of factorization in Z and $Z[x]$, which leads them to understand that a factorization of an element reduced to a product of irreducibles is unique up to associates and the reordering of the factors. We aim to answer the following questions: *1) Which aspects of the task sequence and which teacher-researchers' pedagogical moves support students' reinvention of unique factorization domains? 2) As students engage in the task sequence, which ways of reasoning are productive to leverage to guide them toward reinventing unique factorization domains?*

Literature Review

One major strand of research in students' reasoning about abstract algebra is based on students' guided reinvention (Freudenthal, 1991) of algebraic structures. Larsen (2009; 2013) developed a local instructional theory by which undergraduate students were guided to reinvent the definitions of group and group isomorphism by reasoning about the properties of the group of symmetries of an equilateral triangle. Larsen and Lockwood (2013) extended this work by developing another local instructional theory by which undergraduate students were guided to reinvent quotient groups. Cook (2012) also developed a local instructional theory for students' reinvention of rings, integral domains, and fields. These studies illustrated how students' intuitive reasoning could be leveraged for reinventing and defining formal mathematical concepts in abstract algebra. Researchers have not yet developed local instructional theories for unique factorization domains. Our study, therefore, contributes to the literature through its design of a hypothetical learning trajectory of how IPSTs reinvent the defining axioms of a UFD.

There has been minimal research done on students' understanding of UFDs. UFDs are special kinds of integral domains that have additional axioms about the existence and uniqueness of the factorization of each nonzero, non-unit element as a product of irreducibles. When exploring students' understandings of the factorization of elements in algebraic structures, researchers have focused on investigating student thinking in UFDs including the integers, Z, and the polynomials with integer coefficients, $Z[x]$. They have demonstrated the productivity of students considering the juxtaposition of elements in these sets, as doing so led them to recognize the overarching structure shared by the sets of integers and polynomials (e.g., Lee, 2018; Lee & Heid, 2018). We leverage this same juxtaposition of integers and polynomials with integer coefficients in this study to help guide students to make conjectures about the properties that integers and polynomials with integer coefficients share, which can be generalized to other integral domains that also satisfy the axioms of UFDs. One of the defining axioms of UFDs is the uniqueness of a nonzero non-unit element's factorization. Conceptualizing the factorization of integers and polynomials as unique has been shown to be a non-trivial endeavor for students (Zazkis & Campbell, 1996) This uniqueness property of irreducible factorizations is essential for secondary teachers to understand, as it is commonly used in problems that relate to the fundamental theorem of arithmetic or the fundamental theorem of algebra. Our task sequence, therefore, focuses on supporting IPSTs in reinventing the existence and uniqueness properties of the factorizations of integers and polynomials that are commonly used in the secondary mathematics content that they may teach.

Theoretical Background

Design Research and RME

We ground our teaching experiment by the design research methodology (Gravemeijer, 1999) that consists of designing and implementing a set of tasks that will be tested and iteratively modified, so that the task sequence aligns with the research team's learning goals. We designed a task sequence consisting of four units. We refer to the collection of four units as a local instructional theory (LIT), and each individual unit as a hypothetical learning trajectory (HLT). Our LIT had the overall goal of guiding students to reinvent the concept of UFD, which is why our task sequence was designed guided by the heuristics of RME (Freudenthal, 1991) which involve implementing: a) experientially real tasks where students can solve context problems that develop their intuitive understanding and guide them to formalize their understanding, b) an emergent model that will serve as a *model-of* students' informal understanding that can become a *model-for* students' formal and generalized understanding of an object, and c) guided reinvention, where students construct their own knowledge through their intuitive understanding with the guide and scaffolding of the tasks and/or an instructor. Each of the HLTs of the LIT, describes the intended learning goals, the learning activities, and the hypothesized student's understanding and learning (Simon, 1995). Once researchers design a proposed task sequence, they implement it in a small-scale teaching experiment where the tasks are administered to pairs of students, who with the help of a teacher-researcher, complete the tasks. The researchers then analyze students' thinking, testing the effectiveness of the tasks depending on how the students engage in such tasks and accomplish the intended goals. Through this process, the HLTs are edited and modified accordingly to help the students meet the hypothesized learning outcomes. Once this occurs, the LIT can be implemented in a larger-scale teaching experiment. Through this process of designing, implementing, and revising HLTs, researchers can discover students' ways of thinking that anticipate formal mathematics and identify ways to evoke and leverage those ways of thinking to support students' development of formal concepts.

