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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-sixth annual Conference on Research in Undergraduate Mathematics Education in Omaha, Nebraska from February 22 - February 24, 2024.

The 26th RUME Conference enabled presenters and attendees the option to participate fully online such that travel was not a requirement, approximately 31% of participants were online.

The program included plenary addresses by Dr. Yvonne Lai, Dr. Michelle Friend, and Dr. Estrella Johnson and the presentation of 154 contributed, preliminary, and theoretical research reports and 97 posters. The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in undergraduate mathematics education, each underwent a rigorous review by three or more reviewers:

- Contributed Research Reports describe completed research studies
- Preliminary Research Reports describe ongoing research in early stages of analysis
- Theoretical Research Reports describe new theoretical perspectives for research
- Posters may fall into any of Contributed, Preliminary, or Theoretical and were presented in poster format. Authors contributed the poster itself, a summary of the work, or both.

The conference was hosted by the University of Nebraska – Omaha's Department of Mathematical and Statistical Sciences. Many members of the RUME community volunteered for the Program Committee where they reviewed many submissions such that every submission was reviewed by at least one member of the Program Committee. The Program Committee aided us in putting together this program and their hard work is greatly appreciated. The Local Organizing Committee were responsible for the smooth running of the presentations and on-site activities, which would not have been possible without the help of many volunteers, and we thank them for their tireless efforts to host a conference that runs smoothly.

Thank you to all of the researchers who submitted such strong proposals and ultimately made the conference a fun and joyous event.

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Learning in Pieces: Horizontal Décalage in Students' Thinking on Homework Tasks

Allison Dorko
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This report (a) documents the phenomenon that student learning from online homework may occur across a series of mathematical tasks and (b) provides a theoretical explanation for the phenomenon by characterizing some of the cognitive mechanisms that underpin it. Specifically, findings indicate one way students learn from online homework is by comparing similar tasks. The results add to knowledge about what and how students learn from homework. They also have implications both for practice and research in terms of how we define and measure learning. In particular, the results support the need to consider and design for learning across multiple mathematical tasks.

Keywords: assimilation, accommodation, online homework, equation solving

Rationale and Overview

Although many university students spend more time doing mathematics homework than they do in class (Ellis et al., 2015; Krause & Putnam, 2016; Lew & Zazkis, 2019), there is a paucity of research about student learning from homework. Research about what students learn from homework and the mechanisms by which they learn it can inform the creation of more effective homework assignments, which, given the amount of time students spend doing homework, could greatly improve student learning. This report, which focuses on student learning while doing online homework, makes several contributions. First, it documents what Piaget called horizontal décalage, or the phenomenon that there is often a temporal delay in learning. Specifically, the paper shows how some students' learning from online homework occurred across a series of mathematical tasks. Second, the paper characterizes some of the cognitive mechanisms that supported students' learning over the series of tasks.

Theoretical Perspective and Theoretical Literature

von Glasersfeld (1995) summarized Piaget's definition of learning as follows: learning "takes place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium" (p. 68). A *scheme* is a construct describing how people organize knowledge. It consists of "(1) recognition of a certain situation; (2) a specific activity associated with that specific situation; and (3) the expectation that the activity produces a certain previously experienced result" (von Glasersfeld, 1995, p.65). The framework was chosen for the study in part because of its clear definition of learning, which afforded identifying when learning happened and what supported that learning.

People employ, construct, and modify schemes via assimilation, perturbation, equilibration, and accommodation. Consider a person engaging with a math task. *Assimilation* occurs if the person "treats new material *as an instance of something known*… [the] cognizing organism fits an experience into a conceptual structure it already has" (emphasis original; von Glasersfeld, 1995, p. 62). If the person is unable to fit the experience into an existing conceptual structure, they may be *perturbed,* meaning they recognize assimilation has failed. von Glasersfeld (1995) gives disappointment or surprise as examples of indicators that a person is perturbed. *Accommodation* is the modification of a scheme to remove perturbation. *Equilibration* describes the "construction process and [a] mechanism of change" by which a person accommodates a scheme and resolves perturbation. One way accommodation could occur is through the person reviewing the situation and identifying characteristics missed in the assimilation attempt; the identified characteristics "may effect a change in the recognition pattern and this in the conditions that will trigger the activity in the future" (von Glasersfeld, 1995, p. 65). A second way accommodation could occur is if the person creates a new recognition pattern (and hence a new scheme).

An accommodation is a semipermanent state; Piaget notes "it can take a certain time for a particular way of operating to spread to other contexts (*horizontal décalage*)" (von Glasersfeld, 1995, p. 71-72). Montangero and Maurice-Naville (1997) note that horizontal décalage "illustrates the limits of the generalization of a mental structure" (p. 89). That is, horizontal décalage can be thought of as the temporal delay in an individual's capacity to apply a scheme to a structurally-isomorphic task (i.e., engage in observer-oriented transfer). The idea that accommodation (learning) takes place over time and across contexts instead of as a single "lightbulb" moment, is the focus of this paper. The theory that learning occurs over time and across contexts is neither new nor unique to Piaget. di Sessa (1993) proposed the knowledge in pieces (KiP) framework, and Wagner (2006) characterized transfer as occurring in 'pieces.' KiP, developed in the context of physics learning, has been employed in empirical mathematics education work (e.g. Izsák, 2005).

Empirical Literature

The findings about student learning from homework indicate students' affinity for learning from examples (Dorko, 2021, 2020; Dorko, Cook, & DeHoyos, 2023; Aichele et al., 2011; Bissel, 2012; Erickson, 2020; Kanwal, 2020; Krause & Putnam, 2016; Lithner, 2003; Weinberg et al., 2012). Students appear to like examples in part because they provide an opportunity to reason via viewing a similar problem. However, the extant literature also indicates the learning from these problems tends to be procedural in nature, with students missing conceptual insights the professor intended.

When learning from online homework, multiple researchers have found students employ immediate feedback to inform their work (Dorko, 2020; Kontorovich & Locke, 2022). That is, students tend to submit an answer to a part of a problem, see if it is correct, and then proceed to the next part of the problem.

Methods

The data discussed are representative excerpts from video recorded sessions of five precalculus students doing online homework. I collected data at a large, public US university during fall 2022 and spring 2023. All five students had the same instructor. The entire data set consists of 27 online homework assignments between the six students, collected over a two-month period in each term. During the sessions, students were instructed to think aloud. Students received immediate feedback on their answers from the homework platform and had unlimited attempts for non-multiple choice problems.

During data collection, the researcher had observed that students might answer a problem correctly, only to struggle with a subsequent problem about the same mathematical idea. This was intriguing because it raised the question why the student did not employ the previously-successful scheme in the new context. Hence the first step in analysis was isolating all series of problems that dealt with the same mathematical idea and taking from that set the data in which a student struggled with some (but not all) of the problems. There were two goals in analyzing the data. The first was to evaluate the extent to which horizontal décalage described the students' struggles over a series of problems. Having determined that it did, the researcher then sought to characterize the mechanism of equilibration.

The criteria for evaluating if a student's work across the series of problems was an example of horizontal décalage was the student modifying a scheme over a series of problems, with evidence for those modifications coming from students' utterances and the answers they submitted. This follows from Piaget's characterization of accommodation as a semipermanent state and the definition of horizontal décalage (above). Evidence that students were perturbed came from their expressing confusion, disappointment, or surprise (von Glasersfeld, 1995). If students approached and solve a problem without difficulty (either explicitly expressed difficulty, or the researcher noticing activity like their writing/erasing in scratchwork), the students were considered to be in an unperturbed state and assimilating the problem to an extant scheme.

To characterize the mechanisms of equilibration, the researcher performed an analysis grounded in the data and informed by the theory. In terms of the theory, she sought evidence that the student had (upon an assimilation failing) explored the context to seek characteristics the assimilation had disregarded. This followed from von Glasersfeld's (1995) description of one way accommodation can occur. However, this explanation did not seem to fit the data under consideration. The researcher did notice that Dylan, the first student whose data she analyzed, made progress on a task in the middle of the series by re-examining his work on previous problems that he identified as similar. In accordance with grounded methods of analysis, the researcher evaluated data from other students to see if *comparison* appeared to be a cognitive mechanism that supported equilibration for other students (it did). This work is ongoing, and subsequent analysis will seek to identify other mechanisms of equilibration. The results shown below provide examples of (a) horizontal décalage as explaining students' difficulty on a series of tasks that employed the same mathematical idea and (b) comparison as a cognitive mechanism supporting equilibration. The illustrative excerpts correspond to the students' work on the following tasks[1], which are given in the order they appeared in the students' assignment. The course coordinator chose the problems because they all focused on the "solve for" idea but had variations in the formulas and task wording that might lead students to see them as different. For example, the GDP and gas mileage part A problems did not involve any constants, while the other tasks did. The gas mileage part B task differed from the other tasks because students had to set a variable as constant. The wordings also differed from "solve for E" to include "solve for t and express t as a function of V" to "to obtain a formula". That is, in some wording, students were told the answer was a function or a formula, while in others, they were not. Hence the tasks had enough variation that students might have to engage in cognitive work to see them as similar.

Gross domestic product (GDP) task. The gross domestic product, P, is calculated as the sum of personal consumption expenditures, C, gross private domestic investment, I, government gross investment G, and net exports E of goods and services. All are measured in billions of dollars. The formula is P = C + I + G + E. Solve the equation for E.

Boat task. The resale value, V, in thousands of dollars, of a boat is a function of the number of years *t* since the start of 2011, and the formula is V = 12.5 – 1.5t. Solve for t in the formula to obtain a formula expressing t as a function of V.

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[1] The tasks are from Crauder et al. (2018). The presentation here paraphrases wording and omits parts of the tasks irrelevant to the report.

Gas mileage task: The distance, d, in mi., that you can travel without stopping depends on the number of gallons g of gas in your tank and the gas mileage, m, in mi/gal. The relationship is d = gm. (A) Solve the equation for m. (B) An engineer wants to ensure the car she is designing can go 415 miles on a full tank of gas. She does not yet know what gas mileage the car will get. Solve the equation d = gm for g using 415 for the distance.

Ant task: A scientist observed that the speed S (cm/sec) at which ants run was a function of T, the temperature (°C). He discovered the formula S = 0.2T – 2.7. Solve for T to obtain a formula expressing the temperature as a function of the speed.

Illustrative Excerpts

Dylan

Dylan's work on a series of three problems provides an example of horizontal décalage. The course content included solving an equation for a particular variable and having the answer be a formula instead of a number. That "solve for *x*" could result in an *x* = … expression in which the right side of the equation contained variables was a new idea for Dylan. By 'new idea', I mean he had been exposed to this meaning of "solve for" in lecture but was grappling with it on his own for the first time during the homework session. Dylan solved the GDP and boat tasks on the first try. On Gas Mileage part A, he solved d = gm for m correctly on his first try. In part B, however, Dylan was perturbed by the fact that a "solve for…" answers was a formula. Dylan's facility with the boat task and part A of the gas mileage task, but perturbation on part B of the gas mileage task, indicated to the researcher that Dylan was in the process of accommodating his "solve for…" scheme. Specifically, the perturbation on part B of the gas mileage task appeared to be an example of horizontal décalage because Dylan had just answered two similar tasks unperturbed. Dylan's thinking about part B of the gas mileage task is below. It provides evidence of how he equilibrated by *comparing* the problem at hand to the previous problems to remove his perturbation.

Dylan: I'm thinking, thinking I don't have enough information but, because I'm missing, missing m.

Interviewer: What do you mean [by] you don't have enough information?

Dylan: I mean well right now I'm thinking I only have one, only have one variable and I need two.

Int.: What makes you think you need two?

Dylan: That's kind of how it works, right? You're given, like you're given a variable, you're given like an equation and then you put in all the numbers.

Int.: So you think the answer, you think it needs to be g equals some number.

Dylan: Yeah. Because it's solving for the equation. But I guess I solved for the equation up here [moves mouse to part A]. So I guess not… if it [Webassign] took this [m=d/g as correct in part A] and it's asking part B the same that it's asking [in] this [part] I mean it's asking it in the same way. [Submits g=415/m, correct]

I take Dylan's comments about not having enough information and needing two variables so he could "put in all the numbers" as evidence of perturbation. I focus on how, following those comments, Dylan reviewed his earlier, correct m=d/g. He noted that his present task was worded the same way as that task; that is, he compared the wording of the tasks and employed the answer from the previously-solved task to support his solution in the unsolved task. I use the word *comparison* to characterize the mechanism of change as Dylan equilibrated[2]. Dylan solved the subsequent Ant Task without perturbation, which we take as evidence that accommodation to his "solve for…" scheme had occurred:

Int.: You have a formula that's got another variable and it's not just like all numbers. You feel okay about that?

Dylan: Yeah, yeah I do this time. Because I remember from up [above] that it worked.

Amy

Amy's work on a series of 4 tasks provides an example of horizontal décalage. Amy struggled with GDP task. She opened the e-book, located the answer to the problem, and said she understood why the answer was correct. Amy copied the book answer, submitted it, and the platform marked it correct. On the boat task, Amy noted the problem was "another 'solve for'." She wrote t = V(t); note this is an equation in which t = … and the right side of the equation contains only variables[3]. After submitting this answer and seeing it was wrong, Amy located the problem's answer in the e-book and commented she had never seen an answer like that before. She said the solution "looked right", though she did not understand why.

Amy: I don't quite get what that's asking for [opens the e-book].

Int.: What are you hoping to find there?

Amy: Something that explains this specific problem. [finds exact problem with answer key, looks at answer] this [answer] is not what I would have thought of… like I would think that you'd solve this equation for what E is, not in terms of variables. In my mind whenever you say solve that means like get an answer for E.

Int.: Get an answer like a number?

Amy: Yes…. [looks over answer in e-book] I understand why that's the answer.

Amy: [Boat task] It's another "solve for"… like "solve for t in the formula, expressing t as a function of V." My mind goes to like this type of answer [types t = V(t)] but I don't know if that's correct because given the past problem it probably isn't. [opens e-book] That's what we've always used for like functions of, whenever you're expressing t as a function of V it's like anything is a function of something else it's always been like, A, like B is a function of A it's like A(B). [submits answer, incorrect] I don't quite understand why this is wrong. [looks at answer in book] I've never seen something like this before. And I don't know how I would get this answer.

Int.: Does anything in that [book answer] make sense?

Amy: I mean yeah because I mean given the whole picture like this [highlights formula answer] does make sense and it is correct… Like looking at it, it looks right. Like if someone showed me this and explained how to get to it I would understand more how it was right.

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[2] A difference in Gas Mileage Parts A and B is that A had three variables while B directed students to use d = 415. We do not have enough evidence to know if this difference caused Dylan's perturbation. Similarly, von Glasersfeld (1995) hypothesized a scheme could be modified to include new recognition criteria or the learner could create a new scheme. It is impossible to know which is the case here (in part because of the difficulty defining the boundaries of a scheme). These concerns are beyond the scope of this report; our goal is to (a) identify comparison as the mechanism of change and (b) show an example of horizontal décalage.

[3] As the interview excerpts indicate, Amy switched the independent and dependent variables here and in her generic A(B); this is immaterial to the point that what she wrote was an equation in which the right-hand side was an expression with variables, not a number.

We take Amy's "that's what we've always used for like functions of, whenever you're expressing t as a function of V it's like… like B is a function of A it's like A(B)" as evidence that she assimilated the boat task to a scheme for "function of" instead of a "solve for" scheme, though she clearly knew (because she had stated) it was a "solve for" problem. We consider this an example of horizontal décalage because she had indicated that she understood why the answer to the previous GDP problem was correct, identified the boat problem as similar to the GDP problem because they were both "solve for" tasks, but did not use her "solve for" scheme. That is, the boat problem context was somehow different enough for there to be a time delay (decalage) in terms of Amy's scheme use. On the next task, Amy correctly solved the problem and stated that while she thought "solves" meant to get a number, she was seeing in the problems that these answers were not numbers. Amy described the answers in these tasks as "rearrang[ing]" the equations. We take those statements as evidence that her "solve for" scheme was being modified from her engagement in the tasks. Like Dylan, the mechanism in equilibration was comparing. Amy compared the gas mileage task to the previous problems:

> *Amy*: [Gas mileage tasks] [types m = d/g] Given like these previous problems I'm starting to pick up that what I think would be the answer, like my original first thought, is not the answer.
> *Int.*: You said this is like, so so far these haven't been what you thought it would be.
> *Amy*: Yes.
> *Int.*: So you still want that m to be some number, like that's your initial instinct?
> *Amy*: Yes.
> *Int.*: But, you said based on the past couple problems, you feel like that's not the way to go on these.
> *Amy*: Yes. I guess like looking at like "solve", like I said "solves" to me means get a number… but because of the previous problems and like the previous explanations via the textbooks, the answers the textbooks have given, and also this thing down here, it's very obvious that like you're not getting a number. This [highlights GDP problem] is technically the same format that this [highlights b in current problem] is asking this in [the "solve for…"], solve the given equation for a variable, I put a number in here, it was not correct, and then it gave me basically this original formula, rearranged, and then the same down here, this original formula, rearranged… that's how I'm getting this, it's just this original formula that's given to you, essentially rearranged… Like solving the given equation for a variable means, I guess in like the simplest form of like terms, like rearranging the original formula to put t on, like by itself, and then all the other ones would be rearranged to equal t.

In the ant task, Amy answered the question correctly on the first answer. When asked how she got the answer, Amy said, "rearranging this [given] formula." We took this as more evidence that she had accommodated her "solve for" scheme.

Discussion, Significance, and Implications for Instruction and Research

A limitation of this work is that it only analyzed data from five students completing somewhat procedural problems. However, these preliminary results support extant classroom practices (a legitimate way research can inform practice). For example, Dylan and Amy learned when “solving for” a variable, the answer can be a formula. This is an example of student learning via online homework, which supports other evidence of online homework as efficacious (e.g. Dorko, 2021, 2020; Ellis et al., 2015). Another implication is that instructors should consider providing students with explicit opportunities to compare problems to identify underlying mathematical structure, if they are not already doing so. This was the mental activity that supported Dylan and Amy’s success on the task series in question. Specifically, it seems important to provide students with multiple tasks that explore the same mathematical idea, followed by an explicit reflection task engaging students in identifying the underlying similarity.

The results here extend prior work about student learning from similar examples in homework (Dorko, 2021). While that work identified one way of learning as students following steps of similar problems, this study found another way students learn from homework is by identifying a series of tasks as similar, then comparing across them to identify an underlying mathematical structure or meaning. That students here learned a new meaning for “solve for…” is also a novel finding because it gives empirical evidence of conceptual learning from homework, somewhat contrasting other findings that students’ learning from online homework is primarily procedural in nature and that students often miss conceptual insights the professor intended they learn (Dorko, 2019; Dorko, Cook, & DeHoyos, 2023). A limitation of the paper is that the tasks were somewhat procedural practice problems. Future research could focus on homework problems that are intended to extend what students might have seen in class.

Various features within the online homework platform supported students’ learning. For example, Amy opened the e-book and made use of the solutions it provided to homework problems. While printed textbooks tend to have solutions to some exercises in the back of the book, the e-book in this course had direct links from some problems to their solutions. The homework platform’s immediate feedback supported Dylan’s reasoning because he knew something had “worked” in a previous problem that the platform marked as correct, and this allowed him to compare subsequent tasks to correct solutions. This finding provides more support for findings that students employ immediate feedback from online homework platforms to guide their work (Dorko, 2020; Kontorovich & Locke, 2022).

The results have implications for how we define and measure learning in classroom and research settings. The evidence shows that learning occurs over time, implying a need for classroom practices that support this. This could include helping students form growth mindsets so they come to see struggling as a necessary and acceptable part of the learning process. Another way to acknowledge learning as happening over time could be extant practices such as mastery-based grading (e.g., allowing a final exam grade to replace a lower preliminary exam score). In terms of research, the horizontal décalage supports designs that examine student thinking about a mathematical idea in multiple contexts because a student might struggle with the idea in one context but not another. This is likely of particular importance in a study that employs a single task-based interview with multiple students, because one question about a given mathematical idea could yield an incomplete picture about the student’s understanding of the idea.

References

Dorko, A. (2021). How students use the ‘see similar example’ feature in online mathematics homework. *The Journal of Mathematical Behavior, 63.* doi: 10.1016/j.jmathb2021.100894

Dorko, A. (2020). Red X's and green checks: A model of how students engage with online homework. *The International Journal of Research in Undergraduate Mathematics Education. 6*(3), 446-474.
Dorko, A. (2019). Professors' intentions and student learning in an online homework assignment. In (Eds.) A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro, *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 172-179). Oklahoma City, OK.
Dorko, A., Cook, J.P., & DeHoyos, I. (2023). Learning from lecture and homework: The case for studying intersections of milieu. *Investigations in Mathematics Learning, 15*(4). pp. 311-330. doi: 10.1080/19477503.2023.2253023
Aichele, D.B., Francisco, C., Utley, J., & Wescoatt, B. (2011). Computer-aided college algebra: Learning components that students find beneficial. *MathAMATYC Educator, 2*(2), 12-19.
Bissel, A.N. (2012). Architecture and impact of an open, online, remixable, and multimedia-rich algebra 1 course. *Journal of Asynchronous Learning Networks, 16*(5), pp. 49-59.
Crauder, B., Evans, B., & Noell, A. (2018). *Functions and change: A modeling approach to college algebra* (6th edition). Cengage.
Ellis, J., Hanson, K., Nunez, G., & Rasmussen, C. (2015). Beyond plug and chug: An analysis of Calculus I homework. *International Journal of Research in Undergraduate Mathematics Education, 1*(2), 268-287.
Erickson, A. (2020). Every mathematics class is online: Students' use of internet resources for self-directed learning. In (Eds.) S. Karunakaran, Z. Reed., & A. Higgins, *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 168-175). Boston, MA.
Izsak, A. (2005). "You have to count the squares": applying knowledge in pieces to learning rectangular area. *The Journal of the Learning Sciences*, *14*(3), 361-403.
Kanwal, S. (2020). Exploring affordances of an online environment: A case study of electronics engineering undergraduate students' activity in mathematics. *The International Journal of Research in Undergraduate Mathematics Education, 6*(1), pp. 42-64.
Kontorovich, I., & Locke, K. (2022). The area enclosed by a function is not always the definite integral: Relearning through collaborative transitioning within a learning-support module. *Digital Experiences in Mathematics Education, 9,* pp. 255-282.
Krause, A., & Putnamn, R. (2016). Online calculus homework: The student experience. In (Eds.) T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown, *Proceedings of the 19th Annual Conference on. Research in Undergraduate Mathematics Education,* pp. 266-280.
Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics, 52,* pp. 29-55.
Montangero, J., & Maurice-Naville, D. (1997). *Piaget or the Advance of Knowledge.* (Cornu-Wells, Trans.). Lawrence Erlbaum Associates.
Von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Routledge.
Wagner, J. F. (2006). Transfer in pieces. *Cognition and instruction*, *24*(1), 1-71.
Weinberg, A., Corey, D.L., Tallman, M., Jones, S.R., & Martin, J. (2022). Observing intellectual need and its relationship with undergraduate students' learning of calculus. *International Journal of Research in Undergraduate Mathematics Education.* DOI: 10.1007/s40753-022-00192-x

Metaphors in Discrete Math: Not All Sameness is the Same

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While sameness is a theme that appears throughout mathematics courses, limited work has examined how multiple types of sameness are understood within the same course. In this paper, we examine survey responses from 49 discrete mathematics students who characterized how they would explain equivalence relations, numerical congruence, and graph isomorphism to a child. Results include language conveying notions of sameness in response to all three prompts but variations in how the sameness was framed for the different concepts. Implications include the need for more work characterizing nuanced differences in students' understandings of similar concepts within specific courses.

Keywords: discrete mathematics, isomorphism, congruence, equivalence relation, sameness

A variety of concepts conveying a notion of sameness have stipulated definitions throughout mathematics and serve purposes that suit their contexts (Rupnow et al., 2022). For instance, isomorphism in abstract algebra and homeomorphism in topology serve similar purposes in conveying structural sameness (in the form of maintaining how elements interact under the defined operations in isomorphism and maintaining continuity in topology). Nevertheless, their definitions differ in order to provide rigor with respect to the relevant objects and to attend to properties that are important to their contexts. However, while some research has examined students' understandings of types of sameness across different courses (e.g., Rupnow et al, 2023), limited work has examined how students attend to multiple types of sameness within the same course context. Thus, we seek to address two research questions:

1. How are discrete mathematics students' characterizations of equivalence relations, congruence, and isomorphism similar?
2. How are discrete mathematics students' characterizations of equivalence relations, congruence, and isomorphism different?

Background Literature

Historically, limited work in mathematics education has attended to sameness as a conceptual area of study; however, notions of sameness have received more attention recently. Students' notions of sameness may impact their understanding of mathematical concepts in ways researchers do not expect, such as claiming multiplication and division are the same binary operation because the same numerical value can be obtained from related operations (Melhuish & Czocher, 2020) or not claiming that having the same graphs indicates having the same functions (Mirin, 2018; Mirin & Zazkis, 2020). However, mathematicians also treat sameness as an understandable concept, including characterizing relative levels of sameness conveyed by concepts (Rupnow & Sassman, 2022). Mathematicians have also highlighted multiple concepts conveying sameness across mathematics, including graph isomorphism, numerical equivalence, and equivalence relations (Rupnow et al., 2022). Much like sameness, equivalence has received more attention of late, including mathematicians' evolving notions of equivalence (Asghari, 2019), characterizations of students' productive interpretations of equivalence (Cook et al., 2021; Cook et al., 2022), and conceptualizations of substitution equivalence (e.g., Wladis et al., 2022).

Moreover, many types of sameness, including equality and congruence, play central roles in the mathematics curriculum. Extensive research on students' understanding of equality has highlighted the value of a relational understanding of equality, in which students understand both sides of the equal sign to have the same value, as this often assists students in understanding algebraic manipulations (e.g., Alibali et al., 2007; Jones et al., 2012; Knuth et al., 2006). Some research has connected students' understanding of sameness to geometric congruence (e.g., Rahim & Olson, 1998; Zazkis & Leron, 1991) or group isomorphism (e.g., Rupnow et al., 2022). Nevertheless, no extant research appears to exist on students' conceptions of equivalence relations, numerical congruence, or graph isomorphism in the context of discrete mathematics.

Theoretical Perspective

Conceptual metaphors is a theoretical perspective intended to provide insight into how individuals' thinking is structured, using their language choices to guide that interpretation (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 1997). Cross-domain conceptual mappings connect the cognitive structure of a target concept (e.g., congruence, isomorphism) to the more developed thoughts in source domains (e.g., same properties, classification mechanism). For instance, "An equivalence relation is a classification mechanism" is a conceptual metaphor that gives information about a target domain (equivalence relation) by relating it to a source domain that is already understood in some way (a classification mechanism). We acknowledge that this theoretical perspective imposes the researchers' views on our participants' statements, but believe this lens permits insight into ways of reasoning about our target concepts.

Prior work using conceptual metaphors include examinations of mathematicians' views of isomorphism and homomorphism in abstract algebra (Rupnow, 2021; Rupnow & Randazzo, 2022; Rupnow & Sassman, 2022). Other work has examined undergraduate students' understandings of bases and linear transformations in linear algebra (Zandieh et al., 2017; Adiredja & Zandieh, 2020) and of isomorphism and homomorphism in abstract algebra (Melhuish et al., 2020; Rupnow, 2017). Here we aim to extend the use of conceptual metaphors to other concepts conveying a type of sameness (Rupnow et al., 2022) in discrete mathematics. In particular, we build upon the metaphors identified in Rupnow (2021) and the metaphor clusters identified in Rupnow and Randazzo (2022).

Methods

Data was collected from surveys sent to the Fall 2022 and Spring 2023 sections of the first author's discrete mathematics course. 21 students completed the survey in Fall 2022, and 28 students completed the survey in Spring 2023. We did not observe major differences between the sections and thus combine our reporting into one dataset of 49 students' responses. Students in this course were mostly computer science majors and included freshmen through seniors. Course topics included sets, equivalence relations, functions, modular arithmetic, combinatorics, graph theory, and coding theory. The course textbook was Dossey et al. (2005). Because the instructor's research interests include understandings of sameness in mathematics, equivalence relations, congruence, and isomorphism were all highlighted as types of sameness when those concepts were discussed in class. The surveys were given the last week of classes in each semester, after students had taken in-term exams assessing equivalence relations, numerical congruence, and graph isomorphism. Initial questions on the survey asked about what it means to be the same in math and in discrete math specifically. However, the analysis here focuses on responses to three questions from the middle of the survey:

1. How would you describe the concept of an equivalence relation to a ten-year-old?

2. How would you describe the concept of numerical congruence to a ten-year-old?
3. How would you describe the concept of isomorphic/isomorphism to a ten-year-old?

Data was analyzed by two researchers who coded independently, then discussed and came to consensus for each response. While students were prompted to provide explanations about concepts that could be understood by a child, which itself encourages providing comparative explanations, we (the researchers) sought to classify the underlying meanings conveyed by these explanations according to our interpretations of the participants' responses. Thus, we used the metaphors for isomorphism (in abstract algebra) highlighted in Rupnow (2021) as an initial codebook but added new metaphors when necessary for our data set. This process aligns with codebook thematic analysis (Braun et al., 2019), in which existing codes are used as a basis for coding but new codes are permitted to be added to adequately capture the nuances in data.

Results

We organize the results of our coding into three distinct subsections separated by topic. The first subsection characterizes metaphors utilized to describe the concept of equivalence relation, the second examines metaphors used to describe congruence, and the third depicts metaphors describing isomorphism. In Table 1, we provide an initial overview of the frequencies of each metaphor divided by concept. Metaphors are organized by cluster, with some belonging to sameness-based clusters such as sameness (*generic sameness*, *same properties*, *same but looks different*, *classification mechanism*, *same remainder/leftovers*), sameness/mapping (*renaming/relabeling*, *matching*), and sameness/formal definition (*structure-preservation*) clusters. Other metaphors belong to the mapping cluster (*generic mapping/relation*, *morphing/transformation*, *invertible*), the formal definition cluster (*literal formal definition*, *computation*), or made *no attempt*/*unclear*. Within each cluster, metaphors are arranged by frequency. It should be noted that while 49 responses were examined, the sum of any single column may be greater than 49 due to some students having more than one code per response.

Table 1. Frequencies of metaphors by concept out of 49 total responses.

Metaphor Cluster	Metaphor Code	Equivalence Relation	Congruence	Isomorphism
Sameness	Generic sameness	15	6	9
	Same properties	2	2	19
	Same but looks different	3	3	19
	Same remainder/leftovers	0	20	0
	Classification mechanism	6	0	0
Sameness/ mapping	Matching	0	0	4
	Renaming/relabeling	0	0	1
Sameness/formal definition	Structure-preservation	0	0	2
Mapping	Generic mapping/relation	4	0	1
	Invertible	1	0	4
	Morphing/transformation	0	0	2
Formal definition	Literal formal definition	20	0	0
	Computation	0	8	0
No attempt/ unclear	Unclear/no attempt	6	13	5

Equivalence Relation

The first metaphors we will examine relate to describing the concept of an equivalence relation. For reference, the definition of equivalence relation given in the course textbook was: "A relation on S that is reflexive, symmetric, and transitive is called an equivalence relation" (Dossey et al., 2005, p. 50) and that "A relation R on a set S may have any of the following special properties:

(a) If for each $x \in S$, $x\ R\ x$ is true, then R is called reflexive.
(b) If $y\ R\ x$ is true whenever $x\ R\ y$ is true, then R is called symmetric.
(c) If $x\ R\ z$ is true whenever $x\ R\ y$ and $y\ R\ z$ are both true, then R is called transitive." (Dossey et al., 2005, p. 49)

Overall, the most common metaphor was *literal formal definition*, followed by *generic sameness* and *classification mechanism*. Many responses invoked a mathematically precise definition when asked to describe equivalence relation to a child. The majority of these responses specifically discussed the necessity of being reflexive, symmetric, and transitive. Some achieved this through mathematically rigorous descriptions, such as the following student who used ordered pair notation: "An equivalence relation is a set of ordered pairs that is reflexive (for each x in the ordered pair, there should be (x,x)), symmetric(if (x,y) is in the set, then (y,x) has to be), and transitive(if (x,y) and (y,z) are in the set, then (x,z) must be as well)." Other students described these concepts in a less formal way, utilizing everyday objects and concepts:

> Equivalence relation is a concept that finds the relation between two sets and is reflexive, transitive, symmetric… Imagine reflexive like standing Infront of a mirror, the picture on mirror is the same as a reflection. Symmetric is like taking a paper and drawing a circle and folding it in the middle, now it is a half circle, but when you open it, the symmetric of the half circle is shown making a full circle. Transitive is like saying, Peter and Tom are friends … Tom and Jerry are friends …which will mean Peter and Jerry can be friends.

This response emphasizes the underlying formal structure of an equivalence relation but utilizes visual descriptions, physical action, and common scenarios to convey the key properties involved instead of relying on formal vocabulary and notation.

Other responses characterized equivalence relations using *generic sameness* language. Most of these responses made statements about finding similarity across objects or groups of items: "An equivalence relation is a way of comparing things to see if they are the same in a certain way" or "an equivalence relation is where you can measure the 'sameness' between 2 sets." These metaphors emphasize commonality across elements or sets, while simultaneously conveying the possible variety in what it can mean to be the same.

Another metaphor that many respondents utilized to describe equivalence relations is that of *classification mechanism*. This particular metaphor is unique to equivalence relation, as it describes how elements can be sorted or binned to form distinct groups within an equivalence relation. Many of the responses described grouping everyday objects by some defining characteristic such as color or shape:

> An equivalence relation is a way to group things together based on certain rules. For example, imagine you have a bunch of toys that you want to group based on their color. If you have a blue toy, a red toy, and a green toy, you can group the blue toy with other blue toys, the red toy with other red toys, and the green toy with other green toys. This is an example of an equivalence relation.

Several responses also depicted equivalence relation using *generic relation*. This metaphor was unique to equivalence relation and conveys a relation between elements or groups, such as "A relationship between two or more things that follows some rules." Others provided an *unclear/no attempt* response, of which many simply stated that they were unsure of how to the explain the concept: "It took me a two whole lectures and a Youtube video to piece all of this together myself; tha[t] ten-year-old will leave that conversation more confused than they started."

Congruence

The second group of metaphors relate to congruence. The definition for congruence given in the textbook was: "Let m be an integer greater than 1. If x and y are integers, we say that x is congruent to y modulo m if $x - y$ is divisible by m. If x is congruent to y modulo m, we write $x \equiv y\ (mod\ m)$; otherwise, we write $x \not\equiv y\ (mod\ m)$. We call this relation on the set of integers congruence modulo m" (Dossey et al., 2005, p. 100). Congruence was most frequently described using *same remainder/leftovers*, followed by *computation* and *generic sameness*. Over 25% of respondents provided an unclear response or made no attempt to depict congruence; this was by far the concept with the most responses not attempting to answer the question.

Same remainder/leftovers was the most common metaphor. Many instances of this metaphor were contextless and directly defined congruence, such as "If one number and another number can both be divided by another third number, and they have the same left overs, they are congruent." Other responses introduced context, such as using candy, specific numbers, or cake as in the following: "let's say you have 2 cakes and divide each cake by the same number each cake, if the number of slices left in each cake is the same, then they are congruent." One respondent focused on the sameness of quotients rather than remainders, stating "If you divide both numbers by a single number, and get the same answer." This response was included in the *same remainder/leftovers* category because we felt they intended to describe what was the same numerically, albeit inaccurately.

Just as *same remainder/leftovers* was a metaphor unique to congruence, so too was *computation*. Respondents who conveyed congruence via *computation* often described the action of adding or subtracting a fixed number so as to maintain congruence. In some instances, this was done explicitly: "you take a number and then are able to either add or subtract by another number as many times as you want." Other cases were more implicit, as below:

> Let's say I have a bucket of apples which can only hold 12 apples; when I fill that bucket up, I have to go empty it somewhere (modulo 12). No matter how many apples I put in that "somewhere," I can't put more than 12 apples in my bucket, and I will never keep 12 apples in the bucket (the number is between 0 and 11). So, if I grab 38 apples, there will be 2 apples left in the bucket at the end.

In addition to metaphors unique to congruence, some responses also used *generic sameness* as a metaphor for congruence. Most of these responses discussed congruence in terms of some connection or flavor of sameness across numerical values: "Numerical congruence are the pairs of numbers that are the same as each other in the context of some dimension of numbers."

Isomorphism

The final concept we discuss is isomorphism. The textbook's definition for isomorphism was: "A graph G_1 is isomorphic to a graph G_2 when there is a one-to-one correspondence f between the vertices of G_1 and G_2 such that the vertices U and W are adjacent in G_1 if and only if the vertices $f(U)$ and $f(W)$ are adjacent in G_2. The function f is called an isomorphism of G_1 with G_2 (Dossey et al., 2005, p. 159). Both *same properties* and *same but looks different* were commonly used as a metaphor for isomorphism, as was *generic sameness*. It is notable that metaphors for isomorphism were more focused on appearance than our other two concepts. Isomorphism also had the greatest variety in metaphor codes.

Many of the metaphors discussed shared properties across groups of objects. As isomorphism in Discrete Math is centered on graphs, a large number of these metaphors described graphs with the same properties: "two graphs will need the same number of points to be there, it will also need the same points to be put connected, finally it will need the same number of lines." Other responses emphasized common properties across other 2D or 3D shapes: "imagine you have a shape made out of blue blocks, and another shape made out of red blocks. They look different, but they have the same number of blocks and the same angles between the blocks." This response seems to emphasize notions of geometrical congruence that need not be present in isomorphic graphs (angle measures) as well as notions that apply to graph isomorphism (same number of vertices).

Another commonly used metaphor for isomorphism was *generic sameness.* Focusing first on contexts regarding graphs, the majority of metaphors in this category emphasized the underlying sameness of two graphs: "isomorphic is being of identical or similar form, shape, or structure." One response pulled from a Chemistry perspective, discussing isomorphism as occurring "When two or more crystals have similar chemical compositions exist in the same crystalline form."

Still another metaphor characterizing isomorphism was same *but looks different*. These types of instances might be viewed as a variation on *generic sameness*, but placed emphasis on the idea that while isomorphic objects have the same underlying structure or properties, they may appear different from a visual standpoint. The context of these metaphors varied; as with *same properties*, many discussed graphs: "for two graphs to be isomorphic they are pretty much the same but with a different shape." Others emphasized the underlying action behind these surface differences through a variety of mediums, including play dough and stick figures:

> Let's say you have a stickman, the vertices are the stickman joints such as the knees, elbows, and shoulders, you can pose the stickman to have his arms up, be sitting down, or running, but he will always be isomorphic to the stickman that is just standing in place, because the arms and legs and head are all still connected to the same points.

While not as frequently utilized, there are several additional metaphors that were unique to isomorphism. The physical action of *morphing* or *transforming* from one shape to another was described as a way to demonstrate isomorphism, such as "if you have some play doh, even if you morph the play doh, you still have the same clump of play doh." Others used *matching* language, emphasizing that elements in one object can be directly matched to elements of another:

> For example, imagine you have two puzzles. One puzzle is a picture of a dog and the other puzzle is a picture of a cat. Even though the pictures are different, the puzzles might have the same number of pieces and the same shape for each piece. If you were to switch the pieces between the two puzzles, you could still put them together in the same way. This means the puzzles are isomorphic[.]

Still others described the *structure-preservation* of isomorphism: "an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping." Interestingly, both instances of this particular metaphor also included the metaphor *invertible*. There was also one mention of *renaming/relabeling* elements so as to demonstrate the concept of isomorphism, placing emphasis on the arbitrariness and interchangeability of labels in a graph: "If we have two graphs, and they look kinda similar, and we take the names of the points away, and can put them back to make the graphs looks the same, they are isomorphic."

Discussion

This study examined discrete mathematics students' explanations of equivalence relations, congruence, and isomorphism. By using the conceptual metaphor lens, we observed that many explanations focused on sameness, including metaphors from the sameness, sameness/mapping, and sameness/formal definition clusters. Some explanations also highlighted metaphors from the mapping and formal definition clusters or made *no attempt*/*unclear*. We recognize that students using sameness-infused language is not surprising, given the context. The students were asked about notions of sameness earlier in the survey, and their instructor highlighted each concept as a type of sameness both in class and in the class notes. What we do find interesting is that students highlighted different metaphors for the three concepts. Sameness was salient but conveyed through different common metaphors (or non-attempts) for each concept: *literal formal definition* and *generic sameness* for equivalence relations, *same remainder/leftovers* and *unclear/no attempt* for congruence, and *same properties* and *same but looks different* for isomorphism.

Then again, perhaps this variation in metaphors should not be surprising, since the instructional emphasis varied in each unit. In the first unit, where equivalence relations were introduced, a key purpose was to familiarize students with the use of mathematical definitions, so extensive time was spent on understanding and verifying whether the reflexive, symmetric, and transitive properties held for each relation. It is possible that this instructional choice obscured the notion of an equivalence relation conveying sameness or serving a classification purpose for some students. For congruence, many students did note the sameness of the remainders, but this also was the concept for which the most students struggled to give a clear description. Perhaps the computation-heavy applications in subsequent lessons (i.e., in the RSA Method) obscured the meaning of congruence itself. Finally, isomorphism had the most variety in metaphors. Notions of sameness came through clearly in students' language for isomorphism, but what was the same varied across lists of properties, a core sameness despite the difference in appearance, and being able to match aspects of different graphs. It is possible that the instructor's more extensive experience in graph theory led to clearer instructional experiences. However, more research is needed on the differential impact of the instructional context versus the concepts themselves.

Additionally, the underlying objects relevant for each concept vary widely in abstractness from almost anything (equivalence relations) to numbers (congruence) to graphs (isomorphism). While previous research has attended to the objects to which sameness-based concepts are applied (Rupnow et al., 2022; Rupnow & Sassman, 2022), this work raises new questions about the direction of the impact of objects and concepts. In particular, do the underlying objects (e.g., graphs) influence our conceptions of the concepts (e.g., isomorphism), the concepts influence our conceptions of the objects, or do the concepts and objects mutually impact each other? For students who later see isomorphism in other contexts (e.g., linear algebra), how might first seeing graph isomorphism impact how they interpret isomorphism of vector spaces?

Finally, the students were prompted to explain in a manner understandable to a 10-year-old. Admittedly, this is an unusual prompt, especially to students who do not intend to teach, but our goal was not to assess the age appropriateness of the explanations. Rather, we believe this prompt permitted a different window into students' understandings than just asking them to describe each concept, as it encouraged a distillation of key idea(s) each student took away. Moreover, it encouraged students to be creative, as displayed in the wide variety of contexts in which students chose to position their explanations (e.g., blocks, candies, Play Doh). Future work can further examine ways to build on the creative examples students highlighted while supporting students to integrate their intuitive understandings with those of the formal definition.

References

Adiredja, A. P., & Zandieh, M. (2020). The lived experience of linear algebra: A counter-story about women of color in mathematics. *Educational Studies in Mathematics, 104*, 239–260. https://doi.org/10.1007/s10649-020-09954-3

Alibali, M. W., Knuth, E. J. Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221–247. https://doi.org/10.1080/10986060701360902

Asghari, A. H. (2009). Experiencing equivalence but organizing order. *Educational Studies in Mathematics, 71*(3), 219–234.

Braun, V., Clarke, V., Hayfield, N., & Terry, G. (2019). Thematic analysis. In Liamputtong P. (Ed.) *Handbook of Research Methods in Health Social Sciences*. Springer: Singapore. https://doi.org/10.1007/978-981-10-5251-4_103

Cook, J. P., Dawkins, P., & Reed, Z. (2021). Explicating interpretations of equivalence in measurement contexts. *For the Learning of Mathematics, 41*(3), 36–41.

Cook, J. P., Reed, Z., & Lockwood, E. (2022). An initial framework for analyzing students' reasoning with equivalence across mathematical domains. *The Journal of Mathematical Behavior, 66,* 100935

Dossey, J. A., Otto, A. D., Spence, L. E., & Vanden Eynden, C. (2005). *Discrete Mathematics* (5th edition). Pearson.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology, 113*, 166-176. http://dx.doi.org/10.1016/j.jecp.2012.05.003

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: The University of Chicago Press.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Melhuish, K., & Czocher, J. A. (2020). Division is pretty much just multiplication. *For the Learning of Mathematics, 40*(2), 38–43.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60,* 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Mirin, A. (2018). Representational sameness and derivative. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 569–576), Greenville, SC.

Mirin, A., & Zazkis, D. (2020). Function sameness: Bringing coherence to implicit differentiation. *For the Learning of Mathematics, 40*(3), 14–19.

Rahim, M. H., & Olson, A. (1998). Qualitative patterns in plane geometry. *Journal of Mathematical Behavior, 17*(3), 373–389. https://doi.org/10.1016/S0364-0213(99)80069-X

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual*

Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education, (pp. 259–273), San Diego, CA.
Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*(2), 100867. https://doi.org/10.1016/j.jmathb.2021.100867
Rupnow, R., & Randazzo, B. (2022). Mathematicians' language for isomorphism and homomorphism. In *Proceedings of the 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 1099–1107). Nashville, TN. https://doi.org/10.51272/pmena.44.2022
Rupnow, R., Randazzo, B., Johnson, E., & Sassman, P. (2022). Sameness in mathematics: A unifying and dividing concept. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-022-00178-9
Rupnow, R., & Sassman, P. (2022). Sameness in algebra: Views of isomorphism and homomorphism. *Educational Studies in Mathematics, 111*(1), 109–126. https://doi.org/10.1007/s10649-022-10162-4
Rupnow, R., Uscanga, R., & Bergman, A. (2022). Connecting sameness in abstract algebra: The case of isomorphism and homomorphism. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1131–1136). Boston, MA.
Rupnow, R., Uscanga, R., & Bergman, A. (2023). Variation in students' thinking about sameness and comparisons to mathematicians. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 27–34). Omaha, NE.
Wladis, C., Sencindiver, B., Offenholley, K., Jaffe, E., & Taton, J. (2022). Conceptualizing mathematical transformation as substitution equivalence: The critical role of student definitions. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 893–900). Boston, MA.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0
Zazkis, R., & Leron, U. (1991). Capturing congruence with a turtle. *Educational Studies in Mathematics, 22*(3), 285–295.

Affordances of Semantic and Syntactic Proof Approaches for Isomorphism and Homomorphism

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Group isomorphism and homomorphism are core concepts in abstract algebra, but limited work has directly examined student conceptions of homomorphism or how students approach finding particular mappings. Based on interviews with two students, we contrast one student who used predominantly syntactic proof approaches for homomorphism with one who used semantic approaches, noting that both experienced success in finding homomorphisms in most cases.

Keywords: isomorphism, homomorphism, semantic, syntactic

Introduction

Students' understandings of multiple areas of abstract algebra content have received attention, including binary operation (e.g., Melhuish & Fagan, 2018; Melhuish, Ellis, & Hicks, 2020), inverses (e.g., Serbin, 2023; Wasserman, 2017), and functions (e.g., Melhuish, Lew, et al., 2020; Uscanga & Cook, 2022). We build on the attention to functions, in particular the focus on student understanding of isomorphism (e.g., Weber & Alcock, 2004; Melhuish, 2018) and homomorphism (e.g., Melhuish, Lew, et al., 2020). Specifically, we examine students' reasoning when creating isomorphisms and homomorphisms from a proof production perspective. We believe this work holds two contributions to the literature. First, this paper extends our currently limited knowledge of how students understand homomorphism and solve problems related to homomorphism. Second, this paper provides insight into the relationship between students' success while problem solving and their proof approaches.

Literature Review

Isomorphism was one of the first abstract algebra topics studied in the undergraduate mathematics education community (e.g., Dubinsky et al., 1994; Leron et al., 1995). This early work characterized properties students attended to when assessing whether groups may be isomorphic, such as checking cardinality (Dubinsky et al., 1994), determining alignment between the orders of elements, and determining whether groups were abelian (Leron et al., 1995).

Other work, whose primary aim was examinations of proof approaches, contrasted undergraduates' and doctoral students' (Weber, 2001; Weber & Alcock, 2004) or undergraduates' and algebraists' (Weber & Alcock, 2004) approaches to determining whether groups were isomorphic. Weber (2001) also examined approaches to proofs focused on homomorphism. Of note, doctoral students and algebraists relied more on semantic reasoning, wherein their knowledge of properties of groups and holistic understanding of theorems were key aspects of their reasoning and seemed to be important aspects in graduate students' success. In contrast, undergraduates tended to reason syntactically, wherein they largely focused on symbol pushing with the formal definition, and experienced more struggles in proving more difficult statements. Moreover, semantic reasoning was linked to Skemp's (1976) relational understanding construct, which emphasizes understanding both how and why notions work as well as what to do to solve a problem, whereas syntactic reasoning was linked to instrumental understanding, which only requires understanding what to do (i.e., procedural knowledge). More recently, in Melhuish's (2018) replication study, undergraduates at times successfully used properties to reason about whether groups were isomorphic but seemed to check these properties procedurally, suggesting the need for further research on connections between semantic reasoning, relational understanding, syntactic reasoning, and instrumental understanding.

However, explicit discussion of students' understandings of homomorphism has been more limited (e.g., Weber, 2001). Moreover, research on approaches to homomorphism have focused on students' and instructors' language (e.g., Melhuish, Lew, et al., 2020; Rupnow, 2021; Rupnow & Randazzo, 2022), rather than centering their approaches to proof, suggesting a need for a concentrated look at proof strategies with homomorphism.

Methods and Theoretical Perspective

This paper is grounded in a wider study examining instruction in introductory (junior level) abstract algebra at a land-grant university in the Mid-Atlantic United States. Here we focus on two participants, Bryce and Blake, who were in a section taught with a mixture of lecture and activity days. The students' names are gender-neutral pseudonyms, and they/them pronouns are used throughout. Participants each engaged in a semi-structured interview (Fylan, 2005) lasting roughly one hour. The interview occurred after instruction and an exam on group isomorphism and homomorphism had been completed. Interviews were audio and video recorded and any written work was collected. The interview questions analyzed here focused on finding mappings between specific groups: $\mathbb{Z}_3$ and $\mathbb{Z}_6$, $\mathbb{Z}$ and $2\mathbb{Z}$, and two groups given in Cayley tables (one representing $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the other $\mathbb{Z}_4$). If students were able to determine one mapping, they were often asked to determine if another isomorphism or homomorphism could be formed.

The interviews and videos were transcribed and coded in alignment with the definitions of semantic and syntactic proof production given in Weber and Alcock (2004). Semantic proof production is characterized by the use of "instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences" that are drawn whereas syntactic proof production is "written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way" or could be viewed as symbol-pushing (Weber & Alcock, 2004, p. 210). For instance, an isomorphism proof in which a student uses properties and theorems, especially by drawing on their knowledge of the groups involved, would be a semantic proof, whereas a proof in which a student directly uses the formal definition (i.e., for group isomorphism, show bijectivity and the homomorphism property) would be a syntactic proof. From this analysis, we address the following research question: What patterns exist between proof production approaches and success in creating isomorphisms and homomorphisms?

Results

We present vignettes of two students who were generally successful at the tasks, but one approached homomorphism tasks syntactically (Bryce) and the other semantically (Blake).

Bryce

$\mathbb{Z}_3$ and $\mathbb{Z}_6$. Bryce quickly said these groups were not isomorphic because they are different sizes: "Okay, so can't be an isomorphism because this has order three and this is order six, so elements in $\mathbb{Z}_3$ would have to map to more than one element in $\mathbb{Z}_6$, so it's not one-to-one." While Bryce attempted to explain this property, we note their explanation seems to conflate function and one-to-one definitions and does not focus on the groups' structures.

Regarding the homomorphism task, Bryce realized that nontrivial homomorphisms should exist because three divides six. They were initially unsuccessful in finding homomorphisms when trying to map individual elements. However, they found success when they created formulas to describe maps rather than mapping elements, beginning with the map $f([x]_3) = [2x]_6$ from $\mathbb{Z}_3$ to $\mathbb{Z}_6$. They started checking the homomorphism property element-by-element, but then realized that they could do this process generally (see Figure 1). Later, they also generalized their map to include "any multiple of 2… it could be like 2, 4, 6, any of those."

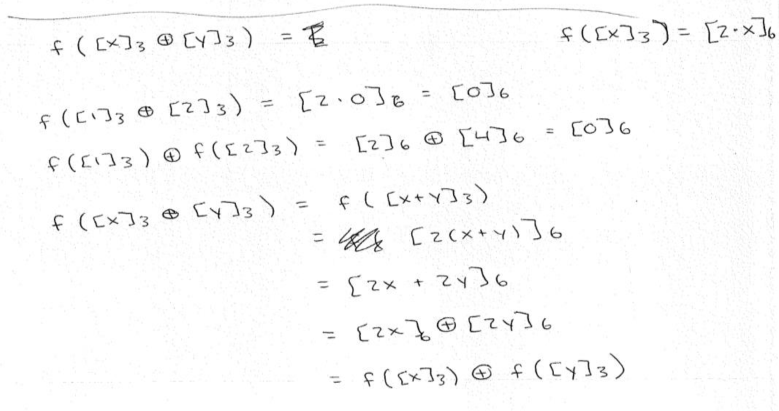

Figure 1. Bryce's Homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_6$.

Going from $\mathbb{Z}_6$ to $\mathbb{Z}_3$, Bryce was successful in finding all homomorphisms. In particular, they mentioned the identity map, the map $f([x]_6) = [2x]_3$ ("2 times x… or any multiple of that") and the trivial homomorphism, which they stumbled upon by considering odd multiples.

B: I don't know if an odd number would work too. So if you were just multiplying by 3 instead… I guess if you multiply by 3 here, that would always work cuz that would always go to 0 mod 3, and any multiple of 3 would reduce down to 0 mod 3. So any multiple of 3 would work…

I: So you're mapping everything to 0.

B: Mmhmm…that would be a group homomorphism. Because if you had 1 mod 6 and 4 mod 6, they were both being mapped to 0. Then 0 mod 3 plus 0 mod 3 is still 0.

Note Bryce was very successful in the syntactic approach of applying the definition to construct homomorphisms but did not appear to be thinking about the groups structurally. Interestingly, they did not explicitly acknowledge that the "multiple of 3" map they constructed is the trivial homomorphism and continued to check the homomorphism property for this map.

$\mathbb{Z}$ and $2\mathbb{Z}$. Bryce noticed that both of these groups are infinite but did not use that to say they were isomorphic. Instead, they set up a specific map and appeared to mentally verify that particular pairs of elements would satisfy the homomorphism property while creating their map.

> So they both have an infinite order to them, and I want to say that [2ℤ] has half the order that the other one does, but when you have infinite orders in your groups that doesn't really matter, so I feel like it could be an isomorphism. I'm not sure how I would set it up, but…[rereading] you'd always have 0 mapped to 0, and then 1 would have to map to 2, cuz that's the next one, and 2 would have to go to 4, so if you just had $f(x)$ where x is in $\mathbb{Z}$, x is an integer, if you map that to $2x$, then it's always going to map to something in $2\mathbb{Z}$. And then 3 would go to 6, so I think that would work for an isomorphism.

Going the other direction, Bryce used $f(x) = x/2$, noticing that this was the inverse of the map previously found. They again noted that "they're infinite, they have the 'same order' [air quotes and sarcastic voice] sort of" and used this to say, "that would make it…one-to-one". Here, they grudgingly accepted the standard mathematical claim that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality and used this to say the map must be one-to-one, while being uncomfortable with this notion of infinity. This is consistent with their tendency to focus on syntactic verification of the existence of these maps, rather than claiming structural sameness between the groups based on theorems.

For the homomorphism task, going from $\mathbb{Z}$ to $2\mathbb{Z}$, Bryce identified maps involving linear functions with even coefficients, noting that these will not be isomorphisms since they are not one-to-one. When asked to consider whether there would be other homomorphisms, Bryce said:

> I don't think so, because… if you did like instead of $2x$ this was $4x$, then that leaves out 2 and -2, -6, anything that's not divisible by 4, but it has to be one-to-one. So I guess there's no other isomorphisms, but there could be a homomorphism. If you had any other even integer here, then it would be a homomorphism, but it would fail to be one-to-one.

Going from $2\mathbb{Z}$ to $\mathbb{Z}$, aside from the previously discovered isomorphism, they did not find any homomorphisms because "you can't do any other even integer down here [in the denominator] because if you had like 4, then you'd run into problems where you're not mapping to an integer." They seemed to be focused on only using maps that were similar to their isomorphism.

Cayley Tables. For the Cayley table task, Bryce made a comment about the tables needing to "match up" and initially said the groups could not be isomorphic because one group has all self-inverses, and the other does not; but they were reluctant to use the term "inverse" and focused on whether the results of particular computations align with the homomorphism definition.

> So to show an isomorphism, usually the Cayley tables at least match up in a way…I mean they look similar. So everything in this one [the first group] is symmetrical around this diagonal. Everything over here is symmetrical around this diagonal. I don't think it would be an isomorphism because over here every element operated on itself gives back the same thing, so that would be your, I don't want to say inverse… And [in the second group] every element operated on itself doesn't always give you the same element, so I don't think they're isomorphic.

Later, they became unsure of this conclusion and proceeded to count the orders of the elements to check. During this process, they noted that one group was cyclic and the other was not.

> You have an order 4 over here and order 2 over here. So they can't match up. Or you don't have any order 4 over here, so that's why they're not an isomorphism. You could also say this one is cyclic, cuz c would generate the whole group, and this one's not cyclic.

For the homomorphism aspect, they knew that "ultimately it has to respect the group operation" and wrote $f(x * y) = f(x) + f(y)$, but otherwise were unsure how to proceed. In this case, Bryce's syntactic approach focused on definitions and maps was not productive, as they were unsure how to visualize setting up homomorphisms using the Cayley tables.

Blake

$\mathbb{Z}_3$ ***and*** $\mathbb{Z}_6$. Blake immediately said that these groups were not isomorphic without giving an explanation. We note this was the fourth of the original set of tasks the students were given, and Blake had previously used the justification of different sizes to explain a lack of isomorphism (for $\mathbb{Z}_5$ to $5\mathbb{Z}$, and $\mathbb{Z}_5$ to $\mathbb{Z}_6$), so we assume they were using the same reasoning here.

Unlike Bryce, Blake found homomorphisms here by considering subgroups: "There is a homomorphism because $\mathbb{Z}$ mod 6 has a three-element subgroup because three divides six so we can just set each element of $\mathbb{Z}$ mod 3 to those elements" [referencing the 0 to 0, 1 to 2, and 2 to 4 mapping written on their paper]. When asked why this would be a homomorphism, Blake struggled to explain, saying "I guess I would have to do it out fully in order to fully convince myself… I would just try to check the $f(ab) = f(a) * f(b)$". However, they never checked this, suggesting that they valued the use of theorems asserting existence more than specific mappings. They also noted the trivial homomorphism would be a homomorphism and believed they had found all homomorphisms "because of the divisors of six and three and matching up the subgroups and stuff." Later in the interview, they were prompted to consider the 0 to 0, 1 to 4, 2 to 2 mapping and reexamine their original map. When asked whether this new map was a homomorphism, Blake again responded with an answer highlighting the structure of the groups.

B: Because 0, 2, and 4 form a subgroup within ℤ mod 6. [Long pause] Because this is isomorphic to ℤ mod 3, so yeah. This looks like it would work. If you add these two, you get the identity back… It looks like I had something along that train of thought before…

I: So is it surprising or not surprising that both of those seem to form homomorphisms?

B: [Blake thinking] I guess it's not surprising because 0, 2, and 4 both form a three-element subgroup of ℤ mod 6 and because three is prime, each of those elements is going to be generators of the whole group, so they're kinda gonna behave similarly. So it's not surprising that you might be able to just switch them around like in this case.

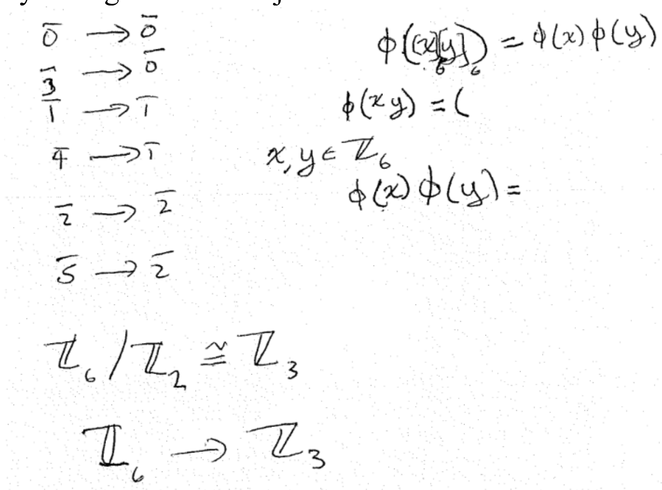

Figure 2. Blake's Homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_3$.

Going from $\mathbb{Z}_6$ to $\mathbb{Z}_3$, Blake again used a semantic approach. They found a map, but when pressed to show it was a homomorphism, they were hesitant and struggled to do this syntactically (see Figure 2). Instead, they invoked the Fundamental Homomorphism Theorem (FHT).

B: There's the trivial homomorphism, there should be another homomorphism as well. Which is…taking the two elements which are order three and mapping it to 1 bar…[writing and thinking] Yeah. I think there is one where you can find two elements in ℤ mod 6 and send them each to one element in ℤ mod 3, but I haven't fully figured out which ones those are, but I think the multiple of 3 will go to 0 bar, and it can work out.

I: Ok. If you can complete that mapping, I would appreciate it.

B: Ok. [writing and thinking] Alright. [turns paper] I just sent all of the elements that are congruent mod 3 to what they're congruent to.

I: Ok. And how do you know that's a homomorphism?

B: Because I know, because of the way modular addition is defined. It should work out pretty nicely in the proof.

I: Just for the sake of doing that once, what would that proof look like?

B: … For some reason, I'm having trouble with this way. But if you go back to the Fundamental Homomorphism Theorem… the kernel of this transformation is isomorphic to ℤ mod 2, and ℤ mod 6 mod ℤ mod 2 is isomorphic to ℤ mod 3 by that theorem. Which means that the function ℤ mod 6 to ℤ mod 3, which has this [air-circling 0 and 3] as its kernel, gives a homomorphism.

The FHT as stated in Blake's textbook is: "Let $f: G \to H$ be a homomorphism of G onto H. If K is the kernel of f, then $H \cong G/K$" (Pinter, 2010, p.151). This theorem is generally used to show the existence of isomorphisms and cannot be used to show a particular map is a homomorphism. Nevertheless, Blake seems to have used the requirements of the theorem to convince themselves that the map they had created would be a homomorphism and provide a kernel as described.

ℤ and 2ℤ. Blake said that these groups are isomorphic because they are both infinite and cyclic, remembering that these properties are sufficient to guarantee isomorphic groups: "I've seen the proof that any two cyclic groups of an infinite size are isomorphic, and then I can just find a generator in each one, and then feel reasonably confident in saying they're isomorphic." This is likely the reasoning they used to set up the map where "every integer k will get sent to $2k$" or "you could do negative $2k$". When pressed, they successfully used the formal definition to check that their map satisfied the homomorphism property (see Figure 3).

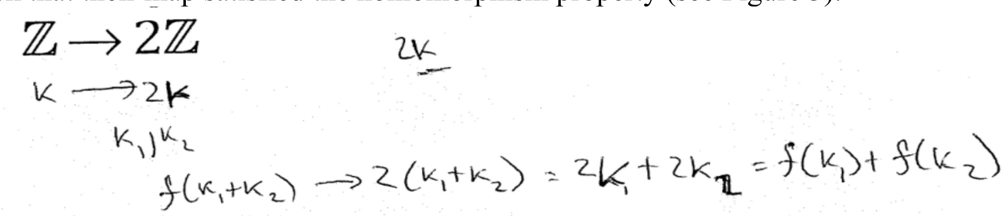

Figure 3. Blake's Isomorphism from ℤ to 2ℤ.

Regarding homomorphisms from ℤ to 2ℤ, they mentioned "you could probably send them to multiples of 4, or multiples of any other even [number]." Going the other way, they suggested the relation is invertible: "Isomorphism is symmetric so it is indeed isomorphic, I think. You can just take the inverse of that function." They also realized the homomorphisms in this direction will be similar: "So you can construct a function like $2x$ again. You just get multiples of 4 which is again infinite and cyclic, but that should be a homomorphism." Again, they used semantic approaches in setting up homomorphisms, as they mention the image will be infinite and cyclic.

Cayley Tables. Blake, like Bryce, claimed that these groups were not isomorphic because the orders of elements did not align. However, they were different in pointing out the isomorphism class of these groups, stating that one is $\mathbb{Z}_4$ and the other is likely the Klein four-group. From here, they began to look at subgroups in order to find homomorphisms and were generally successful going from $\mathbb{Z}_4$ to the Klein four-group. Blake again attempted to use the FHT for justification, but they were not confident in their answer.

B: So, one of these has to be ℤ mod 4 I think.

I: Why is that?

B: …I think that there are only two groups of order 4. I'm not completely certain about that but I think I'm gonna go with that… [writing and thinking] You might be able to map ℤ mod 4 onto a two element subgroup of this other one which might be the Klein, or something. I don't really remember. But you could probably choose just a two element subgroup. a seems to be the identity here and each element squared returns a, so yeah. You could just choose like a, b or a, c or a, d and map them [gesture] onto one another.

I: Ok. So how do you know that would work?

B: [Blake writing and thinking]…I'm not sure it's going to work. Because I was going to say something about the Fundamental Homomorphism Theorem but then I remembered that this function then has to be onto. And if we're doing anything like that, we're gonna not hit every element in this group, so I'm not completely confident that'll work.

As we can see, Blake used the structure of the groups, focusing on order of elements and subgroups, to determine a homomorphism, despite their lack of confidence in the mapping.

Discussion

We focused on two students' approaches to three tasks. For the isomorphism tasks, both students correctly applied their knowledge of the properties of the groups to identify whether or not the groups were isomorphic. For $\mathbb{Z}_3$ and $\mathbb{Z}_6$, this amounted to noticing the groups had different sizes, though Bryce expanded to say that this means you cannot create a one-to-one map. For $\mathbb{Z}$ and $2\mathbb{Z}$, both students knew these groups are infinite and were able to set up at least one specific isomorphism in each direction. However, Blake used the fact that both groups are also cyclic to confirm that they should be isomorphic. For the Cayley table task, both students referenced the orders of the elements to justify a lack of isomorphism. However, the proof approaches on the homomorphism tasks were more different. In particular, both semantic and syntactic proof approaches were successfully employed to draw conclusions, and both were linked to struggles at times. Bryce used syntactic approaches to the homomorphism tasks and was both successful and comfortable working with generalized algebraic notation in the definition. In contrast, Blake mostly used semantic approaches to the homomorphism tasks, wherein they focused on the properties and structures of the relevant groups.

While syntactic reasoning generally aligned with instrumental understanding, like Melhuish (2018), we note that semantic approaches did not always seem to suggest a relational understanding. In particular, both students used semantic approaches to finding isomorphisms, but Bryce appeared to be checking memorized properties, more in alignment with instrumental understanding, while Blake seemed perpetually attuned to structure, suggesting a relational understanding. For the homomorphism tasks, Bryce mostly used syntactic approaches in a manner demonstrating instrumental understanding. In contrast, Blake behaved more like the graduate students of Weber (2001) in their semantic approaches, displaying a relational understanding, wherein they intently considered the structures of groups.

We note that the task of finding homomorphisms between groups presented in a Cayley table was particularly difficult. This was also the only task featuring a non-cyclic group. Moreover, while their course had engaged with isomorphism tasks where the groups were presented in Cayley tables, they had not done such a homomorphism task. Rather, this required the students to really understand what homomorphisms do structurally and use properties of the groups since they could not write an algebraic formula for the maps. Thus, instrumental understanding was less likely to be sufficient for this task, and this manifested in only Blake making headway on the homomorphism part of the task. Future research could examine how students' reasoning as portrayed through language relates to their proof approaches and success on tasks, thus binding together Weber and Alcock (2004), Melhuish (2018), and this work with that of Melhuish, Lew, et al., (2020), Rupnow (2021), and Rupnow and Randazzo (2022).

References

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Fylan, F. (2005). Semi structured interviewing. In *A handbook of research methods for clinical and health psychology* (pp. 65–78).

Judson, T. W. (2019). *Abstract Algebra: Theory and Applications.* Retrieved from http://abstract.ups.edu/download/aata-20190710-print.pdf

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9–38.

Melhuish, K. & Fagan, J. (2018). Connecting the Group Theory Concept Assessment to core concepts at the secondary level. In Nicholas H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 19–45). Cham, Switzerland: Springer Nature. https://doi.org/10.1007/978-3-319-99214-3_2

Melhuish, K., Ellis, B., & Hicks, M. (2020). Group theory students' perceptions of binary operation. *Educational Studies in Mathematics, 103*, 63–81. https://doi.org/10.1007/s10649-019-09925-3.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60*, 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Pinter, C. C. (2010). *A Book of Abstract Algebra* (2nd ed.). Dover Publications, Inc.: Mineola, New York.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62,* 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Rupnow, R., & Randazzo, B. (2022). Mathematicians' language for isomorphism and homomorphism. In *Proceedings of the 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 1099–1107). Nashville, TN. https://doi.org/10.51272/pmena.44.2022

Serbin, K. S. (2023). Prospective teachers' unified understandings of the structure of identities. *The Journal of Mathematical Behavior*, *70*, 101066.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20–26.

Uscanga, R., & Cook, J. P. (2022). Analyzing the Structure of the Non-examples in the Instructional Example Space for Function in Abstract Algebra. *International Journal of Research in Undergraduate Mathematics Education*, 1-27. https://doi.org/10.1007/s40753-022-00166-z

Wasserman, N. H. (2017). Making sense of abstract algebra: Exploring secondary teachers' understandings of inverse functions in relation to its group structure. *Mathematical Thinking and Learning*, *19*(3), 181-201.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209–234.

A Teaching Experiment for U-Substitution Based on Quantitative Reasoning

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Prior work has created approaches to calculus based on crucial quantitative reasoning. For integration, however, the major topic of u-substitution has generally not been fully detailed in these paradigms. This paper presents a study where students were taught u-substitution from a quantitative perspective based on a three-part quantitative structure: differential quantity, integrand quantity, bounds quantity. The students reasoned about the quantitative conversions in flexible ways, and used various quantitative relationship types in their reasoning. However, reasoning about the differential quantity was difficult, and a new type of "collapse" metaphor was identified. By the end, the students had all developed a good quantitative basis for u-sub.

Keywords: calculus, integrals, u-substitution, quantitative reasoning, teaching experiment

Research has strongly established quantities-based meanings for calculus concepts as essential for robust student understanding and for productive usage outside of math classes (Byerley, 2019; Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). For integrals, this means moving away from the purely "area under a curve" meaning in favor of a "sum of small bits" meaning (Ely, 2017; Jones, 2015; Sealey, 2006). Several studies have examined teaching integrals and students' reasoning and modeling with integrals through this meaning (Bajracharya et al., 2023; Blomhøj & Kjeldsen, 2007; Chhetri & Oehrtman, 2015; Dray & Manogue, 2023; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). However, the major "u-substitution" method has largely been absent from explicit research examination in these paradigms, despite its importance as the first in a long line of substitution techniques, and as a means in the sciences and engineering for converting between quantitative expressions (e.g., see Koretsky, 2012). Treating u-substitution quantitatively can allow students both to effectively *use* it and to understand the mechanisms for *why* it works. Very recent theoretical work has examined u-substitution through a quantitative paradigm (Jones & Fonbuena, 2024), and here we contribute by extending this theoretical work to an empirical examination of students learning u-substitution in this way. Our research questions were: (1) In teaching u-substitution through this paradigm, what quantitative reasoning did students use? (2) What understandings did they construct for the individual parts of u-substitution and for the overall u-substitution process?

Brief Review of Closely Related Literature

Research work has put forward a quantitative structure for definite integrals called *adding up pieces* (AUP) (Jones, 2013; Jones & Ely, 2023). AUP is comprised of three parts: *partition*, *target quantity*, and *sum* (see also Dray & Manogue, 2023; Sealey, 2014; Von Korff & Rebello, 2012). To explain these, consider the example of a solar panel generating energy (in kJ), where power is defined as the rate of energy generation (kJ/hr) over time (hr): $E = P \cdot t$. If power is constant, Oehrtman and Simmons (2023) call this a *basic model*. If power varies over time, this basic product cannot be used to find the total energy. Rather, time must be *partitioned* into essentially infinitesimally small pieces, denoted by the differential *dt* (Ely, 2020). In other words, *dt* is a very tiny Δt (Ely & Ellis, 2018). While differentials can be rigorously formalized by limits or hyperreals (see Jones & Ely, 2023), the more loosely-defined "essentially infinitesimal" is quite common across the sciences and is the crucial idea for quantitative reasoning, even if no specific threshold is defined for passing from "Δ" to "*d*" (Amos & Heckler, 2015; Hu & Rebello, 2013; Pina & Loverude, 2019; Thompson & Dreyfus, 2016; Von Korff & Rebello, 2014).

With infinitesimal partition pieces, power can be considered essentially constant over each *dt* piece, so the basic model can be applied to find the tiny bit of energy produced within each *dt*: $dE = P \cdot dt$. Here, energy is called the *target quantity*, and the application of the basic model to an infinitesimal piece is called the *local model* (Oehrtman & Simmons, 2023). While integrals in AUP can utilize any quantitative relationship (Simmons & Oehrtman, 2017), in this paper we primarily focus only on quantitative relationships defined through a product: $Q3 = Q1 \cdot Q2$.

With a conceptualization of tiny bits of the target quantity in each infinitesimal partition piece, *sum* refers to the literal summation of these target quantity bits to obtain the total target quantity amount. AUP follows Leibniz's convention (Katz, 2009) in using the integral symbol, ∫, as a literal "sum" symbol. For example, $\int_{t1}^{t2} P(t)dt$ is the sum of infinitesimal bits of energy (each produced by $P \cdot dt$) across the *dt* pieces between $t = t_1$ and $t = t_2$, yielding the total energy.

Finally, to introduce some terminology, because time is an input for the varying power, $P(t)$, as well as for the target quantity, $E(P, t)$, we call it the main *input quantity* (Jones & Fonbuena, 2024). Because power becomes the integrand in the integral, we call it the *integrand quantity*.

Theoretical Perspective: Quantitative Reasoning Applied to U-Substitution

Thompson (Smith & Thompson, 2007; Thompson, 1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure" (1990, p. 12). Quantitative structures are based on quantitative relationships, defined as "the conception of three quantities, [any] two of which determine the third" (1990, p. 12) (Figure 1a, below). In particular, a product-based definite integral captures a quantitative relationship between an *input quantity*, an *integrand quantity*, and a *target quantity* (Figure 1b). Our previous work applied quantitative reasoning to u-substitution (Jones & Fonbuena, 2024), which we recap here through the solar panel example.

Suppose the sun rises at 6am ($t = 0$) and reaches its zenith at 12pm ($t = 6$). A horizontal solar panel would have less power in the early morning, but power would increase to a maximum at noon. We use $P(t) = 250 \sin\left(\frac{\pi}{12}t\right)$ kJ/hr as a reasonable model for this situation. Based on the explanations in the previous section, this means that for each *dt* interval, the local model would be $dE = 250 \sin\left(\frac{\pi}{12}t\right) \cdot dt$, and the total energy would be given by $E = \int_0^6 250 \sin\left(\frac{\pi}{12}t\right) dt$.

To now move to u-substitution, note that while energy can be tracked over time, the power is mostly due to the sun's *angle* with the panel, so it might make sense to switch from time as the input to the sun's *angle*. This is exactly the quantitative question we claim u-substitution is built on: What if we want to track the target quantity in terms of a new input quantity (e.g., θ), rather than the original input (e.g., t)? To start the conversion, note there is a relationship from $t \to \theta \to P$, or $P(\theta(t))$, which is called *nested multivariation (MV)* (Jones, 2022). In the "quantity triangle" (Figure 1c), this nested MV is located along the edge of the triangle between the original input and integrand quantity. Time and angle are related by $\theta = \frac{\pi}{12}t$, meaning we can

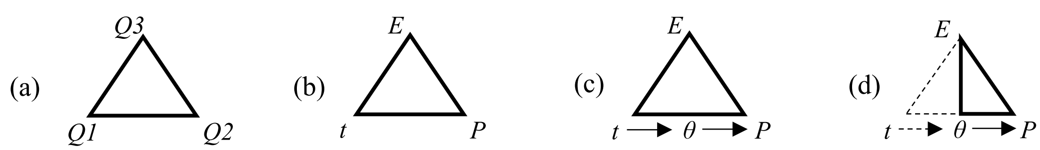


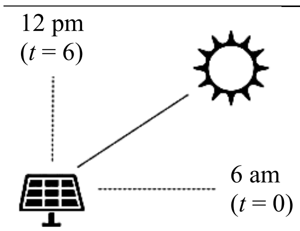
Figure 1. A basic quantitative relationship (a); and u-substitution as the transformation of the original integral relationship (b), through the nested MV between t, θ, and P (c), to a new integral relationship (d)

convert $P(\theta(t)) = 250\sin(\theta(t))$ to simply $P(\theta) = 250\sin(\theta)$. This relation also means that a change in angle is always $\pi/12$ as big as the change in time, so $d\theta = \frac{\pi}{12}dt$, or $dt = \frac{12}{\pi}d\theta$. The local model can now be fully described in terms of θ: $dE = 250\sin(\theta) \cdot \frac{12}{\pi}d\theta$. Finally, the sum over *dt* pieces from $0 \le t \le 6$ hr is equivalent to a sum over *dθ* pieces from $0 \le \theta \le \frac{\pi}{2}$ rad. Taken together, $E = \int_{t=0}^{t=6} 250\sin\left(\frac{\pi}{12}t\right)dt$ transforms to $E = \int_{\theta=0}^{\theta=\pi/2} 250\sin(\theta)\frac{12}{\pi}d\theta$. Our previous theoretical work defined u-substitution quantitatively as *transforming the original relationship by shifting the vertex from t to θ, to form a new relationship* (Figure 1d).

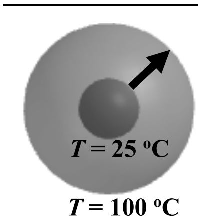
How is this transformation enacted? Our previous work also detailed a three-part process to do so: *differential, integrand,* and *bounds* (Jones & Fonbuena, 2024). These parts match the *partition, target quantity,* and *sum* structure of AUP. *Differential* means converting infinitesimal pieces of the original input to infinitesimal pieces of the new input (e.g., $dt \to [conversion]d\theta$). *Integrand* means using the nested MV structure to redefine the integrand quantity in terms of the new input (e.g., $P(t) \to P(\theta)$), allowing the target quantity to be determined by the new input (e.g., $E(P,t) \to E(P,\theta)$). *Bounds* means re-describing the sum over the original input pieces to a sum over the new input pieces (e.g., a sum over $0 \le t \le 6$ hrs to a sum over $0 \le \theta \le \pi/2$).

Methods

For this study, we designed an experimental lesson consisting of two u-substitution tasks using real-world contexts (Figure 2). The first author was the researcher-instructor for the lesson. In Task 1, the goal was for students to construct an integral and then enact the u-substitution conversions outlined in our theoretical perspective section. In Task 2, the goal was similarly for students to identify the three-part transformation of (a) *bounds*: converting between the sum over $25 \le T \le 100$ and the sum over $10 \le r \le 15$, (b) *integrand*: converting between $4\pi(\sqrt{T}+5)^2$ and $4\pi r^2$, and (c) *differential*: using the relation $dr = \frac{1}{2\sqrt{T}}dT$ to convert between *dr* and *dT*.



Task 1: A solar panel generates energy in joules, J. Power is the rate at which energy is produced (joules per hour, J/hr). Note that power depends on the location of the sun and is maximized when the sun is directly overhead (12 pm). We'll use $P(t) = 250\sin\left(\frac{\pi}{12}t\right)$ as a reasonable model, because it is zero at 6 am (*t* = 0), increases from $0 \le t \le 6$ hours, and is maximized at 12 pm (*t* = 6).
(a) Construct an integral (in time) for the total energy generated over these 6 hours.
(b) The sun's angle is what really matters for power. What if we tracked the total energy produced with respect to angle instead? How does that change the integral?



Task 2: The gas inside a spherical balloon is heated from the inside from *T* = 25 ºC to *T* = 100 ºC. As the gas expands, the sphere grows [a dynamic visual is provided]. Suppose the radius (in cm) is given by $r(T) = \sqrt{T} + 5$. Note that at a given moment, for a tiny change in temperature, the tiny bit of increased volume is essentially the product of the current surface area and the corresponding tiny change in the radius.
(a) Consider the following two integrals, and explain how they each find the total change in volume: $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}}dT$ and $\int_{r=10}^{r=15} 4\pi r^2\, dr$.
(b) Explain how these two integrals are equivalent to each other.

Figure 2. Abbreviated versions of the two interview lesson tasks

To ensure participating students had the needed background, we recruited from a first-semester calculus class (*not* taught by the authors) that incorporated quantitative meanings for calculus concepts. We recruited when the unit on integration had started, so they would have basic knowledge about definite integrals through an AUP perspective. We recruited six students and paired them into three groups of two students each: Luke and Liam, Matt and Mark, and Ruby and Rafael. We analyzed the data as follows. First, for quantitative reasoning, we coded excerpts according to whether they referred to (a) the quantities, (b) the units, (c) relationships between quantities, and (d) operations done on quantities. We also coded for contraindications of quantitative reasoning, such symbol manipulations without attention to quantities. Second, to connect their reasoning to u-substitution, we coded excerpts according to the three parts in our breakdown: (a) *differentials*, (b) *integrand*, and (c) *bounds*. We looked for connections between their quantitative reasoning and each of the parts. Finally, the researcher-instructor asked the students at times to recap what they understood about u-substitution and what it meant. We summarized these general understandings for each student about u-substitution.

Results

Research Question #1: Students' Use of Quantity and Quantitative Relationships

We first describe how our students used quantities and quantitative relationships. One important aspect was what Redish (2005) called loading meaning onto symbols, which means the students directly interpreted the symbols and expressions as referents to the quantities in a productive way. To illustrate, consider these excerpts from Matt and Ruby:

> *Matt*: This [points to sin(π/12 t)], is showing how much joules you're getting per hour. And this [points to dt] is showing small time, in hours.

> *Ruby*: That equation [i.e., expression, points to $\frac{1}{2\sqrt{T}}\,dT$] is essentially equal to bits of radius.

Next, recall that Thompson's (1990) definition of quantitative relationships is of three quantities, where any two determine the third. Our students certainly used this type of relationship across the interviews for time-power-energy, angle-power-energy, temperature-area-volume, and radius-area-volume. However, our students also exhibited different types of relationships outside of this classic definition. The first type was "two-quantity relationships."

> *Rafael* (Task 2, after converting to radius): Well, it's just a much more direct relationship between radius and volume. The relationship between temperature and volume isn't as direct.

> *Liam* (Task 1, in converting the differential): We also want the relationship between dt and $d\theta$.

In fact, these two-quantity covariational relationships (Carlson et al., 2002) were a key part of the students' work. While each context held four quantities (e.g., time, angle, power, energy), decomposing into two-quantity relationships helped them track the conversions (cf. Jones, 2022).

Also, some three-quantity relationships the students imagined did not have Thompson's triangle format (Figure 1). Rather, some relationships followed the nested MV structure instead.

> *Liam*: At that time… The angle of the sun at a certain time, and like the amount of energy produced.

> *Mark*: Our tiny change in temperature… how this temperature affects the radius at that specific temperature and all that stuff we're adding up… we're adding up different volumes.

Liam described a relationship of *time* → *angle* → *energy*. Mark described a relationship of *temperature* → *radius* → *volume*. Importantly, though, note that the students' descriptions go from *original input* → *new input* → ***target*** *quantity*, skipping the integrand quantity. This result conflicts with the study's intended *input* → *new input* → ***integrand*** *quantity*, suggesting the lesson did not properly scaffold this particular nested MV relationship entailed in u-substitution.

Research Question #2: Students' Quantitative Conversions

We now explain the results pertinent to each of the three parts in the u-substitution structure: *differential, integrand,* and *bounds*. First, the *integrand* and *bounds* were mostly unproblematic for the students. The students readily converted the summation bounds from one quantity to the other and demonstrated understanding the quantitative equivalence. For example Matt explained,

> *Matt*: Your degree Celsius is going to start at 25 degrees and going to 100 degrees… It's kind of the same thing as radius… We're both [i.e., temperature and radius] changing at the same time. It's just you calling this one in terms of temperature instead of radius, even though they both happen at the same time… It's the same thing as showing when your radius is changing from 10 to 15.

The students also readily converted the integrand quantity from a dependence on the original input to the new input. These conversions, though, tended to be based on symbolic appearance at first, and were only later justified through quantities. Yet, we view this symbolic work as fine, so long as the quantitative relationships were understood, as demonstrated in these excerpts:

> *Mark*: Basically, because this *is* [emphasis in original] the angle right here [points to $\frac{\pi}{12}t$]. Like, that's the sine of theta [$\sin(\theta)$], it's our power output and this [again points to $\frac{\pi}{12}t$] would be what θ would be equal to.

> *Luke*: [We have] the relationship between *r* and *T* [i.e., the function $r(T) = \sqrt{T} + 5$], so we know that, like, for whatever *T* we put in there [i.e., into "$4\pi\left(\sqrt{T}+5\right)^2$"], it's gonna come up to the right *r* to get the same result as this one [points to the integral with "$4\pi r^2$"].

While *differential* and *integrand* were fairly unproblematic, one of the more important results was that the differential quantitative conversion was particularly cognitively demanding. The students spent significant time there and needed to carefully reason about the relationships. The first big difficulty was assuming that the differentials, such as *dt* and *dθ*, could be directly substituted. In fact, all three groups initially did so in Task 1 (Figure 3). It seemed based on the idea that all "infinitesimals" must be equivalent, since they are all so small. This idea is similar to Oehrtman's (2009) "collapse" metaphor, though it is different in that instead of collapsing to "nothing", these infinitesimals seem to all collapse to a single "infinitesimal sameness."

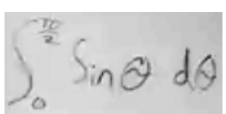
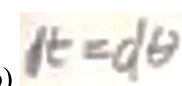
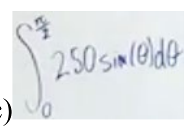
(a)  (b)  (c) 

Figure 3. Evidence from each group of initially equating dt and dθ

To help, the interviewer pointed out this implicit assumption and asked if the differentials were actually equal. The students reasoned about the quantities in order to realize they were not.

> *Luke*: I think they'd be proportional, but I don't know if they'd be exactly the same… One hour is equal to *π*/12, but in my mind they don't mean the same thing [*dt* and *dθ*]. Because *π*/12 is a ratio.

The interviewer followed up by asking if 1/100th of an hour was equal to 1/100th of a radian. This helped the students see and resolve the issue. This excerpt shows how one group began to do so:

> *Rafael*: *dt* is infinitesimal, getter smaller over a period of 6 hours. Where that's, where *dθ* is getting infinitely smaller over the range of *π*/2 [i.e., 0 to *π*/2].

Based on this reasoning, Rafael set up a proportion: $\frac{dt}{6} = \frac{d\theta}{\pi/2}$. He rearranged this relation to produce $d\theta = \pi/12\ \ dt$. We note that once this issue was resolved in Task 1, it did not reappear in Task 2. The students referred back to Task 1, now explaining that the differentials might be different sizes, and that some comparative relation between them needed to be identified.

A separate issue with differentials was interpreting "*d*" to just mean "derivative." This is sensible because the derivative was the other major concept they had experienced that used the differential notation "*d*", as in *dy/dx*. When interpreting *d* in this way, the students operated computationally and lost sight of the quantities. For example, in Task 2, Ruby interpreted "*dr*" as the derivative of the function $r(T) = \sqrt{T} + 5$, leading her to assert that $dr = \frac{1}{2\sqrt{T}}$. She then created the differential-less integral: $\int_{25}^{100} 4\pi\left(\sqrt{T} + 5\right)^2 \left(\frac{\sqrt{T}}{2T}\right)$. When asked about it, Rafael realized there was essentially no "local model" in the integral because of the absence of a *dT*.

Rafael: This isn't, this is just for one temperature. It's not 'as we're getting smaller.'

This realization helped the students see the need to think of *dr* and *dT* as representing *quantities*.

Of course, using derivatives was not always a problem and was sometimes necessary and useful. The students needed to derive $r(T) = \sqrt{T} + 5$ to find the relationship $dr = 1/(2\sqrt{T})\, dT$. Even so, they still tended to lose sight of the quantities within that computation and had to reconceptualize the quantitative relationship in the differential equation after the fact.

Our last result to highlight here is that the students successfully enacted the three parts of u-substitution (*differential, integrand, bounds*) in different orders, suggesting that these do not need to be enacted in a specific order. Table 1 shows the orders the groups used for each task.

Table 1. The three-part u-substitution conversion enacted in different orders by the groups across the tasks

	Task 1: Solar panel	Task 2: Sphere volume
Group 1	*bounds → integrand → differential*	*integrand → bounds → differential*
Group 2	*integrand → differential → bounds*	*integrand → differential → bounds*
Group 3	*integrand → bounds → differential*	*integrand → differential → bounds*

Research Question #2: General Understanding of U-Substitution

By the end of the interview lesson, all of the students had developed an understanding of the three-part structure of u-substitution: *differential, integrand, bounds*. They expressed the need for the two integrals to be equivalent in all aspects, as seen in Mark's representative explanation.

Mark: If we're going to go from one relationship to the other, like radius to temperature, or from, you know, time to degrees or I guess radians, we had to change our function, bounds, and our differential. We had to make sure that they were still equivalent statements.

The students also understood the need to find a relationship between the original input and the new input. As stated before, though, the nested MV relationship from these to the integrand quantity was somewhat blurred and pointed to a needed revision in the lessons.

We thus claim that this lesson was generally successful in helping the students (a) to reason about the quantitative nature of converting from one input quantity to another, and (b) to develop a personal understanding of the *differential-integrand-bounds* structure for u-substitution. However, we describe here one other unanticipated issue that cropped up during the lesson that points to a needed revision. While the students were reasoning quantitatively and understanding the u-substitution process, a couple of students (Ruby in particular) occasionally expressed confusion about what the point of doing these conversions would be.

Ruby: Well, I feel like, I mean, there must be a reason that we're doing it [converting from time to angle], but it seems like a lot of work, um, for not really anything.

After the completed process, though, Ruby had some recognition why it might be useful.

> *Ruby*: I think that this [seeing the two completed integrals] solves my problem, that this [the integral in time] is harder than this [the integral in angle]… Over here [the integral in angle] we have, we have the numbers plugged in in a way that will solve that for us [i.e., an easier antiderivative].

While Ruby began to the see the computational benefits of u-substitution, the lesson still did not help the students see possible scientific motivation for converting between quantities. For example, in Task 2, radius information might be easier to come by than temperature information.

Discussion

The major contribution of this study is to help induct u-substitution within a quantitative reasoning paradigm for calculus. This study extended prior theoretical work (Jones & Fonbuena, 2024) to an empirical analysis of students learning u-substitution in this way, which provided several insights into a quantitative u-substitution. First, the students used several distinct quantitative relationships, including classic three-quantity relationships (Smith & Thompson, 2007; Thompson, 1990), two-quantity covariational relationships (Carlson et al., 2002), and nested MV relationships (Jones, 2022). However, interestingly, the identified nested relationship was not the intended *original input → new input → integrand quantity*. This result suggests the need to scaffold this particular MV relationship to make it more salient in the lesson.

In the three-part structure of u-substitution (*differential, integrand, bounds*), the quantitative treatment of the differential was the most cognitively demanding. Crucially, we observed a new type of "collapse" distinct from Oehrtman's (2009). Rather than collapsing to "nothing," the differentials seemed to collapse, in the students' minds, to a single infinitesimal "sameness." It took careful work to consider the relationship between changes in the original input versus the new input to identify that even at very small scales there is a conversion factor between an infinitesimal bit in one quantity versus an infinitesimal bit of another. This strengthens the case that gaining the ability to reason about increasingly small increments of a quantity in calculus is important for quantitative reasoning (Ellis et al., 2020; Ely & Ellis, 2018).

Another implication from our study for teaching u-substitution is the need to ensure a motivating "why" for doing it (Harel, 2013). A couple students were unsure of the benefit of converting from time to angle, or from temperature to radius. One motivation, of course is the computationally practical "easier antiderivative" used in typical pure-math approaches. But we believe it is important to also weave in the scientific reasons for converting between quantities at times (e.g., see Koretsky, 2012). For example, it is possible that certain data is more available, or easier to obtain, which would definitely be the case for radius versus temperature in Task 2.

Overall, we believe this study to be a proof-of-concept of teaching u-substitution through a quantitative paradigm and teaching it in a way that illuminates the quantitative structures inherent in it. We believe that doing so can help students construct the *differential-integrand-bounds* structure that would allow flexibility in using u-substitution. However, beyond simply enacting the substitutions, our students also came to understand the mechanisms for why the substitutions work, and could reason about the conversions in context. Further, we believe that understanding u-substitution in this way can provide a strong basis for understanding future substitution techniques, such as trigonometric substitution or multivariable change of variables. For example, trigonometric substitution uses a different nested MV relationship where the new input ($\sin(\theta)$) *precedes* the original input (x), as in $\sin(\theta) \to x \to f(x)$. Thus, we claim that understanding these relationships can help students in future math learning and science usage.

References

Amos, N. R., & Heckler, A. F. (2015). Student understanding of differentials in introductory physics. In A. Churukian, D. Jones, & L. Ding (Eds.), *Proceedings of the 2015 Physics Education Research Conference* (pp. 35-38). College Park, MD: American Association of Physics Teachers. https://doi.org/0.1119/perc.2015.pr.004

Bajracharya, R. R., Sealey, V. L., & Thompson, J. R. (2023). Student understanding of the sign of negative definite integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00202-y

Blomhøj, M., & Kjeldsen, T. H. (2007). Learning the integral concept through mathematical modelling. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the 5th congress of the European Society for Research in Mathematics Education* (pp. 2070-2079). ERME.

Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *The Journal of Mathematical Behavior, 55*, article #100694. https://doi.org/10.1016/j.jmathb.2019.03.001

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378. https://doi.org/10.2307/4149958

Chhetri, K., & Oehrtman, M. (2015). The equation has particles! How calculus students construct definite integral models. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th annual Conference on Research in Undergraduate Mathematics Education* (pp. 418-424). SIGMAA on RUME.

Dray, T., & Manogue, C. A. (2023). Vector line integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00206-8

Ellis, A. B., Ely, R., Singleton, B., & Tasova, H. I. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics, 104*, 87-103. https://doi.org/10.1007/s10649-020-09951-6

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152-167. https://doi.org/10.1016/j.jmathb.2017.10.002

Ely, R. (2020). Teaching calculus with infinitesimals and differentials. *ZDM--The International Journal on Mathematics Education, 53*(3), 591-604. https://doi.org/10.1007/s11858-020-01194-2

Ely, R., & Ellis, A. B. (2018). Scaling-continuous variation: A productive foundation for calculus reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, & M. Wawro (Eds.), *Proceedings of the 21st annual Conference on Research in Undergraduate Mathematics Education* (pp. 1180-1188). SIGMAA on RUME.

Harel, G. (2013). Intellectual need. In K. R. Leatham (Ed.), *Vital directions for mathematics education research*. Springer.

Hu, D., & Rebello, N. S. (2013). Understanding student use of differentials in physics integration problems. *Physical Review Special Topics: Physics Education Research, 9*(2), article #020108. https://doi.org/10.1103/PhysRevSTPER.9.020108

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R. (2022). Multivariation and students' multivariational reasoning. *Journal of Mathematical Behavior, 67*, article #100991. https://doi.org/10.1016/j.jmathb.2022.100991

Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00203-x

Jones, S. R., & Fonbuena, L. (2024). U-substitution through quantitative reasoning: A conceptual analysis. In *Proceedings of the 26th annual Conference on Research in Undergraduate Mathematics Education (forthcoming)*. SIGMAA on RUME.

Katz, V. J. (2009). *A history of mathematics* (3rd ed.). Pearson Education.

Koretsky, M. D. (2012). *Engineering and Chemical Thermodynamics* (2nd ed.). Wiley.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426. https://doi.org/10.5951/jresematheduc.40.4.0396

Oehrtman, M., & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00209-5

Pina, A., & Loverude, M. E. (2019). Presentation of integrals in introductory physics textbooks. In Y. Cao, S. Wolf, & M. B. Bennett (Eds.), *2019 PERC Proceedings* (pp. 446-451). AAPT.

Redish, E. F. (2005). Problem solving and the use of math in physics courses. In *Proceedings of World view on physics education in 2005: Focusing on change*.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 46-53). PMENA.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*(1), 230-245. https://doi.org/10.1016/j.jmathb.2013.12.002

Simmons, C., & Oehrtman, M. (2017). Beyond the product structure for definite integrals. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th annual Conference on Research in Undergraduate Mathematics Education* (pp. 912-919). SIGMAA on RUME.

Smith, J. P., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). Erlbaum.

Stevens, B. N., & Jones, S. R. (2023). Learning integrals based on adding up pieces across a unit on integration. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00204-w

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. San Diego State University.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274. https://doi.org/10.1007/BF01273664
Thompson, P. W., & Dreyfus, T. (2016). A coherent approach to the Fundamental Theorem of Calculus using differentials. In R. Göller, R. Biehler, & R. Hochsmuth (Eds.), *Proceedings of the Conference on Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 355-359). KHDM.
Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics: Physics Education Research, 8*(1), article #010125. https://doi.org/10.1103/PhysRevSTPER.8.010125
Von Korff, J., & Rebello, N. S. (2014). Distinguishing between "change" and "amount" infinitesimals in first-semester calculus-based physics. *American Journal of Physics, 82*(7), 695-705. https://doi.org/10.1119/1.4875175

Physics Students' Challenges Coordinating Multiplication with the Definite Integral Concept

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Integration plays a significant role in applied problems, including those set in a wide range of physical phenomena. Past research has shown that students often experience difficulties applying calculus knowledge to physics, engineering, and other subjects. More specifically, recent research has identified students' difficulties constructing the "product layer," $f(x) \cdot dx$*, as a core part of the definite integral. In this study, we extend research on students' understandings of the definite integral by focusing on how they reason about multiplication with quantities, physical units, and the construction of the "product layer."*

Keywords: Calculus, Integration, Product layer, Knowledge-in-Pieces

Introduction and Literature Review

Research on the teaching and learning of calculus has focused primarily on limits and differentiation (e.g., Larsen et al., 2017; Rasmussen et al., 2014), but there is an emerging body of literature on integration. Among other things, research on integration has observed that the "product layer," $f(x) \cdot dx$, can pose particular challenges for students. This same literature has not examined closely how students coordinate reasoning about multiplication with quantities and reasoning about integrals. This gap limits our ability to help students develop more robust understandings of the "product layer" and of the definite integral.

