Students' Treatment of the Negation of Logical Implication

Rachel Arnold
Virginia Tech

Vladislav Kokushkin
Colorado State University

Anderson Norton
Virginia Tech

Joseph Antonides
Virginia Tech

Understanding how students reason about the negation of logical statements is essential in supporting their mathematical development. Prior literature suggests that undergraduate students experience difficulties with the precise negation of logical implications. Many respond with the opposite statement "P implies not Q" when asked for the negation of "P implies Q." We investigate the challenges that introductory proofs' students experience when negating implications, and the reasoning they demonstrate in addressing these challenges. Our findings indicate quantification contributes significantly to their difficulties with negation. Further, nuances in students' quantification may validate an implication's opposite as its negation.

Keywords: logical implication, negation, quantification, transition to proof

Understanding the negation of a statement is an essential skill in logical reasoning. For example, Dubinsky et al. (1988) assert that "in order to understand what something is, it is essential to understand what it is not" (p. 46). Prior research has identified the negation of mathematical statements—particularly logical implications (LIs)—as a persistent challenge for undergraduate students (Barnard, 1995; Griggs & Cox, 1983; Sellers, 2020). The negation of a statement is the statement that contradicts the original statement and excludes any middle ground (Dawkins, 2017). In other words, an LI and its negation have opposite truth values in all cases. However, students will sometimes present a contrary statement as a negation without satisfying the law of excluded middle. In particular, students might argue that the negation of P→Q is the statement P→~Q (Epp, 2003), which we refer to as the *opposite* of P→Q. The primary objective of our study is to understand students' ways of reasoning when negating an LI and the distinctions they make between counterexample, negation, and opposite.

The negation of a universally quantified LI is the assertion that a counterexample exists in the universe. To negate the LI P(x)→Q(x), one must specify the logical properties of a counterexample (P(x) and ~Q(x)) and additionally attend to the quantification of x.

Although students may generate specific counterexamples to statements like "If a number is a multiple of 3, then it is a multiple of 6" (Dawkins & Hub, 2017), there seems to be a significant cognitive gap between identifying a counterexample to an LI and articulating its precise negation. Prior research has documented students' persistent struggles with quantification in general (e.g., Dawkins & Roh, 2020; Dubinsky et al., 1998; Shipman, 2016), which suggests quantification may be a primary source contributing to this gap. However, there may also be other factors influencing the ways they perceive negation, opposite, and counterexample. To understand the nuances of students' treatments of the negation of an LI, we analyzed the reasoning demonstrated by two undergraduates enrolled in a Fall 2022 introductory proofs course taught by the first author. Our research questions were:

1. *Following research-based instruction in an introductory proofs course, what challenges do students experience in negating an LI; and what reasoning do they demonstrate as they address these challenges?*

2. *In particular, how do students experience and address the challenges identified in prior research on treating negation as the opposite statement?*

Framework: Epistemological Obstacles Associated with Negating LIs

Our framework presents the persistent challenges students experience with negating an LI as epistemological obstacles (EOs). The term EO originates in Bachelard (1938), and Brousseau (2002) later described EOs as obstacles that remain in response to best instructional practice. We further characterize EOs as experienced in best instructional interactions, by the student or the teacher. EOs are *persistent*; they cannot be resolved in a single lesson. Here we provide the supporting literature for existing EOs that are pertinent to this study.

Students make meaning of quantified statements in various ways (Sellers et al., 2021). Sometimes, they even interpret the language of quantification in ambiguous ways (Dawkins & Roh 2020; Epp, 1999). Other times, quantification might not be explicitly stated at all, so the intended quantification of a statement remains "hidden" (Shipman, 2016). When expressing LIs, we often say "P→Q" to mean "for all x, P(x)→Q(x)." It may seem that instructors could resolve the issue by consistently and explicitly stating quantification, but subtle differences in meanings of quantification persistently remain hidden for students. Dubinsky et al. (1988) described quantification as an "insurmountable barrier for students in developing a sophisticated understanding of limits and continuity" (p. 44). This description fits our characterization of EOs, and following Shipman (2016), we refer to this EO as hidden quantification.

As we have noted, a statement and its negation have opposite truth values, so negating a statement and generating counterexamples are inherently related (Shipman, 2016). Generating a counterexample for a given statement can be challenging, and when attempting to prove a universally quantified statement false, students will often claim that a single counterexample is insufficient (Zaslavsky & Ron, 1998). Sellers (2020) conjectured that this claim might relate to the recognition that a singular example is insufficient for proving a universally quantified statement is true. It may seem counterintuitive that the negation of a universally quantified statement is an existence statement, and the negation of an existence statement is a universally quantified statement. Thus, we see quantification of the negation as another potential obstacle.