Defining and Conjecturing through Guided Reinvention

In order to leverage students' ways of thinking, students need to be engaged in mathematical activities that can help them come up with ideas that describe their informal activity. The tasks in an LIT contribute to this process, but the teacher-researcher in charge of guiding the LIT has the role of evoking and leveraging those mathematical practices. In our case, the participants engage in the process of reinventing an abstract algebraic concept. Dawkins (2015) defined a successful reinvention as "an instance of psychological explication whenever students already possess some intuitive or less formal understanding from which they construct formalized meanings" (p. 67). To achieve this reinvention, the students engage in mathematical activities of defining and conjecturing where students can sense mathematics as a human activity (Zandieh & Rasmussen, 2010). To guide students in the process of conjecturing, researchers can generate examples and counterexamples that can help students "enrich their concept images and enable them to judge the probable truth of conjectures" (Selden, 2012, p. 403). For definitions consisting of axioms, students axiomatize while transitioning from a model-of to model-for (Larsen, 2013), where students start to formalize their intuitive reasoning into a more structured and refined conjecture. Vroom and Alzaga Elizondo (2023) described college instructors' ways to guide students in the process of refining and editing a definition by suggesting an edit with implicit/explicit mathematical reasoning and by presenting students with problematic situations. We explore students' conjecturing and defining activities, as well as the pedagogical moves that scaffold those activities, in this study.

Methods

In this study, we designed RME-based HLTs with the goal of guiding IPSTs to reinvent the definition of UFD and to connect their knowledge to the teaching of mathematics. The HLTs include task sequences that are designed to guide students to reinvent the definition of reducibles and irreducibles (Unit 1), reinvent the definition of UFD (Unit 2 & 3), and apply their knowledge to pedagogical scenarios in hypothetical classroom situations (Unit 4). In this paper, we focus on the IPSTs' work in Units 2 and 3. The IPSTs were given experientially real tasks to guide their reinvention of the two defining axioms of UFD following the two HLTs that we initially hypothesized in the task design stage of this research. They were asked to investigate composite integers and factorable quadratics using algebra tiles and generate conjectures about the reducibility of elements in Z and $Z[x]$ in Unit 2. They were asked to examine different factorizations of the same elements that are essentially the same and conjectured about the uniqueness of factorization in Z and $Z[x]$ in Unit 3.

We tested the HLTs by administering tasks to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in a Hispanic-serving institution in the Southern US. Six mathematics graduate student IPSTs participated in this study: Josie and Raul (Group A), Javier and Roberto (Group B), and Kim and Taylor (Group C). They took abstract algebra courses at the undergraduate and/or graduate levels but had not learned UFD in their coursework. Each group participated in five 90-minute sessions of teaching experiments with the research team including a teacher-researcher (TR), secondary instructor, and research assistant. The TR's role was to guide the session, provide students with prompts to work on, and ask follow-up questions to better understand and clarify the IPSTs' thinking. The research assistant operated the video camera and took field notes. The secondary instructor observed the sessions, took field notes, and asked questions to the IPSTs when needed. Groups A and B participated in person, and Group C participated on Zoom due to their availability. In-person groups were given printouts with physical algebra tiles, and the virtual group was provided with an online whiteboard with tasks and virtual algebra tile manipulatives.

We collected and transcribed the video recordings of fifteen 90-minute teaching experiment sessions. The groups' written work on handouts and virtual whiteboards were collected. We used inductive coding (Miles et al., 2013) to analyze how IPSTs engaged in conjecturing and defining as they were guided to reinvent the definition of UFD. We focused our analysis on their ways of reasoning that were leveraged by using the algebra tiles and by interacting with the TR.