We emphasize three themes in the literature on definite integrals. First, students often interpret the definite integral as "area under the curve," as a "derivative/antiderivative" relationship, or as "adding up pieces/multiplicatively-based summation" (e.g., Jones, 2015a, 2015b; Ely & Jones, 2023; Jones & Ely, 2023). These researchers have argued that first two interpretations provide limited help to students when making sense of the definite integral in problem situations and that the third interpretation is more helpful when applying the integral to problems in physics, engineering, and other fields.

Second, students often have difficulties interpreting the "product layer," $f(x) \cdot dx$, as expressing multiplication (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). Sealey (2014) presented the most direct framework for characterizing student understanding of Riemann sums and definite integrals. This framework consists of five layers: orienting, product, summation, limit, function. She reported that conceptualizing the "product layer" was the most complex part of constructing definite integrals to model problem situations. She concluded that students had trouble not with calculations but with "understanding what is being multiplied together and what quantity is produced from that multiplication" (p. 240).

Third, students often have difficulties with modeling quantities such as distance and work with rectangular areas (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011; Christensen & Thompson, 2010). Thompson and Silverman (2008) pointed out that for students to perceive the area under a curve as representing a quantity other than area (e.g., velocity, force), it is important for students to consider the quantity being accumulated as a sum of infinitesimal elements formed multiplicatively. As a result, they proposed a model which emphasizes two "layers" of integration: the multiplicative layer when the bits are formed and the summation layer when the bits are accumulated.

In the present study, we examined how students construct the "product layer," which can be used to relate quantities and to build new quantities in the given problem situation. In addition, we focused on knowledge resources students used in the process of forming the "product layer." We asked the following two research questions:

1. *How do students use/construct the "product layer" to relate quantities and form new quantities in problem situations?*
2. *What knowledge resources do students apply when using/constructing the "product layer" to relate quantities and form new quantities in problem situations?*

Theoretical Framework

In this section we present the theoretical framework which best spans the results obtained in the study. It combines mathematical structure (as perceived by experts) with cognitive components evidenced by students.

Vergnaud (1983, 1988) analyzed mathematical structures for multiplication with quantities and distinguished two subtypes of multiplication situations. Each component in a product is associated with a *measure space* and the relation between the components is either an *isomorphism-of-measure-spaces* situation *(I-O-M)* or a *product-of-measure-spaces* situation *(P-O-M)*. The former means that the product structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). An example is the product structure which has the form $\frac{meters}{second} \cdot seconds$. The latter means the product structure consists of the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). An example is the product structure which has the form $area\ (cm^2) \cdot height\ (cm)$. These subcategorization of problem situations helps us delineate definite integral problems which possess two different product structures.

For the cognitive component, we draw from the knowledge-in-pieces epistemological perspective. The perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006). It has since been applied to various topics in mathematics, including whole-number multiplication (Izsák, 2005), functions (e.g., Izsák, 2004; Moschkovich, 1998), and integrals (Jones, 2013). From this perspective reasoning is supported by diverse, fine-grained knowledge resources and more novice knowledge evolves into more expert knowledge through processes such as the construction of knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some recourses and loosing connections among others. In the present study, we were particularly interested in knowledge resources students evidenced when reasoning about multiplication in the context of definite integrals and across both *I-O-M* and *P-O-M* situations.

Methods

We conducted three one-to-one, semi-structured, think-aloud interviews to assess how five students enrolled in the first-semester calculus-based physics course at the selective university reasoned about the integral concept in different problem situations, with particular attention to whether and how they constructed the product layer. Each interview lasted approximately 1 hour.

The *first* interview examined physics students' reasoning about multiplication in I-O-M and P-O-M problem situations. This interview consisted of three situations. None of the tasks mentioned integrals, but for each situation one could first multiply and then add to construct a total. We paid close attention to how students assigned units of measurement to the quantities and how they used the later to identify unknown quantities in each problem situation. The *second* interview examined physics students' reasoning when connecting definite integrals to I-O-M and P-O-M problem situations. In addition, we examined how students identified multiplicative structures and related them to graphical representations. The *third* interview intended to examine how students reasoned about the definite integral concept in I-O-M and P-O-M situations, but we were only able to pursue the I-O-M situations in the allotted time. Again, we paid close attention to how students reasoned about multiplicative structures and graphical representations as a part of the definite integral concept.

We recorded the interviews using two cameras, one to capture the student (body movements, hand gestures, etc.) and the interviewer and one to capture the student's written work. We collected all written work at the end of each interview. We used a computer to synchronize the two video recordings into a single file and transcribed the interviews verbatim.

We watched videos for all five students side-by-side with the transcripts, checking the transcripts for accuracy. All students had trouble at one point or another coordinating multiplication with the "product layer" and definite integrals. After multiple viewings of the interview data, it became clear that Iliana's data was particularly useful for our research questions because she first struggled to reason about multiplication in the context of the integration tasks, but then constructed the product layer appropriately. Her case is all the more noteworthy because she self-reported strong performance in calculus (Calculus, A+; AP Calculus AB, 5; AP Calculus BC, 5). We generated summaries of the Iliana's reasoning that attended to her spoken language, hand gestures, and drawings as she worked on each task.

Results

Iliana's Work

Due to space limits, we focus on Interview 3. During Interview 1, Iliana relied on recalled formulas for transforming units when determining what to multiply — for instance, she recalled that $ft/sec \cdot \sec = ft$ and that $cm^2 \cdot cm = cm^3$. The main additional result from Interview 2 was that, in the context of a problem that provided equally-spaced, cross-sectional areas of a liver and asked for an approximate volume, Iliana multiplied cross-sectional areas by lengths to construct small pieces of volume which she then added, appropriately. At the same time, when using integral notation to express her reasoning, she included both Δx to denote small lengths and dx (Figure 2). When asked to interpret dx, she explained:

Iliana: I think maybe that just like everything's like in respect to x, I don't know, honestly, like that's always like added at the end and it's like I, we've definitely like gone over exactly why.

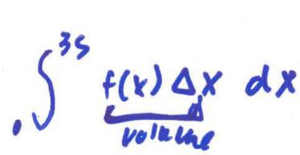

Figure 2. Iliana's definite integral representation (Interview 2)

Thus, in the Liver task Iliana perceived products in terms consistent with normatively correct Riemann sums and integrals but, for her, the dx notation did not indicate a factor to multiply. Interview 3 would reveal additional challenges experienced by Iliana.

Data. Task 1 from Interview 3 asked Iliana to approximate the total force from water pressure on a tank wall. The task included a picture of the 3ft-by-4ft wall and stated that the pressure on the wall varied with the depth, x, according to $P(x) = 15x$. A normatively correct integral would be $\int_0^3 4 \bullet 15x dx$. The physics course had yet to cover the relationship between pressure and force, and we anticipated that the context would be novel for Iliana. Indeed, she said she had to think about how pressure relates to force. She then integrated $15x$ "on the bounds from zero to x, x being whatever the depth is" to calculate the "sum of all the surface areas." Finally, she explained that she thought the pressure pointed down to the bottom of the tank (Figure 3). For this reason, she did not "feel the 4 feet of the width has anything to do with that."

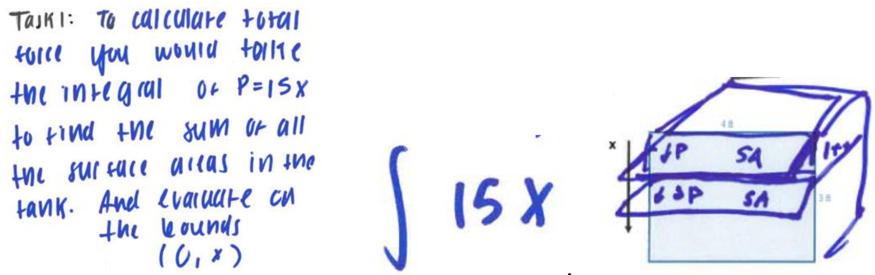

Figure 3. Iliana's work on calculating the total force (Task 1)

When the interviewer asked about force acting in another direction, Iliana continued to view pressure vertically. At the same time, she was not confident and reported that she did not know if force and pressure were related through a derivative/antiderivative, where the width of the wall would come in, and how to interpret the phrase "pressure across a surface area." Finally, Iliana sketched an appropriate graph for $P(x) = 15x$ but did not discuss area under the curve.

Analysis. Iliana's work on Task 1 provided access to how she approached integration in a novel situation, not in one where she recalled certain multiplicative relationships or formulae. An interconnected set of ideas (i.e., uncertainty about the pressure and force relationship, downward pressure) impeded her construction of products used in normatively correct Riemann sums and definite integrals. First, although the problem statement asked explicitly about pressure on the wall, Iliana consistently focused on downward pressure. This orientation side-stepped the need to consider small pieces of wall area, $4dx$, as one factor in a product. Second, similar to her work in Task 2 summarized above, Iliana did not view multiplication by dx as transforming units. Rather, she integrated the pressure function, $P(x) = 15x$, to get a total pressure. Third, similar to her work on the Liver task discussed above, she did not treat dx as a conceptually essential piece of the definite integral. In fact, she omitted the notation. Thus, how she oriented pressure in the situation directed her attention away from multiplying small areas of the wall ($4dx$) with pressures ($15x$), and it was unclear whether she noticed the absence of unit transformation through multiplication. If transformation of units through multiplication were central to her understanding of definite integrals, integrating a pressure function to get total pressure might have indicated to her that something was off.

Data. Task 2 asked Iliana to consider an automobile accelerating at $a(t)$ (m/sec^2) for 9 seconds and to find the velocity after 9 seconds. First, Iliana recalled an appropriate velocity formula, $v = v_0 + at$. She substituted 0 for v_0 and 9 for both a and t, arriving at 81 *m/s*. She then wrote $v = 9a$ (Figure 4). When asked how she knew to multiply, Iliana responded:

Iliana: a is the acceleration, which is just equal to, over here, like the amount that the, the amount that the, the like velocity is changing like per second, which in terms of units is meters per second second. Because meters per second is the unit for velocity, and then you add an extra second in the denominator because it's the change of velocity per second. So then when you multiply by time, which is just seconds, it cancels out and you have this, the unit for velocity again.

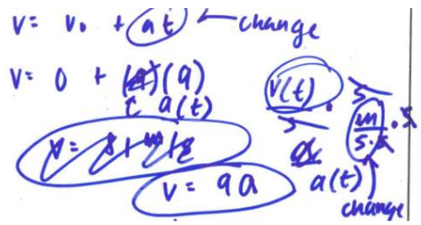

Figure 4. Iliana's work on calculating the velocity value for 9 seconds (Task 2)

Finally, Iliana mentioned the derivative/antiderivative relation between the acceleration and velocity quantities: "velocity is the antiderivative of acceleration", that is, "given like acceleration, let's say, $8x$, you can think that integral of acceleration evaluated on the time period zero to t, where t is the number of seconds that the system did."

Analysis. Iliana's work, including her discussion of anti-derivatives, suggested prior experience with problems relating velocities and acceleration. She introduced multiplication through a recalled formula and formal cancellation of units. From her discussion, it appeared that she recalled appropriate units for acceleration and for velocity and then reasoned about how to get from one to the other by cancelling units of time. Given that at one point she substituted 9 for both a and t, it is likely that she presumed a to be constant. This could explain why she did not use multiplication to construct small changes in velocities which could then be added, in contrast to the way she did use multiplication to construct small pieces of liver volume.

Data. Task 3 asked Iliana to explain the meaning of the definite integral and its interpretation in Task 1 and 2. Iliana stated that "it's harder than I thought to quantify what [the integral] exactly means." She then explained that the integral means "the sum of like a function per like change" and that "d just means like the change." (Figure 5). She added that "from a to b" means the summation of "those like areas of $f(x)$ per each change, like d of x." She concluded by saying "you are thinking of each small area and then adding them up." An exchange later she added, "Like change for each x, like per x, that's the change."

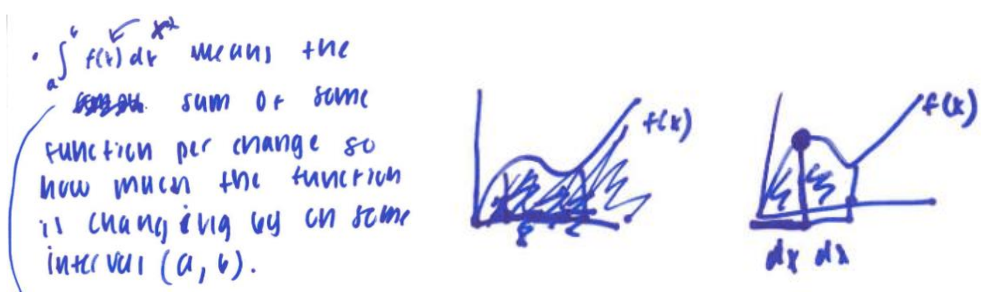

Figure 5. Iliana's definition of the definite integral and graphical representation of $f(x) \cdot dx$ (Task 3)

The interviewer then asked Iliana to explain her understanding of areas. Iliana graphed a function, shaded the area below it, and said "the summation kind of means like, you want to find what's under here, right?" When interviewer then asked Iliana where she saw dx and the product $f(x)dx$, Iliana drew horizontal brackets to indicate instances of dx and a large point on her graph to indicate $f(x)dx$ (Figure 5).

Analysis. Iliana demonstrated several ideas relevant to explaining definite integrals. These included summation of pieces, attending to changes in x, and areas. At the same time, there was no evidence that products were central for her. Similar to her work on the Water Pressure task, she talked about summing the function $f(x)$, not $f(x) \bullet dx$, and her discussion of change for each x and change per x suggested that she thought more in terms of a correspondence between values of $f(x)$ and instances of dx than a product of the two. Her plots of dx and $f(x)dx$ provided further evidence that she did not connect this notation to products and small areas.

Data. Next, Iliana interpreted the definite integral in the context of the acceleration task. First, she wrote appropriate integral notation and stated: "It's actually really the same thing, a of t per change in time. And then you're summing for, to evaluate the change of velocity per the amount of time done basically." Then she produced a graph corresponding to her explanation and drew the large point to indicate where she saw the expression $a(t)dt$ (Figure 6, middle).

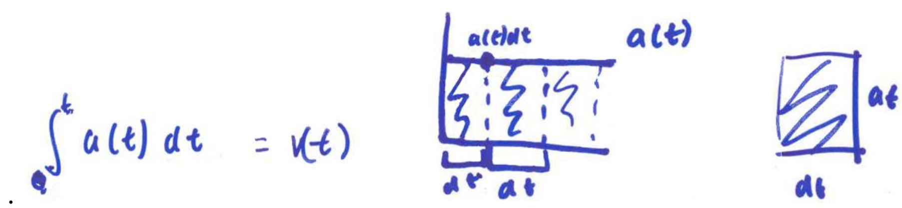

Figure 6. Iliana's meaning of the definite integral (Task 3)

Right after this answer, Iliana expressed some confusion about the dt term in her expression:

Iliana: Now, I'm confused what dt exactly is like representing. Because like sometimes it just feels like it's just there and doesn't really represent something. But like, obviously I know it does…. it's just like something that's kind of there, I don't know.

When the interviewer asked "What are you indicating when you add the squiggly lines?", Iliana responded the "area under the line" which was the "change in velocity for that time period."

Then the interviewer asked the following question, which made multiplication explicit:

Interviewer: Does this area that you indicated here (points to the squiggly lines, Figure 6, middle), and that expression $a(t)$ multiplied by dt has any relation or no for you?

After thinking for few seconds, Iliana changed her reasoning:

Iliana: Well, that is the area. Okay. Maybe this is making a little more sense now actually. Dt multiplied by a(t) gives you this area because it's just a box and you know that like to find an area of a rectangle is just like the base times height or whatever, you know. So, when you multiply those together, that gives you an area and the summation is just saying the sum of each of those areas for one, some time period. So, this isn't a point (points to $a(t)dt$). This is all of this, (points to shaded region) the a of t d of t is the squiggly lines.

Iliana shifted her reasoning and produced a new rectangular area picture (Figure 6, right), where she indicated $a(t)$ to be the height of the rectangle and dt its width. She reported that connecting $a(t)dt$ to an area was new to her. Finally, Iliana stated that integration will give her the velocity quantity and pointed to the sum of areas under the curve as the total velocity value on that time interval determined by the bounds in the integral notation. There was not time to return to the Water Pressure problem.

Analysis. Initially, Iliana interpreted the integral of $a(t)dt$ in ways similar to her interpretation of $f(x)dx$. She thought of a summation of pieces that corresponded to instances of dt but that did not think of those pieces as products for which dt was a factor. Just as she interpreted $f(x)dx$ as point, so again did she interpret $a(t)dt$ as a point. In both cases, associating dx and dt with intervals on the horizontal axis was insufficient to cue multiplication for Iliana. Once the interviewer introduced interpreting $a(t)dt$ as a product, Iliana generated a normatively correct interpretation that coordinated multiplication with the definite integral for the first time during the interviews.

Remaining Four Students. The remaining four students also demonstrated a range of resources for reasoning about multiplication—including formal unit cancelation manipulations—and evidenced a range of challenges reasoning about multiplication in the context of definite integrals. For the Water Pressure task, a second student integrated $15x$, one student considered integrating P/A, one student considered the product $15x \cdot 4x$, and one student avoided integrals altogether and reasoned instead about average pressure. We also observed further instances in which students did not associate dx notation with small horizontal lengths.

Discussion and Conclusion

The present study extends prior reports (e.g., Sealy, 2014) that students have trouble with the "product layer" when constructing definite integrals. In particular, we considered the possibility that students' difficulties might reflect where and how they use knowledge resources related to multiplication, as well as how they understand integral notation. Data on Iliana provided particularly good access to knowledge resources used by one high-achieving student. We found that Iliana had a range of resources for modeling problem situations with multiplication—including recalled formulas and strategies for transforming units—as well as a range of understandings about integrals—including thinking of areas under curves and derivative/anti-derivate relationships. We are not claiming that this is an exhaustive list of Iliana's resources related to multiplication and integration, only that they were ones central to her progress by the end of Interview 3. Iliana's work on the Liver problem demonstrated a case where she drew on prior knowledge about multiplication to form small pieces of volume that she then added. Thus, in some contexts she reasoned in ways consistent with Riemann sums but was not able to connect this reasoning to normatively correct integral notation. A main issue was her uncertainty about the meaning of the dx notation. Iliana's work on the Water Pressure problem demonstrated a case where the way she perceived novel quantities (i.e., downward pressure) directed her attention away from products. Her challenge was compounded by the fact that she did not perceive a product in the integrand that transformed units, in this case units of pressure. Finally, at the end of Interview 3, and in response to some direct suggestions by the interviewer, Iliana perceived dt for the first time as a factor in a product and combined thinking about products, transformation of acceleration into bits of velocity, and integral notation appropriately. Such complex coordination provides new evidence for why forming the "product layer" can be challenging for students. Our results are based on a small sample, but they do suggest that research on integration should pay more explicit attention to the knowledge resources students have for modeling situations with multiplication and how those combine with understandings of the definite integral.

Acknowledgments

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References

Christensen, W. M., & Thompson, J. R., (2010). Investigating student understanding of physics concepts and the underlying calculus concepts in thermodynamics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*. Mathematical Association of America.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction, 10*(2/3), 105-225.

diSessa, A. A. (2006). A history of conceptual change research: Treads and fault lines. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 265–282). New York, NY: Cambridge University.

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152-167.

Ely, R., & Jones, R. S. (2023). The Teaching and Learning of Definite Integrals: A Special Issue Guest Editorial. *International Journal of Research in Undergraduate Mathematics Education, 9,* 1-7.

Izsák, A. (2004). Teaching and learning two-digit multiplication: coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning, 6*(1), 37-79.

Izsák, A. (2005). "You Have to Count the Squares": Applying Knowledge in Pieces to Learning Rectangular Area. *Journal of Learning Sciences, 14*(3), 361-403.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2)*,* 122-141.

Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann-sum based conceptions in students' explanations of definite integrals. *International Journal of Mathematics Education in Science and Technology, 46*(5), 721-736. https://doi.org/10.1080/0020739X.2014.1001454

Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education, 9*(1)*,* 8-35.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). National Council of Teachers of Mathematics.

Moschkovich, J. (1998). Resources for refining mathematical conceptions: Case studies in learning about linear functions. *The Journal of the Learning Sciences, 7*(2), 209-237.

Nguyen, D.-H., & Rebello, N. S. (2011). Students' understanding and application of the area under the curve concept in physics problems. *Physical Review Special Topics – Physics Education Research, 7*(1), 010112.

Rasmussen, C., Marrongelle, K., & Borba, M. (2014). Research on calculus: What do we know and where do we need to go? *ZDM Mathematics Education, 46,* 507-515.

Sealey, V., (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33,* 230-245.

Thompson, P., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*(1-2), 124-147.
Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America. http://pat-thompson.net/PDFversions/2008MAA Accum.pdf.
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.
Vergnaud, G. (1988). Multiplicative Structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.

What is the Correct Amount of Change? A Case Study on Kala's Covariational Reasoning

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Students' understanding of the mathematics of change has been a crucial topic in the teaching and learning of precalculus. Various studies have indicated that covariational reasoning – reasoning about two quantities changing together—is critical for constructing productive meanings for rate of change. The Bottle Problem is one way teachers have supported students' covariational reasoning. We report on a case study of one student – Kala – an Applied Precalculus student. Through analysis of her work on the Precalculus Content Assessment (PCA), course assignments, and a cognitive interview, we report on her reasoning with amounts of change—a level of covariational reasoning. As a result, we highlight the importance of using appropriate amounts of change to analyze situations covariationally. This study contributes to literature on students' development of productive amounts of change reasoning.

Keywords: Covariational Reasoning, Precalculus, Cognition, Bottle Problem

Supporting students in constructing productive meanings for rates of change has been an aim for researchers for decades (Thomspon & Carlson, 2017). From middle school on, various materials aim to support students in reasoning with rates (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis et al., 2015; Johnson, 2015; Tasova, 2021; Yu, 2022). The Bottle Problem (e.g., Carlson et al, 2002) is one well known example. Nevertheless, students still enter their post-secondary courses struggling with ideas of rates of change (Carlson, Oehrtman, & Engelke, 2010). The Precalculus Content Assessment (PCA) is one assessment that not only helps identify students' precalculus content knowledge, but also attends specifically to students' covariational reasoning—reasoning about two quantities changing in tandem (Carlson et al., 2002) (heretofore CR). Our research topic was to explore students' ways of reasoning when they attempt to reason about changing quantities quantitatively but struggle to construct or interpret representations. This interest led us to explore Kala's reasoning via her and her classmates' results on the PCA and the Bottle Problem. Our results provide insights into the importance of selecting appropriate *amounts of change*—one of the levels of covariational reasoning. We conclude by discussing the implications of her reasoning to the research on CR.

Background and Theoretical Framework

Covariational Reasoning and the Bottle Problem

Carlson et al. (2002) illustrated their framework for covariational reasoning with the Bottle Problem, a problem used and adapted in several settings with various populations. In Carlson et al.'s (2002) version of the Bottle Problem, students receive an image of a cross-section of a spherical bottle with a cap and asked to imagine the bottle filling with water and to "sketch a graph of the height as a function of the amount of water in the bottle." Their resulting framework consisted of five mental actions: *coordination of change*, *directional change*, *amounts of change*, *average rate of change*, and *instantaneous rate of change*. For example, students who coordinated the quantities might say that the height is changing as the amount of water is changing. Students with *directional reasoning* might say that the height is increasing as the amount of water is increasing. Students with *amounts of change reasoning* might say that for equal changes in the amount of water added, the water is increasing by increasing amounts. From these descriptions, we can recognize increasing precision about how the quantities change.

In this study, we were interested in exploring more in depth about how students develop CR. Johnson, Castillo-Garsow, Moore and colleagues (e.g., Castillo-Garsow, Moore, & Johnson, 2013; Johnson, 2015; Moore et al, 2019) have already done work in this area, reporting how students visualize quantities changing (such as the speed of the change or considering discrete vs. continuous change) and how their meanings for graphs can impact their reasoning. Another relevant result is from Stevens (2023), in which students compared the steepness of lines on a graph (i.e., the slope) to determine increasing rates of change.

Precalculus Content Assessment (PCA)

The goal of the Precalculus Content Assessment (PCA) is to provide a means to evaluate how effective a curriculum and its instruction are in setting students up to be successful in calculus (e.g., effectiveness of College Algebra (Carlson, 2010)) and link between AP Calculus and PCA (Meylani, 2011). In Carlson's (2010) study of 902 precalculus students, the average score on the PCA was 41% with a reliability coefficient of 0.73. The average score on the PCA for the covariation reasoning abilities was 50% with a reliability coefficient of 0.46. Though these reports pull out students' covariational reasoning problems (which we also do in this study), it is important to note that a factor analysis on the PCA indicated the use of a single total score on the PCA may be the "most psychometrically defensible method of scoring the instrument" (Jones, 2021, p. 78).

Intellectual Need

Because this study is attempting to explore in depth a student's CR, Harel's (2008) *intellectual need* became a key idea in the analysis of the data in identifying reasons a student might engage in covariational reasoning. Harel (2008) defined *intellectual need* as the need to reach an equilibrium (i.e., no state of perturbation) by learning a new piece of knowledge. For instance, when choosing to graph a straight line or a curve, a student may recognize a need to attend to the rate at which quantities are changing in the situation to reach a state of equilibrium.

Methods

Subjects and Setting

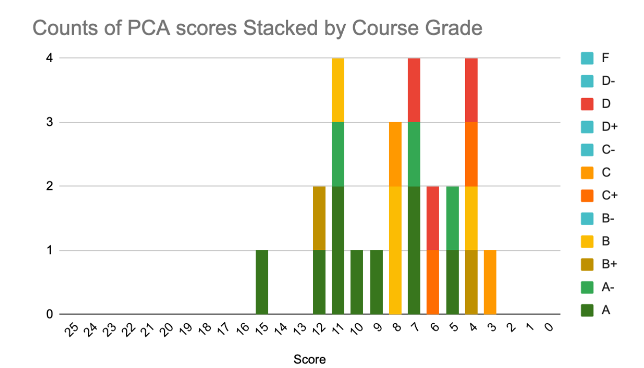
This study was conducted with an Applied Precalculus class of 39 students at a medium-sized university in the northeastern US. The coordinated course had students with various majors (but not Mathematics or Engineering). The students were assigned four different assignments throughout the semester, one of which included an extended version of the Bottle Problem from Moore's *Advancing Reasoning* NSF project (see results for details) and a prompt to submit a video explaining their reasoning. 25 of the students completed the optional PCA in class at the end of the semester. One student, Kala, volunteered to be part of a semi-structured clinical interview (Clement, 2000) after the semester ended in which she, once again and with no advance knowledge, went through a subset of the problems from the four classroom assignments.

Analysis

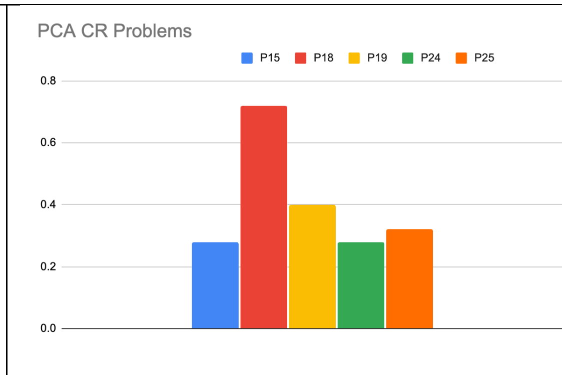
We recorded PCA responses and scanned student work, we downloaded the classroom assignments from the course's LMS where students had uploaded pdfs and video files, and we video-recorded and transcribed the interview (camera and screen recording of iPad). We analyzed the PCA results by question (correctness, summary data, distribution of responses), and then focused on the problems Carlson, Oehrtman, & Engelke (2010) identified as relevant to CR (heretofore PCA CR problems). We used Carlson et al.'s (2002) CR framework to analyze the student work on those problems, and we then compared student's work with Kala's work. We then analyzed Kala's student work and video associated with the Bottle Problem assignment, coding her *amounts of change reasoning*, which we analyzed further using Kala's interview on the Bottle Problem.

Results

Kala's Overall PCA Results Compared with Classmates



(a)



(b)

	Total PCA by problem	PCA CR by problem	Total PCA by student	PCA CR by student
max.	76%	72%	60%	100%
min.	0%	28%	12%	0%
average	31.2%	40%	31.2%	42.8%
mode	32%	28%	44%	40%
median	32%	32%	28%	40%
var.	0.0496	0.0344	0.016	0.0610
st.dev.	0.2227	0.1854	0.1264	0.2470

(c)

Problem	Non-CR categorizations
15	rate of change, general non-linear, graphical, contextual
18	algebraic, contextual
19	rate of change, general non-linear, graphical
24	general non-linear, graphical
25	rational, algebraic

(d)

Figure 1: (a) Total PCA scores with course grade (b) % correct for PCA CR problems (c) Summary statistics by problem/student for the total PCA and for PCA CR problems and (d) non-CR categories for PCA CR problems.

The PCA results from the 25 students' scores are summarized in Figure 1a. Kala was one of four students who scored 4 points (out of 25) and received a B+ in the course. Figure 1b shows the results of the student responses on the PCA CR problems. Kala was one of the 7 students (i.e., 28% of the class) who got Problem 15 (P15) correct on the PCA, but all her other PCA CR problem responses were incorrect. Problem 15 was the most like the Bottle Problem, because it involved choosing a geometric 3D shape that could represent a given height vs. volume graph (other PCA CR classifications in Figure 1d). Figure 1c provides PCA summary statistics. In the first two columns, we see 76% of students answered the most correctly answered PCA problem correctly, while it was 72% for the max PCA CR problem. Also, students performed better on average on the PCA CR problems than the total PCA assessment (31.2% vs 42.8%). Kala scored a 16% on the PCA and a 20% on the PCA CR problems alone, both below average scores.

Kala's PCA CR Problems Results Compared with Classmates

Kala only got one of the five PCA CR problems correct (P15). In this section, I (i) state how she answered the problems (ii) compare her responses to her peers' responses, and (iii) report on whether (and how) the content was covered in the Applied Precalculus course she took.

On P15, students chose a shape that represented a given volume-height graph. Most students chose shapes that had the opposite rate of change as what was represented in a given volume-height graph. Kala (no written work) was part of the 28% of students who answered correctly. In class, students did a very similar task (i.e., the Bottle Problem).

On P18, students needed to determine the *direction of change* and an extreme function value from given formula. P18 was the second most correctly answered PCA problem. Most students substituted values to solve this problem. Kala (no written work) got P18 incorrect. For P19, 40% of students correctly described the behavior of a function when given a graph. Only one student (not Kala) incorrectly identified the *directional change*, but the most common response included the opposite rate of change. Kala and one other student identified a constant rate. In class, students determined *directional change*.

For P24 and P25, students described the end behavior of a function given a graph and a rational function definition, respectively. 28% and 32% of students correctly identified end behavior, respectively. Kala did not correctly identify any end behavior statements for either problem. End behavior was not explicitly taught in the course.

From Kala's work on these problems, we identify the following characteristics. Kala did not demonstrate a way to reason about the *direction of change* when given a function definition (P18), but she could when given a graph (P19). Kala did not demonstrate a way to reason about the end behavior of a function when given a graph or a function definition (P24/P25). Kala inconsistently reasoned about rates of change when given a graph (P15/P19). To gain more insight into Kala's reasoning, we turn to her work on Assignment 3 and her cognitive interview.

Assignment 3 (The Bottle Problem) Results

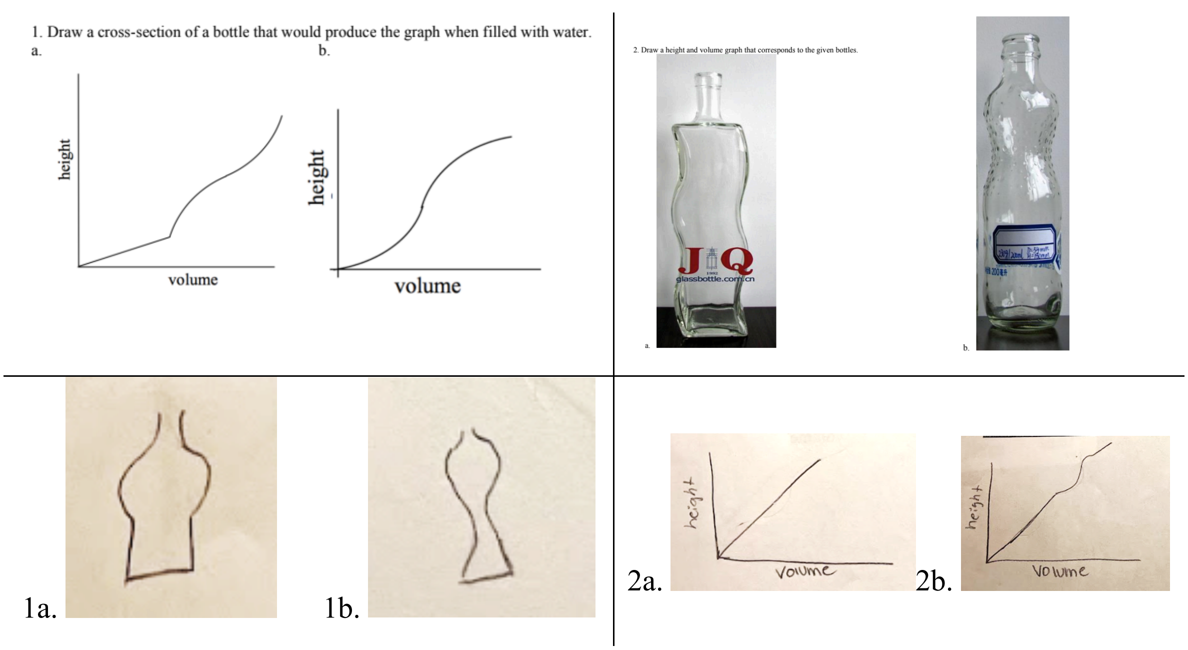
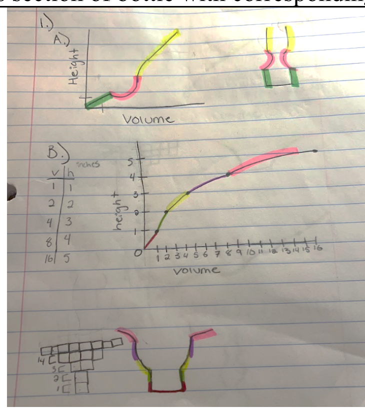


Figure 2: (top) Assignment 3: Part 1 and (bottom) Kala's solutions.

1a. Height-Volume graph with corresponding cross section of bottle.
1b. Height-Volume graph; the water doubles in volume for each additional inch of height.
1c. Height-Volume graph; for each inch in height the bottle increases, the volume of the bottle increases by two more inches cubed in volume that the previous increase.
2. Cross section of bottle with corresponding Height-Volume graph.



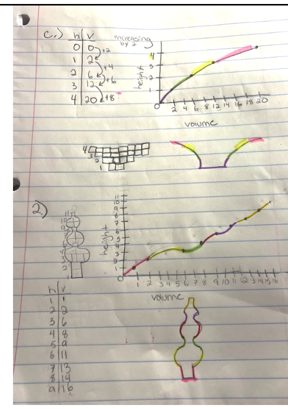


Figure 3: Kala's Assignment 3: Part 2 solutions with summary of problem statements in top row.

During the semester, students worked on a modified version of the Bottle Problem. Part 1 (Figure 2) is above with Kala's solutions. For 1a, her cross section of the bottle is a normative solution. For 1b, there is extra curvature in her bottle not represented in the given graph. For 2a and 2b, her graphs are potentially missing curvature in her graph to represent the inward curvature underneath the cap of the bottle. We return to 2b, as does Kala, in her interview.

In analyzing Kala's submitted work (Figure 3) and video from Part 2, we note three aspects. First, in Part 2: Problem 1a, Kala highlights a single upward curvature in her graph with a corresponding pink cross section of her bottle in which the height has *both* an increasing and decreasing rate of height with respect to volume. In describing the pink section in her graph, she stated it is "dip[ping] down" and "the volume is decreasing, while the height is still increasing." When relating that to the pink section in her bottle, she stated "because the volume is decreasing, so the bottle gets skinnier at that part". Similar work is in Part 2: Problem 2 (e.g., green section). Thus, we infer Kala associated a thinner portion of a bottle with a "dip" in her graph.

Second, Kala uses unit squares to represent equal unit cubes of volume. For example, in Part 2: Problem 1c, in which the volume of the bottle increases by two units for each additional inch in height, Kala created a table of volume values and then drew a corresponding unit cube sketch using her table values. She described her construction of this "cube diagram" as follows: "So we start with two, we do for our one inch of height, we have two blocks of volume. And then two inches, we have a total of six." The two and six correspond to the volume values in her table for a height of one and two inches, respectively. Similar work is in Part 2: Problem 2.

Third, after plotting points and creating sketches using unit squares using tables she constructed, she inconsistently connects the points and creates final bottles. For instance, Part 2: Problem 1b does not have a smooth curve in her graph, whereas Part 2: 1c does. Similarly, Part 2: Problem 1b does not have a smooth curve in her final bottle, whereas Part 2: 1c does. Her interview provides more details on how she chose to connect points on her graph.

Kala's Interview on the Bottle Problem

Figure 4 is Kala's final work on the Bottle Problem (left) and her additional work done throughout the interview on the Bottle Problem (right). Note that the bottle is the same one from

Part 1: Problem 2b (Figure 1) and that the resulting graphs are very similar. However, new for the interview was her use of a "cube diagram", a table of values, and plotted points (methods she used during Part II (Figure 3)). In this section, we focus explicitly on Kala's CR given (i) we have insights into her methods for constructing graphs from the Bottle Problem assignment and (ii) the interview enabled more in-depth questioning given its semi-structured nature.

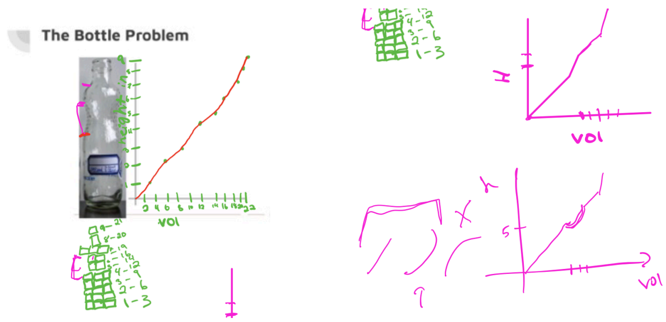


Figure 4: Kala's final work on the Bottle Problem during her cognitive interview.

First, we note Kala's quantification of volume and height using a diagram of the cross section of the bottle. Her use of her "cube" method to quantify volume enables her to draw a diagram in which she can measure volume via counting the cubes. She also defines "layers" of cubes, each layer corresponding to one inch of height. In the green "cube diagram" in Figure 3, Kala uses this diagram to measure the volume in the bottle for each additional inch of height (e.g., "1-3" indicates at 1 inch of height, the volume is 3 unit cubes, and then 2-6, 3-9, 4-12, 5-14, and so on).

Second, we note that Kala attempts to describe how her two quantities, height and volume, are changing together in her graph. For instance, she describes the initial straight segment emanating from the origin (in all her graphs, but her final graph, specifically) the following way: "what this is kind of showing that it's increasing at the same time, as the height is increasing, the volume is increasing at the same time". This corresponds to the second level of CR.

Third, we note that Kala recognizes that the "dips" in the cross section of the bottle impact how much volume can fit within a single layer. For instance, when describing what happens after the straight section of the bottle, she states "the height of the water is gonna increase, and the volume-there's less room for water." She uses that observation to justify why her *graph* should not continue to be straight at these "dips". She states, "the height is staying the same but the volume is gonna decrease, because there's less room so we kind of dip-kind of like dip a little bit and then come back up." Here, although she states the *volume* is going to decrease, she is referencing her layers in her bottle, noting that the layers are going from three unit cubes down to two unit cubes. She also clarifies later that it was the one inch increment that "is staying the same" for the height and that "the height is still increasing as we go up, since we're moving up the bottle since we are filling it up." Thus, though not explicitly using *amounts of change* language, we see Kala attending to and comparing changes in height and changes in volume.

Lastly, we note the perturbation Kala experienced when trying justify straight segments versus curved segments. To discuss this perturbation, we focus on when Kala is referencing the bottle from four to six inches, which corresponded to the pink region labeled on the given cross section of the bottle (i.e., the "dip"). In her cube diagram, this region corresponded to two layers of two unit cubes each. In her graph, this corresponded to the curve immediately after the straight portion of her graph (see bolded region and the three curves she considered in Figure 4 (right)).

When considering using straight lines in her graph, her reasoning is as follows:

Kala: So we're at five inches, which is where that fifth layer shrinks to two, and then I have fourteen cubic whatever is of volume, and then the volume, the height is still increasing, so we jump to six, but it's only fourteen units of volume.

From this quote, we see that Kala identified that there is less volume gained per inch of height in the layers (that each have two cubes) than the first four layers (that each have three cubes). She concluded that the line segment corresponding to that "dip" in the bottle has a steeper slope than the line segment corresponding to the first section. This use of "increasing rate" implying "steeper" straight line segment is what Stevens (in press) described.

When considering curved lines, she hesitated to associate it only with the curve of the bottle, referencing her bottle from Assignment 3: Part 1 2a (Figure 2), where the curvy bottle still resulted in a graph with a straight line. She stated she felt like the bottle was "trying to trick me and make me want it [the graph] to be curvy" because "the bottle is curvy". In the end, she chose her final graph to have a curved segment in that region vs. a straight segment, though admitting, "So I honestly don't know which is the move-but let's go curvy".

In summary, Kala wanted to justify a "curvy" graph, but given her reasoning with the bottle's cross section, she could not. In the discussion, we discuss why Kala's reasoning with the diagram was both quantitative and limited, and its implications for students' *amounts of change* CR.

Discussion and Implications

Across Kala's work, we noted ways of reasoning about *directional change* but a struggle to reason about rates of change relative to her peers. From her PCA results and assignment, there was not much evidence about how she determined rates of change, only that she seemed to connect the idea to "dips" in the bottle in the Bottle Problem. However, in her interview, where she used her quantitatively appropriate "cube diagram" method, we find Kala's *amounts of change* reasoning had one major limitation: she chose height intervals that spanned across two different rates of change (i.e., decreasing rate then increasing rate). This decision resulted in *equal* changes in volume for a successive change in height in her cube diagram given the symmetry of the dip in the bottle. Because this constant change matched the constant changes in volume she identified for successive changes in height for the straight segment of the bottle, and because that constant change in volume is how she justified a straight line segment in her graph, she concluded that the "dip" should do the same (but steeper to accommodate less volume added). However, this conclusion perturbed Kala, who still drew a curve on her graph.

To justify appropriate curvature, Kala would have needed to identify smaller amounts of change in height, successive changes in height that started at the beginning of the inward curvature of the bottle and symmetric with the outer curve of the bottle (e.g., Figure 5). Understanding the necessity of this careful construction of *amounts of change* is crucial for making appropriate quantitative conclusions about the CR of the quantities. From Kala's work, we see the importance of supporting this intellectual need for the careful construction of *amounts of change*. This need goes beyond a need to compare amounts of change in one quantity for equal successive changes in another quantity, because as Kala illustrated, she had that goal.

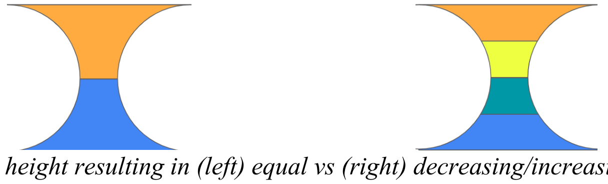


Figure 5: Equal changes in height resulting in (left) equal vs (right) decreasing/increasing changes in volume.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5).

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, *33*(3), 31–37.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Corbin, J. M., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage.

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in creating multiplicative rates of change. *Journal of Mathematical Behavior*, *39*, 135–155.

Harel, G. (2008). DNR persepctive on mathematics curriculum and instruction, part II: With reference to teachers' knowledge base. *Zentralblatt Fuer Didaktik Der Mathematik*, *40*, 893–907.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, *89*(1), Article 1. https://doi.org/10.1007/s10649-014-9590-y

Jones, B. L. (2021). *A Psychometric Analysis of the Precalculus Concept Assessment*. Brigham Young University.

Meylani, R., & Teuscher, D. (2011). Precalculus concept assessment: A predictor of AP calculus AB and BC scores. In *Proceedings of the thirty-third annual conference of the North American chapter of the international group for the psychology of mathematics education*. University of Nevada Reno, Nevada.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *Journal of Mathematical Behavior*, *56*, 100844. https://doi.org/10.1016/j.jmathb.2019.01.008

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Stevens, I. E. (2023). Implications of faster/slower language on undergraduate precalculus students' graphing. In Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno. (pp. 894–903)

Tasova, H. (2021). *Developing middle school students' meanings for constructing graphs through reasoning quantitatively* [Unpublished doctoral dissertation]. University of Georgia.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Yu, F. (2022). Promoting quantitative reasoning in Calculus: Developing productive understandings of Rate of change with an adapted Calculus 1 curriculum. In Karunakaran, S.

S., & Higgins, A. (Eds.).*(2022). Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education. Boston, MA.*

Whiteness-at-work in Mathematics Department Initiatives to Ameliorate Racialized Gatekeeping in Calculus

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Racialized gatekeeping in calculus courses is a national, systemic concern. However, research on equitable calculus instruction has focused on classroom-level changes, providing limited guidance for department-level reforms necessary to implement change in large calculus programs. To extend such work, we present a case study of a mathematics department at a large university engaged in improving its calculus program to better serve Black and Latin students. Informed by critical whiteness studies, we explore how whiteness was maintained amidst this equity-oriented initiative. Findings exhibit disciplinary-specific forms of whiteness that stymied departmental reform. Implications are provided for equity-oriented, department-level change.*

Keywords: whiteness, mathematics faculty, calculus, testing

Study Purpose & Background

Across the nation, undergraduate calculus serves as a source of racialized attrition from STEM majors (Chen, 2013). In response, mathematics education research has increasingly explored racialized aspects of undergraduate mathematics to offer guidance for advancing equity through instruction (Larnell, 2016; Leyva et al., 2021). However, disciplinary and departmental factors can hinder the implementation of such antiracist instructional recommendations (Ching & Roberts, 2022). The prevalence of faculty's colorblind beliefs that mathematics is a neutral space (McNeill et al., 2022) can impede sociopolitical noticing in the classroom (Louie et al., 2021), making it difficult for instructors to enact equitable instruction. Furthermore, racially-conscious instructors can find their agency to implement instructional changes limited by the coordination of calculus courses across multiple sections. Prior research has explored department-level change efforts in coordinated calculus courses, but such interventions have largely focused on implementing generally supportive practices (e.g., incorporating active learning pedagogies; Williams et al., 2022). Such practices leave the racialization of undergraduate mathematics uninterrogated, including the role of calculus as a gatekeeper to STEM majors, and therefore fall short in improving course experiences among racially minoritized students (Leyva et al., 2022).

Higher education research shows that departmental changes are mediated by faculty's racial beliefs as well as historically white institutional structures and policies. During equity-oriented systemic change efforts, these manifestations of whiteness can contribute to internal contradictions that can cause interventions to fall short of their antiracist aims (Dowd & Bensimon, 2015). Critical analysis of race in such research has been reserved for student experiences, leaving little guidance about how to counteract whiteness in faculty politics. Such analyses are necessary in undergraduate mathematics where disciplinary values, which reflect the values of white, male elites who historically codified academic mathematics (McNeill & Jefferson, in press), also mediate departmental change. Systemic solutions to address racialized gatekeeping in calculus must anticipate and counter ideological forces of whiteness in both the discipline and educational structures to actualize equity-oriented aims. This requires understanding the functions of whiteness in disciplinary values and institutional structures.

This paper presents a case study of a mathematics department at Wesselman University (pseudonym), an institution engaged in reforming its calculus program to better serve Black and Latin* students. Informed by critical whiteness studies, we assume that whiteness is omnipresent and explore its workings in this change initiative through the following research questions: (1) How is whiteness functioning in the mathematics department's initiatives to ameliorate racial inequities in calculus courses?; and (2) What role does mathematics, as a discipline, play in the reproduction and disruption of whiteness within calculus courses and the department?

Theoretical Perspectives on Whiteness

Whiteness is a set of ideologies that reinforce white supremacy, antiblackness, and systemic racism (Bonilla-Silva, 2006). For example, of particular relevance to the present analysis is the ideology of "quantity over quality" (Jones & Okun, 2001), which describes placing a high value on outputs, especially those counted or quantified. This ideology is prevalent in mathematics as a quantitative field, privileging abstract and generalizable approaches to produce expedient solutions over problem-solving approaches that attend to social context and purpose. Consequently, mathematical procedures are leveraged in situations like warfare with limited ethical consideration (Ernest, 2018) and, furthermore, dehumanize mathematics students by prioritizing the productivity of their labor over the learning experience (Ladson-Billings, 1997). Characteristics of whiteness such as these structure norms in undergraduate mathematics classes that reinforce racialized access to participation and recognition (Leyva et al., 2021).

We engage two theories of whiteness to guide our analysis. First, the theory of white institutional space (Moore, 2008) characterizes how organizations and institutions, such as law schools (the context in which the theory was developed), enact whiteness to maintain racial inequity. Moore characterized white institutional spaces as having four features:

> (1) Racist exclusion of people of color from elite law schools and positions of power in legal institutions, which results in the accumulation of white economic and political power, (2) The development of a white frame that organizes the logic of these institutions and normalizes white racial superiority, (3) The historical construction of a curricular model based on the thinking of white elites, and (4) The assertion of law as a neutral and impartial body of doctrine unconnected to power relations. (p. 27)

A white frame can be understood as a set of perspectives, shaped by ideologies of whiteness, that are used to make sense of everyday situations and shape individuals' inclinations to action.

Historically, whiteness has flexibly adapted to maintain dominance in a dynamic U.S. racial context (e.g. Bonilla-Silva, 2006). Whiteness-at-work (Yoon, 2012), the second guiding perspective for our analysis, characterizes these adaptations. Whiteness-at-work describes how individuals ascribe to ideologies that embody contradictions and paradoxes, thus creating the appearance of having progressive racial values while reinforcing racial oppression. For example, a white teacher expressing a desire to 'call out' colleagues on their racially- oppressive beliefs while simultaneously avoiding workplace conflict demonstrates whiteness-at-work (Yoon, 2012).

Together, whiteness-at-work and white institutional space complemented each other to guide our analysis of whiteness in the Wesselman mathematics department. Namely, we examined how whiteness-at-work among faculty gives rise to contradictions that impede equity-oriented reform and thus maintain workings of the mathematics department as a white institutional space. An illustrative example of whiteness-at-work maintaining a white institutional space in a mathematics department is a predominantly white faculty ensuring a short list of job candidates is racially diverse and then eliminating a Black candidate because her job talk used mathematics to interrogate racialized policing (McNeill & Jefferson, in press).

Methods

Research Context and Participants

Wesselman University is a large, elite, predominantly white, research-intensive university in the southern U.S. During the 2022-2023 academic year, the mathematics department had approximately 30 tenure-track faculty, 20 postdoctoral faculty, 10 senior lecturers, and 25 doctoral students. The demographics of the entire teaching staff were approximately 70% white and 85% male. During the same year, the domestic undergraduate population was roughly 40% white, 20% Asian American, 10% Black, 10% Latin*, 5% multiracial, and 5% unknown race.

In 2020, the department and university identified a need to improve the calculus program, particularly to ameliorate racial disparities in calculus grades and attrition from STEM majors. Before changes were initiated, weekly course meetings consisted of three lectures taught by the instructor (a graduate student, postdoc, or lecturer) in groups of 40 students, and a discussion taught by a graduate teaching assistant in groups of 20 students. All class meetings, including discussions, were led in a lecture-based format. Assessment structures included four midterm exams and a final exam. Exams took place in large lecture halls that held multiple sections of the course. Hats, backpacks, and other items were not allowed. Instructors were directed to "actively proctor" exams. Questions were not permitted. Students experienced exams as high-stress environments and frequently reported that they did not have enough time for the exam.

The research team (the three authors) began a collaboration with the Wesselman mathematics department in Spring 2021. Our partnership was structured in three phases: (1) Documenting Black and Latin* students' experiences in calculus; (2) Exploring instructor and stakeholder sensemaking about racial equity in calculus courses and related interventions, and (3) Facilitating professional development with calculus staff. The present analysis is based on data from the second phase, during which several interventions were being piloted. These included some implementation of active learning, the introduction of undergraduate course assistants who led small study groups outside of class, and replacing the fourth midterm with a "redemption test" that could replace a midterm grade. Indicators suggested that these interventions improved, but did not eliminate, racial disparities in grades. The mathematics department continued to function as a white institutional space despite these interventions. Corresponding to Moore's characteristics of white institutional spaces, this is demonstrated by: (1) A mathematics faculty that is approximately 70% white; (2) Shared departmental values that reflect whiteness, such as wanting to treat all students the same; (3) A curriculum supported by textbooks written almost entirely by white men; and (4) A shared perception of mathematics as socially-neutral.

Prior to the beginning of the Fall 2022 semester, instructors teaching calculus I courses were invited to participate in the study at an in-person meeting of calculus staff, and again through email. Faculty who were stakeholders in calculus reform efforts, such as the department chair, the coordinator for calculus courses, the director of undergraduate studies, and members of the "rethinking calculus" committee, were also invited to participate through email. All who expressed interest participated. In this report, we describe participants in aggregate to preserve anonymity given the unique departmental context. The 22 participants consisted of 11 calculus instructors (five lecturers and six doctoral students) and 13 stakeholders (seven tenure-track faculty members, five lecturers, and a postdoctoral scholar), with two participants serving in both instructor and stakeholder roles. Participants were majority white and men.

Data Collection & Researcher Positionality

Data consists of interviews with instructors and stakeholders as well as class observations and departmental artifacts. All interviews were semi-structured, audio recorded, and transcribed verbatim. Interview questions solicited instructor and stakeholder perspectives on: (1) the role of race in students' experiences and instructors' decisionmaking; (2) traditions in and changes to the calculus program; and (3) participants' role and agency in the mathematics department.

Interviews averaged one hour in length, with stakeholders participating in one interview and instructors participating in four. While the topics discussed with instructors and stakeholders were similar, interviews with instructors were more oriented towards instructional sensemaking. For example, the third interview asked instructors to read excerpts from interviews with Black and Latin* students from the first phase of the study, and consider to what extent students in their course(s) might have similar experiences. One abbreviated excerpt is presented below.

> When I'm under [test] time pressure stress, I just start making stupid mistakes. But then when I do take my time, I don't finish my problems on time… When I just saw that [exam] grade, I was like, "Oh…that's really going to hurt my GPA."... That was what compelled me to drop the class... It was just a matter of weighing what was going to push me farther in life… I don't want to be another Black kid just growing up to be nothing. I feel like breaking those social norms or those social views of being Black, that just pushes me to be successful… And so me not doing well in calculus wasn't really an option, per se.

Interviews with stakeholders had a more expansive focus beyond the classroom and into the departmental context. For example, stakeholder participants explored connections between interview topics and their involvement with the calculus program (e.g., as a member of the "rethinking calculus" committee).

Secondary data sources included classroom observations and departmental artifacts. Each instructor was observed four times. Observations were documented with field notes and audio recordings. Departmental artifacts included course syllabi, instructional policy documents, and reports produced by department members that proposed or evaluated interventions.

Two authors collaborated with the mathematics department to build rapport with participants and enhance our understanding of the research context. The second author, a white woman, participated on a search committee. The first author, a white transgender person, taught a calculus course and served as an instructor participant in the present study. The first author conducted all data collection for which they were not a participant, allowing them to provide instructional feedback and raise topics from informal collegial conversation during interviews. The third author, a Latino man, provided an outside lens for analysis informed by his research expertise on equity issues in undergraduate mathematics education. Collectively, we were reflective about how our respective experiences of privilege and oppression shaped our perceptions of whiteness in mathematics. The team resisted deficit interpretations of participants' reflections while maintaining critical awareness that participants served as representatives of a discipline and institution that are historically entrenched in racism.

Data Analysis

Theoretical perspectives of whiteness-at-work and white institutional space guided data analysis that followed an open, axial, and selective coding scheme (Strauss & Corbin, 1998). We used three sets of open codes. One set of open codes captured white racial frames (Moore, 2008), such as paternalism and individualism (Jones & Okun, 2001), used in participants' sensemaking about race, change initiatives, and their role in the calculus reform. This set of open codes contributed to answering our first research question about how whiteness influenced reform initiatives. Another set of open codes flagged organizational tensions. These included collegial conflicts or lack of support, organizational barriers that impede change, and mixed messaging from faculty leadership. The focus on organizational tensions highlights contradictions and paradoxes that exemplify whiteness-at-work (Yoon, 2012), also addressing our first research question. An additional set of open codes flagged instances where the mathematics discipline shaped participant sensemaking about racial equity and related change initiatives in calculus. This included using mathematical language, applying quantitative or formal logic, and appealing to mathematics epistemological values (e.g., abstraction, generalizability; McNeill & Jefferson, in press). This set of open codes supported our exploration of the second research question about how white frames and organizational actions are shaped by the mathematics discipline.

We used two sets of axial codes. One set flagged connections between white racial frames and organizational tensions to address our first research question about whiteness-at-work. Another flagged connections between white racial frames and mathematics, addressing our second research question about mathematical influences on departmental reform in calculus. Selective codes extracted contradictions that exemplify departmental whiteness-at-work.

Findings

For brevity, this report presents one form of whiteness, "quantity over quality," through two contradictions that exemplify whiteness-at-work in calculus reform. We use testing as a focal context to illustrate the maintenance of whiteness amidst equity-oriented reform.

The department privileged quantitative evaluations of student performance through testing over qualitative aspects of learning despite little inquiry into the validity of quantitative evaluation measures. Such valorizing of quantitative evaluations was based on the departmental beliefs that quantities provided a neutral measure of merit. Since evaluation structures were constructed to ensure some failure, this departmental practice functioned as a meritocratic mechanism for denying resources (e.g., access to more advanced courses) among those deemed unfit and therefore reproduced racialized gatekeeping in calculus.

Contradiction 1: Trying to Reduce DFW Rates While Maintaining Historical GPAs

Whiteness-at-work was evident in the department's effort to maintain the historical GPAs while expressing a desire to improve student success in calculus. As one stakeholder articulated,

> If we define success as getting… a C or higher… pretty reliably, 80% of students succeed in calculus. Then there's the 20% who aren't succeeding and we want to figure out why, so that we can propose solutions to help those students succeed. We don't expect everybody to succeed… but 20% seems like a high number.

Improving the proportion of students who receive a C or higher in the course would require that students receive higher grades on tests, given that test scores constituted 85% of a student's grade. Thus, maintaining historical GPAs conflicted with commitments to improving success rates.

Test design was a central context that contributed to whiteness-at-work, wherein whiteness in the form of "quantity over quality" framed the department's underemphasis on qualitative aspects of learning. The appropriateness of a test was determined primarily by the range of scores it produced, with less oversight given, for example, to the alignment of test questions with course learning goals. As one stakeholder shared,

> There were no TA meetings, there was no emphasis or monitoring, mentoring of first-time instructors. There were no discussions about where everybody was in the course… It was this idea that the principle of academic freedom meant that you could teach whatever you wanted, however you wanted, as long as the average of a test exam was X. And nobody really monitored... What was the validity of X? How do we get X? How do we design a test?

A focus on exam averages rather than content supported reductionist views of tests as either easy or hard. To produce tests that were the desired difficulty, instructors described constructing test questions from textbook problems beyond those assigned for homework. One instructor shared that "some problems… are super similar to the problem set, which will be more easier ones," and "If you go to every section… to the 60s and 70s, although those ones aren't [assigned], I do look at some of those to try to see about slightly harder ones to turn into a more harder exam problem." The same instructor described receiving feedback from students that "people wanted to see… more exam-type problems in class," suggesting that students perceived that class did not prepare them for the exams. Such test-design strategies reflected the sentiment that a stakeholder perceived as common on student evaluations: "assessments are harder or different than what is being taught in the class or what is being practiced on out-of-class assignments."

Adherence to historical averages was a central priority, despite the department's professed commitment to advancing equity and the qualitative misalignments between presented, practiced, and tested content. Following the first midterm, one instructor was advised to produce lower test scores on subsequent exams. In a later discussion with a stakeholder about implementing this advice, the impact on STEM persistence among Black students, who had received lower test averages as a group, was framed as a lesser concern than reproducing historical averages.

Instructor: In terms of test writing for the next time… I'm imagining what the score distribution might have been if I had just made a harder test, and I'm imagining that [racial] gap between 83 and 70 moving to 75 and 62, or something like that… given that [Calculus I] is a prerequisite and a required class for so many students' majors, it feels like shifting the test average for Black students in my class down to a 62 potentially could have some real ramifications on their careers.

Stakeholder: Given the fact that the Black students are getting lower averages than the white kids, then if you make your exam… harder… to lower your average, then you're going to put these kids in dangerous territory. On the other hand, I think that you probably don't want to be in a position where we are just starting tweaking with how we teach calculus in your course. I don't think it would be a problem if your course has an average 85 or whatever, but from the point of view of how things are currently set up in the math department that would spark a lot of probably unpleasant conversations. Right? You don't want to be there. It's not up to you to be tackling this problem right now.

Whiteness in the form of "quantity over quality" was visible in the department's adherence to historical GPAs. This prevented addressing testing structures that were an established source of STEM attrition among Black and Latin* students, thus reflecting whiteness-at-work in the contradictions between faculty's commitments to equity and the reproduction of DFW rates that the department wished to address.

Contradiction 2: Teaching and Learning Priorities Disincentivized by Evaluation Practices

Instructors and stakeholders expressed beliefs that students motivated by test scores would engage in rote learning. This was misaligned with desires for students to engage in collaborative mathematical sensemaking. The department's evaluation practices strongly emphasized scores. At the same time, reducing the percentage of students' grades that were determined by tests or redesigning tests to better incentivize desired forms of learning were not widely-considered interventions. Instead, instructors and stakeholders described a need for students to hone test-taking skills, in some cases advocating for more class time devoted to test-taking strategies. Whiteness in the form of "quantity over quality" can be seen in the department's loyalty to traditional quantitative assessment strategies, even when such assessments were at odds with qualitative aspects of their teaching goals as well as Black and Latin* students' wellbeing.

When responding to the interview prompt about testing, GPAs, and racialized pressure (see "Methods" section), an instructor claimed that prioritizing test scores hindered student learning.

> *Instructor*: This always makes me cringe. It [the interview prompt] was just like "This is really going to hurt my GPA."
>
> *Interviewer*: What's cringey about that to you?
>
> *Instructor*: Oh, I guess it's the fixation on grades, just this shows that this is in many students' eyes, and maybe rationally... "It's my job here to be the good grade getter."
>
> *Interviewer*: You feel like it gets in the way of learning?
>
> *Instructor:* I think so.

Simultaneously, preparation for tests often included instructional messages like the following:

> You have to put your economics hat on when you take tests and do a cost-benefit analysis… Time is currency that you have, the good that you are going to get back is points. Don't invest too much of your currency into a not-valuable commodity.

In emphasizing test-taking strategies in these ways, the department explicitly directed students to focus on aspects of the course that were viewed as unproductive for learning.

The misalignment between learning goals and the emphasis on testing limited the efficacy of an intervention designed to better support Black and Latin* students. Structured study groups led by undergraduate course assistants were implemented to foster peer support and a sense of belonging, particularly for racially minoritized students. A report on the efficacy of the piloted course assistant program found it to be successful in achieving these goals. However, attendance waned because students did not perceive that attending study groups helped them with exams. Commenting on this pattern, the report stated, "We believe that it would be a dereliction of our duty as instructors to teach our students that the only calculus problems that are worth thinking about and doing are the ones that could appear on an exam." In response, the report suggested calculus staff should "step up our messaging to the students about the purpose of the [study group] tasks" rather than consider how assessment structures could better incentivize the socially-supportive study group learning. In this way, this intervention embodied a paradox: aiming to ameliorate racial disparities in grades by encouraging Black and Latin* students to engage in mathematical work that students perceived to not significantly influence their grade.

Discussion & Implications

Our analysis highlighted two contradictions depicting whiteness-at-work. Whiteness functioned through a focus on quantitative measurables (i.e., grades), and a de-emphasis of qualitative aspects of calculus teaching and learning. While interventions introduced did improve calculus courses and reduce racial disparities in grades, such improvements were incremental and preserved functions of the mathematics department as a white institutional space. Such incremental changes reflect how institutions of mathematics education appear to be progressive, and at the same time, reflect forms of self-correction that reproduce racism (Martin, 2019).

Critical analysis of local data is important to produce change within mathematics departments (Bressoud & Rasmussen, 2015; Felix et al., 2015). Although student evaluations, test scores, and overall GPAs can serve as such local data, faculty must approach these data with equity-mindedness (Dowd & Bensimon, 2015) to mitigate possibilities for white racial frames, such as values around quantitative evaluation, to drive reform efforts and, thus, stymie their equity-oriented goals. Institutions invested in department-level equity reform should consider ways to integrate support and accountability for faculty within mathematics departments to ensure that implemented interventions are aligned with broader equity goals.

References

Bonilla-Silva, E. (2006). *Racism without racists: Color-blind racism and the persistence of racial inequality in the United States.* Rowman & Littlefield Publishers.

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the AMS*, 62(2), 144-146.

Chen, X. (2013). STEM Attrition: College Students' Paths into and out of STEM Fields. Statistical Analysis Report. NCES 2014-001. National Center for Education Statistics.

Ching, C. D., & Roberts, M. T. (2022). Crafting a racial equity practice in college math education. *Journal of Diversity in Higher Education*, *15*(4), 401.

Doane, A. W. (2003). Rethinking whiteness studies. In Doane. A. W. & Bonilla-Silva, E. (Eds.) *White out: The continuing significance of racism.* Psychology Press.

Dowd, A. C., & Bensimon, E. M. (2015). *Engaging the" race question": Accountability and equity in US higher education.* Teachers College Press.

Ernest, P. (2018). The ethics of mathematics: is mathematics harmful?. In *The philosophy of mathematics education today* (pp. 187-216). Springer, Cham. Chicago

Felix, E. R., Bensimon, E. M., Hanson, D., Gray, J., & Klingsmith, L. (2015). Developing agency for equity-minded change. *New Directions for Community Colleges*, 2015(172), 25-42.

Jones, K., & Okun, T. (2001). White supremacy culture. Dismantling racism: A workbook for social change.

Larnell, G. V. (2016). More than just skill: Examining mathematics identities, racialized narratives, and remediation among black undergraduates. *Journal for Research in Mathematics Education*, *47*(3), 233-269.

Leyva, L. A., Amman, K., Wolf McMichael, E. A., Igbinosun, J., & Khan, N. (2022). Support for all? Confronting racism and patriarchy to promote equitable learning opportunities through undergraduate calculus instruction. *International Journal of Research in Undergraduate Mathematics Education*, 8(2), 339-364.

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *The Journal of Higher Education*, 92(5), 784-814.

Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM–Mathematics Education*, 53(1), 95-107.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459-478.

McNeill, R. T., Jefferson, A. (in press). "Prove Yourself": Exploring Epistemological Values in Mathematics Department Support and Oppression of Black Women Faculty. To appear in the *Journal of Research in Mathematics Education*.

McNeill, R. T., Leyva, L. A., & Marshall, B. (2022). "They're just students. There's no clear distinction": a critical discourse analysis of color-evasive, gender-neutral faculty discourses in undergraduate calculus instruction. *Journal of the Learning Sciences*, 31(4-5), 630-672.

Moore, Wendy L. (2008). *Reproducing Racism: White Space, Elite Law Schools, and Racial Inequality.* Lanham, MD: Roman & Littlefield.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research techniques.* Thousand Oaks, CA: Sage publications.

Williams, M., Apkarian, N., Uhing, K., Martinez, A. E., Rasmussen, C., & Smith, W. M. (2022). In the driver's seat: Course coordinators as change agents for active learning in university Precalculus to Calculus 2. *International Journal of Research in Undergraduate Mathematics Education*, 8(1), 121-148.
Yoon, I. H. (2012). The paradoxical nature of whiteness-at-work in the daily life of schools and teacher communities. *Race Ethnicity and Education*, 15(5), 587-613.

How Students Reconcile Incongruous Mathematics Self-Efficacy

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Incongruous mathematics self-efficacy (IMSE) is when a student's self-efficacy does not match their performance. We present a subset of results from a larger qualitative study examining IMSE in collegiate intermediate algebra. We used qualitative interviews and surveys to longitudinally follow students for their entire semester in intermediate algebra and present the cases of three students with IMSE (over confidence). We show that participants' over confidence seemed to stem from them doubting their performance would be predictive of their future success; participants did eventually lower their self-efficacy in response to repeated low performance. Results have implications for college mathematics instructors and for the study of self-efficacy.

Keywords: Beliefs, College algebra, Incongruous mathematics self-efficacy, Mathematics self-efficacy

In this paper, we present a subset of the results from a larger study that sought to examine and describe instances of incongruous mathematics self-efficacy (when students' mathematics confidence does not match their mathematics performance). For this paper, we present the cases of three students who had incongruous mathematics self-efficacy in their college intermediate algebra class, and we describe how, throughout the course of the semester, the students reconciled their self-efficacy to their actual performance. Previous research on incongruous mathematics self-efficacy is limited and no previous research has examined how students reconcile their self-efficacy with their performance. This work has important implications for self-efficacy theory at the undergraduate level as well as for college instructors who teach introductory classes and want to help their students better calibrate their work and effort to be successful in their college mathematics classes.

Conceptual Framework and Literature Review

Mathematics self-efficacy (MSE) is defined as a person's beliefs about their ability to learn or perform mathematics (Bandura, 1986). MSE is individualized to the student. That is, a student might make the statement "I am good at math" and this could mean that they are good at algebraic computations or Calculus, depending on the level of mathematics they have taken. A person's self-efficacy is an individual judgement that only they can make; it is a belief about themselves constructed from their perceptions and experiences. These beliefs are subjective. In spite of this, MSE is a reliable predictor of performance and perseverance, suggesting that a large percent of the time students' MSE judgements are probably aligned with their abilities (Hackett, 1985; Hackett & Betz, 1982; Hackett & Betz, 1989; Hutchison et al., 2006; MacPhee et al., 2013; Marra et al., 2009; Multon et al., 1991). However, previous research has found that sometimes self-efficacy beliefs and external measures of ability do not match (Chen, 2006; Dassa & Nichols, 2019; Pajares & Miller, 1994; Schraw, 1995; Sheldrake, 2016; Stolp & Zabrucky, 2009). This has been called the "feeling-of- knowing accuracy" (Schraw, 1995, p. 401), a student's calibration (Pajares & Miller, 1994), or incongruous mathematics self-efficacy (IMSE) (Russell & Phelps-Gregory, 2022), the term we will use in this paper.

Students who have IMSE can either be “under confident” (their self-efficacy is lower than their external performance would suggest) or “over confident” (their self-efficacy is higher than their external performance would suggest). Confidence and self-efficacy are not the same (confidence is a broader domain general feeling and self-efficacy is a person’s specific subject-based beliefs in their performance) (Morony et al., 2013). However, for ease of communication, we will use these terms instead of under-efficacious and over-efficacious. IMSE can have multiple drawbacks on students’ learning. For example, over confidence might lead to students not studying or asking for help (Bandura, 1986).

Previous research on IMSE is lacking but does suggest that it exists. Several studies of incongruous self-efficacy (not in mathematics) have been conducted with prospective teachers. These studies have found that many prospective teachers demonstrate over confidence in their teaching efficacy, rating their teaching ability as higher than outside observers rate it (Dassa & Nichols, 2019; Wyatt, 2014). In terms of mathematics, previous research has shown evidence IMSE exists (Champion, 2010; Chen, 2006; Labuhn et al., 2010; Lopez & Lent, 1992; Russell & Phelps-Gregory, 2022; Sheldrake, 2016; Tellhed et al., 2017). However, this research is often limited in sample size and, in many cases, did not set out to identify IMSE. For example, Chen (2006) examined seventh graders’ MSE assessments and their teachers’ assessments of their abilities in mathematics problem solving. Chen found that students’ judgements were less aligned with their performance than their teachers’ judgements, suggesting at least some of the students had IMSE (but describing IMSE no further).

Why might IMSE develop and how might students hold these beliefs in spite of their performance? Previous research has confirmed Bandura’s (1986) theory that MSE develops based on four sources: (1) mastery experiences (performance on assessments and in mathematics courses), (2) vicarious experiences (judgements based on comparison with peers or colleagues), (3) social persuasions (verbal encouragement or discouragement from parents or teachers), and (4) affective or physiological state (a physical response, like crying during a mathematics test) (see, e.g., Hackett and Betz, 1982; Lopez and Lent, 1992; Zeldin et al., 2008). It is likely that these four sources also influence the development of IMSE though little research has examined this. One study, Sheldrake (2016), found that students with congruous self-efficacy appeared to have mostly based their self-efficacy on past performance. In contrast, Sheldrake found that under confident students appeared to have based their MSE on perceived peer comparison, perceived teacher encouragement, and interests in mathematics. The over confident students appeared to base their beliefs on their high perceived belief in the utility of mathematics. This work suggests that students may use different sources to form their MSE and which source they use may determine if they develop IMSE or not. However, further work is needed to examine this relationship. In addition, no work has examined how students continue to hold these beliefs in spite of their performance (or adjust their beliefs in response to continued performance measures that run contrary with their beliefs). In this paper, we sought to examine this. Our research question was:

How do undergraduate intermediate algebra students who have IMSE (over confidence) reconcile their self-efficacy beliefs with their performance during the semester?

Methods

To identify students with IMSE, we conducted a larger study. Participants in the larger study were undergraduate students enrolled in one of many intermediate algebra courses at a Midwestern public university. Students in this course who consented were given a survey to identify their MSE and then students with high and low MSE were invited to participate in two interviews and to complete two additional surveys, administered at various points throughout the semester. Fifteen participants were invited to interviews in the larger study, 11 identified as women and 4 identified as male, non-binary, or transgender. To protect the identities of the participants, we have given them all pseudonyms (using female or non-binary pseudonyms), and we will use non-binary pronouns (they/them) for all participants.

Interview 1 included questions designed to better understand students' self-efficacy generally and their self-efficacy for intermediate algebra in particular. In it, we asked the students questions such as "How would you describe yourself as a math student?" and "On your [first] survey you responded by saying [paraphrase], can you tell me why you feel that way?" Survey 2 was an online survey administered immediately after the first exam, before students received their grades for the exam. In it, participants were asked about how they thought they had performed on the first exam and how they expected to perform in the course overall. Interview 2 occurred after students had received their exam grades and before the next exam; we again asked participants about their MSE beliefs as related to their experience with the first exam and about their MSE for the course overall. There was overlap in what the interviews covered because they were designed to examine changes in participants' MSE throughout the semester. Finally, survey 3 was administered electronically at the end of the semester once the students had received their final grades. This survey included questions about the participants' final grades as well as how they felt about their MSE after the course was over and how it compared to their experiences during the semester. The final survey also asked students questions about how their second and third exams went, including questions for students to describe how they felt leading up to, during, and after each exam compared to the grade they received. Having these final pieces of data enabled us to compare students' MSE beliefs longitudinally throughout the course and to compare how those beliefs relate to their final grade.

The interviews were transcribed and coded for analysis. The codes were based on a pilot study with additional codes developed inductively based on the data (Lichtman, 2012). To ensure trustworthiness, we used peer debriefing, member checks, and a coding reliability check with a third (non-author) researcher (Campbell et al., 2013; Lincoln and Guba, 1985). Once coding was complete, we developed emergent themes based on the coding. We defined cases of IMSE as when a participant's predicted grade was at least one full letter grade (10 percentage points) apart from their actual grade. That is, a student who predicted they would receive a B- but received a C- or lower would be identified as having IMSE. For this paper, we focus specifically on three participants who demonstrated IMSE for the course. The three participants were over confident, expecting to perform better in the course than they actually performed. We will describe how, over the course of the semester, these three participants reconciled their MSE with their ongoing lower than expected performance with the larger goal of better understanding IMSE and how MSE develops and changes as a result of repeated performance measures.

Results

Michelle, Lisa, and Adrian were all over confident in their ability to succeed in intermediate algebra. Michelle predicted (at the start of the semester) they would receive an A in the course but their final grade was a B. Lisa predicted they would receive an A- but received a C. And Adrian predicted they would receive a low B but failed the course with an F. We will examine each case individually with the goal of examining how each participant reconciled their MSE with their performance in the course. Across all three cases, we will see that participants did lower their MSE during the semester, though not always in line with their actual performance. And participants did use their algebra performance to judge their MSE; however, they often had reasons to doubt their performance would be predictive of their future success which may explain why they maintained their over confidence.

Michelle

When describing why they felt confident about their math work, Michelle said it was because they felt the material was mostly review. They said, "[I feel] pretty confident, especially in the class I am in now because the stuff we're doing, I have already done." When asked about understanding the material Michelle said, "Just from my academic background, I feel that I'll be able to get through the course fairly easy because I've done math similar to this before." In spite of this, by the end of the course, Michelle's performance was lower than they initially predicted.

We can see their grade predictions throughout the course in Table 1. They initially predicted they would receive an A, changed to an A or B after exam 1, to a B or A (with the B first) in interview 2, and ended the semester with a B. In interviews, Michelle remained confident even after exam 1 because they attributed their low score on exam 1 to them forgetting their calculator. That is, they said, "I didn't get as good as the grade I would have gotten I believe with a calculator." They also said, "I thought I would have done way better and like looking at the exam doing it… it's just little tedious mistakes, which I believe from not having a calculator kind of screwed me over." However, as the course went on, Michelle continued to perform lower than they expected. After the first exam, when asked what they would do differently, Michelle said "I would bring a calculator, and maybe go slower… maybe some of the mistakes I could have prevented but I honestly think most of them were from the calculator." After the third exam they stated, "I was nervous about this exam and ended up making dumb mistakes. So, I was disappointed with this score (71%) because it was low, because of little mistakes and not actually because I don't know the concepts."

Table 1. Over Confident Grade Predictions Over the Semester.

Participant	First Prediction (before exam 1)	Survey 2 Prediction (before grade for exam 1)	Interview 2 Prediction (after exam 1 grade)	Final Grade
Lisa	A-	A	B or C	C
Michelle	A	A or B	B or A	B
Adrian	Low B	High C, Low B	C	F

In general, Michelle initially seemed to be over confident because they believed the material was review. However, they then seemed to maintain their overconfidence because they blamed their lower performance on what they termed "little mistakes" and forgetting a calculator. This type of reasoning, blaming something external like a calculator, is often called an external attribution (Kelley & Michela, 1980). In the end, Michelle did reconcile their performance with their MSE by slowly lowering their MSE (suggesting a B was possible by Interview 2) but they never entirely gave up their prediction of receiving an A. That is, Michelle seems to be a case of remaining slightly over confident throughout the semester, with some reconciliation happening around Interview 2.

Lisa

Like Michelle, Lisa also believed the math in the intermediate algebra course was review. Lisa said "I've noticed, especially in [intermediate algebra], I feel like it's the same stuff I learned in Algebra 2 [High School]. So, I feel like I'm reviewing a lot of the time." And "It kind of just feels like I've been reviewing, I mean I know it's only like four weeks in… So, I don't really know what to expect in the future, but as of right now, I feel like it's all review." Based on this initial belief, Lisa predicted they would receive an A- in the course. Table 1 shows their grade predictions throughout the course of the semester. We see that after exam 1 (before they received their score) their MSE had increased, and they believed they would receive an A. However, by Interview 2, they had lowered their MSE closer to their actual final grade. In Interview 1, they said they felt "pretty confident" in the course but, when asked how they felt in the second interview, they said "not very confident." Despite this, Lisa continued to be somewhat surprised by their exam scores throughout the semester; for example, regarding the third exam Lisa said, "I thought I'd do better on this exam."

Lisa's initial over confidence seemed to stem from their belief that the course was review and this over confidence seemed initially reinforced by the exam. In addition, like Michelle, Lisa also demonstrated an external attribution for her lower performance, and this seemed to help maintain her over confidence. In particular, Lisa partially blamed the instructor for their lower-than-expected scores. Lisa said, "[My professor] doesn't teach in a way that I have normally been taught to do math and this unit is something that I've never done before like rational expressions and equations. So, I've been struggling." Lisa also stated of their instructor "I don't really think my teacher has covered enough of [the material] for me to understand." As the semester went on, Lisa also felt the material became more difficult, which may be why they lowered their MSE. These lower-than-expected scores were motivating to them, causing them to study harder which helped them sustain higher MSE (believing the extra studying would raise their performance). Regarding the final exam they stated, "I definitely studied a lot more leading up to the final after I received my previous test grades." Despite this, Lisa received a C in the course and, in the final survey, expressed not understanding the material as well as they thought.

In the end, Lisa did not reconcile their performance with their MSE until Interview 2, where they predicted a C was possible and described themselves as not very confident. Lisa seems to be a case of being over confident in the beginning but, as the difficulty of the material increased and they continued to perform poorly, reconciling their MSE and performance by lowering their MSE. In addition, we also see that external attributions for poor performance helped Lisa maintain their over confidence.

Adrian

Adrian demonstrates a different reason for over confidence than Michelle and Lisa. Adrian had actually taken the course before, referencing that this was their third or fourth time through this same material. From the beginning, they expressed a lot of doubt in their abilities to succeed in math courses in general; they specifically said that they were not confident, yet were hopeful, that they would pass intermediate algebra at the college level. Adrian initially predicted they would receive a low B in the course.

As shown in Table 1, between Interview 1 and Interview 2, Adrian's MSE did fall but did not align with their final course performance. In fact, Adrian's MSE never fully aligned with their performance. There seemed to be several reasons for this. First, Adrian responded to lower-than-expected performance by working more and getting help. They said, "I attempted to study more, by working out more problems and getting help from others in the class." Overall, Adrian seemed to believe working out more problems and working with classmates would result in better performance though, sadly, it did not. Adrian also believed that their homework and quiz scores could make up for poor exam performance, leading them to believe they could still perform well in the course. They said, "I know that I absolutely will not ace any exams, but I can probably still pass [the course] with the quizzes and homework." Unfortunately, this is not true with the grade calculations of the course.

Finally, one other reason for Adrian's high MSE seemed to be that, during the course, they were diagnosed with ADHD for the first time. In the second interview, Adrian seemed more hopeful that the course would go well because they finally felt like they could learn to navigate having ADHD and then do better in the class. Adrian said, "Currently I'm trying to find a medication that works for my ADHD and getting accommodations... So that may help once I find the right medication [and] actually get the accommodations." Adrian seemed to attribute their past low performance externally, to ADHD, and seemed to believe that fixing this ADHD would help them succeed. However, in the end, Adrian was not able to pass the course.

Across the Cases

Looking across the three cases, two themes emerge. First, all three students did lower their MSE during the semester. That is, their continued lower than expected performance did lead to lower confidence in their ability to be successful in intermediate algebra. However, this reconciliation was slow (happening around interview 2 and not always ever fully aligning with their final grade). Second, the reason this reconciliation was slow varied by participant but included students externally attributing their poor performance (it was the teacher, the lack of a calculator, or ADHD) as well as believing that changes (studying more or getting ADHD treatment) would raise their future performance. Overall, we see that all participants doubted that their initial exam scores were predictive of future performance (allowing them to maintain over confidence). Their MSE only started to lower once they had repeated lower performance.

Discussion

Previous research has shown IMSE can exist though this research often lacked detail, was not at the collegiate level, or was not in mathematics (Chen, 2006; Dassa & Nichols, 2019; Wyatt, 2014). Our work shows that IMSE does exist in collegiate mathematics and describes qualitatively what that IMSE can look like. In addition, no previous research had examined how IMSE was reconciled with performance. Our study shows that over confident students do slowly reconcile their IMSE with their performance by lowering their MSE. However, this can be slow because students externally attribute their low scores and have reasons to doubt their performance scores will be predictive (because they are changing their behavior).

Recall that Sheldrake (2016) found that students with IMSE may base their MSE on sources other than mastery experiences (performance). Sheldrake found, in particular, that over confident students based their MSE beliefs on the perceived utility of mathematics. Unlike in Sheldrake's study, our participants never mentioned the utility of mathematics as a reason for their MSE; however, given they were taking a mathematics course required for graduation, they were presumably aware of the utility for their degree. Also in contrast with Sheldrake, we did not find that participants were basing their MSE on sources other than performance; that is, they were not basing their MSE on social comparison, vicarious experience, or affective/ physiological state. Instead, we found that participants with IMSE were using performance as a source of MSE but had reasons to doubt their performance measures were predictive of the future because they externally attributed their low performance and changed their behavior. However, these external attributions and behavior changes did not lead to them being more successful.

Limitations and Future Work

The first limitation of this study is the small sample size. Having only fifteen participants, all from the same university, limits our ability to extrapolate results. The choice of a small sample size allowed us to collect detailed qualitative data at five points in time throughout the semester. However, while we were able to identify three cases of over confidence, we were unable to identify cases of under confidence. It is unclear if this is because this is rare or if participants with under confidence simply did not participate in the study. This could be a topic for future research where larger sample sizes might help better identify such participants.

An additional limitation is the self-reporting nature of the surveys. Self-reporting is necessary to capture the MSE beliefs of students at a point in time and is common in previous research on MSE (Dassa & Nichols, 2019; Hackett, 1985; Pajares & Miller, 1995). However, one issue that can arise is students' desire to give the "correct" answer, how they think they "should" feel. To account for this, in the interviews, we asked students to describe and elaborate on their MSE beliefs multiple times and then used member checking to ensure trustworthiness. However, it is always possible that students did not share the full truth with us as researchers.

Implications for Practice

The findings in this study indicate that instructors should be aware of students who are under or over confident and potentially take measures to help students calibrate their MSE beliefs with their performances. This is not to say that we need to go around lowering or raising students' MSE beliefs, but rather that we should try to help students become aware of their own abilities in realistic ways. Perhaps we can design instructional material that includes components of introspection alongside the content. Such introspection could increase the likelihood that students will adjust their MSE beliefs accordingly.

Another aspect of this study that is informative for instructors is that MSE beliefs can change over the course of the semester. What this means for instructors is the need to "keep a pulse" on their students and their MSE beliefs. This is especially true with repeated performances that are not what the student initially expected. Instructors could help students by encouraging them to have a personal meeting with the instructor after the first or each exam. This session would not have to be very long, but it would be a time to address the student's performance on the exam, any questions they missed, and the long-term effects of the exam and their current homework or quiz performances on their grade. This counseling session could help students to have a better understanding of the situation they are in and calibrate their MSE and performance better.

Overall, our results show that IMSE exists, is a result of specific reasons from the students' course (e.g., initially seeming easy), and is based on factors similar to MSE but with students perhaps interpreting these factors differently than instructors. Learning about IMSE serves as an opportunity for instructors to better understand their students and help their students better understand themselves and what leads to their success. If we can continue the pursuit of helping students achieve their goals and learn about themselves, we can increase student success in mathematics.

References

Bandura, A. (1986). Self-Efficacy. In A. Bandura (Ed.) *Social foundations of thought and action* (pp. 290 – 453). Englewood Cliffs.

Campbell, J. L., Quincy, C., Osserman, J., & Pedersen, O. K. (2013). Coding in-depth semistructured interviews: Problems of unitization and intercoder reliability and agreement. *Sociological Methods & Research*, *42*(3), 294 – 320. DOI: 10.1177/0049124113500475

Champion, J. (2010). *The mathematics self-efficacy and calibration of students in a secondary mathematics teacher preparation program.* [Doctoral Dissertation, University of Northern Colorado]. https://www.proquest.com/docview/516275195?pq-origsite=gscholar&fromopenview=true

Chen, P. (2006). Relationship between students' self-assessment of their capabilities and their teachers judgements of students' capabilities in mathematics problem solving. *Psychological Reports*, *98*, 765 – 778.

Dassa, L., & Nichols, B. (2019). Self-efficacy or overconfidence? Comparing preservice teacher perceptions of their content knowledge and teaching abilities to the perceptions of their supervisors. *The New Educator*, *15*(2), 156 – 174.

Hackett, G. (1985). Role of mathematics self-efficacy in the choice of math-related majors of college women and men: a path analysis. *Journal of Counseling Psychology*, *32*(1), 47 – 56.

Hackett, G., & Betz, N. (1982). Mathematics self-efficacy expectations, math performance, and the consideration of math-related majors. *Proceedings of the Annual Meeting of the American Educational Research Association*. https://files.eric.ed.gov/fulltext/ED218089.pdf

Hutchison, M. A., Follman, D. K., Sumpter, M., & Bodner, G. M. (2006). Factors influencing the self-efficacy beliefs of first-year engineering students. *Journal of Engineering Education*, *95*(1), 39 – 47.

Kelley, H. H., & Michela, J. L. (1980). Attribution theory and research. *Annual Review of Psychology*, *31*, 457 – 501.

Labuhn, A. S., Zimmerman, B. J., & Hasselhorn, M. (2010). Enhancing students' self-regulation and mathematics peformance: the influence of feedback and self-evaluative standards. *Metacognition Learning*, *5*, 173 – 194.

Lichtman, M. (2012). *Qualitative research in education: a user's guide* (3rd ed.). Los Angeles, CA: Sage Publications.

Lincoln, Y., & Guba, E. (1985). *Naturalistic inquiry.* Beverly Hills: SAGE Publications, Inc.

Lopez, F. G., & Lent, R. W. (1992). Sources of mathematics self-efficacy in high school students. *The Career Development Quarterly*, *41*, 3 – 12.

MacPhee, D., Farro, S., & Canetto, S. S. (2013). Academic self-efficacy and performance of underrepresented STEM majors: gender, ethnic, and social class patterns. *Analyses of Social Issues and Public Policy*, *13*(1), 347 – 369.

Marra, R. M., Rodgers, K. A., Shen, D., & Bogue, B. (2009). Women engineering students and self-efficacy: a multi-year, multi-institution study of women engineering student self-efficacy. *Journal of Engineering Education*, *98*(1), 27 – 38.

Morony, S., Kleitman, S., Lee, Y. P., & Stankov, L. (2013). Predicting achievement: confidence vs self-efficacy, anxiety, and self-concept in Confucian and European countries. *International Journal of Educational Research*, *58*, 79 – 96.

Multon, K. D., Brown, S. D., & Lent, R. W. (1991). Relation of self-efficacy beliefs to academic outcomes: a meta-analytic investigation. *Journal of Counseling Psychology*, *38*(1), 30 – 38.

Pajares, F., & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: a path analysis. *Journal of Educational Psychology*, *86*(2), 195-203.

Pajares, F., & Miller, M. D. (1995). Mathematics self-efficacy and mathematics performances: the need for specificity of assessment. *Journal of Counseling Psychology*, *42*(2), 190 – 198.

Russell, B., & Phelps-Gregory, C. M. (2022). Global and task-specific mathematics self-efficacy and performance: A qualitative study in intermediate college algebra. *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Schraw, G. (1995). Measures of feeling-of-knowing accuracy: a new look at an old problem. *Applied Cognitive Psychology*, *9*, 321 – 332.

Sheldrake, R. (2016). Differential predictors of under-confidence and over-confidence for mathematics and science students in England. *Learning and Individual Differences*, *49*, 305 – 313.

Stolp, S., & Zabrucky, K. M. (2009). Contributions of metacognitive and self-regulated learning theories to investigations of calibration of comprehension. *International Electronic Journal of Elementary Education*, *2*(1), 7 – 31.

Tellhed, U., Bäckström, M., & Björklund, F. (2017). Will I fit in and do well? The importance of social belongingness and self-efficacy for explaining gender differences in interest in STEM and HEED majors. *Sex Roles*, *77*, 86 – 96.

Wyatt, M. (2014). Using qualitative research methods to assess the degree of fit between teachers' reported self-efficacy beliefs and their practical knowledge during teacher education. *Australian Journal of Teacher Education*, *40*(1), 117 – 145.

Zeldin, A. L., Britner, S. L., & Pajares, F. (2008). A comparative study of the self-efficacy beliefs of successful men and women in mathematics, science, and technology careers. *Journal of Research in Science Teaching*, *45*(9), 1036 – 1058.

Knowledge Resources for Multiplication and Challenges Reasoning About the Product Layer

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Research on students' construction and interpretation of integrals has grown significantly over the past decade. Some work has identified classes of interpretation for definite integral notation that include areas under curves, antiderivatives, and adding up pieces. The present study contributes to recent research on challenges students experience with the "product layer," $f(x) \cdot dx$, *when reasoning about adding up pieces. In particular, we present a case study in which 1 college student in a first-semester, calculus-based physics course drew on multiple knowledge resources associated with multiplication as she tried to construct a definite integral in a situation that for her was novel. Our results suggest that students' understandings of multiplication with quantities is understudied in the literature on integration and an important direction for further research.*

Keywords: Calculus, Integration, Product layer, Knowledge-in-Pieces

Introduction

An earlier generation of research on the teaching and learning of calculus focused primarily on limits and differentiation (e.g., Larsen et al., 2017; Rasmussen et al., 2014). Although research on integration goes back several decades (e.g., Orton, 1983), research on the topic has accelerated considerably in the last decade (e.g., Jones & Ely, 2023). We concur with Oehrtman and Simmons' (2023) recent observation that research to date has focused on "students' meanings associated to components and relationships within the standard definition of a limit of Riemann sums" (p. 36). Less research exists on students' moment-to-moment reasoning when constructing integrals to model problem situations. Furthermore, in our reading, existing literature has hinted at challenges students can experience coordinating multiplicative relationships between quantities with understandings of integration. This same literature, however, has stopped short of examining such challenges closely.

In the present study, we analyze in detail the reasoning of Larisa, one college student drawn from a larger study in which we examined how first-semester, calculus-based physics students coordinated reasoning about multiplication with quantities with reasoning about definite integrals. The analysis provides an existence proof that such coordination can be complex and, thus, suggests that inattention to students' reasoning about multiplication with quantities is understudied in the literature on integration and an important direction for further research. We asked the following research questions:

1. *What knowledge resources did Larisa evidence when reasoning about multiplication with quantities?*
2. *How did Larisa coordinate her multiplication resources with understandings of definite integrals to solve what was for her a novel problem?*

The first question asks about resources related to multiplication broadly and independently of integration. The second question avoids situations where Larisa might simply recall previously learned relationships—for instance, that velocity is the integral of acceleration.

Literature Review

Although definite integrals are defined as limits of Riemann sums, researchers have debated the role of these sums in students' reasoning. Jones (2013, 2015a) drew on Sherin's (2001) symbolic forms to classify how nine college students interpretated definite integral notation. He reported three primary forms. For the *perimeter and area* form students interpreted the integrand, the limits of integration, and the x-axis as boundaries of an enclosed region *not* partitioned further or approximated. For the *function matching* form, students interpreted the differential, *dx*, as indicating that the function to be integrated, *f*(*x*), is the derivative of some other function that is yet to be determined. For the *adding up pieces* form, students interpreted the $f(x)dx$ notation as indicating small amounts of a target quantity that are then added. Finally, if one interprets $f(x)dx$ as a product, then one can think of a multiplicatively-based summation (e.g., Jones 2015a). Thus, this last form is well-aligned with the definition of a Riemann sum. (Thompson & Silverman, 2008, developed an alternative approach to integration based on *accumulation from rate*.)

We highlight four themes related to the *adding up pieces* form. First, even after instruction, Riemann sums often remain peripheral to students' understanding of definite integrals (e.g., Jones, 2015a, 2015b; Jones et al., 2017; Rasslan & Tall, 2002). Second, students can be confused by using rectangular areas to model other quantities, such as distances (Thompson et al., 2013). Third, within the traditional $\int_a^b f(x)dx$ notation students can have trouble interpreting $f(x)dx$ as expressing multiplication (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). Sealey provided the most direct result: She decomposed reasoning about definite integrals into five layers—orientation, product, summation, limit, and function—and reported that college calculus students had the most difficulty reasoning about $f(x)dx$ in the product layer. She concluded that students had trouble not with calculations but with "**understanding what is being multiplied together and what quantity is produced from that multiplication**" (p. 240). Fourth, there is debate over the range of situations to which the multiplicatively-based summation and Riemann sum interpretations can apply. Meredith and Marrongelle (2008) claimed that (a) multiplicatively-based summation only makes sense when the quantity being integrated can be conceived of as a rate and (b) thinking about the electric field at a point, q_1, due to a bar charge is better thought of as summation of small effects. We question whether this really is a counter-example: The small effects in this linear situation can be thought of as the accumulation of the rate of field strength per unit of charge, $\frac{k}{r^2}$, multiplied by the charge in small segment of the bar, Δq, a distance *r* from q_1.Thus, conceiving of the quantity being integrated as a rate applies more broadly than previously acknowledged.

The present study builds most directly on Oehrtman and Simmons' (2023) recent discussion of *Emergent Quantitative Models* for definite integrals, which consists in turn of three primary models. A *basic* model is based on constant values for quantities. A *local* model is an adaptation of a basic model to a small portion of an object or event in which quantities may be approximated as constant. A *global model* is derived from the accumulation of local models. These researchers presented examples to demonstrate that student reasoning can involve varied interactions among basic, local, and global models. Examples include making different choices for (a) which factor in a product to treat as infinitesimal and (b) partitioning. At the same time, they stopped short of considering students' ecology of knowledge resources for multiplication.

Theoretical Framework

Our theoretical frame combines mathematical structure (as perceived by experts) with cognitive components evidenced by students. Vergnaud (1983, 1988) analyzed mathematical structures for multiplication with quantities and distinguished two subtypes of multiplication situations. In an *isomorphism-of-measure-spaces* situation (I-O-M) the multiplicative structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). An example is a measure space for time in seconds, a measure space for distances in meters, and the relationship $\frac{meters}{second} \cdot seconds$. In a *product-of-measure-spaces* situation (P-O-M) the multiplicative structure consists of the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). An example is a measure space for area in cm^2, a measure space for height in cm, and the relationship $area\ (cm^2) \cdot height\ (cm)$. The distinction between I-O-M and P-O-M situations highlights two different ways one might interpret $A \bullet B$ as a basic model.

For the cognitive component, we draw from the knowledge-in-pieces epistemological perspective. The perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006) and has been applied to various topics in mathematics, including multiplication (Izsák, 2005; Izsák et al., 2021), functions (e.g., Moschkovich, 1998), and integrals (Jones, 2013). From this perspective, reasoning is supported by diverse, fine-grained knowledge resources and novice knowledge evolves into expert knowledge through processes such as the construction of new knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some resources and loosing connections among others. In the present study, we examined knowledge resources Larisa cued and evidenced when reasoning about multiplication with quantities in the context of definite integrals.

Methods

We conducted one-to-one, semi-structured interviews (e.g., Bernard, 1994; Ginsburg, 1997) to assess how physics students reason about multiplication with quantities and the integral concept in both I-O-M and P-O-M situations. In Spring 2023, we recruited five students enrolled in first-semester, calculus-based physics at a selective university. Each participant completed three, 1-hour interviews spaced a few weeks apart.