The challenge of quantifying negations sharpens when we consider that the negation of a universally quantified statement encompasses *any* possible counterexample. That is why when proving a universally quantified statement, it is insufficient to show that a particular case is not a counterexample, or to simply modify the statement to bar that case (Shipman, 2016). It is also why the opposite of a statement does not suffice as its negation. Consider the universally quantified (hidden) LI, P→Q. Its opposite would be the universally quantified statement that P→~Q; but both statements could be false because they leave middle ground untouched (Dawkins, 2017). Because students ostensibly conflate the negation of a statement and its opposite—even after instruction in introductory proofs courses (Arnold & Norton, 2017)—we refer to it as another EO. Research suggests relying on Euler diagrams to sort out reasoning with LIs (Hub & Dawkins, 2018), but even Euler diagrams may have their limitations (Antonides et al., in review).

Methodology

The data reported here come from a larger study investigating the EOs that students experience, and ways of addressing them, in transition-to-proof courses. The class was taught by the first author in the Fall 2022 semester. Emphasizing active learning, the class was designed around four main topics: LI, quantification, mathematical induction, and functions. We identified

EOs suggested by existing literature and adopted or created instructional tasks for addressing those EOs head-on. Four students were interviewed three times by Norton and Kokushkin, with each video-recorded interview lasting 30-45 minutes. Students were given a LiveScribe pen and were asked to think aloud. The interview tasks corresponded to the four topics of the course; the third interview included tasks on mathematical induction and functions, as well as a stimulated recall interview (SRI) on a subset of tasks (chosen subjectively) from the first two interviews.

All interview data were coded to capture students' EOs. Our codebook (Figure 1) included codes both from existing literature and emergent codes that could not be explained using existing codes. This paper centers around the NO code, informed by existing literature; emergent codes were NF and LIv (Figure 1). Following the initial round of coding, we engaged in thematic analysis, analyzing contiguity of codes with the NO code.

Code	**Description (Source)**
LIv	Eliminating the vacuous case (Emerged)
NF	Difficulty with the difference between negation and disproof (Emerged)
NO	Treating the negation as the opposite, e.g., $P \rightarrow \sim Q$ (Dawkins & Hub, 2017; Epp, 2003)
Qh	Hidden quantification (Shipman, 2016; Durand-Guerrier, 2003; Ernst, 1984)
Qneg	Uncertainty about quantifying the negation (Dawkins & Roh, 2016; Shipman, 2016)
Reu	Interpreting logic via Euler Diagrams (Hub & Dawkins, 2018; Dawkins & Roh, 2021)

Figure 1. Codes.

We report on interview data from two students, Shivani and Carmen, a sophomore and senior majoring in Computational Modeling and Data Analytics and Aerospace Engineering, respectively. Shivani and Carmen were chosen because of the salience of their data vis-à-vis the negation-opposite conflation, and because their cases represent qualitatively distinct issues that may arise for students as they reason about the negation of an LI.

1. Consider the implication: $P(x) \rightarrow Q(x)$. What does it mean? How would you quantify it?
2. Consider the statement: "For all x, $P(x) \rightarrow Q(x)$." How would you negate it?

Figure 2. Interview task.

Results

In this section, we present the details of Shivani's and Carmen's reasoning about an LI and its negation in response to the interview tasks shown in Figure 2.

Shivani's Treatment of Negation

When first asked what the unquantified LI $P(x) \rightarrow Q(x)$ meant to her (see Figure 2, Task 1), Shivani did not demonstrate an intellectual need for making the quantification explicit. She responded, "If P, then Q," and then elaborated to say, "So if P, if P is true then Q is true."

When the researcher asked how Shivani might quantify $P(x) \rightarrow Q(x)$, she drew a correct Euler diagram (Figure 3) and said, "So there exists… an x in the subset P. There exists an x... in... in P if uh... If there exists an x in P then x also exists in Q.” Shivani saw $P(x)\rightarrow Q(x)$ as referring to a specific x and therefore quantified it existentially. Moreover, she restricted the universe to consider only those x in the truth set of P. She also seemed to quantify the *hypothesis* rather than the LI itself: “If there exists an x in P.”