Results

Reducibility of Elements in Z, $Z[x]$, and UFD

In the first task set of Unit 2 (see Figure 1), the IPSTs observed side lengths of rectangular arrays of algebra tiles that represent 12 and its factors. They were asked to keep reducing the side lengths of rectangular arrays and to conjecture about the reducibility of integers. For instance, groups arranged tiles to represent $12 = 3\times4$ or $12 = 2\times6$ then reduced the composite factors of 12 into other rectangular arrays such as $4 = 2\times2$ and $6 = 3\times2$. The IPSTs transitioned their use of the emergent model (i.e., algebra tiles) from being a model of factoring integers to a model for abstracting the reducibility property of composite integers. IPSTs initially conjectured *reducible integers can be reduced to a product of irreducible integers*. Their reasoning was leveraged when prompted to think of counterexamples to their conjectures. When needed, the TR suggested what to consider such as 0 and units in Z (1 and -1). This also involves referencing the definition of irreducibles from the previous unit where units and zero are not included. The IPSTs found that *the zero and units cannot be represented as a product of irreducibles because they are not irreducibles by the definition*. This investigation allowed them to revise their conjectures by limiting the scope of integers in the statement: *all non-zero, non-unit integers can be reduced to a product of irreducible integers*. Examining precise definitions of irreducibles and products (of elements) was productive to leverage their reasoning in revising their conjectures. For instance, they observed 2×1 arrangement of algebra tiles and asked if this could be a product of irreducibles. It allowed them to consolidate their definitions of irreducibles (e.g., irreducibles do not include units and zero) and products (e.g., a product can be a single element). In sum, IPSTs' initial conjectures about the reducibility of integers were elicited and leveraged by the prompts in the task sequence with support from the TR when needed. Counterexamples and formal definitions were useful resources for their progression in the practices of conjecturing and revising their conjectures.

Task Set 1	1. Look at the side lengths of the rectangles of 12. Are those side lengths (factors) of 12 reducible or irreducible? How do you know? 2. If the side length is a reducible, arrange the tiles representing that side length into a rectangular array. Are those side lengths of the new rectangle reducible or irreducible? 3. What do you notice about the reducibility of integers? Make a conjecture. 4. Can we always write an integer as a product of irreducibles? Try to find a counterexample where an integer cannot be factored into a product of irreducibles.
Task Set 2	1. Look at the side lengths of the rectangles that represent these quadratics. Note that we consider polynomials with integer coefficients only: (a) $3x^2 + 9x$, (b) $x^2 + 3x + 3$, (c) $2x^2 + 4x$, (d) $x^2 + 4x + 3$, (e) $x^2 + 5x + 7$. Are those factors of the quadratic reducible or irreducible? How do you know? 2. If the side length is a reducible polynomial, arrange that side length polynomial into a rectangular array. Are those new side lengths reducible or irreducible? 3. Create a conjecture about the reducibility of quadratics. 4. Can we always write a quadratic as a product of irreducibles? Try to find a counterexample where a reducible cannot be factored into a product of irreducibles. 5. What about a higher degree polynomial? Can we write any polynomial as a product of irreducibles? Try to find a counterexample where a reducible cannot be factored into a product of irreducibles.
Task Set 3	1. The Common Cores State Standards of Mathematics suggest that students should "Understand that polynomials form a system analogous to the integers." Thus, polynomials have similar structure to the integers. In what ways are polynomials and integers similar? 2. Integers and rings of polynomials with integer coefficients are special kinds of integral domains with special properties about their elements' reducibility. Let's make a conjecture about the reducibility of integers and polynomials. What similarities do you see between these examples of the side lengths of the rectangular array that represents an integer and the side lengths of the rectangular arrangements of algebra tiles that represent quadratics?

Figure 1. Unit 2 task sequence

Task Set 2 used the same structure of the task sequence to guide IPSTs to conjecture about the reducibility of quadratics over $Z[x]$ (see Figure 1). This time, they used algebra tiles to represent quadratics and their factorizations if they are reducibles. For instance, IPSTs reduced $3x^2 + 9x$ into $3x \times (x + 3)$, then further reduced $3x$ into $3 \times x$ using algebra tiles (see Figure 2). The progression in their reasoning seemed similar to those from the previous unit: generating initial conjectures, looking for counterexamples (e.g., units and zero), and refining their conjectures by eliminating units and zero from the initial conjectures. When examining the array of $x \times 1$ that cannot be reduced further, the IPSTs needed the precise definition of units in a ring to determine if x is a unit in $Z[x]$ or not. The TR guided the IPSTs to reference the definition and pushed them for precision in their conclusion that x is not a unit because its multiplicative inverse $1/x$ does not exist in $Z[x]$.