The first interview asked students to reason about three situations—two I-O-M situations about estimating distance from varying velocity and one P-O-M situation about estimating volume from varying cross-sectional areas. The tasks were ones for which one could first multiply and then add to construct a total, but there was no mention of integration. We paid close attention to how students assigned units of measurement to quantities when multiplying. The second interview asked students to reason about two P-O-M problem situations (one estimating volume from cross-sectional areas and one about person-hours) and then to interpret the $\int_a^b f(x)dx$ notation in the context of each. We paid close attention to how students reasoned about units and graphical representations when interpreting integrals. The third interview asked students to reason about two I-O-M situations (one about water pressure on wall and one about acceleration from rest) and then to interpret the $\int_a^b f(x)dx$ notation in the context of each. Again, we paid close attention to how students reasoned about units and graphical representations when interpreting integrals.

We recorded the interviews using two cameras, one to capture the student (body movements, hand gestures, etc.) and the interviewer and one to capture close ups of the student's written work. We collected all written work for later analysis at the end of each interview. We used a computer to synchronize and combine the two video recordings file into a single file. In addition, we transcribed the interviews verbatim.

We watched videos for all five students side-by-side with the transcripts. After multiple viewings of the interview data, we narrowed our attention to Larisa and studied her as a particularly good case for our research questions. She provided evidence for several resources related to multiplication with quantities but, at the same time, coordinating those resources with her understandings of definite integrals was not straight forward, particularly when working on the Water Pressure task in Interview 3. We then traced Larisa's spoken language, hand gestures, and drawings in more detail to make sense of her reasoning.

Results

Larisa was majoring in engineering and self-reported good grades (B range) in her prior precalculus and calculus courses and on the calculus AB exam. We summarize a few key points from Interview 1 and Interview 2 before discussing in detail her performance in Interview 3.

In Interview 1, Larisa used the trapezoid method on a velocity graph to estimate distance traveled and focused on computations—for instance, when asked where she saw distance in her work, she canceled units in $\frac{feet}{\cancel{second}} \cdot \cancel{seconds}$. She appeared to understand such cancelation as a formal method for determining units for the product, but she did not connect her symbol manipulation to her graph. She also gave a fluent explanation for how to estimate the volume of liver when given cross-sectional areas in cm^2 and spaced 2 *cm* apart. In particular, she recalled multiplication from the formula "length times width times height gives you volume." In Interview 2, Larisa considered a similar liver problem when given cross-sectional areas in cm^2 and spaced 1/3 *inches* apart. This time, she discussed the integral as giving the area under a curve; but, her explanation for the role of multiplication was imprecise. At one point she commented "if you multiply the *x* axis by the *y* axis….you get the whole area under [the graph]." At another point, she commented "I just kind of know you have to write the *dx*. I kind of, like I haven't done integrals in a while, so I kind of forget why it's exactly that way, but it's like this means integral. So this means area under the curve." Finally, she mentioned "the integral is the opposing, I guess the inverse of the derivative." Thus, in Interview 1 and Interview 2, Larisa articulated two meanings for integrals (areas under curves and anti-derivatives) and recalled how multiplication related quantities in familiar situations; but, she did not provide evidence for interpreting $f(x)dx$ as a product.

For Interview 3, we adapted the Water Pressure task (Figure 1) from prior studies (e.g., Oehrtman & Simmons, 2023; Sealy, 2014). In so doing, we deliberately omitted direct references to multiplication and integration because we wanted to see how students would reason about these spontaneously in what for them was a novel task. We also omitted units for pressure and for force, because we did not want students to rely on formal cancelation of units. Using Oehrtman and Simmons' (2023) terms, the basic model would be *pressure* • *area* = *force*, or $P \bullet A = F$, and the corresponding local model would be $15x \bullet 4\Delta x$. Larisa did not produce a complete correct solution, but the multiplication knowledge resources she cued could play a central role in one or another of at least three different correct solutions. The first solution follows Oehrtman and Simmons' local model and computes $\int_0^3 15x \bullet 4dx = 270$. The second solution considers the average pressure on the wall. Because the pressure varies linearly with respect to depth, the total force on the wall is the pressure at the average depth multiplied by the area of the wall. In this case, 22.5 • 12 = 270. The third solution considers the force per 1-foot width of wall multiplied by the width of the wall in feet. In this case, $\int_0^3 15x\,dx$ • 4 = 67.5 • 4 = 270.

> Pressure, P, applied across a surface area, A, creates a total force, F. Consider a vertical side wall of a tank with a width of 4 feet and a maximum depth of 3 feet (see picture below). Assume that the tank is full of water. The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where x is the depth of the water (the deeper the water, the greater the pressure). Given this information, describe a method for approximating the total force from the water pressure on the wall.

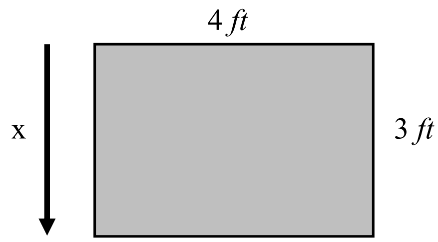

Figure 1. The Water Pressure task

Data

Upon reading the task aloud, Larisa reported that she did not recall working similar tasks in her prior experience. She graphed pressure as a function of depth, shaded the area under her graph, and stated that "taking like the integral of this bottom piece here would probably get you the total force kind of applied throughout" (Figure 2). She then computed the area of the triangle, $\frac{3 \bullet 45}{2} = 45 + \frac{45}{2} = 67.5$, and stated she was not sure what units to attach to force. Finally, she computed the area of the wall and multiplied 67.5 • 12, stating that 67.5 was the "average pressure." She concluded with the following explanation for how she saw multiplication in the situation:

> *Larisa*: Some pressure applied across a surface area creates a total force. That sounds like a very English way of saying like *P* times *A* equals *F*. Um, I think I would definitely like think you need to take the average in order to at least have it be an easier problem, because I don't know how you would do it if you were just kind of going like piece-by-piece as the pressure is increasing. That seems difficult and weird.

Analysis

Larisa relied on the text to cue multiplication in a novel situation and articulated an appropriate basic model, *P* • *A* = *F*. She also introduced both holistic and local approaches to the task. She initiated the holistic approach by cuing the notion of an average value, which she incorrectly associated with the integral or area under her graph of the pressure function. She then multiplied what she took to be the average pressure, 67.5, and multiplied by the area of the wall. This reasoning overlapped with the second solution discussed above when she multiplied by 12 and with the third solution discussed above when integrating the pressure function from 0 to 3.

When reasoning that integrating a pressure function would give an average pressure, Larisa did not seem aware that the units for the region under her graph would be different than those for pressure. This complemented the data from Interview 2 that multiplication with quantities was not well-coordinated with reasoning about integrals. At the same time, Larisa's discussion of “going like piece-by-piece” suggested that she gave initial consideration to a solution based on adding up pieces but predicted that this approach would be “difficult and weird.”

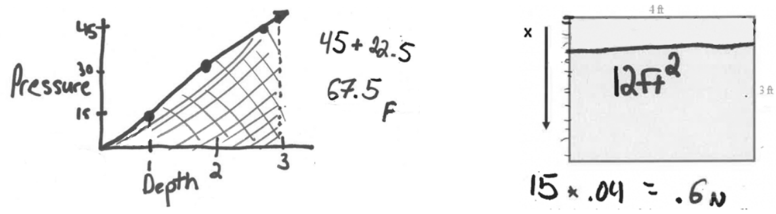

Figure 2. Larisa’s work on Water Pressure task (converted to grayscale)

Data

Larisa continued to compare her approach based on averages to an approach based on adding up pieces. On the one hand, she recalled how the meaning of the integral was explained in her past calculus classes: “That’s like how they explain integrals in class. They’ll explain it as like a ton of really small sections, and then you just add them all up.” At the same time, she remained skeptical: “I think if you were to do the force on like one of those really, really, really small sections, it would just be really small, because the surface area would be basically nothing. So you kind of need it to be a larger scale thing.” At the interviewer’s request, Larisa drew a horizontal line on the wall to indicate what she meant by a “small section” and wrote an appropriate multiplicative expression (i.e., the local model) for the force: $15 * .04 = .6\ N$ (Figure 2). She explained:

> *Larisa*: Like at 1 foot, and it’s 0.01 part of a section. So it would be 15, which is the pressure at that depth, and then times .01 times 4, so like .04, and then that would equal .6 Newtons. And that’s like just a really small amount. And like you can go smaller than that, and you like, it like just infinitely gets smaller to the point where like it doesn’t matter, nobody’s going to ask you that question. Because like you’re asking about just like the pressure on like a crack in the glass, and that’s not, that’s not helpful.”

Analysis

Larisa clearly recalled from past instruction the adding up pieces interpretation of the definite integral, and she gave a clear explanation for a normatively correct local model where she assumed a constant pressure over a thin, rectangular area. At the same time, even though she knew such reasoning would be encouraged in class, she rejected it in favor of “a larger scale thing.” Thus, Larisa’s appropriate local model was in tension with her sense of what could contribute to a consequential effect. Part of the issue may have been lack of experience accumulating many small contributions into a larger effect.

Data

An exchange later, the interviewer recalled Larisa’s comment about an English way of indicating multiplication and asked for more detail. Larisa responded:

> *Larisa*: Multiplication makes sense because I don’t, because addition doesn’t because, and I don’t know the units, but like in order to get a new unit, it would need to be multiplication or division. Because addition would just be like, you can’t add pressure and surface area. It’s not like terms. You would need to multiply them to create a new term, which would be force.

She reiterated that “total” cued multiplication and explained that division would be for a “specific point.” A few exchanges later, she restated that the area under her graph for the pressure function gave an average pressure and provided another example analogous to her 67.5 • 12 calculation—she considered 1 foot of water depth and multiplied the triangular area under the pressure graph by the area of a 1 *ft*-by-4 *ft* section of wall, 7.5 • 4.

Finally, after working for a while on another task, Larisa returned to the Water Pressure task, wrote "$\int of\ P$", computed $\frac{15x^2}{2}$, and explained that she was trying to see if this would also solve the problem. She evaluated her expression at $x = 3$ and got 67.5 once more.

Analysis

Similar to her work in Interview 1, Larisa demonstrated connections between multiplication and transformation of units when she said, "You would need to multiply them to create a new term, which would be force." We think it likely she was thinking about "new terms" when she performed her 67.5 • 12 and 7.5 • 4 calculations, even if she was unsure which units to use for pressure and for force. At the same time, she did not reference transformation of units when thinking about area under her graph of the pressure function. She characterized this area as the average pressure early on in her work and restated this interpretation after discussing multiplication to create a "new term." Somehow notions of area under a curve did not cue for Larisa a similar transformation of units.

Conclusion

To answer our first research question, Larisa demonstrated a variety of knowledge resources associated with multiplication, including some specific results (e.g., recalled *speed* • *time* = *distance* and *area* • *height* = *volume* relationships) and some general associations with finding totals, transforming units, and taking averages. She used these resources to distinguish multiplication situations from those calling for addition, subtraction, or division. Because each of these multiplication resources could support one or another of the three solutions we mapped for the Water Pressure task, none was inherently correct or incorrect. What mattered was how she employed subsets of the these resources when reasoning. To answer our second research question, Larisa mentioned antiderivatives in passing when working on the Water Pressure task, and discussed antiderivatives to a greater extent when working on further tasks about relationships among distance, velocity, and acceleration. More often, she attended to areas under curves, but did so in different ways. In Interview 1, she clearly broke up areas into trapezoidal pieces and added, but in Interview 3 she did not partition the area under her graph for the pressure function. Instead, motivated by what she perceived to be an efficient solution, she computed integrals as areas of triangles. From our perspective, when interpreting these areas as averages, Larisa combined using the total area of the wall from the second solution we mapped with the force per 1-foot width from the third solution we mapped. Again, using averages was not inherently correct or incorrect. Most striking was her use of multiplication to construct the normatively correct local model which she then rejected as less useful than a holistic approach. Larisa' use of relevant resources in combinations that do quite work and competition between a normatively correct local model with others of her resources underscore the complexity of coordinating reasoning about multiplication with reasoning about integration.

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References

Bernard, H. (1994). *Research methods in anthropology* (2nd ed.). Sage.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10*(2/3), 105–225. doi: 10.1080/07370008.1985.9649008

diSessa, A. A. (2006). A history of conceptual change research: Treads and fault lines. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 265–282). Cambridge University. doi: http://dx.doi.org/10.1017/CBO9780511816833.017

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior*, *48*, 152–167.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *The Journal of the Learning Sciences*, *14*(3), 361–403. doi: 10.1207/s15327809jls1403_2

Izsák, A., Beckmann, S., & Stark, J. (2021). Seeking coherence in the multiplicative conceptual field: A knowledge-in-pieces account*.* *Cognition and Instruction*, *40*(3), 305–350.

Jones, S.R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*, *32*(2), 122–141.

Jones, S.R. (2015a). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior*, *38*, 9–28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S.R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann-sum based conceptions in students' explanations of definite integrals. *International Journal of Mathematics Education in Science and Technology*, *46*(5), 721–736. https://doi.org/10.1080/0020739X.2014.1001454

Jones, S., Lim, Y., & Chandler, K. (2017). Teaching integration: How certain instructional moves may undermine the potential conceptual value of the Riemann sum and the Riemann integral. *International Journal of Science and Mathematics Education*, *15*, 1075–1095.

Jones, S.R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 8–35.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526–550). National Council of Teachers of Mathematics.

Meredith, D., & Marrongelle, K. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics*, *76*, 570–578.

Moschkovich, J. (1998). Resources for refining mathematical conceptions: Case studies in learning about linear functions. *The Journal of the Learning Sciences*, *7*(2), 209–237. doi: 10.1207/s15327809jls0702_3

Oehrtman, M., & Simmons, C. (2023). Emergent models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 36–61.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1–18.

Rasmussen, C., Marrongelle, K., & Borba, M. (2014). Research on calculus: What do we know and where do we need to go? *ZDM–Mathematics Education*, *46*, 507–515.

Rasslan, S., & Tall, D. (2002). Definition and images for the definite integral concept. In A. Cockburn & E. Nardi (Eds.). *Proceedings of the 26[th] annual meeting of the International Group for the Psychology of Mathematics Education*. PME.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*, 230–245.

Sherin, B. (2001). How students understand physics equations. *Cognition and Instruction*, *19*(4), 479–541.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43–52). Mathematical Association of America. http://pat-thompson.net/PDFversions/2008MAA Accum.pdf.

Thompson, P., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools*, *30*(1–2), 124–147.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). National Council of Teachers of Mathematics; Erlbaum.

Moving Between Abstraction Levels by Linking Recursion and Induction

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The relationship between mathematical induction (MI) and recursion compels us to ask how we could leverage recursive functions to bolster students' understanding of MI. We describe task-based interviews that utilized concurrent interactions with MI tasks and recursive functions that mirrored those induction tasks via a character-based user-interface. To gain insights into how students' conceptions of MI and recursion co-evolved as they interacted with these tasks, it was necessary to accommodate these multiple concurrent contexts by extending Hazzan's (1999) Reducing Abstraction framework. Our extended framework, called the Navigating Abstraction framework, documents ascending, descending, and transferring abstraction levels across contexts. Viewing the data through this lens allowed us to illustrate the way in which these three mechanisms together play a crucial role in students joint development of their understandings of both recursion and induction.

Keywords: Mathematical Induction, Recursion, Proof, Reducing Abstraction

In this paper, we study how students cope with the abstraction required to understand both mathematical induction (MI) and recursion by leveraging the link between them. Accordingly, we designed and implemented an instructional sequence of parallel recursion-MI tasks. In order to determine the abstraction level at which students understood recursion and MI, it was necessary to extend the Reducing Abstraction framework (Hazzan, 1999, 2003). Our extended framework, called the *Navigating Abstraction* framework, provided a lens for examining how students' understanding of recursion and induction co-evolved as they ascended and descended levels of abstraction as well as transferred levels of abstraction across contexts. We use the term *recursive program* to describe a computer program whose primary purpose is educational, but whose primary mechanism is recursion. Our study addresses the two research questions:

1. How does interaction with a recursive program (which mirrors an MI task) affect students' conception of MI?
2. How does concurrent interaction with both a recursive program and MI task affect student's understanding of the relationship between them?

Literature and Framework

Mathematical Induction involves a logical statement which is applied as a technique for proving statements involving countably infinite sets. In fact, taken separately, the abstract nature of proofs (Davis et al., 2009; Healy & Hoyles, 2000; Knuth et al., 2009), complex logical statements (Durand-Guerrier, 2003; Roh, 2010; Stylianides et al., 2004), and infinity (Davis et al., 2009; Wijeratne & Zazkis, 2015; Dubinsky et al., 2005) have all been shown to entail conceptual hurdles, resulting in a substantial body of literature on MI highlighting obstacles to its conceptualization (Baker, 1996; Brown, 2008; Dubinsky, 1989; Harel & Sowder, 1998). Research on students' understanding of MI (Brown, 2008; Davis et al., 2009; Harel, 2002) suggests that the inductive step may pose an obstacle to understanding MI. Furthermore, findings from these studies indicate that understanding the inductive step of MI requires a transition from perceiving MI as a process relating finitely many cases to interpreting MI as an infinite iterative process (Brown, 2008). Additionally, the statement of MI is a complex conditional statement regarding the truth value of a predicate function defined on the positive integers, and studies have found that conditional statements are often misinterpreted (Hoyles & Küchemann, 2002; Stylianides et al., 2004).

Given that the Reducing Abstraction framework (Hazzan, 1999, 2003) has been effectively used in both computer science education research (Hazzan & Hadar, 2005; Rich & Yadav, 2020; Sakhnini & Hazzan, 2008) and in mathematics education research (Hazzan, 1999; Hazzan & Zazkis, 2003; Raychaudhuri, 2014), we felt it was well suited for our Computer Science/Mathematics context.

Hazzan's (1999, 2003) framework is built on a variety of interpretations of abstraction that collectively form the basis of the framework. Each interpretation can be applied to examine students' understanding of concepts in pure mathematics, applied mathematics, computation, and data structures. We summarize the characterizations of each of her interpretations of abstraction below:

1. *Abstraction level as the quality of the relationship (QR) between the object of thought and the thinking person* (Hazzan, 1999) refers to the interpretation that abstraction is not an inherent property of a concept, but rather the relationship between the person thinking about the concept and the concept itself. For example, when students are confronted with a problem and are "fumbling in the dark without any mental object to hang onto" (Hazzan, 2003, p. 107), they may subconsciously reduce the level of abstraction by turning to a familiar concept, experience, or object.
2. *Abstraction level as reflection of the process-object duality (POD)* (Hazzan, 1999) is built on earlier work on the process object distinction (Dubinsky, 1991; Sfard, 1991*)*. This construct refers to the interpretation that conceiving a mathematical concept as a process is at a lower level of abstraction than examining it as a mathematical object. When a concept is understood as an object, one can examine the relationships between two concepts. However, if it is perceived as a process, the concept is expressed using canonical procedures (Hazzan, 2003).
3. *Abstraction level as the degree of complexity (DC) of the mathematical concept (Hazzan, 1999)* refers to the interpretation that the more complex an idea is, the more abstract it is. One may reduce the level of abstraction of an object by reducing its complexity. For example, a student may replace a set with an element of the set.

To demonstrate the utility of Hazzan's framework, we view the following results from existing MI literature through the interpretations the framework provides. (1) We could interpret Harel and Sowder's (1998) findings that students viewed MI as generalizing a conclusion by computing a few cases as reducing the abstraction level as the quality of the relationship (QR) because they were choosing to rely on more familiar computations of concrete examples rather than the unfamiliar concept of MI. (2) We can interpret Brown's (2008) observation that students had ignored some aspects of an MI task and "reduced the task to an algebraic" (p.13) process as reducing abstraction as a reflection of process-object duality (POD). Finally, (3) interpreting the results of Baker (1996) through the lens of reducing abstraction, we can conclude that by omitting the base case, students removed a conjunction from the hypothesis of MI. This rendered the complex statement comprising MI, which is of the form " if [P and (Q implies R)] then S" to a simpler statement "If [Q implies R], then S", thereby reducing the abstraction level of MI as the degree of complexity (DC). The above three examples demonstrate established results from MI literature can be viewed through the lens that the reducing abstraction framework provides.

During our analyses of students working on parallel MI-recursion tasks, it became clear that (a) students sometimes engage with recursion and MI at different levels of abstraction and (b) as students used parallel contexts, their movement between levels of abstraction was not limited to reducing levels of abstraction. These observations catalyzed our expansion of the framework. We call this expanded framework the *Navigating Abstraction* framework. The expanded framework utilizes constructs not included in Hazzans (1999) framework. More specifically, we say that one *ascends levels of abstraction* when they move from a lower to a higher level of abstraction and *descends levels of abstraction* when they move from a higher to a lower level of abstraction. Additionally, we identify instances when a student navigates through different aspects of the task but remains on the same level of abstraction. When such a phenomenon occurs, we say that the student has *transferred abstraction levels* from one aspect to another. Therefore, our framework provides a lens through which we can make sense of the interplay between a student's movement between levels of abstraction and between the MI and recursion aspects of our tasks. This allows us to document how interacting with the tasks in this study facilitates the development of students' understandings of MI and recursion and the connection between them.

The framework for analyzing the level of abstraction as QR with respect to each aspect of a task is shown in Figure 1. There are analogous figures for navigating abstraction via the other interpretations in the framework, but they are too large to fit within the margins of this paper.

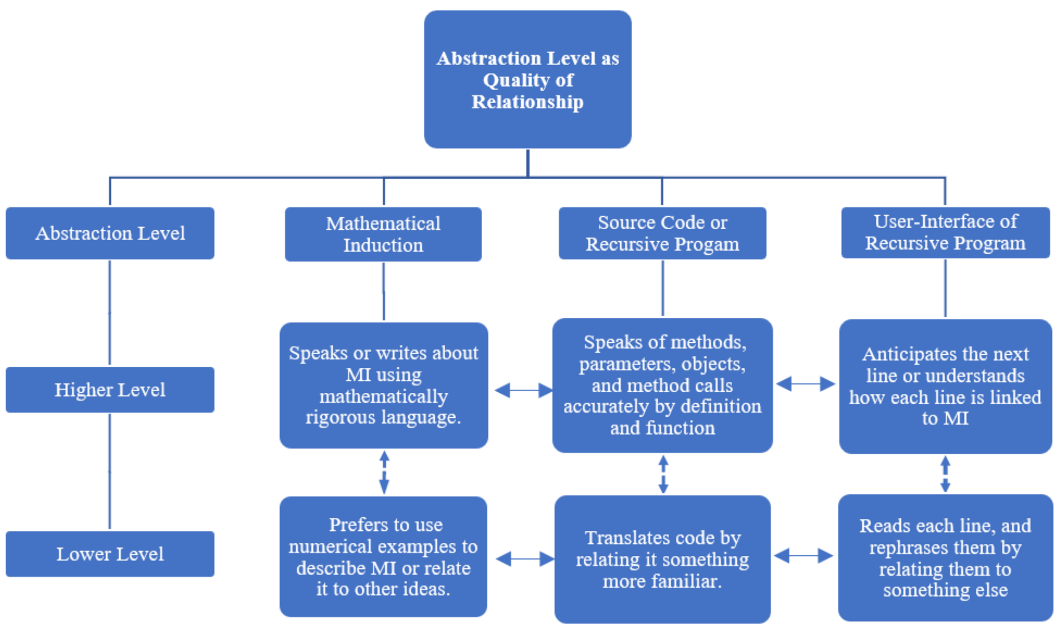

Figure 1. Framework for analyzing abstraction levels as QR with respect to the three aspects of each task.

Methods

Mario, Harry, and Sarah were senior math majors at a public university who took part in a 90-minute, semi-structured, task-based interview study, based on the exploratory teaching experiment methodology (Steffe & Thompson, 2000). The first author of this manuscript functioned as the teacher/researcher and performed the interviews.

The interviews consisted of tasks where MI was presented along with a user-interface and the source code for the corresponding recursive program, to which students could attend at their own pace. Students were prompted to think aloud while they worked, especially if the direction of their gaze changed from screen to paper, or vice-versa. While we found that the *Navigating Abstraction* framework was productive for accounting for our entire data set, due to space constraints, we limit ourselves to discussing Mario's work on Task 1 (The sum of the first n integers is $\frac{n(n+1)}{2}$) and Task 4 (Every positive integer can be written as a sum of powers of 2).

Results

To identify movement between abstraction levels, we focused on the utterances and inscriptions that indicated the participants were directing attention to mathematical induction (MI), the source code (RS), or the output of the recursive program (RP). We focus on two episodes that illustrate how Mario both transferred and navigated between abstraction levels. These examples simultaneously illustrate our *Navigating Abstraction* framework and provide insights into how a student might leverage the connections between MI and recursion to strengthen their understanding of both.

Mario Ascends Abstraction Levels

When Mario worked on Task 1 before accessing its corresponding recursive program, he only spoke about the computation required to validate the equation on which induction was to be performed. Mario's work, which can be seen in Figure 2, does not include explanations or justifications. Later in this manuscript, we will see that Mario understood MI the full logical complexity of its statement, but there was no indication that he was evaluating the validity or truth of any statement; therefore, he did not understand that MI involved a predicate. As seen in Figure 2, Mario considered the entire hypothesis of MI from his written work, but never states the conclusion of MI to finish the proof. We interpret this interaction as working at a lower level of abstraction as POD while working at a higher level of abstraction as DC.

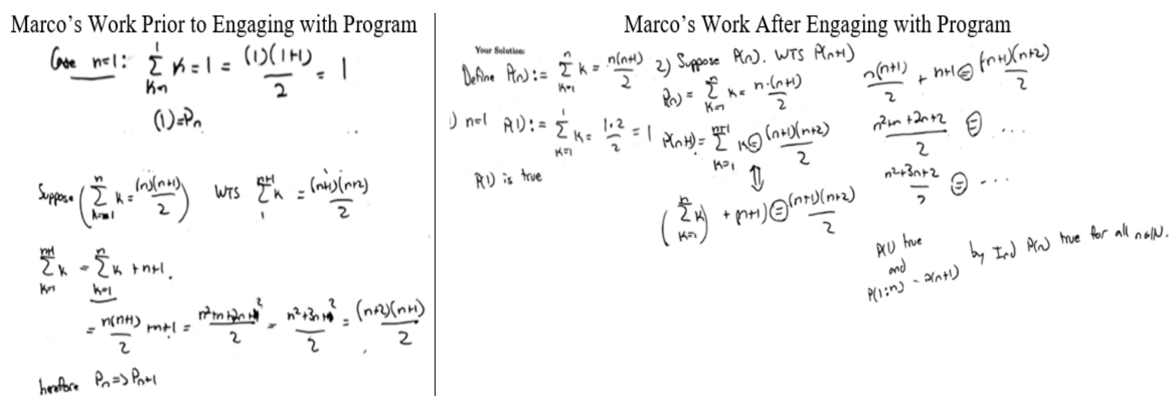

Figure 2. Marco's work on Task 1 before (left) and after (right) working with the recursive program.

The excerpt below highlights how he first makes sense of the MI task:

Mario: What I want to show is that the sum from 1 to n + 1 is n times $n(n+1)/2$. If I can show that this (the sum from 1 to n is $n(n+1)/2$) implies this (the sum from 1 to n +1 is $(n+1)(n+2)/2$). I have the sum that looks like the thing that I'm assuming I know what it is, and if I manipulate it, I should get the thing that I want.

While engaging with the RP aspect of Task 1, Mario first interpreted $P(n)$ not as a predicate, but as the partial sum from 1 to n, and then corrected himself: "Oh. It (the program) defines the statement ($P(n)$) to mean that those two things (left and right side of the equation) are equal. That's not the same thing." Mario expressed a shift in his thinking from $P(n)$ as an algebraic expression in n, to an algebraic statement in n which is either true or false. In other words, Mario was starting to conceptualize $P(n)$ as a predicate which indicates that he ascended levels of abstraction as POD with respect to the RP aspect of Task 1.

Mario chose to re-attempt the MI aspect of Task 1 and defined $P(n)$ as a function of n at the onset of the task, shown in Figure 2. After verifying the base case, he wrote "$P(1)$ is true", indicating that he now thought of $P(n)$ as a Boolean-valued function. Mario had transferred his perception of $P(n)$ as a predicate from the RP aspect of the task to the MI aspect of the task.

Furthermore, as seen at the bottom right of Figure 2, Mario finished the proof by writing the entire statement of MI, which was not included in the program's output or source code. Now that Mario is considering the entire conditional statement when working with the MI aspect of Task 1, we can deduce that he has ascended levels of abstraction as DC.

Mario navigates between abstraction levels

When attending to Task 4, Mario asked a question that he had not asked during any of the other tasks.

Mario: What's my $P(n)$ statement? My first thought is maybe I don't want to use induction on n, but maybe I do. I guess I'll try induction on n and see if that's the right thing. So what's my $P(n)$ statement? n is equal to a sum (pause) is equal to the sum over i equals something to something else.

Although Mario had not hesitated to define the predicate during the other tasks, his work in Figure 3 exhibits his indecision. In the previous tasks, MI was applied to a single-variable equation, but the equation generated by the statement in Task 4 had three unknowns. Mario's question demonstrates an unfamiliarity with the type of MI statement in Task 4. His discomfort with this new type of problem may explain his decision to work out some examples, as shown in Figure 3. Computing examples requires substituting variables with numerical values, consequently decreasing the number of variables. Thus, we could interpret Mario's decision to compute examples as working at a lower level of abstraction as QR with respect to the MI aspect of Task 4.

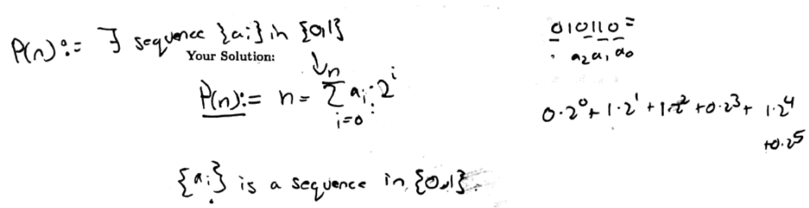

Figure 3. Mario's work prior to interacting with the recursive program.

To see more examples, Mario decided to use the recursive program for Task 4. By delegating the calculation of examples to the RP aspect of Task 4, Mario transferred the level of abstraction as QR from the MI aspect to the RP aspect of Task 4.

Mario: Okay, so we start, I guess an interesting case might be 15 because that's when we overflow the ones into a bunch of zeros.

Mario fluidly transitioned between the MI task and the recursive program's user-interface and source code. After each interaction with the program, Mario progressed further with the MI task. The excerpt below exemplifies the connections Mario was making between the MI aspect of Task 4 and the recursive function, Power2, located in the source code.

Mario: PowerTwo returns the string, which is yeah, the, the sum of powers of two that equal the number that you give it. So, how does it know to stop when n is greater or equal to

0?.. Oh, I have an idea. So, it recursively divides your, the integer by two. Okay and so I can do that too.

Interviewer: So, what gave you that idea?

Mario: Yeah, the PowerTwo. As long as we have a positive integer. Okay, it recursively divides by two and looks at it whether it's zero or one. So, I guess that is how you get the binary decomposition of a number. You keep dividing by two and you see if the thing you get is even or odd. So yeah, and this method uses that recursive loop to construct the sequence.

After connecting Power2 to a proof strategy, Mario resumed writing his proof. Note that Mario interpreted the code in Power2 in terms of its function as a program, indicating that he was working at a higher level of abstraction as QR with respect to RS aspect of Task 4.

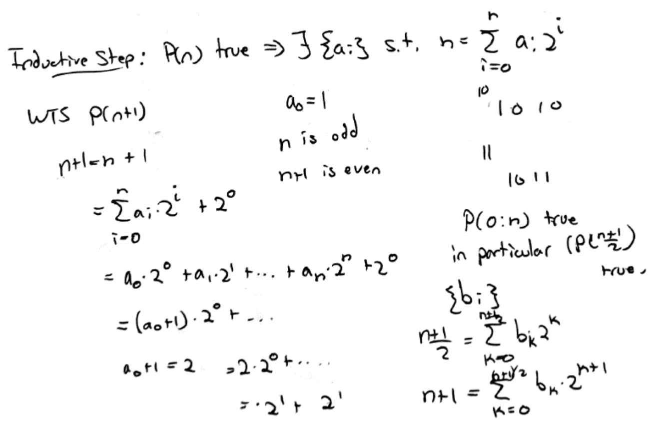

Figure 4. Mario's Proof of the Inductive Step of Task 4.

In his proof, shown in Figure 4, Mario divides $n+1$ by 2, then uses strong induction to represent $(n+1)/2$ as a sum of powers of 2. The sophistication of using two cases along with strong induction indicates an increase of rigor in his proof, suggesting that Mario transferred the higher level of abstraction as QR from the RS aspect to the MI aspect of Task 4. Hence, Mario ascended levels of abstraction as QR with respect to the MI aspect of Task 4.

During the final segment of the interview, the interviewer asked Mario how he conceived of mathematical induction. His response indicates that completing the tasks allowed him to make connections between recursion and induction that shifted his perception of MI.

Mario: Something I noticed. Instead of starting with *n* and going to *n* +1, you are decomposing *n* +1 into *n*…which reminds me a lot of recursion. Then induction is still *P*(*n*) implies *P*(*n*+1), but the way to do it is decompose *P*(*n*+1) into *P*(*n*) and that reminds me a lot of recursion.

To Mario, making the link between induction and recursion provided him with an additional strategy when approaching MI tasks. Specifically, he explains "if you know how to do an easy problem and you know how to decompose a hard problem into an easy problem, then you know how to do the hard problem."

Summary

During Task 1, Mario ascended abstraction as POD from the MI aspect of the task to the RP aspect of the task and then transferred the higher abstraction level from the RP aspect of Task 1 to the MI aspect of Task 1. Additionally, as shown in Figure 5, after interacting with the program, Mario ascended abstraction levels as DC with respect to the MI aspect of the task.

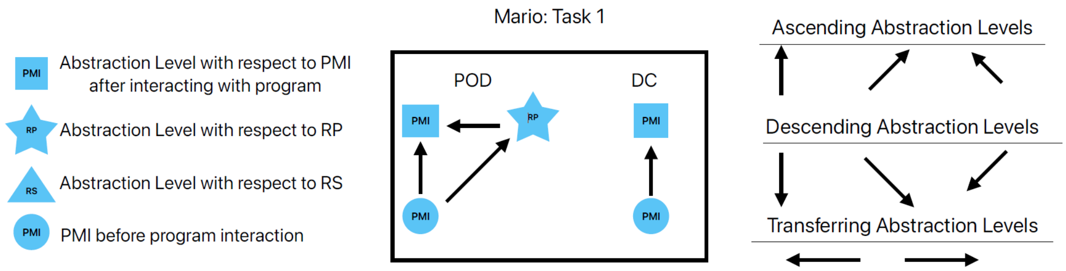

Figure 5. The Navigating Abstraction Framework applied to Mario's work on Task 1

During Task 4, Mario found it necessary to reduce the abstraction level as QR before ascending abstraction levels with respect to the MI aspect of the task. As shown in Figure 6, Mario transferred the lower level of abstraction as QR with respect to MI by interacting with the user-interface of the program and leveraged the source code to ascend levels of abstraction.

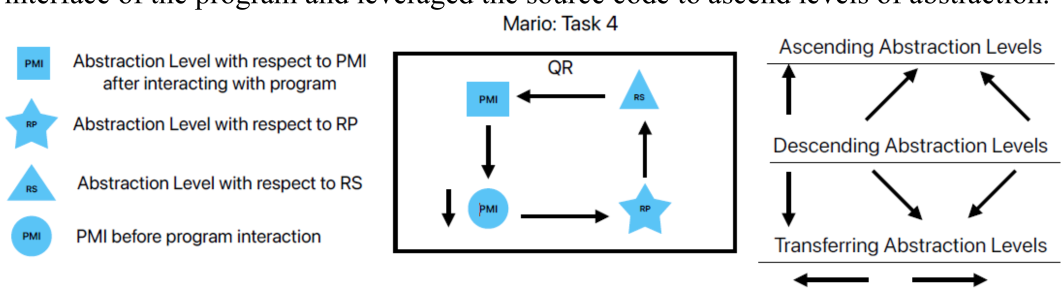

Figure 5. The Navigating Abstraction Framework applied to Mario's work on Task 4..

Discussion

Modifying the reducing abstraction framework to highlight the link between a proof by mathematical induction and a recursive program allowed us to recognize instances in which students transferred an abstraction level from their mathematical proof-work to their interaction with the recursive programs. Whereas Hazzan's (1999) Reducing Abstraction framework was a useful lens in determining when students reduced the abstraction level to cope with a concept, it did not provide the tools to account for how students moved *between* levels of abstraction or contexts. By extending the Reducing Abstraction framework to our *Navigating Abstraction* framework, we were able to recognize and examine how students navigated between abstraction levels and transferred abstraction levels to better understand both recursion and MI.

This framework provided a lens to interpret how students leveraged the back-and-forth navigation between induction/recursion and lower/higher levels of abstraction to ascend levels of abstraction, thereby allowing us to gain insights into how students understanding of induction and recursion co-evolved. This, in turn, helps demonstrate both the educational potential of using recursive programs to reinforce students' understanding of induction as well as the potential for using induction to enforce their understanding of recursion. Hence, the *Navigating Abstraction* framework is itself a research contribution in that it characterizes the movement between abstraction levels while simultaneously providing a bridge between mathematical and computational levels of abstraction.

References

Baker, J. D. (1996). *Students' Difficulties with Proof by Mathematical Induction.*

Brown, S. (2008). *Exploring Epistemological Obstacles to the Development of Mathematics Induction*.

Davis, M., Grassl, R., Hauk, S., Mendoza-Spencer, B., & Yestness, N. (2009). *Learning proof by mathematical induction*.

Dubinsky, E. (1989). Teaching mathematical induction II. *Journal of Mathematical Behavior*, *8*(3), 285–304.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Kluwer.

Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005). Some Historical Issues and Paradoxes Regarding the Concept of Infinity: An Apos-Based Analysis: Part 1. *Educational Studies in Mathematics*, *58*(3), 335–359. https://doi.org/10.1007/s10649-005-2531-z

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, *53*(1), 5–34. https://doi.org/10.1023/A:1024661004375

Harel, G. (2002). *The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based*. 47.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research on collegiate*.

Hawthorne, C., & Rasmussen, C. (2015). A framework for characterizing students' thinking about logical statements and truth tables. *International Journal of Mathematical Education in Science and Technology*, *46*(3), 337–353.

Hazzan, O. (1999). Reducing Abstraction Level when Learning Abstract Algebra Concepts. *Education Studies in Mathematics*, 71–90.

Hazzan, O. (2003). How Students Attempt to Reduce Abstraction in the Learning of Mathematics and in the Learning of Computer Science. *Computer Science Education*. https://doi.org/10.1076/csed.13.2.95.14202

Hazzan, O., & Hadar, I. (2005). Reducing Abstraction when Learning Graph Theory. *Journal of Computers in Mathematics and Science Teaching*, *24*(3), 255–272.

Hazzan, O., & Zazkis, R. (2003). Reducing Abstraction: The Case of Elementary Mathematics. In *International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 39–46). International Group for the Psychology of Mathematics Education. https://eric.ed.gov/?id=ED501007

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428. https://doi.org/10.2307/749651

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*, 193–223.

Knuth, E. J., Choppin, J. M., & Bieda, K. N. (2009). Proof: Examples and Beyond. *Mathematics Teaching in the Middle School*, *15*(4), 206–211. https://doi.org/10.5951/MTMS.15.4.0206

Raychaudhuri, D. (2014). Adaptation and extension of the framework of reducing abstraction in the case of differential equations. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 35–57. https://doi.org/10.1080/0020739X.2013.790503

Rich, K. M., & Yadav, A. (2020). Applying Levels of Abstraction to Mathematics Word Problems. *TechTrends*, *64*(3), 395–403. https://doi.org/10.1007/s11528-020-00479-3

Roh, K. H. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the epsilon-strip activity. *Educational Studies in Mathematics*, 263–279.

Sakhnini, V., & Hazzan, O. (2008). Reducing Abstraction in High School Computer Science Education: The Case of Definition, Implementation, and Use of Abstract Data Types. *Journal on Educational Resources in Computing*, *8*(2), 5:1-5:13. https://doi.org/10.1145/1362787.1362789

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of Research Design in Mathematics and Science Education*, 267–306.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*(1), 133–162. https://doi.org/10.1023/B:EDUC.0000017671.47700.0b

Wijeratne, C., & Zazkis, R. (2015). On Painter's Paradox: Contextual and Mathematical Approaches to Infinity. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 163–186. https://doi.org/10.1007/s40753-015-0004-z

Towards Systematically Examining Students' Epistemic Grounds of Developing Mathematical Generalization: A Preliminary Framework

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The practice of generalization is a fundamental aspect of mathematics that requires further investigation, particularly in advanced contexts. This paper has sought to address the research gap by presenting a preliminary framework that captures how and on what basis undergraduate students develop and evaluate mathematical generalization. I illustrate each aspect of the framework using examples from two focal students' generalization of the concept of a group in Abstract Algebra, uncovering important mechanisms and thinking behind their emergent generalizing ideas. The framework has potential to be used more broadly in other mathematical contexts and to inform instructional designs aimed at promoting students' generalization skills. Key takeaways and potential directions for future research are discussed.

Keywords: Student Thinking/Cognition, Abstract Algebra, Generalization

Generalization is a fundamental practice of the mathematics community (e.g., Aigner, 2003; National Council of Teachers of Mathematics, 2014). While a significant amount of research has investigated younger students' generalizing activities (Carraher, Martinez & Schliemann, 2008; Radford, 2006), much remains to be understood about how undergraduate students generalize formal, axiomatic mathematics concepts (Reed, 2018). In addition, despite the fact that frameworks exist for distinguishing different types of generalization (e.g., Radford, 2003) and for characterizing generalization processes (Ellis et al., 2017), many aspects on *how* and *on what basis* students develop and evaluate the logical validity of mathematical generalization are still incomplete. This paper seeks to contribute to the generalization literature by providing a preliminary framework on the epistemic grounds upon which students develop and evaluate mathematical generalization in advanced settings. For the purpose of this paper, I define *epistemic grounds* to be the foundation or basis upon which knowledge and beliefs are justified or supported. Some examples of epistemic grounds include evidence from empirical data, logical arguments, personal experience, and testimony from reliable sources.

Given that Abstract Algebra is a natural extension of arithmetic and generalization naturally manifests itself in its content (Fraleigh, 2003), I intentionally situate my work in this area. Widely acknowledged as challenging (Melhuish, 2015), Abstract Algebra demands a significant leap to formalism and axiomatization that many mathematics majors find intimidating (Huang, 2022). It is my hope that by investigating students' generalization of a fundamental Group-theory concept, this work can contribute to a more solid understanding of generalization and the ways in which students reason about axiomatic constructs in Abstract Algebra and beyond.

Theoretical Background

Studies have drawn attention to how generalization is vital in fundamental areas such as algebraic reasoning (Amit & Neria, 2008; Stacey, 1989), problem solving (Sriraman 2003), and functional thinking (Ellis 2011; Rivera & Becker 2007). Radford's (2003) categorization of factual, contextual, and symbolic generalization reflects the broader trend among researchers to classify forms of generalization. According to Radford (2003), factual generalization is abstracted from "actions undertaken on objects at the concrete level" (p. 47); contextual generalization occurs when students manage to objectify an "operational scheme that acts on abstract — although contextually situated — objects" (p. 54); and symbolic generalization capitalizes on symbols in the generalizing activity. At the undergraduate level, Harel and Tall (1991) categorized three types of generalization: expansive generalization involves expanding the scope of an existing schema without reconstructing it; reconstructive generalization entails reconstructing relevant schema to adapt to the expansion; disjunctive generalization occurs when students create a new, “disjoint” schema to cope with new problems (Harel & Tall, 1991, p. 39).

In contrast with the common approach of evaluating students’ generalization with respect to certain predetermined benchmarks (e.g., see above), Ellis (2007) proposed to understand generalization from an actor-oriented perspective. This perspective foregrounds actions that students take in the generalization process and views these actions from the perspective of the students (Reed, 2018). According to Ellis (2007), there are three major categories of generalizing actions: relating, searching, and extending. Relating involves recognizing and creating mathematical relationships between objects or situations; searching is an active investigation of whether certain mathematical properties persist across different cases; and extending entails expanding a “pattern or relationship into a more general structure” (Ellis, 2007, p. 241). While Ellis’ (2007) taxonomy originated from observations of middle-school students’ generalizing activities, it has been applied to the undergraduate level (e.g., Reed & Lockwood, 2018) and it lays the foundation for the Relating, Forming, Extending (R-F-E) framework (Ellis et al., 2017).

The R-F-E framework draws from both cognitive and sociocultural research to create a more fine-grained model of students’ generalizing activities (Reed, 2018). This framework distinguishes inter-contextual generalization from intra-contextual generalization, submitting that the former involves establishing similarities across situations and the latter entails forming and extending similarities within one task (Ellis et al., 2017, p. 680). As the primary form of inter-contextual generalization, *relating* centers on developing meaningful mathematical relations and it can be further divided into three subcategories: relating situations, relating ideas or strategies, and recursive embedding (Ellis et al., 2017). As a major form of intra-contextual generalization, *forming* is about perceiving regularities across one’s own activity and it consists of four subcategories: associating objects, searching for similarities, isolating constancy, and identifying a regularity. *Extending* is another form of intra-contextual generalization, which involves applying established patterns and regularities (e.g., those from relating or forming) to novel contexts. According to Ellis et al. (2017), extending includes continuing an existing pattern beyond familiar contexts, operating on an identified relationship, transforming a generalization, and removing particular details so as to express an identified regularity more generally.

While the above perspectives and frameworks offer insights into characterizing generalization, they fall short in addressing the underlying mechanisms that drive students’ generalization process and a majority of the existing frameworks are too broad to meaningfully inform instruction. Indeed, Radford’s (2003) and Harel and Tall’s (1991) works are valuable in distinguishing different forms of generalization, but their broad categorizations obscure students’ actions and fail to shed light on the essential skills or practices required of developing robust mathematical generalization. Ellis’ (2007) taxonomy foregrounds students’ generalizing actions and the later R-F-E framework (Ellis et. al., 2017) provides fine-grained analysis; yet they do not capture the details or nuances in students’ reasoning nor do they provide insights into why or on what basis students spontaneously think or behave in a certain way in the generalization process.

A Preliminary Framework on the Epistemic Grounds of Mathematical Generalization

Through extensive micro-level analysis of undergraduate students’ generalization of the concept of a Group in Abstract Algebra, I developed a preliminary framework that aimed to capture how and on what basis students developed and evaluated their emergent generalizing ideas. The framework was created in the context of design-based research (Bakker, 2018), where I iteratively conducted four design-enactment-reflection-refinement cycles (Lewis et al., 2020) with volunteer participants recruited among undergraduates at UC Berkeley. I integrated Ellis' (2007) actor-oriented perspective with my observer-researcher lens in the data analysis process: I carefully analyzed video-recordings of the students' generalization, envisioned myself in the students' positions, and made informed hypotheses about how and why the students used certain strategies to develop or justify their emergent ideas. The resulting hypotheses became the initial draft of the framework shown in Figure 1, which was later revised according to results from a bottom-up approach in analyzing the video-recordings as well as students' written works. The fact that I have similar undergraduate mathematics experience as those of my research participants (as I earned a B.S. in Mathematics from a similar institution) and a relative expertise in the content (as demonstrated by my recent A in a graduate-level Abstract Algebra course) positions my data analysis and the creation of the framework as accounting for both students' and experts' perspectives. More information about the design and methods can be found in Huang (2022). For the sake of consistency and due to space constraints, I will illustrate the proposed framework using examples from two focal undergraduate participants only. Eden and Jane (pseudonym) participated in this study as a pair for two hours in 2022, and I had been a teaching assistant for their Calculus course for 13 weeks prior to their participation.

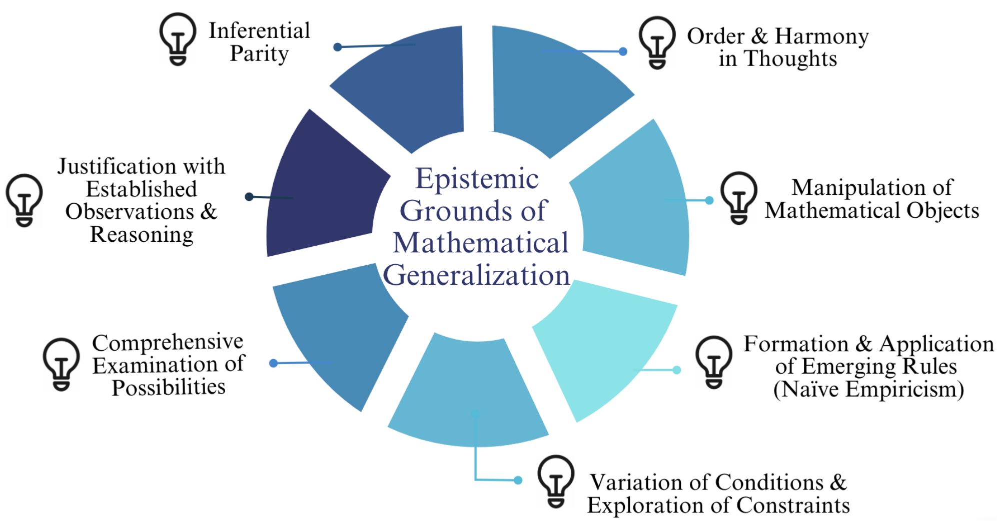

Figure 1. A preliminary framework on the epistemic grounds of developing mathematical generalization

Order and Harmony in Thoughts. This fundamental ground for developing generalization is closely related to the notion of equilibrium in Gestalt psychology (Koffka, 2013) and the common attribution of understanding to structured, harmonious mental organization (e.g., Sierpinska, 1994). According to Piaget (1976), the equilibrium of cognitive structures can be achieved through mental operations of assimilation and accommodation, which will then lead to order and harmony in people's minds. In a similar vein, Sierpinska (1994) proposes that understanding entails "finding a unifying principle, a relation that 'founds' what we want to understand" (p. 33), and that order and harmony as well as the feeling that "it fits" constitute the "most obvious criterion" of understanding (p. 32). In the process of zooming in and out of different algebraic systems and formulating similarities across different cases, Eden and Jane bounced ideas off of each other and made multiple utterances that conveyed similar meanings as

"Oh, it makes sense". I believe these utterances signify moments when they felt a sense of order, harmony, and coherence in their thoughts, and thus accepted their emergent mathematical ideas.

Manipulation of Mathematical Objects. This epistemic ground of generalization centers on an important epistemological practice of generating and validating mathematical knowledge by means of *doing* mathematics. The (manual and mental) manipulation of mathematical objects can involve a variety of techniques and tools, including physically manipulating artifacts, performing algebraic operations, constructing geometric figures, visualizing abstract properties, applying analytic techniques, and many more. Overall, the manipulation of mathematical objects is a crucial epistemic ground for developing generalization, as it allows students to generate and justify new insights, techniques, and results, providing a solid foundation for students to explore mathematical structures and relationships. It is important to note that besides attending to students' "mental math", I intentionally leveraged the embodied aspect of mathematical manipulation (which is often overlooked in undergraduate math classrooms). More specifically, I provided physical artifacts to invite rotation and reflection of geometric figures as a means to explore example Groups (e.g., Dihedral Groups), and scaffolded with semiotic tools such as tables to support students' reflexive abstraction (Vygotsky, 1978) of defining Group properties.

Formation & Application of Emerging Rules (Naïve Empiricism). This epistemic ground for developing generalization involves discerning an emerging mathematical rule from a limited number of empirical cases and utilizing it without rigorous justification. Historically, humans have made extensive use of this approach, due to various constraints such as limited access to data, insufficient analytical tools, and cost and time constraints (Gibson, & Papafragou, 2016). In the context of abstracting properties of multiplication in number systems (and later operations across example Groups), Eden and Jane concluded that "for three numbers in multiplication, order doesn't matter [or more precisely, it is associative]" after randomly picking some set of integers and checking that $(3 \times 4) \times 5 = 3 \times (4 \times 5)$ and $(7 \times 9) \times 100 = 7 \times (9 \times 100)$. As another example, Eden used $2 \heartsuit 6 = 5$ (marked blue in Fig. 2) to challenge Jane's result of $6 \heartsuit 2 = 4$ (marked red in Fig. 2) in the Symmetry Group of an equilateral triangle, based on their established observation that "order doesn't matter [or more precisely, operation is commutative]" for previous example Groups. But making $6 \heartsuit 2 = 6 \heartsuit 2 = 5$ raised another problem, as Jane quickly noticed that there was already a 5 in the same row of the operation table (marked green in Fig. 2). Jane's refutation was supported by their agreed-upon observation that each element appears exactly once in each row and each column of an operation table. Note that

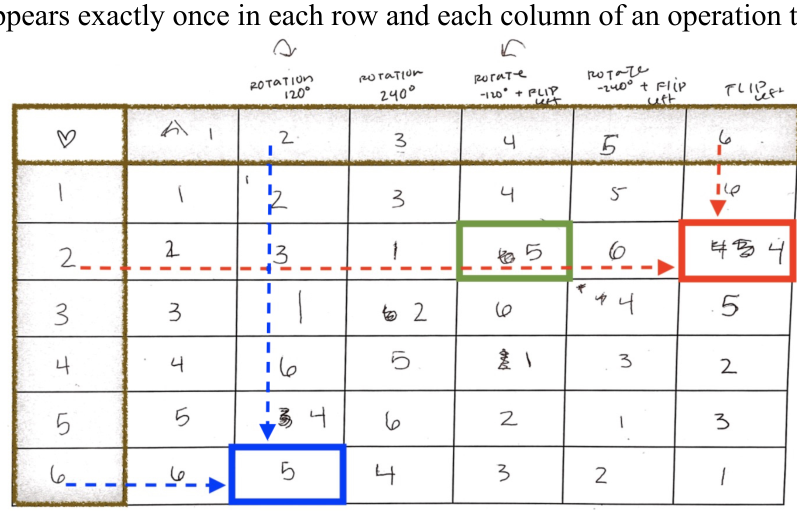

Figure 2. Formation and application of emerging rules: the case of 6♡2

the three emerging rules (i.e., multiplication is associative, commutativity is a defining similarity, operation table satisfies the "Sudoku property") played a crucial role in Eden and Jane's generalization of a Group, although they made no attempts to justify or refine these rules. Since students tend to search for similarities across their mathematical activities and generalize inductively from concrete cases, this epistemic ground is closely related to the idea of forming in the R-F-E framework (Ellis et al., 2017), Radford's (2003) notion of factual generalization, as well as Pólya's (1945) heuristics of "consider special cases" and "look for patterns". Despite its common usage, however, relying on naïve empiricism can lead to over-generalization or underestimation of the underlying mathematical structures, resulting in incorrect conclusions (Bollen et al., 2015). Thus, more sophisticated approaches (e.g., see below) are recommended to extend empirical findings to more generalizable and robust mathematical principles.

Variation of Conditions & Exploration of Constraints. This epistemic ground foregrounds an essential aspect of developing mathematical generalization, that of exploring conditions under which a generalization holds true and identifying any constraints that may apply. This is similar to Pólya's (1945) "variation of problems" heuristic in that both involve changing conditions; yet Pólya's (1945) heuristic focuses on generating new problems to help solve the original problem, while this epistemic ground aims to support the creation of more precise, powerful generalization by specifying its scope and constraints. During Eden and Jane's investigation of "order" (a student-generated vocabulary that bootstrapped their later differentiation between commutativity and associativity), I as their researcher-tutor probed: "We agree that 'order doesn't matter' for two elements in the first few examples, but what about three elements? Will adding parentheses in any way we want affect the result of operations?" In a swift response, Eden bounced back with an unanticipated question: "Does it have to be a set of three elements?" In retrospect, I realize that Eden's question was intended to vary the condition and facilitate an exploration of relevant constraints within their domain of investigation. After knowing that they could certainly investigate more than three elements at the same time (which broadened the scope of my original question), Eden and Jane went on to test operations with three and then four elements, creating a richer foundation for them to unpack underlying principles (e.g., associativity for three elements implies associativity for more elements) and generalize defining properties for Groups. Note that by varying conditions and exploring constraints, students not only generalize actions taken for specific numbers but also abstract the actions themselves. This suggests that they move beyond the realm of concrete examples and consider a broader set of possibilities; in the words of Radford (2003), students are engaged in contextual (and possibly symbolic) generalization.

Comprehensive Examination of Possibilities. In developing generalization, this epistemic ground concerns rigorously examining different situations as a means to assess the validity and soundness of an emerging generalizing idea, which is in mutual reinforcement with the previous epistemic ground. For example, when Eden and Jane tried to test whether "order matters" (or more precisely, whether associativity holds) for more than three numbers in multiplication, they attempted to exhaust all possible ways of adding parentheses in their randomly chosen expression $4 \times 2 \times 3 \times 5$. As a summary, they examined $(4 \times 2) \times 3 \times 5$, $4 \times (2 \times 3) \times 5$, and $4 \times 2 \times (3 \times 5)$, which all turned out to be 120. It is important to note that the first attempt $(4 \times 2) \times 3 \times 5$ was equivalent to $(4 \times 2 \times 3) \times 5$ and that those *were* all the possibilities they could thought of at the moment, despite the fact that advanced thinkers may find another possibility of $4 \times (2 \times 3 \times 5)$ left untested. Through comprehensive analysis of all possible cases, students can identify patterns and connections that may not be immediately apparent, contributing to their more robust understanding of the underlying principles. Since both *comprehensive examination of possibilities* and *variation of conditions & exploration of constraints* demand active investigation of whether certain emerging mathematical properties persist across different situations, the two epistemic grounds are inseparably linked to the idea of searching in Ellis' (2007) taxonomy. Moreover, as students often use logic to systematically exhaust all possibilities, it naturally leads to the next epistemic ground.

Justification with Established Observations & Reasoning. This epistemic ground highlights the importance of justification and proofs in developing mathematical generalization (or more generally in *any* mathematical activity). With the help of existing observations and logical reasoning, mathematics students learn to justify conjectures by showing that it is a logical consequence of certain principles and assumptions. As a standard practice in mathematics, justification involves higher-order thinking skills such as deductive reasoning, analyzing, and critical thinking, providing a rigorous and reliable way to establish the truth of mathematical statements. An example of this could be found in Eden and Jane's scrutiny of "order" in the modulo 5 system under addition. Knowing that "order doesn't matter" when they add integers, Jane argued that "order doesn't matter" in the modulo 5 system either because "You can mod 5 with like any number, and if you are gonna be adding something to it, order does not matter." Jane managed to extend her understanding of the set of integers to the new context of the modulo 5 system, and it is clear that her use of logic was mathematically correct. Note that Eden and Jane had not successfully distinguished commutativity and associativity at this point; the reasoning was valid regardless of the specific "order" they referred to. This example illustrates how justification is closely intertwined with the notion of extending in the R-F-E framework (Ellis et al., 2017), as it operates to establish a universal structure that transcends specific cases, personal viewpoints, and historical eras. Moreover, this epistemic ground bears resemblances to Pólya's (1945) heuristics of "using direct reasoning" as well as Radford's (2003) contextual and symbolic generalization in that all four prioritize logical reasoning in mathematical pursuits.

Inferential Parity. This epistemic ground is taken from Abrahamson (2014) to allow for a more detailed elaboration on how students reconcile their intuitive generalization with formal mathematics concepts. According to Abrahamson, inferential parity refers to the idea that students perceive a formal structure as "bearing the same information about the target property of the source phenomenon as their informal judgment" (2014, p. 13). Through the lens of inferential parity, researchers can capture moments when students recognize a connection between their informal knowledge and the established mathematical ideas, shedding light on how students utilize their intuitive generalization to navigate through formal mathematics. An illustrative example could be found when Eden and Jane compared their generalized concept with the formal definition of a Group. "A Group is an ordered pair where G is a set and ★ is a binary operation on G," read Eden, "that's kind of what we have here, right?" Jane nodded, pointing to the first row of their similarity table (marked blue in Fig. 3). Moments later, they skimmed through the defining properties of identity and inverse in the formal definition, prompting Eden to exclaimed: "Ohhh! That's cool! We are doing that!" They stared at their self-generated terminologies of "prime" (see Fig. 3, which bears the same meaning as identity or "e") and "undoing #s" (see Fig. 3, which bears the same meaning as inverse or " a^{-1} "), turned to each other, and then laughed. In the meantime, Jane copied "e" and " a^{-1} " from the formal definition to the left of their corresponding terminologies (marked red in Fig. 3). A later reflective interview revealed that the experience of struggling through unfamiliar mathematics (with support) and eventually uncovering how their intuitive generalization connected to (or more precisely, encapsulated) the formal, textbook definition was a valuable learning experience, leading to Eden and Jane's renewed sense of agency and ownership in their mathematical learning.

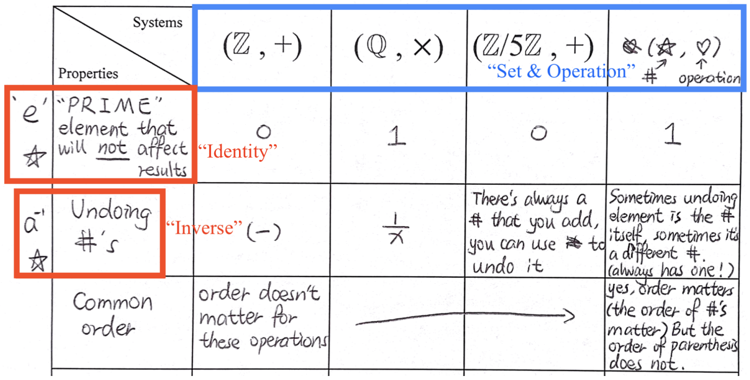

Figure 3: Inferential parity: connections between the formal and the informal

Concluding Remarks

While a fair amount of research has investigated younger students' generalizing activities, existing frameworks and perspectives often fall short in more advanced contexts and they fail to address the specific grounds upon which students develop mathematical generalization. This paper has sought to address the research gap by presenting a preliminary framework that captures *how* and *on what basis* students develop and evaluate their emergent generalizing ideas at the undergraduate level. The seven epistemic grounds highlighted by the framework are: Order & Harmony in Thoughts, Manipulation of Mathematical Objects, Formation & Application of Emerging Rules (Naïve Empiricism), Variation of Conditions & Exploration of Constraints, Comprehensive Examination of Possibilities, Justification with Established Observations & Reasoning, and Inferential Parity. I provided illustrations of each epistemic ground with examples from two undergraduate students' generalization of the concept of a Group in Abstract Algebra. It is important to note that this paper not only uncovers important mechanisms and thinking behind students' emergent generalizing ideas in Abstract Algebra, but also develops theoretical constructs that characterize students' generalization of axiomatic mathematics concepts and that hold potential to be applied more broadly to other mathematical contexts.

In addition to the framework presented in this paper, there are several key takeaways and potential directions for future research. First, while the proposed framework is grounded in literature and my observations in students' generalization of an Abstract Algebra concept, it is preliminary in nature and it requires refinement and empirical validation from other advanced content areas (e.g., Real Analysis). It is hoped that this paper can inspire further investigation into mathematical generalizations that are of a formal, axiomatic nature. Second, I do not claim that the identified epistemic grounds are mutually exclusive; further research is needed to better understand the interrelations among them for different students and different contexts. Third, the framework identifies important generalization-related mechanisms that could potentially inform or even guide instructional designs; future research could explore the framework's effectiveness in facilitating students' critical reflection upon and the development of their generalization-related skills and practices. Finally, it is worth emphasizing that mathematical generalization is a complex and multifaceted practice that demands students to engage in knowledge integration, problem solving, and critical thinking. By enhancing our understanding of the epistemic grounds that underpin this practice, we can better support students in developing the essential skills and knowledge needed for success in (both formal and informal) mathematics and beyond.

References

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1-16. https://doi.org/10.1016/j.ijcci.2014.07.002

Aigner, M. (2003). *Proofs from THE BOOK*. Springer-Verlag.

Amit, M., & Neria, D. (2008). "Rising to the challenge": Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM*, *40*, 111-129.

Bakker, A. (2018). *Design Research in Education: A Practical Guide for Early Career Researchers* (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Bollen, K. A., Cacioppo, J. T., Kaplan, R. M., Krosnick, J. A., Olds, J. L., & Podsakoff, P. M. (2015). Social, behavioral, and economic sciences perspectives on robust and reliable science. *Psychological Science, 26*(6), 821- 826.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM*, *40*, 3-22.

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *The Journal of the Learning Sciences*, *16*(2), 221-262.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308-345.

Ellis, A. B., Lockwood, E., Tillema, E. & Moore, K. (2017). Generalization across domains: The relating-forming-extending generalization framework. In Galindo, E., & Newton, J., (Eds.). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Fraleigh, J. B. (2003). *A first course in abstract algebra*. Pearson Education India.

Gibson, E., & Papafragou, A. (2016). Naïve Empiricism and Anomalous Observations: A Reply to Lynch and Baker. *Journal of Experimental Psychology: General, 145*(5), e37-e43.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic. *For the Learning of Mathematics, 11*, 38-42.

Huang, S. (2022). Bootstrapping students' emergent ideas: Case study of students' guided (re) invention of abstract math concepts. *US-China Education Review*, *12*(6), 210-230.

Koffka, K. (2013). *Principles of Gestalt psychology* (Vol. 44). Routledge.

Lewis, D. R., Carlson, S., Riesbeck, C., Lu, K., Gerber, E., & Easterday, M. (2020). The Logic of Effective Iteration in Design-Based Research. In Gresalfi, M. and Horn, I. S. (Eds.), *The Interdisciplinarity of the Learning Sciences, 14th International Conference of the Learning Sciences* (ICLS) 2020, Volume 2 (pp. 1149-1156). Nashville, Tennessee: International Society of the Learning Sciences.

Melhuish, K. M. (2015). *The design and validation of a group theory concept inventory.* (Doctoral dissertation, Portland State University).

National Council of Teachers of Mathematics. (2014). *Principles to actions*. National Council of Teachers of Mathematics.

Piaget, J. (1976). Piaget's theory. In B. Inhelder, H. H. Chipman, & C. Zwingmann (Eds.), *Piaget and his school* (Springer Study Edition, pp. 11-23). Springer. https://doi.org/10.1007/978-3-642-46323-5_2

Pólya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to

students' types of generalization. *Mathematical thinking and learning*, *5*(1), 37-70.
Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In *Proceedings of the 28th conference of the international group for the psychology of mathematics education, North American chapter* (Vol. 1, pp. 2-21).
Reed, Z. (2018). *Undergraduate students' generalizing activity in real analysis: Constructing a general metric*. (Doctoral dissertation, Oregon State University).
Reed, Z., & Lockwood, E. (2018). Generalizing in combinatorics through categorization. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, S. Brown (Eds.), *Proceedings of the 21st annual conference on research in undergraduate mathematics education* (pp. 311-325). San Diego State University.
Rivera, F. D., & Becker, J. R. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior, 26*(2), 140-155.
Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.
Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations: The problem-solving experiences of four gifted students. *Journal of Secondary Gifted Education, 14*, 151-165.
Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics*, *20*(2), 147-164.
Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Harvard University Press.

Faculty Perceptions of Making Connections: A Story of Obligations and Constraints

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Mathematics is central to STEM coursework, yet limited research has examined how instructors in math and other STEM courses facilitate connections between courses for their students. Based on four focus group meetings with ten faculty members, we characterize factors that assist or hinder making connections between courses and examine their relationship to professional obligations. Results include the largely negative impact of institutional obligations on faculty's willingness and ability to make connections. Implications include the need for institutions to give opportunities to make personal connections across departments and incentivize the development of connections.

Keywords: interdisciplinary connections, professional obligations, instructor decision-making

Introduction and Background Literature

Faculty in science, technology, engineering, and mathematics (STEM) disciplines recognize that math is relevant to their students and their teaching, as evidenced by requirements for students to take specific math classes for their major and the related role of math as gatekeeper to STEM (e.g., Bressoud et al., 2015). Nevertheless, while examinations of better ways to connect math and science or engineering have occurred at administrative or departmental levels (e.g., Robayo et al., 2022; Vroom et al., 2022), limited work has directly attended to individual faculty members' experiences. We thus seek to address the following research question: What factors assist or hinder connections that faculty can make between courses/content areas?

Meaningfully connecting content between disciplines is challenging, as jargon forms barriers to non-members of those communities of practice (Wenger, 1999). Separations exist between STEM faculty members and, by extension, their courses. Thus, research has begun examining how content like definite integrals and bases are communicated and used in different disciplines and how differences impact students' understanding (e.g., Jones, 2015; Serbin & Wawro, 2022).

Furthermore, STEM educators have placed increasing emphasis on math skills or concepts undergirding scientific content and ways instruction can better relate these content areas. Foci include interpretations of graphs in math and science (e.g., Christensen & Thompson, 2012; Rodriguez et al., 2018; 2019) and connections between content in particular courses, such as calculus and biology or chemistry (e.g., Jones, 2019a; 2019b; Williams et al., 2021), calculus or differential equations and physics (e.g., Jones, 2015; Roundy et al., 2015; Schermerhorn & Thompson, 2019a; 2019b; Smith et al., 2013), and linear algebra and physics (e.g., Schermerhorn et al., 2022; Serbin et al., 2020; Wawro et al., 2020). Some work also seeks to connect higher level math to science, including relating discrete math or abstract algebra to chemistry (e.g., Bergman & French, 2019; Bergman, 2020) and biology (Robeva et al., 2010). Considering the many interfaces of math and science, it is understandable that STEM majors may struggle to draw on appropriate math coursework in their science and engineering courses and that math faculty may be unaware of how their content is used in other courses. This paper builds on these cross-disciplinary efforts by examining how the use of focus groups centered on connections between courses' content could be used to illuminate current connections made between courses and gain insight into faculty's perspectives on what could support making connections in future.

Theoretical Perspective

To conceptualize factors that impact making connections between content areas, we draw on professional obligations as characterized by Herbst and Chazan (2011). This perspective is grounded in the theory of practical rationality, wherein mathematics teachers' instructional actions are taken to be sensible based on the norms and obligations that motivate them and theorizes what norms and obligations are impactful to teachers' decision making in different contexts. Whereas norms characterize the typical roles a teacher would play in instruction, professional obligations are characterized as "justifications (or refutations) that participants might give to actions that depart from a situational (or contractual) norm" (p.450). They also characterized four categories of professional obligations: disciplinary, individual, interpersonal, and institutional. Disciplinary obligations relate to providing a valid representation of mathematics as an area of knowledge and a practice. Individual obligations include the need to attend to students' identities and needs. Interpersonal obligations relate to shaping interactions and classroom discourse. Institutional obligations relate to attending to the broader aspects of schooling as an institution, including policies, exams, and curriculum.

While these obligations were originally characterized in the context of K-12 mathematics teaching, here we modify these obligations slightly to attend to teaching in a university setting and do not limit ourselves to the perspective of math teachers. We believe this is appropriate for examining obligations attended to when making connections between one's own course and other courses, because knowing the perspective of both math and non-math teachers is important for symmetric connections to be built. This changed emphasis leads us to slightly reframe the interpersonal obligations as interpersonal and personal obligations to emphasize any interpersonal contexts, including those outside the classroom with colleagues and self-reflection on one's obligations to one's own work-life balance and mental health needs.

Methods

Data was collected from four focus group meetings of ten faculty at one regional university in Spring 2023. Two meetings were held in February and two in April. Each participant was intended to participate in one February and one April meeting, but two participants were unable to attend in April. Gender-neutral pseudonyms, departments, and which meetings were attended are in Table 1. Faculty were solicited from those who completed a survey on their background and perceptions of connections between their courses and mathematics. In the focus groups, each question was asked to the group and each participant was given an opportunity to respond. After everyone had responded, everyone was given an additional opportunity to respond to what others had said. Questions focused on perceived connections between their course and other courses, supports used and desired for making connections, the connections between their courses and math, and supports they suggested or desired for helping students struggling with math content or skills. The focus groups were audio and video recorded and transcribed prior to analysis.

Table 1. Participants' backgrounds and meetings attended.

Pseudonym	Department	February Meeting	April Meeting
Addison	Earth, Atmosphere, and Environment	B	B
Bergen	Curriculum and Instruction	A	A
Carey	Physics	A	A
Darien	Mathematics	B	A
Elliot	Earth, Atmosphere, and Environment	B	A
Freddie	Mathematics	A	A

Gayle	Mathematics	B	none
Harper	Computer Science	B	B
Indigo	Leadership, Educational Psychology, and Foundations	A	B
Jun	Mechanical Engineering	B	none

Data was analyzed by two researchers who coded independently, then discussed and came to consensus on each section of the transcript. We did not enter analysis with a codebook, but used the notion of obligations to guide what we attended to. Both researchers open coded the two February meetings in alignment with descriptive coding (Saldaña, 2016), came to consensus on a codebook, recoded transcripts in alignment with the new codes, and came to consensus on coding. We then, after minor adjustment, coded the two April meetings, added new codes as needed, and came to consensus. Finally, we classified the codes into obligation categories, using three of the obligations of Herbst and Chazan (2011) directly and a modified version of the fourth. This process aligns with codebook thematic analysis (Braun et al., 2019), as we had some sense of the categories but not the particular codes we would use when entering analysis, and had the flexibility to add codes and categories to capture new ideas as analysis progressed.

Results

We present results according to alignment with each of the obligations: Disciplinary, Individual, Interpersonal and Personal, and Institutional. Codes are italicized.

Disciplinary Obligations

Disciplinary obligations in this study manifested as participants' desire to teach the content of their discipline, though this could hinder or assist making connections to other courses. The instructors agreed on the *importance of connections* across content. For example, Bergen stated, "I think it really behooves us as professors, as instructors to make those connections, because if we don't, then we're really not making it relevant to our students…. all the different topics, the fields connect." The instructors also discussed the importance of *showing students connections*, as students can struggle with finding the connections on their own. For instance, Freddie noted:

> …they don't see the connections. They've seen that trig; they've seen the quadratics. But outside of maybe a question at the end of a section, at the end of a unit....in math class…they're focused more on the math rather than the physics. And then when they get to you, it's like "Oh, wait. I know the math. I just got to figure out that physics equation that my math teacher gave me. Where does that come from?"

In addition, Addison recalled student feedback about examples that stated, "I wish those types of examples were in calculus. It didn't make sense why I was learning this".

Instructors also noted hindrances in terms of student knowledge. One issue was that *students had not seen content* required for their course. As Freddie noted, "The problem with the students who are taking that class, they've never seen parametric equations. They've never seen the idea that time controls this, the horizontal and vertical. No clue about it." Another perceived issue was that *students do not retain knowledge* from previous courses. Carey noted: "You can tell there's a light that's like 'Oh, yeah. I've seen that. I remember trig functions at some point.' But how much they remember about trig functions and inverse trig function[s] and that sort of thing is another question." Relatedly, participants perceived *students as not having conceptual mastery of the skills* required from previous courses. In an algebra-based physics course that Carey teaches, they noticed, "So in principle, it's stuff that they should have been exposed to, at least in high school, but it's really nothing new…. I think one of the challenges I'm having is that they don't have the mastery, so they're – in a company of mathematicians, I hate to say that math is a distraction. But it's kind of a distraction from the physics." Instructors also stated actions they take to help combat these issues, such as *helping students review*. Harper shared:

> And so it's never the case, where you walked into a class and everybody knows every single thing from every single prerequisite. And sometimes you have to take a little detour. But you can't spend a lot of time there, simply because that's not the course.…If they're really largely, all of them, wholesale not getting it… then there's no point in going on with the lecture. You have to take a detour in real time and try to give them a week's worth of whatever math in about 15 minutes, hopefully, as a refresher.

Individual Obligations

We classified as individual obligations anything that pertains to the student's mathematical identity and their needs going into their classes. Many aspects related to affective considerations, such as *students being anxious about math*. Bergen said of their students, "They are still terrified of math, oftentimes more so after taking the math classes." Relatedly, Elliot claimed that *helping students reduce anxiety* should be a goal of math courses: "So on the philosophical side, I think that one of the goals of the math classes should be to help students get over their fear of math, because many of them come in with that fear and they are disinclined to pursue occupations, majors, whatever, that have any kind of quantitative aspect to them." Others, like Darien, taught particular tools to reduce anxiety: "We go over developing growing mindset, trying to get them to know that math is for everyone and there is not people that are good or bad at mathematics." Darien also relayed a time that focused on community building and *motivated their students*:

> I had an intervention…So instead of going into content that day, we just sat.…And I just told them, 'What do you want from me?' ... Because I am not going to be your teacher this whole time, during these four years.…And they were like, 'Oh yeah. If we tell you to change something, it's not going to affect us directly.…We have to learn how to move to other courses. We have to improve our skills. We have to study more.' So they were… brainstorming of what they could do to just be better at math. And I've been having good outputs off that: attendance was better, and they were more motivated…after that.

Other aspects related to knowledge of students. Addison noted *student backgrounds are sometimes unknown* based on students transferring and lack of knowledge of courses in other disciplines: "I learned from googling precalc, so I thought it was all algebra, but I was wrong. There's some precalc in there too. Yeah but it is hard when we have transfers and not everyone has to take the full calculus sequence because we don't require everyone to take it." Others, like Indigo, built on their *knowledge of students' backgrounds*: "because most of my teachers in training are not statisticians and not trained in that, I take that as my job to inform them."

Finally, some topics related affect and content, such as the *need for students to be motivated to do the work*. Elliot said, "And I feel like they get a lot of, they're told that this is a mathematical relationship between variables, but they don't grasp, like, 'What are the variables that I might actually work with, and why do I care about all this?'" Darien seemed to suggest that student motivation is misdirected at procedural rather than conceptual understanding: "We are trying to help them understand concepts, but they don't want to because they think that that's not important…it's like, 'Tell me what I need for those skills that I need so I can pass my next calc courses.'" Elliot suggested *building community* with alumni to impact motivation:

> Whatever discipline you're in, having students exposed to people who actually use math in a career… I do think there are more than a few who don't think it's important for

where they see themselves going.…the more they can see people that they can model and see 'I could be like that. That person is like and was like me.' Because when employers come in, a lot of time, they're alumni. They will literally say, 'I was in that chair. I sat in that chair in this class, and I felt exactly the same way you do…this is where I am now.'

Interpersonal and Personal Obligations

We characterized interpersonal and personal obligations as shaping classroom discourse and interactions outside the classroom as well as one's own views. When instructors are *not familiar with other course content*, that can impact assumptions. For example, Addison states:

My hydrogeology class is a math class and that's how I start my class, by saying it is. And I'm a little embarrassed to say, yeah, I've never really spoken to math teachers about what I teach in it. And to the point when I did your survey, I google[d] to find out what classes some of those concepts were taught in because I couldn't remember.

Additionally, Addison noted the impact of their *personal experiences with course material* and a career outside academia on their teaching: "I had to take through Calc 3 when I was an undergrad and forgot it, a lot of it going on. And I didn't really need it in the sense of knowing those foundations. I worked after my bachelor's degree and most students are just going to go out and work. They don't need to know all that specifics." Based on Addison's experience of not using advanced math in their career, they saw limited value in their students having that knowledge. Instructors' *limited time* affects how much they can put into a course. Darien shared their experience with a new course: "I don't feel I'm doing a great job, especially because I'm always running out of time…So I come up with examples that are similar to a textbook…, just to explain the concept and continue with the next class." Freddie noted a lack of familiarity with colleagues: "I guess I'm still the newbie here. I don't know much of anybody." Not knowing colleagues meant they *did not know who to talk to* about connecting content.

However, instructors did not exclusively note challenges. Freddie was *familiar with* math and science K-12 *content* and stated, "I do the proportions that they're going to mostly see in the book, but I also make sure I fold in some of the gas laws…Charles's law or Boyle's laws." In some cases, like Freddie's, instructors may already know other course information. For those that do not, Elliot recommended ways to *gain information about other courses*: "I've thought for years that mathematicians should maybe come into our classes and say, 'All right, here's the nuts and bolts of this,' or the people who do the applied math should provide examples for the people who are teaching the math and say, 'This is how we do this.'" Others noted positive ways their *personal experiences of courses* impacted instruction, such as Carey:

For introductory physics, one of the things I try to do…is to try and use math terminology that I think they would be familiar with from a math class…so the slope of your position-versus-time graph is the velocity. So we look at our position-versus-time graph and we start, 'Oh, if I want to calculate the average velocity, how do I do that?' Then I look at this and it's like, 'Oh, we measure how much this way. Oh, this is like a rise. And now we measure how much this way. And this is like a run. And, oh, if we take the rise over the run, right, oh, that's the slope, right?' And with some of the trig stuff…I specifically use term opposite and adjacent and hypotenuse and similar sort of terms that hopefully they'll be familiar with from math and might start to engage their memory a little bit.

Instructors also noted *time saving strategies* they can use. One of the most common strategies was looking online for resources. Carey had knowledge of a research field that would be likely to have resources: "Physics for the Life Sciences basically is this whole thing, and there's people out there who've had curricula and how they set things up." Additionally, Bergen shared advice on how to *build relationships with colleagues* to encourage collaboration:

> It's interesting that the advice I was given when I first got here 13 years ago. Department chair at the time, he said, 'I want you to join [committee].'…And that was one of the best decisions or… the best voluntelling…Because then I got to…meet people from all over campus...and that's the best way to make you feel like you belong some place.

Institutional Obligations

We characterized institutional obligations as attention to institutional policies impacting promotion and curriculum; most of these obligations hindered making connections across courses. Many instructors noted that *finding others who want to collaborate was difficult* and in many *ways the university did not seem to value the need for connections*. Addison emphasized the need for working with others to be incentivized: "I have to not get dinged for taking the time to reach out to people because, yeah, unfortunately, research is really the key." Similarly, Elliot stated, "Until I see or feel that there's a lot of incentives to do that, and that would also require…changing the whole structure, not of departments but how the university sees these classes." *Others* also need to *value collaboration for connections to be built*, as Carey experienced: "Several of us in the physics department tried to have this conversation with the biology folks.…they sent over their advisor, and basically, he said, 'Oh, they just need to take the MCAT,' which wasn't very helpful for us.…at least based on what he said, physics has no relevance to biology other than doing well on the MCAT." University policies also seemed to encourage competition rather than cooperation. This was especially impactful in *course design* and *credit hour requirements*. For example, Darien and Addison discussed whether a new or different calculus course should be a prerequisite for Addison's course:

> *Addison:* If I didn't let students who either were co-taking calculus or had failed and they're going to retake it, take my class, I wouldn't have a lot of students.…So I have to show the applied and I'm not sure I would be good at teaching those foundational things beyond… "Don't you remember this? It's okay if you don't. We're going to keep going."
>
> *Darien:* Do you think that an applied calculus class will help with that as a prerequisite? Because we don't have an applied class right now. We just have a business and social science class.
>
> *Addison:* In principle, yes. Do I want to make them take another class? No.

In addition to limiting prerequisites from other disciplines to save space for major courses, starting *new courses* could be extremely *competitive*. Harper shared:

> The single most impactful thing that the university could do in order to help the issue you raised is to change the manner in which new courses are being developed or proposed. It's not designed as such but what it actually is is a land rush. It's the first one to market gets to claim the course…That's not thoughtful, it's competitive.

Finally, when creating requirements for courses, instructors build on their own personal experience, but *courses change over time*. For example, consider the following conversation:

> *Harper*: Okay. Do they do proofs in Calc? Calc 1 and Calc 2?
>
> *Interviewer*: No.
>
> *Harper*: Things have changed since I took Calc 1 and Calc 2.
>
> *Gayle*: Depends on who teaches Calc a little bit.
>
> *Harper*: Which is to say that a lot of time has passed since I took Calc 1 and Calc 2, but things have definitely changed.

On the somewhat positive side, Gayle noted they had *collaborated* with Carey *on a co-requisite course*, but also acknowledged some challenges: "For freshman engineering students…

usually like the Calc schedule gets all wonky because we'll end up doing either Rolls [Theorem]… in week 2 or something just so that they have it and like they can then use that for kinematics … so they apply it and do physics labs and things like that." Gayle noted that this arrangement between physics and calculus could lead to an unusual calculus schedule, where applications that typically follow differentiation techniques instead are addressed early in the semester, though this was done to benefit students' understanding in their physics course.

Discussion

This study examined disciplinary, individual, interpersonal and personal, and institutional obligations as perceived by faculty, especially as they support or hinder making connections between courses. Participants highlighted disciplinary obligations as motivation for them to make connections but noted obstacles like a perceived lack of student knowledge or mastery, preventing students from fully understanding the connection or preventing faculty from attempting to make connections. Individual obligations manifested through faculty's attention to students' content backgrounds and affective needs. The supports and hindrances in this case mirrored each other, as faculty highlighted anxiety as a common issue but also shared ways to address this; similarly, they highlighted both a lack of knowledge about students' backgrounds and ways this knowledge could be leveraged if known. Interpersonal and personal obligations emphasized the instructors' knowledge and relationships and their impact on making connections between courses. In these obligations also, supports and hindrances mirrored each other, with some participants extolling the power of connecting with colleagues while others wondered how to make those personal connections; similarly, faculty's prior experiences with courses could either incentivize or prevent making connections. Finally, institutional obligations highlighted the need to attend to existing policies. In most cases, these obligations served as hindrances to making connections, as making connections is not valued for tenure and promotion and policies fostered competition between departments over course design and maximizing credit hours.

We believe applying the lens of professional obligations (Herbst & Chazan, 2011) to our data provides three useful insights. First, our participants were likely a best-case scenario for making connections between math and other areas; they were willing to take time to attend focus groups and saw value in making connections. However, even they highlighted hindrances to connections across all four types of professional obligations, especially institutional obligations. Second, multiple instances arose where faculty did not see the purpose of having certain prerequisites, whether because they viewed the information as irrelevant for their students, likely to drive students away from their own courses, or unlikely to be remembered when needed in subsequent courses. Considering calculus's current role as the gateway to STEM (Bressoud et al., 2015) and other studies with faculty questioning the importance of math courses for learning STEM content (e.g., Rupnow et al., 2023), such perceptions from other disciplines are important to consider to ensure math courses are performing their expected role in the curriculum and valued by other disciplines. Finally, our participants had almost exclusively negative things to say about the impact of the institution on the making of connections. Some policies may be difficult to change (e.g., activities valued for tenure, budgetary formulas) whereas others would require less effort (e.g., providing collaboration spaces for likeminded faculty). We encourage future work on the impact of these policies and how they could be changed to support connections across courses.

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References

Bergman, A. M. (2020). *Identifying a starting point for the guided reinvention of the classification of chemically important symmetry groups* (Order No. 27995592). Available from ProQuest Dissertations & Theses Global. (2427325634). https://www.proquest.com/dissertations-theses/identifying-starting-point-guided-reinvention/docview/2427325634/se-2

Bergman, A. M., & French, T. A. (2019). Developing an Active Approach to Chemistry-Based Group Theory. In *It's Just Math: Research on Students' Understanding of Chemistry and Mathematics* (pp. 213-237). American Chemical Society.

Braun, V., Clarke, V., Hayfield, N., & Terry, G. (2019). Thematic analysis. In Liamputtong P. (Ed.) *Handbook of Research Methods in Health Social Sciences*. Springer: Singapore. https://doi.org/10.1007/978-981-10-5251-4_103

Bressoud, D., Mesa, V., & Rasmussen, C. (2015). Preface. In Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and recommendations from the MAA National Study of College Calculus* (pp. v-viii). MAA Press.

Christensen, W. M., & Thompson, J. R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics-Physics Education Research*, *8*(2), 023101.

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior*, *38*, 9-28.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405–462, Article 2. https://doi.org/10.54870/1551-3440.1225

Jones, S. R. (2019a). What education research related to calculus derivatives and integrals implies for chemistry instruction and learning. In *It's just math: Research on students' understanding of chemistry and mathematics* (pp. 187–212).

Jones, S. R. (2019b). Students' application of concavity and inflection points to real-world contexts. *International Journal of Science and Mathematics Education*, *17*(3), 523-544.

Robayo, B. J. S., Apkarian, N., Johnson, E., Elizondo, T. A., Ellis, B., & Robbins, C. (2022). Institutional and departmental change: Responding to crisis. In E. Johnson, N. Apkarian, K. Vroom, A. Martinez, C. Rasmussen, & D. Bressoud (Eds.), *Addressing Challenges to the Precalculus to Calculus II Sequence through Case Studies: Report Based on the National Science Foundation Funded Project Progress through Calculus,* (pp. 65–74).

Robeva, R., Davies, R., Hodge, T., & Enyedi, A. (2010). Mathematical biology modules based on modern molecular biology and modern discrete mathematics. *CBE—Life Sciences Education, 9*(3), 227–240. https://doi.org/10.1187/cbe.10-03-0019

Rodriguez, J. M. G., Santos-Diaz, S., Bain, K., & Towns, M. H. (2018). Using symbolic and graphical forms to analyze students' mathematical reasoning in chemical kinetics. *Journal of Chemical Education*, *95*(12), 2114-2125.

Rodriguez, J. M. G., Bain, K., Towns, M. H., Elmgren, M., & Ho, F. M. (2019). Covariational reasoning and mathematical narratives: Investigating students' understanding of graphs in chemical kinetics. *Chemistry Education Research and Practice*, *20*(1), 107-119.

Roundy, D., Weber, E., Dray, T., Bajracharya, R. R., Dorko, A., Smith, E. M., & Manogue, C. A. (2015). Experts' understanding of partial derivatives using the partial derivative machine. *Physical Review Special Topics-Physics Education Research*, *11*(2), 020126.

Rupnow, R., Remus, M., Scheuermann, N., & Bergan-Roller, H. (2023). Values, methods, and facts: How do math and science align and differ from the perspectives of mathematicians and biologists? In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 208–216). Omaha, NE.

Saldaña, J. (2016). *The coding manual for qualitative researchers.* Thousand Oaks, CA: SAGE Publications, Inc.

Schermerhorn, B. P., Corsiglia, G., Sadaghiani, H., Passante, G., & Pollock, S. (2022). From Cartesian coordinates to Hilbert space: Supporting student understanding of basis in quantum mechanics. *Physical Review Physics Education Research*, *18*(1), 010145.

Schermerhorn, B. P., & Thompson, J. R. (2019a). Physics students' construction of differential length vectors in an unconventional spherical coordinate system. *Physical Review Physics Education Research*, *15*(1), 010111.

Schermerhorn, B. P., & Thompson, J. R. (2019b). Physics students' construction and checking of differential volume elements in an unconventional spherical coordinate system. *Physical Review Physics Education Research*, *15*(1), 010112.

Serbin, K. S., Robayo, B. J. S., Truman, J. V., Watson, K. L., & Wawro, M. (2020). Characterizing quantum physics students' conceptual and procedural knowledge of the characteristic equation. *The Journal of Mathematical Behavior*, *58*, 100777.

Serbin, K.S., & Wawro, M. (2022). The inextricability of students' mathematical and physical reasoning in quantum mechanics problems. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-022-00174-z

Smith, T. I., Thompson, J. R., & Mountcastle, D. B. (2013). Student understanding of Taylor series expansions in statistical mechanics. *Physical Review Special Topics-Physics Education Research*, *9*(2), 020110.

Vroom, K., Elizondo, T. A., Voigt, M., & Larsen, S. (2022). Variations in approaches to the content of the calculus sequence. In E. Johnson, N. Apkarian, K. Vroom, A. Martinez, C. Rasmussen, & D. Bressoud (Eds.), *Addressing Challenges to the Precalculus to Calculus II Sequence through Case Studies: Report Based on the National Science Foundation Funded Project Progress through Calculus,* (pp. 31–44).

Wawro, M., Watson, K., & Christensen, W. (2020). Students' metarepresentational competence with matrix notation and Dirac notation in quantum mechanics. *Physical Review Physics Education Research*, *16*(2), 020112.

Wenger, E. (1999). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

Williams, K. R., Wasson, S. R., Barrett, A., Greenall, R. F., Jones, S. R., & Bailey, E. G. (2021). Teaching Hardy-Weinberg equilibrium using population-level Punnett squares: Facilitating calculation for students with math anxiety. *CBE—Life Sciences Education*, *20*(2), ar22.

“Close but not Exact”: An Alternative to Introducing Derivatives

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In this paper, I report on the findings of two students engaged in a teaching experiment on instantaneous rate of change (derivatives). While derivatives are a quantification of how two quantities covary, many students tend to have static images of the derivative (Zandieh, 2000). This teaching experiment was designed to support the students in building a meaning for instantaneous rate of change based in quantitative and covariational reasoning. The results of this study provide empirical evidence of the benefit of alternative teaching methods to the current standard curriculum.

Keywords: Derivative, Quantitative Reasoning, Teaching Experiment

Research on Calculus education has been extensive and covers many topics. In particular, there is a large body of research on the learning and teaching of derivatives and instantaneous rate of change (e.g., Orton, 1983; Ferrini-Mundy & Graham, 1991; Zandieh, 2000; Yu, 2020). One typical method of introducing the derivative concept involves sliding secant lines towards one of the endpoints to get continuously better approximations of a tangent line, the slope of which represents a quantity we call *instantaneous rate of change*. However, researchers have indicated that students often struggle with connecting this sliding secant line imagery with rate of change (Ferrini-Mundy & Graham, 1991; Zandieh, 2000; Ubuz, 2007). This idea of quantities changing continuously and covarationally is central to the idea of derivative, yet the research is clear that students often do not associate quantities as covarying when they think about derivatives (Zandieh, 2000; Byerley et al., 2012; Yu, 2020). Due to these issues, this manuscript reports on the results of a teaching experiment designed to support students in developing a productive understanding of instantaneous rate of change.

Theoretical Background

If we examine the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$, the derivative involves an average rate of change between the input values x and $x + \Delta x$ ($\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$) where the variation in the value of x gets smaller and smaller ($\lim_{\Delta x \to 0}$). In the standard presentation, where a sliding secant line converges to a tangent line, students need not attend to what the average rate of change describes about a given context. This is evident in the literature (Zandieh, 2000; Byerley et al., 2012) when students interpret the value of a derivative (e.g., $f'(3) = 7$), as being only associated with the slope of a tangent line and not necessarily attending to how quantities may be covarying. Therefore, it stands to reason that students should simultaneously attend to x and $f(x)$ covarying and how choosing smaller and smaller input intervals can net us what we call an instantaneous rate of change.

Relevant Literature

Ely and Ellis (2018) proposed that leveraging what they call *scaling-continuous covariational reasoning* could support Calculus students in developing productive ways of thinking. Building off this, Ely and Samuels (2019) provide empirical evidence of alternative ways of teaching the idea of derivative with their "Zoom in Infinitely" method. These studies indicate that supporting students in reasoning about how two quantities' values covary together can help them better understand the derivative concept. This study provides another example of teaching one aspect of the derivative (the limiting portion) using an alternative method that deliberately focuses on supporting students in reasoning quantitatively. Compared to the works of Ely and Samuels, the students are not zooming in on an existing graph of a function. Instead, the students create a piecewise linear function and continually choose smaller and smaller intervals to model an object's movement. This study's research question is: What understanding of the derivative concept do individual students develop during an instructional sequence designed to support students' quantitative and covariational reasoning?

Methodology

This study was part of a larger study conducted by engaging students in individual teaching experiments (Steffe & Thompson, 2000). The students selected were enrolled in a Calculus 1 for Engineers course before learning about secant lines and instantaneous rate of change. The teaching experiment involved six sessions (1 pre-interview and 5 teaching sessions) that focused on characterizing and advancing students' ways of thinking about rate of change. This document reports on two students in the fourth teaching session, where the teacher worked with the students to develop a basis for a productive understanding of instantaneous rate of change. In this session, the way of thinking that the instructor wanted students to develop was the idea that "One can obtain better approximations of the (instantaneous) rate of change at a given value of the input variable by determining average rates ($\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$) of change on smaller and smaller intervals ($\lim_{\Delta x \to 0}$) that include that value of the input variable."

Background

Before the fourth teaching session, the third session involved having the students program (writing a piecewise linear position function) to model the movement of a runner by choosing smaller and smaller intervals (Yu, 2023). That session was primarily focused on developing a productive meaning for average rate of change as an imagined constant rate of change that would provide the same net changes in the output quantity over the same input interval. Additionally, it also seeded the fundamental idea of this paper by providing an animation of their programmed function that showed their model's movements as getting more accurate to the actual runner (Figure 1).

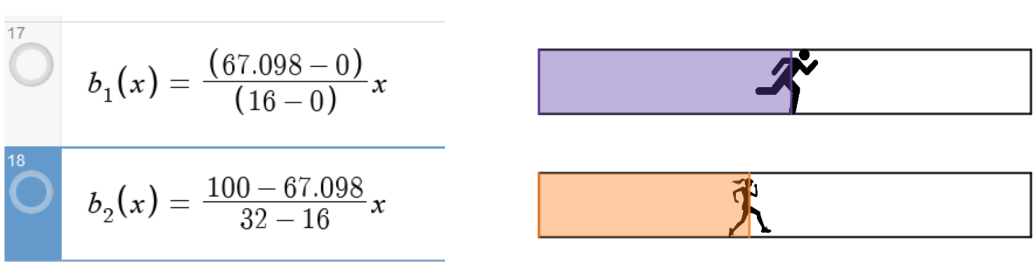

Figure 1: Example of Programming the Bottom Runner using the Average Speed of the Top Runner

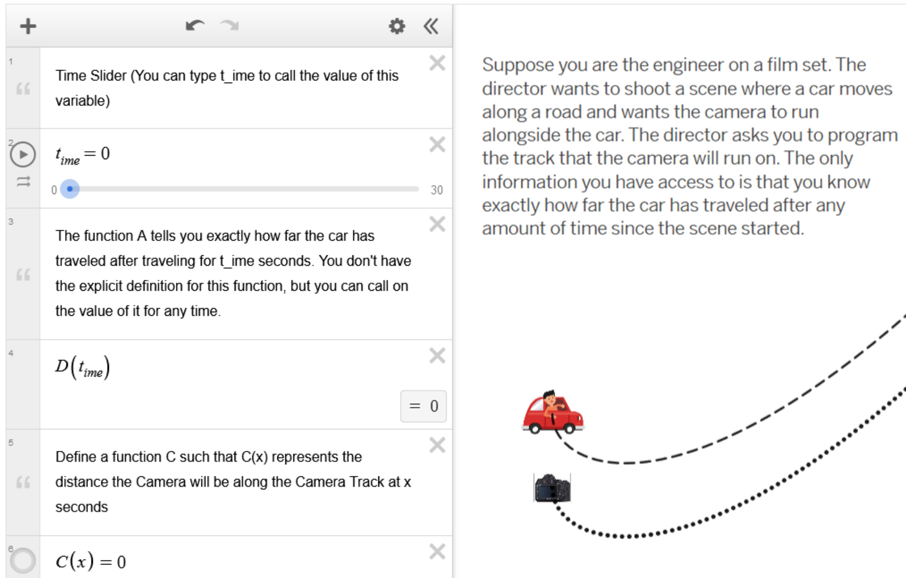

Figure 2: The Camera Task

In the fourth session, students worked through a Desmos Applet and were tasked with modeling the motion of a camera to match a car's motion (Figure 2). While this task is similar to the third teaching session, one important note is that the Desmos Applet provided no indication to construct a piecewise linear function using average rates of change. Instead, students were prompted to discuss their thinking and how they would attempt to mirror the car's movement. Later in the session were questions on how they might determine the speed at a particular time value and approximate future values of a function using that speed.