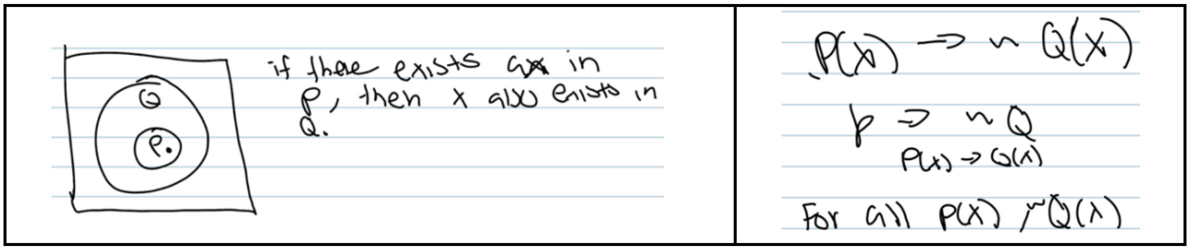

Figure 3. Shivani’s Euler Diagram (left) and her corresponding writing (right).

When posed with Task 2 (see Figure 2), Shivani again omitted quantification saying, “So then it would be like P(x) implies $\sim Q(x)$ [writing ‘$P(x)\rightarrow\sim Q(x)$’]. That would be a negation because the concl… hypothesis has to be true.” Her emphasis on the hypothesis being true coupled with her earlier treatment of x, seems to indicate that, for Shivani, her negation requires x to be in the truth set of P. In an effort to unhide quantification, the researcher asked Shivani how she would quantify x. She responded,

> Since this is for all, then you would have to prove like, there exists one x value that makes this statement true [pointing to her written negation ‘$P(x)\rightarrow\sim Q(x)$’ (Figure 3)]... Because then that would be the counterexample because you wanna disprove it…Because you only need one counterexample to prove that for all x.

For her, the existentially quantified LI $P(x)\rightarrow\sim Q(x)$ represented the existence of a counterexample. When the researcher asked how her negation might be quantified if the original LI was existentially quantified, Shivani adjusted her negation to “For all P(x), $\sim Q(x)$.” Shivani demonstrated her understanding of how quantification of the original LI affected the quantification of her negation. And, her use of “for all P(x)” seemed to indicate that Shivani was again requiring x to belong to the truth set of P. Regardless of how the original LI was quantified, when quantifying x in her negation, x belonged to the truth set of P, not simply the universe.

Carmen’s Treatment of Negation

Like Shivani, Carmen initially quantified the LI $P(x)\rightarrow Q(x)$ existentially saying, “This means that there is some point in P, some point x in P, that is also in Q.” Note that Carmen also restricted the universe to the truth set of P, the values for which she saw the LI as relevant. Then, the researcher asked Carmen to reason about Task 2 (see Figure 2). She responded as follows.

> So we’re saying that for every single x, in the set of P... So if... for all values of x, P(x). So this would mean that P would be a subset of Q… [draws an Euler diagram] So I can negate this by saying... what is it? It’s P(x)... So we would negate this by keeping the... hypothesis true, saying that P(x) is still true. And then... you can negate this by saying and not Q(x). I think that’s how you negate this. Because saying that... um... Yes. And the ‘and’ is important, because this means that P(x) has to be true, and not Q(x) has to be

true. So you're saying that you are in P(x), but you're not in Q(x), so that would be negating the P(x) and being in Q(x).

Carmen correctly stated the negation as P(x) and ~Q(x), emphasizing the importance of the connective "and." However, she did not explicitly quantify x, so the researcher prompted her.

If we're saying for all x, and you want to negate it... I... think the negation... because you're not trying to find like a counterexample, you're just trying to negate it. I think you could say for every x… in P... and that's what you could say for every like for every x... in this set of, like a universal set, I guess. Maybe not... For every x in P... I'm not good at writing them out. Because I think… that with negations, you're trying to find a way that the original statement, like, not necessarily the opposite, but just like an alteration, I guess, to the original statement, or I don't think you're like looking for a counterexample. Because that would just be proving that the original statement will be false. So if negating it, you… it's still for every x, in... in the universe of... for x is in the universal set.

Carmen struggled to quantify her negation, expressing that negating the LI was different than finding a counterexample or disproving it. She felt that the negation should be a statement about every x (questioning whether x is in P or x is in the universe). Ultimately, Carmen settled on universal quantification, but never stated her full negation. So, the researcher (R) probed Carmen's (C) distinction between disproving the LI and finding its negation.

C: To show that that's false, I would just find one x ...um, that is in P(x), but it's not in Q(x).

R: And that's not necessarily the same as the negation?

C: I don't think so. Because that's just a counterexample. Yes, I think the negation is just kind of like doing the opposite of what it's asking for. It's not necessarily like… it's not asking you to prove that something is false or to make it or it's like asking you to rewrite this statement that would, in a way that would make it false, not asking you to prove that it is false. I guess...