TR: ... why is x not a unit?

Kim: x is not a unit because, um, its multiplicative inverse, it does not have one that is within that ring. Because the power of x would need to be outside of the natural numbers, so it would not be the unit.

In this task set, the IPSTs' reasoning was leveraged along the task sequences with the support of the TR and the emergent model of the algebra tiles. In particular, those supports were productive in identifying non-unit and non-zero elements in $Z[x]$ that were less explicit for them than those in Z. So, the IPSTs concluded the conjectures about the reducibility of integers in Z and polynomials in $Z[x]$ with essentially the same structure: *A non-zero, non-unit element in R can be written as a product of irreducible elements in R.*

In Task Set 3, the IPSTs were asked to identify common algebraic properties of Z and $Z[x]$ and write a conjecture of the reducibility in UFDs that works for both Z and $Z[x]$. The TR supported the IPSTs to recall and synthesize algebraic properties of Z and $Z[x]$ and formalize their final conjecture. In sum, the task design in Unit 2 used algebra tiles, and the TR's pedagogical moves leveraged the IPSTs' reasoning of generating, revising, and abstracting conjectures of the reducibility in the respective algebraic structures.

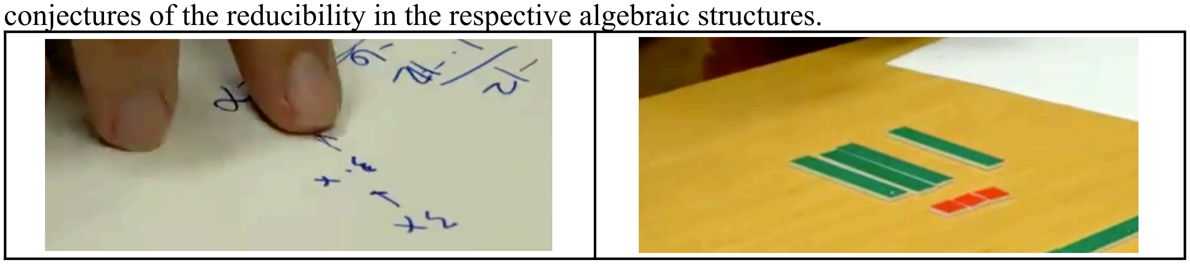

Figure 2. Group A used algebra tiles to represent the factorization of $3x = 3\times x$

Unique Factorization of Elements in Z, $Z[x]$, and UFD

Unit 3 tasks began with investigating two factorizations of an integer or a quadratic that differ by the order of factors in the products or by -1. Then, the IPSTs were guided to reinvent the concept of the uniqueness of factorizations, i.e., determine when two factorizations are essentially the same. For example, when asked to convince why two factorizations $(x + 3)(x + 2)$ and $(-x - 3)(-x - 2)$ are essentially the same, the IPSTs showed algebraic procedure to convert one into the other by using $(-1)^2 = 1$. They initially conjectured *two factorizations of a quadratic in* $Z[x]$ *are essentially same when a factor in one factorization is the negative of the respective factor in the other factorization.* This reasoning was leveraged in the following set of tasks that guided them to reinvent the definition of associates. Later, it allowed the IPSTs to use the formal language to capture their ideas about the relationship between factors in two essentially the same factorizations. When defining associates, the TR helped the IPSTs to generalize 1 and -1 to units for a general integral domain when they wrote the formal definition. The TR suggested to generalize their initial definitions, asked a leading question, and provided scaffolding. For instance, Javier (Group B) first wrote a conjecture that non-zero and non-unit elements a and b in an integral domain are associates if and only if their greatest common factors are 1 and -1. The TR leveraged his reasoning about associates with common factors of 1 and -1 but also pointed out a limitation of the conjecture by providing counterexamples.

TR: […] the only divisor I guess, the 1 and -1, then we could have something like $x + 3$ and $x + 2$ are associates? […] So you're right in that they do have this shared factor of 1 and -1, and it could be that it's just $a = 1\times b$ or it could be that $a = 1\times(-b)$. So it's looking like there's this shared kind of structure here, where a is equal to something times b. So now we have to figure out, what's that something?