Teaching Session 4 – Scott

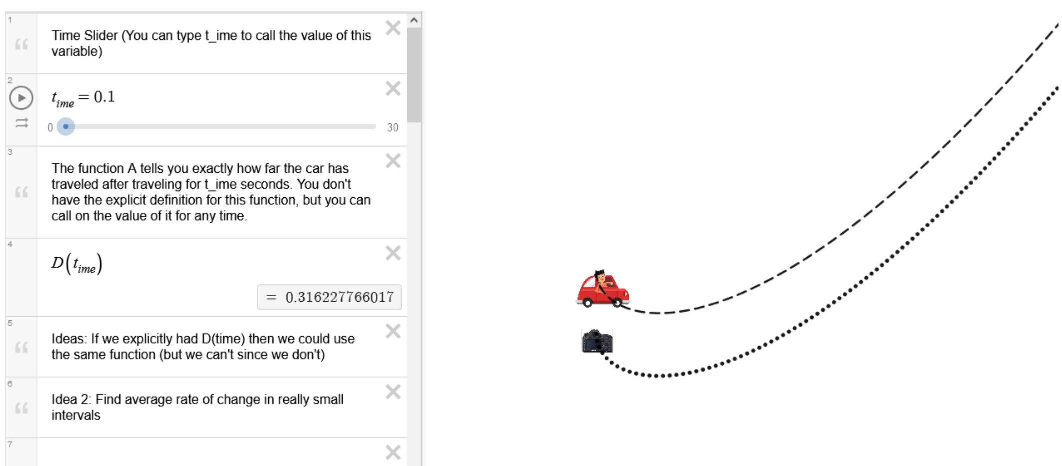

Figure 3: The Camera Problem and Scott's initial ideas

The camera problem prompted students to program a camera's movement on a track to mirror the movement of a car on a parallel track. Scott initially stated that he wanted to determine the equation of the line that would create the same path the camera was on. In other words, Scott wanted a function where the physical shape of that function's graph matched the shape of the track he saw on the Desmos applet. However, he realized that having the function whose graph matched the physical shape of the track path would not model how the camera's distance traveled would vary as time varied. Scott then suggested using the given distance function of the car, D, but soon recognized he could not do this since he did not have the function definition for D (Figure 3). Scott then decided that we could "find the average rate of change in really small intervals where it would be pretty close but not exact." As Scott started to define a function, C, where he used the car's average speed and multiplied it by an amount of time, he verbalized that he wanted to use the same method as the Runner Task (Figure 1) by using average rates of change. Scott's choice to utilize average rates of change is essential to note since the questions accompanying the Desmos applet did not have any prompts to define piecewise linear functions or use average rates of change to model the camera's motion. Scott broke up the 30-second interval into one-second intervals (Figure 4). As he defined his piecewise linear function, he checked his work by playing the animation to ensure that the camera moved according to his expectations.

$t_{IntNumber}$	m	d_{car}
1	$\frac{D(1)-D(0)}{1}$	0
2	$\frac{D(2)-D(1)}{1}$	$D(1)$
3	$\frac{D(3)-D(2)}{1}$	$D(2)$
4	$\frac{D(4)-D(3)}{1}$	$D(3)$
5	$\frac{D(5)-D(4)}{1}$	$D(4)$
6	$\frac{D(6)-D(5)}{1}$	$D(5)$
7	$\frac{D(7)-D(6)}{1}$	$D(6)$
19 more rows Show all		
27	$\frac{D(27)-D(26)}{1}$	$D(26)$
28	$\frac{D(28)-D(27)}{1}$	$D(27)$
29	$\frac{D(29)-D(28)}{1}$	$D(28)$
30	$\frac{D(30)-D(29)}{1}$	$D(29)$

Figure 4: Scott programming the camera's distance traveled over time using the car's average speed in each one-second interval

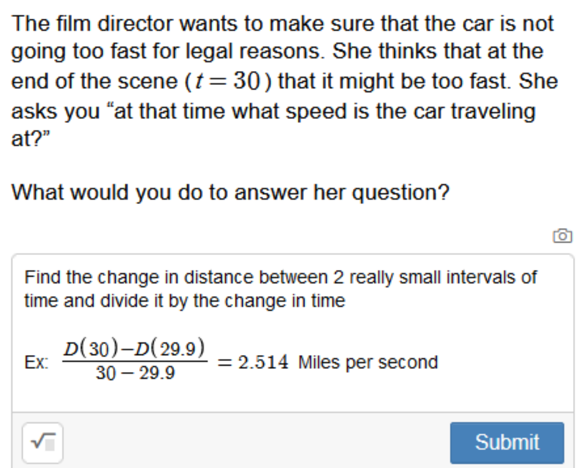

Figure 5: Scott's response to finding the speed of the car at a given input value

After programming the function to model the camera's distance traveled along the track with respect to time elapsed, Scott was asked a series of questions on determining the car's speed at a particular value of time (Figure 5). Scott stated that he would "find the change in distance between 2 really small intervals of time and divide it by the change in time." He later generalized this statement: "I would just take 2 values very close to that time, and I would calculate the rate of change between them." There are several aspects of Scott's worth response worth highlighting:

1. The sizes of the time intervals he chose to determine the speed at $t = 30$ (Figure 5) were different from the size he used to program the camera's motion (Figure 4). For

programming the camera's distance traveled, he used the car's average speed in 1-second intervals, and for the speed at $t = 30$, he used the car's average speed in a 0.1-second interval. This suggests that Scott's choice in the "2 values very close to that time" was arbitrary and that no specific interval size was needed.

2. Scott verbalized that using average rates of change over a small time interval "would be pretty close but not exact" to modeling the car's motion as time passed. As he answered each of the prompts about determining the car's speed at a particular time, he never verbalized a need to justify that they were close approximations. It is likely that because the animation displayed the motion of the camera and the car was essentially the same; he might have believed that his approximations were good enough and that the difference between the actual and the approximated distance functions was insignificant.

Teaching Session 4 – Hans

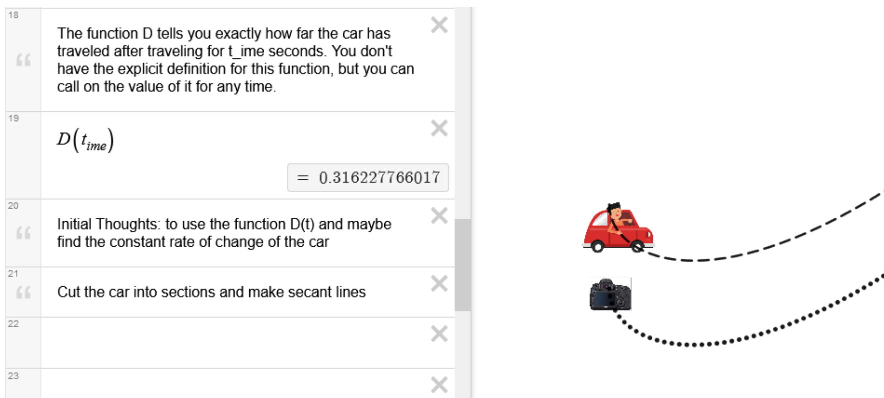

Figure 6: The Camera Problem and Hans' Initial Ideas

Like Scott, Hans' initial approach to the camera problem was to determine the equation of the line whose graph matched the path of the road. He then suggested finding "the constant rate of change of the car." However, after playing the animation, Hans recognized that the car was moving at varying speeds, so he proposed that we "cut the car into sections and make secant lines" (Figure 6). Hans verbalized that this was similar to the third teaching session, and he wanted to build a piecewise linear function using the car's average speed over small time intervals to model the camera's motion.

Hans employed 3-second intervals (Figure 7) and did not play the animation until he finished defining the piecewise function. After playing the animation, he noticed that the camera and the car's movements did not align adequately in the first few seconds. He then suggested, "we could do better if I made it like 1 second [the interval size] instead." Rather than having Hans rewrite his piecewise function, the interviewer asked hypothetical questions such as "suppose we wanted to do 1-second intervals, what would we need to change?". Hans replied that he would have to change everything since "you wouldn't be able to use like 3 seconds…like how these are done. I would have to re-do them to resize these rates of change." After this round of questioning, Hans demonstrated that he could obtain better approximations of the car's distance traveled with respect to time elapsed by using average rates of change over a smaller time interval.

	$t_{IntNumber}$	m	d_{car}
	1	$\frac{(D(3)-D(0))}{3-0}$	0
	2	$\frac{D(6)-D(3)}{6-3}$	$D(3)$
	3	$\frac{D(9)-D(6)}{9-6}$	$D(6)$
	4	$\frac{D(12)-D(9)}{12-9}$	$D(9)$
	5	$\frac{D(15)-D(12)}{15-12}$	$D(12)$
	6	$\frac{D(18)-D(15)}{18-15}$	$D(15)$
	7	$\frac{D(21)-D(18)}{21-18}$	$D(18)$
	8	$\frac{D(24)-D(21)}{24-21}$	$D(21)$
	9	$\frac{D(27)-D(24)}{27-24}$	$D(24)$
	10	$\frac{D(30)-D(27)}{30-27}$	$D(27)$

Figure 7: Hans' programming the camera's distance traveled over time using the car's average speed in each three-second interval

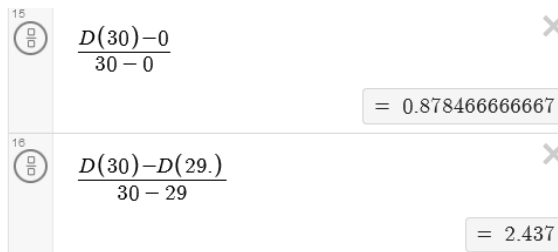

Figure 8: Hans moving from $\frac{D(30)}{30}$ *to* $\frac{D(30)-D(29)}{30-29}$

The interviewer then asked Hans how he might determine the speed at $t = 30$. Hans initially stated "$\frac{D(30)}{30}$ since it is the rate of change for 30… err… it's just a rate, not a rate of change." He later clarified that $\frac{D(30)}{30}$ "is a rate [and not a rate of change] because it remains a constant value of 0.878 like it would be the constant rate at 30." The interviewer then wrote $\frac{D(30)}{30}$ as $\frac{D(30)-0}{30-0}$ and asked what $\frac{D(30)-0}{30-0}$ would represent in this context. Hans identified that $\frac{D(30)-0}{30-0}$ would be the car's average speed between 0 and 30 seconds, and then quickly realized that $\frac{D(30)}{30}$ would not be the car's speed at 30 seconds. Afterward, he suggested we could determine the average speed between 29 and 30 seconds since "it's in the region that we want it in like 30, so it's close enough to the speed" (Figure 8). On the other tasks that prompted Hans to determine the speed at a given time value, Hans' responses indicated that he would determine the average rate of change over a small interval near the requested time (Figure 9).

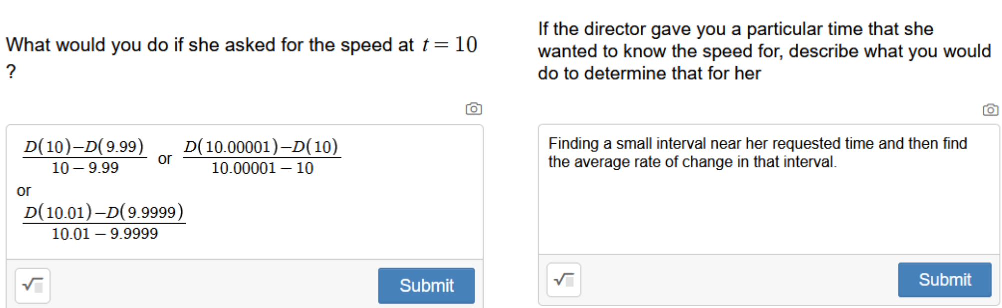

Figure 9: Hans' explanation of how to find a speed at a given input value

Several aspects of Hans' responses are important to discuss.

1. Hans thought about $\frac{D(30)}{30}$ as a "rate not a rate of change." His expression, $\frac{D(30)}{30}$, suggested that a fraction with a singular distance divided by a singular time would determine a

"rate," and something of the form $\frac{y_2-y_1}{x_2-x_1}$ would determine a "rate of change" since the numerator and denominator represented a change in one quantity's value. It was likely through reconceiving $\frac{D(30)}{30}$ as being equivalent to $\frac{D(30)-D(0)}{30-0}$ (which he identified as the car's average speed between 0 and 30 seconds) that supported Hans in conceiving $\frac{D(30)}{30}$ as the car's average speed over 30 seconds.

2. Another possibility for his response of $\frac{D(30)}{30}$ as the car's speed at 30 seconds is that he considered that a rate was "distance over time," and so he used the distance at 30 seconds, $D(30)$ and divided it by the time value of 30 seconds. In either case, what supported Hans in shifting his conception of $\frac{D(30)}{30}$ was the prompting to explain what each portion of his expression represented. Additionally, writing an equivalent expression in the form that he was familiar with (e.g., recognizing that $\frac{D(30)-D(0)}{30-0}$ would represent an average speed) aided him in reflecting on the quantities he attempted to represent.

Like Scott, Hans also demonstrated an understanding of the arbitrariness of choosing how small an interval had to be when determining the speed at a given input value. When determining how he might determine the speed at $t = 10$, Hans initially wrote $\frac{D(10)-D(9.99)}{10-9.99}$ but also noted that he could choose other sized intervals like the subsequent ones he wrote in Figure 9.

Discussion

The findings of this study present a possible alternative to how Calculus instructors can introduce the idea of instantaneous rate of change compared to the traditional sliding secant line methods used by popular textbooks (e.g., Stewart). In this study, both students benefitted from connecting their mathematical expressions with an applet to model their mathematics. Additionally, the applet provided a medium for the students to visualize how choosing small enough input intervals could be useful in approximating an instantaneous rate of change. Compared to the standard receding-secant line method, where the secant lines converge to one single point (in other words, a demonstration of approximating the instantaneous rate of change at only one particular value), the students in this study were engaged with developing an entire rate of change function as they constructed their piecewise linear functions. As a result, both students in this study demonstrated that they likely internalized this process of approximating an instantaneous rate of change since they described the process as involving an average rate of change over a small enough input interval.

While motion-based contexts should not be the only examples Calculus students should interact with, they can provide a tangible situation that students can associate their mathematical understandings with (Berry & Nyman, 2003). This study builds off the existing literature (Ely & Samuels, 2019) of alternative ways to introduce the idea of derivatives to support students in quantitative reasoning.

Limitations and Future Directions

While the results of this study show promising results for alternative teaching methods, the results can only be known to be true for the students in the study. Future studies can continue to investigate how we can support students in reasoning covariationally in Calculus and how to implement these alternative methods in the classroom.

References

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior*, *22*(4), 479–495.

Byerley, C., Hatfield, N., & Thompson, P. W. (2012). Calculus students' understandings of division and rate. In *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education*.

Ely, R., & Ellis, A. B. (2018). Scaling-continuous variation: A productive foundation for calculus reasoning. In S. Brown, G. Karakok, K. Roh & M. Oehrtman (Eds.), *Proceedings of the 21st Annual Conference for Research in Undergraduate Mathematics Education*

Ely, R., & Samuels (2019). "Zoom in Infinitely": Scaling-continuous Covariational Reasoning by Calculus Students. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 180-187).

Ferrini-Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, *98*(7), 627-635.

Orton, A. (1983). Students' Understanding of Differentiation. *Educational Studies in Mathematics*, *14*(3), 235-250.

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology*, *38*(5), 609-637.

Yu, F. (2020). Students' Meanings for the Derivative at a Point. Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). (2020). *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education.* (p, 672-680)

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 4* (pp. 103–127). American Mathematical Society.

Bridging the Mathematical and Social Dimensions of Undergraduate Calculus: Students' Perspectives on a Program of Weekly Guided Collaborative Problems Solving

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This study explores undergraduate Calculus students' perspectives on the Course Assistants (CA) program at a southern US university, where upper-class undergraduates guide weekly Math Lab sessions of collaborative problem-solving. Using sociocultural theories and qualitative methods the research investigates what aspects of Math Lab students and CAs found useful or not. Students' interviews and CAs' weekly reflections indicate that, on the one hand, students by and large appreciate the experiences of collaborating over the weekly problems with their peers and the CAs, the social networking it provided, and the mathematical confidence it helped build. In our own words, students appreciated how the academic and social dimensions of math learning came together. However, many students and CAs are skeptical about the connections between the weekly tasks and success in the course exams. Relatedly, students often complained about Math Lab being another structured hour in an already time-consuming course and about varying student engagement.

Keywords: undergraduate calculus; collaborative problem solving; course assistants

Research and professional reports of undergraduate mathematics education often state the promises and challenges of shifting learning towards more active approaches (where *active learning* is conceived broadly and inclusively). For example, research reports in the proceedings of the 2022 Research of Undergraduate Mathematics Education conference (RUME 2022) addressed a variety of issues around active learning including instructors' workshops (Archie et al., 2022), instructors' dispositions (Ireland et al., 2022), online active learning (Kerrigan et al., 2022), linguistic diversity (Rios, 2022), and more. Such RUME research reports often cite as a starting point The Common Vision (Saxe & Braddy, 2015) and empirical evidence about the impact of active learning in undergraduate STEM courses (Freeman et al., 2014; Laursen, 2019; Laursen et al., 2014; Theobald et al., 2020). Side by side with these encouraging outcomes, we also know that K-16 efforts to implement collaborative math learning approaches (the specific type of active learning which is the focus here) are complex to incorporate and facilitate, sometimes fall short, and might even amplify exclusionary dynamics (Ehrenfeld & Horn, 2020; Louie, 2017; Reinholz et al., 2022; Rios, 2022; Shah & Lewis, 2019). In this paper we explore such tensions in the context of a Course Assistants (CA) program, in which upper class undergraduate students facilitated weekly collaborative problem solving sessions (known as Math Lab) in Calculus courses at a private southern US university. First, we provide a general overview of the program. Then, we share our sociocultural theoretical perspective, an overview of the data and methods of analysis we used, and finally, the main themes that emerged from the data. The overall question we address in this report are *What aspects of Math Lab students and CAs found more or less useful, how, and why?*

Research Context: The CA Program

The overarching goal of the CA program was to support undergraduate Calculus students mathematically and socially during this transition to college. Students worked in small groups of 3-9 with a CA in weekly Math Lab sessions on calculus problems which were longer and more conceptual in nature than typical homework and exam questions. Those problems were written collaboratively by the course instructors, with direct relation to the weekly materials. Socially and mathematically, we hoped that students' experiences in the Math Lab would support them in developing collaborative study skills, conceptual understanding, and a sense of community and belonging in the course. After a pilot semester in Spring 2022, the Fall 2022 program started in one course with 20 course assistants (CAs) who facilitated 40 Math Lab groups which served ~250 students. The Spring 2023 program expanded to three courses with 20 CAs (17 returning and 3 new) who facilitated 40 Math Lab groups which served ~250 students. Math Lab was required only for students in the first Calculus course of their sequence. The rationale behind not requiring it in the following course was that by the time students are taking a second semester course, they have more practice in figuring out what kind of support they need and should have more agency to decide that for themselves.

Theoretical Framework: Sociocultural Theories of Mathematics Learning

Sociocultural theories of mathematics learning extend a cognitive focus on individual learners, instead focusing on the participation of learners in social practices within a particular context (Danish & Gresalfi, 2018; Sfard, 1998). They inform our work by providing core assumptions about students' mathematical competencies and identities as constructed through interactions within context rather than predetermined, acontextual, and act as individuals' static traits (Ehrenfeld & Heyd-Metzuyanim, 2019; Gresalfi et al., 2009; Gresalfi & Hand, 2019). These theoretical assumptions often imply a tendency towards qualitative methodologies that allow a deeper look at how participants engage within their learning environments, navigating and reconciling the mathematical and social aspects of learning.

Methodology

In this section we describe the data we collected along the year, including students' interviews and recordings of the CA training across two semesters (see Table 1 for a summary). We then describe how we reviewed and interpreted the data, with students' interviews being the primary data for this report.

Data Collection

In Fall 2022, the study was approved by the IRB around week 7 of the semester. At this point we started to video record the last 7 weekly CA training sessions. In Spring 2023, we video recorded 13 of the 14 weekly CA training meetings in the Department of Mathematics. Finally, we interviewed nine students about their Math Lab experiences. The students were not chosen by us individually but rather answered our general call for interviews. Some of the students participated in the program in Fall 2022 and decided to register or not in Spring 2023 as well. Some of the students repeated their course with Math Lab at their second round and spoke about their experience in the same course with and without Math Lab. One student simply took both the course and the Math Lab for the first time in Spring 2023

Table 1. Summary of data collected for this study.

Data for this study		
<u>Semester</u>	<u>Data</u>	<u>Including…</u>
Fall 2022	7 (of 14) weekly CA meetings	Video + groups audio
Spring 2023	13 (of 14) weekly CA meetings	Video + groups audio
Spring 2023		Zoom interviews

	End of semester interviews with 9 students	(primary data for this paper)

Data Analysis

For the purposes of this paper we primarily reviewed the end of Spring 2023 semester interviews with the nine students. As secondary data we also used recordings of the weekly CA meetings where students reflected on their Math Lab facilitation. We collected codes and then themes (e.g., Charmaz, 2006; Strauss & Corbin, 1990), allowing our reading of the data to suggest the themes, rather than searching for themes identified a priori. In practice, this meant assigning initial codes to episodes where students and CAs were talking about aspects of Math Lab as more or less useful for the students. Note that after reviewing all interviews and about half of the CA training data we reached a point of saturation where codes repeated, and we almost did not encounter new ones. At this point we had an initial set of 10 codes under Strengths describing participant-identified dimensions of utility of the MathLab, and a set of 10 codes under Weaknesses describing less useful participant-identified aspects of Math Lab. In the final phase we merged and reduced each category of codes into three themes that were common in the data, significant for the students, and provoked generative points for this discussion.

Themes Emerging From the Data

In this section we first describe and illustrate three strengths of the program from the students' and CAs' perspectives: the social support and network that Math Lab provided, the opportunities to develop mathematical competence and confidence through conversations about math problems, and the affordances of near-peer math support. These repeating themes sketch a collective image of what was found conducive to success in Calculus by the interviewed participants. All three themes illustrate how students found useful bringing together the social and academic dimension of mathematics learning. Some also contrast this interrelatedness with typical mathematics lectures and university life. Then, we describe and illustrate three problems, areas for revision, or less useful aspects of the program, from the students' and CAs' perspectives: The fact that it requires extra time in courses that are already dense and demanding, that students and CAs are not seeing the weekly Math Lab assignment as directly connected to the tests, and that for some, fully participating within the group was a challenge. Finally, we briefly discuss the implications of these themes, how they add to a collective image of needs, pressures, and valued support for success.

Strengths of the Program and Things Students and CAs Found More Useful

Social support and networks. Many students and CAs described their Math Lab groups as a friendly and fun environment, where positive relationships were established. For some, these relationships stayed within the scope of the Math Lab meetings. For example, Student 3 said that "our group like even though we didn't really talk outside of Math Lab we always had the really like friendly funny conversations during math lab we were always cracking jokes and everything." For others, those relationships extended onto the general course, or more broadly to social life on campus. For example, when asked what the highlight of Math Lab was for her, Student 4 said:

> [The highlight is] the people that you meet. Especially first semester just because like you know during that time everything is so stressful and especially if you don't know people in the class. And like maybe you are someone that struggles, [...] or like just like have

> questions sometimes. I think that like being in the Math Lab group is important because you have like just a few- handful people you know you can reach out to like if you need help. And I feel like that is better because sometimes being like in a class with 30 students is little intimidating when you like want to speak to other people and ask where they are at, but the Math Lab group kind of makes it a little bit more like a social- like add a social aspect which I think is important especially when you are a first semester freshman...

All in all, often students and CAs saw their Math Lab groups as a friendly and welcoming environment, where positive bonds were created (sometimes contrasting it with other courses without Math Lab). These relations were of different type and scope: with some being more local in nature, just within Math Lab (as Student 1 described), some extended to the course classroom (as Student 6 described), and for some (like Student 4) those relations continued to students' ongoing academic and social campus life. Importantly, we know from the data and the literature that improvements in sense of belonging and deeper social connections in turn translate to better learning outcomes. The next theme further illustrates how students experienced the convergence of the social and the mathematical.

Communicating about math and developing competence & confidence. When we asked students what their Math Lab looked like, they often described positive experiences of doing math together, communicating and reasoning collaboratively about the content. Sometimes they also connected this type of math to their sense of confidence and competence, or to the advantages of explaining math out loud or being exposed to the ways other students are thinking about the weekly problems. For example, Student 2, who described herself as someone who "hates math" and wished she wouldn't have to take any math courses, told us that the best thing about Math Lab was "I think the most beneficial thing for me in Math Lab was like knowing that I can do these sort of problems, this is a good confidence thing..." In another example, Student 6 shared:

> we would just go problem by problem and a lot of- we would be like "hmm: what did people like get for solutions on problem 1." And if there was like difference in opinions we would hmmm like go to the whiteboard and work out our work or show our work on paper or ipad and discuss and like discuss collaboratively about different methods for this solutions and what do we think we should put on the group doc we submitted.

Similarly, Student 8 shared how they were "talking it through" and her insight that "going through your line of reasoning could help you better understand how you look at math and how others look at math, and that can inform a more solid foundation…" as well as how sharing her thinking with the group and being validated by the CA was gave her a "confidence boost." Finally, Student 9 concluded his Math Lab experience with the insight that it is really good to have a dedicated time for groupwork because "sometimes you can learn more from your fellow students in a way that you can't from your instructor or the TA."

Near-peer support. While many students appreciated the opportunities to do math with their peers, the Math Lab experiences were different from a regular student learning group at least in one significant way: having their near-peer support. Near-peer support was discussed both in terms of the CAs, and sometimes also in terms of their upper-level Math Lab friends (e.g., freshmen and sophomores in the same groups). In addition, near-peer support was mostly discussed in terms of math support during Math Lab, but it also extended to other aspects of students' life like choosing courses and applying for summer jobs. Near-peers were also mentioned in contrast to TAs and to instructors, as being more approachable and as allowing to form closer mentoring relationships. The CAs reflections pointed out similar narratives. Capturing nicely the near-peer theme, Student 6 articulated the experience of being with a near-peer CA as being both “formal and informal,” She said: “My CA did a really good job about making it like formal but also informal at the same time. Kind of like just a different setting than the actual class which I think was helpful.” Describing the work with the same CA, student 8 explained the importance of the CA being someone who took this class recently:

> [The CA] was awesome she was really [unclear] and she understood the importance of the emotional support aspects, and like, someone who took the course and she is sophomore so she is like close to our age actually one of the kids in our lab group is older than her, and it felt like a peer relationship and she was very open if you guys need anything at all i'm always here for advice.

Others often mentioned their CAs for their math support (Student 1), their ability to get to know students and how they feel (Student 4), and their support with academic issues beyond math content (Student 3). Student 3, who is a student of color who had in the Fall semester a CA of color, described how the two are still in touch one semester after the course ended. She particularly appreciated how beyond the math assignments, the CA “loved to like… she was always talking about, like, doing internships and encouraging us to, like, apply ourselves outside the math lab and do different programs and stuff and so, um, yeah. She was awesome and we all realized that very early on.”

In sum, the near-peer support was a recurring motif in students’ and CAs’ descriptions of Math Lab, and this position seemed to offer some affordances that are different from the support typically provided by same-level peers, by TAs, and by instructors.

Weaknesses of the Program: Things Students and CAs Found Problematic, Less Useful, or Requiring Revision

Math Lab takes extra time in already time-demanding courses. With the Math Lab (and its weekly problems), the TA discussions, and the daily homework, Calculus students at the university have a relatively extended set of obligations. For example, the first thing that Student 1 told us about, was that she was surprised by the time demands of the calculus sequence. In the context of an already-demanding course, the Math Lab could seem like an added burden to students having an experience like that of Student 1. Similarly, Student 6 explained that “for some students it can be hard to justify like showing up somewhere for another extra hour a week especially when we are all kind of like busy.” For Student 3, who was generally happy with her Math Lab experience in the Fall, time constraints were the main reason she decided not to continue with Math Lab for the Spring when it was not required. Just as Student 1, she explained at the very start of the interview that “math is a lot of work here, it's a lot of homework and I feel like I have math every single day with the math lab and lecture and discussion....” Then, when we asked why she decided on not going to Math Lab on MATH 1201 she added:

> Math is basically like everyday here with the lectures and discussions so I thought like if I didn't have to sign up for Math Lab to begin with, so maybe it would help my schedule just a little bit. Because I just remember last semester like trying to balance Math Lab with like clubs and everything was a lot of work so I thought like it would be better to allocate that time to something else, some organization or something else on campus.

As we illustrate in the next section, others questioned the usefulness of Math Lab because they did not see the direct connections between the weekly problems and the exams.

Students and CAs are not seeing connectedness to exams. The concern that the weekly Math Lab problems are not useful for students because they are not similar in nature to the daily homework question nor to the exam (e.g., longer, more conceptual, open ended, use tech like Desmos), came up across students' interviews and CAs' discussions. For example, when we discussed metacognition in the CA training, we asked the CAs when and how they were noticing students' frustration with the weekly problems. One CAs answered as follows.

> I feel like where I see the peak of frustration so like- my group they work on their homework as well, so like they go through the problem set and the homework because it is due the same day. And so their frustration is often times the weekly [Math Lab problem] is different than the [non Math Lab] homework, like significantly different. So in their minds it's like how is this benefiting me because I have to worry about my homework right now which is like my grade that has like a higher grade [weight], I see more of the correlation from this to my exam vs. like some of the... [...] they wouldn't be writing something like that [on the exam]. I guess it's gonna be more you're given a function and then you write the derivative something like that. So they get frustrated because they are comparing both of them [non Math Lab homework and weekly Math Lab problem] because they are doing it at the same time. I don't know how to explain like why the dynamic should be different because you don't do the same thing. That's not gonna benefit you.

These conversations were typical in the CA training where CAs described their students' perspective about the tasks, their own perspectives, and their concerns with regards to responding to students complaints. And while in some interviews with students we did hear some appreciation to the conceptual nature of the weekly assignments, students mostly expressed dismay with not seeing connectedness to exams. For example, when asked to elaborate on different aspects of Math Lab, Student 1 said:

> I feel like the actual Math Lab tasks were very- they felt very like disconnected at times from what we were actually learning. Because none of the stuff we did in Math Lab was never on any test like I feel like it always required some outside knowledge [...]

In sum, many MATH 1200, 1201 students and their CAs identified and struggled with the incoherence between the Math Lab and other course activities, including those who appreciated the opportunity to do math with peers and the CA and generally enjoyed their Math Lab. It is worthwhile to note that MATH 1300 students generally responded better to the open and conceptual nature of the problems, and more easily saw the connections between the Math Lab assignments and the course.

Limited student engagement within the Math Lab group. While we aspired to provide students with productive and inclusive mathematical work in small groups, this was not always exactly what played out in practice. Students and CAs often described less effective ways of working together (and not together) and encountered different forms of disengagement. Interpretations of disengagement ranged from deficit-oriented (e.g., a student described some students as "piggybacking" the group submission) to acknowledging that it is challenging to learn how to do math together, and it might require time and support for students to "put themselves out there."

We start by illustrating how students themselves pointed out certain group dynamics as troubling. Some students described them as a constant concern across the semester, and others as growth pains at the beginning of the semester. For example, when Student 2 was asked what makes a good Math Lab, she mentioned having regular attendance and that it should be clear that "if you are in a group you participate in doing the work." She then elaborated, expressing frustration that some students were "piggybacking" on the submission, showing up but not really participating. Student 8 shared a similar feeling. However in her case group dynamics shifted after the first 2-3 weeks:

> I was worried in the beginning like not everyone putting in effort but as we went on I think everyone started to see how valuable the assignments were and how like talking it through and going through your line of reasoning could help you [...] The first like two or three meetings it was kind of mostly 2 other girls and me sort of carrying stuff and everyone else were kind of silent. I don't know exactly why it changed but it did. And then everyone started contributing very equally and I felt more comfortable coming in with questions rather than answers.

The narrative of group growth across the semester towards more effective and inclusive participation repeated in interviews with the CAs at the end of Fall semester and in their end of year feedback. These accounts were typical in the CA training, and we encouraged CAs not to share just "success stories" but also everyday problems. Our hope was that they will develop sensitivity to notice non-productive and exclusionary patterns of student engagement, and will improve their capacity to respond to them.

Conclusions and Implications

Students and CAs found several aspects in this program useful, including the development of social support networks, enhanced mathematical competence and confidence through collaborative problem-solving, and the value of near-peer support from CAs. These strengths illustrate that students found important the convergence of the social and academic dimensions of mathematics learning, which they often noted as missing in their undergraduate math journey. With regards to areas of revision, students felt that Math Lab added to the time demands of an already challenging course, and they struggled to see direct connections between Math Lab assignments and course exams. These themes could be interpreted either from the CA program perspective or from a general Calculus program perspective. From the CA program perspective, one attempt we made in the Spring to bridge for students the Math Lab weekly tasks with other course activities for students was to include some related homework/exam-like exercises with each conceptual lab problem that are rooted in the same mathematical concepts but are not for submission. From a general program perspective, if conceptual understanding is a goal of Calculus programs overall, building coherence across component of the course might also imply broader changes in the course. In that sense, the themes we presented are not only addressing students' perspective on this specific CA program, but they also direct us to the general needs of undergraduate Calculus students. These findings highlight the value and complexities of implementing undergraduate Calculus collaborative math learning approaches that bridge the academic and social dimensions of mathematics learning.

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References

Archie, T., Daly, D., Laursen, S. L., Hayward, C. N., & Yoshinobu, S. (2022). Investigating the effectiveness of short-duration workshops on uptake of inquiry-based learning. In Karunakaran, S. S., & Higgins, A. (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis.* Thousand Oaks, CA: Sage.
Danish, J. A., & Gresalfi, M. (2018). Cognitive and Sociocultural Perspectives on Learning. In F. Fischer, C. E. Hmelo-Silver, S. R. Goldman, & P. Reimann (Eds.), *International Handbook of the Learning Sciences* (1st ed., pp. 34–43). Routledge.
Ehrenfeld, N. & Heyd-Metzuyanim, E. (2019). Intellective identities and the construction of a hybrid discourse: the case of an ultra-Orthodox Jewish mathematics classroom. *International Journal of Science and Mathematics Education, 17*, 739–757.
Ehrenfeld, N., & Horn, I. S. (2020). Initiation-entry-focus-exit and participation: a framework for understanding teacher groupwork monitoring routines. *Educational Studies in Mathematics, 103*, 251–272.
Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111*(23), 8410-8415.
Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational studies in mathematics, 70*, 49-70.
Ireland, J., Dorko, A., Cook, J. P., Tallman, M., Oehrtman, M., & Jaco, W. (2022). Conceptions of active learning held by undergraduate mathematics instructors participating in a statewide faculty development project. In Karunakaran, S. S., & Higgins, A. (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.
Laursen, S. (2019*). Levers for change: An assessment of progress on changing STEM instruction.* Washington, DC: American Association for the Advancement of Science. https://www.aaas.org/resources/levers-change-assessment-progress-changing-stem-instructio n
Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry based learning in college mathematics: a multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406–418.
Rios, J. (2022). Active learning in the language diverse undergraduate math classroom. In Karunakaran, S. S., & Higgins, A. (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education. Boston*, MA.
Kerrigan, J., Tabanli, S., Cochran, G., Charnley, M., Mulvey, S. (2022). Transitioning active learning STEM recitations online. In Karunakaran, S. S., & Higgins, A. (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.
Reinholz, D., Johnson, E., Andrews-Larson, C., Stone-Johnstone, A., Smith, J., Mullins, B., Fortune, N., Keene, K., & Shah, N. (2022). When active learning Is inequitable: Women's participation predicts gender inequities in mathematical performance. *Journal for Research in Mathematics Education, 53*(3), 204–226. https://doi.org/10.5951/jresematheduc-2020-0143.
Saxe, K., & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025.* The Mathematical Association of America.
Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational researcher, 27*(2), 4-13.

Shah, N., & Lewis, C. M. (2019). Amplifying and attenuating inequity in collaborative learning: Toward an analytical framework. *Cognition and Instruction, 37*(4), 423-452.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research (Vol. 15).* Newbury Park, CA: Sage.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., ... & Grummer, J. A. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences, 117*(12), 6476-6483.

The College Mathematics Beliefs and Belonging Survey: Instrument Development and Validation

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This contributed report describes the development and validation of a new measure—the College Mathematics Beliefs and Belonging (CMBB) survey. The CMBB provides a contemporary measurement of undergraduate students' perceptions of their mathematical proficiency and reasoning, beliefs about mathematics, and sense of belonging in mathematics. Primarily first- and second-year undergraduate students in five courses at a large public university in the United States completed multiple surveys to provide the data used for survey development. Confirmatory factor analysis (N = 935) and a reliability analysis indicate that the CMBB is a survey with fifteen factors that adequately measure various aspects of perceived mathematical proficiency and reasoning, beliefs, and sense of belonging. The CMBB survey is intended for use by both researchers and instructors to assess undergraduate students' perceptions across these three domains with the aim of improving students' experiences in college mathematics courses.

Keywords: proficiency, mathematical reasoning, beliefs, belonging, factor analysis

For decades, mathematicians, mathematics educators, and psychologists have explored individual differences and instructional factors that affect students' achievement and motivation in postsecondary mathematics. There is a long tradition of postsecondary mathematics instructors identifying habits, beliefs, and abilities that are viewed as critical for effective mathematical practice. This collection of habits, beliefs, and abilities is sometimes referred to collectively as "mathematical maturity" (Faulkner et al., 2019; Garrity, 2011; Lew, 2019; Steen, 1983). In a first step toward delineating specific key habits, abilities, and beliefs, mathematicians engaged in scholarly teaching have produced practitioner-focused reports documenting developmental goals for postsecondary mathematics students, often through the work of professional societies such as the Mathematical Association of America (MAA). The MAA Committee for the Undergraduate Program in Mathematics has long been active in this work, producing the series of Curriculum Guides to Majors in the Mathematical Sciences (MAA, 2015) and The Curriculum Foundations Project (Ganter & Barker, 2004), created by the Curriculum Renewal Across the First Two Years subcommittee. This work has been complemented by a growing body of research in undergraduate mathematics education involving a range of themes including student thinking, student-instructor interaction, and effective pedagogy, among others (e.g., Bressoud et al., 2015; Carlson & Rasmussen, 2008; Laursen & Rasmussen, 2019).

Psychologists have also pursued parallel lines of investigation. Psychological research has revealed that students' persistence in STEM courses is highly influenced by their beliefs about math, perceptions of their own learning, and their academic motivation more broadly (e.g., Hulleman & Harackiewicz, 2009; see Eccles & Wigfield, 2020, for a review). However, many existing measures of students' beliefs, motivation, and sense of belonging in mathematics based on psychological research have paid insufficient attention to how mathematicians and mathematics educators conceptualize student learning in the context of college mathematics.

The goal of the current investigation was to develop an instrument to measure college students' beliefs about mathematics, and beliefs about their own mathematics learning, that have been identified as effective for mathematics practice by scholarship, research, and theory from multiple communities of practice. The *College Mathematics Beliefs and Belonging* (CMBB) survey can be used by researchers and instructors to assess student attitudes and beliefs in undergraduate mathematics courses, including students' views about their own mathematical proficiency, their beliefs about the nature of mathematics, aspects of what motivates them in mathematics learning, and multiple facets of their sense of belonging in mathematics. By considering the contributions of multiple communities of practice, we believe that the CMBB survey is representative of a wide range of perspectives that reflect the increasing diversity of undergraduate students taking introductory mathematics courses at the postsecondary level.

Theoretical Frameworks

Mathematical Proficiency and Reasoning

First, we sought to capture student individual differences in theoretically meaningful aspects of their mathematical proficiency and reasoning. One influential framework is the 5-strand model from the National Research Council's (NRC) "Adding It Up: Helping Children Learn Mathematics" report (Kilpatrick et al., 2001). The 5-strand model is comprised of five theoretically unique, but related, aspects of students' knowledge: Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition. The NRC Committee proposes that these five strands of mathematical proficiency and reasoning are separable yet interwoven constructs, with each supporting the others and productive disposition bringing together all strands. This view is also aligned by psychological research on conceptual and procedural understanding in mathematics (e.g., Rittle-Johnson & Siegler, 2021) and academic motivation (e.g., Eccles & Wigfield, 2020).

In addition to these five strands, we reintroduce Gray and Tall's (1994) conceptualization of proceptual thinking, a separate construct that focuses on the fundamental integration between understanding a procedural process and understanding conceptual mathematical relationships. Proceptual thinking reflects a key aspect of how mathematicians perceive mathematical symbols. Gray and Tall argue that mathematicians' perception of symbols includes understanding processes that a symbol represents, the mathematical object a process produces, and an understanding of the symbol flexible enough to include both the processes and object.

Beliefs About Mathematics

Postsecondary students enter college with a range of beliefs about the nature of mathematics (see Philipp, 2007). For example, some students think of mathematics as a collection of relatively independent rules and procedures (Schifter, 1990). Another common belief is that mathematics is fundamentally a form of logical thinking (Dossey, 1992). Finally, some students consider efficiency to be a key characteristic of doing mathematics (Boaler et al., 2015). Many of these beliefs held by students do not accurately reflect mathematicians' conceptions of mathematics (Gold et al., 2017; Hersh & John-Steiner, 2010). Thus, we aimed for our contemporary scale to reliably assess how undergraduate students conceptualize the nature of mathematics.

Students also possess motivational beliefs about mathematics (Kloosterman, 2002; Royster et al., 1999), including the usefulness of mathematics (utility value; Eccles & Wigfield, 2020), confidence in their own mathematical proficiency (self-efficacy; Bandura, 1997), and their own and others' abilities to learn and understand mathematics (growth mindset; Yeager et al., 2019). Importantly, motivational beliefs often uniquely predict academic outcomes, over and above mere content knowledge. For example, classroom-based experimental studies reveal that when students can see the utility of their college STEM classes, they are more likely to persist in STEM majors years later (Asher et al., 2023). Additionally, mathematical confidence is a driving factor in persistence in college mathematics, especially among women (Ellis et al., 2016). Finally, students who understand that mistakes are a part of learning are more likely to persist in advanced mathematics and earn higher grades in mathematics courses (Yeager et al., 2019).

Given these considerations, our view is that understanding the multidimensional nature of student beliefs of mathematics is pivotal to fostering student learning, understanding differences between groups of students (e.g., gender differences) within a course, and assessing the impact of different pedagogical approaches. Thus, we aimed to include a broad range of belief-oriented items—encompassing both beliefs about the nature of mathematics and motivational beliefs about mathematics—in our survey.

Centering Equity and Inclusion through Sense of Belonging

A growing body of work has adopted sociocultural frameworks that acknowledge the importance of the broader learning context—such as belonging—in shaping student learning and engagement (Allen et al., 2016). Sense of belonging is defined as "the feeling that a member fits in, belongs to, or is a member of the academic community in question" (Good et al., 2012, p. 700). Among college students, sense of belonging affects mathematics motivation differently for different students. For example, Good and colleagues (2012) found that when women experienced a decline in sense of belonging across the academic year, they were less likely to intend to enroll in additional courses in mathematics.

There is also substantial overlap between sense of belonging and research regarding inclusive and equitable teaching practices and the development of supportive learning communities. Leyva et al. (2022) analyzed Black and Latine students' perceptions of practices in calculus instruction intended to be supportive for all students, finding that these were generally perceived as necessary but insufficient to create equitable opportunities and content access. Following reflection and re-evaluation of data from the MAA's National Studies of College Calculus, Hagman (2019) articulated that diversity, inclusion, and equity practices constituted a critical factor for student success; this factor was not considered explicitly in the original findings of the MAA studies. Further, many mathematicians and mathematics educators have reported on their efforts to build inclusive and welcoming communities within mathematical contexts (Cunningham et al., 2021; Hardin & Shahriari, 2022; Karaali, 2022).

Initial Instrument Development

Our instrument development process unfolded in two distinct stages. First, we developed a set of preliminary survey items based on an extensive literature review, piloted the survey with a small sample of students, and conducted a preliminary analysis to assess the factor structure of these survey items. Second, based on results of our preliminary analysis, we revised the survey items and collected data to assess the validity and reliability of the measures.

Item Development and Pilot Data Analysis

The 5-strand model combined with proceptual reasoning combine to provide six strands of proficiency: Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, Productive Disposition, and Proceptual Thinking. For each of these strands, the two lead authors wrote specific items guided by both the construct definitions and the specific content knowledge that students in undergraduate mathematics courses should master, with a focus on courses taken by first-year students. All items were designed as Likert-type items on a 1–6 scale (strongly disagree to strongly agree, with no neutral option). All items in the initial version of the survey are available on our Open Science Framework (OSF) page https://osf.io/683ek/?view_only=cadf80e3935d41caa88699d9f1f3e4a8.

We also sought to characterize students' feelings of belonging, membership, and acceptance in the mathematics community (Good et al., 2012). Items in this section were modeled after Good et al.'s (2012) scale of sense of belonging in mathematics, questions asked by Piatek-Jimenez (2008) in their interviews of women in mathematical careers, and themes described in Oppland-Cordell and Martin (2015). The remaining items were written by the two lead authors to capture students' feelings of belonging and community when working with peers in the classroom and students' own identification with mathematics.

In Summer 2022, we conducted an Exploratory Factor Analysis (EFA) on the initial survey data. The EFA showed that we should remove most reverse-coded items and clarify the item questions regarding beliefs about mathematics. The EFA for the first version of our survey did show that there were two factors clearly loading on proficiency/beliefs and belonging; thus, we determined it was appropriate to revise the survey and collect new data. The final survey items are available on our OSF page.

Reliability and Validity Study on Final Survey Items

To establish reliability and validity, we adopted the approach set forth in the Standards for Educational and Psychological Measurement in Education (AERA, APA, & NCME, 2014) in which the importance is placed on gathering multiple sources of validity evidence relevant to the interpretation of an instrument. The Standards recommend considering five types of validity evidence: test content, response processes, internal structure, relation to other variables, and consequences of testing. Validity evidence from item content was addressed by drawing on literature in the content domains of interest and by the composition of a multidisciplinary team of content experts. Our multidisciplinary team of mathematicians, cognitive developmental psychologists, mathematics educators, and an engineer required several discussions about test content as we aimed to have a shared understanding of how the items informed each scale and the overall purpose of the survey from our multiple perspectives.

Validity Evidence: Response Process and Test Content

Six undergraduate college students were recruited for 30- to 45-minute cognitive interviews in Fall 2022. A cognitive interview is an interview procedure meant to explore a participant's comprehension of an item or task (Leighton, 2017). When participants articulate their thoughts in response to an item survey (i.e., a "talk-aloud" concurrent verbal report; see Ericsson & Simon, 1993), researchers can determine if their interpretation of an item matches the construct it was designed to measure. The interviews indicated that college students interpreted most of the items as they were intentionally designed.

Reliability and Validity Evidence: Internal Structure

To provide validity evidence for internal structure, we conducted independent analyses on the data we collected at the beginning and at the end of the Fall 2022 semester. In one dataset, we conducted an exploratory factor analysis (EFA). The EFA was used to guide a confirmatory factor analysis (CFA) on a second dataset.

Revised survey administration. We administered the CMBB survey to students in five university mathematics courses (Contemporary Mathematics, College Algebra, Precalculus, Calculus I, Calculus III) that primarily serve first- and second-year undergraduates. The surveys were administered using an online survey software (Qualtrics); they were advertised and distributed by course instructors on our behalf.

Participants. After removing participants who did not pass an attention check item or had less than 30% survey response completion, our analytic sample included 1,135 students who completed the pre-survey and 935 students who completed the post-survey. Approximately half of the students in the pre-survey (49%) and the post-survey (53%) reported currently being in a mathematics-intensive major. More women than men completed the survey: 56–57% women, 34% men, 0.2–0.4% transgender, 1% nonbinary, and 5–6% other. In both the pre-survey and post-survey, the racial and ethnic distribution of the data was highly similar with 80–82% reporting as White, 8–10% Black/African American, 1% American Indian or Alaskan Native, 5% Asian, 0.5% Native Hawaiian and Other Pacific Islander, and 5% reporting as other. In both the pre-survey and post-survey, the first-generation student composition is representative of the university at which the data were collected, with 26–27% of the students reporting that their parents do not hold at least a 4-year college degree.

Exploratory factor analysis. First, we carried out a parallel analysis leading to an EFA on the pre-survey data. Computations were done in RStudio version 2022.07.1+554 using the base R package. The parallel analysis suggested an 18-factor model, where this large number of factors aligned with the design of the instrument, therefore orthogonal (varimax) rotation was used in the EFA to minimize correlations between factors. In the resulting EFA with 18 factors, two of the factors had no items load above .3, and one of the factors had only one item load at .31. Combining the EFA results with validity evidence considerations, we proceeded with a 15-factor model on 56 items. The loading matrix and item correlations for the EFA analysis are available on our OSF page.

Confirmatory factor analysis. Next, to confirm that our hypothesized 15-factor model fit the data, we conducted a CFA on the post-data using maximum likelihood estimation with the *lavaan* package version 0.6-12. The CFA model is represented in Figure 1. To interpret our goodness-of-fit indicators, CFI and RMSEA, we used equivalence testing to establish comparison and cutoff values (Marcoulides & Yuan, 2016). Using equivalence-testing based cutoffs, we find that an RMSEA of .047 is identified as a "Close" fit while a CFI of .92 is identified as a "Fair" fit (our T-size RMSEA and CFI values were .048 and .91, respectively). Thus, we conclude from our equivalence testing that our 15-factor model is a good fit for the data. Note that the subscales within each factor are also aligned with the constructs in our literature review.

Reliability and relationships among factor scores. To assess reliability, we examined Cronbach's alpha for items within each subscale (factor). Alpha scores ranged from .74–.95 with one scale having an alpha of .66. Thus, most subscales had high internal consistency. To provide additional validity evidence, we calculated correlations between the item subscales associated with each factor (see Figure 2).

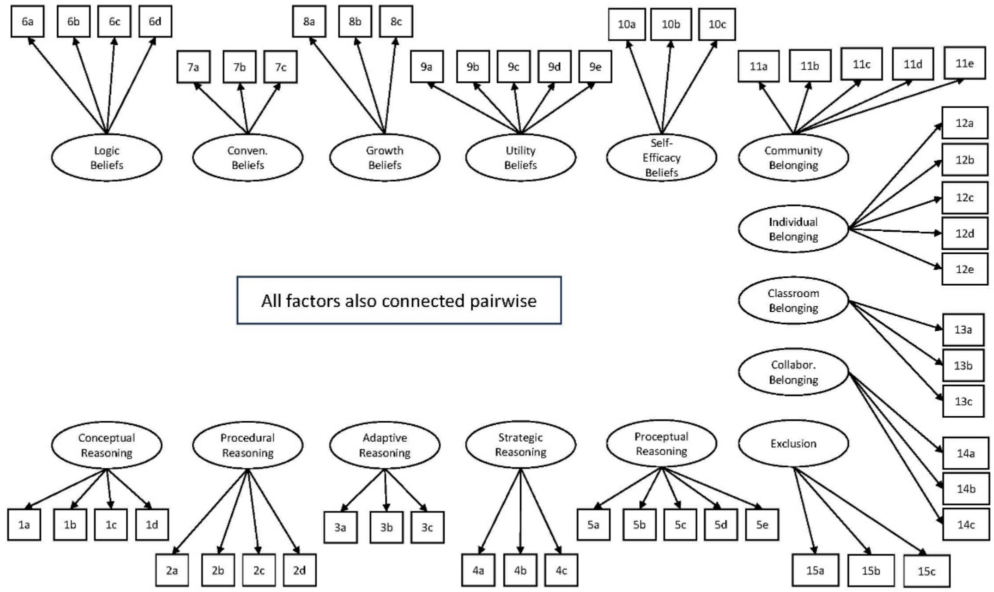

Figure 1. This graphic presents our CFA model, including the 15 factors and the items that load onto them.

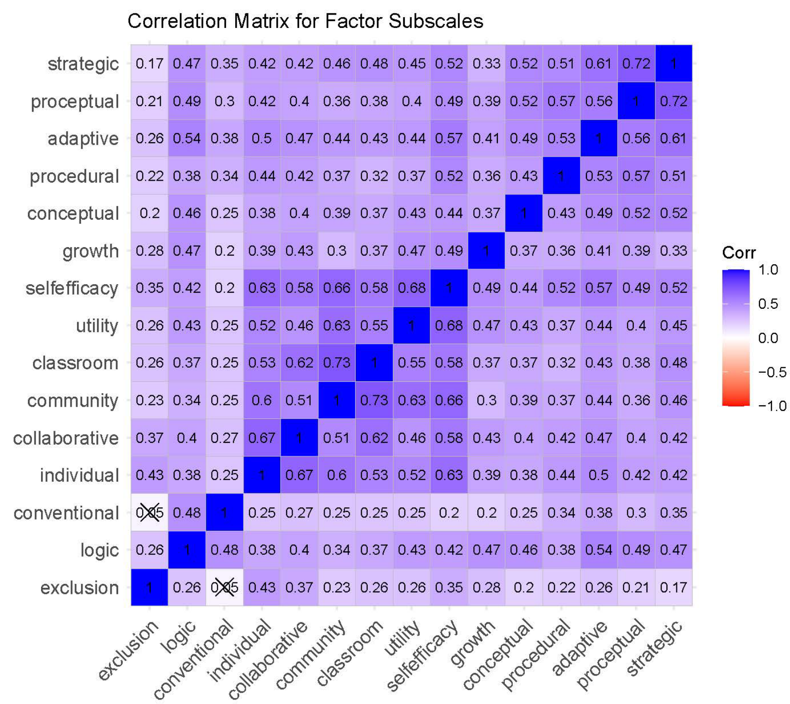

Figure 2. Correlation matrix for factor subscales. All correlations are significant at the alpha = .05 level except those indicated with an X.

Importantly, our analysis reveals strong correlations among items tapping components of belonging (Community, Individual, Classroom, Collaborative), suggesting that these subscales do, indeed, measure interrelated aspects of students' sense of belonging in mathematics. In contrast, belonging subscales are less strongly correlated with students' beliefs about the nature of mathematics and their self-perceived proficiency. Likewise, subscales that measure students' self-perceived proficiency (Conceptual, Procedural, Strategic, Adaptive, Proceptual) are strongly related to each other as well as the Self-Efficacy subscale and the Logic beliefs subscale. In sum, the general pattern of correlations across subscales reflects our expected theoretical structure.

Discussion and Conclusions

Researchers and educators interested in understanding how instructional factors in postsecondary mathematics education shape students' beliefs about mathematics and their own approaches to mathematics need a valid and reliable tool for measuring these beliefs. Our team successfully developed a contemporary instrument, the College Mathematics Beliefs and Belonging (CMBB) survey, to assess a variety of related theoretically meaningful constructs that capture various students' experiences in college mathematics courses. The survey assesses students' self-perceived proficiency in a way that is aligned with theory and prior research delineating the multiple facets of student reasoning that are crucial for engaging in college mathematics. The survey also measures a variety of beliefs, including students' beliefs about the nature of mathematics, the utility of the mathematics they are learning, their own abilities to learn mathematics, and the possibilities for growth in mathematical knowledge. In sum, the College Mathematics Beliefs and Belonging (CMBB) survey captures meaningful variability in students' beliefs and is a valid tool for gathering information about college students' beliefs and sense of belonging in mathematics. It is our hope that postsecondary mathematics instructors and education researchers can use this tool to examine and improve students' experiences in mathematics courses.

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References

Allen, K. A., Vella-Brodrick, D., & Waters, L. (2016). Fostering school belonging in secondary schools using a socio-ecological framework. *The Educational and Developmental Psychologist, 33*(1), 97–121. https://doi.org/10.1017/edp.2016.5

American Educational Research Association, American Psychological Association, & National Council for Measurement in Education. (2014). The standards for educational and psychological testing. American Educational Research Association.

Asher, M. W., Harackiewicz, J. M., Beymer, P. N., Hecht, C. A., Lamont, L. B., Else-Quest, N. M., Priniski, S. J., Thoman, D. B., Hyde, J. S., & Smith, J. L. (2023). Utility-value intervention promotes persistence and diversity in STEM. *Proceedings of the National Academy of Sciences of the United States of America, 120*(19).

Bandura, A. (1997). Self-efficacy: The exercise of control. W H Freeman/Times Books/ Henry Holt & Co.

Boaler, J., Williams, C., & Confer, A. (2015). Fluency without fear: Research evidence on the best ways to learn math facts. *Reflections, 40*(2), 7–12.
Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and Recommendations from the MAA National Study of College Calculus.* MAA Press.
Carlson, M. P., & Rasmussen, C. (2008). *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (Vol. 73). Mathematical Association of America.
Cunningham, E. P., Seashore, K. H., & Eaton, K. E. (2021). Graphs about us: Building classroom community through Mathematics. *PRIMUS*, 1–15. https://doi.org/10.1080/10511970.2021.1940401
Dossey, J. A. (1992). The nature of mathematics: Its role and its influence. In D. A. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 39–48). New York: MacMillan.
Eccles, J. S., & Wigfield, A. (2020). From expectancy-value theory to situated expectancy-value theory: A developmental, social cognitive, and sociocultural perspective on motivation. *Contemporary Educational Psychology, 61*, 101859. https://doi.org/10.1016/j.cedpsych.2020.101859
Ellis, J. M., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PLOS ONE, 11*(7), e0157447. https://doi.org/10.1371/journal.pone.0157447
Ericsson, K. A., & Simon, H. A. (1993). *Protocol Analysis: Verbal Reports as Data.* National Geographic Books.
Faulkner, B., Earl, K. A., & Herman, G. L. (2019). Mathematical maturity for engineering students. *International Journal of Research in Undergraduate Mathematics Education,* 5(1), 97–128. https://doi.org/10.1007/s40753-019-00083-8
Ganter, S. L., & Barker, W. (Eds.). (2004). The Curriculum Foundations Project: Voices of the Partner Disciplines. In *The Mathematical Association of America.* Mathematical Association of America.
Garrity, T. (2011). Using Mathematical Maturity to Shape our Teaching, our Careers and our Departments. *Notices of the American Mathematical Society, 58*(11), 1592–1593.
Gold, B., Behrens, C. E., & Simons, R. A. (2017). *Using the philosophy of mathematics in teaching undergraduate mathematics.* Mathematical Association of America.
Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology, 102*(4), 700–717.
Gray, E., & Tall, D. (1994). Duality, Ambiguity, and Flexibility: A "Proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116.
Hagman, J. E. (2019). The Eighth Characteristic for Successful Calculus Programs: Diversity, Equity, & Inclusion Practices. *PRIMUS, 31*(1), 70–90.
Hardin, J., & Shahriari, S. (2022). Community, collaboration, and climate. PRIMUS, 33(5), 445–462. https://doi.org/10.1080/10511970.2022.2082611
Hersh, R., & John-Steiner, V. (2010). *Loving and Hating Mathematics: Challenging the Myths of Mathematical Life.* Princeton University Press.
Hulleman, C. S., & Harackiewicz, J. M. (2009). Promoting interest and performance in high school science classes. Science, 326(5958), 1410–1412. https://doi.org/10.1126/science.1177067

Karaali, G. (2022). Whose Math and for What Purpose? A Community Seminar on Identity, Culture, and Mathematics. PRIMUS, 33(6), 569–601.
Kilpatrick, J., Swafford, J. O., & Findell, B. R. (Eds.). (2001). *Adding It Up: Helping Children Learn Mathematics.* National Academy Press.
Kloosterman, P. (2002). *Beliefs about Mathematics and Mathematics learning in the secondary School: Measurement and implications for motivation*. In Springer eBooks (pp. 247–269).
Laursen, S. L., & Rasmussen, C. (2019). I on the Prize: Inquiry Approaches in Undergraduate Mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*, 129–146.
Leighton, J. P. (2017). *Using Think-Aloud interviews and cognitive labs in educational research*. Oxford University Press.
Lew, K. M. (2019). How do mathematicians describe mathematical maturity? In *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*. Mathematical Association of America.
Leyva, L. A., Amman, K., McMichael, E. A. W., Igbinosun, J., & Khan, N. (2022). Support for All? Confronting Racism and Patriarchy to Promote Equitable Learning Opportunities through Undergraduate Calculus Instruction. *International Journal of Research in Undergraduate Mathematics Education*, *8*, 339–364.
Marcoulides, K. M., & Yuan, K. (2016). New ways to evaluate goodness of fit: A note on using equivalence testing to assess structural equation models. *Structural Equation Modeling, 24*(1), 148–153. https://doi.org/10.1080/10705511.2016.1225260
Mathematical Association of America. (2015). *Curriculum Guide to Majors in the Mathematical Sciences.* Mathematical Association of America.
Oppland-Cordell, S., Martin, D. B. (2015). Identity, power, and shifting participation in a mathematics workshop: Latin@ students' negotiation of self and success. *Mathematics Education Research Journal, 27*, 21–49.
Philipp, A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), Second handbook of research on mathematics teaching and learning. Reston: National Council of Teachers of Mathematics.
Piatek-Jimenez, K. (2008). Images of mathematicians: a new perspective on the shortage of women in mathematical careers. ZDM Mathematics Education, 40, 633–646. https://doi.org/10.1007/s11858-008-0126-8
Rittle-Johnson, B., & Siegler, R. S. (2021). The relation between conceptual and procedural knowledge in learning mathematics: A review. In *Psychology Press eBooks* (pp. 75–110).
Royster, D. C., Harris, M. K., & Schoeps, N. (1999). Dispositions of college mathematics students. *International Journal of Mathematical Education in Science and Technology, 30*(3), 317–333.
Schifter, D. (1990). Mathematics process as mathematics content: teachers. In *Fourteenth PME Conference with the North American Chapter Twelfth PME-NA Conference* (Vol. 1). Program Committee of the 14th PME Conference.
Steen, L. A. (1983). Developing Mathematical Maturity. In: Ralston, A., Young, G.S. (eds) *The Future of College Mathematics*. Springer, New York, NY.
Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics.* Cambridge: Cambridge University Press.
Yeager, D. S., Hanselman, P., Walton, G. M., Murray, J. S., Crosnoe, R., Muller, C., Tipton, E., Schneider, B., Hulleman, C. S., Hinojosa, C., Paunesku, D., Romero, C., Flint, K., Roberts,

A., Trott, J., Iachan, R., Buontempo, J., Hooper, S. Y., Carvalho, C. M., . . . Dweck, C. S. (2019). A national experiment reveals where a growth mindset improves achievement. *Nature, 573*(7774), 364–369.

How do calculus instructors frame tasks for introducing derivatives symbolically? Identifying Calculus-specific instructional situations

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In this study, I describe how seven U.S. college calculus instructors framed instructional tasks for introducing derivatives symbolically to students. During one-on-one interviews, the instructors were presented with before and after student conceptions and were asked to propose tasks for introducing derivatives symbolically using the limit definition of derivative, both at a point and as a function. The instructors' task framings reveal the types of mathematical problems students are expected to work on with respect to the content at stake: symbolic definitions of derivatives. Although the instructors heavily relied on calculating situations to introduce derivatives symbolically, some also used graphing, exploring, installing, and proving situations, which shows the high variability of task framing in calculus even when student conceptions before and after working on the tasks are predetermined.

Keywords: Calculus teaching, instructional tasks, derivatives, instructional situations

Despite the continuing research on students' understandings of calculus concepts, there is less research on the teaching of calculus (Larsen et al., 2017). The scarce research in this area focuses on the student outcomes of instructional treatments (e.g., Borji et al., 2018; Hähkiöniemi, 2006) or tasks that instructors use in their assignments, textbooks, and exams (e.g., Tallman et al., 2016; White & Mesa, 2014). I focus on teaching of calculus by investigating instructional tasks, or activities that are used during instruction "to focus students' attention on a particular mathematical idea" (Stein et al., 1996, p. 460). As basic units of instruction and "objects of students' activity" in mathematics classrooms (Ni et al., 2018; Sullivan et al., 2009, p. 859), instructional tasks are often seen as the conceptual bridge between teaching and learning (Christiansen & Walther, 1986; Stein & Lane, 1996).

I contribute to this line of research by focusing on the ways college Calculus I instructors shape, or *frame*, students' mathematical work during instruction of one specific piece of content: derivatives. By focusing on these subject-specific *framings* in tasks, we learn about the learning opportunities that are offered to students during instruction (Herbst et al., 2018). Here, I address the following research question: How do college Calculus I instructors frame instructional tasks to introduce derivatives symbolically?

Theoretical Frameworks

Framing

The concept of framing originates from the work of social scientists and sociologists Gregory Bateson (1904-1980) and Erving Goffman (1922-1982). Bateson (1974/2003) referred to framings as definitions of reality, determined by the culture, that allow people to interpret and respond to objects and events during social interactions. For Goffman, framing is simply one's answer to the question "What is it that's going on here?"; the answer, implicitly or explicitly, informs the person about acceptable and unacceptable (re)actions and behaviors during a social event (1974/1986, p. 8). To unpack college calculus instructors' framing of instructional tasks for teaching derivatives, I divide Goffman's question of 'what is it that is going on here' into two questions: 1) What is it that is going on here with respect to the content presented in the task? and 2) What is it that is going on with respect to interaction with other actors? I refer to the answers to these questions as *Framing for Interaction with Content* and *Framing for Social Interaction*. Although students make sense of what they should and should not do during a lesson by answering both questions, I only focus on the former here due to space reasons.

To answer the question about framing of a task for interaction with the content, I rely on Herbst and colleagues' (2020) notion of instructional situations (or situations in short) within the notion of didactical contract that govern the teacher-student relationship in the teaching and learning of mathematics (Brousseau, 1984). Instructional situations, as subject-specific types of mathematical problems in a course of studies, communicate teacher's and students' appropriate and customary units of work regarding the knowledge at stake (Herbst et al., 2020, p. 5). Given an instructional situation (e.g., finding the equation of a line in an algebra task), students know what kind of problem they are presented with and what kind of mathematical work and interactions they should prototype (Herbst & Chazan, 2012). The eight generic types of problems they have identified so far in geometry and algebra include: *graphing*; *calculation*; *exploration/conjecturing*; *doing proofs*; *generating a new definition or installing a new concept*; *installing a new theorem, property, or formula*; *solving equations with known methods*; and *solving word problems* (see Herbst et al., 2010).

Conceptions of Derivatives

To ground instructors in the students' conceptions of derivatives, I used Zandieh's (2000) framework for the concepts of derivative (Table 1). Zandieh (2000) organized students' conceptions by representation (graphical, verbal, physical, symbolic) and process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of* finding the limit of the ratio as an *object*. Here, I focus on instructors' tasks that were proposed to introduce derivatives symbolically at the limit layer and at the function layer.

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical	Verbal	Physical	Symbolic
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

Methods

Using an in-take survey of 48 calculus instructors, I purposefully selected eight interviewees with different patterns of inquiry (see Gerami, 2023 for details): Adrian, Barry, Gopher, Justin, Matthew (He/Him pronouns), Monica (She/Her pronouns), Max (They/Them pronouns), and Alex (all pronouns). The participants were tenured and had more than four semesters of experience teaching Calculus I with inquiry. My focus on teaching experience and using inquiry was intentional; as my aim was to have instructors suggest diverse instructional tasks based on targeted students' conceptions of derivatives during the interviews, I assumed that experienced instructors who employ inquiry-based teaching methods for calculus are more likely to know of and consider student thinking when proposing tasks, and that they would propose a broader range of instructional tasks compared to those who teach calculus more traditionally (e.g., following conventional textbooks, lecturing). I conducted four semi-structured interviews (1-2 hour long each) with each instructor, as they proposed up to eight instructional tasks for introducing derivatives. I structured the interviews based on Zandieh's (2000) framework. During each interview, the instructors were asked to propose two tasks that targeted students' conception at the limit and the function layer within one representation. They could use their teaching materials to propose already used tasks or design new tasks for the future. Here, I report the findings based on the tasks proposed in response to Prompt 7 and 8:

> **Prompt 7:** Assume that you have already taught about the difference quotient ($\frac{f(x)-f(x_0)}{x-x_0}$ or $\frac{f(x+h)-f(x)}{h}$) and want students to learn about the derivative of a function at a point ($f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ or $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$). Propose a task where students have to figure out the derivative of a function at a point.
>
> **Prompt 8:** Assume that you have already taught about the derivative of a function at a point ($f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ or $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$) and want students to learn about the derivative function ($f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$). Propose a task where students have to represent the derivative function.

Because calculus-specific instructional situations have not been identified in the literature, I used inductive/deductive hybrid thematic analysis (Proudfoot, 2022) to identify them. The analysis entailed using pre-ordinate themes "through the application of an explicit theoretical framework developed through engagement with the literature" (the deductive element) to generate themes from the data (the inductive element; Proudfoot, 2022, p. 1). For the deductive portion, I used the generic types of problems: *graphing*; *calculation*; *exploration/conjecturing*; *doing proofs*; *generating a new definition or installing a new concept*; *installing a new theorem, property, or formula*; *solving equations with known methods*; and *solving word problems* (Herbst et al., 2010). For the inductive portion, I identified the emerging instructional situations by focusing on the mathematical work that instructors expected of their students would do within a generic type of problem. I also listened to the interviews and read the transcripts to find relevant information that instructors mentioned but did not include in the task description. After three rounds of coding, I identified 36 calculus-specific instructional situations across *all* eight tasks.

Findings

Figure 1 lists the 12 calculus-specific instructional situations that instructors used in their two tasks proposed for teaching derivatives symbolically at a point and as a function (Prompt 7 an 8).[1] All seven instructors who proposed tasks used one or two specific calculating situations to frame their tasks: *Calculating derivative at a point using a limit definition* (C6) and *Calculating derivative function $f'(x)$ using a limit definition* (C11). Four instructors utilized other types of situations to frame their tasks. Alex, Max, and Adrian used graphing situations at different layers

[1] Each instructional situation is identified by a letter representing the generic situation type (C for Calculating, E for Exploring/Conjecturing, G for Graphing, I for Installing, and P for Proving), and a number to differentiate it from other situations in the same category. Although Figure 1 only includes the situations that appeared in the tasks for introducing derivatives symbolically, I kept the original numbering system to keep my findings across research reports consistent.

(secant line [ratio, G1], tangent line [limit, G2], derivative function by plotting [function, G3]). Alex, Max, and Monica used the tasks to install ideas and formulas (e.g., power rule, I7). Finally, Max was the only instructor who used exploring and proving situations (E11 and P1).

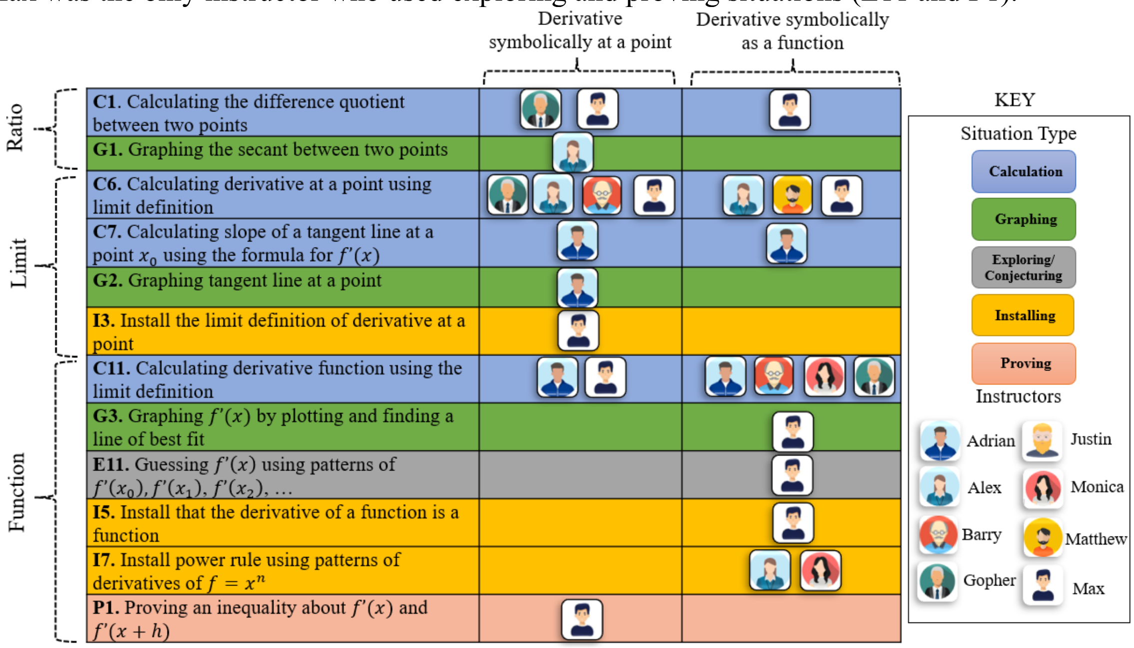
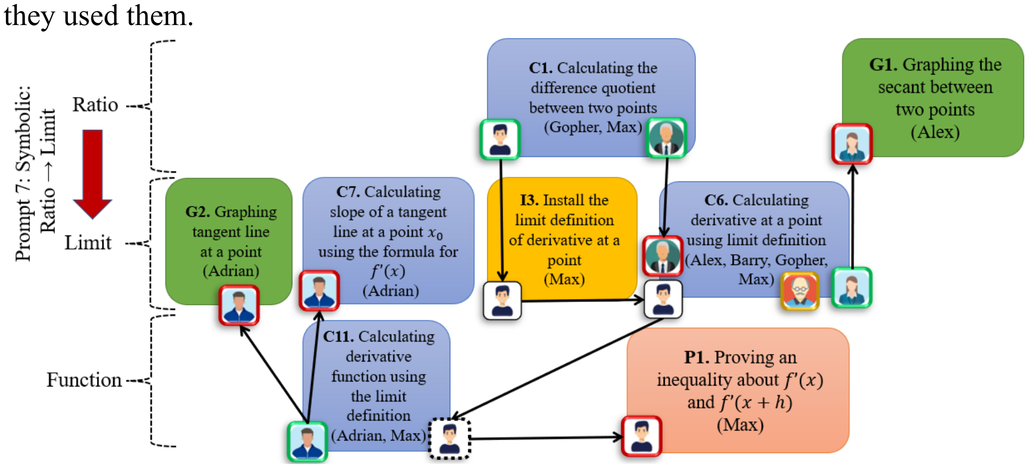


Figure 1. Identified instructional situations for introducing derivatives symbolically at a point and as a function and instructors who used them. Situations are organized by layer and generic type.

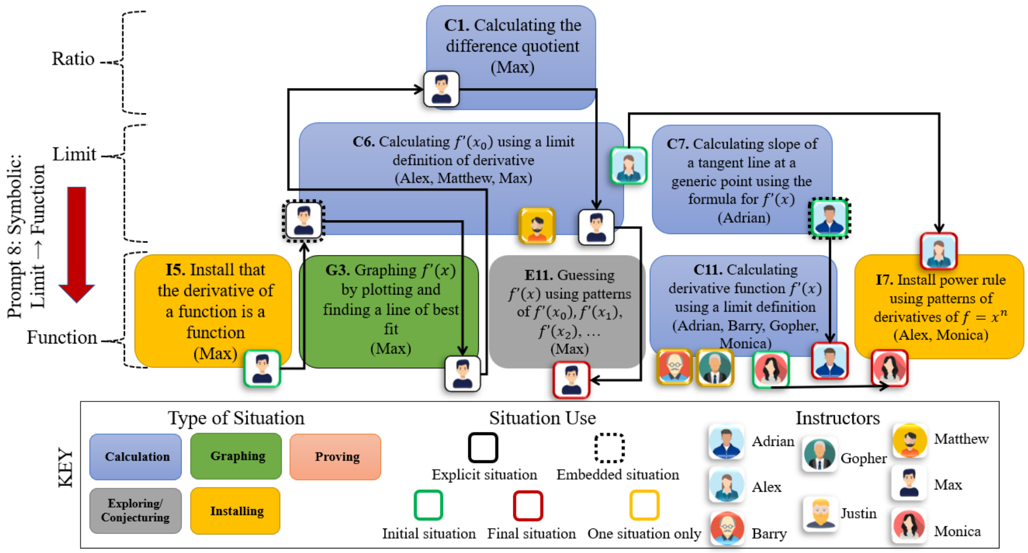
Justin did not propose any tasks for Prompt 7 and 8 (he stated that he does not explicitly talk about the difference quotient in the notation proposed by the prompts or the limit definition of derivative). Monica and Matthew did not propose tasks for Prompt 7, as they did not teach about derivative symbolically as a point, but they did propose tasks for Prompt 8. Next, I provide an overview of how the instructional situations were used in the tasks proposed for each prompt.

Tasks Proposed for Prompt 7: Ratio → Limit

Figure 2a shows the instructional situations that five instructors used to frame their tasks in the order they used them.



(a)



(b)

Figure 2. Calculus-specific instructional situations in the order they appeared in the tasks proposed for (a) Prompt 7 and (b) Prompt 8. The arrows show the order in which the situations were used by each instructor. A situation is explicit if students know what type of problem they are working on after being presented with a task, whereas a situation is embedded within an explicit (larger) situation if, while working on the explicit problem, students find out that they must solve another problem in order to solve the explicit situation.

Max and Gopher started their tasks at the ratio layer, before the limit/target layer, by asking students to: *calculate difference quotients between two points*, x_1 and x_2, or x_0 and $x_0 + h$ (C1). Gopher ended his with the situation of *calculating the derivative at a given point* x_0 at the limit layer (C6), which was the situation that Barry started and ended his task with. Barry, Gopher, and Max expected their students to use formal limit notation to calculate $f'(x_0)$ at a given point x_0 ($\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$). Max continued their task with four more situations after calculating difference quotients between two points (x_0 and $x_0 + h$, with decreasing h, the ratio layer; C1). First, Max *installed the limit definition of derivative at a point* ($\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$; I3). While other instructors installed the limit definition of derivative at a point *themselves* before the tasks (most probably via lecture), Max expected students to come up with the definition as part of the task by asking them: "What we can do to determine a value at $x = 0$? What concept from calculus helps us analyze questions about arbitrarily small sizes?". Next, they asked their students to: 1) use their newly defined derivatives to *calculate* $f'(x_0)$ *at some given points* (C6), and 2) to show an inequality $f'(x) < f'(x+h)$ for $h > 0$ (*P1. Proving an inequality is true for a specific function*). The proving situation has embedded calculating situations—*calculating* $f'(x)$ *and* $f'(x+h)$ *using the limit definition of derivative* (C11).

Alex started their task at the target/limit layer with the same situation that Barry, Gopher, and Max also used in their tasks: *calculating* $f'(x_0)$ *at some given points* (C6). However, unlike the other instructors, they did not use formal limit notation; instead, they expected their students to assume the denominator goes to zero when calculating the difference quotient $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$. Alex explained that, at this point in their course, the limits are replaced with arrows (→) instead of equal signs ($\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to c$ as $\Delta x \to 0$, instead of $\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = c$). To assist students with confirming the value of the difference quotient with Δx going to zero, Alex then asked students to "Describe what happens to the secant lines as Δx gets closer to zero?", which is a situation at the ratio layer: *graphing secant lines* Δx*-units distanced to the right side of a fixed point* x_0*, assuming* Δx *gets closer to zero* (G1).

Adrian approached the task differently than the rest of the instructors by starting his task at the function layer (the layer after the target/limit layer): *calculating* $f'(x)$ *using the limit definition of derivative* $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (C11). He then asked his students to g*raph a tangent line at a given point* $(x_0, f(x_0))$ (G2) and *find its slope* (i.e., the derivative at a point or $f'(x_0)$; C7), which are two situations at the limit/target layer. I capture the latter situation C7 separately than those used by Barry, Gopher, Max and Alex (*C6. Calculating* $f'(x_0)$ *at a given point* x_0 *using the limit definition of derivative at a point* or *Calculating the difference quotient assuming the denominator goes to zero*) because Adrian expected his students to substitute x_0 in the formula of $f'(x)$ found in his task's initial situation, instead of using the limit definition to calculate derivative at the point.

Tasks Proposed for Prompt 8: Limit → Function

Figure 2b shows the instructional situations that the seven instructors used to frame their tasks in the order they used them. At the function/target layer, *calculating* $f'(x)$ *using the limit definition of derivative* (C11) was the most common situation, used by four instructors—Barry, Gopher, Adrian, and Monica—with Barry and Gopher solely using this situation to frame their tasks. In Adrian's task, the situation was embedded within another—*calculating slope of a tangent line at a point* x_0 (C7), as students were expected to calculate and use the formula for $f'(x)$ to find the slope of the tangent line at x_0. Monica asked her students find $f'(x)$ by calculating the difference quotient, $\frac{f(x+h)-f(x)}{h}$, and assuming that *h* goes to zero. After finding $f'(x)$ for $f = x$, $f = x^2$, and $f = x^3$, she used an installing situation, expecting students to *install the power rule using patterns of derivatives of* $f = x^n$ *for* $n \geq 1$ (I7): "What is the general formula for the derivative of x^n?"

Matthew and Alex started their tasks with a situation within the limit layer, where students were expected to *calculate* $f'(a)$ *for a generic variable (a) either using the limit definition of derivative at a point using the formal or informal definition of derivative with limit of difference quotient* ($\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ for Matthew, and $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to c$ *as* $\Delta x \to 0$ for Alex; C6). Although the instructors were prompted to introduce students to the derivative symbolically as a function, Matthew only used the situation of calculating $f'(a)$ for a generic value of a at the limit layer, not $f'(x)$. Because Matthew used a linear function in the task—$f(x) = 5x - 3$, with $f'(x) = 5$ for all values of x, it is possible that he equated finding $f'(a)$ to $f'(x)$, which, mathematically, is true. However, from the task's description, it is not obvious whether students would conclude that the "derivative 5" is in fact a function. Alex continued their task like Monica's by installing the power rule using patterns of derivatives of $f = x^n$ *for* $n \geq 1$ (I7).

To frame their task, Max used five explicit situations across all three layers. Max started the task by asking students to review their work for Task 7, in which students found the derivative at various points. They then encouraged students to *install that derivative of a function is a function* (I5) by asking: "Because we have a derivative value at every function input value, what do we have?". Next, Max asked students to *graph* f' *by plotting derivative values at different points and finding a curve of best fit* (G3). Although students used the derivative values they found at various points in Task 7, I captured *calculating $f'(x)$ using the limit definition of derivative* $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (C6) as an embedded situation before the graphing situation to count for the work students should have done to complete this part. Next, Max used a situation at the ratio layer by asking students to *calculate the difference quotient* $\frac{f(x_0+h)-f(x_0)}{h}$ (C1) for various values of x_0, which was immediately followed by asking students to *calculate $f'(x_0)$ using the limit definition of derivative* $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ (C6) for their chosen x_0 values. Max ended his task at the function/target layer with an exploring/conjecturing situation: *guessing $f'(x)$ using patterns of $f'(x_0), f'(x_1), f'(x_2), \ldots$* (E11).

Discussion

In this report, I identify the ways seven college Calculus I instructors framed their instructional tasks to introduce the symbolic representations of the derivative at a point and as a function, that is the limit definitions of derivative. While all instructors used calculating derivative at a point and/or calculating $f'(x)$ using a limit definition, four instructors also relied on other types of situations: graphing, exploring/conjecturing, installing, and proving. Although I did not review the findings from the rest of their tasks in this report, the proving situation (P1), the installation of power rule (I7) and installation of the limit definition of derivative at a point (I3) were only used to frame tasks for Prompt 7 and 8. Moreover, the proving situation was the only proving situation used across all tasks, which suggests that the instructors avoided using proving situations when introducing derivatives. More specifically, regarding the installation of power rule (I7), it is noteworthy that two instructors (Alex and Monica) used the symbolic representations to install one specific derivative rule—the power rule. Lastly, although some instructors used similar calculus-specific instructional situations within the three generic problem types, only two instructors proposed the same task consisting of all the same instructional situations (Barry and Gopher for their second tasks). This illustrates instructors' varied ways of shaping students' work when introducing derivatives, which shows that inquiry methods are implemented considerably differently among instructors. Nonetheless, most instructors relied on explicit, rather than implicit situations, which shows their inclinations towards scaffolded inquiry, rather than open-problem or discovery-based inquiry.

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References

Bateson, G. (1974/2003). A theory of play and fantasy. In Caro, I., & Read, C. S. (Eds.), *General semantics in psychotherapy Charge: Selected writings on methods aiding therapy* (pp. 267–283). Institute of General Semantics.

Borji, V., Font, V., Alamolhodaei, H., & Sánchez, A. (2018). Application of the complementarities of two theories, APOS and OSA, for the Analysis of the university students' understanding on the graph of the function and its derivative. *EURASIA Journal of Mathematics, Science and Technology Education*, *14*(6), 2301-2315. https://doi.org/10.29333/ejmste/89514

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H.-G. Steiner (Ed.), *Theory of mathematics education* (pp. 110–119). Occasional paper 54. Bielefeld, Germany: IDM.

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A.G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 243-307). D. Reidel.

Gerami, S. (2023, February). Calculus I Instructors' Use of Representations in Instructional Tasks: Introducing Derivatives with Inquiry. Research in Undergraduate Mathematics Education (SIGMAA-RUME). Omaha, NE.

Goffman, E. (1986). *Frame analysis: An essay on the organization of experience*. Northeastern University Press. (Original work published 1974)

Hähkiöniemi, M. (2006). The role of representations in learning the derivative. In M. Johnsen-Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 73–80). PME.

Herbst, P., Boileau, N., & Gürsel, U. (2018). Examining the work of teaching geometry as a subject-specific phenomenon. In *International perspectives on the teaching and learning of geometry in secondary schools* (pp. 87-110). Springer, Cham.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM*, *44*(5), 601-612. https://doi.org/10.1007/s11858-012-0438-6

Herbst, P., Chazan, D., & Boileau, N. (2020). Framing, normativity, and serviceability in teachers' decision-making during lessons. In *Proceedings of the 14th International Congress on Mathematical Education (ICME-14).* Shanghai, China.

Herbst, P. with González, G., Hsu, H. Y., Chen, C., Weiss, M., and Hamlin, M. (2010). Instructional situations and students' opportunities to reason in the high school geometry class. Manuscript. *Deep Blue at the University of Michigan*. http://hdl.handle.net/2027.42/78372

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.) *Compendium for research in mathematics education* (pp. 526-550). National Council of Teachers of Mathematics.

Ni, Y., Zhou, D. H. R., Cai, J., Li, X., Li, Q., & Sun, I. X. (2018). Improving cognitive and affective learning outcomes of students through mathematics instructional tasks of high cognitive demand. *The Journal of Educational Research*, *111*(6), 704-719.

Proudfoot, K. (2022). Inductive/Deductive hybrid thematic analysis in mixed methods research. *Journal of Mixed Methods Research.* https://doi.org/10.1177/15586898221126816

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.
Sullivan, P., Clarke, D., & Clarke, B. (2009). Converting mathematics tasks to learning opportunities: An important aspect of knowledge for mathematics teaching. *Mathematics Education Research Journal*, *21*(1), 85-105.
Tallman, M. A., Carlson, M. P., Bressoud, D. M., & Pearson, M. (2016). A characterization of calculus I final exams in US colleges and universities. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 105-133. https://doi.org/10.1007/s40753-015-0023-9
White, N., & Mesa, V. (2014). Describing cognitive orientation of Calculus I tasks across different types of coursework. *ZDM – Mathematics Education*, *46*(4), 675-690. https://doi.org/10.1007/s11858-014-0588-9
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-127.

Student Reasoning and Cooperative Learning while using a Dynamic Geometry Environment in Taxicab Geometry: An APOS Perspective

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The goal of this research is to investigate ways a dynamic geometry environment (DGE) and interactions during cooperative learning can leverage student understanding in geometry. Data from 18 students enrolled in a College Geometry course were collected by video recording in-class group work while students explored concepts in Taxicab geometry in a DGE. The textbook from this course and its activities are based on Action-Process-Object-Schema (APOS) Theory. As such, APOS Theory was used as a theoretical framework to analyze student reasoning during these activities. For this report, results are presented for one group of students and their discussion while working on an activity which encouraged the exploration of the mathematical definition of a circle in Taxicab geometry in a DGE. Trends emerged about how group structure while working in DGEs may influence interactions and outcomes for students. Some pedagogical suggestions are provided based on the results of this study.

Keywords: Geometry, Taxicab, Dynamic Geometry Software, Circle, Definitions

Introduction

By exploring concepts and definitions in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands et al., 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry, which is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegal et al. (1998) and Dreiling (2012) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it more accessible for students to reason and abstract concepts. Further, the use of a dynamic geometry environment (DGE) for the teaching and learning of geometry is encouraged (Liljedahl, P., 2020; Hollebrands 2003; Hollebrands et al., 2010; Glass & Deckert, 2001; Contreras, 2013; Kemp, 2018). Since properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition" (Fischbein, 1993, p. 141), rather than a geometric representation influencing the definition. As DGEs offer opportunities for students to interact with accurate constructions rather than drawings on paper, students can explore and focus on relationships between concepts and their definitions. This can guide students to differentiate between drawings and constructions, abstract properties and relationships, and develop a higher level of geometric reasoning and understanding. Cooperative learning, which involves students working in groups to complete a common goal, can help students to maximize their own and each other's learning (Johnson & Johnson, 1999). In cooperative learning settings, students are also more likely to reflect on the procedures they perform (Vidakovic, 1993). Researchers also emphasize the importance of cooperative learning on psychological health and social competence, as this can encourage students to value themselves and increase independence (Johnson & Johnson, 1999).

For this report, results are presented on the following research questions: (a) How do students in a College Geometry course develop their understanding of the definition of a circle in Taxicab geometry using a DGE? (b) How does the use of cooperative learning and DGEs help students to construct knowledge in geometry?

Theoretical Framework

Action-Process-Object-Schema (APOS) Theory is a constructivist framework based on Piaget's *reflective abstraction*. According to this framework, there are four different stages of cognitive development (Action, Process, Object, and Schema) and mechanisms to move between these levels (e.g., interiorization, encapsulation) (Arnon et al., 2014; Dubinsky, 2002; Asiala et al., 1996). In APOS Theory, an *Action* is an externally driven transformation of a mathematical object. For example, an action conception of **Circle** may be exhibited by a student being prompted to identify a point on a circle when given a specific center and radius. Once an individual performs an Action enough and reflects on this, they can *interiorize* this Action into a *Process*. That is, a student demonstrating a process conception of **Circle** can imagine how to draw a circle given any center and radius, understanding that the circle is made up of an infinite number of points that satisfy a property. A Process is *encapsulated* into an *Object* once the individual is aware of it as a totality on which other actions can be performed. As an example of this for **Circle**, a student may exhibit this level of cognitive development when they construct a perpendicular bisector using circles, generally understanding how properties of circles justify why this creates a set of points equidistant from the endpoints of a segment. Once a student constructs an object in APOS Theory, it may be necessary to *de-encapsulate* it. For example, Kemp and Vidakovic (2019) present results of students de-encapsulating their object conception of **Distance** to coordinate their **Euclidean distance** and **Taxicab distance** processes. The entire collection of Actions, Processes, Objects, and other Schemas that are connected to the original concept that form a coherent understanding in the mind of the individual is called a *Schema* and is uniquely formed based on experiences (Dubinsky, 2002). The concepts involved with the *circle schema* are identified to be **Distance, Radius, Center,** and **Locus of points** (Kemp, 2018; Kemp & Vidakovic, 2018, 2019, 2021b, 2023). As a part of these students' developing *circle schemas,* we analyze their conception of the definitions of these concepts as they emerged during a class activity. Detailed descriptions of the levels of cognitive development in APOS Theory associated with these and other concepts in Euclidean and Taxicab geometry are provided in Kemp (2018) and Kemp and Vidakovic (2018, 2019, 2021a, 2021b, 2023).

There is some research that utilize APOS Theory in relation to DGEs (Hollebrands, 2003; Patsiomitou, 2019; Trigueros et al., 2022; Kemp, 2018), but a search in the literature implies a need for more. Further, the vast majority of this research focuses on Euclidean geometry. As exploration in non-Euclidean geometry can offer opportunities for students to refine their understanding of concepts, it is important to investigate how exploration in non-Euclidean geometries in DGEs can also contribute to this refinement. Results presented in Hollebrands (2003), where high schoolers' use of a DGE and how these students reasoned about transformations in geometry was investigated using APOS Theory, is used as a model in this report as the author provides a framework based in APOS Theory for student understanding as it evolves while using DGEs. The author indicated student understanding of the domain of a transformation may have been influenced by their interactions with the computer as the students interiorized the actions they performed on the computer, which contributed to their ability to form explanations of the transformations. Without the opportunity to use a DGE, students in the control group of this study were only able to examine and perform transformations on the limited number of samples provided to them and did not develop as dynamic of an understanding of these transformations. Adapted from Hollebrands (2003) for the context of this study, an action conception of *circle* in Taxicab geometry in a DGE may be exhibited by a student counting units to find a point on a circle given a specific center and radius. Once a student interiorizes this action, they can consider all points on the circle and operate with the theoretical definition of a point on a circle, rather than particular points on the screen. They can also imagine drawing a circle given any center and radius. An object conception of *circle* in Taxicab geometry in a DGE may be demonstrated by a student considering general properties and behaviors of circles in Taxicab geometry without relying on the specific examples in their DGE. As a note from Hollebrands (2003), students with a process conception of a transformation can anticipate the results of a transformation without having to perform the actual transformation on the computer.

Methodology

This study was conducted in a College Geometry course at a large university. There was a prerequisite for the class of an introduction to proofs course, and the enrollment in the class was comprised of seven undergraduate math majors and eleven graduate students who were pre-service or in-service secondary teachers. The textbook used for the course was College Geometry Using the Geometer's Sketchpad (Fenton & Reynolds, 2011) and is based on APOS Theory and the Activities, Classroom discussions, and Exercises (ACE) Teaching Cycle (Asiala et al., 1996). It is noted the course utilized Geometer's Sketchpad (GSP), which is a platform no longer supported for use. As GSP is a DGE and operates similarly to other DGEs, the methodology and results of this study can be transferrable to other DGEs. Concepts learned throughout the course in Euclidean geometry were introduced in Taxicab geometry during the last four 75-minute classes of the semester. All 18 students enrolled in the course volunteered to participate in this study, and audio and video recordings from in-class group work, work completed in GSP, and written work or notes during these class sessions for all 18 students were collected as data. The activity corresponding to the results presented in the current report was intended to guide students through the construction of a circle in Taxicab geometry in GSP and generalize this construction. Examples of a circle in Euclidean geometry and a circle in Taxicab geometry are illustrated in Figure 1. The activity is described below:

1. Plot points at $P(3,4)$, $A(2,2)$, $B(3,7)$, $C(2,5)$, and $D(5,5)$. By counting the number of blocks from P to A, we find that the *taxi-distance* PA is 3 units. Find the taxi-distances PB, PC, and PD. Two of these points are the same taxi-distance from P as A is. Which two?
2. The set of all points that are at the same taxi-distance from P form a *taxi-circle* centered at P. In part (a), three of the points lie on a taxi-circle of radius 3 centered at P. Find several additional points on this taxi-circle. Describe the set of all points that are at a taxi-distance of 3 units from a fixed point P. How is the shape of a taxi-circle different from (or similar to) the shape of an ordinary Euclidean circle?
3. If you are given a point $Q(x_q, y_q)$ and a radius r, how could you quickly sketch a taxi-circle of radius r centered at Q?

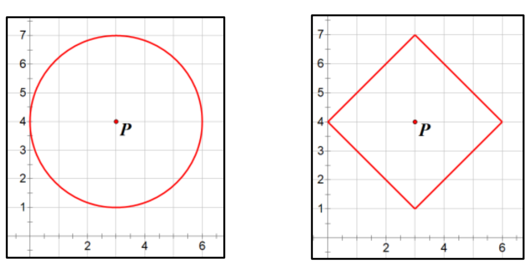

Figure 1. Geometric representations of a Euclidean and Taxicab circle each with center $P(3,4)$ and radius 3.

The design of this activity guides students through performing actions by prompting them to calculate the distance between points in Taxicab geometry to find examples and non-examples of points lying on a circle centered at (3,4) with radius 3. The students are prompted to find additional points that are 3 units away from the center and describe the set of all points that make up this circle. This is intended to guide students to interiorize the actions performed previously in the DGE into a process by focusing on the theoretical definition of a circle rather than the points on their screen. Students are then asked to compare their geometric understanding of a circle in Taxicab geometry to a circle in Euclidean geometry to aid in generalizing properties of a circle (i.e., all points are equidistant from the center, a circle looks different in Taxicab and Euclidean geometry based on how distance is defined), indicative of an object conception of *circle*.

The 18 students enrolled in the course were divided into six groups of three prior to this activity and had already worked through several activities about distance in Taxicab geometry. Students were used to working in groups and in GSP as these were regular and integral parts of the course. The students had access to a class set of laptops that had GSP installed on them and were at liberty to either each use their own laptop or share a laptop to work on the DGE activities in their groups. The classroom was set up with six square tables with two seats on two of the sides opposite one another, and students were free to move chairs around to work together in different ways in their groups. Using the adapted framework from Hollebrands (2003), data was analyzed from each group to investigate their individual understanding in terms of APOS Theory, their use of the DGE, and how their interactions within the group influenced both of these things. This report focuses on data collected from one of the groups, comprised of one undergraduate student, Ally, and two graduate students who were pre-service secondary teachers, Amy and Brianna, during the activity described above. This group was chosen in this report as they were representative of groups that worked cooperatively towards a shared goal and demonstrated various levels of cognitive development of the concept of **Circle**.

Results

As the prompts for this activity were intended to guide students through various levels of cognitive development, the data collected during this activity provided a rich understanding of how these levels can be exhibited by students while working in a group in a DGE. Ally, Amy, and Brianna all worked on their own laptops while communicating regularly with one another and often talking out loud to themselves to try and articulate their thinking. Ally's work in GSP for this activity is shown in Figure 2 below as a representative of the group. It is noted that students had access to a tool in GSP that was designed to measure the distance between two points in Taxicab geometry. This tool would also display the algebraic representation of the Taxicab distance between two points, seen in the top left of Figure 2.

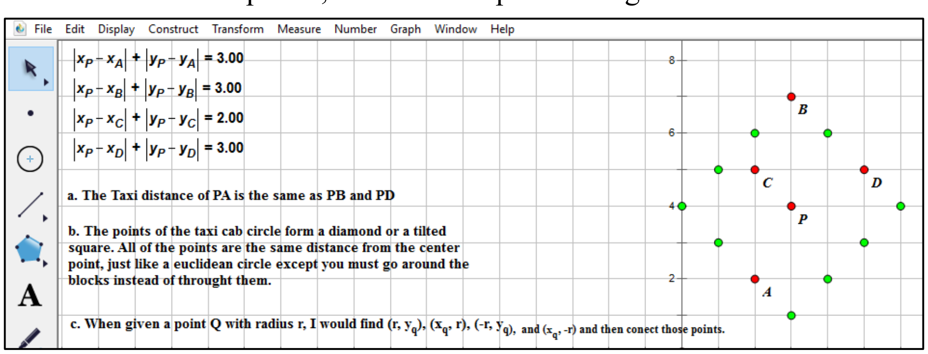

Figure 2. Work in GSP submitted by Ally.