Carmen understood that to disprove the LI, she needed just one counterexample. Unlike disproving, finding the negation for her entailed "rewriting" the statement to "make it false." To understand her emerging distinctions between negation, counterexample, and opposite, the researcher followed up with Carmen two months later during her SRI, presented here.

C: With a counterexample, you choose one specific value. So I think that's the difference between being the opposite and negation. So you have one specific value that you're looking at, but with a negation, you're saying that something is the opposite. You keep it arbitrary, because you kind of want to show like, for all umm so but with like a counterexample, you just say, okay, my k value or whatever equals a specific number. And that's why it's, it's false, because I can show that it doesn't work. Like for this one specific case.

R: Okay. Does that mean that negation and opposite are the same thing?

C: Um, I don't think... I know that they're not the same thing. Because opposite would be like hot and cold, right? ... But, I think a negation... I'm trying to remember...

R: What if I want to negate "hot"? How would I negate "hot"?

C: You can negate it in many ways. You could say warm, cold... or warm, hot, you could say, cold or freezing or stuff like that. So that's a negation because it's not hot. But if you want to just say it's opposite of hot, then you have to go to cold.

R: So if I want to do a counterexample of "hot", I would say, I would find an example of something that's not hot… If I wanted to do the opposite of "hot', I would show everything is cold…And if I wanna show the negation of 'hot", I would show everything is "not hot".

C: Yes.

Carmen used the example "not hot" to explain her distinctions between counterexample, opposite, and negation. In suggesting "you can negate it in many ways," Carmen seems to be referring to all possible counterexamples. Carmen still universally quantified her negation.

Discussion

The above data suggest that students' reasoning about the negation of an LI can be influenced by multiple factors, including students' difficulties with (hidden) quantification, reliance on Euler diagrams, reducing the universal set to the truth set of the hypothesis, and uncertainty about the distinctions between negation and proving false. In what follows, we present the discussion of three main findings that emerged from our thematic analysis.

Difficulty with (Hidden) Quantification

Consistent with prior literature, our participants experienced challenges in recognizing the hidden quantification of $P(x) \rightarrow Q(x)$. Shivani omitted quantification of both the LI and its negation. Carmen explicitly quantified the LI, but neglected the quantification of its negation. This finding confirms that quantification may significantly influence students' ability to articulate the precise statement of negation.

Ultimately, both Shivani (when prompted) and Carmen existentially quantified $P(x) \rightarrow Q(x)$. Shivani supported her reasoning by drawing an Euler diagram and saying "if there exists an x in P." Carmen did not rely on drawings. Nevertheless, her language for quantifying x ("there is some point in P, some point x in P, that is also in Q") referenced an implicit use of an Euler diagram as well. This may indicate that students interpret $P(x) \rightarrow Q(x)$ as referring to a specific x, whereas $P \rightarrow Q$ might be universally quantified. The necessity to refer to a specific x may result in overlooking the vacuous case, which is crucial in distinguishing between the negation and the opposite (Arnold & Norton, 2017). This observation echoes our previous finding that reliance on Euler diagrams may hide the vacuous case (Antonides et al., in review).

Reducing the Universal Set to Only Those Values in Truth Set of the Hypothesis

Prior research has documented students' struggles with the vacuous case. Hoyles and Küchemann (2002) found that students often view the vacuous case as irrelevant. Dawkins et al. (2023) reported a tendency for students to treat an LI as a statement only about the truth set of its hypothesis. Furthermore, Norton et al. (2022) explicated the nuances of students' apparent conflation of the truth of an LI with the truth of its hypothesis. Our analysis suggests that students may omit the vacuous case not merely because of an apparent conflation, but rather because they focus only on values for which they view the LI as relevant. Reducing universe to the truth set of the hypothesis yields students' treatment of the negation as its opposite logically correct—a finding not previously reported.

Both students focused on x values for which P(x) was true (coded as LIv). Shivani said, "if there exists an x in P" and "for all P(x), ~Q(x)," while Carmen stated, "there is some point in P, some point x in P" and "for every x in P." For us, the LIv code emerged as students' elimination of the vacuous case. However, in furthering our analysis, this treatment is closely related to the above prior research. Students' focus on only values for which the hypothesis is true is consistent with their prior experience of using LIs (via *modus ponens*) to deduce new conclusions. This tendency could explain why students seemingly conflate the negation of P→Q with its opposite P→~Q. In fact, when students restrict the universe U to the truth set of P, the negation "$\exists x \in U$ such that P(x) and ~Q(x)" is logically equivalent to its existentially quantified opposite "$\exists x \in P$ such that P(x)→~Q(x)." This opposite statement is indeed the negation.