Group B wrote a formal definition of associates using the idea of $a = ub$ where u is a unit in R that generalizes their initial conjecture to work for arbitrary integral domains (see Figure 3a). Following the problems in Task Set 2, the IPSTs revised their conjectures of the uniqueness of factorizations by using the definition of associates: *Two factorizations of an element in an integral domain are essentially the same if one factorization can be converted to the other by rearranging the order of factors or by replacing factors with their associates*. This involves using symbolic notations representing two possible factorizations $a_1 \cdot a_2 \cdots a_n$ and $b_1 \cdot b_2 \cdots b_m$ to express their reasoning of the uniqueness up to the associations and the order of factors. The TR helped the IPSTs to come up with precise subscripts in their writing. For instance, the TR suggested different subscripts n and m for the last factors in the given factorizations, a_n and b_m and different subscripts i and k for arbitrary factors from each factorization that are associates to each other, a_i and b_k. The TR pointed out an unintended assumption of correspondence between a_i and b_i when using the same subscripts for associates that are not necessarily ordered the same way (see Figure 3b).

TR: I like how you symbolized that, the $r = a_1 a_2 \cdots a_i$ and then the $u_1 b_1 u_2 b_2 \cdots u_i b_i$. So you're kind of pairing the a_1 with the b_1 and the u_1. So you're numbering them, that's kind of implying there's this correspondence between the a_i and the b_i.

This pedagogical move of the TR elicited the IPSTs' attention to the precision in mathematical language and leveraged their reasoning of what it means two arbitrary factorizations are essentially the same in the context as well as how to express precisely. The IPSTs' reasoning about the reducibility and the uniqueness of factorizations in Z and $Z[x]$ were leveraged in Task Set 3 where they were asked to synthesize two conjectures they made about the existence and uniqueness of factorizations of elements as a product of irreducibles to create the two axioms in the definition of UFD. The TR guided them to formalize their statements for the axioms in this process and helped them with making sense of the definition they created by using examples and non-examples (e.g., $Z[\sqrt{5}i]$) of UFD.

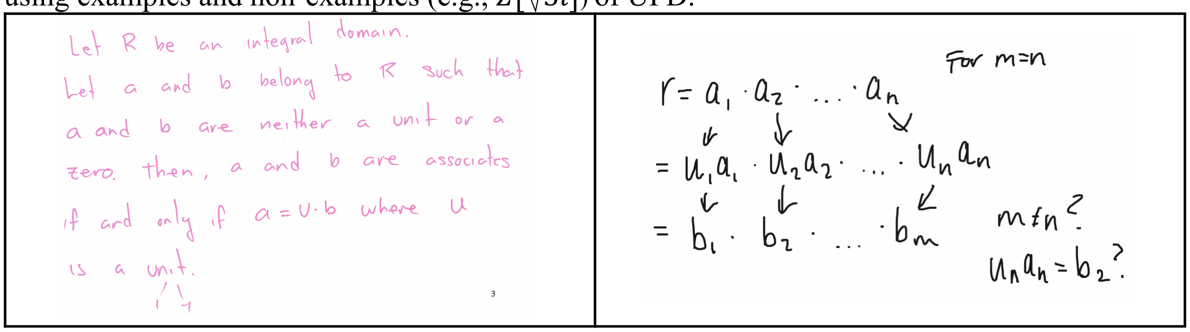

Figure 3. (a) Roberto's final definition of associates (left), (b) Kim's use of symbolic notations for two arbitrary factorizations that are essentially the same (right).

Conclusion

The findings of this study showed evidence of HLTs (Simon, 1995) in reinventing the definition of UFD by leveraging students' reasoning of the reducibility and the unique factorization of elements in Z and $Z[x]$. The sequence of experientially real tasks provided guiding prompts for the IPSTs' mathematical activities of conjecturing and defining in the progression of reinventing the two defining axioms of UFD. The emergent model of algebra tiles served as a *model-for* structuring and formalizing their intuitive reasoning about factorizations of integers and quadratics. The TR's pedagogical moves provided sociomathematical scaffolding that facilitates the IPSTs' horizontal and vertical mathematization (Gravemeijer & Doorman, 1999) of their informal reasoning and mathematical language. We plan to continue the design research on refining the hypotheses of student reasoning and task design of the HLTs in the following iterations of teaching experiments and classroom implementation.

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