After they read part (2) of the problem which asked students to plot several other points that are three units away from P, Brianna was looking at her graphical representation of the problem and said, "wait, why is PB the same?... Just cause it's three straight up?" to which Amy replied, "'Cause it's like the radius, yeah... of that circle." Shortly after this exchange, Amy began to investigate why the distance from P to A was three and the following interaction occurred:

Amy: 'Cause if that was a triangle, then the length of that hypotenuse be… wait, because PA is like… sides one and two.

Brianna: But we're just finding other like… how many points?

Ally: So, we just need to form like a taxi-circle.

Brianna: But I think we're trying to find other points that we think lie on the taxi-circle.

Ally: (4, 6), umm…

Brianna: Wouldn't $(3,1)$? 'Cause it's the same as PB, just the opposite direction? Like that's the radius, right?

Here, Amy had imagined a right triangle with PA as a hypotenuse and was expressing a comparison between the distance between P and A in Euclidean geometry ("hypotenuse") and in Taxicab geometry ("sides one and two"). This is evidence that Amy was exhibited a process conception of **Distance** by imagining this triangle and comparing the geometric representations of Euclidean and Taxicab distance between P and A, consistent with Kemp and Vidakovic (2023). Ally seemed to indicate she understood they were constructing a circle ("just need to form…a taxi-circle") and used this theoretical definition to anticipate all the points they were finding on the circle they were constructing, indicative of a process conception of **Circle** based on our framework adapted from Hollebrands (2003). Ally also seemed to recognize Brianna was not processing her comment that they are forming a circle and began to list specific points to help the group move into this reasoning. Brianna visualized the vertical distance PB as a radius and essentially reflected this distance ("the opposite direction") over the horizontal line through $(3,4)$ to obtain the point $(3,1)$. Aligning with Hollebrands (2003), since she anticipated the results and implications of a transformation on this radius without having to perform the actual transformation, we interpret this as her exhibiting evidence of a process conception of **Distance** and **Radius**.

As the group continued to explore in GSP while working on the activity, they began to anticipate the shape of the object they were constructing.

Ally: It forms a diamond.

Amy: So, it's not a circle.

Ally: This radius is the same…Well, an ordinary circle is round…I mean they're similar because… the radius is always the same.

Brianna: It's like the distance from…they're all the same distance from the center?

Amy: Well, this is the same distance too, but it's the same taxi-distance.

Brianna: But they're like a different type of distance cause you can go like up, you can move different ways, it's not like straight to it… like a clock.

Ally: They're similar cause its…the radius is all the way around [makes a circular motion with pointer finger].

Brianna: But it's like different.

Amy: They all are like… the same distance.

Brianna: The same distance, but it's like a different type of distance. I'm going to say it's a different type of distance than the radius of a circle.

In this cooperative learning exchange, the group seemed to be maximizing their own and each other's learning as described by Johnson and Johnson (1999) and were all reflecting on the procedures they had performed, consistent with Vidakovic (1993). All three students were working to articulate their thinking, bouncing phrasing and ideas off one another. At first, Amy implied that because this object was a "diamond" [instead of round as in Euclidean geometry], it was not a circle. As the conversation progressed, she heard what her groupmates said and indicated she processed their comments by referring to "the same taxi-distance" in her construction in the DGE. Brianna used a metaphor to explain that a circle in Euclidean geometry is round as a result of using a radius that is a straight segment, like the hand of a round clock. This is evidence Brianna was operating at a process conception of **Circle**, at least within Euclidean geometry, as she was able to generalize properties of the construction of a circle. Amy and Brianna ended up writing extremely similar responses in GSP for this prompt. Seen in Figure 3, Amy [and Brianna] wrote "they are all the same distance from the center, but it's a different type of distance than the radius of a circle." In other words, they seemed to agree that the way distance is measured is the cause of the difference in appearance of a circle in Euclidean geometry and in Taxicab geometry, but indicated they equated the radius of a circle with its representation as distance between two points in Euclidean geometry and that the distance between the same two points in Taxicab geometry is different than a radius.

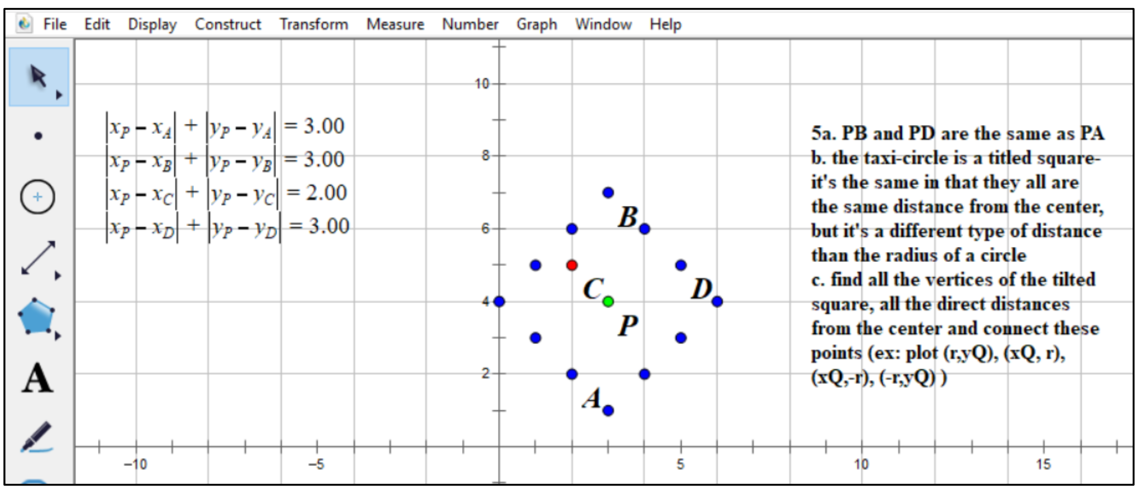

Figure 3. Work in GSP submitted by Amy.

Ally explained that a Euclidean and Taxicab circle were similar in the way they are constructed and compared these constructions across geometries ("they're similar because…the radius is the same…the radius is all the way around.") As she motioned with her pointer finger in a smooth, circular motion it is possible she was imagining a continuous, dynamic procedure defining the locus of points of a generalized circle by rotating a radius 360 degrees. Note she was speaking about an arbitrary circle in both Euclidean and Taxicab geometry at the same time. This is reiterated in her GSP work in Figure 2, where she writes, "All of the points are the same distance from the center point, just like a euclidean circle except you must go around the blocks instead of throught (*sic*) them." It is interpreted that by "go around the blocks," Ally was talking about distance in Taxicab geometry and by "through them," she was talking about distance in Euclidean geometry. As Hollebrands (2003) describes an object conception as a student considering general properties and behaviors rather than relying on the images on their screen, Ally demonstrated here she was exhibiting an object conception of **Circle**.

In the third prompt which asked students how they might quickly sketch an arbitrary circle in Taxicab geometry, the group began discussing geometrically how they would do so. While attempting to articulate this, they shifted over to attempting to write an algebraic explanation for this process. Specifically, for a circle in Taxicab geometry centered at $Q(x_q, y_q)$ with radius r, they determined the points (r, y_q), (x_q, r), $(-r, y_q)$ and $(x_q, -r)$ would be on the circle and by connecting these points, they will have constructed the entire circle. Brianna wrote in GSP, "I would find the 4 vertices (or points that are a straight/direct distance from the center) and then connect them to find the other points that lie on the taxi-circle." Brianna indicated here that she could imagine and understood all the points she would draw would be on the circle, indicating she was operating with at least a process conception of **Circle**. Amy wrote, "find all the vertices of the tiled square…and connect these points." This is notable because it may be the case that Amy was uncomfortable with calling the object a circle, perhaps an example of the tendency to neglect the definition of a concept under the pressure of [assumed] figural constraints, described by Fischbein (1993). Although the procedure determined by the group would only be true for a circle centered at the origin, this indicates a conceptual understanding in terms of identifying the intended points. Right after they wrote these coordinates, the class period ended. If they had more time, it is possible that they would have been able to test their conjecture of these coordinates and adjust them.

Discussion

As they worked to abstract properties, Amy, Brianna, and Ally bounced reasoning off one another. Ally demonstrated operating at a variety of levels of conception of **Circle**, moving between these levels during conversations to either explain a concept to a group member or try and abstract an idea. Brianna and Amy both provided evidence of operating at the action and process levels of conception of **Circle** and made connections through their conversation and explorations in the DGE. In this cooperative learning setting, all three students provided evidence that they were working toward a shared goal. They leveraged their exploration individually on their own laptops in their discussion and used comments from one another to help guide their exploration. When one group member was confused, they tended to try and clarify or help explain something. In this way, the group seemed to support the notion that the activity encouraged them to help one another as it would also benefit themselves, maximizing their own and each other's learning, while increasing their independence (Johnson & Johnson, 1999). As this group all explored on their own laptops, they often were not looking at the same screen while referencing the same objects. While this did not seem to hinder their conversation, it is possible this could deter some groups from conversing meaningfully about the activity. Some pedagogical suggestions based on this research include structuring DGE work in groups to encourage individual exploration while explicitly working toward a shared goal. By doing this, it is possible students will leverage their understanding to help explain concepts to one another and maximize the entire group's understanding, as the participants in this study did. Further, offering more opportunities for students in College Geometry courses to explore non-Euclidean geometry earlier in the course and in DGEs may be beneficial to their understanding of Euclidean concepts as they study them. Future research may investigate different types of structure in DGE group work and how this affects individual understanding. For example, what affordances or hindrances might arise if students all shared one laptop and were looking at the same screen rather than exploring individually? As this study utilized APOS Theory, DGEs, and cooperative learning, it may help fill a gap in the literature of undergraduate mathematics education.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior, 16*(4), 399-431.

Contreras, J. N. (2013). Fostering mathematical creativity through problem posing and modeling using dynamic geometry: Viviani's problem in the classroom. *Journal of Mathematics Education at Teachers College*, *4*(2).

Dreiling, K. M. (2012). Delving deeper: Triangle construction in taxicab geometry. *The Mathematics Teacher*, *105*(6), 474-478.

Dubinsky, E. (2002). Reflective abstraction in advanced mathematical thinking. In Tall, D. (Eds.), *Advanced mathematical thinking* (Vol. 11, pp. 95-126). Springer, Dordrecht.

Fenton, W., & Reynolds, B. E. (2011). *College geometry: Using the Geometer's Sketchpad*. Wiley.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Glass, B., & Deckert, W. (2001). Connecting research to teaching: Making better use of computer tools in geometry. *The Mathematics Teacher*, *94*(3), 224-229.

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55-72.

Hollebrands, K. F., Conner, A., & Smith, R. C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. *Journal for Research in Mathematics Education*, *41*(4), 324-350.

Jenkins, T. L. (1968). Euclid, you must be kidding. *Mathematics Magazine*, *41*(1), 34-37.

Johnson, D. W., & Johnson, R. T. (1999). Making cooperative learning work. *Theory into practice*, *38*(2), 67-73.

Kemp, A. (2018). *Generalizing and Transferring Mathematical Definitions from Euclidean to Taxicab Geometry*. (Doctoral Dissertation, Georgia State University).

Kemp, A., & Vidakovic, D. (2018). A student's use of definitions in the derivation of the Taxicab circle equation. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 21st annual conference on research in undergraduate mathematics education* (pp. 1394-1399).

Kemp, A., Vidakovic, D. (2019). The transfer and application of definitions from Euclidean to taxicab geometry: Circle. In Weinberg, A., Moore-Russo, D., Soto, H., & Wawro, M. (Eds.), *Proceedings of the 22nd annual conference on research in undergraduate mathematics education* (pp. 324-331).

Kemp, A., & Vidakovic, D. (2021a). Ways secondary mathematics teachers apply definitions in Taxicab geometry for a real-life situation: Midset. *The Journal of Mathematical Behavior*, *62*, 100848.

Kemp, A., Vidakovic, D. (2021b). The circle schema: An example of schema interaction in college geometry. In Karunakaran, S. S.& Higgins, A. (Eds.), *2021 research in undergraduate mathematics education reports* (pp. 147-155).

Kemp, A., & Vidakovic, D. (2023). Students’ understanding and development of the definition of circle in Taxicab and Euclidean geometries: an APOS perspective with schema interaction. *Educational Studies in Mathematics*, *112*(3), 567-588.
Liljedahl, P. (2020). *Building thinking classrooms in mathematics, grades K-12: 14 teaching practices for enhancing learning*. Corwin press.
Patsiomitou, S. (2019). Hybrid-dynamic objects: DGS environments and conceptual transformations. *International Journal for Educational and Vocational Studies*, *1*(1), 31-46.
Siegal, M., Borasi, R., & Fonzi, J. (1998). Supporting students' mathematical inquiries through reading. *Journal for Research in Mathematics Education*, *29*(4), 378-413.
Trigueros, M., Orozco-Santiago, J., & Martínez-Planell, R. (2022). Learning two-variable functions using 3D dynamic geometry. In *Fourth conference of the International Network for Didactic Research in University Mathematics*.
Vidakovic, D. (1993). *Cooperative learning: Differences between group and individual processes of construction of the concept of inverse function*. Purdue University.

Instructors’ Grounding Metaphors for Quotient Groups

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In this report, we examine the use of grounding metaphors across three abstract algebra instructors in their discussion of quotient group instruction. We identified three primary categories: construction metaphors, equivalence metaphors, and positional metaphors. The construct metaphors were further refined into building metaphors and demolition metaphors. In this paper, we provide an overview of the types of grounding metaphors in usage and provide insight into how the different instructors took on these metaphors when describing quotient groups and their instruction. We conclude with some considerations as to how these metaphors provide insight into quotient structure and considerations for future research.

Keywords: Metaphorical Thinking, Abstract Algebra, Instruction, Quotient Groups

Quotient groups have been identified as one of the most important, but also one of the most difficult topics in Abstract Algebra (Melhuish, 2019). Most of the research conducted regarding quotient groups has focused on how students think about quotient groups (e.g., Asiala et al., 1997; Melhuish et al., 2023). This literature base suggests students might develop only partial understandings such as relying on coset generation procedures (Hazzan, 1999) or seeing elements of quotient groups as sets and elements but not both (Asiala et al., 1997; Siebert & Williams, 2003). Some researchers have suggested that meanings related to partitioning are more productive as they are compatible with viewing quotient groups as the preimage of a homomorphism (Melhuish et al., 2023) and can support a richer understanding of why only certain subgroups can form quotient groups (Larsen & Lockwood, 2013). This work suggests that partitioning might serve as one important grounding metaphor for quotient groups. However, there is significantly less known about the way in which instructors conceptualize quotient group, what other metaphors may exist, and how they are taught. The goal of this proposal is to contribute to the literature on the teaching of quotient groups by addressing the use of metaphors in instruction. The research questions being addressed is:

What grounding metaphors about quotient groups are used during abstract algebra instructors’ descriptions of their teaching?

Literature Review

There is a great deal of consistency in the teaching of proof-based courses with the genre of chalk-talk dominating the way in which most of the courses are taught (Melhuish et al., 2022). That is, instructors lecture, writing formal mathematics on the boards, verbalize what is written and provide additional, often informal, insights, and ask students some form of questions (e.g., Artemeva & Fox, 2011; Paoletti et al., 2018). This consistency may mask important variations in instruction. For example, Pinto (2019) compared two mathematicians teaching a real analysis course using the same lesson plans. Despite this common material, they differed in the general enactment of the lesson including the timing of when parts were enacted and relevant to this paper, with their use of metaphorical imagery.

Pinto (2019) discussed how one of the mathematicians, Amit, used metaphorical language and imagery to unpack the definition of the derivative. It was stated that Amit wanted to help the students “obtain a picture of it” (p. 10). Thus, there was use of visual images and colloquial language including phrases such as “line”, “swing,” and “trap.” This contrasted with Yoav, who, rather than using imagery-based language approached the definition of a derivative term by term.

Rupnow (2021) similarly found a range of conceptual metaphors used by the mathematicians in their discussion of homomorphisms and isomorphisms. These metaphors spanned ideas of sameness and mappings and specific metaphors used by the mathematicians varied both in when and how often they occurred. Olsen et al. (2020) identified that instructors also use a number of metaphors conveying ideas of what it is to do mathematics during lectures. Oehrtman’s (2009) analysis of students’ calculus metaphors indicated that their experiences likely shaped their metaphors. Works such as Oehrtman (2009), Pinto (2019), and Rupnow (2021) would suggest that as instructors teach similar content – quotient groups in this case – it should be expected that there is variation in the types of metaphors being used. Furthermore, students understanding of concepts can be conjectured to be tied to the metaphors provided during instruction. Although, we caution that the exact mechanism of this link at the undergraduate level is currently understudied.

Theoretical Perspective

In consideration of metaphors, it is often taught that a metaphor is a comparison between two things without using the word that like or as. However, modern linguistics considers this to be an antiquated and incomplete view of metaphor as it treats metaphor as a consequence of language. Rather, they take the view that metaphors involve more than language, but rather are tied to meaning and cognition. Despite disagreement among linguists in what constitutes a metaphor, they tend to agree that metaphor more broadly is a way of viewing one construct or object through the lens of another (Cameron, 1999). For the purposes of this study, in order to understand the concept of metaphor, we are drawing on the work of Lakoff and Núñez (1997, 2000) defining a metaphor to be a “cross-domain conceptual mapping” (Lakoff & Núñez, 1997, p. 32). Additionally, Lakoff & Núñez define two types of conceptual metaphors: grounding metaphors and linking metaphors. In this report, we focus on grounding metaphors. When using a grounding metaphor, a person is taking an experience from everyday life and mapping or projecting the experience onto the concept they are attempting to understand or communicate. We will be using this construct as a means to discuss the types of metaphors being used in teaching quotient groups and examining how quotient groups are understood from a metaphorical point of view.

Methods

The results of this study are a part of a larger dissertation study aimed at exploring quotient group instruction. Data collection took place across seven universities in the United States. The data examined here came from the first of a series of two semi-structured interviews in which the goal of the interview was to examine instructional practices surrounding the teaching and learning of quotient groups. A subset of the participants was purposefully chosen for this analysis based on their professed methods of teaching. Of the instructors selected, one utilized a research-based inquiry curriculum emphasizing partitioning, one primarily lectured and relied on physical manipulates and built up from ideas of cosets, while the last instructor primarily lectured and built quotient groups from more general ideas of equivalence classes.

Data Collection Methods

The goal of the initial interview was to examine instructional practices of those who teach abstract algebra with respect to their instruction on quotient groups. Prior to the discussion of teaching practice, each of the instructors were asked two questions about how they themselves conceptualized quotient groups and how they wanted their students to conceptualize quotient groups. The rest of the questions were focused on how they conducted instruction including questions regarding typical lessons, examples, proofs, assessments, and their view of their role in the learning process. Each of the interviews lasted approximately one hour and was videorecorded and transcribed. These responses were in turn used to inform the design of a follow up interview designed to illicit further discussion on decision making.

Data Analysis Methods

The first research open coded the transcripts (Braun & Clarke, 2006) identifying grounding metaphors. These metaphors were identified based on the participants' the use of "actionable words" or words that evoke a sense of action (i.e., the use of action verbs, gerunds, or participles) or physical classroom experiences that occurred. Once the initial list of grounding metaphors was generated, a second researcher read through each of the transcripts to offer critiques, pushback, and additional metaphors that may have been missed. Any disagreements were discussed between the researchers and worked through. The list of metaphors found in the results section reflects a subset of the agreed upon metaphors.

Results

We identified three overarching categories of metaphors. The first major class was quotient groups through the lens of construction-based metaphors. This metaphor class was further divided into building and demolition metaphors. Additionally, there were equivalence-based metaphors and positional metaphors. Table 1 provides a summary and definition of these metaphors. In the first column the metaphor class is listed along with examples of key phrases that fell into that class. A definition and example of each class is also provided in Table 1.

Table 1. Summary of Grounding Metaphors Class Used to Understand Quotient Groups

Metaphor Class	Class Definition	Class Example
Building Metaphors ● Building ● Clumping ● Sticking Together ● Partitioning*	A class of metaphors that describes the creation of quotient groups through building ideas	"we're doing these things and *building* these, these subsets"
Demolition Metaphors ● Partitioning* ● Collapsing ● Breaking/ Splitting ● Subdividing	A class of metaphors that describes the creation of quotient groups through demolition-based building ideas.	"…think about *collapsing* every one of those parallel lines on to this intersection point."
Equivalence Based Metaphors ● Setting equal to; setting to	A class of metaphors that grounds ideas of treating elements as if they are the same	"you're factoring out by Z mod 3, then you're looking at integers, and you can *ignore.* You can *ignore* multiples of

• Ignoring • Regarding as		three. Any things that differ by a multiple of three are essentially the same thing when you're looking at a quotient group."
Positional Metaphors • Becoming • Going Around • Lying	A class of metaphors rooted in the idea of appearance or location. Often involve type of movement, but it is not necessary	"you think what would happen if you were to collapse every one of those circles to a point, what would you get each of those circles, is going to *become* one point."

In the remainder of the results, we share some examples from three instructors to illustrate how these metaphors are used in their instruction descriptions and how the different types of metaphors support each other in grounding ideas of quotient structures.

Dr. A: Clumping and Collapsing

When explicitly asked how Dr. A wanted their students to think about quotient groups, they stated that they wanted them to understand quotient groups as "clumping." In their explanation of what they meant by clumping, Dr. A stated

> So, this is a clock. So, this is going to be like Z_{12} you start with 0,1,2,3, 4, … ,11 and then you go 12, 1,2,3,4,5 so let's look at a quotient group of Z_{12} Mod four times Z_{12}. So, I start with Z_{12} and then, when I have four, I am back to zero. Go around again when I get to eight if I cut my string correctly to get back to zero. And so, then I've got Z_4, but I can see that all of these things, one I yeah and all of these are one item, and all of these are these are one. […] okay so I'm *grouping* or *clumping,* and I'm setting to zero.

During this illustration, Dr. A conveyed a building metaphor through the use of the word clumping. In this activity, they physically constructed piles of numbers (cosets) through this clumping or gathering action. To further explain this metaphor, they also drew on the positional metaphor of "going around" and equivalence-based metaphors in the use of the language such as "setting to zero" and "these are all one."

As Dr. A continued to explain how they wanted their students to understand quotient groups and the ideas of clumping, they turned to another example and rather than using a building metaphor they instead used, "collapsing", which is a demolition metaphor. Dr. A drew the image that is seen in Figure 1 and stated:

We'll take the cosets of $y = x$ and it's always fun to watch them try to figure out what a coset is but they usually eventually get to the parallel lines which is correct. And so, all the cosets are parallel lines and now, when you think of this every one of those parallel lines is going to get clumped to one point, and so you can draw the counter diagonal and think about *collapsing* every one of those parallel lines on to this intersection point, and so you can see that if I take **R** cross **R** and mod by the diagonal, or, I guess, we do (x, y) such that x equals y. That you get something that is isomorphic to **R.**

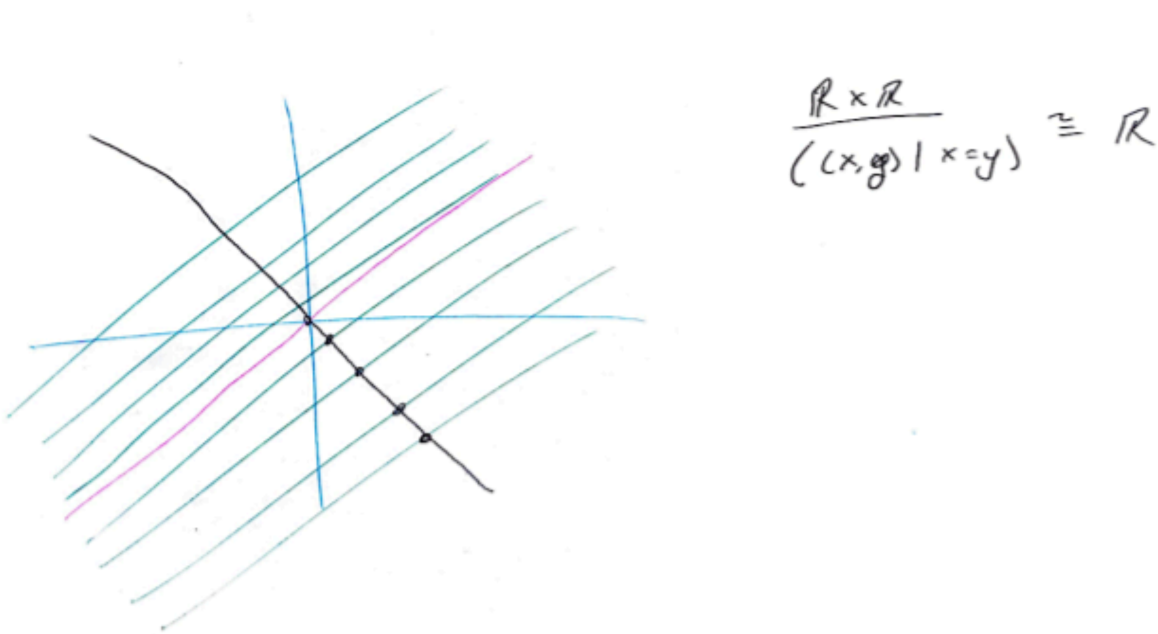

Figure 1

The main imagery that Dr. A utilized in this example is the idea that each of the parallel lines will fall or collapse. However, in this second except, we see that the collapsing is to a specific point thus furthering the metaphor of collapsing by giving a directionality of where the collapsing is going. It is worth noting that in this excerpt, the phrase clumped is used. However, unlike in the previous excerpt, "clumped" appears to describe the state of the parallel lines after they were collapsed rather than the action being applied to the parallel lines. Thus, this metaphor contrasts with the previous example as the mental imagery involved invoked different actions. In the first example, Dr. A invoked imagery of creating piles (and maintaining the subset structure with elements) whereas in this second example the collapsing moves from cosets as collections of elements to their role as single element losing the set structure through the act of setting to zero (emphasizing the representative element).

Dr. B: Partitioning to Build and to Deconstruct

Dr. B was similarly asked how they wanted their students conceptualize quotient groups and they explained that they wanted their students to understand quotient groups as "partitioning" stating "... so this idea of partitioning I think is powerful for thinking about the structure where you're building this, these weird things that are sets and elements kind of at the same time." In this explanation of partitioning, Dr. B communicated that partitioning was a way of building. This class of building continued in their explanation as they made statements such as "building up cosets" and "understanding the machinery involved in building."

In a similar manner to Dr. A, Dr. B also made a transition from building metaphors to demolition construction metaphors. However, unlike Dr. A, the demolition metaphor connected to the same action metaphor: partitioning. While explaining an activity regarding the partitioning of D_4, Dr. B stated

> Is there a way that we could *partition* up this D_4 table into something that acts like evens and odds, right? […] I always expect that the students are going to just *break it up* into like the rotations and the flips. […] I always have groups of students who break it up in other ways, but. They always break up in ways that that work. They're doing something really reasonable, and how they're subdividing these up, and it's now kind of laying this foundation that like one of these subsets has to act kind of like an identity type of thing, like the evens did. We're seeing every element end up somewhere […] This is kind of emphasizing the partitioning into new elements. And that physical color gives you something that you can think of this all as like one object over altogether. So, we spent a lot of time doing that.

In this explanation, Dr. B is described the creation of quotient groups through actions such as "subdividing" or breaking up. However, this is still describing the understanding of partitioning because they first posed the task of partitioning the group D_4, yet they hold the expectation that students will break things up. Thus, this explanation yielded partitioning as a demolition metaphor in this case, but ultimately as both a demolition and building metaphor as it had previously been described as building. Additionally, within this explanation, Dr. B drew on the use of a positional metaphor through the use of the phrase "end up somewhere" as they are discussing this transformation of the elements within the original group structure. In this positional metaphor elements are moving from the original group to their assigned coset.

Dr. K: Equivalence-Based Metaphors

Dr. K emphasized quotient groups as an extension of equivalence classes. In looking at Dr. K's conception of quotient groups, he stated:

> It's when you're factoring out by something. It's that's a factor you can ignore. So simplest examples of course, are modular arithmetic. So, if you're factoring out by *Z* mod 3, then you're looking at integers, and you can ignore -- you can ignore multiples of three. Any things that differ by a multiple of three are essentially the same thing when you're looking at a quotient group.

Within this explanation, Dr. K used primary metaphorical expression: ignore. This expression is used to convey the mathematical idea of equivalence and that "you have a set of objects" in which things are regarded as the same. Dr. K further elaborated on these ideas as they explained one of the examples they liked to use in their practice.

> I do use examples which are not which are not finite examples. Things like the fact that the nonzero complex numbers are a group under multiplication. S^1 is a subgroup, so then what is S^1 this time? Meaning the unit circle, of course. And what is the nonzero complex numbers mod S^1, right? And what is? And also, the positive reals are a subgroup, so what's the nonzero complex numbers mod the positive? These are very geometric things

> for them to think about. And modding out by S^1 really mean it also reinforces this idea. Well, now we're regarding two nonzero complex numbers as the same if they lie on the same circle about the origin and your quotient group, then is effectively the positive risk, so I like to talk about those examples somewhat.

Additionally, in this example, sameness was used in conjunction with the positional metaphor "lie." Unlike the previous positional metaphors, lie takes on a more static nature in that it lacks any type of movement. However, it is an important metaphor to note in that it serves as a justification for how Dr. K judged sameness in this example in that two points were considered to be the same if they were on (*lie)* on the same circle.

Discussion

We shared data from these three instructors because during their description of their teaching of quotient groups, they each seemed to emphasize a different aspect of quotient group understanding. During our analysis, there were four primary classes that emerged as instructors discussed quotient groups: building, demolition, equivalence, and position. As we considered these metaphors and the use of them, we hypothesize that each type of metaphor may reflect different ways of thinking about quotient groups. Dr. A emphasized *clumping* and *collapsing*. These types of actions emphasize an outcome where elements are clumped into sets then collapsed into a single element. These actions may reflect the duality between elements and sets. Dr. B's metaphors were rooted in *partitioning* from original group to quotient group traversing the duality between elements being members of the original (*subdividing)* and these elements now being cosets in the quotient group *(building)*. Finally, Dr. K leveraged ideas of equivalence to emphasize *ignoring* which may reflect the way that a coset can be thought of in terms of its representative element. In some ways these differences may emphasize the different ways of conceptualizing Zn elements as subsets, set with representative element, and representative element only.

For the scope of this paper, we focused on the primary metaphors shared by these instructors. In all three cases the primary metaphors focused on the element and set relationships rather than the group operation. We also note that positional metaphors could be found in all three instructors' language but did not appear as prevalent. Other metaphor classes that appeared in the data but that are not discussed in this report include the personification of normality and its role in "breaking" or "busting" groups or maintaining "machinery." Future research will involve testing this taxonomy with other instructional interviews and expanding classifications to include other elements of quotient groups such as normality. Additionally, future work will also examine the use of a second type of metaphor - linking metaphors - that instructors use as they discuss quotient groups as these were at times used in tandem with some of the grounding metaphors that the instructors were using. Both aspects of this future work will be carried out with the remaining thirteen participants within the larger dissertation study.

At this point, we conjecture that during quotient group instruction there is a lot of imagery that is at play due to the language that instructors use. However, students may or may not be influenced by their instructor's metaphor. Finally, we note that it is unlikely that quotient groups can be conveyed fully with any particular metaphor or class of metaphors. The language used by these instructors reflects attention to a number of salient dualities and therefore grounding metaphors for quotient groups are likely needed in clusters to fully account for the concept.

References

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication*, *28*(4), 345-379-379. https://doi.org/10.1177/0741088311419630

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., & Oktaç, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *The Journal of Mathematical Behavior*, *16*(3), 241-309. https://doi.org/10.1016/S0732-3123(97)90029-8

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Cameron, L. (1999). Operationalising 'metaphor' for applied linguistic research. In L. Cameron & G. Low (Eds.), *Researching and Applying Metaphor* (pp. 3-28). Cambridge University Press.

Hazzan, O. (1999). Reducing Abstraction Level When Learning Abstract Algebra Concepts. *Educational Studies in Mathematics*, *40*(1), 71-90.

Lakoff, G., & Núñez, R. E. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In *Mathematical reasoning: Analogies, metaphors, and images.* (pp. 21-89). Lawrence Erlbaum Associates Publishers.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *Journal of Mathematical Behavior*, *32*(4), 726-742. https://doi.org/10.1016/j.jmathb.2013.02.010

Melhuish, K. (2019). The Group Theory Concept Assessment: a Tool for Measuring Conceptual Understanding in Introductory Group Theory. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 359. https://doi.org/10.1007/s40753-019-00093-6

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, *67*, 100986. https://doi.org/https://doi.org/10.1016/j.jmathb.2022.100986

Melhuish, K., Guajardo, L., Dawkins, P. C., Zolt, H., & Lew, K. (2023). The Role of Quotient Group Meanings in a Theorem and Proof Comprehension Task. *Educational Studies in Mathematics*, *113*(3), 499-517.

Oehrtman, M. (2009). Collapsing Dimensions, Physical Limitation, and Other Student Metaphors for Limit Concepts. *Journal for Research in Mathematics Education*, *40*(4), 396-426. http://www.jstor.org/stable/40539345

Olsen, J., Lew, K., & Weber, K. (2020). Metaphors for learning and doing mathematics in advanced mathematics lectures. *Educational Studies in Mathematics*, *105*(1), 1-17. https://doi.org/10.1007/s10649-020-09968-x

Paoletti, T., Krupnik, V., Papadopoulos, D., Olsen, J., Fukawa-Connelly, T., & Weber, K. (2018). Teacher questioning and invitations to participate in advanced mathematics lectures. *Educational Studies in Mathematics*, *98*(1), 1-17. https://doi.org/10.1007/s10649-018-9807-6

Pinto, A. (2019). Variability in the formal and informal content instructors convey in lectures. *Journal of Mathematical Behavior*, *54*. https://doi.org/10.1016/j.jmathb.2018.11.001

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior*, *62*. https://doi.org/10.1016/j.jmathb.2021.100867

Siebert, D., & Williams, S. R. (2003). *Students' Understanding of Z[subscript N]* International Group for the Psychology of Mathematics Education, Honolulu, HI. https://libproxy.txstate.edu/login?url=https://search.ebscohost.com/login.aspx?direct=true&db=eric&AN=ED501121&site=eds-live&scope=site

Instructor-Student Interactions in Geometry Courses for Secondary Teachers: Results from A National Survey

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We report preliminary results of selected questions from a national survey of instructors of geometry courses for secondary teachers about the nature of instructor-student interactions. Survey responses (n= 118) are used to indicate six latent constructs describing aspects of instructor-student interaction that in turn quantify hypothesized characteristics of two didactical contracts, which we call inquiry in geometry and study of geometry. We found that instructors whose highest degree is in mathematics education are less likely to rely on a study of geometry contract than instructors whose highest degree is in mathematics. Also, instructors who have previously taught high school geometry are less likely to lecture.

Keywords: geometry, secondary teacher education, survey, didactical contract, instruction

Objectives

The work reported contributes to describing instruction in undergraduate mathematics education. Based on the responses from 118 instructors to 24 survey items, we describe how instructors relate to students in geometry courses taken by prospective secondary mathematics teachers (GeT courses, hereafter), including whether and how they incorporate students' input in lectures, how they handle student difficulties, and how they handle student contributions. After testing a measurement model of constructs that inform the extent to which instructors lead students in the study of geometry or in inquiry in geometry, we report on how indicators of these constructs relate to each other, and whether characteristics of the instructors (including whether their highest degrees are in mathematics or in mathematics education, and whether they have taught high school geometry in the past) predict scores in any of those latent variables.

Literature Review

The mathematical preparation of prospective secondary teachers (PST, hereafter) is an important area for investigation in the RUME community (e.g., Lai et al., 2023; Serbin & Bae, 2023). Whereas scholars and practitioners have written about the mathematics curriculum of teacher preparation for more than a century (Schubring, 1989), the empirical study of mathematics instruction in those courses has lagged for most of our field's history, along with the lag in the study of mathematics instruction at the undergraduate level. Speer et al. (2010) had noted how limited scholarship on mathematics instruction at the undergraduate level had been. Yet, a more recent review by Melhuish et al. (2022) updated that assertion, noting that the study of instruction at the undergraduate level has captured much more interest between 2010 and 2020. A variety of methods have been used in these studies, including, in particular, some instructor surveys of instructional practices (e.g., Johnson, et al., 2018, 2019). Though a main interest in the analysis has been to report on the incidence of lecture in instruction, researchers have also cautioned that the incidence of lecture is not necessarily an indicator of the absence of student-centered instruction (Smith et al., 2014). In advanced mathematics classes such as abstract algebra, however, studies of instructors' beliefs have suggested that mathematicians value lecture as an instructional method to prepare future mathematicians (Melhuish et al, 2022).

For the specific case of the preparation of PST, one might expect instruction could be different, especially considering the emphasis that has been given to inquiry-oriented instruction in the last couple of decades (Abell et al., 2018; Mahavier, 1999). Yoshinobu and Jones (2011) had singled out preservice teachers among those who could benefit the most from inquiry-based learning; and Laursen et al. (2016) documented important gains for PST who had learned mathematics through inquiry. Important questions to ask include: But to what extent do prospective secondary mathematics teachers participate in inquiry-oriented instruction?

GeT courses are salient locations where the incidence of lecture as well as of student-centered instructional practices could be inspected to answer that question. Though small-scale research has been done in GeT classes (e.g., Blanton, 2002), little research has looked at undergraduate geometry instruction at scale thus far. Wong (1970) was an early survey of institutions and the curriculum offered in GeT courses. Grover and Connor (2000) reported on a survey of ~100 GeT course instructors and included one question aimed at pedagogical practices. The responses showed that though only 7.1% of instructors described their courses as consisting of only lectures, only up to 34.3% included classroom discussions facilitated by instructor. Though responses to just one survey question are hardly enough to describe instruction, no other instructor survey has been conducted after Grover and Connor (2000) to expand or update what we know about instructional practice in GeT courses since. The answers from that one question, however, suggested that to understand instructional practice in more detail, we could use an analysis of the components of lecture and inquiry to develop an instrument that more accurately served for instructors to describe what they do in their classrooms.

An important precursor of the work reported here was Shultz's (2020) INQUIRE survey, which explored the extent to which undergraduate mathematics instructors engaged in practices that could be used to describe inquiry. Shultz's (2020) INQUIRE instrument defined latent constructs that could indicate various components of inquiry-oriented instruction described in the literature on inquiry-based learning. Shultz (2020) organized those constructs using the edges of the instructional triangle (Cohen et al., 2003). For example, *interactive lecture* and *hinting without telling* were two constructs identified to measure the extent to which instructor-student relationships (the instructor-student edge of the instructional triangle) were inquiry-oriented. Rather than relying on single questions to indicate a construct, the INQUIRE instrument included 5 items to indicate interactive lecture and 3 to indicate hinting without telling. Among important findings from Shultz (2020) are that the various constructs that can be associated with inquiry-based instruction portray a more complex distribution across instructors who claim to engage in inquiry. Shultz found evidence that lower-division undergraduate mathematics instructors might cluster in four different groups, depending on the scores on various of those constructs. Our GeT Instructor survey also used the instructional triangle to organize various aspects of instruction as latent constructs to be indicated by survey items. In this study we focus on the instructor-student edge of the instructional triangle, and we inquire on the incidence of constructs characteristic of inquiry as well as those which are characteristic of traditional study of geometry.

Theoretical Framework

We build on a theoretical framework about mathematics instruction that combines Cohen et al.'s (2003) instructional triangle and Brousseau's (1997) didactical contract. Specifically, Cohen et al. (2003) conceptualize instruction as a system of relationships among instructor, students, and content that take place in environments. The latter are institutional environments, namely mathematics departments and teacher education programs in colleges and universities. Herbst et al. (2023) further elaborate the content vertex of the instructional triangle to account for the fact that whereas students mainly relate to the content in terms of the work they are asked to do, instructors also relate to the content in terms of the instructional goals that such work is designed to support the acquisition of (Figure 1). This distinction is especially important in inquiry classrooms as the work students are asked to do may not too obviously disclose what the knowledge at stake is (e.g., Hitchman, 2017).

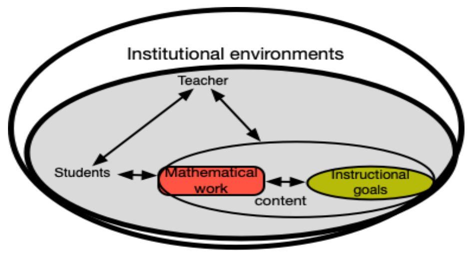

Figure 1. Elaboration of the instructional triangle

The specific ways in which those relationships are entertained call for the use of the notion of didactical contract. The literature has often used holistic names such as “school mathematics” and “inquiry classrooms” (Cobb et al., 1992) or “lecture-based” or “student-centered” to distinguish between types of teaching (Mesa et al., 2020). The notion of didactical contract (Brousseau, 1997), which Herbst et al. (2023) interpret as a system of norms that underpin how relationships among instructor, students, and content take place, serves us to operationalize those nominal distinctions into sets of possible norms that might characterize those relationships. Leading to the development of a survey that could help us elicit descriptive information about GeT instruction, we hypothesized that features of inquiry-oriented instruction could be considered possible norms of a didactical contract (inquiry in geometry) and that features of what often is called traditional or lecture-based instruction could also be identified to characterize a different didactical contract (the study of geometry). We did not expect that the didactical contract in any individual GeT class could be simply classified as either inquiry or study, but rather designed the survey so that we could measure instructors’ recognition of each of the various norms that describe instructor-student relationship in both contracts. The present study reports on instructors’ recognition of the various norms that characterize study and inquiry contracts. We hypothesized that the study contract would rely on norms such as LECTURE (the instructor is expected to introduce any new content), RIGHTANS (the instructor is expected to provide the right answers to students who have difficulties), and STALKTOINS (the instructor is expected to take students’ public comments as directed to the instructor). And we hypothesized that inquiry contracts might rely on other norms including INTLECTURE (students are expected to participate in lectures), HINTNOTELL (the instructor is expected to hint without telling when students have difficulties, and STALKTOCLASS (the instructor is expected to take students public contributions as directed to the whole class). These norms were used to create the items in the survey with which we expected to answer three questions: (1) How likely is it that students participate when new knowledge is being installed? (2) How do instructors respond to individual student difficulties with class work? and (3) How do instructors make use of individual student contributions to the whole class? Further, we expected that constructs that describe a study contract would correlate with each other and the same would happen with variables that describe an inquiry contract. And we wondered the extent to which responses to those questions were predicted by individual characteristics of the instructors, specifically whether their highest degree was in mathematics or mathematics education and whether they had prior experience teaching high school geometry.

Methods

The GeT Instructor Survey

The GeT Instructor survey was designed to describe instruction across geometry courses for secondary teachers taught in mathematics departments across universities in the US. Broadly conceived, it aims to measure the incidence of various instructor-centered and student-centered practices as well as various types of students' engagement with content, including geometry and geometry knowledge for teaching. Some of those items ask instructors to report the extent to which they engage students in tasks of teaching geometry (such as providing feedback on students' written work). Data collection has been ongoing; the present report provides initial gleanings from the analysis of some of the constructs being measured.

Herbst et al. (2024) analyzed the GeT Instructor survey responses concerning students' interaction with content. In that analysis, we estimated the relationship among four factors that capture instructors' descriptions of the nature of their students engagement with content: (1) students *study* geometry (Study), (2) students *inquire* into geometry (Inquiry), (3) students engage in tasks of teaching geometry (ETT), and (4) students engage with dynamic geometry software (DGS). Analysis showed significant correlations between Inquiry and ETT, between Inquiry and DGS, and between ETT and DGS. A structural equation model showed that DGS fully mediates the relationship between Inquiry and ETT. The present report concentrates on another of the instructional triangle's edges: The instructor-student relationship.

In its initial design, the GeT Instructor survey included 24 items to indicate 7 constructs which could be used to describe the instructor-student relationship in the instructional triangle. Each of those items asked participants to indicate their level of agreement using a 6-point Likert scale (ranging from Strongly Disagree to Strongly Agree) in response to provided sets of statements that could describe the respondents' practice in the GeT class. For example, item 811204 presented the statement "While introducing new material, I called upon the students to ask questions about the material being covered" which we hypothesized would indicate the construct interactive lecture (INTLECTURE). Of the seven hypothesized constructs, only six could be measured with the designed items (items for the seventh construct did not meet standard requirements in a confirmatory factor analysis). The six constructs helped provide answers to the three first research questions. Besides, we expected that correlations among the six constructs might align with the different contracts: Study (constructs LECTURE, RIGHTANS and STALKTOINS) and Inquiry (constructs INTLECTURE, HINTNOTELL, and STALKTOCLASS). The GeT Instructor survey hypothesized other constructs as useful to answer questions about the other relationships represented in Cohen et al.'s (2003) instructional triangle. We do not report on those questions and constructs in the present report.

Sample

To reach widely across GeT Instructors in the US, we obtained lists of all the universities and colleges across the US and checked whether they had a secondary teacher preparation program and whether their mathematics departments offered a GeT course required for prospective teachers. This canvassing yielded (n=670) mathematics departments; emails were sent to department heads (or their secretaries) asking them to forward a link to the survey to the instructor who had taught the course last. By the time of this analysis, our effective sample size consisted of 118 GeT instructors who completed all items of a Qualtrics survey, including the GeT Instructor survey and a background questionnaire. Our sample participants confirmed they had taught a geometry course required for secondary mathematics teachers in the last ten years. The participants comprised approximately 55.9% male instructors and 39.8% female instructors. Approximately 72% had their highest degree in mathematics, while 25.4% had their highest degree in mathematics education. And 30.5% had prior teaching experience in high school geometry. A significant 85.6% of participants held either tenure or tenure-track faculty positions, while 10.2% occupied non-tenure roles including lecturers and graduate students.

Results

Descriptives of the raw scores for the seven hypothesized constructs are provided in Table 1. We performed a Confirmatory Factor Analysis (CFA) to evaluate how well the observed items measure each of the seven hypothesized constructs. Notably, items designed to indicate one construct, that instructor functions as an older peer (OLDERPEER), exhibited low correlations among them and small item loadings which we took as evidence that the items did not represent a single latent construct. After excluding these items, we re-evaluated CFA with the remaining items that had item loadings above the threshold of approximately 0.3 onto the six hypothesized constructs. The results indicate an acceptable model fit, with the Root Mean Square Error of Approximation (RMSEA) at 0.080, the Comparative Fit Index (CFI) at 0.854, and the Tucker-Lewis Index (TLI) at 0.818.

Table 1: Descriptive Statistics of the Seven Hypothesized Constructs

	N	Mean	Median	Min.	Max.	Std. Dev.
LECTURE	118	3.997	4.2	1	6	1.287
INTLECTURE	118	4.883	4.8	2.6	6	0.711
OLDERPEER	118	3.547	3.5	1.75	5.75	0.824
RIGHTANS	118	3.031	3	1	6	1.021
HINTNOTELL	118	3.723	3.7	1	6	0.925
STALKTOINS	118	2.989	3	1	4.7	0.879
STALKTOCLASS	118	4.381	4.5	1	6	1.039

The CFA analysis revealed notable patterns of correlation among the constructs, primarily distinguishing between constructs hypothesized as characteristic of the Study contract (LECTURE, RIGHTANS, and STALKTOINS) and those hypothesized as characteristic of the Inquiry contract (INTLECTURE, HINTNOTELL, and STALKTOCLASS). The highest correlations were observed between LECTURE and RIGHTANS (.550), RIGHTANS and STALKTOINS (.240), as well as LECTURE and STALKTOINS (.229). These findings suggest a strong connection between presenting traditional lectures, providing correct answers when students have difficulties, and taking student contributions as a dialogue between the student and the instructor. Conversely, in the realm of the inquiry contract, we observed significant correlations between HINTNOTELL and STALKTOCLASS (.207), and between INTLECTURE and STALKTOCLASS (.119) (see Table 2). These findings indicate relationships among delivering interactive lectures, fostering classroom discussions, and affording students opportunities to solve their own problems. Additionally, the correlation between LECTURE and INTLECTURE was significant (.161, $p < 0.05$). This may indicate instructors' inclination towards utilizing lectures, regardless of the specific type of lecture.

Table 2: Correlation Matrix of the Six Factors in CFA

	811100	811200	812100	812200	813100
811100					
811200	.161*				
812100	.550***	.037			
812200	-.111	.037	-.013		
813100	.229*	.004	.240**	-.054	
813200	-.198	.119*	-.177	.207*	-.069

* p < 0.05, **p<0.01, ***p<0.001

We also conducted a comparative analysis of latent variable means (LVM) among instructors from various demographic backgrounds to explore the potential association between these demographics and scores on specific constructs. Specifically, we focused on instructors' highest degree to answer whether having a highest degree in either mathematics (M) or mathematics education (ME) could predict whether the instructor's GeT class might follow more of a study or inquiry contract. To express the between-group differences, we set the LVM in the first group (mathematics) to zero and estimated the LVM in the mathematics education group (see Table 3).

Table 3: Latent Variable Mean (LVM) Difference -Between Demographics Groups

	Highest Degree in Mathematics (M) (N=85) or Mathematics Education (ME) (N=30)		Did Not Teach HS Geometry (N) (N=82) or Taught HS Geometry (Y) (N=36)	
	LVM Difference (LVM in ME after setting M to 0)	p-value	LVM Difference (LVM in Y after setting N to 0)	p-value
LECTURE	- .755***	.0009	-.462*	.04
INTLECTURE	.133	.18	.121	.19
RIGHTANS	-.543**	.002	-.285	.15
HINTNOTELL	.312	.08	.182	.25
STALKTOINS	-.334*	.02	-.171	.10
STALKTOCLASS	.390*	.04	-.054	.81

* p-value < 0.05, **p<0.01, ***p<0.001

A Wald test and chi-square difference test revealed significant differences in latent variable means between groups of instructors according to highest degree for constructs LECTURE, RIGHTANS, and STALKTOINS. Thus, instructors holding their highest degrees in mathematics education are less likely to manage a Study contract, in which traditional lectures are given, correct solutions are offered when students encounter difficulties, and student contributions are seen as one-on-one dialogues with the instructor, as compared to instructors with highest degrees in mathematics. There is some, but not enough evidence to say that instructors whose highest degree is in mathematics education are more likely than instructors with degrees in mathematics to engage in practices aligned with Inquiry. Furthermore, we examined instructors who had or had not taught high school geometry. After setting the LVM in the second group (had not taught high school geometry) to zero and estimating the LVM in the first (had taught) which represent between-group differences, we observed a significant mean difference in the LECTURE construct. Instructors with prior experience teaching high school geometry seem to be less prone to employing traditional lectures in their instruction.

Conclusion

A few observations about the contracts that we call Study and Inquiry can be made as regards how these contracts characterize the instructor-student relationship. The survey successfully deconstructs Study and Inquiry into six constructs (3 for study and 3 for inquiry) that are well indicated by several items. It thus can provide a more nuanced image of what the study and inquiry contracts mean. In particular, as related to the popular conflation of inquiry with no lecturing and the defense that some instructors have offered of the possibility to combine lecturing with inquiry (e.g., Alcock, 2018), the survey provides other well-indicated constructs that can be used to inspect the incidence of inquiry practices. Though the full survey is designed specifically for GeT courses, the specific constructs used to understand the instructor-student relationship are indicated with items that depend very little on the nature of the content being transacted (though they are about mathematics instruction); thus, researchers investigating instruction in other courses of study might be able to use same survey items.

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References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *MAA instructional practices guide.* Mathematical Association of America.

Alcock, L., (2018). Tilting the classroom. *London Mathematical Society Newsletter, 474,* 22- 27

Blanton, M. L. (2002). Using an undergraduate geometry course to challenge pre-service teachers' notions of discourse. *Journal of Mathematics Teacher Education, 5*(2), 117-152.

Brousseau, G. (2006). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (Vol. 19). Springer Science & Business Media.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, *29*, 573-604.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, *25*(2), 119-142.

Grover, B. W., & Connor, J. (2000). Characteristics of the college geometry course for preservice secondary teachers. *Journal of Mathematics Teacher Education*, *3*, 47-67.

Herbst, P., Brown, A., Ion, M., & Margolis, C. (2023). Teaching geometry for secondary teachers: What are the tensions instructors need to manage? *International Journal of Research in Undergraduate Mathematics Education*, https://doi.org/10.1007/s40753-023-00216-0.

Herbst, P., Jeon, S., & Brown, A. (2024, April). *How do geometry courses for teachers support learning to teach? Results from an instructor survey*. Paper presented at the Annual Meeting of the American Educational Research Association, Philadelphia, PA.
Hitchman, T. (2017). Euclidean Geometry: An introduction to mathematical work. *Journal of Inquiry-Based Learning in Mathematics,* 45. http://jiblm.org/downloads/dlitem.php?id=108
Mahavier, W. S. (1999). What is the Moore method? *PRIMUS, 9(4), 339–354.*
Johnson, E., Keller, R., & Fukawa-Connelly, T. (2018). Results from a survey of abstract algebra instructors across the United States: Understanding the choice to (not) lecture. *International Journal of Research in Undergraduate Mathematics Education*, *4*, 254-285.
Johnson, E., Keller, R., Peterson, V., & Fukawa-Connelly, T. (2019). Individual and situational factors related to undergraduate mathematics instruction. *International Journal of STEM Education*, *6*, 1-24.
Lai, Y., Strayer, J., Anhalt, C., Bonnesen, C., Casey, S., Kohler, B., Lischka, A., Ross, A., Young, C. (2023). What instructional factors do prospective secondary teachers attribute to their learning? In Cook, S., Katz, B. & Moore-Russo D. (Eds.). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 556-565). Omaha, NE.
Laursen, S. L., Hassi, M. L., & Hough, S. (2016). Implementation and outcomes of inquiry-based learning in mathematics content courses for pre-service teachers. *International Journal of Mathematical Education in Science and Technology*, *47*(2), 256-275.
Mesa, V., Shultz, M., & Jackson, A. (2020). Moving away from lecture in undergraduate mathematics: Managing tensions within a coordinated inquiry-based linear algebra course. *International Journal of Research in Undergraduate Mathematics Education*, *6*, 245-278.
Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, *67*, 100986.
Schubring, G. (1989). Pure and applied mathematics in divergent institutional settings in Germany: The role and impact of Felix Klein. In D. Rowe and J McCleary (Eds.), *The History of Modern Mathematics, Volume II: Institutions and Applications* (pp. 170-220). Academic Press.
Serbin, K. & Bae, Y. (2023). How a Graduate Mathematics for Teachers Course "Helped Me See Things from a Different Perspective." In Cook, S., Katz, B. & Moore-Russo D. (Eds.). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (pp.* 343-351). Omaha, NE.
Shultz, M. (2020). *The rationality of college mathematics instructors: The choice to use inquiry-oriented instruction* (Doctoral dissertation). University of Michigan, Ann Arbor.
Smith, M. K., Vinson, E. L., Smith, J. A., Lewin, J. D., & Stetzer, M. R. (2014). A campus-wide study of STEM courses: New perspectives on teaching practices and perceptions. *CBE—Life Sciences Education*, *13*(4), 624-635.
Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*(2), 99-114.
Wong, R. (1970) Geometry preparation for high school mathematics teachers. *The American Mathematical Monthly, 77*(1), 70-78.
Yoshinobu, S., & Jones, M. (2011). An overview of inquiry-based learning in mathematics. *Wiley encyclopedia of operations research and management science*, 1-11.

Exploring Graphical Reasoning from Revised Responses to Function Composition Tasks

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Students learn about function composition, $(g \circ f)(x)$, in secondary school. From two given equations, one might identify the composite function algebraically via substitution, $g(f(x))$. But what about when functions are given as graphs? This study aims to explore how students reasoned graphically with their revised responses to function composition tasks. We previously identified common types of resultant graphs participants generated in trying to sketch – in a short time period – the composition of various graphically-depicted functions. This paper specifically examines the ways in which students revised their graphs, when given more time to do so and after having engaged in trying to compose functions in a wider variety of contexts, and the reasoning that they provided for these changes. As such, we extend prior work by exploring how students reason about function composition when provided unlimited time constraints on their activity.

Keywords: function composition; graphical reasoning

Students are exposed to function composition in secondary school (cf., CCSSM, 2010). Primarily, they are taught that $(g \circ f)(x) = g(f(x))$ – meaning, for two functions defined by equations, one can procedurally determine the composite function via algebraic substitution. Yet, such algebraic approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition (e.g., Ayers et al., 1988; Moore & Bowling, 2008). For example, in our prior work we showed how, when given 30 seconds to sketch a graph of the composite of two graphed functions, students did. In this study, we extend our prior work by exploring the ways students reason graphically about the composition of two functions when given time to revise their original sketched graphs, after having sketched graphs to represent the composition of functions represented graphically, algebraically, and tabularly.

Research Question, Background, and Framework

In this study, we seek to answer the following research question: *After having several experiences thinking about function composition, how and why do students revise their responses to graphical function composition tasks?*

Function and Function Composition in Extant Literature

Much literature in mathematics education has been devoted to the function concept (e.g., Breidenbach et al., 1992; Freudenthal, 1983; McCallum, 2019; Mirin et al., 2021; Paoletti et al., 2018). This has included exploring students' conceptions about real-valued functions (from $\mathbb{R}$ to $\mathbb{R}$) through the notion of covariational reasoning (e.g., Paoletti & Moore, 2017). In this context, a deep understanding of function demands students understand how quantities co-vary together – meaning, how changes in one quantity correspond to changes in the other quantity. Studies have shown that students can experience difficulties in understanding the relationship between co-varying quantities and their graphs (e.g., Carlson et al., 2002; Schoenfeld, 1985). More broadly, Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus.

Yet when it comes to function composition, very little research exists – and what does exist primarily uses algebraic perspectives (e.g., Ayers et al., 1988; Clark et al., 1997; Moore & Bowling, 2008). There is a dearth of literature on function composition, especially through a graphical lens. To address this gap, in a prior analysis (Chen et al., 2023), we identified common types of resultant composite function graphs participants generated in 30 seconds when given graphs, equations, and tables. Although we recognize that students primarily have difficulty with sketching graphs of composite functions given two graphs, in this analysis, we aim to gain further insight into students' reasoning about graphical representations of composite functions. This was done by examining the ways in which students revised their graphs after having further experiences with function composition via equations and tables – which are more familiar ways of thinking about function composition.

Graphical and Algebraic reasoning

A key premise underlying this research is that algebraic and graphical reasonings are different, where each provide distinct conceptions that are complementary for developing deep mathematical meanings. Broadly speaking, algebra is a mathematical field that examines particular structures, based on a set of objects (e.g., $\mathbb{R}$), and a binary operation(s) defined on them (e.g., $+$). Whereas geometry explores spaces that are related with distance, shape, and size (e.g., the Euclidean plane). To highlight this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, graphically, we depict a function with a graph (typically in the Euclidean plane), e.g., $f(x)$ is a line. Moreover, not only do these fields explore different things, but key ways of reasoning differ between them. Driscoll et al. (1999; 2007), for example, differentiated algebraic from geometric "habits of mind." In our context, for simplistic purposes, we use *graphical reasoning* to mean reasoning that relies on figures and shapes in the plane and their properties; *algebraic reasoning*, by contrast, relies on equations.

Methodology and Data Sources

To explore how students reason graphically about function composition, we had university students (n=143) in two mathematics and mathematics education programs (primarily from precalculus classes) sketch the composite function $(g \circ f)(x)$ given two given functions f and g. Participants responded to six tasks (yielding 858 responses), and the task order is important: they went from least familiar to most familiar (in the first four the functions were given as graphs, the next involved two equations, and the last gave a table of values). Students were given the opportunity to revise their original answers after a first attempt at all six tasks – meaning they could sketch a new graph for each of the six tasks after having been refreshed on what were likely more familiar function composition tasks (i.e., the latter using equations and tables).

Task Design

On Desmos, participants were asked to sketch within a limited timeframe (30 seconds, to capture their intuitions and reasonings), the composite function $(g \circ f)(x)$ for six pairs of functions. The majority of these pairs were represented graphically (Figure 1 displays an example Desmos page; Figure 2 displays the other 5 tasks). The purpose in providing participants with somewhat unusual, and non-algebraic forms of the functions, was to deter them from immediately converting to an algebraic equation to complete the composition. Doing so increased the likelihood that students would have to try to reason about function composition graphically based on the shapes of (or relationships represented in) the graphs. In the last two tasks, functions were given in the more familiar forms of an equation and a table. In this paper, we analyze their second attempts – which included either a new composite function graph or a modification of their original graph, as well as a written justification for their changes. Notably, their engagement in these revised tasks happened after participants: i) had more experience with graphing tasks; and ii) had reviewed function composition with – presumably – more familiar modes of reasoning (i.e., equations and tables).

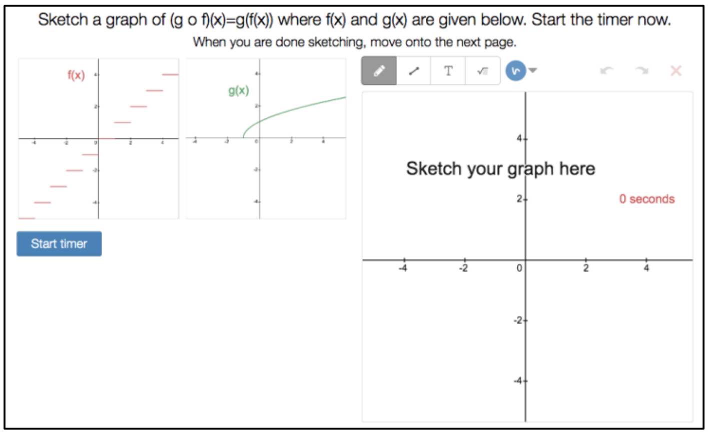


Figure 1. Task 2 in Desmos

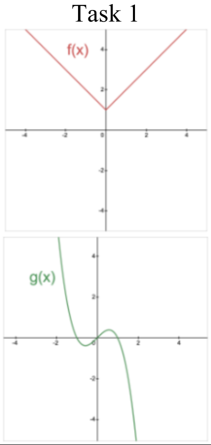
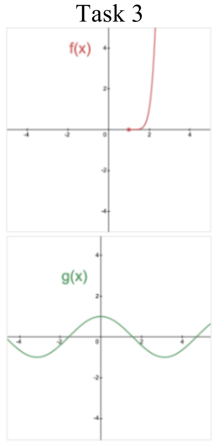
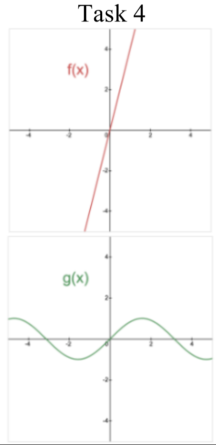
Task 1	Task 3	Task 4	Task 5
			$f(x) = x^2$ $g(x) = \sqrt{x}$ Task 6 (Presented in Table form) $f(x) = \{(1,4),(2,3),(3,1),(4,-1)\}$ $g(x) = \{(1,-2),(2,0),(3,1),(4,-1)\}$

Figure 2. Given functions for Tasks 1, 3, 4, 5, 6

Analysis and Coding

This study applied grounded theory as the data analysis technique (Stough & Lee, 2021). Out of 85 altered responses (from the possible 858), our analysis focused on (*n*=73) responses, from 43 distinct participants, in which participants provided (non-blank) justifications. Our analysis identified normative reasoning provided for changes to the initial graphs that were shared across participants. Notably, we aimed to create coding categories that could be applied across the entire set of tasks. We anticipated the revised graphs to include both incorrect and correct graphs of composite functions, and a diverse range of reasoning students may have used. We coded changes between their original and revised graphs as beneficial (+) if they went from incorrect to correct; non-beneficial (-) indicated no progress toward correctness. We generated and refined all categories of coding as a group; when there were uncertainties in how to code a graph, we resolved such cases together and made refinements to categories as needed.

Due to space constraints, we do not include all codes used in our analysis. However, Table 1 provides codes and definitions for the majority of codes – and, in particular, those we report on in this paper. Specifically, these are the codes used to determine students' reasoning for modifying their initial responses. Clarifying examples are given in the results section.

Table 1. Categories of students' reasoning for changing their graph

Code	Description of the reasoning for how or why participants made changes to the composite function $g(f(x))$ sketched
MIXED PROPERTY (MIXED PROP)	One graph put together with the property of the other graph
ALGEBRAIC THINKING (ALG)	Algebraic thinking involving equations or identifying types of functions (e.g., polynomial functions)
DISCRETE POINTS (DP)	Pointwise thinking involving specific points or identifying specific discrete points of the graph (e.g., $f(1)=2$, $g(2)=4$, so $g(f(1))=4$)
TIME	Citing a need for more time on the original tasks

Results

Table 2 provides an overarching summary of the reasoning used by those students who altered their answers after their first attempt. ALGEBRAIC THINKING (ALG) and MIXED PROP were the two dominant responses, so those will be the focus of our results. It is important to note, however, that TIME was also a major factor, with 19 out of 73 responses citing a need for more time on the original tasks as the reason for them altering their original response. Those who ran out of time did not provide any additional reasoning for their answer changes, thus indicating that one reason students revised their graphs was that they needed more time to think about how to compose the two functions. Since a total of 18 out of 73 total changes were beneficial, it reinforces the idea that additional time may provide more opportunities for participants to reason through the challenges and showcase their thinking. There were also a small number of responses (11) that fall under OTHER codes. However, the four most influential categories (ALG, MIXED PROP, DP, and TIME) will be the focus of this report.

Table 2. Summary of data for those with answer changes on Tasks 1-6

Code	Task 1-4	Task 5	Task 6	Total
ALG	4(+)/2(-)	3(+)/6(-)	2(-)	7(+)/10(-)
MIXED PROP	5(+)/14(-)	-	-	5(+)/14(-)
DP	1(+)/2(-)	-	4(+)	5(+)/2(-)
TIME	1(+)/13(-)	2(-)	3(-)	1(+)/18(-)

Note. The symbols (+)/(-) show beneficial/non-beneficial changes in their responses.

Given Graphs

In Tasks 1-4, students were given *graphs* of $f(x)$ and $g(x)$ and asked to sketch the composite function. In these tasks, about 9% of the responses were changed (49 out of 572). See Figures 3 and 4 for (-) and (+) examples (original graph in blue; revised graph in red).

Of those changed responses, 39% of them used MIXED PROP as their reasoning for how they generated their revised graph (19 out of 49). This seems sensible since MIXED PROP implies that students are utilizing and combining physical characteristics of the two graphs – though it was only beneficial in 5 of the 19 cases. On these tasks, 29% (14 out of 49) of responses mentioned timing issues. Perhaps this is because students are accustomed to approaching function composition algebraically, and thus required more time to orient themselves to these graphical representations. Notably, even with the additional time to revise, students still had difficulty – only 1 of the 14 responses that cited time issues made progress toward a correct composite function on these graphical tasks. 12% of students (6 out of 49) still attempted to use ALG to understand these graphical tasks, and this group seemed to make the most progress toward correct graphs. Finally, 6% of students used DP to make sense of the graphical representations (3 out of 49).

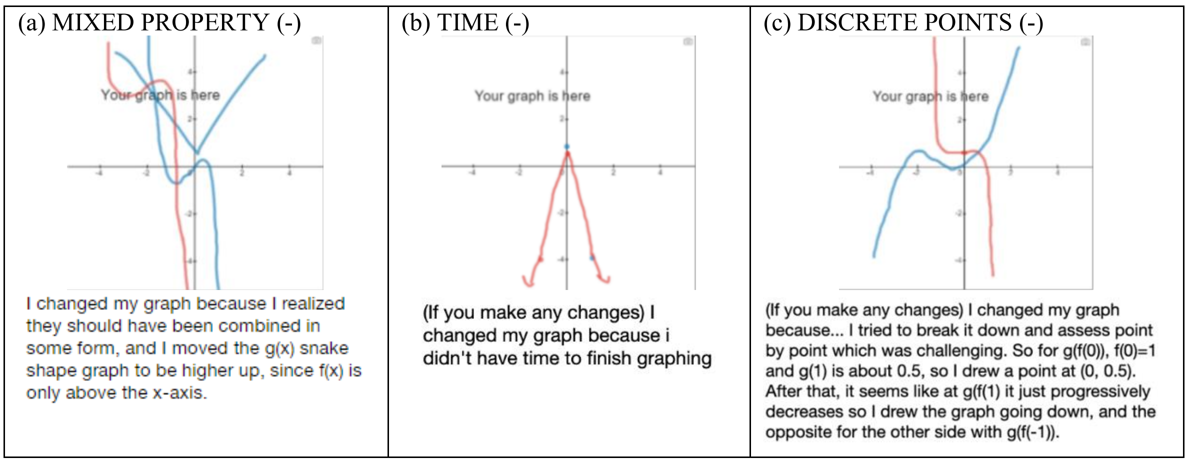


Figure 3. (Still) Incorrect examples from Task 1 of (a) "mixed property" reasoning, (b) "time" reasoning, and (c) "discrete points" reasoning (original graph in blue; revised in red)

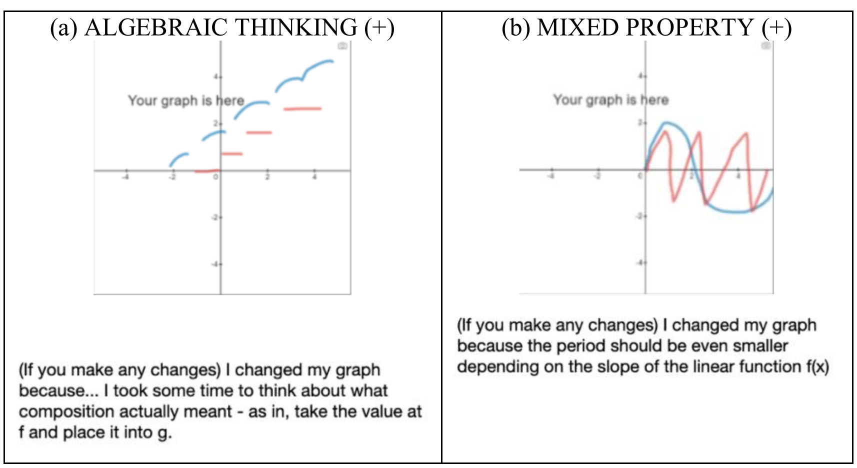


Figure 4. Correct examples from Task 2 of (a) "algebraic thinking" reasoning, and Task 4 of (b) "mixed property" reasoning (original graph in blue; revised in red)

Given Equations

In Task 5, students were given *equations* for $f(x)$ and $g(x)$ and were asked to sketch the composite function. In this task, about 8% of the responses were changed (11 out of 143). See Figure 5 (original in blue; revised in red). Of those changed responses, 82% of them used ALG as their reasoning (9 out of 11), and 18% cited TIME as their reason for changing their response

(2 out of 11). This result is unsurprising, since this task gave students equations, which lends to a more algebraic approach. Of those who changed their answers in Task 5, 27% (3 out of 11) changed their answer so that it improved. All 3 of the improved responses were coded as using ALG for their reasoning, as exemplified in Figure 5a. Interestingly, however, in 4 (of 11) cases students changed their answers from the correct response to the incorrect response. Each was coded as ALG and all were nearly identical to the response in Figure 5c, where x^2 and $\sqrt{x}$ were seen to "cancel" algebraically to x.

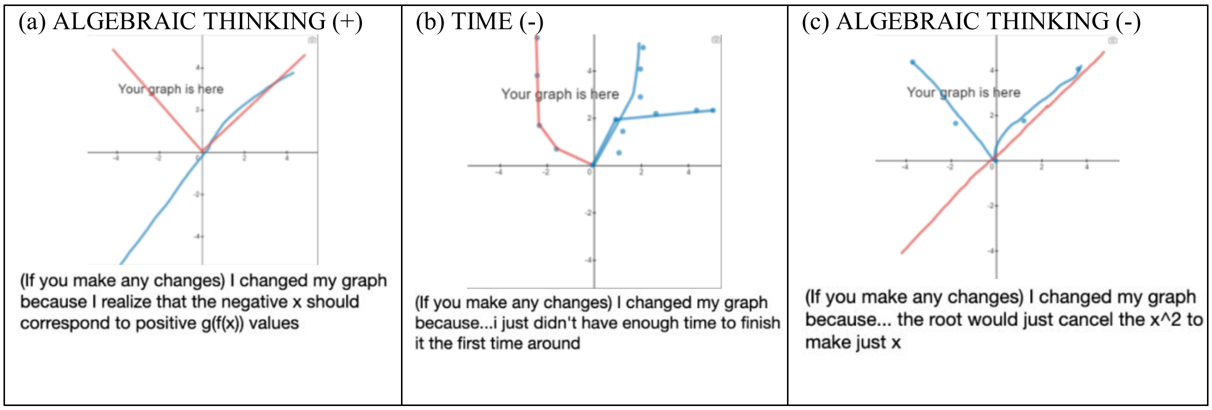


Figure 5. Examples from Task 5 of (a) (c) "algebraic thinking" reasoning, and (b) "time" reasoning (original graph in blue; revised in red)

Given Tables

Task 6 gave students *tables of values* and students were asked to sketch the composite function. In this task, about 9% of the responses were changed when students were given this opportunity (13 out of 143). See Figure 6 for examples (original in blue; revised in red). About 31% of responses (4 out of 13) used DP to explain their changed answers, which makes the most sense based on a table of values. Indeed, all 4 made improvements (see Figure 6a). (Notably, even if students decided to connect the correct set of points with a curve, we considered the composite graph (mostly) correct; the correct graph should just be the discrete set of points.) 23% (3 out of 13) cited an issue with TIME as their reason for changing their answers. 15% (2 out of 13) used ALG to change their answers.

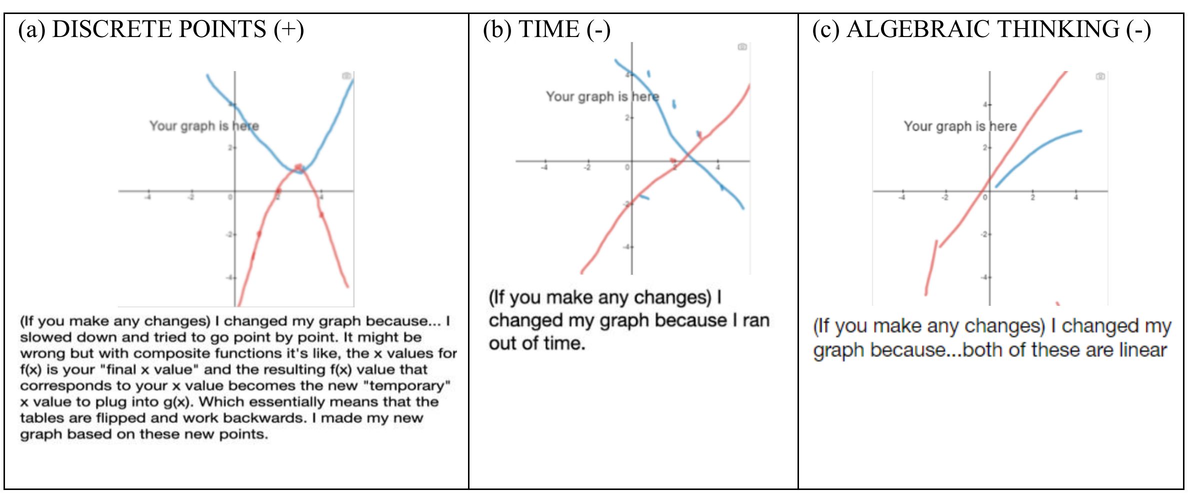


Figure 6. Examples from Task 6 of (a) "discrete points" reasoning, (b) "time" reasoning, and (c) "algebraic thinking" (original graph in blue; revised in red)

Conclusions and Implications

In this section, we highlight four important takeaways for the reader. First, perhaps unsurprisingly, with more time, students performed better. Approximately 25% of the students who edited their responses improved upon their original answer in some way (18 out of 73). This might imply that some students' original wrong responses were due to lack of time rather than lack of ability to perform the composite function task.

Second, when given the opportunity to revise, students seemed to rely primarily on an algebraic approach. Approximately 23% of revised responses cited ALG as their method (17 out of 73). This could imply that more time simply allowed students to rely on the more familiar algebraic manipulations – though, even still, less than half were able to determine the correct composite function. This result seems to suggest that students have not been supported in developing ways of reasoning graphically about function composition, and the extra time serves only to allow for attempts to reason algebraically by identifying approximate equations with which to work.

Third, we also saw that there can be a negative impact to an algebraic approach. All of the students who changed their answers from correct to incorrect (all on Task 5) used algebra in their reasoning for their revisions. This implies that students are using the familiar algebraic approaches even when it actually hinders their natural understanding of the problem! The dual-edged nature of algebraic thinking evident in these revised responses seems to affirm the broad aims of this work – exploring graphical reasoning approaches that support students in developing a richer conception of function composition.

Fourth, overall, the most productive approach in revised responses was based on pointwise thinking (DP) – though most of these were in Task 6. That is, using specific values of x, to evaluate and plot $(x, g(f(x)))$, was helpful reasoning that often led to a correct or mostly correct graphically composed function. Additionally, MIXED PROP seemed to be a promising approach on some "given graphs" (Tasks 1-4) – although identifying which properties of the two functions to combine represents an additional challenge. Both show potential for helping students develop graphical reasoning about function composition.

Collectively, the results presented here along with the results we presented in Chen et al. (2023) highlight that most college students have not been provided opportunities to develop rich meanings for function composition; function composition problems are difficult regardless of if there is a limited time frame and unfamiliar representations (Chen et al., 2023) or if there is an unlimited time frame and students have experienced more familiar representations (this report). Hence, more work is needed to identify further productive intuitions and ways of reasoning students may have that might be helpful to build on. We conjecture novel representations and digital tools might afford learners an opportunity to develop a richer conception of function composition, such as conceiving of function composition graphically or as representing a covariational relationship between multiple quantities. We call for future research to explore this possibility, as well as others, for supporting students in understanding function composition.

References

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education, 19*(3), 246-259.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378.

Chen, Y., Wasserman, N., & Paoletti, T. (2023). Exploring geometric reasoning with function composition. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (RUME)* (pp. 145-153). Omaha, NE: RUME.

Clark, J. M., Cordero, F., Cottill, J., Czarnocha, B., DeVries, D. J., St. John, D., Tolias, G., & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule? *Journal of Mathematical Behavior, 16*(4), 345-364.

Common Core State Standards in Mathematics (CCSSM). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics.

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness and periodicity. *Focus on learning problems in mathematics*, 5(3), 119-132.

Driscoll, M. J. (1999). *Fostering algebraic thinking: A guide for teachers, grades 6-10.* Portsmouth, NH: Heinemann.

Driscoll, M. J., DiMatteo, R. W., Nikula, J., & Egan, M. (2007). *Fostering geometric thinking: A guide for teachers, grades 5–10.* Portsmouth, NH: Heinemann.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structure*. D. Reidel.

McCallum, W. (2019). Coherence and fidelity of the function concept in school mathematics. In H.-G. Weigand et al. (Eds.), *The Legacy of Felix Klein* (pp. 79-90). ICME-13 Monographs. Springer.

Mirin, A., Milner, F., Wasserman, N., & Weber, K. (2021). On two definitions of 'function'. *For the Learning of Mathematics, 41*(3), 22-24.

Moore, K. C., & Bowling, S. A. (2008). Covariational reasoning and quantification in a college algebra course. *Proceedings of the 11th Annual Conference on Research in Undergraduate Mathematics Education*. RUME.

Paoletti, T. & Moore, K. C. (2017) The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior*, 48, 137-151.

Paoletti, T., Stevens, I. E, Hobson, N. L. F., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics*, 97(1), 93-109.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Stough, L. M., & Lee, S. (2021). Grounded theory approaches used in educational research journals. *International Journal of Qualitative Methods*, *20*, 16094069211052203.

Undergraduate instructor's perceptions of barriers to implementing culturally sustaining practices

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Rutgers University

Undergraduate math education is highly inequitable. One potential strategy to improve the unfair experiences and outcomes of historically marginalized groups in mathematics is to use culturally sustaining practices, which relate math content to student's culture and everyday lives. However, there is limited work exploring the applicability of culturally sustaining practices in the undergraduate mathematics setting. This study interviews seven math undergraduate instructors to better understand their perceived barriers to implementing culturally sustaining practices.

Keywords: undergraduate, equity, culturally sustaining practices

Introduction

Undergraduate mathematics education is highly inequitable. Black and Latin* students are frequently weeded out of introductory undergraduate math courses like calculus, prohibiting access to advanced level mathematics courses (Ellis et al., 2016; Larsen et al., 2017). This disproportionate exclusion of Black and Latin* students can result in isolating experiences for the few students who do persist to upper-level courses, who have diminished access to same-gender, same-race peer support (Borum & Walker, 2012).

One potential strategy to ameliorate the differential experiences and outcomes experienced by minoritized groups in mathematics is to relate math content to students' culture, experiences, and everyday lives (Civil, 2016). The framework of culturally sustaining pedagogy (CSP) (Paris, 2012) proposes a step towards equitable instruction that embeds course content in students' lives while providing them with political consciousness. However, there is limited work exploring the applicability of culturally sustaining practices in the undergraduate mathematics setting, with the exception of Adiredja and Zandieh (2020). Since most existing scholarship on CSP has been in the K-12 context, it is unclear the extent to which the same strategies work in the university setting. Thus, in order to better understand how to implement culturally sustaining practices in college classrooms, we might want to understand the barriers specific to the university context.

Accordingly, this study addresses the following research question: What do instructors of math undergraduates perceive to be the barriers to implementing culturally sustaining practices? Addressing this question presents three contributions. First, this work links the expertise gained from scholarship in K-12 equitable practices in mathematics to the undergraduate setting, where we have not taken up as much of the work from our colleagues in K-12. Second, there is little existing work documenting culturally sustaining practices in undergraduate classrooms. This work will explore instructors perceived barriers to implementing CSP. Finally, as Chazan et al. (2016) note, much of the work in mathematics teaching documents instructors' beliefs without considering the influences that institutions and various stakeholders can have on instructors' choices. This work answers Chazan et al.'s call for greater attention to institutional and greater societal context on the choices teachers make.

Literature Review

An increasing amount of scholarship has documented inequitable experiences in college mathematics experienced by Latin* and Black students and students of under-represented genders in mathematics (for a review, see Adiredja & Andrews-Larson, 2017; Leyva et al., 2021; McGee & Martin, 2011).

Leyva et al. (2021) document instructional practices and broader ideologies of mathematics ability (e.g. mathematics ability as innate; white and Asian people are better at math than Black or Latin* students) that create hostile classroom environments for historically marginalized students. While some work has illustrated cases of successful Black (Borum & Walker, 2012) and Latin* (Leyva, 2016) students, I am not aware of work that considers how more equitable instruction interacts with broader institutional and ideological factors. In other words, given various institutional obligations, how far can an instructor go in implementing culturally sustaining practices?

When done well, culturally sustaining practices can encourage critical dialogue, improve student performance on exams, and promote reflection on previously held social assumptions (for a review, see Aronson & Laughter, 2016). Indeed, Hubert (2014) found that students appreciated seeing their home lives reflected in the classroom, which in turn promoted engagement with mathematics content. However, implementing CSP is far from straightforward. Even for those instructors who would like to adopt CSP practices, some struggle with developing authentic connections between school mathematics and students' everyday lives (McCulloch & Marshall, 2011; Wager, 2012).

Unfortunately, there is still little work documenting supportive practices in undergraduate math education that disrupt commonplace racialized and gendered experiences (Leyva et al., 2022). Leyva et al. (2022) document supportive practices intended "for all" students that are necessary yet insufficient to supporting students of historically under-represented groups in mathematics. While instructional practices such as creating space for questions and mistakes and extending out of class support are important, they are limited in their ability to confront the broader influences of racism and patriarchy on classroom experiences. The limitations inherent to practices that do not explicitly address raced and gendered harm in the classroom calls for further work on practices like CSP that do attempt to directly address these harms.

Theoretical Frameworks

Culturally Sustaining Pedagogy

This study uses the theory of culturally sustaining pedagogy (Paris, 2012), reformulated from Ladson-Billings' culturally relevant pedagogy (1995, 2021). Ladson-Billings developed culturally relevant pedagogy based on her studies of effective instructors of African American students. She found that effective instructors accomplished three key objectives, promoting: student achievement, cultural competence, and critical consciousness.

CSP emerged from an acknowledgement that cultural relevance itself is insufficient. Instead, Paris (2012) noted the importance of going beyond curricula that uncritically cite students' cultures towards one that sustains them. As Ladson-Billings (2021) notes, the wide popularization of culturally relevant pedagogy has led to uses and abuses of the theory. As conceptualizations of culturally relevant pedagogy morph over time, the theory changes and gains new subtleties. Culturally sustaining pedagogy combines students' cultural knowledge with the aim of critical consciousness, or an understanding of current sociopolitical realities. In this study, I used CSP to identify a set of recommendations to discuss with college instructors.

Theory of Practical Rationality

To make sense of why my participants would or would not adapt elements of CSP in their teaching, I used Herbst and Chazan's (2003) theory of Practical Rationality of Teaching. The theory considers the different societal and institutional obligations that affect instructors' choices. Chazan et al. (2016) contend that instructors cannot simply do as they please, identifying various obligations instructors must consider: professional obligations, obligations to the discipline of mathematics, obligations to students as individuals, and obligations to society and its imperatives. In this study, I use Practical Rationality to identify instances when different obligations conflict with or support CSP. Like Herbst and Chazan, I make no moral judgments about the choices instructors make in prioritizing certain instructional commitments over others. The theory also does not make any normative judgments about whether an instructor's perceived obligations are truly obligatory. Instead, it is sufficient that an instructor perceives an obligation as relevant to the instructional choices they make.

Methods

Participants were selected among current doctoral students in a mathematics department at a medium-sized public university in the northeast. My participants were PhD students in Mathematics with at least one semester of undergraduate teaching experience as an instructional teaching assistant (TA). I excluded participants who only had experience as grading TAs or whose only teaching experience was prior to starting their PhD. The choice of target population of mathematics PhD students as opposed to current mathematics professors was intentional. First, mathematics PhD students represent the cohort of future college-level mathematics educators. Additionally, as a younger cohort, current mathematics PhD students may be more amenable to CSP practices. While general support of equitable practices is not a requirement to participate in the study, I anticipated that participants would provide far richer insights about the possibilities of equitable instruction if they themselves were invested in such practices.

Data Collection

I invited participants to engage in a single one-hour semi-structured interview about how they might adapt various culturally sustaining practices from the K-12 mathematics context to the undergraduate classroom. The interview was divided into roughly two parts. In each section, I asked the participant to consider a recommendation from New York State Education Department's (NYSED) Culturally Responsive-Sustaining Education Framework. The first section reviewed a selection of the recommendations for making a welcoming and inclusive environment. The second half of the interview discussed practices related to identifying inclusive curriculum and assessment. I picked these themes as they seemed the most relevant and potentially challenging for the undergraduate math context. The remaining themes were: fostering high expectations and rigorous instruction and engaging in ongoing professional learning and support. While there is no agreed set of recommendations for what constitutes CSP in mathematics education (Thomas & Berry III, 2019) the NYSED recommendations provide actionable recommendations constructed by reputable scholars in the field.

Each question opened with an elicitation in which I read one of the NYSED recommendations for culturally sustaining practices in K-12 mathematics to the participant. The participant was then asked the following questions:

1. What are some of the ways you do or could adapt these recommendations to your classroom?
2. Are there any constraints that would make it hard to adapt this recommendation?

3. What kind of resources would you need to implement this recommendation?
4. What factors get in the way of implementing this practice? OR What supports have made these practices possible for you?

Throughout data collection, I engaged in preliminary data analysis via in-process memos in which I pre-coded (Saldaña, 2009) data, documenting any initial impressions or findings.

Coding

Data analysis merged inductive analyses in first cycle coding and deductive methods in the second and third cycles. During the data collection and pre-coding stage, I observed that participants frequently noted moments of conflict in which it became difficult to reconcile certain CSP recommendations with other obligations that they held. I decided to use Herbst and Chazan's theory of Practical Rationality to foreground instructional obligations. To capture these moments of tension and the underlying values that were in conflict, I blended versus and values coding methods (Saldaña, 2009) for my first cycle of coding. I also coded for the various institutional obligations that participants indicated in response to my follow-up questions about barriers to their practice. In summary, the first cycle yielded two categories of codes: codes indicating participants' values, some of which were paired as versus codes, and codes indicating barriers towards CSP practices.

After first cycle coding, I reanalyzed the data using sensitizing concepts (Charmaz, 2014) from the theory of culturally sustaining pedagogy and aligned asset pedagogies. I categorized the values and barrier codes from the first cycle of coding based on their alignment with CSP. By the third cycle of coding, the code corpus consisted of: versus codes indicating tensions between different instructional commitments, values codes that justified instructional moves, and codes aligned with the literature on culturally sustaining practices. However, there were notable tendencies in the ways in which these coding schemes overlapped. Frequently the versus codes contained two different values codes. Furthermore, the values codes could be roughly organized on whether they aligned with culturally sustaining practices. I then used concepts from broader critical theory (e.g. asset-based pedagogies, resource pedagogies) as parent codes to organize the values codes into those that were broadly aligned with CSP and those that were not.

Findings

Conflicting Commitments

In some cases, participants indicated commitments that conflicted with a desire to implement practices that were culturally sustaining. One of the NYSED recommendations advised instructors to implement restorative justice circles to address harm done in the classroom. Restorative justice is an alternative that focuses on redressing harms caused instead of punishing the offender. I provided a brief description of restorative justice and offered academic dishonesty as an example of a potential harm. I then asked how participants could make sense of this recommendation in their own practice and any supports they would require. In response, one participant, Michael, indicated:

> Going along the lines of the example that you gave, the professor needs the flexibility to be able to adjudicate these things on their own. At [this university], we have a great bit of discretion, as TAs or as professors, about how we handle instances of academic integrity. So at [this university], the structure is actually pretty fertile for us to do this sort of thing. But at other institutions, they are very, very, very strict. To give an example, [at other university] … I am pretty sure that if you cheat on the exam, they just expel you. […] So in order to allow [for restorative approaches], you need to have this discretion, because

> otherwise it's simply impossible. You don't want to be in a situation where common sense or sympathy as a human being is telling you, “look, I can handle this in a better way.” But you have your own job to be concerned with, and, unfortunately, if they're going to come after you if you do something, regrettably, you're gonna have to do what's in the interest of your job.

Michael indicated that professor discretion was necessary in order to implement a restorative approach to academic dishonesty. Ultimately, even if an instructor’s “common sense or sympathy” led them to favoring a restorative approach, the possibility of institutional repercussions could require one to “do what’s in the best interest of [their] job.” I interpreted Michael’s reference to common sense or sympathy as his perceived obligation to the student conflicting with his professional obligations to abide by institutional rules. From Michael’s perspective, attempting to adopt the culturally sustaining pedagogical practice of restorative justice may be rendered an impossibility depending on institutional factors.

Like Michael, Owen described institutional obligations that impeded his ability to implement CSP. In reference to the recommendation to build rapport with students and elicit their opinions and concerns, I asked Owen what constraints he felt in adopting the recommendation. He noted that the institutional requirement of having to escort students to the bathroom during exams and practice of failing 30-40% of business students taking calculus make it difficult to build rapport with students. He mentioned apologizing to students to push back against these constraints. He empathized with the business students and recalled attempting to push back against institutional practices:

> And I think I did do that one day. I think I do try to bring this up sometimes like, “I don't want you to be judging yourself over what happens in the classroom. Because that's sort of what's going on. You're getting grades and you're going to get into this school or not and be allowed to go on or not. And that's stupid and frustrating. I don't believe in it.” I don't think that's a huge affront to the system. I think it'd be much more dangerous to start criticizing the professor. I mean, not that I'm opposed to that, but I'm just saying. You can get fucked up for that. And the other thing is also, in terms of like maintaining rapport, you don't want to tell somebody what they're doing is pointless or stupid…

Owen felt it was hard to build rapport with his students if he had to patronise them by walking to the bathroom or punishing them with bad grades. Interestingly, one of Owen’s perceived commitments appears to be the disruption of the very institutional practices that hinder student autonomy and obstruct CSP. While following his professional obligation to follow the professor’s instructions and escort the student to the bathroom, he also attempted to abide by his obligation toward the student by making a joke about the situation and apologizing to them.

In Michael’s case, he felt that institutional barriers could obstruct his attempts to follow CSP practices. If it came down to choosing between making the best professional decision or acting out of sympathy towards the student, he felt that one must choose what’s in the best interest of their job. On the other hand, Owen, by apologizing to students or directly calling institutional practices “stupid and frustrating,” described methods for subverting institutional barriers in favor of aligning with more CSP-like practice. For Owen, the presence of constraints against student agency seemed to generate novel obligations to acknowledge the constraints and be candid with students.

Limited knowledge of how to implement culturally sustaining practices

In some cases, participants indicated that their lack of knowledge of how to properly implement certain culturally sustaining practices served as a barrier. One of the presented recommendations is to "feature and highlight resources written and developed by traditionally marginalized voices that offer diverse perspectives on … identities traditionally silenced or omitted from curriculum" (New York State Education Department, n.d.) In response, Max commented:

> The main thing that prevents me from doing this is that I don't put enough time into it. If I'm preparing for class, I'm mostly thinking about how to explain—I'm gonna use the example of derivatives again—rather than looking back and saying "can I find a useful historical example of a woman or disabled person who contributed to this area?" and highlight that. I've had a couple of professors who have taken extra time to highlight, for example, women who have contributed to an area of math that we're learning about and I've enjoyed that part. It's always nice when there's a little break in the lemma-lemma-theorem-proof structure to talk about history. That's definitely enjoyable, but it's just extra effort for the professor. […] I try to set pretty high honesty requirements of myself, and I'm not a historian. So one constraint would be I want to tell my students the truth about what happened and what was important in the history of math. I guess that just adds to the amount of time it would take as input to make this work well is that I don't want to just read a BuzzFeed article about Mayan mathematical practices and just quote that without understanding what I'm really talking about. […] If my PhD had a requirement to take a math history class to graduate, then I would love that, and I would know a lot more about that by the time I was teaching it.

Max appears to be supportive of the recommendation, recalling previous "enjoyable" experiences in which professors highlighted historical context in their courses. While he does refer to this work as "extra effort" by the professor, he later indicates that he would appreciate having a requirement embedded in his PhD program to take a math history course. While Max may not view this recommendation as essential, he appears open-minded towards it. However, Max mentions that the amount of time it would take to fully research and understand historical examples of the work of under-represented mathematicians would be substantial. Part of his hesitance stems from wanting to present historical content accurately and his awareness that he may lack the sufficient background knowledge as he is "not a historian."

While Max does open by citing time as a primary constraint, these instances appear to be rooted in the time it would take for Max to prepare and learn any material on his own. He closes by mentioning a previous course in his PhD program would alleviate some of his concerns. It appears that Max's commitment towards presenting material accurately conflicts with the recommendation to include under-represented perspectives in mathematics, given the lack of prior knowledge that PhD students like Max tend to have.

Discussion

The data support two main claims. First, instructors perceive various commitments that may conflict or align with culturally sustaining practices. Second, instructors also indicate that a lack of knowledge of how to properly implement certain CSP practices serves as a barrier to implementation.

In their work documenting the limitations of "for all" practices, Leyva et al. (2022) argue that equitable instruction must also directly confront broader racialized and gendered discourses that permeate mathematics classrooms. If we are to ask instructors to directly subvert institutional ideologies, it would be crucial to understand the various perceived commitments instructors may be balancing as employees of an inequitable institution. Regardless of whether these commitments are commonly perceived or even misplaced, these commitments serve as barriers to instructors who may desire to implement equitable practices but feel unable to.

Previous work described a need for instructors to have deep pedagogical content knowledge (Enyedy & Mukhopadhyay, 2007) as well as knowledge of how students use mathematics in their everyday lives outside of school (Wager, 2012). This data suggests that instructors also felt a lack of historical knowledge about mathematics as a discipline provided a barrier to enacting CSP effectively. The instructors perceived institutional obligations and obstructions even when armored with appropriate training and knowledge would be an interesting direction for future research.

This study explored the question: what do instructors perceive to be barriers to implementing culturally sustaining practices? Interviews with seven math doctoral students who have taught undergraduates, revealed two main findings. Instructors balance conflicting obligations, some in service of culturally sustaining practices and others that directly conflict with CSP. Second, instructors expressed insufficient knowledge and training to implement some CSP practices. Given the need for instructors to disrupt institutional discourses about mathematics, it is important to better understand how instructors may operate and balance conflicting commitments to equitable practices and to the institution to which they are accountable. Furthermore, this work is part of a larger call to situate instruction in its institutional context (Chazan et al., 2016) and potentially understand the possibilities for instructional disruptions given inequitable institutional contexts. Previous literature (e.g. Leyva et al., 2022) has viewed oppressive ideological and institutional influences in tandem with instructional ones. One potential route for further work would be an understanding of the mechanism behind instructional disruptions and how they interact with racialized and gendered institutional and ideological discourses. For example, if an instructor effectively provides counter-narratives against a discourse that mathematics talent is innate, what are the limitations and affordances of a single instructor doing this amidst an institution that continues to perpetuate the dominant discourse?

Limitations

This study presented only a narrow slice of the many perspectives instructors might hold. Of our participants there were: four white men, one Asian man, and two white women. It is possible that the types of responses we would have received would have different had our sample population included more participants of under-represented identities in mathematics. However, generalizing to the population of future professors was not the goal of this study. Additionally, there is a possibility, using a different theoretical lens, that instructors did not truly perceive the obligations they shared with me as legitimate concerns and instead used them as part of a rhetorical strategy to uphold systems of white supremacy in mathematics (e.g. Bonilla-Silva, 2001). While this may have been the case for any of the participants, I argue that it is nonetheless valuable to understand the types of commitments instructors indicate may be in conflict with equitable practices such as culturally sustaining pedagogy.

References

Adiredja, A. P., & Andrews-Larson, C. (2017). Taking the sociopolitical turn in postsecondary mathematics education research. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 444–465. https://doi.org/10.1007/s40753-017-0054-5

Adiredja, A. P., & Zandieh, M. (2020). The lived experience of linear algebra: A counter-story about women of color in mathematics. *Educational Studies in Mathematics*, *104*(2), 239–260. https://doi.org/10.1007/s10649-020-09954-3

Aronson, B., & Laughter, J. (2016). The Theory and Practice of Culturally Relevant Education: A Synthesis of Research Across Content Areas. *Review of Educational Research*, *86*(1), 163–206. https://doi.org/10.3102/0034654315582066

Bonilla-Silva, E. (2001). *White supremacy and racism in the post-civil rights era*. Lynne Rienner Publishers.

Borum, V., & Walker, E. (2012). What Makes the Difference? Black Women's Undergraduate and Graduate Experiences in Mathematics. *The Journal of Negro Education*, *81*(4), 366–378. JSTOR. https://doi.org/10.7709/jnegroeducation.81.4.0366

Charmaz, K. (2014). *Constructing grounded theory* (2nd ed.). SAGE Publications, Inc.