Universal Quantification as a Distinction between Negation and Proving False

Carmen's unquantified negation, P and ~Q, was correct. But, when prompted to quantify it, Carmen chose universal quantification (and further elaborated in her SRI). She consistently made distinctions between negation and proving false. She explained that negation is an "alteration" of the original statement "that would make it false." Perhaps because the original statement was about all values in the universal set and the negation was an "alteration" of this statement, Carmen reasoned that the negation should also be universally quantified. While existential quantification is indeed an assertion about the state of the universe, because Carmen associates "there exists" with a specific x, she may have felt that existential quantification of the negation would not be a broad enough statement about the universe.

When the researcher asked Carmen to distinguish between opposite and negation in her SRI, she gave the example of "not hot," saying "you can negate it in many ways" (e.g. warm or cold). This language may indicate that Carmen sees negating as a collection of all the ways one might say a statement is false. If Carmen was viewing negation as the collection of *all* possible counterexamples, then this might also explain her choice of universal quantification.

Implications and Future Research

We have discussed how EOs related to attending to quantification, treating the vacuous case, and making distinctions between negation and proving false may significantly influence students' treatment of negation. Nevertheless, there are other potential factors that may contribute to students' conception of negation. Such factors include treating an LI as an action (Norton et al., 2023), students' difficulties with transforming LIs (e.g., Durand-Guerrier, 2003), understanding the Principle of Universal Generalization (Norton et al., 2022), and others. Future research should shed light on the effect of these factors on students' reasoning about negation.

Our findings suggest potential instructional interventions. Teachers might draw attention to the distinction between using an LI to draw conclusions (modus ponens) and determining the validity of an LI. In particular, they might emphasize that the latter requires considering all values in the universal set U, not just values in the truth set of P. To make explicit their tendency to narrow their focus to only those values in P, students might compare and contrast the logical difference between "$\exists x \in U$ such that P(x)→~Q(x)" and "$\exists x \in P$, such that P(x)→~Q(x)."

Acknowledgements

This project is supported by NSF under grant number DUE-2141626. Special thanks to advisory board member, Paul Christian Dawkins, whose feedback elucidated many of the ideas we present in this paper. We also appreciate Ph.D. student Matthew Furuta Park's efforts in bringing together a large base of research literature which assisted in the framing of our work.

References

Antonides, J., Norton, A., & Arnold, R. (in review). Linking structures across logic and space: The role of Euler diagrams. Manuscript submitted for publication.

Arnold, R., & Norton, A. (2017). Mathematical actions, mathematical objects, and mathematical induction. *Proceedings of the 20th annual conference for Research in Undergraduate Mathematics Education*, San Diego, CA.
Balechard, G. (1938). La formation d'esprit scientifique: Contribution à une psychanalyse de la connaissance. Vrin.
Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990*, 79–117.
Barnard, T. (1995). The impact of 'meaning' on students' ability to negate statements. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th International Conference on the Psychology of Mathematics Education*. (pp. 3-10). PME.
Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education, 3*, 496-522.
Dawkins, P. C., & Hub, A. (2017). Explicating the Concept of Contrapositive Equivalence. *North American Chapter of the International Group for the Psychology of Mathematics Education*.
Dawkins, P. C., & Roh, K. H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education, 6*, 1-22.
Dawkins, P. C., Roh, K. H., & Eckman, D. (2023). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. *Journal of Mathematical Behavior, 70*, 101043.
Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics*, 8(2), 44–51.
Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective, *Educational Studies in Mathematics, 53*, 5–34
Epp, S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12, 1999 Yearbook* (pp. 188-197). NCTM.
Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886–899.
Griggs, R. A., & Cox, J. R. (1983). The effects of problem content and negation on Wason's selection task. *Quarterly Journal of Experimental Psychology, 35*(3), 519–533.
Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*, 193–223.
Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. *Proceedings of the 45th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 69-78). University of Nevada, Reno.
Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, 1-27.
Sellers, M. E. (2020). Students' quantifications, interpretations, and negations of complex mathematical statements from calculus [Doctoral dissertation, Arizona State University].

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *Journal of Mathematical Behavior*, *61*, 100802.

Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and Its Applications, 35*, 41–49

Zaslavsky, O., & Ron, G. (1998). Students' understandings of the role of counterexamples. In A. Olivier & K. Newstead (Eds.) *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, ed. A.* (pp. 25-32.). PME.