Chazan, D., Herbst, P. G., & Clark, L. M. (2016). Research on the Teaching of Mathematics: In D. H. Gitomer & C. A. Bell (Eds.), *Handbook of Research on Teaching* (5th Edition, pp. 1039–1098). American Educational Research Association; JSTOR. http://www.jstor.org.proxy.libraries.rutgers.edu/stable/j.ctt1s474hg.22

Civil, M. (2016). STEM learning research through a funds of knowledge lens. *Cultural Studies of Science Education*, *11*(1), 41–59. https://doi.org/10.1007/s11422-014-9648-2

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 Times More Likely to Leave STEM Pipeline after Calculus Compared to Men: Lack of Mathematical Confidence a Potential Culprit. *PLOS ONE*, *11*(7), e0157447. https://doi.org/10.1371/journal.pone.0157447

Enyedy, N., & Mukhopadhyay, S. (2007). They Don't Show Nothing I Didn't Know: Emergent Tensions between Culturally Relevant Pedagogy and Mathematics Pedagogy. *The Journal of the Learning Sciences*, *16*(2), 139–174. JSTOR.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics*, *23*(1), 2–14.

Hubert, T. (2014). Learners of mathematics: High school students' perspectives of culturally relevant mathematics pedagogy. *Journal of African American Studies*, *18*(3), 324–336.

Ladson-Billings, G. (1995). But that's just good teaching! The case for culturally relevant pedagogy. *Theory Into Practice*, *34*(3), 159–165. JSTOR.

Ladson-Billings, G. (2021). Culturally relevant pedagogy 2.0: A.k.a. The remix. *Harvard Educational Review*, *Spring 2014*(84), 74–86.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. *Compendium for Research in Mathematics Education*, 526–550.

Leyva, L. A. (2016). An intersectional analysis of latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education*, *9*(2), 81–121.

Leyva, L. A., Amman, K., Wolf McMichael, E. A., Igbinosun, J., & Khan, N. (2022). Support for all? Confronting racism and patriarchy to promote equitable learning opportunities through undergraduate calculus instruction. *International Journal of Research in*

Undergraduate Mathematics Education, *8*(2), 339–364. https://doi.org/10.1007/s40753-022-00177-w
Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing Racialized and Gendered Mechanisms of Undergraduate Precalculus and Calculus Classroom Instruction. *Cognition and Instruction*, *39*(1), 1–34. https://doi.org/10.1080/07370008.2020.1849218
McCulloch, A. W., & Marshall, P. L. (2011). K-2 teachers' attempts to connect out-of-school experiences to in-school mathematics learning. *Journal of Urban Mathematics Education*, *4*(2), 44–66.
McGee, E. O., & Martin, D. B. (2011). "You Would Not Believe What I Have to Go Through to Prove My Intellectual Value!" Stereotype Management Among Academically Successful Black Mathematics and Engineering Students. *American Educational Research Journal*, *48*(6), 1347–1389. https://doi.org/10.3102/0002831211423972
New York State Education Department. (n.d.). *Culturally responsive-sustaining education framework*.
Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational Researcher*, *41*(3), 93–97. https://doi.org/10.3102/0013189X12441244
Saldaña, J. (2009). *The coding manual for qualitative researchers*. SAGE Publications, Inc.
Thomas, C. A., & Berry III, R. Q. (2019). A Qualitative Metasynthesis of Culturally Relevant Pedagogy & Culturally Responsive Teaching: Unpacking Mathematics Teaching Practices. *Journal of Mathematics Education at Teachers College*, *10*(1), 21–30.
Wager, A. A. (2012). Incorporating out-of-school mathematics: From cultural context to embedded practice. *Journal of Mathematics Teacher Education*, *15*(1), 9–23. https://doi.org/10.1007/s10857-011-9199-3

Resource Use and Student Engagement in Multivariable Calculus

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This article examines the relations between content, resource use, and the pedagogy of instructors in multivariable calculus. Content included topics that typically appear in multivariable calculus which were grouped into these general areas: introductory ideas, differential calculus, integral calculus, and vector calculus. The resources used included digital resources, 3D-printed models, and other physical resources. Pedagogy practices included both collaborative and individual in-class learning activities, instructor demonstrations, and homework. We also consider the instructor's assessment of the activities they use. Data were obtained from three instructors working at three different types of institutions in the United States: a community college, a small selective undergraduate college, and a large comprehensive research university. The results provide information about instructional needs, as instructors implement resources in multivariable calculus, and about patterns related to the content they cover, the technology they use, and their pedagogical practices.

Keywords: Multivariable calculus, digital resources, TPCK, 3D-printed models

Introduction

Learning multivariable calculus (MVC) is a challenge for students. After operating primarily in two-dimensional worlds to study algebra, pre-calculus, and single-variable functions in calculus, students move to MVC, which typically involves two-variable functions. This introduces a new situation for students, as they must now generalize from the two-dimensional context of their previous studies to the MVC context. Doing so requires students to model and visualize the mathematical objects and their relations in three-dimensional (3D) space. As an instructional aid in this regard, instructors can leverage resources to help students bridge the gap from two to three dimensions, providing learning opportunities that are closer (Parzysz, 1988) to MVC topics than drawing on a chalkboard. To clarify, by resources we mean digital technology, digitally generated technology (e.g., 3D-printed models), or other physical artifacts.

However, the use of resources is not ubiquitous in MVC instruction. This leads to questioning how to best support instructors' use of resources for teaching MVC. In order to provide this support, it is imperative to first try to understand instructors' choices of resources and how resource selection may depend on the topic at hand or a particular pedagogy. One way to initiate this quest is to investigate the relationship between MVC instructors' use of resources for topics in the MVC course and their pedagogical intentions when leveraging resources to engage students in certain types of activities. For this case study, we consider data obtained from three MVC instructors who reported using the digital resource CalcPlot3D, 3D-printed models, as well as other physical resources in their MVC instruction.

To understand the relation between content, resource use, pedagogical practices, and instructor self-assessment of activity quality, we use the following research questions.

- Which topic areas of MVC seem more amenable to resource adoption? What types of resources are adopted?
- Which topic areas of MVC allow for student-centered activities?
- Which topic areas of MVC have activities that instructors assess as being of good or excellent quality?
- Are any patterns noted above that are specific to certain instructors in the sample?

Literature Review

The survey study by Martínez-Planell and Trigueros (2021) shows there is an increasing number of studies dealing with the teaching and learning of multivariable calculus (MVC). However, there is not much research related to the use of digital resources in MVC (Trigueros et al., 2023). There are some studies that report improved learning with the use of physical manipulatives, including tangible surfaces and 3D-printed models (e.g., McGee et al., 2012, 2015; Paul, 2018; Sherer et al., 2013; Wangberg, 2020; Wangberg et al., 2013). A few articles consider the use of digital technology like GeoGebra, Maple, and Mathematica as an aid in visualization, to foment student discussion, and to help bridge the gap between single variable and multivariable calculus (Alves, 2012, Ingar 2014). Some of the studies using digital technologies focus on specific topics or areas of MVC, for example, Trigueros et al. (2023) for basic ideas of two-variable functions, Rojas et al. (2019) for directional derivative, Ingar and Silva (2019) for extrema of two-variable functions, Alves (2014) for using second-order Taylor polynomials for approximating two-variable functions near critical points and relating this to the second derivative test, Henriquez (2006) for multivariable integration, particularly for drawing complex regions and finding limits of integration, and VanDieren et al. (2020) for the cross product. There are also some articles discussing the effect of using digital resources for an entire MVC course (e.g., Carvalho & Pereira, 2011; Gemechu et al., 2018; Habre, 2001). Some more recent articles consider the use of virtual or augmented reality (Cheong, 2023, Jones et al., 2022; Kang et al., 2020; Karabina et al., 2023). Yet, overall, there is much to learn about how resources, especially digital resources, can support the teaching and learning of MVC. Further, all these studies primarily focus on student learning. There is only one published study we know about that considers the use of resources for teaching MVC from the point of view of the instructors' needs (Martínez-Planell & Moore-Russo, 2023). This research is an extension of that previous study, where more data were obtained about the relation between content, pedagogy, and resources for teaching MVC.

Theoretical Framework

We use the Technological Pedagogical Content Knowledge (TPCK) model (Mishra & Koehler, 2006; Koehler & Mishra, 2009) as a theoretical framework for this study. This framework of instructor knowledge builds on the Pedagogical Content Knowledge model initially introduced by Shulman (1986) while also considering the role of technology in instructional decisions and actions. Koehler and Mishra (2009) situate the flexible knowledge needed for teaching as a "complex interaction among three bodies of knowledge: content, pedagogy, and technology" (p. 60). We adopt Mishra and Koehler's (2006) terms "pedagogy" and "content" to mean respectively the "processes and practices or methods of teaching and learning and how [they] encompass educational purposes, values, and aims (p. 1026)" and "the actual subject matter that is to be learned or taught (p. 1025)" respectively. We also adopt their TPCK stance on technologies as being both more traditional items (e.g., pencils and chalkboards) and what they refer to as newer technologies, including digital resources that include computers, tablets, and mobile devices as well as the digital applications they use.

Mishar and Koehler (2006) do not treat technology, pedagogy, and content as mutually exclusive domains. Instead, there is an interplay between the three that is noted in Figure 1 by the overlapping intersections in the TPCK Venn Diagram. In fact, certain resources are more likely to be employed for instructional tasks due to the affordances and capabilities of the resources (Koehler & Mishra, 2008), the fit between the characteristics of the resources and the characteristics of the task at hand (Goodhue & Thompson, 1995), and the instructor's perceptions of how useful and easy-to-use the resources are (Davis, 1989).

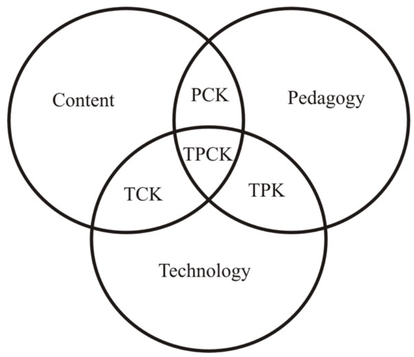

Figure 1. TPCK: Technological pedagogical content knowledge.

Methodology

We used a convenience sample of three MVC professors who employ the CalcPlot3D digital platform, digitally generated 3D-printed models, and other physical artifacts as aids in their MVC instruction. The professors taught at three different universities in three different states in the USA and had taught MVC anywhere from 12 to almost 30 times over their respective academic careers. Professor A teaches at a public community college with over 10,000 students in the Northeast. Professor H teaches at a small selective undergraduate college with about 4,000 students in the Midwest. Professor T teaches at a large public comprehensive research university with over 35,000 students in the East. The three were participating for a second year in a project involving CalcPlot3D and 3D-printed models. The data obtained came directly from the professors, and informal interviews were used to clarify their responses when needed.

The data were instructors' responses to a list of 35 common MVC topics, which was grouped into the following general *content areas* during analysis: introductory information, differential calculus, integral calculus, and vector calculus. The professors were given this list and asked to respond to questions about the types of resources they used in teaching each topic, describe the activities in which the resources were used, and rate the quality of each activity. An activity was the unit of analysis. The professors were asked to self-analyze their instruction using the following codes under each italicized coding category: *resource type* (digital resources, 3D-printed models, other), *pedagogical practice* (instructor demonstration, individual in-class student activity, collaborative in-class student activity, student homework), and *instructor rating* (i.e., evaluation of each activity as excellent, pretty good, needs work, no longer used).

Results

Table 1 displays the counts for several coding categories. However, we will primarily discuss the more qualitative aspects of the data in this section based on the research questions.

Table 1. Activity counts by pedagogical practice, instructor rating, and instructor for each content area and resource type.

Content Area	*Resource Type* Digital Resource	 3D-Printed Model	 Other Physical Resource
Introductory Information	**39 activities** *pedagogical practice* 5 ind act, 37 demos, 10 hwk* *instructor rating* 9 excellent, 30 good	**3 activities** *pedagogical practice* 3 collab *instructor rating* 2 excellent, 1 good	**11 activities** *pedagogical practice* 3 collab, 4 demos *instructor rating* 7 good, 4 not used
Differential Calculus	**19 activities** *pedagogical practice* 2 collab, 1 ind, 18 demos, 4 hwk *instructor rating* 9 excellent, 9 good, 1 needs work	**10 activities** *pedagogical practice* 10 collab *instructor rating* 7 excellent, 2 good, 1 needs work	**4 activities** *pedagogical practice* 1 collab, 3 hwk *instructor rating* 1 excellent, 3 good
Integral Calculus	**21 activities** *pedagogical practice* 21 demos, 1 hwk *instructor rating* 3 excellent, 14 good, 4 needs work	**6 activities** *pedagogical practice* 6 collab *instructor rating* 3 good, 3 not used	**0 activities**
Vector Calculus	**21 activities** *pedagogical practice* 21 demos, 4 hwk *instructor rating* 3 excellent, 15 good 3 needs work	**4 activities** *pedagogical practice* 3 collab, 1 demo *instructor rating* 2 excellent, 2 not used	**0 activities**

* Instructor H allows students to use CalcPlot3D for all homework, but she did not detail how this resource is leveraged. For this reason, we did not count her homework assignments.

We start with the first research question, dealing with TCK, by considering the activities as well as the *content areas* that were more amenable to resource adoption. There were 138 activities reported by the three instructors. Of that total, there are 53 activities dealing with introductory ideas (38%), 33 with differential calculus (24%), 27 with integral calculus (20%), and 25 with vector calculus (18%). Of the 138 activities, 100 (72%) use digital technology (CalcPlot3D), 23 (17%) use 3D-printed models, and 15 (11%) use other resources. This shows there was a clear preference among the participating instructors for using digital technology.

As for *resource type*, the 100 activities that used digital resources were spread across all four *content areas*. The most activities (39%) were employed for introductory ideas with a near even split for differential (19%), integral (21%), and vector (21%) calculus covering every individual topic area except limits and continuity and the chain rule.

Ten of the 23 activities (43%) used with 3D-printed models were for differential calculus. This suggests that instructors find differential calculus more natural or amenable to the design of activities based on 3D-printed models. There are two topics for which the three instructors used 3D-printed models (functions of several variables, partial derivatives), and five topics for which two of the three instructors used 3D-printed models (maxima and minima, Lagrange multipliers, triple integrals in rectangular, cylindrical, and spherical coordinates). There were six other topics for which a single instructor used 3D-printed models. This suggests that the use of 3D-printed models and their implementation should be further explored.

Eleven of the 15 activities were used with resources other than digital or 3D-printed addressed introductory information. The other resources include K'nex rods, a desktop coordinate system, large vector props made from wooden dowels, extendable vectors, a hill outside class, and static images on paper. For the most part, the use of these resources predated the adoption of digital and digitally based resources and attest to instructors' initial TPCK and motivations for engaging in a project to work on using digital and digitally generated resources.

For the second research question, we consider *pedagogical practices* to determine which class activities (by *content area*) allow for more student-centeredness (i.e. individual and collaborative in-class activities). Eleven of the 53 activities for introductory ideas (21%), 14 of the 33 differential calculus activities (42%), 6 of the 27 integral calculus activities (22%), and 3 of the 25 vector calculus activities (12%) were student-centered. All three instructors reported valuing student engagement, but their TPCK allowed them to design more student-centered activities for differential calculus than other areas, especially for vector calculus.

There were only two topics for which all three instructors designed in-class (individual or collaborative) student-centered activities (functions of several variables, partial derivatives) and five topics for which two of the three instructors designed student-centered activities (maxima and minima, Lagrange multipliers, and triple integrals in rectangular, cylindrical, and spherical coordinates). These topics suggest commonalities in instructors' PCK. At the same time, there were 11 topics for which a single instructor designed student-centered activities.

As for homework, which is also student-centered, but not completed in class, Instructor H shared that she allowed students to use CalcPlot3D on all homework, but she did not provide explicit details. There were 6 topics where the other two instructors expected students to leverage digital resources to complete their homework (equations of lines and planes; vector functions and space curves; functions of several variables; tangent plane and linear approximations; maxima and minima; Lagrange multipliers). There were also 13 other topics in which one of the instructors explicitly stated how he expected students to leverage digital resources. Note that none of the six common topics fall under the integral or vector calculus general topic areas.

We now consider the activities that involve instructor demonstrations, which are less student-centered than the other in-class activities just reported, and which also stem from instructors' TPCK. Demonstrations include 41 of the 53 activities for introductory ideas (77%), 18 of the 33 for differential calculus (55%), 21 of the 27 for integral calculus (78%), and 22 of the 25 for vector calculus (88%). So, the activities designed for introductory ideas, integral calculus, and particularly, vector calculus are mainly instructor demonstrations.

There were 13 topics where the three instructors used classroom demonstrations (vectors; equations of lines and planes; cylinders and quadric surfaces; vector functions and space curves; motion in space-velocity and acceleration; functions of several variables; partial derivatives; tangent planes and linear approximations; directional derivatives and the gradient vector; maxima and minima; Lagrange multipliers; vector fields; fundamental theorem of line integrals). There were also 14 other topics in which exactly two of the three instructors used classroom demonstrations. So, there is an ample collection of available demonstrations, as the most common pedagogical practice used with digital technology implementation, which might speak to instructors' TPK.

For the third research question, we consider the topics for which *instructor ratings* were that their activities were of good or excellent quality. Not many activities (only 20%) were self-rated as excellent. Of these, nine dealt with introductory topics (33%), 12 with differential calculus (44%), 3 with integral calculus (11%), and 3 with vector calculus (11%). So, we see that a higher proportion of differential calculus activities rated as "excellent", and there is a lack of highly rated activities in integral and vector calculus.

In examining instructors' PCK, there was only one topic for which all three instructors self-rated their activity as excellent (partial derivatives), and five topics for which two of the three instructors self-rated their activity as excellent (motion in space-velocity and acceleration; functions of several variables; directional derivatives and the gradient vector; maxima and minima; Lagrange multipliers). At the same time, there were 11 other topics for which an activity was self-rated as excellent by a single instructor.

We now move to the fourth research question which considers if any patterns were noted specific to certain instructors in the sample. First, *pedagogical practices*, as determined by student-centeredness (collaborative and individual in-class activities), vary not only by *content area* but also by instructor. For example, in introductory information, only one of the 27 activities of instructor H (4%) was student-centered, while in differential calculus, five of the 13 activities (38%) reported by the same instructor were student-centered. So, the *pedagogical practice* seems to depend on the *content area*. Similarly, in the areas of integral and vector calculus, instructor T designed six of his 13 activities (46%) to be student-centered (collaborative) activities, while none of the nine activities of Instructor A in these areas were student-centered. So, *pedagogical practices* also seem to depend on the instructor.

The choice of *resource type* also seems to be instructor dependent. For example, Instructor H used CalcPlot3D 23 times when discussing introductory ideas, while Instructor T used it only four times. Similarly, Instructor T, when teaching smaller classes, used 3D-printed models in 12 activities, while Instructor A used them for three activities.

Discussion and Summary

Martínez-Planell and Moore-Russo (2023) found that concern for students' learning and the conviction that geometrical understanding is crucial in multivariable calculus motivated their sample of instructors to consider a variety of educational resources. It was found that the instructors were sensitive to students' needs and that over time their teaching shifted from being instructor-centered (demonstrations presented in class) to being more student-centered (activities to engage students often in collaborative group work). The present study examines instructors' TPCK by relating the resources instructors use, the pedagogy employed, and the content at hand.

To answer the first research question, we observed that the area of introductory ideas is more amenable to resource adoption in the sense that it was more frequently the target of activities, mainly for demonstrations using digital technology. This might relate to instructors' TCK. This area of content is also more likely to leverage resources other than digital or 3D-printed for in-class activities. In the case of 3D-printed models, we found that it seems to be more natural for the instructors in our study to use them for differential calculus. For the second research question, mainly focusing on PCK, we observed that all areas allow for student-centered activities through the use of resources. However, there were more student-centered activities for differential calculus followed by the area of introductory ideas. The areas of integral calculus, and particularly vector calculus, are comparatively lacking in the design of student-centered activities. Activities for 3D-printed models, although fewer, tend to be more student-centered than other activities. As regards the third research question, we found that differential calculus has more activities assessed as excellent, followed by the area of introductory ideas. The areas of integral and vector calculus do not show comparable development. For the fourth research question, we found that content addressed, pedagogy used, and resources employed vary by instructor. The general patterns that we found are that digital technology is preferred to other resource use; the use of in-class demonstrations by the instructor vastly outnumbers the use of student-centered activities; there are fewer activities for 3D-printed models, although they tend to be more student-centered; and the area of vector calculus is vastly overlooked in comparison with the others. These patterns suggest that MVC instructors' TPCK is not uniform across content area nor across resource type.

There were six topics for which only one instructor used 3D-printed models, 11 topics for which only one instructor designed student-centered activities, and 11 topics for which an activity was self-rated as excellent by only one instructor. This suggests that sharing materials might help instructors incorporate 3D-printed models into other topics, make the MVC course more student-centered, and disseminate activities that have been rated as excellent. Consistent with Martínez-Planell and Moore-Russo (2023), who argued that instructors could benefit from support on how to teach MVC to engage students by leveraging instructional resources, we found that establishing a means to facilitate instructors sharing student-centered activities, activities they self-rate as excellent, and activities that employ 3D-printed models, could help others and improve MVC teaching. This study makes a case for establishing a means to do so.

In the areas of integral calculus and vector calculus there are fewer activities, very few of which were student-centered or rated as excellent. Correspondingly, most of the relatively few integral and vector calculus activities are instructor demonstrations. This perceived inadequacy or imbalance in these two areas of MVC in comparison with the areas of introductory ideas and differential calculus could be a result of several factors: a consequence of the content itself; a consequence of these areas (integral and vector calculus) being taught towards the end of the course, after instructors have learned to use resources for the introductory and differential calculus areas (so they have yet to get there); or a consequence of a lack of digital capabilities or other tools to properly treat these two areas. More research is needed to better understand this issue. Documenting that these two areas need more work in terms of activities, especially those that are student-centered and of high quality is a contribution of our study.

In the MVC course, there were relatively many activities to address introductory ideas. However, not many of them were student-centered or rated excellent; in fact, classroom demonstrations constituted a large percentage of these activities. While this suggests the possibility of re-examining existing teaching activities in order to increase student engagement with the content, there might be more of a function of time constraint issues exerting pressure on instructional choices more than pedagogical beliefs playing a role. Further study is needed here.

In terms of content, this study suggests that more activities that use resources to improve the teaching of integral and vector calculus are needed. In regard to pedagogy, there is a preference for instructor demonstrations and not enough highly rated student-centered activities. For resources, we observe that more work is needed to explore the pedagogical potential of 3D-printed models. Overall, this study contributes to a better understanding of resource use in MVC, while also contributing to a better understanding of MVC instructors' TPCK.

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References

Alves, F.R.V. (2012). Discussão do uso do GeoGebra no contexto do cálculo a várias variáveis [Discussion of the use of GeoGebra in the context of calculus of several variables]. *Revista do Instituto GeoGebra de São Paulo*, *1*(2), 5-19.

Alves. F. R. V. (2014). Análise e visualização de formas quadráticas na determinação de pontos extremantes. *Debates em Educação Científica e Tecnológica, 4*(01), 128-149. https//doi.org/10.36524/dect.v4i01.69

Both Carvalho, N. T., & Pereira, R. (2011). O software "Maple" no estudo de funções de várias variáveis [The "Maple" software in the study of functions of several variables]. *Educação Matemática em Revista, 17*, 52-60.

Cheong, K.H., Chen, J.S., Kang, K., & Yeo, D.J. (2023). Supporting students' visualization of multivariable calculus partial derivatives via virtual reality. *Mathematics, 11*, 831. https://doi.org/10.3390/math11040831

Davis, F. D. (1989). Perceived usefulness, perceived ease of use, and user acceptance of information technology. *MIS Quarterly, 13*(3), 319–339. https://doi.org/10.2307/249008

Gemechu, E., Kassa, M., & Atnafu, M. (2018). Matlab supported learning and students' conceptual understanding of functions of two variables: experiences from Wolkite university. *Bulgarian Journal of Science and Education Policy (BJSEP), 12*(2), 314-344.

Goodhue, D. L., & Thompson, R. L. (1995). Task-technology fit and individual performance. *MIS quarterly*, 213-236. https://doi.org/10.2307/249689

Habre, S. (2001). Visualization in multivariable calculus: the case of 3D surfaces. *Focus on Learning Problems in Mathematics, 23*(1), 30-48.

Henriques, A. (2006). *L'enseignement et l'apprentissage des integrales multiples: Analyse didactique intégrant l'usage du logiciel Maple* (Unpublished doctoral dissertation). Université Joseph Fourier, France.

Ingar, K. V. (2014). *A Visualização na Aprendizagem dos Valores Máximos e Mínimos Locais da Função de Duas Variáveis Reais* (Unpublished doctoral dissertation). Pontifícia Universidade Católica de São Paulo, Brazil.

Ingar, K. V., & Silva, M. J. F. (2019). Estudio de la derivada parcial por medio de las aprehensiones en el registro gráfico de funciones de dos variables con estudiantes de ingeniería. *Educação Matemática Pesquisa, 21*(1), 49-68. https://doi.org/1.23925/1983-3156.2019v21i1p049-068

Jones, S. R., Long, N. E., & Becnel,J. J. (2022). Design of virtual reality modules for multivariable calculus and an examination of student noticing within them. *Research in Mathematics Education*. https://doi.org/10.1080/14794802.2022.2045625

Kang, K., Kushnarev, S., Wei Pin, W., Ortiz, O., & Shihang, J. C. (2020). Impact of virtual reality on the visualization of partial derivatives in a multivariable calculus class. *IEE Access, 8*, 58940-58947.

Karabina, B., Long, L., & García, A. (2023). Investigating the impact of augmented and virtual reality on learning in a multivariable calculus classroom. *World Academy of Science, Engineering and Technology International Journal of Educational and Pedagogical Sciences 17*(8).

Koehler, M.J., & Mishra, P. (2008). Introducing TPCK. AACTE Committee on Innovation and Technology (Ed.), *The Handbook of Technological Pedagogical Content Knowledge (TPCK) for Educators* (pp. 3-29). Mahwah, NJ: Lawrence Erlbaum Associates.

Koehler, M. & Mishra, P. (2009). What is Technological Pedagogical Content Knowledge (TPACK)? *Contemporary Issues in Technology and Teacher Education, 9*(1), 60-70. Waynesville, NC USA: Society for Information Technology & Teacher Education. Retrieved August 28, 2023 from https://www.learntechlib.org/primary/p/29544/.

McGee, D., Moore-Russo, D., Ebersole, D., Lomen, D., & Marin Quintero, M. (2012). Visualizing three-dimensional calculus concepts: The study of a manipulative's effectiveness. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22*(4), 265-283. https://doi.org/10.1080/10511970.2010.494652

McGee, D. L., Moore-Russo, D., & Martínez-Planell, R. (2015). Making implicit multivariable calculus representations explicit: A clinical study. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 25*(6), 529-541. https://doi.org/10.1080/10511970.2015.1038858

Martínez-Planell, R., & Moore-Russo, D. (2023). Multivariable Calculus Instructors' Reports of Resource Use: Resources Used and Reasons for Their Use. In S. Cook, B. Katz, and D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 53-62. Omaha, Nebraska

Martínez-Planell, R. & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM-Mathematics Education, 53*(3), 695-707. https://doi.org/10.1007/s11858-021-01233-6

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Parzysz, B. (1988). "Knowing" vs "seeing". Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19*(1), 79-92. https://doi.org/10.1007/BF00428386

Paul, S. (2018). 3D printed manipulatives in a multivariable calculus classroom. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *28*(9), 821-834. https://doi.org/10.1080/10511970.2018.1445675

Rojas Flórez, L. C., Mejía Velasco, H. R., & Esteban Duarte, P. V. (2019). El Cálculo y su Enseñanza, *Enseñanza de las Ciencias y la Matemática, 12*, 13-26. Cinvestav-IPN: México. https://recacym.org/index.php/recacym

Sherer, G., Kustusch, M. B., Manogue, C. A., & Roundy, D. J. (2013). The partial derivative machine. In P. V. Engelhardt, A. D. Churukian & D. L. Jones (Eds.) *Physics Education Research Conference Proceedings*, pp. 341–344.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Trigueros, M., Orozco-Santiago, J., & Martínez-Planell, R. (2023). Learning two-variable functions using 3D dynamic geometry. In M. Trigueros, B. Barquero, R. Hochmuth, & J. Peters (Eds.), *Proceedings of the Fourth Conference of the International Network for Didactic Research in University Mathematics* (INDRUM 2022, 19-22 October 2022) (pp. 213-222). Hannover: University of Hannover and INDRUM.

VanDieren, M., Moore-Russo, D., & Seeburger, P. (2020). Technological pedagogical content knowledge for meaningful learning and instrumental orchestrations: A case study of a cross product exploration using CalcPlot3D. In J. P. Howard & J. F. Beyers (Eds.), *Teaching and learning mathematics online* (pp. 267-293). Boca Raton, FL: CRC Press.

Wangberg, A., & Johnson, B. (2013). Discovering calculus on the surface. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 23*(7), 627-639. https://doi.org/10.1080/10511970.2013.775202

Wangberg, A. (2020). Fostering student discovery and conjecture in multivariable calculus. *Journal of Humanistic Mathematics*, *10*(2), 315-347. DOI: https://doi.org/10.5642/jhummath.202002.15

“I don’t see myself as an expert at all”: Authority in Facilitating Instructional Change

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Of the many challenges in shifting towards inquiry-oriented instruction from a lecture-based approach is understanding the role of the teacher in this new paradigm. One aspect of that role involves navigating novel classroom authority dynamics as students bear more authority to create and justify mathematical ideas. Supporting teachers as they navigate this shift is one of the many roles of effective professional development (PD). This study of one PD facilitator’s authority as he worked to support instructors’ inquiry-oriented instruction (IOI) revealed a new type of authority, called pedagogical expertise authority, that may be of particular importance to understanding how to best support IOI. Additionally, the results of this study suggest the impacts of facilitators’ beliefs and teaching experiences on long-term professional development.

Keywords: authority, professional development, inquiry-oriented instruction

Best practices for supporting instructors as they take up student-centered instructional practices have been a central focus in the RUME community in recent years (e.g., Johnson, 2019; Andrews-Larson et al., 2019; Henderson et al., 2011). Due to the shift in authority relations inherent to this change, specific attention must be paid to supporting instructors as they begin to teach in ways that invite students to bear authority in classroom discourse. This study investigates one facilitator’s authority during a semester-long professional development (PD) aimed at supporting instructors’ implementation of the Inquiry Oriented Abstract Algebra (IOAA) curriculum materials (Larsen et al., 2013), with a particular eye toward the potential impacts of the facilitator’s teaching experience on his practice.

Background & Theoretical Framing

Because fostering students' mathematical autonomy is one central goal of inquiry-oriented instruction, PD must support teachers as they adjust their behaviors and roles in the classroom to share authority with students. To create more opportunities for students to autonomously build and justify their own mathematical ideas, as well as develop more productive attitudes and mathematical identities, instructors must move away from a classroom authority tradition based on their own power and mathematical expertise (Langer-Osuna, 2017; Wilson & Lloyd, 2000; Amit & Fried, 2005). Sharing authority among teachers and students is a mutual goal of inquiry pedagogies (Gerson & Bateman, 2010; Hicks et al., 2021), but perspectives differ on what precisely constitutes shared authority. In the most hierarchical sense, a teacher must relinquish authority to students (e.g., Bleiler-Baxter et al., 2023; Fried, 2020), or otherwise minimize his or her authority to create and assess mathematical ideas (e.g., Gerson & Bateman, 2010). More progressive conceptions focus not on the teacher's abdication of authority but rather on how it can be used to promote students' authority (e.g., Bishop et al., 2022; Oyler, 1996). In IOI, teachers are tasked with authoritative roles like guiding discussion and leveraging students' ideas to advance the mathematical agenda of the class while simultaneously bolstering students' authority to create and justify their own mathematics. PD facilitators may become key agents in supporting instructors as they adjust their teaching practices to share authority with students in this way.

Since both facilitators and teachers are tasked with guiding others' learning, the practices of teachers who become PD facilitators are a rich territory for research. To this end, I lift a framework for analyzing classroom authority dynamics to the professional development context. I take specific direction from Gerson and Bateman (2010), viewing authority as an enacted relationship between individuals where an authority bearer makes a claim to legitimacy based on their position within a hierarchy, their expertise, or the norms and practices of the mathematical community, and another receives that authority by changing their behavior, i.e., what is said, what is written, how they are thinking. In addition to Gerson and Bateman's ownership and mathematics expertise authorities (summarized in Table 1), I add the construct of *pedagogical expertise authority.* This construct aims to capture those authority relations arising in PD when instructors' claims about mathematics and pedagogy are legitimized by their expertise as teaching professionals, rather than as mathematicians.

Table 1. Authority types and subtypes with definitions.

Type	Sub-type	Legitimized by . . .
Hierarchical	Institutional	position as instructor of facilitator
	Granted	permission from instructor or facilitator
	Ownership expertise	one's creation of an idea or solution
Expertise	Mathematics expertise	one's mathematical expertise
	Pedagogical expertise	one's pedagogical expertise
Mathematical	Mathematics Community	shared norms, ideas, and practices
	Justification	mathematical & logical reasoning

Context & Methods

The professional development in this study took place in the context of an online working group (OWG) within the TIMES project (Teaching Inquiry-oriented Mathematics: Establishing Supports, NSF Awards: #1431595, #1431641, #1431393). The OWG consisted of three instructors new to teaching IOAA, Elena, Laura, and Roger, and one facilitator, Mickey (all pseudonyms). The instructors were introduced to the curriculum materials in the summer workshop and met synchronously online via Google Hangouts once per week to reflect on the curriculum and their implementation with the facilitator. Participants were regularly asked to share reflections on their instruction and investigate the curriculum tasks from a mathematical perspective. Additionally, the facilitator met with one of the TIMES project's principal investigators (PI) each week in a series of Debrief meetings, wherein the facilitator and PI discussed his goals for facilitation, beliefs about teaching IOAA, and updates on the OWG's proceedings. Both Debrief and OWG meetings were recorded for retrospective analysis.

To identify Mickey's orientations to authority in his facilitation, three sets of video data were analyzed: the OWG meetings, the Debrief meetings, and Mickey's teaching with the IOAA curriculum materials in the previous year. This analysis was guided by iterative thematic analysis (Braun & Clarke, 2021) and had four main phases, summarized in Table 2.

Results & Discussion

Mickey's decision-making practices appear to be influenced by an aim to shift traditional authority relationships in both the online working group and the classroom context. Mickey first demonstrates this intention in an explicit discussion of classroom authority in the early weeks of

Table 2. Summary of analysis phases.

Phase	Description
One	Created analytic notes of OWG meetings; organized into five columns: Video File Name, Timestamp, Quote, Speaker(s), and Notes. Memos focused on Mickey's utterances during and about facilitating discussions regarding IOI.
Two	Created analytic notes of Debrief meetings with identical structure to OWG notes, focusing on subject-matter already prevalent in Phase One data log.
Three	Built descriptive themes of Mickey's prevailing behaviors and attitudes based on commonalities between phase one and two logs.
Four	Analyzed videos of whole-class activities from Mickey's teaching, identifying confirming and conflicting instances of Mickeys previously identified orientations.

the OWG meetings. In the following OWG episode, Elena had been reflecting on her pedagogical choice to give her students the definition of bijection when they could not recall it independently.

Elena: I didn't feel like that was something they could just "inquiry out"' – it wouldn't just happen.
Mickey: Sure, but the thing is is that, holistically, are you doing that more often than not? Because . . . What you're seeing is authority- or power-shift.
Elena: Right.
Mickey: Right. Where is the authority lying? Is it lying with you? Well then, they're only going to do things that satisfy you, instead of doing things that satisfy themselves, you know, for them to learn.

This episode suggests a belief that shifting authority to be shared between teacher and students promotes learning. Mickey's last statement implies that if authority lies with the instructor, then students' engagement will not result in learning. For Mickey, students must take up authority in order to learn. By explicitly motivating participants to attend to these authority dynamics within their own classrooms, Mickey demonstrates his orientation toward authority. This is best characterized as a belief or value in shifting authority structures so that participants, be they students in a classroom or instructors in professional development, rely on their own mathematical or pedagogical expertise to reason about their quandaries, rather than turning to Mickey for his. This practice of shifting authority arises in both the online working group and the classroom when Mickey shares expertise authority with participants while carefully leveraging his authority to advance the agendas in both contexts.

Expertise Authority

Throughout his teaching and his facilitation, Mickey rejects the notion that he is an expert. This refusal to bear expertise authority was displayed in the OWG when Mickey directly discussed his lack of expertise and consistently elicited participants' contributions before making his own, thus sharing pedagogical expertise and ownership expertise authorities among participants. This attitude toward expertise authority is made more apparent in discussions from the debrief meetings and episodes from Mickey's teaching.

In the online working group, Mickey displayed his orientation toward sharing authority by refusing to position himself as the expert in the room, consistently eliciting participant thinking *first* before sharing his own ideas, and sometimes refusing to give advice at all. Often, when a participant asked Mickey a direct question, he would redirect it to the entire group with phrases like, “So that [question] is for everybody, I can answer later,” and “I think this [topic] merits at least a three-to-five-minute discussion. I’ll add a disclaimer that I have no idea how to handle that situation.” In a quintessential episode, Elena has asked Mickey how he would handle the situation of having an exceptionally bright student eager to share all of the answers: “If you're looking at me to be the expert in this situation, I don't have a clue. I have conjectures and thoughts about this, but I want to open it up to everybody in the room.”

Mickey directly states that the participants should not view him as an expert. Furthermore, by choosing to give the opportunity to answer Elena's question to the whole group, Mickey receives the pedagogical expertise authority of the remaining participants. This is a clear invitation for participants to bear pedagogical expertise authority to reason about Elena's quandary, as well as ownership expertise authority to create and discuss possible solutions. It can be inferred that Mickey’s belief about this practice of sharing expertise authority as essential for participants' learning motivates his decision-making as a facilitator.

Mickey's orientation toward avoiding expertise authority is also apparent in discussions from the Debrief meetings, where he overtly describes his relationship to his own expertise. An instance of dialogue that captures this is:

> It’s hard even with them sometimes, because when I ask a question, they're seeking “Is this the answer you wanted?” and like, a colleague of mine introduced me the other day as a pedagogical expert and . . . I am so not that.

Even more explicitly, he says: “I don't consider myself an expert in inquiry-oriented [sic] by any means.” Rather, than viewing himself as an expert in IOI, Mickey is aware that his position as facilitator garners him institutional authority, but chooses to leverage that authority to share pedagogical and ownership expertise authorities with students. This is indicative of Mickey's orientation toward the importance of sharing expertise authorities in professional development.

Likewise, in discussing his own views on assessment during a debrief meeting in Week 8, Mickey mentioned that his OWG seems to struggle consistently with student learning assessments, trying to lean on his experience for guidance. He said, “This keeps coming up over and over and over again, and I keep asking them, ‘What are the goals of your course?’ For me, your assessment should match your goals.” Mickey shared his personal teaching philosophy but rejected any positioning of himself as the arbiter of what kinds of assessments were best. He encouraged his instructors to create their own standards for assessment, rather than to look for the “correct way” to assess their students based on his expertise. This clearly demonstrates Mickey's goal of sharing expertise authorities. His comments about answering participants’ questions based on his expertise authority emphasize his intention to shift traditional authority structures within his facilitation to create dynamics of shared authority.

Mickey's teaching actions provide additional context for these facilitation practices and the authority beliefs that inform them. Because of the contextual differences between the classroom and the online working group, it is no surprise that this orientation arose differently during Mickey's teaching in the year prior. In the classroom, Mickey's orientation toward traditional authority dynamics can be characterized by his intentional and particular distribution of opportunities to bear expertise authorities among his students through a process-over-product value selection of contributions. Additionally, he invites students to bear mathematical authority to evaluate each other’s contributions.

It is often the case that students with correct solutions most consistently bear granted authority to contribute to discussion. Mickey subverts this authority tradition by minimizing the contributions of students with correct solutions and granting authority to students with partial or in-progress solutions to contribute their thinking. In the classroom, this careful distribution of opportunities to bear authority arises when Mickey leverages his institutional authority to prompt students to share their thinking, especially when he asks students to share what he calls “productive failures”. One explanation of productive failures that Mickey gives in class is:

> “Try stuff. In Algebra, you're going to cook a lot of equations up. Some of them may be fruitful, some of them may not be. The biggest thing, though, is to explore. Because, some help, some don't. Learning from your mistakes, having that productivity when you're failing, is a huge part of this.”

Mickey regularly granted students the authority to share these productive failures, and it should be noted that students were expected to discuss these processes as part of their course grade. By granting students authority to contribute their in-progress mathematical ideas, Mickey subverted the tradition that only correct solutions can legitimize a students' mathematics expertise authority, shifting toward a dynamic where students are enculturated into the mathematical community through opportunities to share their informal, in-progress reasoning. In an exemplifying episode, the class is attempting to prove a conjecture about a generating set for a symmetry group. The student who made the conjecture begins to share his completed proof, and Mickey interrupts him to call on a student who is still working on the conjecture, saying, ``Hold on, hold on, let's see, Madison, can you help me out with this?'' Mickey has not received the mathematical authority the student with the correct solution is attempting to bear, and instead grants authority to Madison to share her reasoning about the conjecture so far. In inviting her to bear granted authority, Mickey creates the opportunity for Madison to bear both ownership expertise and mathematics expertise authorities. This is supportive of Madison's, and arguably many other students', mathematical autonomy because it values the process of creating a proof over the final product, which helps students to develop productive beliefs about their position in the mathematical community (see Serbin et al., 2020, p. 4). This support is also created by centering Madison's in-progress thinking, which aids other students in gaining insight into their classmate's reasoning.

Mickey also shifts away from the tradition that the instructor bears the institutional and mathematics expertise authorities to evaluate students' contributions by inviting students to bear authority to evaluate each other’s ideas. Following the episode above where all students have been prompted to share work for task 3 on the whiteboard, Mickey asks, “So, what do we think?

And remember, we're in a non-judgmental phase in our lives right now, so keep your comments very productive, okay?" By empowering his students to evaluate each other's contributions through constructive feedback, Mickey creates opportunities for students to bear mathematical authority to evaluate each other's work and ownership expertise authority to respond to evaluations of their own ideas. This supports their mathematical autonomy by engaging students in reasoning about the contributed solutions and giving them ownership of mathematical ideas.

Mickey's refusal to bear pedagogical expertise authority in the online working group parallels his refusal to bear mathematical expertise authority to evaluate students' contributions in the classroom, instead inviting students to evaluate each other's contributions. Similarly, inviting OWG participants to share their thinking before sharing his own appears in much the same way as his classroom practice; in both instances, Mickey invites participants to share expertise authorities specific to the learning environment. The practice of sharing authority in both settings demonstrates Mickey's intent to shift traditional authority dynamics. This practice arises as an artifact of Mickey's implicit belief that sharing authority supports the development of subject-specific autonomy in participants.

Authority to Advance the Agenda

Although Mickey refused to bear the expertise authority his participants often requested of him, there was tension between Mickey's authority values and his responsibility to advance the agenda of the OWG. Advancing the agenda required Mickey to bear institutional authority both as the facilitator in the OWG and as the instructor in the classroom.

Mickey often prompted participants and guided group discussion to meet the goals of the OWG, bearing institutional authority in order to advance the agenda of the OWG. This is evidenced on the individual level when Mickey directly asks participants for their thoughts on the mathematics, as seen in the following exchange between Mickey and Roger during a group discussion of the Sudoku task (Larsen et al., 2013).

Mickey: Turning toward the content in Lesson 4, why is it, mathematically, that in every group Cayley table, each symmetry appears exactly once in each row and each column?
Roger: It has to do with cancellation?
Mickey: Can you expand on that? Pardon the pun.

When Mickey bears his authority as the facilitator to prompt Roger to share his thinking, he advances the professional development agenda by guiding participants to engage with the curriculum materials from a mathematical perspective. He also directly prompts the participants to reflect aloud on their teaching experiences. After Laura had discussed a difficulty she was having with using one of her student's contributions to formalize a mathematical idea, Mickey prompted her to dig deeper, saying "Now that you can reflect on it, what went through your mind at that moment when you were handling the situation?" and later directing the remaining participants to discuss potential pedagogical moves to apply to Laura's quandary. By exercising his institutional authority as the facilitator in this way, Mickey advances the agenda of the OWG by inviting participants to engage in pedagogical reasoning.

So, while Mickey shares authority often by refusing to be the bearer of expertise authority in the Online Working Group, he is willing to bear the kind of institutional authority associated with the facilitator's responsibility to advance the agenda to meet the OWG's goals. Although these actions do not depart drastically from hierarchical authority relations, they are consistent with a motivation to support participants' pedagogical reasoning and autonomy, which is in line with both Mickey's goals and the intent of the instructional support model.

Similarly, Mickey must bear institutional authority in the classroom to direct students' activity by providing tasks and granting students authority to share their solutions. Similar to the OWG, Mickey's orientation toward his own institutional authority is somewhat unsurprising in the classroom context, but by using it to grant authority to students to contribute, Mickey acts with the goals of supporting the development of his students' mathematical autonomy and advancing the mathematical agenda. What is most striking is the way this agenda informs Mickey's choice of which students are granted authority to contribute. By de-emphasizing the mathematical authority of students with correct answers, Mickey creates more opportunities for a wide range of students to bear ownership and mathematics expertise authorities, thereby promoting their mathematical autonomy.

Conclusions

The results of this case study suggest strong parallels between the teaching and facilitating practices of this PD facilitator: across contexts, Mickey balanced the goal of sharing authority with participants with his responsibility to advance the agenda of the group, pointing to the possibility that teachers' beliefs can transcend the classroom and influence the way they facilitate professional development. Mickey's emphasis on OWG participants bearing pedagogical expertise authority to reason about their instructional practices may have arisen from his experiences with balancing authority in an IOAA classroom, where, unlike in other types of inquiry-based learning, the instructor's responsibility to advance the mathematical agenda demands that they carefully leverage their own authority in ways that encourage, rather than stifle, students' authority.

More broadly, this study speaks to the importance of facilitator's prior teaching experience, attitudes, and beliefs about authority, in creating shared authority dynamics that can support the professional development of new instructors. Viewing both instruction and PD facilitation through this lens of shifting authority dynamics can be useful in future research on both IOI and professional development, especially with the added construct of pedagogical expertise authority. Future work should investigate the effects on teaching when participants are bearers of pedagogical expertise authority in professional development, as well as their affect, attitudes, and conceptualizations of shared authority in IOI.

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References

Amit, M., & Fried, M. N. (2005). Authority and Authority Relations in Mathematics Education: A View from an 8th Grade Classroom. Educational Studies in Mathematics, 58(2), 145–168.

Andrews-Larson, C., McCrackin, S., & Kasper, V. (2019). The next time around: Scaffolding and shifts in argumentation in initial and subsequent implementations of inquiry-oriented instructional materials. The Journal of Mathematical Behavior, 56, 100719.

Bishop, J. P., Koehne, C., & Hicks, M. D. (2022). By Whose Authority? Negotiating Authority for Modes of Activity in Mathematics Classrooms. For the Learning of Mathematics, 42(2), 35–41.

Bleiler-Baxter, S. K., Kirby, J. E., & Reed, S. D. (2023). Understanding authority in small-group co-constructions of mathematical proof. The Journal of Mathematical Behavior, 70, 101015.

Braun, V., & Clarke, V. (2021). One size fits all? What counts as quality practice in (reflexive) thematic analysis? Qualitative Research in Psychology, 18(3), 328–352.

Fried, M. N. (2020). Authority and Mathematics Education. In S. Lerman (Ed.), Encyclopedia of Mathematics Education (pp. 69–72). Springer International Publishing AG.

Gerson, H., & Bateman, E. (2010). Authority in an agency-centered, inquiry-based university calculus classroom. The Journal of Mathematical Behavior, 29(4), 195–206.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. Journal of Research in Science Teaching, 48(8), 952–984.

Hicks, M. D., Tucci, A., Koehne, C., Melhuish, K., & Bishop, J. (2021). Examining the Distribution of Authority in an Inquiry-Oriented Abstract Algebra Environment. Research in Undergraduate Mathematics Education Reports, 109–116.

Johnson, E. (2019). Chapter 6. Undergraduate mathematics instruction: Not as bad as you'd think? In C. Andrews-Larson & B. Ruedi (Eds.), Levers for Change: An assessment of progress on changing STEM instruction. (pp. 100–122). American Association for the Advancement of Science.

Langer-Osuna, J. M. (2017). Authority, Identity, and Collaborative Mathematics. Journal for Research in Mathematics Education, 48(3), 237–247.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. The Journal of Mathematical Behavior, 32(4), 693–711.

Oyler, C. (1996). Making Room for Students: Sharing Teacher Authority in Room 104. Teachers College Press, 1234 Amsterdam Avenue, New York, NY 10027 (clothbound: ISBN-0-8077-3546-9; paperback: ISBN-0-8077-3545-0).

Serbin, K. S., Sara Brooke, M., Marilin Annie, K., & Johnson, E. (2020). Social norms conducive to women's learning in inquiry-oriented abstract algebra. Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 568–572.

Wilson, M. (Skip), & Lloyd, G. M. (2000). Sharing Mathematical Authority with Students: The Challenge for High School Teachers. Journal of Curriculum and Supervision, 15(2), 146–169.

The Nature of an Online Work Group Reveals the Teaching Knowledge of Inquiry-Oriented Linear Algebra Instructors through Their Goals for Instruction

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Postsecondary instructors interested in inquiry-oriented instruction of linear algebra participated in a sequence of eight one-hour online work group meetings with other inquiry-oriented linear algebra instructors and facilitators. Recordings were analyzed for how two participants referenced goals for instruction in discussions of implementing a new instructional unit on subspaces. We identified four goals for the instruction of teaching subspaces. We discuss the intersections of several goals that exist due to the tension caused by real-world contexts and abstract mathematical concepts. The instructors presented resolutions to the tension by utilizing varying teaching knowledge. Based on the results, we make suggestions for those who want to transition to inquiry-oriented instructional approaches.

Keywords: teaching knowledge, goals for instruction, inquiry-oriented linear algebra, online work group

Inquiry-Oriented Linear Algebra (IOLA) is a reform-oriented instructional approach that derives from Realistic Mathematics Education (RME). This instructional approach encourages teachers to support students in their reinvention of mathematical concepts through inquiry (Freudenthal, 1991; Kelley & Johnson, 2022). Research has shown that authentic engagement with mathematics through this instructional approach, can benefit student achievement and possibly incite equitable outcomes among students (Burke et al., 2020; Freeman et al., 2014; Haak et al., 2011; Kogan & Larsen, 2014). This instructional practice is difficult to enact, however, because instructors may not fully utilize the teaching knowledge needed to inform their practice. This is especially true for novice instructors implementing inquiry-oriented approaches (Wagner et al., 2007). If long-lasting instructional change is needed for the desirable outcomes available by IOLA, then teachers need to shift their instructional approach (Cohen, 1990; Henderson et al., 2011) by growing teaching knowledge. Thus, researchers have declared "a need for professional development programs that foster the development of undergraduate mathematics instructors' pedagogical reasoning" (Andrews-Larson et al., 2019, p. 129).

This lays the groundwork for the following problems: what is being done to address the professional development gap, what teaching knowledge IOLA instructors possess, and how do we capture it. Thus, an Online Work Group (OWG) for postsecondary mathematics instructors is examined in this study. The OWG was part of the IOLA-X project and was initially created to provide IOLA instructors a chance to collaborate with other instructors who are interested in their continuous pursuit of enacting IOLA. These IOLA instructors are guided by facilitators who are researchers of IOLA-X. Instructors from various universities across the United States joined this OWG for eight sessions across the Spring 2022 semester. They worked on an IOLA task unit "subspace" and discussed their teaching practices with the other researchers, instructors, and facilitators of the OWG. The researchers of the IOLA-X project took their contributions to inform their continual effort to adjust their curriculum and to create teacher notes for other IOLA instructors. The instructors' contributions can be valuable to capture how the OWG makes way for discussion of teacher practice and to also collect teacher knowledge IOLA instructors possess. Here, our research question is "How does the OWG serve opportunities for instructors to exhibit knowledge for teaching in inquiry-oriented ways?"

Literature Review

Linear Algebra is a postsecondary mathematics course that is often a requirement for students in STEM-related majors. As a result, many students at some point enroll in this course. IOLA is one instructional approach to active learning (Freeman et al., 2014) and inquiry-based mathematics education (Laursen & Rasmussen, 2019). Laursen and Rasmussen (2019) discuss this approach to mathematics education as "student engagement in meaningful mathematics, students' collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 140).

As stated previously, inquiry-oriented instructional approaches are difficult to implement. Mostly because instructors at first may not possess the reasoning and knowledge necessary for enactment (Andrews-Larson et al., 2019). The knowledge we are referring to is mathematical subject matter knowledge (Ball et al., 2008; Hill et al., 2008) and pedagogical content knowledge (Ball et al., 2008; Hill et al., 2008; Shulman, 1986). Shulman in 1986 first introduced the idea of Pedagogical Content Knowledge (PCK) as the subject-matter knowledge for teaching. This includes at the time the unnamed domains of PCK which are knowledge of content and teaching (KCT) and knowledge of content and students (KCS). These domains capture knowledge for teaching such as an instructor's knowledge of the best representation to present to students or knowledge of what ideas or conceptions students will bring to the table. The domains are not restricted to the teaching of a specific content area.

Ball and colleagues (2008) expanded on the work of Shulman in their framework of Mathematical Knowledge for Teaching (MKT). Their framework includes PCK as half of their domains of MKT. The other half is subject-matter knowledge, in other words, mathematical knowledge that is unrelated to the practice of teaching. Teaching knowledge for mathematics instructors has been studied for decades as either declarative or "knowledge-in-use" (Andrews et al., 2022). These studies capture how experienced teachers approach their instruction. Although there is little evidence that more experience means more teaching knowledge (Andrews et al., 2022), there are studies that point to the differences in teaching knowledge between experts and novices (Auerbach et al., 2018). Thus, analyzing the teaching knowledge of experienced IOLA instructors can prove to be worthwhile as they highlight areas of instruction novice IOLA instructors may not consider.

Theoretical Framework

Wagner and colleagues (2007) studied the MKT and PCK regarding the challenges of a novice instructor teaching an inquiry-oriented differential equations course. As a result, Wagner et al. (2007) identified four types of instructional goals in the context of inquiry-oriented instruction at the undergraduate level: classroom orchestration goals, cognitive/learning goals, assessment goals, and content goals. These goals encompass their framework called *goals for instruction.* Each goal is summarized as follows:

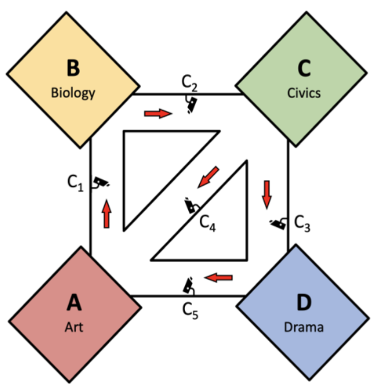
1. Classroom orchestration goals: How instructors orchestrate, intervene, and redirect the discussions and negotiate an agenda with emerging ideas.
2. Cognitive/Learning goals: What student understanding, questions, and activities look like.
3. Assessment goals: How to assess student learning, what the evidence of understanding is, and how to design a pace or curriculum.

4. Content goals: What and how specifically mathematical concepts should be learned.

Using the work from Wagner et al. (2007) as a priori scheme, this proposal identifies how instructional goals were discussed by experienced IOLA instructors in the OWG.

Study Context: Inquiry-Oriented Linear Algebra and Online Work Group

The IOLA-X project focuses on developing student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra (Wawro et al., 2013). There are five main phases in the design research spiral: *Task Design*, *Paired Teaching Experiment* (*PTE*), *Classroom Teaching Experiment* (*CTE*), *Online Work Group* (*OWG*), and *Web* (Wawro et al., 2023). The participants of our study come from the OWG in the fourth phase of the research project. The main purpose of the OWG for the IOLA research team is to learn from instructors how IOLA is implemented in various classrooms with various student populations and to gain insights to develop instructor notes and revise tasks (Wawro et al., 2023). At the center of the OWG for this study was the discussion of a unit of tasks about subspaces and Table 1 illustrates the overview of the subspace unit. In this unit, the tasks were contextualized in a problem about students walking in one-way hallways past cameras monitoring their traffic (See Figure 1). To draw out the feedback from the instructors, the facilitators managed mathematical discussions about the tasks as well as facilitated discourse about the preparation and implementation of the tasks. Through examining discussion and input from the experienced undergraduate instructors participating in the OWG, questions, and thoughts about the goals for instruction and challenges with implementation naturally arose.



The hallways in one wing of Ida B. Wells High School were changed to one-way corridors to promote social distancing during a pandemic (red arrows in the diagram). These hallways connect classrooms A-D (the Art room, Biology lab, Civics room, and Drama room) as shown in the diagram. Each hall has a security camera that allows Principal McDaniel to monitor student movement through the hallways (cameras 1-5, as shown in the diagram). As a further precaution, each wing is isolated from the rest, so the students in a wing stay within that wing and no students from any other part of the school will enter the west wing.

SCENARIO ONE: TRACKING MOVEMENT

Before the school year starts, Principal McDaniel goes into the school when it is empty to spend a day learning how to use data from the camera system. Her daughter Ella comes with her, and she asks Ella to help her test the system by walking between rooms – so long as she stays in the building and follows the one-way hallway rules. Ella decides to start in the Art room, passes Camera 1 (C_1) as she walks from the Art room toward the biology lab, and then continues walking past Camera 2 (C_2) as she walks from the biology lab to the Civics room. Afterwards, Principal McDaniel sees that the camera system has recorded the number of people who walked by each camera with the vector $p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Figure 1. The hallway scenario of Tasks 1-2 in the IOLA subspace unit

Methods

Our primary data source was the recorded videos of the OWG meetings—held and recorded via Zoom, and group artifacts such as Google Slides and Jamboards that served as secondary data sources. In this OWG, there were six members: F1 and F2 (facilitators), R1 and R2 (IOLA researchers), and I1 and I2 ("pure" participants who are experienced inquiry-based instructors but not IOLA researchers). This study focuses on the pure participants, so the participation and contributions of I1 and I2 serve as the main data of our study. I1 is an associate professor at a small private college in the Northwestern United States and I2 is a senior instructor at a large public university in the Northeastern United States and they all taught linear algebra at their universities at the time of the OWG sessions. Other than pure participants, this team involves one graduate student (F2), two associate professors (F1 and R1), and one full professor (R2).

Table 1. The IOLA subspace unit overview

Task	Driving Question	Mathematical Formalism
Hallway Task 1: Finding Paths	What are the possible paths from room A to room C and from room C to room C?	Closure (of "inputs")
Hallway Task 2: Managing Populations	What are the possible paths that induce a specific change in room populations? What are the possible paths that leave the room populations unchanged?	Correspondence between ("input" and "output") vectors
Hallway Task 3: New Wings	What are the possible paths for a different wing of Ida B Wells High School (defined by a matrix) that will leave populations unchanged?	Null spaces as a type of closed "input" spaces and column spaces as a type of closed "output" spaces

Each OWG meeting was approximately one hour and there were eight meetings throughout the Spring 2022 semester. Thus, a total of around eight hours of OWG meetings were conducted and recorded over several days. In the first four videos, the OWG members discussed the IOLA subspace tasks, either as if they were students or teachers, so they shared the mathematical progression of an IOLA subspace unit comprised of three tasks. The subspaces unit focused primarily on notions of closure of sets of vectors under vector addition and scalar multiplication, as well as null and column spaces. So, in the first four meetings, participants worked on the mathematical problems as a group and then discussed mathematical goals, approaches, and links to other ideas and topics. The remainder of the meetings took place throughout the participants' implementation of the sequence, with each participant reporting on how the implementation went, what they liked, how their students reasoned about tasks, what they would change or what they would do differently.

We first analyzed all eight videos in terms of goals for instruction. To do so, videos were transcribed by Otter, an online artificially intelligent transcription application. Both authors separately coded all transcripts using Nvivo software. In this analysis, the four goals of instruction were the code schemes. We coded for all the participants of the OWG—even though this report focuses on two pure participants. While coding, we assigned four codes, which mean four goals, at the level of a single turn of talk. Then, we compared codes to reach agreements to build inter-rater reliability. We identified common themes within each code and found out that there were many intersections between two or more goals. We decided to dig into the intersections more precisely—and so analyzed what kinds of pieces of knowledge of IOLA instructors were discussed, considering the goals for instructions and main themes of OWG meetings. For this report, we selected several examples of what I1 and I2 shared and contributed.

Results

Generally, in the OWG meetings, the pure participants discuss how to manage discussions of contextualized tasks about closures and subspaces (classroom orchestration goal), what kinds of discussion topics and communication emerged in engaging in IOLA tasks (cognitive/learning goal), curricular trajectories and mathematical content relevant to subspaces reorganized by instructors (content goal), and pacing, timing, testing, and grading of inquiry-based teaching (assessment goal). Within the findings, the main notable pattern in this OWG is there are many intersections of two or more goals for instruction, except for assessment goals. Also, it is interesting these intersections are rooted in some tensions between RME-based context and abstract mathematical concepts in implementing IOLA tasks. The following examples address those intersections.

I1: Yeah, my only hesitancy in all of this is the fact that you know, in the other task sequences, we have this clear and direct parallel between the intuitive contextual setting and the formal definitions. And this one, we're a little loosey-goosey in a couple of places, and I just, I don't know how that's gonna translate. Like, are they like, are they going to internalize what has been their tendency to think only about scalars that are, you know, positive whole numbers in the first place, right? And so, is this going to somehow reinforce that? Um.

Here, I1 expresses her concern of how students will take up subspaces according to the “loosey-goosey” definition in the task. That it may be difficult to align the abstract with what students would develop intuitively as the formal definition of subspace. This excerpt also is an intersection between classroom orchestration, cognitive/learning, and content goals. This intersection illustrates challenges for the instructor to discern what content ought to be taught (content goal), how those jive with the class activities (classroom orchestration goal) and concerns that students may "think only about scalars that are...positive whole numbers" in the context of the problem and if the task will continue to "somehow reinforce that" knowledge (cognitive/learning goal).

Similarly, another instructor participant, I2 also talks about the transition from the task activity to the introduction to the abstract version of subspace definition.

I2: So, I do want to say like, so it seems like we, the intent or how people have been talking about this is that we're going to use these like non-negative integers for the exploration stuff, but then tell the students to use real numbers for the actual subspace definition.

This is where I2 has the intersection between the classroom orchestration goal and a content goal. In terms of classroom orchestration, I2 anticipates how the task will be used for "exploration stuff" and also plans when there will possibly be direct instruction to then “just tell the students” to use real numbers for the actual subspace definition. Also, in terms of the content goal, I2 discusses what mathematical concept—the actual subspace definition, should be brought up during instruction using the IOLA task. Like above, throughout the overall OWG meetings, I2 expresses some tension in the negotiation between the real-world context of the task and the abstract mathematical concepts.

The intersections between classroom orchestration, cognitive/learning, and content goals stand out in OWG discussions of Task 3, where the concept of subspace is introduced. Task 3 starts from a new 5×7 matrix that represents a new scenario of camera record in another school wing, and the set U is all the possible camera data vectors which leaves the number of students—in each room unchanged. In the last part of Task 3 (See Figure 2), the set U is meant to be connected to the concept of subspace, and then null space. Here, I1 and I2 communicate with each other by discussing their anticipation of implementing Task 3.

I2: Yeah. So, in the previous prompt, they have to, you know, U as defined as, you know, actual students and actual cameras, right? So that means the entries on U are the entries in the vectors and U, I think, have to be non-negative integers. It's then, closed under scalar multiplication for those scalars for non-negative integer scalars. And then we change the scalars in the box definition, but we don't change U. So, our students going to be confused about, I think, "no" is a reasonable answer to C. Right? I think they might say "no", because it is not closed under vector scalar multiplication because I can't scale by a fraction, or a non-negative number, and I'll get something that is not in U because we didn't change U. So, either. So do either. So, I feel like the changes have to be synced, right? Like the change to U. And the change to the scalars needs to be synced, otherwise C turns into false.

I1: I assume that's what we want. Is, is that not what we wanted? like to just point out that like, well, U is closed to being a subspace, but because the scalars need to be any real numbers. It's not? Maybe I misunderstood? The…

I2: I don't know. That was one I thought. I kind of wanted there to be like a thing you found was a subspace, punch line, students like that. I mean, maybe there's used to it.

Recall: Definitions for closure of a set under vector addition and scalar multiplication:

- A set of vectors S is called **closed under vector addition** when $v,u \in S \Rightarrow v + u \in S$.
- A set of vectors S is called **closed under scalar multiplication** when $v \in S \Rightarrow k*v \in S$ for any scalar k.

New Definition: Subspaces

- A non-empty set of vectors in R^n is called a **subspace of** R^n if it is closed under vector addition and scalar multiplication (where, <u>for our purposes, we assume scalars come from the real numbers</u> R).

5. Several statements are written about U and F below. Circle ones that are true, and modify the ones that are false so that they become true statements.

a. U is a sub<u>set</u> of R^7.	a. F is a sub<u>set</u> of R^7.
b. U is closed under vector addition	b. F is closed under vector addition
c. U is closed under scalar multiplication (scalars $\in R$)	c. F is closed under scalar multiplication (scalars $\in R$)
d. U is a sub<u>space</u> of R^7	d. F is a sub<u>space</u> of R^7
e. U is all of R^7	e. F is all of R^7
f. <add your own statement here>	f. <add your own statement here>

Figure 2. The statements about subspaces in Task 3

I1 and I2 presented contrasting approaches to the IOLA instructions, especially in terms of the tensions between the RME context and formal concepts in the subspace. It seems I1 liked to engage her students in conversation and whole classroom discussions related to the tensions in the subspace tasks. From a previous OWG session, I1 remarked on her experience in a previous "stellar class" with their discussion of span. When her students discussed span, she "...thought it was like, superfluous, but it turned out not to be." As it turned out, there was a tension or confusion caused by the RME-based scenarios in IOLA tasks so students only used positive whole numbers as scalar multiples. She faciliated a classroom discussed where she "freaked out" her students by introducing other real numbers such as e and pi. That led to her students and she having "deeper and deeper" communicatively engaged conversations, and so "that meant content coverage was sacrificed a little. And I decided I was okay with that." On the other hand, I2 seemed to prefer focusing on what may confuse students so he wanted to reduce confusion beforehand. The second instructor wrestles with what content should he bring into discussions in his class between himself on the teacher side and the mathematician side. This wrestling particularly happened when he addressed more formal mathematical concepts such as closure under scalar multiplication, non-emptiness, and dimensions of a subspace—they usually have the potential to conflict with the context of the IOLA tasks. He stated, "I feel it makes me a bad mathematician. Probably a better math teacher to not show them that."

Discussion

The results of this analysis demonstrate several points. Firstly, the goals for the instruction framework can be used as a coding scheme that highlights teaching knowledge instructors utilize as they reflect on their instruction. Secondly, both participants of the study demonstrated often overlapping goals for their instruction. This is most likely due to the varying knowledge they rely on to inform their instruction. We saw from Participant I1 a demonstration of pedagogical knowledge being utilized as she anticipated ways to host a discussion with her students to untangle the difference in the contextualized and abstract definition of subspace. The other participant, I2, mostly relied on his mathematical subject-matter knowledge and his knowledge of students to hypothesize ways to deliver content in the most streamlined manner possible. These insights of teaching knowledge these two instructors possessed were made possible due to the semi-casual nature of the OWG. The instructors participated the most in the work group, yet the facilitators actively engaged in questioning instructors on their decision-making all the while. It is because of the structure a lot of varying teaching knowledge is revealed.

It was through discussion between the participants that revealed tension in implementing the subspace task due to its incomplete definition. The two participants presented different approaches to the IOLA instruction as it related to ironing out this wrinkle. Our two participants highlighted how experienced IOLA instructors will utilize various domains of teaching knowledge while balancing their knowledge of the principles of Realistic Mathematics Education to problem-solve. Thus, we think postsecondary instructors can have different avenues to becoming IOLA instructors so that their approaches to resolving tensions would be different. Therefore, it will be important for both novice and experienced IOLA instructors to have a professional development space—that may look like OWG—to unpack their speculations and approaches and then move forward. After implementation, reflection on instruction is also vital for developing teaching knowledge of oneself and others.

In terms of teacher knowledge—in addition to reflection on teaching practice, the OWG provided an opportunity for instructors to communicate with other instructors and also with the curriculum developers on the insights of the instructional design. This process of examining tasks and reflecting on their implementation is especially vital for IOLA instructors, so the OWG serves the place for them to analyze and reflect on the curriculum they implement in their classrooms. As the instructors examine instructional task designs after listening to what other instructors unpack from their implementation, their approaches can adjust based on their previous examination. As a result, they were able to build their teaching knowledge as it relates to adjusting curriculum to serve their student populations. That demonstrates the importance of reflection for IOLA instructors. Sharing approaches to task implementation and analysis is beneficial, yet it becomes more powerful for other instructors if it sparks reflection on practice.

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References

Andrews, T. C., Speer, N. M., & Shultz, G. V. (2022). Building bridges: A review and synthesis of research on teaching knowledge for undergraduate instruction in science, engineering, and mathematics. *International Journal of STEM Education*, *9*(1), 1–21. https://doi.org/10.1186/s40594-022-00380-w

Auerbach, A. J., Higgins, M., Brickman, P., & Andrews, T. C. (2018). Teacher knowledge for active-learning instruction: Expert–novice comparison reveals differences. *CBE Life Sciences Education*, *17*(1), 1–14. https://doi.org/10.1187/cbe.17-07-0149

Andrews-Larson, C., Johnson, E., Peterson, V., & Keller, R. (2019). Doing math with mathematicians to support pedagogical reasoning about inquiry-oriented instruction. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-019-09450-3

Ball, D. L., Phelps, G. C., & Thames, M. H. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Burke, C., Luu, R., Lai, A., Hsiao, V., Cheung, E., Tamashiro, D., & Ashcroft, J. (2020). Making STEM Equitable: An active learning approach to closing the achievement gap. *International Journal of Active Learning, 5*(2), 71-85.

Cohen, D. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*(3), 311. 329

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111*(23), 8410-8415.

Freudenthal, H. (2002). *Revisiting mathematics education: China lectures*. Kluwer Academic Publishers

Haak, D. C., HilleRisLambers, J., Pitre, E., & Freeman, S. (2011). Increased structure and active learning reduce the achievement gap in introductory biology. *Science, 332*(6034), 1213-1216.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching, 48*(8), 952–984.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400. https://doi.org/10.5951/jresematheduc.39.4.0372

Kelley, M. A., & Johnson, E. (2022). The inquirer, the sense maker, and the builder: Participant roles in an online working group designed to support. *Journal of Mathematical Behavior, 67*, 1-14. https://doi.org/10.1016/j.jmathb.2022.100984

Kogan, M., & Laursen, S. L. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative higher education, 39*(3), 183-199.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129-146.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.2307/1175860

Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *The Journal of Mathematical Behavior, 26*(3), 247-266.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry-oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Wawro, M., Andrews-Larson, C. J., Plaxco, D., & Zandieh, M. (2023). Inquiry-oriented linear algebra: Connecting design-based research and instructional change research in curriculum design. In R. Biehler, G. Guedet, M. Liebendörfer, C. Rasmussen, & C. Winsløw (Eds.), *Practice-Oriented Research in Tertiary Mathematics Education* (pp.1-17). Springer.

Availing Orientations and Facilitating Behaviors: An Emerging Framework for Change Leaders

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Scholars and practitioners in higher education recognize that transformational change of organizations—especially departments and institutions—is difficult but essential to achieve needed, national-scale improvements in access, quality and equity in STEM instruction and career development. Based on studies of change projects in college mathematics education and gender equity on STEM faculties, we identify and describe a suite of common leadership approaches among change agents who led these projects. We propose that these approaches function as constructs for an emerging framework about change leadership. By observing how change agents lead complex change projects in higher education, we seek to develop theory about leadership for organizational change and to offer practical guidance to such leaders.

Keywords: institutional change, departmental change, leadership

RUME scholars are well familiar with calls for mathematics departments across the U.S. to improve student success in introductory mathematics courses (PCAST, 2012). Concerns about passing rates in these courses, coupled with student dissatisfaction with an overpacked curriculum that has little connection to their interests, are not new problems (e.g., Seymour & Hewitt, 1997; Seymour & Hunter, 2019). In response to these ongoing problems, professional societies have called for increased uptake of active learning, which is known to improve student outcomes (e.g., Abell et al., 2018; Freeman et al., 2014; Saxe & Braddy, 2015).

Such efforts to improve undergraduate teaching and learning in mathematics are not new. Decades ago a set of major, NSF-funded calculus reform projects sought to create a "lean and lively" calculus (Douglas, 1986). These largely focused on creating and disseminating new curricula and lab activities as the primary lever for change. These materials offered more challenging and relevant problems and often deployed small group work as a primary teaching strategy. Research on college STEM education was in its infancy and researchers were not ready to provide guidance for this kind of work in higher education. In retrospect, we can recognize this approach as insufficient: by focusing on new materials, calculus reform engaged a subset of faculty, but could be largely ignored by many others. High-quality instructional materials are necessary but not sufficient to motivate reform (Henderson et al., 2011).

Today, research-based materials and classroom approaches are widely available for many college courses (Fairweather, 2008). In addition to high quality instructional materials, current innovations are embracing inclusive active learning via sustained professional development. Helping individuals to develop the classroom skills, foundational knowledge of learning, and availing beliefs is essential support for their effective use of research-based curricula (Yoshinobu et al., 2022). Yet relying on individuals to take up innovative materials, one by one, is a slow route to widespread change. The most promising reform approaches today focus on deploying these resources—research-based materials and effective professional development—in a department-wide context with explicit attention to local culture and norms (Laursen et al., 2019).

In this paper we offer some insights derived from our work as scholars examining organizational change in higher education, particularly at the level of departments and institutions. Based on our studies of two quite different sets of change projects that use different interventions to accomplish distinct change goals, we identify a suite of common leadership approaches among change agents who led these projects. We classify these approaches, discuss how they manifest in change projects, and propose that they function as constructs for an emerging framework about change leadership. By asking how change agents lead complex change projects in higher education, we seek to develop theory about leadership for organizational change and to offer practical guidance to those doing such work.

Conceptual Foundations

While leadership and institutional change has long been a focus of scholarly inquiry in higher education (e.g., Eckel et al., 1999; Birnbaum, 1991), this area of research has yet to be broadly taken up in STEM reform contexts (Reinholz et al., 2020). In situating our theory-building contribution to this emerging field of inquiry, we draw on the distinction between a global change theory and a local theory of change (Reinholz & Andrews, 2020). A global change theory is an overarching, meta-level framework of ideas that provide backing or justification for the change process, typically empirically informed and grounded in ideas from fields such as sociology, psychology, or management. Examples include the Four Frames model (Bolman & Deal, 2008), the Networked Improvement Community (Bryk et al., 2015), and the River model (Elrod & Kezar, 2015). Such frameworks offer 10,000-foot views that neither refer to specific change projects nor outline specific interventions but may be used to explain or predict.

In contrast, a local theory of change is a project-specific description that links the overall goals and rationale to desired outcomes, planned activities, and indicators or metrics to explain how the intended local change is expected to come about. Thus, a local theory of change is pragmatic and action-oriented compared to a formal change theory. It may be graphically represented in a logic model or driver diagram (Kinzie & Kuh, 2017), and represents a ground-level view of a particular change plan for a particular place and time.

The framework for change leaders that we begin to develop here takes a 100-foot perspective situated somewhere between a global change theory and a local theory of change. As we detail below, the framework constructs include availing orientations and facilitating behaviors. We use these constructs to characterize change leaders' work, not to prescribe specific interventions or actions. They are aspirational, intended as touch points for decision-making. Local context and goals will drive how they manifest or take shape. In sum, if the local theory of change describes *what* work is to be done, our change leaders' framework offers advice on *how* to accomplish it.

Study Contexts and Data Sources

The authors have separately studied institutional and departmental change in different higher education contexts: efforts to change undergraduate instruction in mathematics departments, and efforts to advance gender equity on STEM faculties. Here we briefly describe these settings and the studies that support our cross-case analysis involving multiple cases of change projects.

Departmental Reform in Mathematics: The SEMINAL Project

The Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) project was an NSF-funded effort to better understand mechanisms for initiating and sustaining department change focused on implementing active learning in undergraduate mathematics classes. The project began with retrospective case studies of six departments that had implemented active learning in their precalculus and calculus courses (Smith et al., 2021b). The second phase of the project consisted of longitudinal case studies of nine mathematics departments as they rolled out their own change initiatives. Each sought to shift department norms to enable greater uptake of active learning in their precalculus and calculus courses, supported by nominal funding and a networked improvement community to help accelerate their local change efforts. The national call for SEMINAL participants drew 37 proposals, reflecting high interest in departmental approaches to change.

Two overarching themes characterize the change interventions chosen by the nine departments. First, many developed approaches to support instructors as they implemented active learning approaches in their classrooms. They created or adapted active learning instructional materials, offered professional development that met instructors where they are, and nurtured communities of practice. Second, departments created or reimagined the structures and policies that shape instruction from outside classroom walls. For example, departments created curricular structures to organize faculty around thematic groups of courses, developed course coordination policies and practices, and curated a wide range of instructional support materials in an accessible, easy-to-use format. All of these efforts sought to lower barriers that can prevent instructors from implementing the desired classroom changes and to establish norms of coordination and collaboration that help to align multi-section courses.

For this paper, we make use of SEMINAL data about the nine Phase 2 departments. Primary data include project reports that summarized interviews with project change leaders, faculty, administrators, student focus groups, and classroom observations, collected annually over three years. The SEMINAL team also interacted with campus team members via webinars and summer workshops. We conducted secondary analyses of these data and drew on these projects' own writing about their work, published in a special issue of PRIMUS (Smith et al., 2021a).

The ADVANCE Institutional Transformation (IT) program

The U.S. National Science Foundation initiated its ADVANCE program in 2000, calling for systemic approaches to address the persistent problem of women's underrepresentation on STEM faculties. Past programs had supported individual women but left untouched the biased institutional processes and hostile environments that generated the career challenges that women widely faced (Rosser & Lane, 2002). Taking a new tack, ADVANCE supported institutions to identify and remediate those structural and cultural barriers to women's recruitment, retention and advancement, and to share their strategies and tactics widely through both scholarly research and practice-focused dissemination. By 2023, NSF had made nearly 70 awards for Institutional Transformation—sizable ($3M) five-year grants to single institutions—and over 250 awards for adaptation, partnership and research that enable others to adapt and adopt the strategies and tactics developed across the ADVANCE community (Laursen & De Welde, 2019).

As a group, ADVANCE IT grantees have developed a variety of strategic interventions to address core challenges that face women STEM faculty. Laursen and Austin (2014, 2015, 2020) studied the strategies developed and tested by early ADVANCE awardees, and categorized them according to the core problem each addressed and the approach taken to address that problem. Laursen and Austin observed four broad strategies, aimed at (1) interrupting implicit bias in evaluation of faculty for jobs, awards or advancement; (2) improving workplace environments; (3) supporting faculty to fulfill both personal and professional responsibilities; and (4) fostering individual success. Within each broad type of strategy, an array of several specific interventions used in distinct settings is richly described in an online toolkit and book (2014, 2020).

For this paper, we draw on secondary analyses of data and insights gained from over two decades of work with ADVANCE as an evaluator and researcher. Primary data include interviews with 19 principal investigators, focus groups with 18 institutional teams, site visits to five campuses, and two working meetings with 27 change leaders, as well as deep analysis of a large library of documents gathered from over 40 ADVANCE IT projects.

Findings: An Emerging Framework for Change Leaders

It is challenging to shift organizational cultures to make the use of active learning the norm rather than the exception, and SEMINAL departments' progress varied from moderate to substantial. Given the often intransigent nature of departmental change, even moderate progress is a marker of success. The same is true of ADVANCE Institutional Transformation projects: institutions' progress on STEM faculty gender equity was variable, nonlinear, and context-dependent. Though they may seem very different on the surface, what these change projects have in common is that each used a variety of strategies as levers for change (Laursen et al., 2015). No single tactic alone did the job; rather, projects made headway through selecting and combining interventions to build a strategic portfolio that fit their local circumstances and conditions.

Building on the strategic choice of interventions, we noticed that change projects were impactful when the interventions within the portfolio were coordinated, synergistic, and guided by shared language and principles. In this analysis, we focus not on the specific change goals (improving student success in calculus; advancing gender equity) or interventions (shared materials, professional development; implicit bias training, partner hiring policy). Rather, we identify approaches that change leaders used in guiding their projects. Just as each project deployed a mix of interventions to accomplish targeted, local goals, likewise change leaders used a diverse toolkit in leading their group. We identify ten leadership approaches seen in change projects: five availing orientations and five facilitating behaviors (Figure 1). Below we describe these, with examples mainly from ADVANCE, then highlight a case study from SEMINAL.

Availing orientations	Facilitating behaviors
Considers a unit of change beyond the individual	Co-opts or plugs into existing structures
Takes a systems approach to change	Makes new ways of working easier
Attends to context and culture	Foregrounds inclusive practices and equitable outcomes
Leverages a theory of change	Addresses people's needs for a sense of purpose and meaning in their work
Promotes a non-prescriptive, asset-based view of people and system components	Regularly communicates with stakeholders

Figure 1. An Emerging Framework for Change Leaders: Availing Orientations and Facilitating Behaviors

Availing Orientations: Mindsets for Change Leaders

Following Muis' (2004) labeling of students' beliefs about learning, we label change leaders' beliefs as "availing" if they avail or advantage the desired change outcomes. This terminology avoids value judgments of beliefs as 'better' or 'more sophisticated.' We call them "orientations" to recognize that they are not dogmas but mindsets: ways to frame or think about a situation.

The first two orientations are foundational for leaders. In *considering a unit of change beyond the individual*, leaders focus on the goal of changing whole courses, curricula, processes, or programs. This does not mean that everything is upended at once! Rather, it portrays the concern of interest as a shared responsibility. While changes in individuals' knowledge, beliefs, skills and behaviors may be needed, leaders emphasize collective decisions and actions rather than calling out individuals' views and behaviors as the source of a problem or its solution.

Taking a systems approach to change acknowledges that the target course or program is a system of interacting parts. People take actions and enact their beliefs as instructors or advisors; physical structures and infrastructures such as physical spaces and add/drop policies steer or limit perceptions and behaviors; assumptions about the needs of client departments or traditions in assessment influence what topics are highlighted, how they are taught, and what is assumed about the needs of the student audience. A course is in turn embedded in the larger systems of the department, college and institution. Taking a systems approach means recognizing those elements explicitly, probing their functions and interactions, asking why things are this way, and looking for ways to rethink, rebuild, or mitigate the negative impacts of different components.

The next two orientations help leaders translate to their own setting these broad, foundational concepts. In *attending to context and culture*, leaders recognize that every organization has a particular context, based on its mission, history, geography, and role in national, state and local education landscapes. Two-year or four-year, public or private, urban or rural, secular or church-related, historically Black or predominantly white: such factors may reflect real differences in how problems appear and what strategies may work. Outside factors also shape a department, such as its relationships with other departments or with high schools whose students enroll in their courses. And context varies over time. For example, economic trends determine if the department can hire instructors or must tighten its belt; and changes to state policy may shift what courses the institution must provide. Contextual factors shape how a problem presents locally, and they mean that strategies cannot be imported wholesale from other institutions.

Thus it is also important for leaders to *develop and leverage a theory of change,* as described under Conceptual Foundations. Elements of a theory of change may draw on scholarship about how people and organizations change, but often more useful to leaders is a local theory that identifies the specific problems to be addressed, articulates the interventions to be tried, and explains the rationale: why will *these* activities help to solve *this* problem? Developing a theory of change forces leaders to articulate their ideas and assumptions before leaping into action, making visible what may be taken for granted. Sharing it can engage others and build buy-in to the rationale and goals of the project. Leaders can also make use of formal change theory to map out and guide interaction among the components of their local theory of change.

The final orientation emerges in part from the others. By conceiving of issues as arising from larger systems, yet accounting for local particularities, change leaders more readily see problems as shared, systems-embedded challenges rather than pointing fingers toward someone "at fault." They learn to spot local resources that can help them engage others or achieve their change goals. By *adopting a non-prescriptive and asset-based view* of the people, programs, and policies that constitute the system, they depersonalize the problem, invite others to be more curious and less defensive, and welcome multiple ideas and strategies for addressing the challenge.

Facilitating Behaviors: Tactics with Many Uses

We label change leaders' behaviors as "facilitating" if they facilitate or advantage progress on desired change outcomes. These are not specific actions, procedures or interventions, but approaches to leadership that work in concert with the availing orientations. In this way, the framework sits between formal change theory and on-the-ground local theory of change.

One facilitating behavior is to *co-opt or plug into existing structures*, especially entrenched infrastructure, relationships, and policies. Within departments, this may mean introducing course coordination or making teaching assignments that support use of common teaching materials and strategies. Co-opting existing structures may be less work than inventing new ones, and the solutions that emerge are more readily sustained if they are already embedded in the workings of the department. Conceptual structures are often useful too, such as important goals or campus-wide initiatives. Campus ADVANCE leaders met with more whole-hearted support from senior administrators when they articulated their faculty equity goals in language that referenced the institutional mission and mirrored its strategic plan, such as elevating scholarly activity, fostering interdisciplinary research and teaching, or coupling excellence to diversity, as they developed interventions to support women as research leaders or to diversify the faculty body.

Making new ways of working easier is a behavior that helps normalize new or revised ways of working. For example, some ADVANCE teams promoted the use of rubrics to fairly evaluate faculty job candidates, and developed templates and training to help search committees craft and use rubrics. These tools lessened committees' work to develop the rubric; using the rubric in turn lessened the work to filter candidates and negotiate ratings, because standards were clear and already agreed upon. Faculty who had experience with rubrics could then lead other search committees, and the rubric became an anchoring structure for new, normative hiring procedures.

A common way that leaders *foreground inclusive practices and equitable outcomes* is through strategic use of data. ADVANCE leaders used institutional data about faculty retention and promotion, for example, to show that gender inequities known from the literature pervaded their own campus, thus disrupting "Lake Wobegon" narratives that such problems happen elsewhere. They learned how to use social science studies to show the systemic roots of sexism and racism to STEM faculty (who are often unschooled in these fields or methods), and they saw values and behaviors shift as people came to understand gendered and racialized institutional practices as due not to individual "bad apples" but as systems built by and for the historic majority. Other ways to foreground inclusion are inviting diverse voices to the table, interrupting microaggressions, or providing opportunities to learn about inclusive teaching and mentoring.

When leaders *address people's needs for a sense of purpose and meaning in their work,* they are recognizing and harnessing the power of the symbolic frame to elevate certain values and give them cultural meaning (Bolman & Deal, 2008). Mathematics department leaders did this when they celebrated early wins in a long-haul change project or recognized project contributors with teaching awards or callouts in the department newsletter or website. ADVANCE leaders did this when they adapted professional development programs to recognize faculty needs for autonomy, linking individuals' goals to collective goals for leadership and inclusion.

Finally, leaders who *regularly communicate with stakeholders* are thoughtful and persistent in identifying what different constituencies need to know about their change initiative. They consider the clarity, coherence, frequency, and consistency of messaging, and how to use local data to inform and persuade different audiences. ADVANCE teams found that strategically tailored communications could build grassroots support, lessen resistance, and recruit allies who shared some degree of common purpose with their work. Speaking to department chairs, for instance, garnered their support when chairs discovered that ADVANCE could help them with mentoring early-career faculty or supporting associate professors in seeking promotion to full.

The Toolkit in Action: Case Study of a Change Project

California State East Bay (CSUEB) faced historically high DFW rates in precalculus (42% DFW) and calculus courses (36% DFW), especially among students from historically under-represented minority (URM) groups. Like many US mathematics departments, CSUEB relied heavily on part- and full-time lecturers. Because many taught at more than one institution, they often felt isolated or disconnected from the department, and largely relied on traditional lecture. After a multi-year transformation effort, CSUEB created and sustained a strong community of practice that includes lecturers and ladder-rank faculty, initiated a supportive course coordination system, and fostered a culture in which active learning is the new normal. With a keen focus on inclusive practices and equitable outcomes, they also lowered DFW rates to the high teens for all students, and all but eliminated the gap in DFW rates between URM and non-URM students.

Ever mindful of the local cultural value on instructor autonomy, the CSUEB team developed a change portfolio with three primary strategies. First, the team created and curated instructional material focused on "big ideas" and conceptual understanding, all aligned to a master syllabus. All materials were available to all instructors to adopt or adapt, but no one was required to use them. Second, they created a new structure of course coordination, based on a dynamic calendar that offered instructors a pacing guide, suggested group tasks, and helpful comments on content emphasis. Again, these resources were options for instructors to adapt or adopt as they saw best, with change leaders taking an asset-based perspective where instructors were seen as caring and talented, rather than in need of "fixing." Third, and linked to course coordination, they developed a community of practice with monthly meetings that offered instructors a network to support their professional growth, to develop shared understandings, and to build knowledge collectively. This went a long way toward addressing instructors' needs and sense of purpose and belonging.

Together, these three strategies reflect a systems approach to instructional improvement. By offering a variety of resources and embedded professional development opportunities, leaders welcomed all and made the desired change easy, resulting in a "new normal" of equitable active learning. CSUEB's story (Oliver & Olkin, 2022) is not a road map for what others should do, but an example of how leaders' approaches can help a department to succeed in its own context.

Discussion and Implications

In this initial framework, we identify some general ways of thinking and doing change. Observed among leaders of real transformational change projects, they show that meaningful change is possible. That is not to say, however, that leading change is easy. Maintaining the availing orientations may challenge deep-seated beliefs and long-held habits, and enacting the facilitating behaviors requires listening deeply and being open to changing one's mind.

Our focus on leaders shares some commonalities with the empirical investigation of mathematics course coordinators by Martinez et al. (2021), which revealed two orientations to the job (resource/ managerial and humanistic/growth). Knowing these orientations, the authors suggest, can help departments hire and support professional development for these key department personnel. Likewise, our framing of change leaders' availing orientations and facilitating behaviors may help change leaders accomplish their goals and may assist institutions to select and cultivate change leaders whose work will improve the academy.

The availing orientations and facilitating behaviors are neither independent nor linearly related. For example, a change agent's non-deficit view prompts her to seek out data and search for inequities with attention to local concerns about which groups may be privileged or excluded. Moreover, as change agents display these orientations and deploy these behaviors, they can nurture similar mindsets and skills among others, thus broadening or deepening their change coalition. Indeed, recognizing these orientations or behaviors may be a way to identify change leaders whose skills can be further cultivated to share in ongoing work.

At this time the framework is incomplete, as we have identified key constructs but have not specified how they relate to each other. In future work we plan to flesh out these relationships, seeking both to contribute to theory about leading organizational change in higher education and to develop the framework as a practical resource for change leaders.

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References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *Instructional practices guide*. The Mathematical Association of America, Inc.

Birnbaum, R. (1991; 2nd ed., 1999). *How colleges work: The cybernetics of academic organization and leadership*. Jossey-Bass.

Bolman, L. G. , & Deal, T. E. (2008). *Reframing organizations: Artistry, choice, and leadership*. Jossey-Bass.

Bryk, A. S., Gomez, L., Grunow, A. L., & LeMahieu, P. (2015). *Learning to improve: How America's schools can get better at getting better.* Harvard Education Publishing.

Douglas, R. G. (Ed.). (1986). *Toward a lean and lively calculus* (Vol. 6). MAA Press.

Eckel, P., Hill, B., Green, M., & Mallon, B. (1999). *Taking charge of change: A primer for colleges and universities*. On Change Occasional Paper 3. American Council on Education.

Elrod, S., & Kezar, A. (2015). *Increasing student success in STEM: A guide to systemic institutional change.* A Keck/PKAL Project at the Association of American Colleges & Universities.

Fairweather, J. (2008). *Linking evidence and promising practices in science, technology, engineering, and mathematics (STEM) undergraduate education*. Board of Science Education, National Research Council, The National Academies.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111*(23), 8410–8415.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching, 48*(8), 952–984.

Kinzie, J., & Kuh, G. (2017). Reframing student success in college: Advancing know-what and know-how. *Change: The Magazine of Higher Learning*, *49*(3), 19-27.

Laursen, S., Andrews, T., Stains, M., Finelli, C. J., Borrego, M., McConnell, D., Johnson, E., Foote, K., Ruedi, B., & Malcom, S. (2019). *Levers for change: An assessment of progress on changing STEM instruction*. American Association for the Advancement of Science. https://www.aaas.org/resources/levers-change-assessment-progress-changing-stem-instruction

Laursen, S. L., & Austin, A. E. (2014). StratEGIC Toolkit: Strategies for Effecting Gender Equity and Institutional Change. www.strategictoolkit.org

Laursen, S., & Austin, A. E. (2020). *Building gender equity in the academy: Institutional strategies for change*. Johns Hopkins University Press.

Laursen, S. L., Austin, A. E., Soto, M., & Martinez, D. (2015). ADVANCing the agenda for gender equity: Tools for strategic institutional change. *Change: The Magazine of Higher Learning*, *47*(4), 16-24.

Laursen, S. L., & De Welde, K. (2019). The changer and the changed: Evolving theories and practices of change in ADVANCE calls for institutional transformation. *Equality, Diversity and Inclusion: An International Journal, 38*(2), 140-159

Martinez, A. E., Gehrtz, J., Rasmussen, C., LaTona-Tequida, T., & Vroom, K. (2022). Course coordinator orientations toward their work and opportunities for professional development. *Innovative Higher Education*, *47*(2), 327-346.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research*, *74*(3), 317-377.
Oliver, J., & Olkin, J. (2020). A community of practice model for infusing active learning in the classroom. *PRIMUS*, *31*(3-5), 252-268.
Presidential Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. Office of the President.
Reinholz, D. L., & Andrews, T. C. (2020). Change theory and theory of change: What's the difference anyway? *International Journal of STEM Education*, *7*(1), 1-12.
Reinholz, D., Rasmussen, C., & Nardi, E. (2020). Time for (research) on change in mathematics departments. *International Journal of Research in Undergraduate Mathematics, 6*(2), 147-158.
Rosser, S. V., & Lane, E. O. N. (2002). A history of funding for women's programs at the National Science Foundation: From individual POWRE approaches to the ADVANCE of institutional approaches. *Journal of Women and Minorities in Science and Engineering*, *8*(3&4), 20 pages.
Saxe, K., & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025*. Mathematical Association of America.
Seymour, E. & Hewitt, N. M. (1997). *Talking about leaving: Why undergraduates leave the sciences.* Boulder, CO: Westview Press.
Seymour, E., & Hunter, A.-B. (Eds.) (2019). *Talking about leaving revisited: Persistence, relocation and loss in undergraduate STEM education.* Springer Nature.
Smith, W., Rasmussen, C., & Tubbs, R. [Guest Editors] (2021a). Infusing active learning in Precalculus and Calculus, *PRIMUS, 31*(3-5), 239-657.
Smith, W. M., Voigt, M., Ström, A., Webb, D. C., & Martin, W. G. (Eds.). (2021b). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning* (Vol. 138). American Mathematical Society.
Yoshinobu, S., Jones, M. G., Hayward, C. N., Schumacher, C., & Laursen, S. L. (2022). A broad doorway to the big tent: A four-strand model for discipline-based academic development on inquiry-based learning. *PRIMUS 33*(4), 329-354.

"More and New Insight": Undergraduate Mathematics Stakeholders See, Imagine, and Evaluate Selves through Collective Poems on Inclusive Teaching

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The use of inclusive teaching practices is novel for many instructors; there is a need to support instructors in envisioning what it looks like to teach inclusively. Related to this learning is a focus on reflection—on one's current self as an instructor and envisioning whom one could become as an inclusive teacher. This study explores reflections made possible by reading and responding to collective transcription poetry. Informed by Gutiérrez's conceptualization of equitable teaching as existing across critical and dominant axes, we took undergraduate mathematics program stakeholders' definitions of inclusive teaching practice and created collective transcription poems. We presented the poems to stakeholders, who then reflected on them. We highlight reflections indicative of stakeholders' current selves and possible selves, and the emergent theme of evaluative selves as ways in which to bridge these two dimensions of self with regard to inclusive teaching. We conclude by sharing directions for future work.

Keywords: inclusive teaching practices, critical methodology, reflections, possible selves, networked improvement community

Nationally, undergraduate mathematics courses have significant room for improvement when it comes to equitable and inclusive teaching and learning (e.g., Theobald et al., 2020). The work of moving toward more diverse, equitable, and inclusive undergraduate mathematics teaching and learning is not easy, nor can it be accomplished via a checklist type of approach. Rather, the work is complex and messy and involves people in professional relationships with one another, forming communities, and making sense of their current practices and how those could change for the better. Those involved in instruction in undergraduate environments bring their own identities into the classroom context (Ceglie & Settlage, 2019); identities can evolve as a person considers whom they are currently (current selves) and explores whom they might become (possible selves) (Gee, 2001; Markus & Nurius, 1986). We seek to understand the nature of these considerations with regard to inclusive teaching practices and values.

This study is part of a larger project studying how to support mathematics departments to positively impact diversity, equity, and inclusion. Each department formed a networked improvement community (NIC; Martin et al., 2020) of six to 11 members who met regularly to consider department data and make plans for action. The project brings a critical lens to this equity-focused work. Thus, we chose critical research methodologies that centered the voices of participants and engaged collaboratively with the participants to elicit their feedback and reactions to our research analyses. Here, we explore how NIC members engaged with poetry based on their definitions of inclusive teaching.

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Literature Review

We use Gutiérrez's work (2002; 2009; 2012) to conceptualize equity in undergraduate mathematics classrooms. Gutiérrez bifurcates issues of equity along the dominant axis and the critical axis. The dominant axis emphasizes the *access* students have to resources that allow them to participate in mathematics and how this access influences their *achievement*. This axis characterizes an approach to improving equity that focuses on helping students navigate and succeed in existing systems. In contrast, approaches aligned with the critical axis challenge these existing systems. The critical axis foregrounds student *identity* and *power* in mathematics, aiming to transform the system to better serve students, particularly those from marginalized communities. Both axes are necessary for equity-oriented reform, although the dominant axis is often highlighted in change efforts and is more commonly drawn upon by undergraduate mathematics stakeholders to justify enhancing diversity in STEM (Tremaine et al., 2021).

Inclusive teaching incorporates instructor mindset and practices along with the establishment of a classroom community in which students are attuned to hearing the voices of peers. Inclusive classrooms involve ongoing dialogue between instructors and students to develop and nurture a climate in which all participants have a voice, feel their voice is heard, and in which connections between content and students' lived experiences are explicit (Dewsbury & Brame, 2019; Freire, 1970; Nieminen & Pesonen, 2022; Saunders & Kardia, 1997).

Instructors may face barriers to implementing inclusive or equity-focused instructional practices, including their epistemological beliefs about mathematics, hesitance to engage in conversations about student identity, and prior schooling experiences (Dewsbury & Brame, 2019; Shultz et al., 2023). Many current instructors did not experience learning in inclusive classrooms and so may struggle to envision and implement inclusive teaching practices (e.g., Dewsbury & Brame, 2019). However, instructor growth to learn about and effectively adopt inclusive teaching practices is quite possible, given appropriate learning opportunities and support, such as faculty learning communities (Corrales et al., 2021) and departmental action teams (Corbo et al., 2015). A key to instructor growth is for instructors to not just hear about inclusive practices, but to actively engage in reflections on their teaching practices as part of intentional improvement efforts (Dewsbury & Brame, 2019; Schön, 1987).

Conceptual Framing: Possible Selves

Possible selves (Markus & Nurius, 1986) has been used as a conceptual framework to describe identity exploration, evaluation, and development (e.g., Dunkel, 2000), including professional identity in education (e.g., Blaney et al., 2022; Park & Schallert, 2020; Quaisley et al., 2023). We draw from possible selves to describe how instructors engaged with the poetry on an individual level. As individuals think about their potential future they draw from a collection of possible selves—who they might become—to guide their behavior. Possible selves are specific to an individual, representing a "cognitive manifestation of enduring goals, aspirations, motives, fears, and threats" (Markus & Nurius, 1986, p. 954) and emerge from whom one was. Possible selves are also influenced by the sociocultural and historical contexts in which the individual resides. Possible selves are a source of motivation for individuals, functioning as "incentives for future behavior," as individuals endeavor to align themselves with possible selves they consider ideal, or to reject behaviors they connect with possible selves to avoid (Markus & Nurius, 1986, p. 955). In this way, individuals engage in reflection, evaluation, and interpretation of the alignment between possible selves and their current self to analyze how the current self measures up to whom they want to become, could become, or fear becoming. Among the collection of possible selves is an ideal self—the self whom one strives to be. Related to our work, instructors may engage with the poems to evaluate or interpret how their current self aligns with the various possible selves they can be as a teacher, specifically related to an ideal self who uses meaningful inclusive teaching practices.

Using possible selves as a conceptual framing, we aim to better understand the ways in which undergraduate mathematics program stakeholders engage with identities that embody or experiment with inclusive teaching practices and values. We investigate the following research question: *In what ways do stakeholders reflect on current and possible selves in the context of inclusive mathematics teaching through engagement with collective poetry?*

Methods

The work of the NICs began in January 2022, and the larger project is collecting a wide array of data from the NICs. We began with the initial reflective journals from members of the Tau University NIC (TU-NIC) and Kappa University NIC (KU-NIC) (all names are pseudonyms). Among other journal prompts, we asked "*Describe what you think it means to teach with inclusive teaching practices.*" Participants consisted of mathematics instructors (including professors of practice and tenure-track faculty members) and administrators. They were given the option to submit their journals anonymously. We extracted the responses to that prompt from all 16 of the journals and coded them using Gutiérrez's (2002; 2009) dimensions of achievement, access, identity, and power. We then engaged in poetic transcription (Authors, 2022; Clarke, 2017; Prendergast, 2009) to capture the four dimensions. We constructed transcription poetry from the full set of excerpts for each NIC without stratification by participant; in this way, each poem contained excerpts from the words of multiple participants in each NIC. This method of *collective* poetic transcription enables maintenance of participant anonymity (e.g., Thunig & Jones, 2020). For TU-NIC, three researchers each wrote individual poems, and then met and created a set of poems. For KU-NIC, one researcher wrote all of the poems.

The collective poems for each institution were then shared with their respective NICs. We captured written feedback via individual Jamboards[2] containing the poems (two from KU-NIC; nine from TU-NIC) and field notes that captured verbal responses to the TU-NIC poems during a regularly scheduled TU-NIC meeting. Notably, participants responded in other ways; Figure 1 provides an example of an artistic response via drawing. Whereas the opportunities for multimodality of response was an intended strength of using the Jamboards to collect reflection data, we are still learning how to analyze non-textual and non-verbal responses and thus have excluded such arts-based modes of reflection from this particular analysis. We approached our analysis thematically, and used open coding on the written Jamboard reactions. We then met and discussed the codes that arose from each NIC's reactions, collectively combined related codes, and came to agreement.

The initial open codes included general reflections, connections to individual and collective practices (current or desired), challenges, posing questions, emotive responses, and responses to poetry as a form. After open coding, we engaged in axial coding, during which we saw connections in the data to possible selves (Markus & Nurius, 1986). Many of the NIC members responded to the poems in personal ways that compared the poems' collective NIC responses to their own values and teaching. Upon making this connection to possible selves, we revisited the

[2] Scholars have used Google Jamboard as a tool for brainstorming and fostering discussion in participatory research projects (e.g. Huỳnh et al., 2022). This study extends such use by engaging Jamboard as a data collection tool focused on NIC members' perspectives. Benefits of Jamboard include its multimodality of engagement and customizable environment.

data and coded it using current selves and possible selves as primary codes and identified evaluative selves as an emergent code that added nuance to our understanding of how stakeholders conceptualized themselves with respect to inclusive teaching.

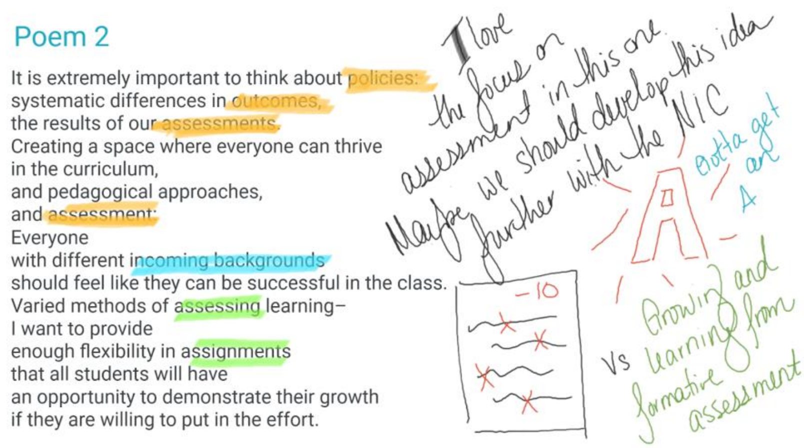

Figure 1. Screenshot of Jamboard response from a NIC participant.

Findings

NIC members engaged in reflections about current and possible selves through their interaction with the poems via the Jamboard platform. We identified three primary shapes taken by the NIC members' reflections—that of the current self, that of the possible self, or that which uses the possible self to evaluate the current self. Below, we detail these shapes, recognizing that all selves are fluid and thus that these reflections represent the specific selves relevant to each participant at the moment in time in which this activity took place (Markus & Nurius, 1986).

Reflections on a Current Self

Reflections on a current self are those in which the individual engaged with the poetry in a way that expressed an awareness of practices that they currently enact in their role as a mathematics instructor or administrator, or asserted a value that they presently hold related to teaching inclusively. Because these poems were constructed through the process of poetic transcription, seeing their colleagues' and their own words about what inclusive teaching can entail prompted moments of resonance or reflection, either with a value or a concrete action, that they currently incorporate into their professional practice. We detail some instances of these reflections below.

Many instances of reflections on current selves were provided in the form of value statements (Seah, 2002), in which an individual connected with the poetry by affirming a value about mathematics education which they read (either explicitly or implicitly) from the poem, or expanding upon the content of the poetry to profess a value which they presently hold in relation to inclusive teaching. For example, in response to the KU poem on Access, Tiersa wrote, "I think it's important to set clear policies, but I want those clear policies to be flexible policies." The poem led them to identify a component of inclusivity that they value: clear policies that maintain a degree of flexibility. In a particularly notable quotation, Connor reflected on the TU Achievement poem by writing that the poem's frequent mention of grading was a "damning perspective," as "learning is more important than grading." This reflection conveys a relativistic value presently held by Connor. In response to the lines, *It is extremely important to think about* *policies: // systematic differences in outcomes*, Skylar agreed and elaborated that this value is "why we have decided to have administrators as part of our NIC membership, because we wanted to examine and change this systemically." In this way, Skylar not only identified a value held by their current self, but connected that current value to a decision.

In some cases, the identification of a resonant value for inclusive teaching was accompanied by a wondering or an experience also embodied by their current self. Kayla, in response to the TU Achievement poem, wrote "understanding experience and barriers is super important—how can we possibly make it equitable?" This statement conveys that Kayla's current self values "understanding experience and barriers," *and* their current self struggles with how *we* on a broader scale might make *it* equitable. Although they did not explicitly define *we* or *it*, they appear to allude to some difficulty or tension associated with the value held by the current self. We see this also from Natalie: they state that "knowing [their] students as individuals is great. But it can become difficult whenever [they] have large class sizes." With this response to the TU Identity poem, they have both identified a value held *and* a barrier faced by their current self. In both of these cases, the ability of their current self to enact a value they hold about mathematics education is tempered by their perception of barriers or lack of agency in achieving it.

Other instances of reflections on current selves took place in the form of an individual directly bringing into their reflection a practice that they currently embody or an experience of their current self which has relevance to the poem. For example, Skylar notes that they "keep trying to build more flexibility into [their] syllabus" while simultaneously juggling continual student requests for "extra grace," a reflection which they wrote next to the lines *Offering exceptions in some cases– \\ a little extra grace?* in the KU Access poem. The question posed by the poem prompted Skylar to reflect on how their efforts to make their classroom—and syllabus—more inclusive do not appear to mitigate the experience of receiving requests for "extra grace" from students.

Reflections on a Possible Self

Reflections on a possible self included those in which the individual imagined or wondered about alternative versions of themselves as an instructor. For instance, some individuals engaged with the poems by contemplating new practices they might try out as instructors who get to know their students despite large class sizes (KU, Skylar) or as "flexible" instructors through the policies they create (KU, Tiersa). Markus and Nurius (1986) also consider past selves—who one was—as a form of possible selves and is another way in which one individual reflected on the poetry. Of the KU poem on Identity, Tiersa wrote: "I feel some shame at the way that I have failed in the past."

Note that these reflections on the possible self sometimes occurred concurrently with reflections on the current self. For example, Taylor reflected on the TU poem on Access: "Structure is important for students in the classroom and this poem makes me see that I need structure too. Note: annotate my stuff for students." Taylor first reflects on their value ("structure is important") and follows up with a new idea for a practice ("annotate my stuff for students") in alignment with a possible self (a more structured or organized instructor).

Individuals also connected with the poetry by describing certain teaching practices as either desirable or incompatible with their current self or practices, or as ones they imagined having difficulty attempting to enact or feeling afraid to enact. For instance, Ethan reacted to the TU poem on Identity by describing a way of engaging with students as incompatible with their personality: "I'm not a social person. It's uncomfortable for me when students are friendly on a personal level. I don't know if I can extend that sort of thing without it seeming fake." Abel reacted to the line "balancing grading policies" in the TU poem on Achievement by revealing a feared possible self; Abel wrote, "This is one of my fears, being fair."

An ideal self was described by multiple individuals as they engaged with the poetry. In response to the lines *Everyone // with different incoming backgrounds // should feel like they can be successful in the class* in the KU poem on Achievement, Skylar identified a desired goal or motivation for instruction: "YES, I really resonate with this goal. It's challenging, but definitely my aim." As these were collective poems, sometimes individuals also described an ideal culture in which each of its members share similar notions of self. For instance, in response to the KU poem on Power, both Skylar and Tiersa expressed their desire for feedback from other instructors and a culture of observation within the department. In this way, Skylar and Tiersa reflected on a personal ideal self—the collegial practitioner self—as well as an alternative community in which each of its members embodied this ideal self.

Evaluative Selves

Evaluative selves are those which exist at a particular intersection of current and possible selves. They took the form of NIC members using notions of a possible self to evaluate the current self, and thus their current values and practices. Markus and Nurius (1986) identify this evaluative bridge between current and possible selves in their work, noting that possible selves provide "a context of additional meaning for the individual's current behavior" (p. 955). In the context of these poetic reflections, we observed participants constructing bridges between possible and current selves by leveraging a possible self to evaluate the actions or values of their current self with regards to inclusive teaching.

For example, we consider Skylar's response to the following lines from the KU Identity poem: *Being attentive // in the problems that I write; // Embedding // culturally relevant examples*. Skylar highlighted these lines and wrote next to them that they "like this idea, and think it's something [they] need to work on." In the same body of text, Skylar reflected on efforts they currently make to incorporate examples that are specific to the KU context, but expressed concern that those examples may still be "exclusionary." Skylar considered two aspects of self—a possible self who is intentional about problem-writing and uses culturally relevant examples, and a current self which tries but may or may not be effectively incorporating these practices. This is a case in which the possible self functioned as an ideal self, utilized as a goal toward which the current self could work. We also saw one example in which a past self was used to evaluate a current self. In reflecting on the TU Achievement poem, Ethan wrote that they recognized from their own reflections the phrase *Hoping // for no disparaging trends // with pass rates or grades* and indicated that they "still really want to know what data from [their] own class would say." By signifying that this experience of the past self remains resonant, a bridge is built through which Ethan frames their current curiosity as something which continues to have value to their current self.

NIC members also engaged in evaluative reflections through the questions they asked in response to their reading of the poetry. These questions called their current self into a place of judgment, without necessarily making a conclusive evaluation of the current self. We view these as equally productive; while they may not have expressed an additional insight into their current values and practices by understanding them through a comparative lens with a possible self, the asking of these questions created space for such evaluation. Tiersa reflected on the KU Identity poem by writing the following questions: "Do I always use the right language? Can students less familiar with English understand my problems?" Implicit in these questions is a possible self which *does* use the "right language" and *does* construct problems understandable by students who are less familiar with English. Also implicit in the nature of questioning is the fact that Tiersa did not equate these attributes with their current self, but rather acknowledged a gap between their current self and this possible self in the context of inclusive teaching. We see something similar from Abel, who reflects on the line *Holding \\ to high standards* by writing "Are my policies applicable and fair to all?" In asking this question, Abel called their current self into a place of judgment—there exists some ideal possible self who *does* have policies which are fair to all, but in asking whether their policies do presently have this quality, they invite into their reflection a comparison of their current self to this ideal possible self.

Conclusion & Implications

Engaging with poetry afforded stakeholders the opportunity to see, imagine, and evaluate themselves with respect to inclusive teaching along Gutiérrez's (2009) dimensions of equity. By seeing the current self within the poems presented to NIC members, individuals felt validation regarding their values, beliefs, and practices surrounding inclusive teaching. Further, imagining possible selves offered individuals the opportunity to grapple with tensions and consider alternative ways of being as practitioners. Some participants also evaluated their current self through comparison to these possible selves, prompting consideration of who an ideal inclusive instructor is and how to become one. This reflection was perhaps a preliminary step toward what Ibarra (1999) describes as trying on a *provisional self*—a self of experimentation, a self that one tries on after observation of and engagement with role models (i.e., people with qualities that align with their ideal self). We found that this dialogue between the current and possible self spring boarded individuals' reflections beyond identifying similar (or dissimilar) values or questioning the feasibility of certain practices and into a problem space for potential experimentation with the current self.

Notably, we also found that the poetry engagement made an impact on some individuals' reflections in unique ways. Consider Taylor's critiques of the TU poem on Access alongside their empathetic connection to students:

> Despite not enjoying the poem as poetry, it did get me to see that doing these things is "easy", but the issue is getting students to make use of our actions. Thank you…If this poem was more structured, more clear, then I wouldn't have made connections for myself and they would have less meaning to me… Structure is important for students in the classroom and this poem makes me see that I need structure too.

Moreover, since this was *collective* poetry, individuals engaged with a variety of different ideas and insights about inclusive teaching. This perhaps gave them a more diverse repertoire of possible selves to construct regarding inclusive teaching. Some individuals expressed that reading collective poetry meant developing a closer understanding of the values of other NIC members, which could perhaps create space for additional community reflection.

Poetic transcription as a critical method highlights not just words or phrases, but also emotions and emphases. Through engaging with collective poetry, participants are invited to reflect on and discuss the ideas *and* feelings of fellow NIC members—and themselves—without placing peers in a vulnerable position. Future work could explore the impact of such reflections on the beliefs, values, and practices of undergraduate mathematics stakeholders. Skylar wrote that the process of reflecting on poetry was "a lot more impactful than [they] expected" and that they "got more and new insight into [their] colleagues' perspectives on inclusive teaching and also [their] own." This insight through novel methodology holds promise for extending our knowledge of how undergraduate mathematics stakeholders see, imagine, and evaluate themselves regarding inclusive teaching practices and values.

References

Blaney, J. M., Wofford, A. M., Jeong, S., Kang, J., & Feldon, D. F. (2022). Autonomy and privilege in doctoral education: An analysis of STEM students' academic and professional trajectories. The Journal of Higher Education, 93(7). https://doi.org/10.1080/00221546.2022.2082761

Ceglie Sr., R. J., & Settlage, J. (2019). Developing as a college science teacher: Using identity to examine transformation. International Journal of Scholarship of Teaching and Learning, 13(2). https://doi.org/10.202429/ijsotl.2019.130213

Clarke, C. L. (2017). Beyond the muskeg: Poetic expressions of a narrative inquiry into curriculum making and identity making on the edges of community (Doctoral dissertation, University of Saskatchewan).

Corbo, J. C., Reinholz, D. L., Dancy, M. H., & Finkelstein, N. (2015). Departmental Action Teams: Empowering faculty to make sustainable change. In A. Churukian, D. L., Jones, & L. Ding (Eds.), 2015 Physics Education Research Conference Proceedings (pp. 91-94). https://www.per-central.org/items/detail.cfm?ID=13919

Corrales, A., Goldberg, F., Price, E., & Turpen, C. (2021). Faculty persistence with research-based instructional strategies: A case study of participation in a faculty online learning community. International Journal of STEM Education, 7(21). https://doi.org/10.1186/s40594-020-00221-8

Dewsbury, G., & Brame, C. J. (2019). Inclusive teaching. CBE-Life Sciences Education, 18(2). https://doi.org/10.1187/cbe.19-01-0021

Dunkel, C.S. (2000). Possible selves as a mechanism for identity exploration. Journal of Adolescence, 23(5). 519-529. https://doi.org/10.1006/jado.2000.0340

Freire, P. (1970). Pedagogy of the oppressed (M.B. Ramos, Trans.). Continuum.

Gee, J. P. (2001). Identity as an analytic lens for research in education. Review of Research in Education, 25(1), 99–125. doi: 10.3102/0091732X025001099

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. Mathematical Thinking and Learning, 4(2&3), 145–187. https://doi.org/10.1207/S15327833MTL04023_4

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." Teaching for Excellence and Equity in Mathematics Education, 1(1), 4–7.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In Herbel-Eisenmann, B., Choppin, J., Wagner, D., Pimm, D. (Eds.), Equity in discourse for mathematics education: Theories, practices, and policies (pp. 17-33). Springer. https://doi.org/10.1007/978-94-007-2813-4_2

Huỳnh, J., To, P. D. N., Lee, C., Dang, T. V., Wu, J. T. C., & Tanjasiri, S. P. (2022). Care during COVID-19: A Virtual Asian American and Pacific Islander Photovoice Project. AAPI Nexus: Policy, Practice and Community, 19(2022-01-02T00: 00: 00+ 00: 00).

Ibarra, H. (1999). Provisional selves: Experimenting with image and identity in professional adaptation. Administrative Science Quarterly, 44(4), 764–791.

Markus, H., & Nurius, P. (1986). Possible selves. American Psychologist, 41(9), 954–969.

Martin, W. G., Lischka, A., Smith, W. M., & Lawler, B. R. (Eds.) (2020). The Mathematics Teacher Education Partnership: The power of a networked improvement community to transform secondary mathematics teacher preparation. Volume 4 in B. Benken (Ed.), Association of Mathematics Teacher Educators Professional Book Series. Information Age Publishing.

Nieminen, J. & H., & Pesonen, H. V. (2022). Politicising inclusive learning environments: How to foster belonging and challenge ableism? Higher Education Research & Development, 41(6) https://doi.org/10.1080/07294360.2021.1945547
Park, J. J. & Schallert, D. L. (2020). Reciprocity between doctoral students' emerging professional identity and their envisionment of a possible future self in real and imagined communities of practice. Learning, Culture and Social Interaction, 26 (100434). https://doi.org/10.1016/j.lcsi.2020.100434.
Prendergast, M. (2009). “Poem is What?” Poetic inquiry in qualitative social science research. International Review of Qualitative Research, 1(4), pp. 541-568.
Quaisley, K., Smith, W. M., Criswell, B., Funk, R., & Hutchinson, A. (2023). From becoming to being: How STEM teachers develop leadership identities. *International Journal of Leadership in Education*, https://doi.org/10.1080/13603124.2023.2292161
Seah, W. T. (2002). Exploring teacher clarification of values relating to mathematics education. Valuing mathematics in society, 93-104.
Saunders, D., & Kardia, D. (1997). Creating inclusive classrooms. Michigan Center for Research on Learning and Teaching. https://crlt.umich.edu/gsis/p3_1
Schön, D. A. (1987). Educating the reflective practitioner: Toward a new design for teaching and learning in the professions. Jossey-Bass.
Shultz, M., Close, E., Nissen, J., & Van Dusen, B. (2023). Enacting culturally relevant pedagogy when “mathematics has no color”: Epistemological contradictions. International Journal for Research in Undergraduate Mathematics Education. https://doi.org/10.1007/s40753-023-00219-x
Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones 2nd, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., Lowe, A., Newman, S., Okolo, V., Olroyd, V., Peecook, B. R., Pickett, S. B., Slager, D. L., Caviedes-Solis, I. W., Stanchak, K. E., Sundaravardan, V., Valdebenito, C., Williams, C. R., Zinsli, K., & Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. Proceedings of the National Academy of Sciences, 117(12), 6476-6483. DOI: 10.1073/pnas.1916903117
Thunig, A., & Jones, T. (2021). ‘Don’t make me play house-n***er’: Indigenous academic women treated as ‘black performer’ within higher education. The Australian Educational Researcher, 48, 397-417.
Tremaine, R. & Hagman, J. E. (2023). Creativity and subjectivity: Two researchers’ poetic re-presentations of critical student perspectives of mathematical success. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (pp. 1117-1124). Omaha, NE.
Tremaine, R., Hagman, J. E., Voigt, M., Damas, S., & Gehrtz, J. (2022). You don’t want to come into a broken system: Perspectives for increasing diversity in STEM among undergraduate calculus program stakeholders. International Journal of Research in Undergraduate Mathematics Education, 8, 365-388.

Analogical Quotient Structure Sense in Abstract Algebra: An Expansion of University Structure Sense

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Despite existing research describing students' understanding of group concepts, little research in undergraduate mathematics education has attended to students' understanding of ring concepts. In this paper, we present an analysis of four students engaged in task-based interviews that provided insights into their understanding of quotient ring when constructed as an analogy to quotient group. Using university structure sense (Novotná & Hoch, 2008) as a foundation, we propose an expansion of structure sense to include attention to various structures (e.g., subgroups and quotient groups), and attention to analogous structure across different contexts (e.g., attending to quotient structure across group theory and ring theory.) Findings suggest that while students uniformly attended to similar structural components when creating the concept of quotient ring, there was variation in the depth of their reasoning about why certain structures exist.

Keywords: Abstract algebra, Analogical reasoning, Quotient groups, Structure sense

Attention to structure is a key component for understanding advanced mathematical concepts. Various studies have investigated students' attention to structure in high school algebra, linear algebra, and abstract algebra. Efforts have also been made to parse students' attention to structure for practical applications to teaching. One approach to parsing students' understanding of and attention to structure is structure sense (Linchevski & Livneh, 1999), which refers to the ability to recognize the same structure flexibly and creatively across various contexts. Although structure sense was originally defined in the context of algebraic equation solving, Novotná and Hoch (2008) extended structure sense to also include university (or abstract) algebra. Specifically, they identified two categories of structure sense: (1) structure sense of elements within a set and their behavior under a binary operation (e.g., understanding closure), and (2) properties of a binary operation (e.g., understanding the relationship between inverses and identities).

The current undergraduate mathematics education literature emphasizes attention to basic aspects of algebraic structure in advanced mathematics, most often with a focus on central algebraic objects (such as group), and the operations associated with them. For instance, Serbin (2023) identified three ways in which secondary pre-service teachers reason about identities and found that a unified understanding of identity could lead to productive reasoning about identities in more abstract contexts. Cook (2014) described students' emerging understanding of ring-theoretic concepts of unit and zero-divisor. Such research has provided a rich understanding of students' sense of basic structure. However, structure sense has not been explicated in the literature for attention to structure in abstract algebra beyond groups, although examples of implicit attention to a wider variety of structure senses exist. For example, Melhuish et al. (2020) presented a case study investigating students' unified understandings of the function and (group) homomorphism concepts and described a variety of understandings, ranging from students who could not recognize homomorphisms as functions at all, to students who could explicitly leverage homomorphisms as functions when approaching tasks.

Given that several important structures exist within abstract algebra (Dubinsky et al., 1994), there is a need to explicate different types of structure sense. However, being that similar structures are also present in many areas across the advanced mathematical curricula (e.g., sub-structures in group theory, ring theory, and topology), there is also a need to describe students' sense of structure *across different contexts.* In this paper, we expand even further upon the proposed extension by Novotná and Hoch of structure sense to abstract algebra to also include structure sense for a wider variety of structures, as well as introducing a mechanism for describing structure sense across contexts. To elucidate this expansion, we pose the following research question: *What are four students' sense of structure associated with quotient groups and quotient rings?*

Theoretical Framing

Expanding on Structure Sense

To describe students' structure sense, we adopt Novotná and Hoch's (2008) structure sense for university (i.e., abstract) algebra as being able to recognize sets of elements together with binary operations and their properties in a range of familiar and non-familiar structures. However, our expansion of structure sense also includes recognizing broader structures found across the advanced mathematics context. For instance, structure sense for 'subgroups' may include the ability to coordinate a subset of a group with the same operation as the parent group. We call the original structure sense in abstract algebra (as discussed by Novotná and Hoch) a *fundamental* structure sense in reference to it being the structure sense for a fundamental object of study in advanced mathematics, such as a group, ring, or topological space. Thus, structure sense need not be constrained to a set and a binary operation, but may be generalized to a set of elements together with some collection of properties that define an object (i.e., a topological space is a set together with a topology defined on the set). In this study, we attended specifically to *quotient structure sense*. Table 1 below summarizes the two types of standard structure senses to be discussed in this paper.

Table 1. Types of Structure Sense in Advanced Mathematics

Fundamental Structure Sense	Recognition of properties inherent to the main object of study (e.g., groups in group theory)	Example: Recognizing the identity element in the dihedral group of a regular polygon.
Quotient Structure Sense	Recognizing properties required to formulate a quotient of a structure, or properties that follow from the definition.	Example: Understanding that the role of a normal subgroup is to allow for well-defined binary operations defined on cosets that respects the original group operation.

As it is central to this paper, we briefly expound upon quotient structure sense. Quotient structures possess a rich and complex structure that involves multiple levels of coordination to grasp in full. For example, there must be a coordination of a parent structure with a particular sub-structure through which an equivalence relation is defined. In the context of abstract algebra, quotient groups and rings require the identification of well-defined binary operations on the quotient structure, thus allowing for the quotient structure to itself be described as an example of the central object of study (e.g., a quotient group is itself a group.)

Analogical Structure Sense

To conceptualize reasoning across different contexts, we view students' comparative activity across contexts as a form of analogical reasoning. We draw upon the Analogical Reasoning in Mathematics (ARM) framework for investigating students' analogical reasoning (Hicks, 2020) which adapts several components of Gentner's (1983) Structure-Mapping Theory to the context of mathematics while also respecting students' potentially idiosyncratic forms of analogical reasoning. Within this framework, analogies are determined by mapping content from a source domain to a target domain. Unlike typical frameworks for analogical reasoning, ARM parses students' reasoning into individual analogical activities, thus allowing for a close examination of one part of a student's analogy (known as an instance of analogical reasoning), followed by a reconstitution of the instances to analyze the student's analogy as a whole. In this way, even if the final result of two students' analogies appear to be the same, their analogies can be distinguished by investigating how the analogy was created.

We introduce *analogical structure sense* to parse students' attention to structure during analogical reasoning across different domains. We define analogical structure sense as the ability to either recognize or create similar structure across two distinct domains. In other words, it is the ability to either (1) reason about which aspects of a known pair of structures are relevant when comparing what is similar and different, or (2) reason about which aspects of a structure in the source domain are crucial to the creation and development of an analogous structure in the target and which aspects of the source are superfluous. Analogical structure sense can be explicated across widely different domains, suggesting that a certain level of abstraction may be required in some cases, such as comparing the subgroup concept and the concept of topological subspace (Hicks, Flanagan & Park, 2022; see also English & Sharry, 1996). In contrast, we refer to a sense of structure as *standard* when it refers to only one domain.

To a degree, standard structure sense already appeals to basic forms of analogical reasoning: a student comparing two examples of groups to identify how they are similar in terms of their structure is reasoning by analogy. When viewed in this way, we recognize that analogical structure sense can be embedded within the original definition. However, for the purposes of this paper, we consider analogical structure in the case where the fundamental structure senses are different between the source and target, such as between groups and rings. Thus, analogical structure sense captures a wider range of structure senses than are possible with the standard definition alone. Table 2 summarizes the two types of analogical structure sense explored in this study.

Table 2. Types of Analogical Structure Sense in Advanced Mathematics

Fundamental Analogical Structure Sense	Recognizing (or creating) similar/different properties inherent to the main objects of study across two domains (e.g., groups and rings).	Example: Articulating that inverses are not necessarily required for multiplication in rings in contrast to inverses always being required for groups operations.
Analogical Quotient Structure Sense	Recognizing (or creating) similar/different properties inherent to quotient structures across two domains.	Example: Considering the need for both additive and multiplicative cosets when defining quotient rings by analogy with quotient groups.

Methods

The data for this study was collected as part of a larger project investigating students' analogical reasoning in abstract algebra. In the Fall of 2019 and the Spring of 2020, emails were sent out to students requesting participation in the study. Four students responded to the request: three advanced undergraduate math majors named Ellen, Nathan, and Brandon (all pseudonyms), and one graduate student in mathematics education named Andrew. Each student was then asked to participate in a series of 5 clinical task-based interviews (Goldin, 2000), with each interview lasting between 60-90 minutes. Each of the interviews was conducted by the first author, who will be referred to throughout as the interviewer. In this paper, we focus our attention on the interview associated with quotient rings in which students were posed the following task: *Make a conjecture for a structure in ring theory that is analogous to quotient groups in group theory.* In addition to this task, students were also asked several questions inquiring into their construction, as well as provided several tasks in which they leveraged their constructed analogue to reason about relevant content, such as generating examples or conjecturing theorems related to the structure.

The analysis of the data occurred in two phases: (1) an analysis of the students' standard structure sense for quotient group, and (2) an analysis of the students' analogical structure sense of quotient structure. The goal was to analyze the students' evoked structure sense, meaning we do not claim that these interviews revealed the full range of the students' structure senses. The video and transcripts of the initial interview were reviewed multiple times, and notes were made whenever any evidence of standard structure sense was present. For example, if the student explicitly mentioned the need for a normal subgroup to describe a quotient group, then evidence of standard structure sense was present. Profiles for each student's senses of structure (totaling 4 profiles) were then written, thus providing a holistic summary of the key points that the student attended to when recalling (or reviewing) the definition of the structure.

To analyze the students' analogical structure sense, the ARM framework was first used to code the students' instances of analogical reasoning while comparing and creating structures in each of the interviews 2-5. Each instance was then scrutinized for evidence of analogical structure sense, meaning whether or not there was evidence that they were reasoning about why certain features of the source should be mapped to the target when comparing or creating structures. In addition, these profiles included descriptions of the concepts that the students attended to while creating the analogous structures. Identifying these concepts often assisted with interpreting the students' analogical reasoning, and these differences revealed a variation in how the students were reasoning by analogy. Following this coding process, a profile was written for each student's sense of analogical structure associated with quotient ring (totaling 4 more profiles overall).

Findings

The four students displayed a range of analogical structure sense, from superficial grasp of the underlying analogous structures to a developing sense of the analogous structure for describing quotient rings. In the section that follows, we describe key holistic observations made about the students' attention to analogical structure for quotient structure. We present these findings in two sections: (1) describing what structures students considered to be relevant to their analogical construction of a quotient structure in ring theory, and (2) describing the variation of depth of the students' reasoning about quotient structure.

Attention to Existence of Analogical Structure: Normal Subrings and Cosets

All students acknowledged that a "normal subring" should be present when constructing the ring-theoretic analogy to quotient groups. Rather than immediately copy over the definition of normal subgroup to the ring context, each participant reflected upon what differences might be present when considering the definition of a normal subring. For example, Figure 1 below exhibits Andrew's constructed definition of a normal subring. Here, Andrew is attempting to reconcile the property of normality for subgroups as a property within the context of rings.

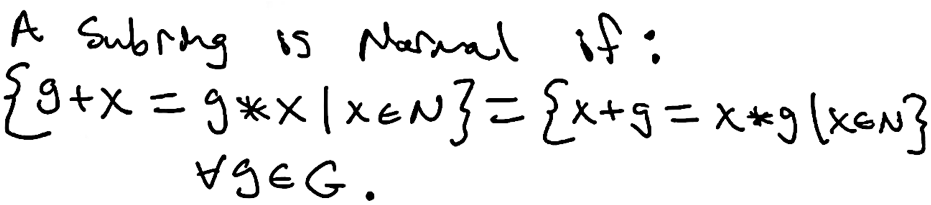

Figure 1. Andrew's definition of a "normal subring."

In the above example, Andrew is exhibiting two important features: (a) he is describing normality as a set of cosets being equivalent, and (b) he is appealing to the existence of two operations by expressing equivalence between additive and multiplicative cosets. This suggests that Andrew's sense of analogical structure included the existence of cosets and the manner in which binary operations were defined on those cosets. Such observations were present across each of the other students as well, meaning that all of the participants uniformly attended to the existence of normal subring, cosets, and adapting binary operations to the quotient ring context in some way.

Although analogical inferences related to the existence of structure across the contexts of group and ring theory may appear superficial on the surface, we contend that our participants' attention to the existence of these structures indicated a productive sense of analogical structure. Furthermore, as we discuss in the next section, the attention to the existence of analogous structure proved to be fertile ground for reasoning deeply about the meaning of those structures.

Variation in Students' Depth of Reasoning: Attending to Meaning of Structure

The students' sense of analogical structure not only entailed the existence of certain components related to quotient structure, but also included the underlying reasons for the existence of those components. However, the depth of reasoning varied from one student to another, thus distinguishing students' analogical structure senses in our analysis. Although all students identified the existence of a normal subring, only some students also investigated further by inquiring into why the normal subring concept was needed at all. In particular, Brandon's proposed description for the structure of a quotient ring was focused on describing a mapping from an element to the equivalence class of the element. Based upon evidence from the initial group interview, Brandon was presumably referring to an analogue for the canonical homomorphism defined for groups wherein every element in the domain is mapped to its coset in the quotient group. When asked what constituted normality in the context of his ring-theoretic analogue for quotient groups, Brandon responded: "I really only remember why we did it for groups." Although he was unable to expound further, he explained that if R' was normal, then the quotient R/R' would be a ring. Thus, Brandon's sense of analogical structure included attention to a quotient structure in fact being an example of the fundamental structure, and thus the normal subring would have to be carefully selected to make this happen.

While Brandon asserted that a normal subring was needed to form a quotient ring, he struggled to provide a description of normality. However, reasoning about the definition of normality for rings appeared in other ways across the students. In particular, each student attended to the need for adapting multiple binary operations to the ring-theoretic analogy for quotient groups. In general, the students made the following observations about binary operations and the quotient ring structure: (1) describing what coset operations look like, (2) determining how the existence of two binary operation affected the normal subring concept. Initially, Andrew focused his attention on just one operation at a time. He stated:

> What I would do, is I would literally just add the second operation to both of these. I'll do it in red what I have. I should probably make it clear though… This [digitally points to $x + H = y + \mathrm{H}$] is a plus operation, and… [writes "$x*H = y*H$"].

He then attempted to reconcile the meaning of normality when using the multiplicative operation: "Oh man, this is gonna be trickier. Because would they be the same? That's harder, let's think about this. I feel like this would be so hard to make happen [digitally points to $x*H = y*H$]." Returning to Andrew's construction of normal subring in Figure 1 above, we see this attention to multiple operations on display as Andrew eventually conjectured that the desired property for a normal subring was that $x + g = x * g$ and $g + x = g * x$

In contrast to Brandon and Andrew, Ellen did not attempt to describe the meaning of normality, nor did she ever formally define a coset in the context of rings. However, Ellen spent significant time reflecting upon how operations would be defined on cosets, thus attending to the need for a set of cosets to form a ring. She proposed the description seen in Figure 2 below. Of particular note was Ellen's attention to a "general" coset of the form aH as seen in the figure. Attention to this general coset included reasoning about a general coset operation, suggesting that Ellen was attending to more than just two operations for the specific context of rings. When asked to explain, Ellen stated: "I'm saying what operation specifically does this need to work for. For rings it's addition and multiplication. So, if you use addition, will that work, and if you use multiplication, will that work."

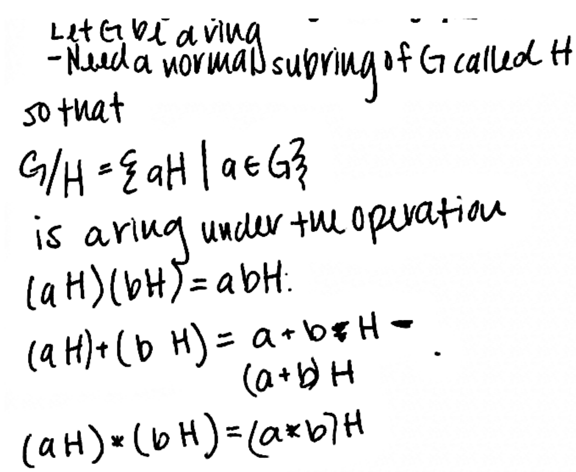

Figure 2. Ellen's proposed quotient ring definition and coset operations.

Summary of Findings

The findings presented above reveal that although all students in this study possessed a strong sense of analogical structure for the *existence* of structural analogies between quotient group and ring, their sense of analogical structure differed greatly in how they attended to the meaning of those structural analogies. Specifically, all students recognized ring-theoretic analogies to normal subgroups, cosets, and the role of binary operations within quotient groups. However, the students' sense of underlying meaning of those structures ranged from deep considerations about the purpose and definition of the structure, to a complete lack of attention to purpose or definition at all.

Discussion and Implications

Our proposed expansion of structure sense, to accommodate a wider variety of structures as well as structures across different contexts, allows for deeper explorations of students' attention to advanced mathematical structures. In this study, we elucidated this framework by investigating students' understanding of a particularly difficult concept: quotient ring. Findings indicate that students do indeed productively attend to several aspects of analogical structure when developing a ring-theoretic analogy to quotient groups, including attention to the need for a "normal subring," and attention to adapting multiple binary operations to the ring context. Overall, while all students were successfully able to make headway into describing a ring-theoretic analogy to quotient groups by spontaneously identifying the so-called "normal subring" concept, the notion of constructing a ring-theoretic analogy to normal subgroups proved to be a difficult task for students.

In contrast to structural similarity, semantic similarity (Holyoak & Thagard, 1989) refers to the degree of similarity between the meanings of objects within a domain rather than the global similarity of the structures as a whole. By investigating students' sense of analogical structure, we discovered that our participants uniformly attended to structural similarity, but varied in terms of their attention to semantic similarity. Although high semantic similarity exists between structures in group and ring theory due to their historical development (see Hausberger, 2018), only a subset of our participants spontaneously inquired into potential semantic similarities, focusing instead on structural similarity.

Our findings warrant suggestions for teaching abstract algebra concepts as well as considerations about the guided reinvention (Gravemeijer, 1999) of ring-theoretic concepts by analogy. The existence of a "normal subring" appeared to be a natural first step for students' construction of the quotient ring concept; however, their understanding of the analogous role that a normal subring should play appeared to be impeded by their understanding of the normal subgroup concept. Thus, our investigation of students' analogical structure sense suggests that, if analogies are to be used during instruction to compare quotient groups and rings, then there is a need to not only emphasize the coordination of structures on their own (e.g., describing quotient groups as a group together with a normal subgroup), but to also emphasize the conceptual underpinnings as to *why* normality is an important component of establishing quotient groups. We assert that this approach could establish an effective first step toward developing curricula for the guided reinvention of the concept of ideal. Furthermore, effectively leveraging of student analogical reasoning could promote productive instances of backward transfer (Hohensee, Willoughby & Gartland, 2022) in which students' understanding of quotient group and normal subgroup could be enriched by attending to the underlying relationship between quotient rings and ideals and making comparisons between the domains. Future research can investigate these matters in greater detail.

References

Cook, J. P. (2014). The emergence of algebraic structure: Students come to understand units and zero-divisors. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 349-359.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267-305.

English, L. D., & Sharry, P. V. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, *30*(2), 135-157.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, *7*(2), 155-170.

Goldin, G.A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177.

Hausberger, T. (2018). Structuralist praxeologies as a research program on the teaching and learning of abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, *4*, 74-93.

Hicks, M.D. (2020). Developing a framework for characterizing student analogical activity in mathematics. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 914 - 921).

Hicks, M.D., Flanagan, K. & Park, M.F. (2023). Explicating the role of abstraction during analogical concept creation. In S. Cook, B. Katz & D. Moore-Russo (Eds.). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education, Omaha, NE* (pp. 806 – 814).

Hohensee, C., Willoughby, L., & Gartland, S. (2022). Backward transfer, the relationship between new learning and prior ways of reasoning, and action versus process views of linear functions. *Mathematical Thinking and Learning*, 1-19.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science*, *13*(3), 295-355.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173-196.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior*, *60*, 1-16.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, *20*(2), 93-104.

Serbin, K. S. (2023). Prospective teachers' unified understandings of the structure of identities. *The Journal of Mathematical Behavior*, *70*, 1-17.

Graduate Students' Pedagogical Mathematical Practices
Used in Approximations of Teaching Practice Tasks

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This research design uses Approximations of Practice (AoPs), simulated practices of responsive pedagogy for prospective teachers to respond to, which lead them to make relevant connections between advanced mathematics courses and teaching practices. Data were collected from students in a Master's level Mathematics for Teachers course. The curriculum and interviews focused on three mathematical content domains: probability, algebra, and analysis. The AoPs were written into a semi-structured interview to demonstrate real-world examples of how teachers may see this material arise in the classroom. The AoPs prompted the participants to interpret and respond to student thinking and structure whole class discussions about students' strategies. Their responses were analyzed using Wasserman's (2022) pedagogical mathematical practices (PMPs). We examined each response to the AoPs to identify the participants' PMPs.

Keywords: Pedagogical Mathematical Practices, Advanced Math, Approximations of Practice

Learning advanced mathematics is an essential part of teachers' preparation, as it can help them understand relations among mathematical concepts and refine their mathematical practices, which can inform various pedagogical practices (e.g., Baldinger, 2018; Serbin, 2021; Zazkis & Kontorovich, 2016; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). The Conference Board of the Mathematical Sciences (2012) recommended that there be opportunities in mathematics courses for prospective teachers to make explicit connections between advanced mathematics and the teaching of secondary mathematics. However, these connections between the curricula and teaching are often left unstated (Wasserman & Weber, 2017), so it is essential to make them more explicit. Researchers and educators have made such explicit connections between advanced and secondary content in their innovative curricula (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Fukawa-Connelly et al., 2020; Goar & Lai, 2022; Wasserman et al., 2017).

Following these efforts, we designed a Master's-level Mathematics for Teachers course on connections from abstract algebra, real analysis, and probability to the teaching and learning of secondary / college mathematics. In this course, we implemented Approximations of Practice (AoPs; Grossman et al., 2009), in which the students, who were preservice or in-service teachers, simulated teaching practices of noticing students' mathematical thinking (Jacobs et al., 2010) and orchestrating class discussions about students' ideas (Smith & Stein, 2018). AoPs are productive in supporting prospective teachers in using their *knowledge* of advanced mathematics in their teaching practices (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Serbin, 2021). AoPs can also be productive for supporting prospective teachers in connecting the *mathematical practices* developed in their mathematics courses to their teaching. Wasserman (2022) intersected mathematical practices (Rasmussen et al., 2005) with pedagogical practices (National Council of Teachers of Mathematics, NCTM, 2014) to develop the theoretical construct of *Pedagogical Mathematical Practices* (PMPs), which are the disciplinary practices common to mathematicians and teachers. There is a need for researchers to identify PMPs that can be enacted in the classroom and AoPs. We address this need with this research question: *What PMPs do graduate prospective and in-service teachers engage in as they work on AoPs?*

Literature Review

AoPs are a valuable tool for engaging prospective teachers in pedagogical situations before entering the classroom. *AoPs* are "opportunities to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2056). AoPs can be enacted by teachers acting out classroom scenarios, interacting with simulated students in mixed reality simulations, interpreting written student work, or writing scripts for a class scenario. AoPs tend to require teachers to simulate a response to student thinking using pedagogical practices, such as noticing students' mathematical thinking (Jacobs et al., 2010), responding to hypothetical class situations (Zazkis & Marmur, 2018), and scripting discussions (e.g., Zazkis et al., 2013).

Instructors can use AoPs in their mathematics content courses and teaching methods courses for future teachers. For instance, Tyminski et al. (2014) used AoPs to support prospective elementary teachers' practice of organizing discussions around the different students' strategies. Campbell and Elliott (2015) created an AoP simulating a class discussion to define trigonometric ratios in a high school geometry class by introducing a problematic mathematical situation. These studies illustrate how engaging prospective teachers in AoPs can give them experience using certain pedagogical practices before they teach.

AoPs can bridge teachers' learning of advanced mathematics to their teaching (e.g., Álvarez et al., 2022; Burroughs et al., 2023; Lischka et al., 2021). Álvarez et al. (2020) suggested that teaching is a form of applied mathematics, so teaching applications should be incorporated into undergraduate mathematics courses' curricula, just as engineering and physics applications often are. Other researchers have used scripting tasks as simulations of practice to help students develop productive mathematical understandings (e.g., Marmur & Zazkis, 2022; Zazkis & Cook, 2018) and make connections between advanced mathematics and the teaching of secondary mathematics (e.g., Fukawa-Connelly et al., 2020; Serbin & Bae, 2023; Wasserman et al., 2017). Other researchers have used noticing tasks (Jacobs et al., 2010) to connect advanced and secondary mathematics. For example, Serbin (2021) demonstrated how prospective teachers used their abstract algebra knowledge as they interpreted and decided how to respond to hypothetical students' mathematical thinking in noticing tasks. Overall, researchers have documented the value of AoPs when embedded in the study of advanced mathematical topics. In this study, we further contribute to the research based on AoPs by analyzing the PMPs of graduate students, who are prospective or in-service mathematics teachers, as they engage in AoPs.

Theoretical Background

"Teaching relies on more than just knowledge" (Wasserman, 2022, p. 29); it also relies on practices. Just as researchers have explicated the intersection of mathematical knowledge and pedagogical knowledge (e.g., Shulman, 1987), Wasserman (2022) conceptualized the construct of Pedagogical Mathematical Practices (PMPs) as an intersection of mathematics and pedagogy with respect to practice. Practices of the discipline are the "regular actions, activities, habits, behaviors, processes, norms, etc. that one engages in while 'doing' that activity" (Wasserman, 2022, p. 5). Wasserman explored this domain of practice through the intersectional lens of mathematical practices (MPs) and pedagogical practices (PPs). MPs are the practices or mathematical habits of mind (Cuoco et al., 1996) that mathematicians engage in, such as identifying patterns, proving, conjecturing, problem-solving, and defining (e.g., Heid et al., 2015; Polya, 1945; Rasmussen et al., 2005). PPs are the practices teachers engage in as they teach mathematics, such as those outlined by NCTM (2014) and Hunter et al. (2016). The intersection of MPs and PPs are *PMPs:* "the regular actions, activities, habits, behaviors, processes, norms, etc., that are productive both as a mathematical practice and as a pedagogical practice for teaching mathematics" (Wasserman, 2022, p. 30). Examples include:

> Acknowledge and revisit assumptions and mathematical limitations; consider and use boundary cases to test and illustrate mathematical ideas; expose logic as underpinning mathematical interpretation; use simpler objects to study more complex objects; avoid giving rules without accompanying mathematical explanations; and seek out and use multiple explanations. (p. 30)

In addition to the utility of this construct for analyzing mathematics teachers' practices, the PMP construct also has useful implications for mathematics teacher preparation. PMPs have the potential for integrating pedagogy into advanced mathematics courses and for integrating advanced mathematics into pedagogy methods courses. This can contribute to preservice and in-service mathematics teachers perceiving more connections between advanced mathematics and the teaching of school mathematics. Given the need for researchers and mathematics teacher educators to support teachers in making such meaningful connections between advanced mathematics and the teaching of school mathematics, there is a need for researchers to identify additional PMPs and explore the contexts in which teachers use them. Our study contributes to this need through our identification of PMPs used by teachers as they engage in AoPs.

Methods

We recruited six participants from a large public research university in the southern US, Jessica, Roberto, Amy, Selena, Linda, and Eduardo (pseudonyms), who were enrolled in an MS in Mathematics program with a Math Education concentration. Jessica and Roberto were in-service teachers with multiple years of mathematics teaching experience. The others were preservice teachers who intended to teach high school or college. All participants had recently completed a Master's-level Mathematics for Teaching course, which addressed content from abstract algebra, real analysis, and probability with connections to the teaching of secondary/ college mathematics. The course instructors often assigned AoPs in class work and assignments.

The second and third authors conducted semi-structured task-based clinical interviews (Drever, 1995; Clement, 2000) with each participant two weeks after they completed their course. The interviews were conducted over Zoom and were recorded and transcribed for retrospective analysis. The participants performed the AoPs shown in Figure 1. These tasks were designed to simulate the practices of noticing/ attending to, interpreting, and deciding how to respond to students' mathematical thinking and leading class discussions about students' work. The four AoPs were each related to the mathematical content that the participants had learned during the Mathematics for Teachers course: conditional probability, the differentiability of functions, the zero-product property, and inverse functions. We designed the interview tasks to elicit evidence of the participants' PMPs used in their responses. We asked the participants questions related to what they noticed in the students' responses, what they interpreted about the students' understandings, how they would respond to the students as their teacher, and how they would facilitate a class discussion about the students' approaches. The first and second authors performed inductive coding on the interview transcripts (Miles et al., 2014), in which we open-coded PMPs that were evident in the responses to the AoP tasks. A participant's response was indicative of a PMP when it satisfied Wasserman's (2022) criteria: the practice was common to mathematicians and mathematics teachers. We used Wasserman's (2022; 2023) PMPs as a list of potential a priori codes to begin the creation of our codebook, and we added newly found PMPs to the codebook. We recoded each response to the AoP tasks in two rounds of deductive coding (Miles et al., 2014). We identified patterns among the coded PMPs across the responses, which provided insight into their similar or varied usage of PMPs.

a. AoP Task 1

In an introductory probability class, your students are discussing the following problem on a coin toss game.

> In this game, two players toss a fair coin 4 times each and compare who gets the most heads to determine the winner. Player A tossed the coin 4 times and got 2 heads. Suppose Player B tossed the coin 2 times and got 2 heads. Player B will toss the coin 2 more times and needs one more head to win the game. What do you think about the probability that Player B will win the game?

Read the following discussion between three students.

Student 1: Well, Player B already got two heads in a row. So, I guess it is much less likely to get a head in the remaining two attempts, because, you know, you would expect two heads on average out of four tossing in anyways, right?

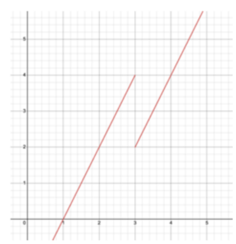
Student 2: We should think about all possible outcomes for tossing the coin four times. Each time, we have two possible outcomes of heads or tails, and repeat it four times, so there are $2 \times 2 \times 2 \times 2 = 16$ possible outcomes in total. These 16 outcomes are all equally likely, so the probability to get each is exactly $1/16$. Player B already got two heads and needs at least one more heads, so the only possible cases are $HHHT$, $HHTH$, $HHHH$. Since we have only 3 cases to win and each has the probability of $1/16$, the probability of winning the game is $3/16$.

Student 3: We know Player B will need at least one head in the remaining two attempts. Since the coin is fair, there will be 25% chance of getting two tails (TT) and 75% chance of getting one or two heads (HT, TH, HH). So, the chance of winning for Player B at this point is 75%.

b. AoP Task 2

In the discussion, three students explained whether the function is differentiable or not and why they think so.

$$f(x) = \begin{cases} 2x - 4 & (x \geq 3) \\ 2x - 2 & (x < 3) \end{cases}$$



Student A: This function is differentiable because I can see the two lines of its graph are parallel, so this function has the same slope everywhere including the point $x = 3$.

Student B: This function is differentiable because I can take the derivative of $2x - 4$, which will be 2 and then it's the same for the derivative of $2x - 2$. You just take the coefficient of the x. So, the derivative of this function is 2.

Student C: This function is not differentiable because its graph is not connected. We know continuous functions are differentiable.

c. AoP Task 3

Suppose you are teaching an Algebra II class on solving quadratic equations. While students work on the task of solving $x^2 + x - 6 = -4$, you see different students use these three approaches.

Student 1	Student 2	Student 3
$x^2 + x - 6 = -4$ $(x-2)(x+3) = -4$ $x - 2 = -4, \quad x + 3 = -4$ $x = -2, \quad x = -7$	$x^2 + x - 6 = -4$ $x^2 + x - 6 + 4 = -4 + 4$ $x^2 + x - 2 = 0$ $(x+2)(x-1) = 0$ $x + 2 = 0, \quad x - 1 = 0$ $x = -2, \quad x = 1$	$x^2 + x - 6 = -4$ $(x-2)(x+3) = -4$ $x - 2 = 2, \quad x + 3 = -2$ $x = 4, \quad x = -5$

d. AoP Task 4

Suppose you are teaching a Calculus I class on derivatives of trigonometric and inverse trigonometric functions. Your students are working on the task of finding the critical points of $f(x) = \cos^{-1}(x)$. You see different students use these two approaches for finding the critical points of $f(x) = \cos^{-1}(x)$.

Student 1	Student 2
$f(x) = \cos^{-1}(x)$ Use the power rule and chain rule: $f'(x) = -1 \cdot \cos^{-1-1}(x) \cdot -sin(x)$ $f'(x) = \cos^{-2}(x) \cdot sin(x)$ $f'(x) = \frac{sin(x)}{\cos^2(x)}$ Set this equal to 0. $\frac{sin(x)}{\cos^2(x)} = 0$ $sin(x) = 0$ Critical points: $x = 0 + k\pi$ for $k \in \mathbb{Z}$	$f(x) = \cos^{-1}(x) = \frac{1}{\cos(x)} = \sec(x)$ $f'(x) = \frac{d}{dx}\sec(x)$ $f'(x) = \sec(x)\tan(x)$ Set this equal to 0. $\sec(x)\tan(x) = 0$ $\sec(x) = 0$ or $\tan(x) = 0$ $\sec(x) = 0$ has no solution. $\tan(x) = \sin(x)/\cos(x) = 0$ Critical points are anything that makes $\sin(x) = 0$ but $\cos(x) \neq 0$ Critical points: $x = 0 + k\pi$ for $k \in \mathbb{Z}$

Figure 1. AoP Tasks Used in Interviews

Results

Overview and Examples of the Preservice and In-Service Teachers' PMPs

In the four AoPs, 15 PMPs were elicited (see Figure 2). PMP 1 was modified from Wasserman's (2022) PMP, "Acknowledge and revisit assumptions and mathematical limitations" (p. 8), and PMP 9 came directly from Wasserman's (2022) findings. We identified 13 new PMPs using inductive coding (see Figure 2). PMPs 1, 2, 3, and 7 were used the most frequently by all participants. PMPs 1, 2, and 3 rely on participants noticing and attending to student thinking. After noticing student thinking, participants frequently validated the mathematical procedure by identifying a mathematical concept that made the work correct or incorrect (PMP 7). An example of using these four PMPs in one AoP can be seen in Roberto's response to AoP 2 (see Figure 1b). The interviewer asked what Roberto noticed about the students' reasoning, and he said:

> Student A says that it has the same slope. I don't know if he's talking about the tangent, but that has nothing to do with this case. It's not really helping to answer the question if it's differentiable. The second student is just taking the derivatives of the piecewise function for each part and getting two, so he's just assuming that this is the same for the derivative, then it's going to be differentiable… differentiable functions are continuous, but the converse of that is not always true.

Roberto acknowledged assumptions (PMP 1) made by the students regarding slope and then invalidated the logic of that claim (PMP 2) by stating it did not answer the question about differentiability. The participant interpreted what the second student was doing in their work (PMP 3) and finally identified the concept of continuous and differentiable functions to invalidate Student C's assumption about differentiability (PMP 7).

Linda used PMP 11 regarding simplifying mathematical definitions into their own words when responding to AoP 2 (see Figure 1b). The interviewer asked if she would ask the students any specific questions to relate it back to the definition. She said,

> I would ask the class to maybe put the definition in their own words because I think sometimes definitions are a little bit fancy. To kind of reason them putting them into their own words would maybe help them connect what they're trying to say to the definition.

Simplifying a mathematical definition in your own words is a valuable tool mathematicians use to better understand a definition. Linda leveraged that PMP in her pedagogical techniques.

An example of PMP 15 was used in Roberto's response to AoP 3 (see Figure 1c). When asked how he would have the students recognize which one was wrong, he responded, "Probably put them in groups and work it out and see what the majority of the group gets or the class gets. If they get it right, I will use that definitely as the answer." PMP 15 was used seven times, mostly in response to the probability group discussion in AoP 1 (see Figure 1a).

Jessica shows an example of using PMP 12 in response to AoP 3, which asked her what she could tell about the students' mathematical understanding (see Figure 1c). Jessica said:

> Student 1 doesn't understand the clear definition of the product. If two things multiplied together, they can't each be -4, because that would be positive 16 there. While Student 3 understands the definition of a product, I feel like they don't understand while $2 \times (-2)$ does work, there's not a value of x that's the same that will work to give you that -4.

Jessica's response identified the procedural error in the student's work (PMP 12), which is a practice that both mathematicians and teachers would do when looking at their own or others' work. Jessica also interpreted the student's thinking (PMP 3), identified errors in their mathematical logic (PMP 2), recognized the concepts that validate a student's use of multiplication and factorization (PMP 7), and checked the student's work (PMP 6). This example shows how multiple PMPs can be used in a short response.

1. Acknowledge assumptions and/or mathematical limitations.
2. (In)validating the logic of a mathematical claim.
3. Interpreting details in someone's mathematical claim.
4. Referencing definition/formulas to verify a mathematical argument.
5. Using multiple representations to make sense of a concept.
6. Check work.
7. Identifying the mathematical concept that validates a mathematical procedure.
8. When multiple mathematical procedures are available choose the most efficient and accurate procedure.
9. Use simpler objects to study more complex objects.
10. Finding a counterexample to a mathematical claim.
11. Simplifying a mathematical definition in your own words to demonstrate understanding.
12. Identifying procedural errors in mathematical work.
13. Interpreting and understanding mathematical symbols and language.
14. Comparing and contrasting mathematical objects or concepts.
15. Combining multiple people's explanations to find a valid mathematical approach.

Figure 2. List of PMPs identified in the data.

Table 1. PMPs Used by Each Participant in Each AoP

	Probability AoP	Differentiability AoP	Equation AoP	Inverse AoP
Amy	1, 2, 3, 4, 5, 6, 7, 11, 12, 14, 15	1, 2, 3, 4, 5, 6, 7, 10, 11, 14	1, 2, 3, 4, 6, 7, 8, 9, 10, 12	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14
Eduardo	1, 2, 3, 7, 14, 15	1, 2, 3, 4, 7, 10, 14	1, 2, 3, 5, 6, 7, 10, 11, 12, 14	1, 2, 3, 5, 7, 9, 12, 13, 14
Jessica	1, 2, 3, 5, 6, 7, 11, 14	1, 2, 3, 4, 5, 10	1, 2, 3, 6, 7, 8, 12, 14	1, 2, 3, 5, 6, 7, 12, 13, 14
Linda	1, 2, 3, 7, 10, 15	1, 2, 3, 4, 7, 11	1, 2, 3, 6, 7, 8, 9, 12, 14	1, 2, 3, 5, 6, 7, 9, 14, 15
Roberto	None	1, 2, 3, 4, 7	1, 2, 3, 6, 7, 12, 14	1, 2, 3, 5, 7, 9, 10, 13, 14
Selena	1, 2, 3, 4, 7, 14, 15	1, 2, 3, 4, 5, 7, 10, 11, 14	1, 3, 6, 7, 12, 14, 15	1, 2, 3, 4, 5, 7, 13, 14

All participants used at least 13 of the 15 identified PMPs throughout the four AoPs, but each participant had some nuances in their use of PMPs. The PMPs used by each participant in each AoP are presented in Table 1. Amy consistently drew on multiple PMPs throughout the whole interview. She used 84 PMPs to answer 11 interview questions. In comparison, Roberto only used 33 PMPs. The lower number of PMPs used could be partly attributed to his uncertainty in his ability to solve the probability problem, which led him to not respond to the probability related AoP. Roberto focused heavily on the PMPs 1, 2, 3, and 7 that involved attending to student thinking and validating the mathematical procedures with the correct mathematical concept. Although he could use other various PMPs, he only leveraged a small subset of PMPs in the interview setting. There are common PMPs, such as 1, 3, and 7, that were used by each participant to respond to all class discussion tasks. Some PMPs were more prevalent in certain AoPs; for example, on the Probability AoP, the participants commonly used PMP 14 and PMP 15. They compared the students' approaches and combined their responses to lead a productive classroom discussion. For the analysis task, PMP 4 was elicited by 5 of the participants. Participants found it helpful to use the definition to answer the discussion task. That analysis-related AoP 2 also showed the highest use of PMP 10, where the participants used a counterexample to prove the student response was incorrect. We can see the highest use of PMP 6 for the algebra-related AoP 3 where 4 teachers used it to respond to the discussion task. This task asked students to find the zeros of a quadratic equation, and checking work was the most used PMP by the participants. PMPs 13 and 14 were the most frequently used for the algebra and analysis calculus-based task, in which students misinterpreted the meaning of the -1 superscript. PMP 13 involved interpreting and understanding mathematical symbols, and PMP 14 involved comparing the students' work as a tool to have a classroom discussion. Thus, certain PMPs were used more frequently depending on the mathematical content addressed in the task.

Participants' Use of Certain PMPs Seemed to Depend on the Type of Pedagogical Task

Each subsequent AoP question was intended to prompt the participants to use the information they identified in a previous task to inform their decision-making for the subsequent class discussion AoP. In the initial questions, where the interviewer asked the participant to identify what the students were doing in their work, the participants relied heavily on PMPs 1, 2, 3, and 7. These PMPs require noticing and interpreting student work. As the tasks progressed to the questions regarding how they would respond to the students or lead a classroom discussion about the students' ideas, the participants drew on their decisions from the prior tasks and utilized more PMPs. This is evident in Amy's response to AoP 1. Her response to the question about what she noticed about students' thinking used PMPs 1, 2, 3, 7, and 14. Her response to a question about how she would respond to the students involved her additional use of PMPs 4 and 10. In her response to the question about how she would facilitate a class discussion about the students' ideas, Amy used PMPs 1, 2, 3, 4, 5, 6, 7, 11, 12, 14, and 15. The subsequent addition of different PMPs used as the questions in each AoP task progressed gives evidence to the effectiveness of the task design in eliciting teachers' use of PMPs. Wasserman (2022) claimed, PMPs "can be used to structure discussions of, and give insight into, observed episodes of teaching and practicum experiences" (p. 13). Our research suggests that PMPs are a tool to help teachers structure their classroom discussions and attend to student thinking. As the tasks progressed, participants made decisions on the information presented in the earlier part of the AoP and used PMPs to structure a productive classroom discussion. The participants frequently used five PMPs for these discussion tasks, PMPs 1, 2, 3, 7, and 14. These PMPs may be particularly conducive for teachers leading classroom discussions about students' mathematical thinking.

Discussion and Conclusion

This study showed how the participating teachers used PMPs in AoP tasks where they engaged in hypothetical classroom situations. The use of PMPs implies that their experience in the course encouraged them to connect their advanced MPs with AoPs in the classroom. Prior to this study, the teachers learned advanced topics in real analysis, abstract algebra, and probability by engaging in MPs like defining concepts in abstract algebra and examining special examples of functions to understand the relationship between continuous functions and differentiable functions in real analysis. They applied those practices in the AoP tasks in this course. Their use of PMPs in the interviews conducted two weeks after the semester indicates that the design of the course contributed to preparing the teachers to carry over their practices from advanced mathematics into their future classroom teaching, which could be further examined by following their classroom teaching as shown in prior studies (e.g., Wasserman & McGuffey, 2021). These findings support the participating teachers' perceived changes in their views on the relevance of learning advanced mathematics to mathematics teaching by providing evidence of their demonstration of PMPs in the AoPs. This study gives evidence of their applications of practices in advanced mathematics to hypothetical classroom scenarios. Future studies are needed to examine how teachers use PMPs in actual classroom settings and to further investigate different patterns of using PMPs by content areas and by individual teachers.

References

Álvarez, J. A., Arnold, E. G., Burroughs, E. A., Fulton, E. W., & Kercher, A. (2020). The design of tasks that address applications to teaching secondary mathematics for use in undergraduate mathematics courses. *The Journal of Mathematical Behavior*, *60*, 100814.

Álvarez, J. A., Kercher, A., Turner, K., Arnold, E. G., Burroughs, E. A., & Fulton, E. W. (2022). Including school mathematics teaching applications in an undergraduate abstract algebra course. *Primus*, *32*(6), 685–703.

Baldinger, E. E. (2018). Learning mathematical practices to connect abstract algebra to high school algebra. In N. H. Wasserman (Ed.) *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers* (pp. 211-239).

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of Teacher Education*, *59*(5), 389–407.

Burroughs, E. A., Arnold, E. G., Álvarez, J. A., Kercher, A., Tremaine, R., Fulton, E., & Turner, K. (2023). Encountering ideas about teaching and learning mathematics in undergraduate mathematics courses. *ZDM–Mathematics Education*, 1–11.

Campbell, M. P., & Elliott, R. (2015). Designing Approximations of Practice and Conceptualising Responsive and Practice-Focused Secondary Mathematics Teacher Education. *Mathematics Teacher Education and Development*, *17*(2), 146–164.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, *547*, 341–385.

Conference Board of the Mathematical Sciences (CBMS). (2012). The mathematical education of teachers II. Issues in mathematics series: Vol. 17 (pp. 383-401). Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375–402.

Drever, E. (1995). *Using semi-structured interviews in small-scale research: A teacher's guide*.

Fukawa-Connelly, T., Mejía-Ramos, J. P., Wasserman, N. H., & Weber, K. (2020). An evaluation of ULTRA; an experimental real analysis course built on a transformative theoretical model. *International Journal of Research in Undergraduate Mathematics Education*, *6*, 159–185.

Goar, J. S., & Lai, Y. (2022). Designing activities to support prospective high school teachers' proofs of congruence from a transformation perspective. *PRIMUS*, *32*(7), 827–842.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055–2100.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23.

Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.). (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations*. IAP.

Hunter, R., Hunter, J., Jorgensen, R., & Choy, B. H. (2016). Innovative and powerful pedagogical practices in mathematics education. *Research in Mathematics Education in Australasia 2012-2015*, 213–234.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Lischka, A. E., Lai, Y., Czap, L., & Strayer, J. (2021). Revealing prospective secondary teachers' mathematical knowledge for teaching in teacher-created representations of practice. *The Mathematician Educator*, *2*(2), 86–106.
Marmur, O., & Zazkis, R. (2022). Productive ambiguity in unconventional representations: "What the fraction is going on?". *Journal of Mathematics Teacher Education*, *25*(6), 637–665.
Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all*. Reston, VA: National Council of Teachers of Mathematics.
Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton University Press.
Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73.
Serbin, K. S. (2021). *Prospective teachers' knowledge of secondary and abstract algebra and their use of this knowledge while noticing students' mathematical thinking* (Doctoral dissertation, Virginia Tech).
Serbin, K. S., & Bae, Y. (2023). How a graduate mathematics for teachers course "helped me see things from a different perspective." *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 343-351). Omaha, NE.
Smith, M. & Stein, M. K. (2018). *5 Practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.
Tyminski, A. M., Zambak, V. S., Drake, C., & Land, T. J. (2014). Using representations, decomposition, and approximations of practices to support prospective elementary mathematics teachers' practice of organizing discussions. *Journal of Mathematics Teacher Education*, *17*, 463–487.
Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, 49, 116–128.
Wasserman, N. H., & Weber, K. (2017). Pedagogical applications from real analysis for secondary mathematics teachers. *For the Learning of Mathematics*, *37*(3), 14–18.
Wasserman, N. (2022). Re-exploring the intersection of mathematics and pedagogy. *For the Learning of Mathematics, 42*(3) 28–33.
Wasserman, N., & McGuffey, W. (2021). Opportunities to learn from (advanced) mathematical coursework: A teacher perspective on observed classroom practice. *Journal for Research in Mathematics Education, 52*(4), 370–406.
Wasserman, N., Villanueva, M., Fukawa-Connelly, T., Mejía-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up to and stepping down from practice. *PRIMUS*, *27*, 559–578.
Zazkis, D., & Cook, J. P. (2018). Interjecting scripting studies into a mathematics education research program: The case of zero-divisors and the zero-product property. *Scripting Approaches in Mathematics Education: Mathematical Dialogues in Research and Practice*, 205–228.
Zazkis, R. & Kontorovich, I. (2016). A curious case of superscript (− 1): Prospective secondary mathematics teachers explain. *The Journal of Mathematical Behavior*, *43*, 98–110.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281.
Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics*, *31*(2), 8–13.
Zazkis, R., & Marmur, O. (2018). Groups to the rescue: Responding to situations of contingency. In N. Wasserman (Ed.) *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*, pp. 363-381.
Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. New York, NY: Springer.
Zbiek, R. M., & Heid, M. K. (2018). Making connections from the secondary classroom to the abstract algebra course: A mathematical activity approach. In N. H. Wasserman (Ed.) *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers* (pp. 189-209).

In Defense of Transformational Activity: Analyzing Students' Productive Reasoning about Equivalence

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Equivalence is a foundational idea in mathematics and a key fixture in the K-16 curriculum. There is considerable evidence, however, that students at all levels experience difficulties with it. A prevailing explanation is that students rely too much on transformations; and yet, transformational activity is absolutely essential: it is the primary means by which one generates more tractable representations that are better suited to the situation at hand. Strikingly, we found no studies that directly examine students' productive uses of transformational activity. To this end, we conducted a series of task-based interviews with undergraduate students in order to illustrate and account for productive instances of transformational activity across undergraduate mathematics. Our findings affirm a hypothesis from the literature that supplementing one's transformational activity with notions of equivalence can support productive reasoning. Additionally, we extend this idea by providing detailed analyses of what these supplementary notions of equivalence entail.

Keywords: equivalence, task-based clinical interviews, conceptual analysis, student thinking

Equivalence is one of the most fundamental notions in all of mathematics and, as such, is prevalent throughout the K-16 curriculum. Despite this importance, research suggests that students at all levels have difficulty leveraging equivalence to solve mathematical tasks (e.g., Chick, 2003; Chesney et al., 2013; Kieran, 1981). Researchers have argued that many of these difficulties are attributable to an overreliance on *transformations*–that is, operations that change a mathematical object into a new (equivalent) form. The prevailing explanation is that attending to transformations can preclude attending to underlying notions of equivalence (e.g., Alibali et al., 2007; Carpenter et al., 2003; Kieran, 1981). That is, "an overemphasis on change overshadows an emphasis on sameness" (Cook et al., 2022, p. 5).

We observe that much of the literature on transformational activity focuses on its role in students' difficulties (e.g., Cook, 2018; Pomerantsev & Korosteleva, 2003; Tall et al., 2014). And yet, transformational activity is absolutely essential in mathematics: it is the primary means by which one generates equivalent representations of an object that are more well-suited to the problem at hand. Clearly, then, transformational activity can–and should–be productive for students, but we found no empirical studies that feature and analyze productive instances of transformational activity. We infer from the literature that supplementing with other ways of reasoning about equivalence can support productive transformational activity, but what these ways of reasoning are and how they might emerge in students' activity is a question that has not yet been examined. To this end, in this study we aim to provide a "positive counterpoint" (Bagley & Rabin, 2016, p. 84) to the large body of work that primarily associates transformational activity with students' difficulties by analyzing episodes in which students use it productively. In doing so, we aim to answer the following research question: *What cross-domain ways of reasoning about equivalence do students demonstrate when engaging in productive transformational activity?*

Literature

This study addresses two gaps in the literature. First, we observe that most of the literature on students' reasoning about equivalence has taken place *within* particular mathematical contexts. At the K-12 level, for example, research has examined equivalence of fractions (e.g., Smith, 1995), numerical expressions (e.g., McNeil, 2008), algebraic expressions (e.g., Solares & Kieran, 2013), and algebraic equations (Knuth et al., 2006). At the undergraduate level, research has included, for example, examinations within the domains of combinatorial equivalence (Lockwood & Reed, 2020), isomorphism (Larsen, 2013), and modular equivalence (Smith, 2006). There is, however, a scarcity of research that has examined how students might reason about equivalence *across* various contexts.

We note that though the literature on students' transformational activity is expansive, it has almost exclusively been associated with students' difficulties (e.g., Godfrey & Thomas, 2008; Kieran, 1981; Pomerantsev & Korosteleva, 2003; Stephens, 2006). Other studies have pointed out how difficulties with transformations can constrain students' abilities to learn about subsequent ideas (e.g., Cook, 2018; Tall et al., 2014). To be clear, this body of literature establishes an important point about transformational activity: an overreliance on it can constrain students' reasoning. We do wish to call attention to the fact that even though the importance of transformational activity is difficult to understate, research that illustrates and analyzes empirical instances of productive transformational activity and what it might entail is scarce. The literature overwhelmingly focuses on students' difficulties. Our efforts here were inspired by Bagley and Rabin (2016), who, upon observing that computational activity has been oft maligned in the linear algebra literature, illustrated how it can be a very useful tool in certain situations. In the same vein, we aim to provide a "positive counterpoint" (Bagley & Rabin, 2016, p. 84) to the treatment of transformational activity in the equivalence literature.

The literature does, however, contain some provisional theoretical suggestions in this respect that shaped the current study. Alibali and colleagues (2007) argued that attending to the equivalence of equations involves "recognition that the transformation preserves the equivalence relation expressed in the first equation" (p. 223). Similarly, Harel (2008) argued that it is important for students to recognize that "algebraic expressions are not manipulated haphazardly but with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant" (p. 14); other researchers have made similar recommendations (e.g., Kieran, 1981; Steinberg et al., 1991). We interpret these comments to suggest that a key component of engaging productively with transformations involves attending to the reasons that the objects being generated by the transformations are equivalent. In the current study, we examine this initial hypothesis by examining the ways of reasoning about equivalence that students demonstrate in conjunction with their transformational activity.

Theoretical Framework

We adopted Cook and colleagues' (2022) framework for analyzing students' cross-domain ways of reasoning about equivalence. The framework is a *conceptual analysis* because it articulates "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 57). Specifically, it outlines ways of reasoning that the authors hypothesize capture meaningful aspects of equivalence as it manifests across mathematical contexts. These include:

- *Common characteristic*: involves attending to equivalence in terms of "a perceived attribute that the objects in question have in common" (Cook et al., 2022, p. 3).
- *Descriptive*: involves attending to the fact that objects "describe the same quantity or serve the same purpose with respect to a given situation" (Cook et al., 2022, p. 3).
- *Transformational*: transformational activity is defined as "a sequence of actions (either already performed or imagined) by which one object might or can be changed into another is enacted or described" (Cook et al., 2022, p. 3).

For example, consider how one might multiply the numerator and denominator of 1/2 by 3 to obtain 3/6. One might supplement this example of transformational activity by explaining that 1/2 and 3/6 are equivalent because they both correspond to the same real number: 0.5 (an example of a *common characteristic* way of reasoning). One might also reason that 1/2 and 3/6 are equivalent by imagining two different ways of shading a circle: both of these fractions correspond to the same amount of shaded area in relation to the area of the whole circle (an example of a *descriptive* way of reasoning because both fractions describe the same quantity of shaded area).

We primarily use Cook and colleagues' (2022) conceptual analysis as a lens through which to build and articulate models of students' ways of reasoning. We note that Cook and colleagues (2022) positioned their framework as their own articulation of key features of the equivalence concept that might be advantageous for students to attend to across contexts. It therefore remains unclear if and how these ideas might emerge when working with students. Put another way, the *first order model* (Steffe et al., 1983) developed by Cook and colleagues (2022) has not yet been used to construct *second order models* (Steffe et al., 1983) of students' reasoning.

Methods

In order to examine students' transformational activity and their associated ways of reasoning about equivalence, we conducted individual task-based clinical interviews (Clement, 2000) with 12 students (due to space constraints, in this proposal we focus only on two interviews). All participants had recently completed a three-course Calculus sequence; each student participated in a single interview ranging from one to two hours in length. Interviews were conducted by the second author. The tasks administered are shown in Figure 1. Students' written work was recorded on an iPad application and was synced to an audio recording.

After completing each task, students were asked to explain their reasoning, particularly how they came to (a) produce one mathematical object from another, and (b) replace one form of a mathematical object with another. For example, students who used row operations to transform the linear system in Task 2 to row echelon form were asked how the new system is related to the original and why it is an acceptable replacement for the original. Interviews were transcribed verbatim and enhanced with screenshots of the students' written work.

We classified an instance of transformational activity as "productive" if the (a) the student successfully used transformations to complete the task, and (b) their answers to these follow-up questions about the objects generated by their transformations involved a description of justification of why they are equivalent. Each enhanced transcript was then independently coded by the first, fourth, and fifth authors using Clement's (2000) interpretive analysis cycles; Cook and colleagues' (2022) framework provided an initial basis for coding. Though the framework was refined as coding progressed, due to space constraints we focus here only on illustrating instances of how students productively supplemented their transformational activity with *common characteristic* and *descriptive* ways of reasoning. The codes for each transcript were then compiled, and coding discrepancies were discussed and revised until a state of negotiated agreement was reached.

Task 1:	Solve the following equation: $3(3x + 1/9) - 2(x - 1/4) = 6(x - 1/36)$
Task 2:	Solve the following system of linear equations: $x_1 + 5x_2 + 2x_3 = 8$ $2x_1 + 4x_2 + 2x_3 = 8$ $x_1 + 5x_2 + x_3 = 7$
Task 3:	Evaluate the following definite integral: $\int_0^1 2xe^{x^2}dx$
Task 4.1:	Consider how we add hours of time on a 12-hour clock. For example, 4 hours from 9:00 is 1:00. We can represent this as $4 \oplus 9 = 1$. Evaluate the following sums: a) $7 \oplus 8$; b) $9 \oplus 9$; c) $10 \oplus 5$; d) $11 \oplus 4$.
Task 4.2:	Consider now how we might multiply hours of time on a 12-hour clock. For example, $3 \odot 7 = 7 \oplus 7 \oplus 7 = 9$. Evaluate the following: a) $2 \odot 10$; b) $3 \odot 11$; c) $4 \odot 8$; d) $5 \odot 9$.
Task 4.3:	Evaluate the following: a) $7 \odot 7$; b) $11 \odot 11$; c) $9 \odot 11$; d) $8 \odot 7$; e) $10 \odot 8$.

Figure 1. Tasks administered during the interviews.

Results

Here we present the results of our analysis of the students' cross-domain ways of reasoning about equivalence. In each subsection, we begin by summarizing the student's relevant *transformational* activity to provide context. We focus on Ethan's demonstration of a *common characteristic* way of reasoning (across linear systems and the integers modulo 12) and Molly's demonstration of a *descriptive* way of reasoning (across fractions and the integers modulo 12).

Transformational Activity with a Common Characteristic Way of Reasoning

Ethan's transformational activity on Task 2 (solving a system of linear equations) and Task 4 (modular arithmetic) was supported by a *common characteristic* way of reasoning about equivalence. Using row operations to transform the augmented matrix for the given linear system into reduced row echelon form (Task 2), Ethan explained that "you do it systematically, […] you want the triangle effect…" and that "…the goal is to get it to reduced row echelon…" (see Figure 2). This action of "reduction" into reduced row echelon (RRE) form is indicative of transformational activity, whereby Ethan translated the system of equations into an augmented matrix, and further produced equivalent augmented matrices.

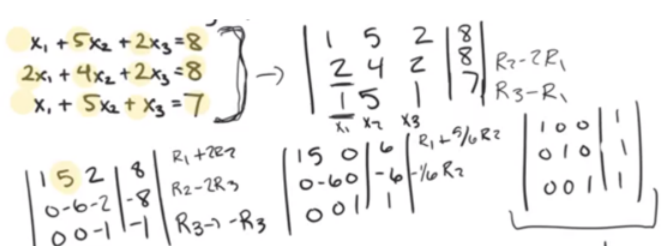

Figure 2. Ethan's written response to Task 2.

The interviewer then asked how the solution to the system represented by the RRE augmented matrix related to the system given in the task. Ethan's response suggested a *common characteristic* way of reasoning:

Ethan: This [RRE] helps complete the original task because […] if you plug these values in [1, 1, 1], they [the equations] would all come out as true. Like $1 + 5 + 2$, that equals 8. […] And then, if you went into that for every single one, you would see that the values would be true. […]

Interviewer: So this solution $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ is not just a solution to this last augmented matrix with the 0s and 1s?

Ethan: No. […] It's a solution for all of them throughout the whole time. If you went back through and made equations with these variables, they would all end up being true. So […] every single step of the way, […] these values would make those equations true.

Interviewer: So, […] even though you're changing-

Ethan: The coefficients?

Interviewer: You're changing the coefficients […],

Ethan: Right. […] So, even though you're changing the system, the values, the corresponding values will still be the same.

Ethan identified a shared attribute among the objects he produced–i.e., the solution of $(x_1, x_2, x_3) = (1,1,1)$–indicating a *common characteristic* way of reasoning about equivalence. More specifically, Ethan was aware that the result of his transformational activity preserved an attribute of the objects he was transforming.

Ethan's initial activity in response to Task 4 (modular arithmetic) centered on multiplying the two integers using the typical multiplication and then repeatedly adding or subtracting 12 until he obtained an integer less than 12 (see Figure 3).

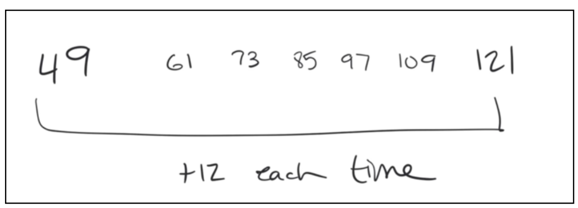

Figure 3. Part of Ethan's written response to Task 4.3.

On Task 4.2, after pointing out that $3 \odot 7$ and $5 \odot 9$ are both 9, the interviewer asked Ethan about the relationship between 21 and 45. Ethan explained that "if you add 12 to 21 you get 33" and "if you add 12 to 33 you get 45." He concluded, "so they are equivalent." On Task 4.3, similarly relating 49 and 121, Ethan said, "we have 49. We want to get to 121, and then […] I'm trying to add 12 to each interval, […] you're adding 12 each time" (see Figure 3). Ethan's transformational activity entailed adding 12 to one integer to produce another equivalent integer.

Ethan eventually pointed out a connection between division and his strategy of repeatedly adding/subtracting 12. He explained that "the reason that it's okay to subtract 12 is because that's basically what you do in division. […] It was how many times can 12 actually go into it, and it came out as, hey, let me just keep subtracting off of it." Shortly thereafter, the interviewer asked Ethan if he could characterize the integers that are equivalent to 4. He initially used his strategy of repeatedly adding 12, yielding 16, 28, and 40, before pivoting to the notion of remainder he had previously described:

Ethan: Their corresponding remainder will always be the original number you were wanting.
Interviewer: And in this case that's …
Ethan: Is 4.
Interviewer: I see. Okay. Even though you're adding 12 repeatedly?
Ethan: Correct. The remainder will always stay constant at that number.

Ethan demonstrated a *common characteristic* way of reasoning by identifying a shared attribute of the equivalent objects he was producing: their remainder after division by 12. Again, Ethan demonstrated awareness that his transformational activity (in this case, adding/subtracting 12) preserved the common characteristic (remainder after division by 12) of the objects in question.

A Descriptive Way of Reasoning about Equivalence

Molly demonstrated evidence of pairing transformational activity with a *descriptive* way of reasoning on Tasks 1 and 4. In her response to Task 1 (solving a linear equation), Molly transformed each fraction into an equivalent fraction with denominator of 36 in order to combine like terms (see Figure 4). The interviewer prompted Molly to explain why she made such replacements:

Interviewer: You rewrote […] -2/4 as -18/36. Can you talk about what you see as the relationship between those two?
Molly: Yeah. So, um, if you multiply -2/4 by 9/9 […] you get -18/36, but it doesn't change the value of the fraction. The fraction stays the same. It's just being written in a different way.
Interviewer: Okay. So, when you say, "The value of the fraction," what do you mean?
Molly: Um ... I mean, it's like if you take a circle and you cut it into 4 and shade 2, and then you cut another circle into 36 parts and shade 18, you'll see the exact same amount is shaded on both circles. […] So, that's how I know that's gonna be the same value.

The interviewer then prompted Molly to draw a picture to accompany her explanation, noting that she could choose another fraction to compare to 2/4 instead of 18/36 for ease of drawing (see Figure 4).

Molly: Like, if you do a circle that has 4, and you shade that, and then you have a circle that has 8, the same amount will be shaded in.
Interviewer: I see. And so, the relationship between 2/4 and 4/8 you're saying is similar to the relationship between 2/4 and 18/36 in here?
Molly: Yeah.

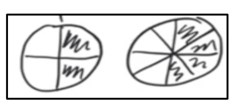

Figure 4. Part of Molly's written response to Task 1 to explain why 2/4 is equivalent to 4/8.

Molly's statement that "if you multiply -2/4 by 9/9 […] you get -18/36, but it doesn't change the value of the fraction" indicates transformational activity because she described a process ("multiply […] by 9/9") by which one fraction (-2/4) could be transformed into another (-18/36).

This transformational activity was supported by a *descriptive* way of reasoning when explaining *why* the transformation was valid. In particular, when describing why these fractions had the same value, Molly appealed to two circles having the same shaded region. This interpretation is further evidenced by Molly's drawing (see Figure 4) to show that 2/4 and 4/8 had the same value, explaining that "the same amount [i.e., the same area] will be shaded in" on both circles.

On Task 4, Molly produced integers she identified as "related" by repeatedly subtracting 12 until obtaining an integer between 1 and 12. Molly wrote" $n = N + x(12)$", where N represented the integer she was starting with, $x(12)$ signified adding a multiple of 12, and n represented the resulting integer. That is, Molly's transformational activity entailed a procedure by which one object could be obtained from a "related" one. The interviewer asked Molly to describe how she would use this process to determine if two given integers were equivalent:

Interviewer: So if I give you... two integers, let's say, um... 412 and 378... and asked you if they are related in the same way that you've been talking about these other numbers, how would you go about figuring that out?

Molly: … I would subtract both by, um, a multiple of 12, and I'll get a number. And then I'll see if n is equal. And if n is equal then I'll know that they, um... are related.

Molly then enacted her procedure of subtracting multiples of 12 from 412 and 378 until she obtained the numbers 4 and 6, respectively.

Interviewer: Okay. Now, what are 4 and 6? ... how are you thinking about those...

Molly: These are what their position on the clock would be.

Interviewer: Okay.

Molly: Um, and since 4 and 6 are not the same position, then, um, 412, um, and 378 are not related.

Here, Molly viewed the resulting objects of her transformational activity in *descriptive* terms: "what their position on the clock would be." Molly was able to judge the equivalence of 412 and 378 by reasoning that since the objects resulting from her transformational activity did not serve the same purpose with regards to telling time (i.e., their positions on a 12-hour clock), the objects were not equivalent.

Discussion

The analysis demonstrated here contributes to the literature in two ways. First, we have expanded the theoretical scope of the framework that we used to execute these objectives. That is, we used Cook and colleagues' (2022) first-order conceptual analysis of equivalence to construct second-order models of students' reasoning, demonstrating that these ways of reasoning can, in fact, account for students' reasoning. Second, our analysis establishes a counterpoint to well-documented difficulties associated with transformational activity in the literature. Previously, analyses of productive instances of students' transformational activity across domains had not yet been documented. In doing so, we affirmed a provisional hypothesis we inferred from the literature (that a key to reasoning productively with transformations includes supplementing with notions of how the objects being generated by these transformations are equivalent), and also explicated what these notions might entail (e.g., demonstrating *common characteristic* and *descriptive* ways of reasoning).

Acknowledgments

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References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221–247.

Bagley, S., & Rabin, J. M. (2016). Students' use of computational thinking in linear algebra. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 83–104.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Chesney, D. L., McNeil, N. M., Brockmole, J. R., & Kelley, K. (2013). An eye for relations: Eye-tracking indicates long-term negative effects of operational thinking on understanding of math equivalence. *Memory & Cognition*, *41*, 1079–1095.

Chick, H. L. (2003). Pre-service teachers' explanations of two mathematical concepts. In *Proceedings of the annual conference of the Australian Association of Research in Education*.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In Kelly, A. E. & Lesh, R. A. (Eds.), *Handbook of research design in mathematics and science education*, 547–589.

Cook, J. P. (2018). An investigation of an undergraduate student's reasoning with zero-divisors and the zero-product property. *Journal of Mathematical Behavior*, *49*, 95–115.

Cook, J. P., Reed, Z., & Lockwood, E. (2022). An initial framework for analyzing students' reasoning with equivalence across mathematical domains. *Journal of Mathematical Behavior*, *66*(100935), 1–15. https://doi.org/10.1016/j.jmathb.2022.100935.

Godfrey, D., & Thomas, M. O. (2008). Student perspectives on equation: The transition from school to university. *Mathematics Education Research Journal, 20*(2), 71–92.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In B. Gold, & R. A. Simons (Eds.), *Proof and other dilemmas: Mathematics and philosophy* (pp. 265–290). Washington, DC: MAA.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326. http://www.jstor.com/stable/3482333.

Knuth, E., Stephens, A., McNeil, N., & Alibali, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 36*, 297–312.

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior, 32*, 712–725.

Lockwood, E., & Reed, Z. (2020). Defining and demonstrating an equivalence way of thinking in enumerative combinatorics. *Journal of Mathematical Behavior*, *58*(100780), 1–4. https://doi.org/10.1016/j.jmathb.2020.100780.

McNeil, N. M. (2008). Limitations to teaching children 2+2=4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development*, *79*(5), 1524–1537.

Pomerantsev, L., & Korosteleva, O. (2003). Do prospective elementary and middle school teachers understand the structure of algebraic expressions? *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1–10.

Smith, J. P. (1995). Competent reasoning with rational numbers. *Cognition and Instruction, 13*(1), 3–50.

Solares, A., & Kieran, C. (2013). Articulating syntactic and numeric perspectives on equivalence: The case of rational expressions. *Educational Studies in Mathematics, 84*, 115–148.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

Steinberg, R. M., Sleeman, D. H., & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for research in mathematics education*, *22*(2), 112–121.

Stephens, A. C. (2006). Equivalence and relational thinking: Preservice elementary teachers' awareness of opportunities and misconceptions. *Journal of Mathematics Teacher Education, 9*, 249–278.

Tall, D., de Lima, R. N., & Healy, L. (2014). Evolving a three-world framework for solving algebraic equations in the light of what a student has met before. *The Journal of Mathematical Behavior*, *34*, 1–13.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45–64). Morélia, Mexico: PME.

The Evolution of Two Undergraduates' Example and Set Use During Conjecturing and Proving

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To better understand how students' example and set use might evolve during their conjecturing and proving activity, we engaged two students in guided reinvention of mathematical statements relating sequence properties during an 11-week teaching experiment. We characterized the students' evolving example/set use in three categories: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences. We exemplify these categories and then discuss some guidance that could explain the students' example/set use.

Keywords: Examples, sets, conjecturing, proving, sequences

Studies show that students find conviction and justify conjectures by using examples (Coe & Ruthven, 1994; Healy & Hoyles, 2000) and some suggest that students need to be supported to recognize limitations of examples (Knuth, 2002; Martin & Harel, 1989; Stylianides & Stylianides, 2009). More recent research examines how example-based reasoning can support deductive reasoning (Aricha-Metzer & Zaslavsky, 2019; Iannone et al., 2011; Komatsu et al., 2017; Ozgur et al., 2019; Pedemonte & Buchbinder, 2011; Sandefur et al., 2013), including how mathematicians and students use examples in their proving activity (Alcock & Inglis, 2008; Ellis et al., 2019; Inglis et al., 2007; Lockwood et al., 2016; Lynch & Lockwood, 2019; Weber, 2008). Furthermore, researchers have drawn on cognitive unity to explore how the development of students' conjectures based on their work with examples is reflected in the students' respective proving activities and proofs (Lin & Wu, 2007; Pedemonte & Buchbinder, 2011). Additionally, Fiallo and Gutiérrez (2017) studied what types of conjectures and corresponding proofs students developed after experimenting with Dynamic Geometry Software. This research points to the potential usefulness of examples for students in their conjecturing and proving activities.

Additionally, a growing amount of literature shows the usefulness of sets for students as they make sense of mathematical statements and engage in proving activities (Dawkins, 2017; Dawkins & Cook, 2017; Hub & Dawkins, 2018). For instance, Dawkins (2017) explained that it can be productive to consider a subset relation when making sense of a mathematical statement in the form *if p, then q*; that is, the set of all objects that fit property *p* is a subset of all the objects that fit property *q*. However, little research has explored how students can use sets to author and prove their conjectures.

To better understand how students' example and set use might evolve during their conjecturing and proving activity, we engaged two students in guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) of mathematical statements relating sequence properties during an 11-week teaching experiment (Steffe & Thompson, 2000). In this study, we examined the evolution of two calculus students' example/set use in response to an intervention we crafted for supporting students' conjecturing and proving activity. We investigated: How can students use examples and sets to support conjecturing and proving? Specifically, how did a pair of students use sets and examples as they were guided to reinvent mathematical statements relating sequence properties?

Theoretical Perspective

We investigated how two calculus students' example and set use evolved as they engaged in conjecturing and proving. By conjecturing, we mean developing a statement that relates mathematical concepts the student suspects are true but does not yet know are true (Lannin et al., 2011). In this study, we are particularly interested in students' conjectures from observing consistent patterns in a finite number of cases (Cañadas et al., 2007) and the generality of the students' conjectures (that is, whether the conjectures were specific to an initial set of finite cases or made a claim about additional cases). The cases that the students in our study observed were instances of sequences (e.g., $x_n = n^2$). Additionally, we use proving to mean conveying a general argument, from the students' perspective, for the statement's truth.

Throughout our data, students organized (and re-organized) examples of sequences into sets. We attended to different sets and set operations to better understand the students' set use. The students' universal set was key to understanding the generality of students' conjectures. By *universal set* we mean the set of all elements under consideration. The students' universal set began as an initial (finite) set of sequences, later expanded to include additional examples, and eventually included all sequences, where the specific examples were just representative members of this infinite set. Additionally, we determined whether students seemed to attend to a union, intersection, subset, complement, and difference (as typically defined in mathematics). For the sake of a relevant example, consider a universal set as the set of all sequences (represented by the shaded region in Figure 1A), which includes all the sequences that are bounded above (purple set in Figure 1), bounded below (green set in Figure 1), or unbounded. The shaded region in Figure 1B represents the *union* of the set of bounded above sequences and the set of bounded below sequences, whereas the shaded region in figure 1C depicts the *intersection* of these two sets. The shaded region in Figure 1D represents the set of sequences bounded below, which is a *subset* of the set of all sequences. The shaded region in Figure 1E represents the *complement* of the set of sequences that are bounded below. The *difference* between the set of bounded above sequences and the set of bounded below sequences is represented by the shaded region in Figure 1F. Finally, Figure 1G highlights that the set of convergent sequences (yellow set) and the set of sequences that are bounded above and not bounded below are *mutually exclusive*. We continue to denote sequences that are bounded above, bounded below, and convergent in this way (i.e., with green, purple, and yellow outlined sets, respectively) throughout our results.

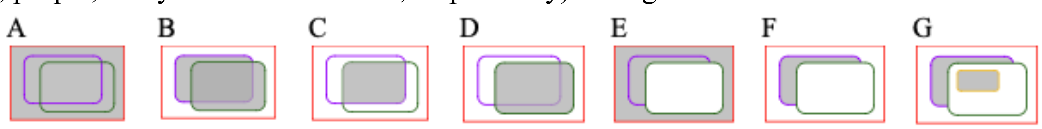


Figure 1. Representation of sets and set operations.

To further understand students' example use, we drew on Ellis et al.'s (2019) Criteria, Affordances, Purposes, and Strategies (CAPS) framework, which offers a way to describe students' example-based activity while investigating and proving conjectures. Specifically, we attended to Purposes and Strategies to understand how students' example-based reasoning supported their conjecturing and proving activity. *Purposes* are the reasons students use examples. For instance, a student might use the example(s) to: form a connection between two concepts (*conjecture development*), confirm their belief in a conjecture's truth (*belief confirmation*), determine a conjecture's truth (*test the truth*), reject a conjecture as true (*refute*), illustrate their claim that a conjecture is true or false (*convey claim*), identify the causal mechanism behind a conjecture (*understand why*), convey an argument supporting their claim that generalizes across cases (*convey a general argument*), and/or respond to a request or prompt (*respond to teacher-researcher*[1]). Strategies are "the range of deliberately strategic approaches students employ both in choosing and using examples" (p. 265). A student can deliberately create a new example that features different properties than previous examples (*seek diversity*), select examples based on particular properties of interest (*attend to properties*), or repurpose an existing example by varying one or more elements (*create systematic variation*). A student can deliberately use examples by searching for a pattern or a mathematical structure to identify general features (*searching for structure*) or developing a formal representation to express what is the same across all examples (*building formality*).

Methods

Data Collection

We conducted a teaching experiment (Steffe & Thompson, 2000) with two students, Lara and Stella (pseudonyms), recruited from a Calculus 1 course at a large university the semester before the experiment. We aimed to experience the students' mathematical learning of and reasoning about sequences and series as they formed conjectures about sequence and series properties, defined related terms, and proved their conjectures. Both students indicated they were freshmen at the time of the experiment, and both used she/her pronouns. Lara was majoring in Biological Chemistry, and Stella was double-majoring in Psychology and Neurosciences. Lara earned a 2.5 (on a 4.0 scale) in her Calculus 1 course, and Stella earned a 4.0. The experiment consisted of eleven 1.5-hour sessions in which the students' collaborative digital work was audio and video recorded. In these sessions, the first author was the teacher-researcher and the third author was the observer.

Realistic Mathematics Education, especially the heuristic of guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999), framed the instruction during the teaching experiment (similar to Lockwood & Purdy, 2019 and Swinyard & Larsen, 2012). We describe the general task design of the teaching experiment since the details are beyond the scope of this paper. The overarching design goal of the experiment was to guide students to conjecture and prove mathematical statements that relate sequence properties (e.g., all convergent sequences are bounded) and statements about series convergence (e.g., the comparison test). To do so, we engaged the students in context problems (Gravemeijer & Doorman, 1999) with sequences (sessions 1-7) and then with series (sessions 8-11). For this report, we analyzed data from Sessions 4-7, when we asked the students to sort sequences based on different properties (increasing, bounded above, bound below, and converging), make conjectures about sequence properties, and justify these conjectures. The first three sessions mainly focused on the students generating and defining sequences. We refer to these sequences as the students' initial set.

Data Analysis

We transcribed all teaching episodes and enhanced the transcripts that we analyzed for this study with embedded pictures of the students' work. We first identified instances of the students discussing connections between bounded, increasing, and/or convergence by reading the transcript and watching the corresponding video. Once we identified these instances, at least two of the three authors participated in coding the data, with any disagreements resolved through

[1] Ellis et al. (2019) refer to this as "placate the interviewer." We used this different phrasing since it better fit our teaching experiment setting.

discussion. In particular, we investigated the students' example use by identifying purposes and strategies using Ellis et al.'s (2019) framework (see previous section). We also identified additional strategies that we did not think were captured by the framework: *classifying* and *using lack of examples*. We coded data as *classifying* when it featured the student checking whether a sequence fit their concept definition and/or concept image (Tall & Vinner, 1981), which often occurred when students considered moving or adding sequences inside (or outside) a category. We coded data as *using lack of examples* when the students reflected on their inability to create an example with specific properties given the properties' parameters. During this pass of the data, we also considered how the students used sets, looking for evidence of the students considering an *intersection*, *subset*, *complement*, *difference*, and/or if sets were *mutually exclusive* (see previous section). Note that the students did not necessarily (if ever) use words like the "union;" instead, these were our way of denoting the sets that the students seemed to consider. We also searched for evidence of what the students considered their universal set (i.e., the students' initial set of sequences, some other finite set of sequences that expanded their initial set, or the set of all sequences). After we coded each episode, we wrote a vignette of the students' example/set use and compiled the vignettes into one document, organizing them chronologically. We then re-read the vignettes, noting changes in the students' example/set use (asking ourselves how the students' example/set use differed from/similar to the previous episode). Through this constant comparison, we noticed three categories that characterize the students' evolving example/set use, which we exemplify next.

Results

We characterized the students' evolving example/set use in three categories: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences.

Classifying examples from the initial set of sequences

The first example/set use category featured the students classifying sequences to develop, test, or refute a conjecture specific to their initial set of sequences. From our perspective, the students used their initial set of sequences as their universal set while they engaged in classifying; their diagram organized their initial set of sequences, and their discussion pertained to these specific sequences. An instance of this example/set use occurred during the fourth session in which the students were first asked to sort their initial set of sequences as bounded above and/or bounded below, or neither by placing their sequences in a corresponding set. In response, the students checked whether each sequence had an upper and/or lower bound (classifying). As they were classifying, Stella stated, "I feel like all of our a_n, $f(a_n)$, all those from that example, are going to be [bounded] both," referencing a set of sequences generated during a previous task. Here, we see that Stella noticed a pattern (searches for structure) to form a conjecture about the placement of some of their sequences from their initial set (conjecture development). From our perspective, she considered some of their initial set of sequences ("all of our a_n, $f(a_n)$…") as a subset of their bounded below and bounded above sequences. Then, to confirm that the pattern held (belief confirmation), Stella and Lara continued to classify the sequences they generated based on whether there was an upper and lower bound.

After the students finished classifying their sequences as bounded above and/or bounded below, the teacher-researcher requested that the students add a place for increasing sequences.

Lara shared that it should be "somewhere in the middle" because there was a sequence that was increasing, bounded below, and bounded above. Stella then suggested that they place it "hereish" (see the blue category in Figure 2A[2]) because "some of the [only] bounded below are increasing." During this, Lara and Stella selected increasing sequences (attended to properties) and determined whether they were bounded below and/or above (searched for structure) to make conjectures about their initial set of sequences that connected increasing and the other properties (conjecture development). The teacher-researcher then responded, "Let's try it!... With the examples or the sequences that we have so far, we'll try to put them in one of these locations. And then if we find that we need to change that box, then we will do that." To respond to the teacher-researcher and to confirm their suspicion about their increasing sequences (belief confirmation), Lara and Stella considered each sequence to check if it was increasing and rearranged the sequences accordingly (classifying). See Figure 2B for their revised map. From our perspective, we see the students suggested that their increasing sequences were a subset of the union of (a) their bounded below and bounded above sequences and (b) the difference in the set of their bounded below sequences and the set of their bounded above sequences.

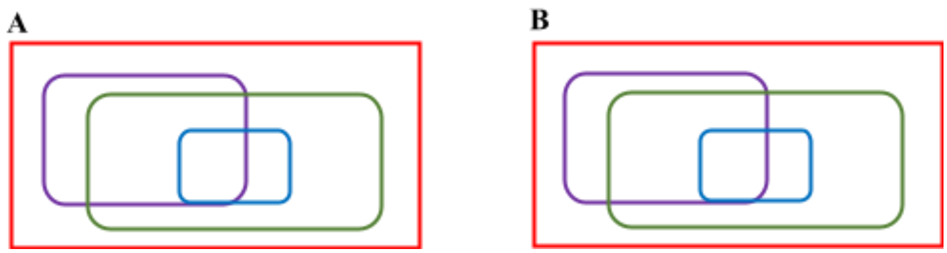
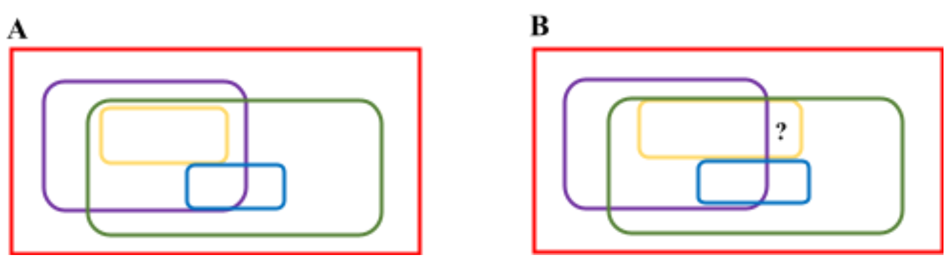


Figure 2. Location of the set of students' increasing sequences (blue set).

Seeking diversity and using lack of examples from an expanded initial set of sequences

The second category of example/set use featured the students seeking diversity in their initial set of examples and using the lack of examples to confirm their belief and understand why. From our perspective, the students expanded their universal set to include more sequences than their initial set while they sought diversity and used the lack of examples. Their discussion involved more than their initial set of sequences; in particular, they attempted to generate new sequences with different properties, and in doing so, their map transformed from organizing their initial set to potentially organizing additional sequences. An instance of this occurred later in the fourth session after Lara and Stella added a location for convergent sequences, which at the time they referred to as sequences "approaching a number" (see yellow set in Figure 3A). The teacher-researcher then asked, "Should [it] be stretched over?," changing the set's placement and adding a question mark to a new area to consider (see Figure 3B).



[2] We slightly altered these pictures from the students' version for clarity and because of space constraints. The students' version included more sequences with different representations (e.g., graphs).

Figure 3. Location of the set of sequences that approach a number.

To respond to the teacher-researcher and test the truth of the teacher-researcher's implied conjecture that there is a sequence that approaches a number, is unbounded above, and is bounded below, the students iteratively chose or created a new object to consider. In particular, they attempted to create a new sequence with the desired properties (seeking diversity, Figure 4A, Figure 4D, Figure 4E see below), selected ones on their map by paying attention to certain properties (attending to properties, Figure 4B), or created a new sequence based on a previous one (systematic variation, Figure 4C). With each new object, they disregarded it since it did not fit with a desired property (classify). Specifically, Figure 4A was bounded above since the intended domain was natural numbers, Figure 4B was not approaching a number, Figure 4C was bounded above, Figure 4D was bounded above and below, and Figure 4E was not approaching a number. After the students continued attempting to identify a sequence with the properties, Lara stated, "Maybe the answer is just no," meaning that there is not a sequence that is approaching a number, unbounded above, and bounded below. We interpret this as Lara reflecting on their inability to create an example with specific properties (using lack of examples) to confirm their initial placement of the converging set as non-overlapping with the difference of bounded above and bounded below (belief confirmation). From our view, the students suspected that the intersection of all convergent sequences, all bounded below sequences, and the complement of all bounded above sequences was empty (mutually exclusive).

Figure 4. Students' new objects.

After the teacher-researcher asked, "Can you think of a reason why we couldn't find a sequence?," the students began to understand why there was not a sequence that was approaching a number, unbounded above, and bounded below (understand why). To do so, Lara and Stella looked within the approaching sequences (attending to properties) and searched for a pattern, during which Lara noticed "no matter which section you take of the graph, it's bounded above and below" (searched for structure).

Attending to properties, searching for structure, and building formality with the set of all sequences

The third category of example/set use featured the students attending to properties, searching for structure, and building formality with the set of all sequences to convey general claims and general arguments. From our perspective, the students used the set of all sequences as their universal set (not just those depicted and organized on their map) while they attended to properties, searched for structure, and built formality. Students used examples and sets in this way during the fifth and sixth sessions, which began when the teacher-researcher asked the students to make conjectures based on their previous explorations. Stella responded, "I think it's true that a function cannot be both bounded above and increasing." After the teacher-researcher asked why not, Stella considered their increasing set and bounded above set (attended to properties) and seemed to notice that the increasing set intersected some of, but not all, the bounded above set (searched for structure), explaining, "So my thinking is, the blue increasing square does not have a category that's just bounded above. So, like increasing and just bounded above.” She then revised her conjecture (conjecture development), saying, "I guess you could say that it has to be bounded below to be increasing". From our perspective, Stella noticed that the set of increasing sequences was a subset of the set of bounded below sequences. After the teacher-researcher requested that they write this conjecture, the students began developing a formal representation in the form of a statement about what is the same across the increasing sequences (building formality), writing: “sequences that are increasing must be bounded below.” During the following session, the teacher-researcher asked for clarity about whether they considered this statement applicable for all or some increasing sequences. The students confirmed that they intended their conjectures to be about “all” increasing sequences rather than some of them or the specific examples in their map, and revised their conjecture to “all sequences that are increasing must be bounded below.”

The teacher-research then requested that they reflect on whether they thought the statement was true or knew it was true. Stella explained, "So I think we know this is true because all of the values are greater than or equal to the given number which would be… I guess it would have to be x equals one in this case.” Here, she noticed a pattern in that all the increasing sequences were bounded below by a value (searching for structure). She did this to understand what about the increasing sequences made them bounded below (understanding why). The teacher-researcher then questioned what she meant by $x=1$. To respond to the teacher-researcher and convey a general argument, Stella continued building formality to her argument revising her justification to : “all of these sequences [increasing sequences] must have a lower bound at a_n where $n = 1$.”

Discussion

In this study, we investigated how students' example and set use might evolve during their conjecturing and proving activity. Our findings suggest that students' example and set use can evolve over time, and specifically, we identified three categories of our participants’ example/set use: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences.

Our findings support previous research that indicates students can productively use examples and sets in conjecturing and proving activities. Our study further contributes to this existing literature by shedding light on how students might use examples and sets in tandem during conjecturing and proving. In particular, the sets served as an organizational tool for students’ examples. These sets evolved from organizing their initial set of sequences to a map that depicted general claims of the set of all sequences.

We also gained some insight into guidance that supported the students' example/set use by attending to the *responding to the teacher-researcher* purpose. We found that the first type of example/set use was supported by requests for students to place sequences into their appropriate categories and/or add new categories. The second type was often supported by questioning whether the students' initial placement of the category should be revised by expanding it to include a new category. Moreover, questions like, "Can you think of a reason why we couldn't find a sequence?" further supported students to understand why. The last type of category was supported by requesting that students write their conjectures and then reflect on whether they thought they were true or knew they were true, and in the case that they thought it was true but were not certain, supporting them to further search for structure. In our future work we hope to further explore such guidance and example/set use.

References

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*(2), 111–129. https://doi.org/10.1007/s10649-008-9149-x

Aricha-Metzer, I., & Zaslavsky, O. (2019). The nature of students' productive and non-productive example-use for proving. *Journal of Mathematical Behavior, 53,* 304–322. https://doi.org/10.1016/j.jmathb.2017.09.002

Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, O. (2007). The conjecturing process: Perspectives in theory and implications in practice. *Journal of Teaching and Learning, 5*(1), 55–72. https://doi.org/10.22329/jtl.v5i1.82

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal, 20*(1), 41–53. https://doi.org/10.1080/0141192940200105

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 496–522. https://doi.org/10.1007/s40753-017-0055-4

Dawkins, P. C., & Cook, J. P. (2017). Guiding reinvention of conventional tools of mathematical logic: Students' reasoning about mathematical disjunctions. *Educational Studies in Mathematics, 94*(3), 241–256. https://doi.org/10.1007/s10649-016-9722-7

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (2019). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior, 53*, 263–283. https://doi.org/10.1016/j.jmathb.2017.06.003

Fiallo, J., & Gutiérrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*(2), 145–167. https://doi.org/10.1007/s10649-017-9755-6

Freudenthal, H. (2005). *Revisiting mathematics education: China lectures* (Vol. 9). Springer Science & Business Media.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1–3), 111–129.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428. https://doi.org/10.2307/749651

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior, 50*, 90–102. https://doi.org/10.1016/j.jmathb.2018.02.001

Iannone, P., Inglis, M., Mejía-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77*(1), 1–14. https://doi.org/10.1007/s10649-011-9299-0

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3–21. https://doi.org/10.1007/s10649-006-9059-8

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379–405. https://doi.org/10.2307/4149959

Komatsu, K., Jones, K., Ikeda, T., & Narazaki, A. (2017). Proof validation and modification in secondary school geometry. *Journal of Mathematical Behavior, 47*, 1–15. https://doi.org/10.1016/j.jmathb.2017.05.002

Lannin, J., Ellis, A., & Elliot, R. (2011). *Developing essential understanding of mathematical reasoning for teaching mathematics in grades Pre-K-8* (R. M. Zbiek, Ed.). National Council of Teachers of Mathematics.

Lin, M. L., & Wu, C. (2007). Uses of examples in geometric conjecturing. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 209–216).

Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education, 2*(2), 165–196. https://doi.org/10.1007/s40753-016-0025-2

Lockwood, E., & Purdy, B. (2019). Two undergraduate students' reinvention of the multiplication principle. *Journal for Research in Mathematics Education, 50*(3), 225–267. https://doi.org/10.5951/jresematheduc.50.3.0225

Lynch, A. G., & Lockwood, E. (2019). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *Journal of Mathematical Behavior, 53*, 323–338. https://doi.org/10.1016/j.jmathb.2017.07.004

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41–51.

Ozgur, Z., Ellis, A. B., Vinsonhaler, R., Dogan, M. F., & Knuth, E. (2019). From examples to proof: Purposes, strategies, and affordances of example use. *Journal of Mathematical Behavior, 53*, 284–303. https://doi.org/10.1016/j.jmathb.2017.03.004

Pedemonte, B., & Buchbinder, O. (2011). Examining the role of examples in proving processes through a cognitive lens: The case of triangular numbers. *ZDM, 43*(2), 257–267. https://doi.org/10.1007/s11858-011-0311-z

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics, 83*(3), 323–340. https://doi.org/10.1007/s10649-012-9459-x

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 160–182). Lawrence Erlbaum Associates.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352. https://doi.org/10.5951/jresematheduc.40.3.0314

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education, 43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*(4), 431–459. https://doi.org/10.5951/jresematheduc.39.4.0431

Coloring the Relationship of Frames and Responses in Teacher Noticing

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This study builds on Louie, Adiredja, and Jessup's (2021) sociopolitical turn on teacher noticing. In this study, I use graduate student instructors' experiences as students and perceptions of their desirable actions to add nuance to the way Louie and colleagues discuss (anti-)deficit frames and noticing. The study uses a novel analytic framework to organize aspects of frames that begin to hint at a complex relationship between deficit and anti-deficit framing and responding.

Keywords: frames, teacher noticing, responding, graduate student instructors

There have been strides in our understanding of how graduate students develop as instructors (Beisiegel et al., 2019; Miller et al., 2018). However, we still require more nuance in the discussions of the development of teaching skills for graduate student instructors (GSIs), including in instructors' noticing within teaching. Typical descriptions of noticing tend to model instructors who respond to students in deficit-based ways as those who frame mathematics and learning in deficit-based ways; and instructors who respond to students in asset-based ways likewise are those who frame mathematics and learning in asset-based ways (e.g., Louie et al., 2021). Although scholars recognize this conception is idealized, there are few examples in the empirical literature of a more complicated relationship between framing and responding. In this proposal, I share results from an interview-based study of GSIs that provide such examples.

In this study, mathematics GSIs were interviewed before their instructor orientation to recount their experiences as students that I will argue inform the GSIs' noticing. This contributed report stems from a larger study of how GSIs develop their frames of teaching and learning within their first semester of teaching as the instructor of record. With the exploration of graduate students, the conceptual contribution of this study is to build on Louie, Adiredja, & Jessup's (2021) teacher noticing framework by coloring the gray relationship between deficit and anti-deficit phases of noticing and articulate a possible spectrum between deficit and anti-deficit frames and teaching responses. The professional contribution of this study is to clarify the professional development (PD) of GSIs by expanding the PD organizers' training for noticing skills.

These research questions guide the study: (1) *How do pre-PD first-time GSIs discuss student learning and their desired teaching practices?* and (2) *How do these discussions clarify our understanding of frames and noticing?*

Motivation from the Literature

Within the circles of research in undergraduate mathematics education, the development of graduate students as instructors is a shared curiosity. Miller et al. (2018) provided a literature review on the growing literature of GSI development which echoed the need for more understanding of how graduate students develop. In the following years, there have been studies that encompass various avenues of GSI development such as the obstacles of mathematics GSIs face in this development (Beisiegel et al., 2019), factors that influence GSI pedagogical empathy (Uhing, 2020), or the different types of programs that prepare GSIs to be instructors of undergraduate mathematics (Bookman & Braley, 2022; Ellis, 2014).

In a different focus, König and colleagues (2022) conducted a systematic literature review of 182 articles focused on teacher noticing conceptualization, study design, and findings. Within these, they found discussions of comparisons between novice and expert teachers (König et al., 2022, p. 13). However, there was no highlighted discussion of how an instructor develops their noticing skills. In particular, GSI and novice instructors' transition from student to instructor has yet to be explored in research of teacher noticing. Following the call for research of mathematics GSI development and this gap within the literature of noticing, this study continues exploration of the specific skill of noticing within the population of first-time teaching GSIs.

Theoretical Perspective

As discussed in König et al. (2022), teacher noticing has different conceptualizations (p. 8). This study follows from Jacobs et al. (2010) in which they deconstructed and extended the facets of noticing into three interrelated skills: attending, interpreting, and responding (AIR). *Attending* means to identify what is most important in the classroom (van Es & Sherin, 2008). *Interpreting* means to assign meaning to objects, such as students' questions, spoken or written solutions (van Es & Sherin, 2008). Lastly, *responding* alludes to the plan or action of answering the interpreted or attended student contribution in the classroom (Jacobs et al., 2010).

Hill and Chin (2018) provide a recounting of the role AIR has had in instruction. For example in classroom management (Star & Strickland, 2008) and responding to meet students' needs (Barnhard & van Es, 2015; Jacobs et al., 2011). They then investigate the interplay between teacher knowledge, instructional practice, and noticing; namely in the realm of "teacher knowledge of students" (Hill & Chin, 2018, p. 1105).

Relatedly, Louie (2018) continued the conversation of *equitable* noticing (Wager, 2014) in which her participant, Amanda Pepper, illustrates the barriers that caused her to still see students' deficits in spite of her substantial skills in noticing that focused on seeing students' mathematical strengths (p. 67). Years later, Louie and colleagues (2021) took a sociopolitical turn in discussing teacher noticing. They build on previous works, especially of Jacobs et al. (2010), to introduce the presence and influence of frames on the AIR teacher noticing framework, dubbing their proposed framework of teacher noticing FAIR. In short, frames (Goffman, 1986) are the interpretive contexts that participants of a given situation use in order to quickly interpret information, filter the details, and decide how to appropriately proceed. It is the vehicle that allows a person to be able to answer the question "What is it that's going on here?" (Goffman, 1986, p. 8) after attending any current situation.

In Goffman's discussion of frames, there is a strong suggestion that the frames one uses strongly dictate their actions. Louie and colleagues take from Greeno (2009) to understand that "frames provide interpretive contexts that support participants in a given situation to understand what kind of task they are engaged in, what kinds of knowledge are relevant or valuable, and what sort of behavior they and others are expected or entitled to engage in" (Louie et al., 2021, p. 3). In illustrating the utility of the FAIR framework, they acknowledge the gray area between deficit and anti-deficit noticing but chose to use two examples with "sharp contrast" to illustrate how consequential framing can be for noticing (Louie et al., 2021, p. 4).

To build their illustration of deficit framing and noticing, Louie and colleagues discuss how "it makes a great deal of sense" for teachers to attend, interpret, and respond in a deficit manner when teachers frame mathematics learning as "absorption of a universal, objective, and fixed body of knowledge" (Louie et al., 2021, p. 5). Here, deficit noticing involves "attending to accuracy and correctness", "interpreting mathematical work [as only] correct or incorrect", and "responding by affirming correct answers and remediating errors (Louie et al., 2021, p. 5).

In building a parallel illustration of the FAIR framework for anti-deficit contexts, Louie and colleagues began with the "AIR" component and ascertained the "F" that shaped the particular noticing. For example, using Oscar's case, they categorized instances of attending into thematic codes first (students, mathematics, and interactions), and then used literature on deficit or equitable frames to theorize the interrelation between framing and these themes. These two illustrations may suggest the same influence that Goffman had, in that deficit framing leads to deficit noticing or anti-deficit framing leads to anti-deficit noticing.

I understand that the FAIR framework is a model of the relationship between frames and noticing, and their discussion focused on the utility of frames in discussing noticing. This discussion does not do their work justice, as Louie and colleagues discuss more than just these illustrations. The best (or worst) case-scenario illustrations of anti-deficit (or deficit) contexts is a helpful model in accounting for frames in noticing, but frames likely cannot dictate all actions and responses. They have done extensive work in illustrating the complex web between the parts of FAIR as well as how the contexts outside of a classroom has a strong influence on the frames of teachers (Louie et al., 2021, p. 11). For now, this report aims to complicate this FAIR framework by focusing on analytically unpacking frames, which colors the gray area that Louie and colleagues mentioned briefly. This is focused on the framing and the responding parts of the FAIR framework and discuss the research implication for the other parts in the conclusion.

Data and Method

The data used in this report comes from a larger study of mathematics GSIs teaching for the first time as instructors of record. Instructors of record are responsible for the content-delivery and assessments of these undergraduate mathematics courses (Rogers & Yee, 2018). This study took place in a large, public, R1 university. More specifically, the interview data used in this report was collected weeks prior to their first PD and official preparation to teach as instructors of record. These four GSIs will be teaching either an "intermediate algebra" course which cover ideas that "are prerequisites for tackling college-level mathematics" or a "college algebra" course which is not commonly deemed as college-level mathematics, like calculus, differential equations, or linear algebra (Burrill et al., 2023, p. 799). Each GSI chose their own pseudonym.

I am a mathematics GSI myself. While the benefits of rapport and confidence helped in acquiring participants, the relatable experience of being nervous about teaching for the first time has an unavoidable influence on the data collection and analysis. For example, the guiding principle that one's past experience of being lost in a college course influences their future desired actions and expectations as instructors of record is a direct result of the author's experiences as a GSI.

The structure of interviews supports analysis of GSIs' preliminary frames, with the caveat that I am limited to only their perceptions of responses as they have yet to teach. The guiding philosophy of the interview was to orient GSIs into their past experiences. By first asking about a college course that they felt "really lost" in, this reveals their desired expectations and actions better than recounting a time where mastery came naturally. This philosophy stems from the author's experience of better articulating what he wants to do as an instructor when the context is bettering a challenging experience. For example, the question "What expectations should the instructor [of the "really lost" class] have had…?" revealed desires of wanting to break the monotony of lectures to implement student-to-student discussions when confusion arises. This can gather information about how they frame learning as collaborative or how instructors should respond to perceived confusion. Additional questions were asked that allows investigation of frames and responding such as “In your view, how do students learn math?” or “when you think of a skillful teacher, what is that person doing?”.

Analysis proceeded by first giving a descriptive summary for each question immediately after each interview. Then, focusing on individual interviews, each response was analyzed closely to highlight ideas about teaching and student learning. Thus, for each interview, a profile of the GSI was built using the guiding questions: “what attitudes does this person have towards student learning?” and “how does this person’s attitudes compare to the (anti-)deficit frame models from Louie et al. (2021)?” In particular to organize data for the latter, a novel researcher analytic framework organized into a table (Table 1) was used, which the larger study this report stems from is trying to elucidate its viability as a researcher tool for frame analysis.

Table 1. Researcher framework for examining instructional frames

Frames provide interpretive contexts that support participants’ **understanding of**…	Frames of teaching provide interpretive contexts that support GSIs perceptions and enactment of…	Frames of learning provide interpretive contexts that support GSIs perceptions and enactment of…
(1) What role(s) they take up	What their role is in the classroom as novice instructors	What do GSIs have to do for students to learn the intended content, practices, and orientations
(2) What knowledge is relevant or valuable	What professional knowledge is relevant in the act of teaching	What content, practices, and orientations should GSIs attend to in the classroom
(3) What sort of interactions they and others are expected or have the right to engage in	What type of interactions (instructor-student, instructor-group, student-student) are favored or useful in fulfilling their role and achieving the intended learning goals	

Here I discuss the formulation of Table 1. The compartmentalization of frames is motivated by Greeno’s (2009) work, which Louie et al. (2021) also use to discuss frames. Its construction was motivated by asking “how can a researcher establish that one instructor’s frame is different from another?” The change of wording from participants’ “understanding of…” to “perceptions and enactment of…” is key, because this framework addresses the participants’ point of view on the three subdivisions as well as the actions that follow. Professional knowledge (Kunter et al., 2013) refers to the many different facets of knowledge an instructor could have such as content, pedagogical content, organizational, pedagogical/psychological, and counseling knowledge. This functions to clarify what Greeno meant by knowledge in a way that still broadly covers the types of knowledge that instructors draw upon for teaching. Content refers to the curriculum of the course, these are what the instructor must cover according to the department. Practices is a general term to refer to the behaviors one can learn as a mathematician like collaboration or justification. Lastly, orientations cover the abstract such as the attitudes towards the subject like confidence and self-efficacy.

I also want to highlight the combined section row for teaching and learning. While the middle column of Table 1 reads as the instructor’s interactions with the broader profession of teaching and pedagogy, the right column serves as the instructor’s intrapersonal connections to the act of teaching. As such, it felt natural to combine the final row as a sort of bridge between the two sub-definitions of frames. Using the guiding questions, the preliminary frames and perceived ways of responding are organized for each GSI.

Results

In organizing these results, Table 1 was used as a way to organize connections, such as "how have these three framed what it means to learn mathematics?" In the responses to "how do students learn math?", there is a common thread among all four GSIs in framing student learning as learning by doing and highlighting the importance of student-student interactions.

Interviewer: In your view, how do students learn math?

Andy: So like the first part of math, like high school math. You can kind of do, so to speak, on your own in the sense that you can repeat it and see the patterns between examples, and from their branch out into other examples, because you can see the structure better. and knowing that structure, and being able to extrapolate on that structure, is still useful to you, an integral in later parts of math. But I think that in proof math it starts to become more social, because you remember certain things from like true, that stood out to you as elegant and useful, and other people remember different things from the lecture that's been as elegant and useful, and you all have your different preferences on what proofs you like to use.

Paul: The biggest component is doing it yourself, as in like doing problems.

Longboat: They learn by exploring it, trying things out, and more importantly, discussing with people [sitting] behind them, the peers, the other students. That's important.

Past this point, differences start to arise. These three GSIs also discussed how learning starts with an introduction that covers the "big picture" concept which then develops into more intuition and understanding by doing more and more examples. For these three GSIs, the instructor's role is explicitly noted as the person who introduces the concepts and chooses the "right" examples. The fourth GSI, Carlos, discussed learning through the lens of how they learn mathematics which differs from the other GSIs since they take it upon themselves to find more challenging problems to work on.

Carlos: I'll talk like from my point of view, like how I like to learn math… I like to go to lecture, ask as many questions as I can during the lecture, … take notes, go home, read the notes, [and] understand what is going on in the notes. And if I don't understand something, I go to office hours, ask questions there that are like more in depth. And then but oh, before that I try to read the book… It's like very good at supplementing the course. Then, also a lot of problems, or like challenging homework problems that are like more challenging than the stuff that I would see inside the classroom. And so I like to work on the homework like if I have a problem to work on, I work on it for like 30 min to an hour. and then if I don't get anywhere. I stop after an hour like after 30 min to an hour. and then ask classmates. and if we don't figure it out, I ask After 30 min to an hour with a classmate, I ask the instructor. during off hours. And that's in my view how this student [points to themselves] learns math.

Now, compiling the responses to questions that center teaching practices, three common teaching practices arose: (a) having awareness of the climate of the classroom, (b) being well-prepared in content knowledge, and (c) creating a collaborative, supportive environment. These practices are then translated into descriptive codes (Miles, Huberman, & Saldana, 2019): climate, content, and collaboration. Below are representative quotes for each of these themes that demonstrate each theme the best but note that each participant spoke of each theme.

Paul: **(climate)** I noticed I liked a lot about [graduate course] and his instructor style. It felt like he was pretty good at noticing when the class was lost. and sort of pausing to take a break and be like, “Okay, check in with your neighbor, talk about what's going on, where or what's gone wrong.” I think that's an important aspect.

Andy: (**content**) I think a skillful teacher is one who has an understanding of their knowledge base, and also the ways that people can misinterpret it when they first see it.

Carlos: (**collaboration**) One thing that I think any class should have, for example, is to make sure that, like you’re a good classmate, not just like a good student. That definitely, I think, should be verbalized in any classes, especially in that one.

Through Table 1 and these themes, these four GSIs have framed teaching and learning as requiring content knowledge to attend to students doing problems, but also requiring sensitivity to the comfort of the students to create a space conducive to learning from each other.

Since these GSIs have yet to enter the classroom as instructors, the study takes the three themes above to allude to their perception of responding in noticing. For an explicit exploration, Longboat has framed the act of learning as an exploration filled with students asking questions and making conjectures and their responding maneuver during this exploration is to be hands off:

> “I explained a bit of a concept, say the derivative. Maybe I don't give the full story yet. I just give them some precursors to it… Then I say ‘for the next 5 minutes, just to the person behind and next to you, ask what's this good for?’ I think these vague, open questions are good because you've given them an idea the students know it has to do with something you've talked about, right? They don't know exactly where we're heading… You expect students to have this mini research mind, they should be in the head that you explore math. You don't just solve it. So trying things out… They shouldn't discuss it with me, because I know what the answer is… So, give them a safe haven with their peers, where they can say anything… and as always allow students to make mistakes.”

Longboat’s response above provides one of the motivating complications to the FAIR framework. In one interpretation of this response, this was seen as dismissive as it disallows students from seeking answers from their instructor. But, in further exploration, this stems from Longboat's insistence that learning does not happen by having answers "thrown" at students and their belief that mathematics (research) is not given, but is discovered through exploration.

Another example, Andy has framed the act of learning with a skeleton metaphor:

> “You have to get some picture of what you’re looking for, and then see it a lot of times to be able to really see the basic skeleton. And then you can try to fill out that skeleton depending on what problem you're working on.”

For them, the instructor’s role is to give an “introductory picture of what that skeleton might look like” and provide a range of examples (simple to difficult, unambiguous to ambiguous) that allow students to add to the skeleton in different ways during different problems. Longboat and

Andy both share the frame that learning occurs when there is student exploration, but they differ in their way of perception of responding to this frame. While Longboat has a hands-off approach, Andy explores a range of examples alongside their students.

Lastly, as another point of comparison, GSIs also learn through exploration of "more challenging" problems, which is expressed in Carlos' large quote above, however Carlos' perceived role as instructor is to be there for students to approach with questions. This parallels Longboat's students' exploration but contrasts Longboat's instructor role, as students are not disallowed from asking questions.

Discussion

This contributed report set out to build on the work of Louie et al. (2021) bringing frames into the AIR teacher noticing framework. The results of the reported study entangle the relationship between anti-deficit or deficit frames with anti-deficit or deficit responding. With Longboat, Andy, and Carlos, a suggestion of a spectrum rises in the gray area between anti-deficit and deficit ways of responding. Even though these GSIs agreed in framing student learning as including exploration of various problems, their ways of responding branched out in different ways. Similar to Louie et al. (2021), it is reductive to label instructors' noticing as purely anti-deficit or purely deficit as it is rarely the case that noticing is (anti-)deficit in every situation. As in Longboat's situation, their way of responding (not discussing with students during their exploration) could be seen as deficit (when viewed as dismissive) or as anti-deficit (when viewed as a total belief that students are capable of exploring on their own).

Implications and Future Direction

This study was only able to focus on their frames and their perceptions of responding. With that, future research should explore where "switching" happens. Since there is a perceived flow from attending to interpreting to responding, future work can explore where the switch from deficit to anti-deficit happens along this chain.

An implication of this study for professional development organizers is to understand the complicated nature of anti-deficit framing and anti-deficit teacher noticing. On one hand, the author now understands that even if novice instructors are taught the same anti-deficit frame to approach teaching and learning, these can morph into different expressions that could be taken as deficit by their students. On the other hand, this complication of the FAIR framework alludes to a spectrum that can form a pathway from deficit responding to anti-deficit responding that PD leaders can utilize to train instructors to provide more equitable experiences for students.

A consequence of this study is that, even before formal professional development or experience as instructors of record, mathematics graduate students have nuanced perceptions and thoughts about their future teaching and what it means to be a skillful teacher. The larger on-going study this report stems from assumes that these perceptions are continuously shaped during their experiences as students in their mathematics courses. As such, more studies in leveraging graduate students' experiences as students to improve, augment, or re-design aspects of the graduate student professional development and preparation to teach undergraduate mathematics.

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References

Beisiegel, M., Gibbons, C., & Rist, A. (2019a). Examining obstacles to mathematics graduate students' development as teachers. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (Eds.). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 490-496).

Bookman, J., & Braley, E. (2022). A survey of programs for preparing graduate students to teach undergraduate mathematics. In S. S. Karunakaran, & A. Higgins (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, Boston, MA (pp. 1200-1201).

Burrill, G., Cohn, H., Lai, Y., Sinha, D. P., Son, J. Y., & Stevenson, K. F. (2023). Listening for common ground in high school and early collegiate mathematics. *Notices of the American Mathematical Society, 70*(5), 798-805. https://doi.org/10.1090/noti2689

Ellis, J. F. (2014). *Preparing future college instructors: The role of graduate student teaching assistants (GTAs) in successful college calculus programs* [Doctoral dissertation, University of California, San Diego]. California Digital Library. ProQuest ID: Ellis_ucsd_0033D_14294. Merritt ID: ark:/20775/bb3619310q. Retrieved from https://escholarship.org/uc/item/9b5813s5

Goffman, E. (1986). *Frame Analysis: An Essay on the Organization of Experience.* Boston, MA: Northeastern University Press.

Greeno, J. G. (2009). A Theory Bite on Contextualizing, Framing, and Positioning: A Companion to Son and Goldstone. *Cognition and Instruction, 27*(3), 269-275. http://www.jstor.org/stable/27739914

Hill, H. C., & Chin, M. (2018). Connections between teachers' knowledge of students, instructions, and achievement outcomes. *American Educational Research Journal, 55*(5), 1076-1112. https://www.jstor.org/stable/26643533

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Educations*, 41(2), 169-202. https://www.jstor.org/stable/20720130

Jacobs, V. R., Lamb, L. L., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97-116). New York, NY: Routledge.

König, J., Santagata, R., Scheiner, T., Adleff, A., Yang, X., & Kaiser, G. (2022). Teacher noticing: A systematic literature review of conceptualizations, research designs, and findings on learning to notice. *Educational Research Review, 36*. https://doi.org/10.1016/j.edurev.2022.100453

Kunter, M., Baumert, J., Blum, W., Klusmann, U., Krauss, S., & Neubrand, M. (Eds.). (2013). Cognitive activation in the mathematics classroom and professional competence of teachers: Results from the COACTIV project. Springer Science & Business Media.

Miles, M. B., Huberman, A. M., & Saldana, J. (2019). *Qualitative data analysis: A methods sourcebook.* SAGE Publications, Inc.

Miller, E. R., Uhing, K., Hass, M., Zigterman, R., Quigley, K., Wakefield, N., & Lai, Y. (2018). Graduate student instructors' growth as teachers: A review of the literature. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown, *Proceedings of the 21st Annual*

Conference on Research in Undergraduate Mathematics Education, San Diego, California (pp. 185-197).

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics, 97*(1), 55-69. https://www.jstor.org/stable/45185389

Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti-deficit noticing. *ZDM Mathematics Education 53*, 95-107. https://doi.org/10.1007/s11858-021-01229-2

Rogers, K. C., & Yee, S. P. (2018). Peer mentoring mathematics graduate student instructors: Discussion topics and concerns. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education,* San Diego, California (pp. 416-423).

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve pre-service mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*(2), 107-125.

Uhing, K. (2020). *Exploring pedagogical empathy of mathematics graduate student instructors* (ProQuest Number: 27956521). [Doctoral dissertation, University of Nebraska-Lincoln]. ProQuest Dissertation Publishing.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*, 244-276. doi:10.1016/j.tate.2006.11.005

Wager, A. A. (2014). Noticing children's participation: Insights into teacher positionality toward equitable mathematics pedagogy. *Journal for Research in Mathematics Education, 45*(3), 312-350.

Raise Your Hand if You Are Overloaded: The Role of Gestures in Proofs Comprehension

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Activities related to reading and understanding mathematical proofs are notoriously challenging for college students. On top of applying advanced logico-mathematical knowledge, these activities require the reader to process and operate on a substantial amount of information. Failure to successfully navigate the incoming stream of information may result in cognitive overload and impede one's ability to make progress on the task. This case study investigates how an undergraduate student, David, used his hands to overcome the cognitive load he experienced when reading a proof of the Two-Color Theorem. The findings suggest that in the absence of other modes of offloading (e.g., using pen and paper, figurative materials, or technology), hand gesturing may serve as a powerful and convenient offloading mechanism.

Keywords: gestures, working memory, proofs

The ability to read and comprehend proofs is critical for undergraduate students' success in advanced mathematics courses. Students are expected to spend substantial time surveying proofs presented in mathematics textbooks and their instructor's lecture notes (Weber, 2004). However, the activity of reading mathematical proofs differs from reading a literary text and requires developing special experience and particular skills. Unsurprisingly, then, numerous studies have documented that college students struggle to read proofs effectively (e.g., Hodds et al., 2014; Inglis & Alcock, 2012; Mejía-Ramos et al., 2012; Selden & Selden, 2003; Weber & Alcock, 2005; Weber 2004, 2015).

The implementation of certain ineffective reading strategies may be associated with the need to operate on an overwhelming amount of information. For example, Selden and Selden (2003) observed that novice readers spend much time paying attention to surface features of proofs, such as algebraic calculations. In another study, Inglis and Alcock (2012) found that undergraduate students tend to read proofs line-by-line, or, in other words, implement a *zooming in* (Weber & Mejía-Ramos, 2011) strategy. This way of reading proofs is the opposite of a *zooming out* strategy – skimming the proof to grasp the main ideas prior to attending to the details. The tendency to read proofs line-by-line or focus on surface details may pose significant cognitive load (Sweller, 1988, 2010) and impede proofs comprehension.

Cognitive offloading refers to the use of embodied activities to reduce the experienced cognitive load (Risko & Gilbert, 2016). Mathematics education literature has documented the beneficial effects of different modes of cognitive offloading, for example, drawings (Stryker et al., 2022), and hand gesturing (e.g., Cook et al., 2013; Goldin-Meadow et al., 2001). Remarkably, all these studies were conducted in the context of elementary school mathematics, and the role of cognitive offloading as manifested via hand gesturing in the context of undergraduate mathematics has been fundamentally under-researched. Moreover, none of the previous studies investigated the role of gestures in understanding written mathematical texts. As such, the goal of this study is to begin filling the gaps in the RUME literature. Specifically, my chief research question is *What is the role of students' gestures in navigating the cognitive load associated with reading and comprehending mathematical proofs?*

Theoretical Framework

In answering this research question, I adopt Piagetian and embodied perspectives. Embodied theories of cognition (e.g., Barsalou, 2008; Lakoff & Nuñez, 2000; Nemirovsky & Ferrara, 2009) propose that cognitive activities are rooted in sensory-motor processes (Wilson, 2002). Likewise, the idea that sensory-motor operations provide a basis for mathematical learning corresponds to Piagetian theories of mathematical development (Piaget, 1972). Sensory-motor activity is a kinesthetic activity involving muscular movements, including hand gestures. This activity can be further internalized and transformed into mental actions and representations. Although Piagetian and embodied perspectives have certain epistemological disagreements (e.g., on how abstraction takes place), these two theories allow for the integration of the constructs of mathematical proofs, gestures, and working memory (WM) by positioning bodily activities at the core of human cognition.

Working Memory

WM is a psychological model for human's ability to simultaneously store and process limited amount of information over a short period of time (e.g., Baddeley, 1992; Baddeley & Hitch, 1974; Daneman & Carpenter, 1980). Although numerous models of WM have been proposed (e.g., Atkinson & Shiffrin, 1968; Baddeley & Hitch, 1974; Cowan, 2012), Pascual-Leone's (1970) neo-Piagetian approach to WM is of primary relevance to my theoretical framework.

Pascual-Leone (1970, 1987) introduced a mathematical model to account for the development and functioning of WM. The proposed model has two levels. The first level contains cognitive units – schemes in the Piagetian sense. The second level includes cognitive operators, which are responsible for the activation of task-relevant schemes and inhibition of activation of irrelevant schemes. An individual's WM capacity (or *M-capacity*) is characterized in terms of the maximal number of schemes they can activate at a time. The cognitive complexity of the task (its *M-demand*) is defined as the number of schemes the task requires to be activated simultaneously. If the M-demand of the task exceeds an individual's M-capacity, cognitive overload takes place, and student's progress on the task can be impeded.

Gestures

Hostetter and Alibali (2008, 2019) proposed the Gesture as Simulated Action (GSA) framework to provide a theoretical account of how gestures emerge from embodied simulations of motor and perceptual states. In line with neurological and behavioral evidence, the GSA framework asserts that thinking and talking about an action activates neural areas that are responsible for planning the action. This activation may be either inhibited by a human or realized as a sensory-motor output. In the latter case, the hand gesture is born.

Mathematical Proofs

Harel and Sowder (2005, 2007) have proposed the notion of *transformational proof schemes.* According to this view, when students read proofs, their reasoning is transformational in nature, because learners are supposed to perform various transformations of mathematical objects of thought to deduce or keep track of a chain of inferences. Although the authors originally defined transformations only as expressed through written and spoken language, I follow Pier et al. (2019) and extend this definition by including transformations manifested via hand gesturing.

Methods

The data presented here are a subset of data collected for a larger project aimed at understanding the role of gestures in offloading cognitive demands on students' WM when they read, present, and construct proofs of mathematical conjectures. In this paper, I report on David, an undergraduate student enrolled in an introductory proof course, working on the proof of the Two-Color Theorem (see Figure 1).

Theorem: *$n \geq 1$ circles are given in the plane. They divide the plane into regions. Then it is possible to color the plane using two colors, so that no two regions with a common boundary line are assigned the same color.*

Proof: call the proposition $P(n)$. Let the two colors be B and W. We will prove the theorem using proof by mathematical induction.

Base case: for $n = 1$, the result $P(1)$ is clear. If there is only one circle, we may color the inside B and the outside W, and this coloring satisfies the requirements of the theorem.

Inductive hypothesis: assume that $P(k)$ holds for a fixed but arbitrary k. So we know that for any k circles there is a coloring which satisfies the requirements of the theorem.

Inductive step: we need to show that the proposition holds for $k + 1$. Consider $k + 1$ circles $\{C_1, C_2, \ldots, C_{k+1}\}$. Ignoring the circle C_{k+1}, we temporarily have k circles $\{C_1, C_2, \ldots, C_k\}$ By the inductive hypothesis, the regions created by $C_1, ..., C_k$ can be colored according to the requirements of the theorem. We color the plane according to this coloring. Now we add the circle C_{k+1} back and do the following

1. For any region which lies inside C_{k+1}, do not change its color;
2. For any region which lies outside C_{k+1}, recolor it with the opposite color.

Now we may check that the new coloring works:

1. Two neighboring regions whose boundary lies inside C_{k+1} have different colors;
2. Two neighboring regions whose boundary lies outside C_{k+1} have different colors;
3. Two neighboring regions whose boundary lies on C_{k+1} have different colors.

This shows that $P(k + 1)$ is true, and so by the principle of induction, the proof follows. ■

Figure 1. The proof of the Two-Color Theorem (as it was presented to the participant).

Procedures

David participated in two pre-interview WM assessments, a proof-based clinical interview, and a stimulated recall interview (SRI).

WM assessments. The participant's WM capacity was measured using two previously validated procedures – the Figural Intersection Test (Pascual-Leone & Ijaz, 1989) and the Direct Following Task (Pascual-Leone & Johnson, 2005). David's scores on the two tests were averaged and resulted in a WM capacity of 8.25[1].

Proof-based interview. The participant was presented with a proof printed out on the piece of paper as it is shown in Figure 1. The student was told that the proof was correct and that his

[1] Average WM capacity ranges between 5 and 9 (Miller, 1956; Pascual-Leone, 1970).

task was to read the proof and do his best to understand it. The student was not given a pen, scratch paper, a calculator, or any other figurative materials. He was not limited in time and was asked to think aloud while surveying the proof. After David indicated that he fully understood the proof, the interviewer asked a set of comprehension questions (data not reported here). The participant's behavior and mathematical reasoning were video- and audio-recorded.

Stimulated recall interview. On a later date, the student was shown selected video clips with himself working on the proof, and the interviewer asked questions about the participant's reasoning and behavior.

Hypothetical Analysis of the M-demand of the Proof

Prior to collecting data, I conducted a hypothetical analysis of the M-demand of the proof. This analysis is based on identifying a sequence of mental actions that need to be carried out to understand the proof. Specifically, I broke the proof into lines containing one or two mathematical statements. Every line was then coded as A_k. For each line, I identified the set of 1) previous lines A_k logically depends on, and 2) contextual schemes that a participant would hypothetically need to activate when reading the proof. These contextual schemes included extra lemmas, definitions, key concepts (e.g., primality or divisibility), and the essential elements of the proof framework (e.g., inductive hypothesis or proof by cases). For the purposes of this project, I use the word "scheme" in a broad sense to capture the information contained in the text of the proof (lines $A_1, \ldots, A_n$) and contextual mathematical knowledge.

Figure 2 illustrates the hypothetical M-demand of the proof of the Two-Color Theorem. It has been hypothesized that to understand the line A_7, the reader needs to activate the schemes A_3, A_4, and A_5. Also, the reader needs to allocate additional cognitive resources to retain the information contained in A_7 itself and to understand why $\{A_3, A_4, A_5\}$ imply A_7 (indicated as an arrow). Thus, the M-demand of line A_7 was assessed as 5 (one unit per scheme plus an arrow). According to the same methodology the M-demand of the lines $A_9 - A_{11}$ was assessed as 9, which is higher than an average WM capacity.

A_3: By the inductive hypothesis, the regions created by $C_1, ..., C_k$ can be colored according to the requirements of the theorem. $[\{A_2, P, IH\} \rightarrow A_3]$

A_4: We color the plane according to this coloring. $[A_3 \rightarrow A_4]$

A_5: Now we add the circle C_{k+1} back and do the following $[\{A_1, A_2, A_4\} \rightarrow A_5]$

A_6: 1. For any region which lies inside C_{k+1}, do not change its color; $[\{A_3, A_4, A_5\} \rightarrow A_6]$

A_7: 2. For any region which lies outside C_{k+1}, recolor it with the opposite color. $[\{A_3, A_4, A_5\} \rightarrow A_7]$

A_8: Now we may check that the new coloring works: $[\{A_3, A_4, A_5, A_6, A_7\} \rightarrow A_8]$

A_9: 1. Two neighboring regions whose boundary lies inside C_{k+1} have different colors; $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \rightarrow A_9]$

A_{10}: 2. Two neighboring regions whose boundary lies outside C_{k+1} have different colors; $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \rightarrow A_{10}]$

A_{11}: 3. Two neighboring regions whose boundary lies on C_{k+1} have different colors. $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \rightarrow A_{11}]$

Figure 2. A line-by-line analysis of the M-demand of the proof of the Two-Color Theorem. The scheme P denotes the proposition of the theorem, IH – the inductive hypothesis, and Cs – the necessity to coordinate cases.

Data Analysis

The analysis began with creating the transcript of the proof-based interview. The videos in conjunction with the written transcript were used to identify gesturing episodes. Given that gestures oftentimes occurred quickly, I re-watched the video recordings and created codes for all gestures. I did this by looking at each gesture and creating a short description considering physical hand movements, the associated mathematical context, and the corresponding speech (if present). Gestures that did not seem to convey any mathematical meaning, were eliminated from the analysis. The remaining gestures were then labeled as (purely) communicative or offloading. In the latter case, the offloaded schemes were specified. The participant's responses during the SRI were used to triangulate my initial interpretations of his gesturing.

Results

I chose the Two-Color Theorem for two main reasons. First, its proof is mathematically accessible for undergraduate students assuming that they are familiar with the method of mathematical induction. Second, my methodology for measuring the M-demand of the proof predicted significant cognitive overload even for the participants with advanced WM capacity.

As was expected, David did not experience difficulties with the task until he reached the end of the proof. However, when reading lines $A_9 – A_{11}$, the participant demonstrated behavioral indicators of cognitive overload. Specifically, he paused reading and reviewed the details of (re)coloring of the plane (lines $A_4 – A_7$) multiple times, asked for a pen, and eventually started assisting himself with active hand gesturing to represent the circles and the associated regions:

Figure 3. David working on the proof of the Two-Color Theorem.

> So, this is a circle [uses his thumb and a pointing finger to form a circle]. And it has a different color from its outer region [points to the inside and outside of the circle, Figure 3, top left]. C_k [meaning C_{k+1}] is put on top of it [uses another hand to represent a circle, Figure 3, top right]. And now it has an opposite color from the original circle [Figure 3, bottom left]. And it is a neighboring region, with the original outer region, which was opposite of its inside [points to the inside of the original circle and to the outer region. Figure 3, bottom left, then bottom right]. They cannot be both opposite to an inside circle… Oh wait, but they switch the color!

The described sequence of gestures allowed David to offload the activation of a few critical schemes and helped him to make progress in understanding the proof. By placing the first circle and pointing to the inner/outer regions, the student offloaded schemes A_3 and A_4. When using his hand to indicate the action of adding another circle on top of the first one, he transmitted the activation of scheme A_5 into his immediate sensory-motor experience and eventually realized that some recoloring was needed, which is one of the key ideas of the proof. Therefore, offloading schemes A_3, A_4, and A_5 allowed the participant to free up WM resources to operate on other pieces of information, such as scheme A_7.

In the SRI, I showed David the described video episode and asked him what was challenging about the proof and what helped him to overcome these challenges. The participant indicated that it was hard to "remember all the configurations and steps" and discussed some offloading effects of hand gesturing:

> …maybe honestly, the working memory tasks helped me realize that if I could like potentially store some of the memory, like, in my hands, and not move them, and now it helped me like, work on other stuff in my head[2]. So, I just froze that piece of the memory in my hands and then continued to like, think about other configurations… Because, um... it was easier to like visualize different colorings of some regions I can actually see. Because I didn't have to remember both the colors and the region, I just had to remember that.

Although David's reflection could be biased by his prior knowledge about WM, which relates to the context of the study, these data provide an important insight into the nature of the offloading power of hand gestures from the participant's perspective.

Discussion and Implications

My work contributes to the existing literature in multiple ways. First, this case study serves as a starting point for the investigation of the role of students' gestures in navigating the cognitive load associated with activities pertaining to high-level mathematical cognition, such as mathematical proofs. In line with the previous research conducted in the context of elementary mathematics (e.g., Alibali & DiRusso, 1999; Cook et al., 2012; Goldin-Meadow et al., 2001; Ping & Goldin-Meadow, 2010; Wagner et al., 2004), the data presented here provide evidence that, in the absence of other modes of offloading, gesturing may be a convenient and powerful offloading mechanism. Using his hands to offload the activation of task-relevant mental

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[2] After the WM assessments, David was curious about what constitutes WM and why it is important in learning mathematics. Apparently, he picked up some WM terminology and used it in the SRI.

schemes, David could reduce the experienced cognitive load and free up WM resources that helped him to make progress in proof comprehension.

Second, prior research on gesturing in proofs has revealed the prevalence of so-called dynamic gestures during proving practices (e.g., Nathan & Walkington, 2017; Nathan et al., 2021; Walkington et al., 2014). These gestures depict real-time transformations of either a single object or multiple mathematical entities, related to each. Dynamic gestures were also evident in my data (for example, gestures depicting the transformations of the regions caused by adding the k+1st circle onto the plane). The findings shed light on the cognitive utility of dynamic gestures in terms of their offloading power.

Finally, the proposed analysis of hypothetical M-demands of mathematical proofs was able to model the experienced cognitive load and predict places of cognitive overload by specifying the number of mental schemes the proof requires to be activated line-by-line. Other scholars used alternative approaches for modeling the cognitive load of a mathematical task. For example, Norton et al. (2023) introduced the notion of unit transformation graphs (UTGs) to illustrate the sequence of mental actions students perform when they are engaged in fractional tasks. Although UTGs provide opportunities for measuring the experienced cognitive load and help to explain how it may be reduced (Norton et al., 2023; Stryker et al., 2022), the applicability of this model is limited, and UTGs become impractical in multidimensional tasks, such as mathematical proofs.

As a closing remark, hand gesturing is only one, although outstanding, example of a cognitive offloading mechanism. In another study, Stryker and colleagues (2022) examined the role of student-generated drawings in offloading cognitive demands of fractional tasks. The results indicated that the possibility to rely on drawings helped the students to free up WM resources and complete a cognitively demanding task. Notably, before starting drawing, the participants actively produced hand gestures. However, the use of gesturing did not help the students to overcome the experienced cognitive overload and finish the task without drawing. This gives some anecdotal evidence that drawing can be considered a “more powerful” offloading tool than gesturing. Nevertheless, future research should directly contrast various modes of offloading.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, 14, 37–56.

Atkinson, R. C., & Shiffrin, R. M. (1968). Human memory: A proposed system and its control processes. In K. W. Spence & J. T. Spence (Eds.), The psychology of learning and motivation: Advances in research and theory (Vol. 2, pp. 89–195). New York, NY: Academic Press.

Baddeley, A. D., & Hitch, G. (1974). Working memory. In *Psychology of learning and motivation* (Vol. 8, pp. 47–89). Academic press.

Baddeley, A. (1992). Working memory. *Science*, *255*(5044), 556–559.

Barsalou, L. W. (2008) Grounded cognition. *Annual Review of Psychology, 59*, 617– 645.

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and cognitive processes*, *27*(4), 594–610.

Cowan, N. (2012). *Working memory capacity*. Psychology press.

Daneman, M., & Carpenter, P. A. (1980). Individual differences in working memory and reading. *Journal of verbal learning and verbal behavior*, *19*(4), 450–466.

Goldin-Meadow, S., Nusbaum, H., Kelly, S., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological Science*, 12, 516–522.
Harel, G., & Sowder, L. (2005). Advanced mathematical thinking at any age: Its nature and its development. *Mathematical thinking and learning*, *7*(1), 27–50.
Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805–842.
Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62–101.
Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic bulletin & review*, *15*(3), 495–514.
Hostetter, A. B., & Alibali, M. W. (2019). Gesture as simulated action: Revisiting the framework. *Psychonomic bulletin & review*, *26*(3), 721–752.
Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358–390.
Lakoff, G., & Núñez, R. E. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York, NY, USA: Basic Books.
Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3–18.
Miller, G. A. (1956). The magical number seven, plus or minus two: some limits on our capacity for processing information. *Psychological Review, 63*(2), 81–97.
Nathan, M. J., Schenck, K. E., Vinsonhaler, R., Michaelis, J. E., Swart, M. I., & Walkington, C. (2021). Embodied geometric reasoning: Dynamic gestures during intuition, insight, and proof. *Journal of Educational Psychology, 113*(5), 929–948.
Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive research: principles and implications*, *2*(1), 1–20.
Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174.
Norton, A., Ulrich, C., & Kerrigan, S. (2023). Unit transformation graphs: modeling students' mathematics in meeting the cognitive demands of fractions multiplication tasks. *Journal for Research in Mathematics Education, 54*(4), 240–259.
Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta psychologica*, *32*, 301–345.
Pascual-Leone, J. (1987). Organismic processes for neo-Piagetian theories: A dialectical causal account of cognitive development. *International Journal of Psychology*, *22*(5-6), 531–570.
Pascual-Leone, J., & Ijaz, H. (1989). Mental capacity testing as a form of intellectual-developmental assessment. *Assessment and placement of minority students*, 144–171.
Pascual-Leone, J., & Johnson, J. (2005). Handbook of Understanding and Measuring Intelligence. India: SAGE Publications.
Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human development, 15*(1), 1–12.
Pier, E. L., Walkington, C., Clinton, V., Boncoddo, R., Williams-Pierce, C., Alibali, M. W., & Nathan, M. J. (2019). Embodied truths: How dynamic gestures and speech contribute to mathematical proof practices. *Contemporary Educational Psychology*, *58*, 44–57.

Ping R.M., Goldin-Meadow S. (2010). Gesturing saves cognitive resources when talking about nonpresent objects. *Cognitive Science,* 34, 602–619.

Risko, E. F., & Gilbert, S. J. (2016). Cognitive offloading. *Trends in cognitive sciences*, *20*(9), 676–688.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for research in mathematics education*, *34*(1), 4–36.

Stryker, R., Kokushkin, V., Kerrigan, S., & Norton, A. (2022). Offloading cognitive demands of fractional tasks on working memory through drawings (pp. 1626-1635). *Proceedings of the Forty-fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Nashville, TN.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12,* 257–285.

Sweller, J. (2010). Cognitive load theory: Recent theoretical advances. In J. L. Plass, R. Moreno, & R. Brünken (Eds.), *Cognitive load theory* (pp. 29–47). Cambridge University Press.

Wagner, S. M., Nusbaum, H., & Goldin-Meadow, S. (2004). Probing the mental representation of gesture: Is handwaving spatial? *Journal of Memory and Language*, *50*(4), 395–407.

Walkington, C., Boncoddo, R., Williams, C., Nathan, M. J., Alibali, M. W., Simon, E., & Pier, E. (2014). Being mathematical relations: Dynamic gestures support mathematical reasoning. Boulder, CO: International Society of the Learning Sciences.

Weber, K. (2004) Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course, *The Journal of Mathematical Behavior, 23*(2), 115–133.

Weber, K. (2015). Effective proof reading strategies for comprehending mathematical proofs. *International Journal of Research in Undergraduate Mathematics Education*, *1*(3), 289–314.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics*, *25*(1), 34–51.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, *76*(3), 329–344.

Wilson, M. (2002). Six Views of Embodied Cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636.

When Cohen's Kappa Is Not Enough: Exploring Methods for Estimating Inter-Rater Reliability for Time Sequential Data

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We identify a current pragmatic and methodological problem facing RUME researchers who study processes that unfold over time, like classrooms and interviews, with qualitative coding techniques. In many cases, existing methods for estimating agreement between raters fail to account for the properties of this kind of data, are not compatible with how the coding schemes are applied, and fail to properly estimate rater agreement. Despite many peer-reviewers requesting estimates of agreement for coded data, there is often no suitable value to report and researchers must make do with claims about coming to consensus. We review methods found in the literature and evaluate their suitability, strengths, and weaknesses. While each reviewed method is appropriate for some aspect of this kind of data, none satisfies all desired criteria.

Keywords: methods, quantitative, qualitative, inter-rater reliability

Qualitative researchers in RUME often use coding techniques (e.g. Clarke et al., 2015; Saldaña, 2016) to analyze recordings, transcripts, or artifacts like written work from educational settings like classrooms or interviews. Because qualitative analysis typically adopts a subjective, interpretive approach to seeking patterns in rich, complex data sources, one should not expect two raters would exactly agree on how to apply a coding scheme to a given record of an educational setting. Yet, there is still an expectation that multiple raters trained on the same codebook will conduct an inter-rater reliability (IRR) analysis and report an estimate. Miscalculating and inaccurately reporting IRR has been identified as a common methodological issue when reporting observational data (Hallgren, 2012). In our view, these methodological oversights are symptoms of qualitative researchers not having adequate access to suitable IRR measures or a clear mapping of which measures are suitable for different research settings. In this paper, we discuss the limitations of common approaches to IRR within RUME. We focus on observational data with particular attention to time-sequenced data, to illustrate the impact of differing types of disagreements between raters and consider the viability of existing approaches to estimating IRR. We conclude that the field lacks suitable estimates of IRR for addressing questions about time-sequenced interactions between teachers and students' reasoning, questions of particular interest to the RUME community.

Why Should RUME Care about IRR?

Seeking an objective numerical metric, such as IRR, may appear in conflict with interpretive paradigms of qualitative analysis; however, a suitable IRR method can serve an important role in supporting claims about the qualitative codebook and the coding process. First, if a suitable threshold is achieved, IRR can support inferences about the level of intersubjectivity (stability) of the codebook and therefore increase trustworthiness of the data analysis process. Second, a suitable measure may give information about when a rater has been adequately trained to use a coding scheme independently. When raters can reliably work independently, the capacity of the research team increases because researchers' confidence that the coding schemes are being consistently applied increases. Relatedly, some researchers intend to disseminate their coding schemes so that other stakeholders, like teachers and school districts, can use them to target and improve qualities of instruction. Third, researchers may wish to track observer drift (Miller, 2018). As raters code independently, they will begin to favor certain codes over others or introduce examples of codes (or additional codes) in distinct ways. It is important to understand how often raters should check in and re-calibrate their usage of the coding scheme. Additionally, raters' convergence or divergence on specific codebooks could become an object of study in itself. Fourth, IRR is essential to mixed methods research, where the resulting frequencies (how much and how often) of code occurrences serve as the input to quantitative methods.

IRR Methods RUME Researchers Use

To situate our discussion within RUME, we conducted a full-text search of the Boston (2022) RUME proceedings and articles published in *IJRUME* using search terms like "agreement" and "rater/coder reliability." The search returned eight proceedings and 16 articles reporting a value for IRR. Only 2 of 8 proceedings and 7 of 16 *IJRUME* articles reported a statistic such as Cohen's κ. A further two articles reported comparable statistics (Fleiss's κ and Krippendorff's α) and two considered correlations between coders. The remaining papers reported an agreement percentage. This brief search suggests that Cohen's κ is the primary method for estimating IRR appearing in published articles in RUME.

What is Cohen's κ and why is it problematic?

Cohen (1960) introduced κ to measure agreement between two raters using a nominal scale to assign analytic units to categories. Cohen's κ (hereafter: κ) is available in many qualitative data analysis software packages. The requirements are fairly stringent: analytic units must be independent, the coding categories must be independent, the codes must be mutually exclusive and exhaustive (MEE), and the raters must be independent. Imagine a recording of a classroom lesson and a codebook containing n different codes, $\{x_i\}$, which do not have a naturally occurring order. The recording is segmented into analytic units. These could be turns of speech or uniform chunks of time, for example. For each analytic unit, Rater 1 can apply x_i (or not) and Rater 2 can apply x_i (or not). For each x_i, and for each segment, there are two agreement states (both apply code x_i, neither apply code x_i) and two disagreement states (one coder applies x_i and the other does not). The counts for each state can be tallied in a contingency matrix and then Cohen's Kappa computes the ratio between (a) the excess probability that two coders agree compared to the probability agreements are due to chance alone and (b) the probability that agreement is not due to chance alone. It is interpreted as the likelihood that raters are coding the same way, corrected for how often the raters (hypothetically) agree by chance alone. Values closer to 1 correspond to stronger inter-rater reliability. Metrics like κ can be suitable for well-structured qualitative data where codes are applied to well-defined analytic units, like turns in a transcript. However, κ rests on quantitative assumptions (e.g., codes are mutually exclusive and represent independent categories) that are not always met by the outputs of qualitative coding.

Four primary categories of observational data can be distinguished: (a) event sequential data encode only the order of events, but not time information about when they occurred, (b) state sequential data assume that every second of the observational record can be mapped to a unique state in the codebook and thus record transitions between states, (c) interval sequential data consist of equally-segmented intervals and apply codes to each segment, and (d) time sequential data (TSD) allow codable events in the observational record to be identified with a rater-defined time interval (Bakeman & Quera, 1995); codes can be applied to momentary behaviors (e.g., a hand being raised) or to behaviors with a duration (e.g., a teacher engaging in motivational talk). Both the start and stop times of each coded event are relevant in TSD, an aspect which distinguishes it from the other categories and makes it particularly resistant to statistical methods like κ. In TSD data, where one codable event begins is partly constituted by when another one ends (or begins). This means not only that TSD violates independence assumptions, but it also introduces low-level disagreements that researchers may not wish to tally. κ assumes that all types of disagreements between raters are equivalent and should be tallied.

Consider the following three kinds of disagreements: (A) *errors of omission/commission*, where one rater misses (or inserts) a codable event relative to the other rater's analysis, (B) *multiple interpretation problems*, where two raters disagree on what code is appropriate for the analytic unit, but agree that something important relevant to the research question is happening, and (C) *boundary problems*, where two raters agree on what code to use, but choose different times to apply it. The severity of omission/commission depends on whether Rater 1 simply did not perceive the event that Rater 2 conceives as codable, or whether Rater 1 disagrees with Rater 2's interpretation of the event as codable. In the first situation, Rater 1 may simply admit they missed something and adopt Rater 2's code. The second situation should count as a disagreement, but probably not the first. After all, this is a primary reason to have at least two raters reviewing data for mixed-methods work. Multiple interpretations problems will arise when codebooks are detailed and complex. Said plainly, the more choices there are for codes, the more likely two raters will disagree on which code to use even if, at a high level, they agree that a) there is a codable event and b) how to interpret the relevance of the event. Boundary problems are common when using coding schemes that allow for codable events to be selected with a rater-defined time interval instead of uniform time intervals. The raters need to decide when each codable event begins and ends, as well as which code(s) to apply. Each of these types of disagreements is less substantial than two raters fundamentally disagreeing on the nature of what is being observed. The problem is that algorithms vary in their capacity to evaluate what kind of disagreements may have occurred, and therefore whether to count them as "true" disagreements. For these reasons, metrics like κ (and percent agreement) may not properly estimate IRR and thus are limited in their suitability for the kinds of qualitative research common in RUME. In fact, there is growing caution Cohen's kappa should not be considered as a standard for IRR in these cases (Walter et al., 2019).

These problems are exacerbated for projects recording temporally-sequenced observations, like interviews or classroom interactions, and can confront the researcher with distasteful choices. To make the discussion concrete, imagine a cognitive task-based interview setting where the project is interested in studying interviewer-student interactions that lead to successful problem solving. In this example, imagine the research question is about how the interviewer's moves support the student's reasoning. The interviewer's moves might get coded for attributes of the questions she asks or mathematical content of the statements she makes. The student's speech, writing, and gestures might get coded with indicators for phases of problem solving, covariational reasoning, and metacognition – as they occur. These codebooks are very complex and may not be hierarchically organized, thereby resisting simplification efforts.

Due to the nature of mathematical reasoning, multiple codable events may take place during overlapping durations of time. For example, while speaking about a graph, the student may be speaking about two quantities covarying, but doing so to reflect on the adequacy of results from their previous work (metacognition) by considering an analogous situation (problem solving heuristics). Thus, this TSD does not satisfy assumptions about mutual exclusivity. To meet the assumptions for κ, the observational record would need to be artificially segmented (i.e., transformed into a non-TSD type), which would disregard detailed interactional information the qualitative codebooks are designed to capture. For example, coding the teacher and student moves every 30s would (inaccurately) concede that the only important aspect is whether a move occurred during the time interval, and then apply that code to the entire interval, which would wash away any interactional information during that interval. In short, researchers want to retain "when" something happened as a property of the data so that time-dependent interactions can be studied.

Considerations for Evaluating a Method's Capacity to Exclude Low-Level Disagreements

In general, estimates of IRR are very sensitive not only to raters' consistency in applying codes to the data but also to their accuracy in segmenting transcripts of videos into analytic units. Ideally, the coding process supports assignment of "clearly defined, disjunct categories" (Reed et al., 2018, p. 208) to each codable event and there is an "intersubjective, transparent basis for forming coding units" (Bilandzic et al, 2001 as cited in Reed et al., 2018, p. 208). This is easy enough when the codebook corresponds to direct observations, e.g., *student raises their hand*. It becomes much more complicated with research questions that use interpretive coding schemes, like those associated with student reasoning or teacher scaffolding moves. As discussed above, there are three kinds of low-level disagreements that researchers may not wish to tally. In this section, we review existing methods for estimating IRR for TSD that treat two of them: errors of

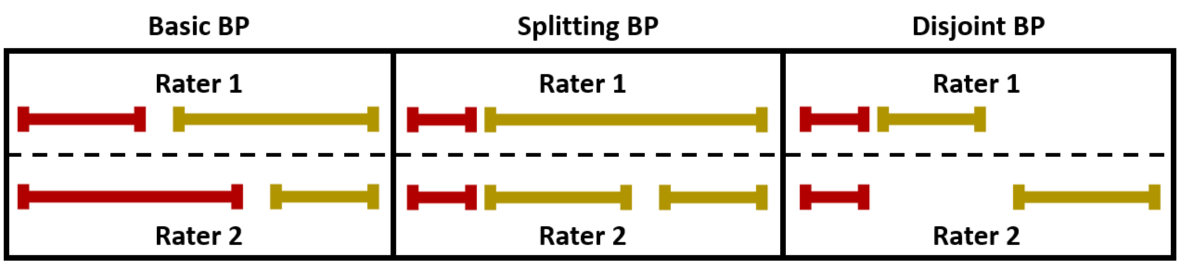

Figure 1 Boundary Problems (BP), which should not count towards disagreements.

omission/commission and boundary problems.

There are three kinds of boundary problems that have implications for the algorithms used to estimate IRR, which we call the basic boundary problem, the splitting boundary problem, and the disjoint boundary problem, diagrammed in Figure 1. The basic boundary problem arises when the raters use non-identical starting or ending boundary points for the codable event. For IRR methods which include some kind of temporal tolerance, a small tolerance can result in false negatives for this kind of boundary disagreement, whereas a large tolerance can result in false positives. The splitting problem occurs when one rater decides to split a code for some non-codable activity that occurred during the interval a codable activity occurs. For example, Rater 1 codes $[t_1, t_2]$ as *quantitative coordination*, but Rater 2 observes that for a duration of time $[t_3, t_4] \subset [t_1, t_2]$ that the student makes a comment about their recent mathematics exam, which is not a codable event, and then goes back to their covariational reasoning activity. Thus Rater 2 produces two codable events, $[t_1, t_3]$ and $[t_4, t_2]$, assigned to *quantitative coordination.* Depending on which IRR method is used, an algorithm could count this as two agreements and one disagreement or as three disagreements. However, the disagreement is closer in nature to deciding the appropriate duration of an analytic unit, which is why we include it as a boundary problem. A suitable method would be sensitive to the splitting problem only in cases where the number of coded events is important to the research question. The disjoint boundary problem happens when a codable event is a very short duration and is couched in an ongoing student activity. For example, the student may exclaim, “wait!”, briefly go silent, and then correct a mistake. Here, the raters may agree on the codable event (*evaluation)*, and which code to apply, but not “when” to give credit for it.

Options for Estimating IRR for Time-Sequenced Data

We sought methods in the literature that aligned with the properties and needs of TSD: (1) they could be modified to accommodate violations of the MEE assumptions (2) the method is properly sensitive to low-level disagreements arising from boundary problems and omission/commission, and (3) the method is adequate to research questions posed about time-dependent interactions. In this section, we review the strengths and weaknesses of three approaches to modifying κ for use with TSD: time-unit kappa, multi-pass, and sequence method. A summary of the methods is provided in Table 1.

Time-unit Kappa. Bakeman et al. (2009) describe the time-unit kappa method as one commonly used in existing IRR literature. This method subdivides the data into very small, uniform, time intervals (typically fractions of a second), and takes the time interval as the analytic unit. As with Cohen’s κ, an agreement matrix is formed by counting agreements and disagreements on the subdivisions and κ is calculated on those matrices. As this approach re-defines the analytic units as small time intervals, disagreement between coders on *number* of coded intervals (as in the splitting boundary problem) are not measured. This method is also often modified with a researcher-specified temporal tolerance, allowing it to potentially be insensitive to disjoint boundary problems. The main drawback of time-unit kappa is that it assumes each tally mark in the matrix corresponds to a codable event, thereby dramatically over-estimating the number of rater decisions. Additionally, long strings of subdivisions would all be (apparently) coded identically, violating the independence assumption of κ. Time-unit kappa may still be useful in some situations. In particular, it addresses the splitting boundary problem, can easily be modified to address the disjoint boundary problem, and would be appropriate for research questions that demand high confidence in code location and duration.

Multi-pass. This family of methods improves upon the time-unit kappa method by identifying agreement between complete coded events instead of breaking them up into time segments. This comes closer to meeting the κ independence requirement. They make multiple passes through the coded data and on each pass they assign agreement or disagreement between raters to each coded event which has not yet been assigned a state. There are three published toolkits that execute multi-pass IRR methods: The Observer (Jansen et al., 2003), Interact (Bakeman et al., 2009), and EasyDIag (Holle & Rein, 2015). The Observer makes two passes to tally agreements based on overlap of coded events (within some researcher-defined tolerance) and then makes two more passes to tally code disagreements. The Interact algorithm works similarly but adds an extra pass to identify errors of commission/omission.

Both The Observer and Interact algorithms state that MEE assumptions should be met and both address the three boundary problems. Both are consistent with research questions that care about the occurrence and order of coded events. These algorithms can still be executed on data which violate the MEE assumption, and so potentially may perform well on data which only minimally violate it (see Jansen et al., 2003). The extent to which they perform well with MEE violations would be testable through simulation methods. However, both algorithms suffer from what Holle and Rein (2015) refer to as the *linking problem*: these methods may produce more entries in the agreement matrix than there are total coded events in the union of the two raters' data. Consider the splitting problem in the middle diagram of Figure 1 from the perspective of Rater 1 and from the perspective of Rater 2. Rater 1's coded event agrees with the first and second events coded by Rater 2 *and* Rater 2's two coded events agree with Rater 1's only coded event. Even though there are only three total coded events, the Observer algorithm would count this as four agreements. Holle and Rein claim that multiple linking can lead to bias towards agreements (2015). Raters who encounter many splitting problems will have artificially inflated reliability.

The EasyDIag algorithm was developed in response to the linking problem in the Observer algorithm (Holle & Rein, 2015). It makes only two passes. In the first pass, it links coded events with sufficient overlap and then removes them from the set. The remaining coded events are considered disagreements. The second pass links the disagreements in order of percent overlap, until the percentage falls below the same threshold. Any remaining coded events are considered errors of omission/commission. While EasyDIag resolves overcounting on the splitting problem, it does not resolve any of the boundary problems. It relies on percent overlap to link coded events and categorize them as (dis)agreements. It would miss boundary problems where the duration of Rater 1's coded event is much longer (or shorter) than Rater 2's coded event. Similarly, it will fail to link coded events with the disjoint boundary problem because it requires some level of overlap between codes. Finally, it will fail to link coded events with the splitting problem because it permits only one link per coded event before removing that coded event from the set of coded events to process. We consider these problems to be potentially resolvable by using a negative threshold, allowing for closely disjoint codes to be associated, which would not be within the range of thresholds (51% to 90%) suggested by the original authors.

Sequence Method. All methods mentioned so far have focused on temporally local agreements between two coders. Bakeman et al. (2009) introduced the GSEQ-DP algorithm which, instead, considers the overall sequence of coded events. This algorithm extends previous work with event sequential data (data viewed as a sequence of coded events, without reference to time stamps). The original algorithm rendered each rater's coded events as a sequence and computed the Levenstein distance between them (Quera et al., 2007). The Levenstein distance measures the minimum number of coded event substitutions, additions, or removals required to transform one sequence of coded events into another. A sequence of transformations is generated in the process and this sequence can be directly mapped to agreements/disagreements (substitutions), commissions (additions), and omissions (removals) in an agreement matrix. κ can then be calculated as usual on the resulting matrix. The GSEQ-DP algorithm connects the sequence-only algorithm to the timing of coded events as follows: Substitutions are free when Rater 1's coded event is temporally close to Rater 2's coded event but a cost is imposed on substitutions that are temporally distant. Imposing the costs ensures the method only identifies agreements between coded events which are temporally close to one another. As with previous methods, the GSEQ-DP algorithm explicitly assume the codebooks satisfies the MEE assumption. Its focus on code sequence resolves the basic and disjoint boundary problems. The splitting boundary problem, though, is still unresolved as the algorithm provides one-to-one mapping between coded events. This method for estimating IRR could be very useful for research questions concerned with the sequencing of coded events, such as, "Does *metacognition* often follow a teacher's *evaluation of response*?" The algorithm could be amended to relax the MEE requirement and to adequately treat the splitting problem by introducing swapping and splitting as valid transformations, bringing in something closer to Damerau-Levenstein distance instead of the Levenstein distance.

Table 1 Summary of methods for estimating IRR for time-sequence data type: criteria not met (✗), met (✓) and could be adapted (!) for requiring codes be mutually exhaustive and exclusive (MEE), the basic boundary problem (BBP), splitting boundary problem (SBP), disjoint boundary problem (DBP), and commission/omission errors (Com/Om)

Method	**No MEE**	**BBP**	**SBP**	**DBP**	**Com/Om**	**1:1**
Time-Unit Kappa	!	!	✓	!	✓	✗
Observer	!	✓	✓	✓	!	✗
Interact	!	✓	✓	✓	✓	✗
EasyDIag	!	!	!	!	✓	✓
GSEQ-DP	!	✓	!	✓	✓	✓

Conclusions

In this methodological review, we problematized existing methods for estimating inter-rater reliability (IRR) for specific kinds of time-sequenced data common to research programs in RUME. Our discussion distinguished between several kinds of low-level disagreements (errors of omission/commission, multiple interpretation problems, and boundary problems) that can occur when analyzing data qualitatively and what we consider true disagreements – which occur when both raters identify the same event but disagree on which code to apply. True disagreements may have implications for the stability of the coding scheme or speak to the need to re-calibrate the raters. The low-level disagreements should be sorted out through algorithms or quarantined for a reconciliation process between raters. We reviewed Cohen's κ and three methods for extending it for suitability by examining their strengths and weaknesses relative to how they handle the differing kinds of disagreements. While these methods have proven appropriate, and useful, to certain kinds of time-sequenced data, using any of these methods with fidelity would require a shift in research questions, a reduction in complexity of the coding scheme, or modification of procedures for applying the coding scheme.

We agree with Jansen et al. that "how one does reliability analysis depends on what research questions one needs to answer eventually" (2003, p. 394). The metric for estimating IRR should reflect the particularities of how the coding scheme is applied. Unfortunately, many of the measures for IRR make assumptions about the data or coding schemes that do not reflect the kinds of data that the RUME community works with nor the complexity of the coding schemes we develop to study problems of teaching and learning. Instead, researchers are restricted from asking and answering certain kinds of questions because the quantitative tools we have available do not suit the properties of the data. Ultimately, the field needs methods for estimating IRR that respect how the coding schemes are applied to the observational records and thereby comport with the research questions posed.

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References

Bakeman, R., & Quera, V. (1995). *Analyzing interaction: Sequential analysis with SDIS & GSEQ*. Cambridge University Press.

Bakeman, R., Quera, V., & Gnisci, A. (2009). Observer agreement for timed-event sequential data: A comparison of time-based and event-based algorithms. *Behavior Research Methods*, *41*(1), 137–147. https://doi.org/10.3758/BRM.41.1.137

Clarke, V., Braun, V., & Hayfield, N. (2015). Thematic Analysis. In J. A. Smith (Ed.), *Qualitative psychology: A practical guide to research methods* (3rd edition, pp. 222–248). SAGE Publications.

Cohen, J. (1960). A Coefficient of Agreement for Nominal Scales. *Educational and Psychological Measurement*, *20*(1), 37–46. https://doi.org/10.1177/001316446002000104

Hallgren, K. A. (2012). Computing Inter-Rater Reliability for Observational Data: An Overview and Tutorial. *Tutorials in Quantitative Methods for Psychology*, *8*(1), 23–34. https://doi.org/10.20982/tqmp.08.1.p023

Holle, H., & Rein, R. (2015). EasyDIAg: A tool for easy determination of interrater agreement. *Behavior Research Methods*, *47*(3), 837–847. https://doi.org/10.3758/s13428-014-0506-7

Jansen, R. G., Wiertz, L. F., Meyer, E. S., & Noldus, L. P. J. J. (2003). Reliability analysis of observational data: Problems, solutions, and software implementation. *Behavior Research Methods, Instruments, & Computers*, *35*(3), 391–399. https://doi.org/10.3758/BF03195516

Miller, S. A. (2018). Measurement (Ch. 2). In *Developmental Research Methods*. SAGE Publications, Inc. https://doi.org/10.4135/9781506331997

Quera, V., Bakeman, R., & Gnisci, A. (2007). Observer agreement for event sequences: Methods and software for sequence alignment and reliability estimates. *Behavior Research Methods*, *39*(1), 39–49. https://doi.org/10.3758/BF03192842

Reed, N., Metzger, Y., Kolbe, M., Zobel, S., & Boos, M. (2018). Unitizing Verbal Interaction Data for Coding: Rules and Reliability. In E. Brauner, M. Boos, & M. Kolbe (Eds.), *The Cambridge Handbook of Group Interaction Analysis* (1st ed., pp. 208–226). Cambridge University Press. https://doi.org/10.1017/9781316286302.012

Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd edition). SAGE Publications.

Walter, S. R., Dunsmuir, W. T. M., & Westbrook, J. I. (2019). Inter-observer agreement and reliability assessment for observational studies of clinical work. *Journal of Biomedical Informatics*, *100*, 103317. https://doi.org/10.1016/j.jbi.2019.103317

Students' Treatment of the Negation of Logical Implication

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Understanding how students reason about the negation of logical statements is essential in supporting their mathematical development. Prior literature suggests that undergraduate students experience difficulties with the precise negation of logical implications. Many respond with the opposite statement "P implies not Q" when asked for the negation of "P implies Q." We investigate the challenges that introductory proofs' students experience when negating implications, and the reasoning they demonstrate in addressing these challenges. Our findings indicate quantification contributes significantly to their difficulties with negation. Further, nuances in students' quantification may validate an implication's opposite as its negation.

Keywords: logical implication, negation, quantification, transition to proof

Understanding the negation of a statement is an essential skill in logical reasoning. For example, Dubinsky et al. (1988) assert that "in order to understand what something is, it is essential to understand what it is not" (p. 46). Prior research has identified the negation of mathematical statements—particularly logical implications (LIs)—as a persistent challenge for undergraduate students (Barnard, 1995; Griggs & Cox, 1983; Sellers, 2020). The negation of a statement is the statement that contradicts the original statement and excludes any middle ground (Dawkins, 2017). In other words, an LI and its negation have opposite truth values in all cases. However, students will sometimes present a contrary statement as a negation without satisfying the law of excluded middle. In particular, students might argue that the negation of P→Q is the statement P→~Q (Epp, 2003), which we refer to as the *opposite* of P→Q. The primary objective of our study is to understand students' ways of reasoning when negating an LI and the distinctions they make between counterexample, negation, and opposite.

The negation of a universally quantified LI is the assertion that a counterexample exists in the universe. To negate the LI P(x)→Q(x), one must specify the logical properties of a counterexample (P(x) and ~Q(x)) and additionally attend to the quantification of x.

Although students may generate specific counterexamples to statements like "If a number is a multiple of 3, then it is a multiple of 6" (Dawkins & Hub, 2017), there seems to be a significant cognitive gap between identifying a counterexample to an LI and articulating its precise negation. Prior research has documented students' persistent struggles with quantification in general (e.g., Dawkins & Roh, 2020; Dubinsky et al., 1998; Shipman, 2016), which suggests quantification may be a primary source contributing to this gap. However, there may also be other factors influencing the ways they perceive negation, opposite, and counterexample. To understand the nuances of students' treatments of the negation of an LI, we analyzed the reasoning demonstrated by two undergraduates enrolled in a Fall 2022 introductory proofs course taught by the first author. Our research questions were:

1. *Following research-based instruction in an introductory proofs course, what challenges do students experience in negating an LI; and what reasoning do they demonstrate as they address these challenges?*

2. *In particular, how do students experience and address the challenges identified in prior research on treating negation as the opposite statement?*

Framework: Epistemological Obstacles Associated with Negating LIs

Our framework presents the persistent challenges students experience with negating an LI as epistemological obstacles (EOs). The term EO originates in Bachelard (1938), and Brousseau (2002) later described EOs as obstacles that remain in response to best instructional practice. We further characterize EOs as experienced in best instructional interactions, by the student or the teacher. EOs are *persistent*; they cannot be resolved in a single lesson. Here we provide the supporting literature for existing EOs that are pertinent to this study.

Students make meaning of quantified statements in various ways (Sellers et al., 2021). Sometimes, they even interpret the language of quantification in ambiguous ways (Dawkins & Roh 2020; Epp, 1999). Other times, quantification might not be explicitly stated at all, so the intended quantification of a statement remains "hidden" (Shipman, 2016). When expressing LIs, we often say "P→Q" to mean "for all x, P(x)→Q(x)." It may seem that instructors could resolve the issue by consistently and explicitly stating quantification, but subtle differences in meanings of quantification persistently remain hidden for students. Dubinsky et al. (1988) described quantification as an "insurmountable barrier for students in developing a sophisticated understanding of limits and continuity" (p. 44). This description fits our characterization of EOs, and following Shipman (2016), we refer to this EO as hidden quantification.

As we have noted, a statement and its negation have opposite truth values, so negating a statement and generating counterexamples are inherently related (Shipman, 2016). Generating a counterexample for a given statement can be challenging, and when attempting to prove a universally quantified statement false, students will often claim that a single counterexample is insufficient (Zaslavsky & Ron, 1998). Sellers (2020) conjectured that this claim might relate to the recognition that a singular example is insufficient for proving a universally quantified statement is true. It may seem counterintuitive that the negation of a universally quantified statement is an existence statement, and the negation of an existence statement is a universally quantified statement. Thus, we see quantification of the negation as another potential obstacle.

The challenge of quantifying negations sharpens when we consider that the negation of a universally quantified statement encompasses *any* possible counterexample. That is why when proving a universally quantified statement, it is insufficient to show that a particular case is not a counterexample, or to simply modify the statement to bar that case (Shipman, 2016). It is also why the opposite of a statement does not suffice as its negation. Consider the universally quantified (hidden) LI, P→Q. Its opposite would be the universally quantified statement that P→~Q; but both statements could be false because they leave middle ground untouched (Dawkins, 2017). Because students ostensibly conflate the negation of a statement and its opposite—even after instruction in introductory proofs courses (Arnold & Norton, 2017)—we refer to it as another EO. Research suggests relying on Euler diagrams to sort out reasoning with LIs (Hub & Dawkins, 2018), but even Euler diagrams may have their limitations (Antonides et al., in review).

Methodology

The data reported here come from a larger study investigating the EOs that students experience, and ways of addressing them, in transition-to-proof courses. The class was taught by the first author in the Fall 2022 semester. Emphasizing active learning, the class was designed around four main topics: LI, quantification, mathematical induction, and functions. We identified

EOs suggested by existing literature and adopted or created instructional tasks for addressing those EOs head-on. Four students were interviewed three times by Norton and Kokushkin, with each video-recorded interview lasting 30-45 minutes. Students were given a LiveScribe pen and were asked to think aloud. The interview tasks corresponded to the four topics of the course; the third interview included tasks on mathematical induction and functions, as well as a stimulated recall interview (SRI) on a subset of tasks (chosen subjectively) from the first two interviews.

All interview data were coded to capture students' EOs. Our codebook (Figure 1) included codes both from existing literature and emergent codes that could not be explained using existing codes. This paper centers around the NO code, informed by existing literature; emergent codes were NF and LIv (Figure 1). Following the initial round of coding, we engaged in thematic analysis, analyzing contiguity of codes with the NO code.

Code	Description (Source)
LIv	Eliminating the vacuous case (Emerged)
NF	Difficulty with the difference between negation and disproof (Emerged)
NO	Treating the negation as the opposite, e.g., P→~Q (Dawkins & Hub, 2017; Epp, 2003)
Qh	Hidden quantification (Shipman, 2016; Durand-Guerrier, 2003; Ernst, 1984)
Qneg	Uncertainty about quantifying the negation (Dawkins & Roh, 2016; Shipman, 2016)
Reu	Interpreting logic via Euler Diagrams (Hub & Dawkins, 2018; Dawkins & Roh, 2021)

Figure 1. Codes.

We report on interview data from two students, Shivani and Carmen, a sophomore and senior majoring in Computational Modeling and Data Analytics and Aerospace Engineering, respectively. Shivani and Carmen were chosen because of the salience of their data vis-à-vis the negation-opposite conflation, and because their cases represent qualitatively distinct issues that may arise for students as they reason about the negation of an LI.

1. Consider the implication: P(x)→Q(x). What does it mean? How would you quantify it?
2. Consider the statement: "For all x, P(x)→Q(x)." How would you negate it?

Figure 2. Interview task.

Results

In this section, we present the details of Shivani's and Carmen's reasoning about an LI and its negation in response to the interview tasks shown in Figure 2.

Shivani's Treatment of Negation

When first asked what the unquantified LI P(x)→Q(x) meant to her (see Figure 2, Task 1), Shivani did not demonstrate an intellectual need for making the quantification explicit. She responded, "If P, then Q," and then elaborated to say, "So if P, if P is true then Q is true."

When the researcher asked how Shivani might quantify P(x)→Q(x), she drew a correct Euler diagram (Figure 3) and said, "So there exists… an x in the subset P. There exists an x... in... in P if uh... If there exists an x in P then x also exists in Q." Shivani saw $P(x) \rightarrow Q(x)$ as referring to a specific x and therefore quantified it existentially. Moreover, she restricted the universe to consider only those x in the truth set of P. She also seemed to quantify the *hypothesis* rather than the LI itself: "If there exists an x in P."

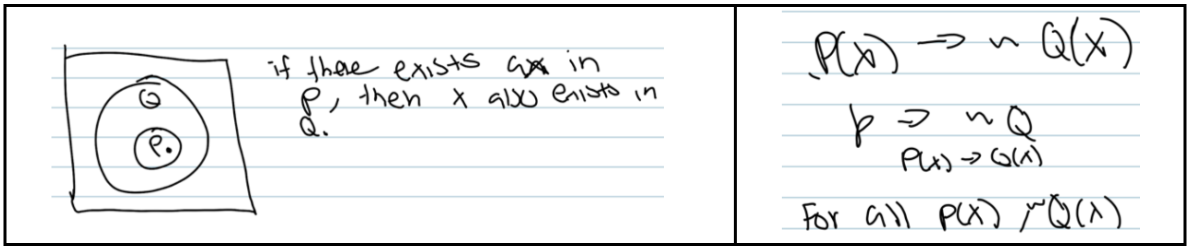

Figure 3. Shivani's Euler Diagram (left) and her corresponding writing (right).

When posed with Task 2 (see Figure 2), Shivani again omitted quantification saying, "So then it would be like P(x) implies ~Q(x) [writing '$P(x) \rightarrow \sim Q(x)$']. That would be a negation because the concl… hypothesis has to be true." Her emphasis on the hypothesis being true coupled with her earlier treatment of x, seems to indicate that, for Shivani, her negation requires x to be in the truth set of P. In an effort to unhide quantification, the researcher asked Shivani how she would quantify x. She responded,

> Since this is for all, then you would have to prove like, there exists one x value that makes this statement true [pointing to her written negation '$P(x) \rightarrow \sim Q(x)$' (Figure 3)]... Because then that would be the counterexample because you wanna disprove it…Because you only need one counterexample to prove that for all x.

For her, the existentially quantified LI $P(x) \rightarrow \sim Q(x)$ represented the existence of a counterexample. When the researcher asked how her negation might be quantified if the original LI was existentially quantified, Shivani adjusted her negation to "For all P(x), ~Q(x)." Shivani demonstrated her understanding of how quantification of the original LI affected the quantification of her negation. And, her use of "for all P(x)" seemed to indicate that Shivani was again requiring x to belong to the truth set of P. Regardless of how the original LI was quantified, when quantifying x in her negation, x belonged to the truth set of P, not simply the universe.

Carmen's Treatment of Negation

Like Shivani, Carmen initially quantified the LI $P(x) \rightarrow Q(x)$ existentially saying, "This means that there is some point in P, some point x in P, that is also in Q." Note that Carmen also restricted the universe to the truth set of P, the values for which she saw the LI as relevant. Then, the researcher asked Carmen to reason about Task 2 (see Figure 2). She responded as follows.

> So we're saying that for every single x, in the set of P... So if... for all values of x, P(x). So this would mean that P would be a subset of Q… [draws an Euler diagram] So I can negate this by saying... what is it? It's P(x)... So we would negate this by keeping the... hypothesis true, saying that P(x) is still true. And then... you can negate this by saying and not Q(x). I think that's how you negate this. Because saying that... um... Yes. And the 'and' is important, because this means that P(x) has to be true, and not Q(x) has to be

true. So you're saying that you are in P(x), but you're not in Q(x), so that would be negating the P(x) and being in Q(x).

Carmen correctly stated the negation as P(x) and ~Q(x), emphasizing the importance of the connective "and." However, she did not explicitly quantify x, so the researcher prompted her.

If we're saying for all x, and you want to negate it... I... think the negation... because you're not trying to find like a counterexample, you're just trying to negate it. I think you could say for every x… in P... and that's what you could say for every like for every x... in this set of, like a universal set, I guess. Maybe not... For every x in P... I'm not good at writing them out. Because I think… that with negations, you're trying to find a way that the original statement, like, not necessarily the opposite, but just like an alteration, I guess, to the original statement, or I don't think you're like looking for a counterexample. Because that would just be proving that the original statement will be false. So if negating it, you… it's still for every x, in... in the universe of... for x is in the universal set.

Carmen struggled to quantify her negation, expressing that negating the LI was different than finding a counterexample or disproving it. She felt that the negation should be a statement about every x (questioning whether x is in P or x is in the universe). Ultimately, Carmen settled on universal quantification, but never stated her full negation. So, the researcher (R) probed Carmen's (C) distinction between disproving the LI and finding its negation.

C: To show that that's false, I would just find one x ...um, that is in P(x), but it's not in Q(x).

R: And that's not necessarily the same as the negation?

C: I don't think so. Because that's just a counterexample. Yes, I think the negation is just kind of like doing the opposite of what it's asking for. It's not necessarily like… it's not asking you to prove that something is false or to make it or it's like asking you to rewrite this statement that would, in a way that would make it false, not asking you to prove that it is false. I guess...

Carmen understood that to disprove the LI, she needed just one counterexample. Unlike disproving, finding the negation for her entailed "rewriting" the statement to "make it false." To understand her emerging distinctions between negation, counterexample, and opposite, the researcher followed up with Carmen two months later during her SRI, presented here.

C: With a counterexample, you choose one specific value. So I think that's the difference between being the opposite and negation. So you have one specific value that you're looking at, but with a negation, you're saying that something is the opposite. You keep it arbitrary, because you kind of want to show like, for all umm so but with like a counterexample, you just say, okay, my k value or whatever equals a specific number. And that's why it's, it's false, because I can show that it doesn't work. Like for this one specific case.

R: Okay. Does that mean that negation and opposite are the same thing?

C: Um, I don't think... I know that they're not the same thing. Because opposite would be like hot and cold, right? ... But, I think a negation... I'm trying to remember...

R: What if I want to negate "hot"? How would I negate "hot"?

C: You can negate it in many ways. You could say warm, cold... or warm, hot, you could say, cold or freezing or stuff like that. So that's a negation because it's not hot. But if you want to just say it's opposite of hot, then you have to go to cold.

R: So if I want to do a counterexample of "hot", I would say, I would find an example of something that's not hot… If I wanted to do the opposite of "hot', I would show everything is cold…And if I wanna show the negation of ‘hot”, I would show everything is “not hot”.

C: Yes.

Carmen used the example “not hot” to explain her distinctions between counterexample, opposite, and negation. In suggesting “you can negate it in many ways,” Carmen seems to be referring to all possible counterexamples. Carmen still universally quantified her negation.

Discussion

The above data suggest that students’ reasoning about the negation of an LI can be influenced by multiple factors, including students’ difficulties with (hidden) quantification, reliance on Euler diagrams, reducing the universal set to the truth set of the hypothesis, and uncertainty about the distinctions between negation and proving false. In what follows, we present the discussion of three main findings that emerged from our thematic analysis.

Difficulty with (Hidden) Quantification

Consistent with prior literature, our participants experienced challenges in recognizing the hidden quantification of $P(x)\rightarrow Q(x)$. Shivani omitted quantification of both the LI and its negation. Carmen explicitly quantified the LI, but neglected the quantification of its negation. This finding confirms that quantification may significantly influence students’ ability to articulate the precise statement of negation.

Ultimately, both Shivani (when prompted) and Carmen existentially quantified $P(x)\rightarrow Q(x)$. Shivani supported her reasoning by drawing an Euler diagram and saying “if there exists an x in P.” Carmen did not rely on drawings. Nevertheless, her language for quantifying x (“there is some point in P, some point x in P, that is also in Q”) referenced an implicit use of an Euler diagram as well. This may indicate that students interpret $P(x)\rightarrow Q(x)$ as referring to a specific x, whereas $P\rightarrow Q$ might be universally quantified. The necessity to refer to a specific x may result in overlooking the vacuous case, which is crucial in distinguishing between the negation and the opposite (Arnold & Norton, 2017). This observation echoes our previous finding that reliance on Euler diagrams may hide the vacuous case (Antonides et al., in review).

Reducing the Universal Set to Only Those Values in Truth Set of the Hypothesis

Prior research has documented students’ struggles with the vacuous case. Hoyles and Küchemann (2002) found that students often view the vacuous case as irrelevant. Dawkins et al. (2023) reported a tendency for students to treat an LI as a statement only about the truth set of its hypothesis. Furthermore, Norton et al. (2022) explicated the nuances of students’ apparent conflation of the truth of an LI with the truth of its hypothesis. Our analysis suggests that students may omit the vacuous case not merely because of an apparent conflation, but rather because they focus only on values for which they view the LI as relevant. Reducing universe to the truth set of the hypothesis yields students’ treatment of the negation as its opposite logically correct—a finding not previously reported.

Both students focused on x values for which P(x) was true (coded as LIv). Shivani said, “if there exists an x in P” and “for all P(x), ~Q(x),” while Carmen stated, “there is some point in P, some point x in P” and “for every x in P.” For us, the LIv code emerged as students’ elimination of the vacuous case. However, in furthering our analysis, this treatment is closely related to the above prior research. Students’ focus on only values for which the hypothesis is true is consistent with their prior experience of using LIs (via *modus ponens*) to deduce new conclusions. This tendency could explain why students seemingly conflate the negation of P→Q with its opposite P→~Q. In fact, when students restrict the universe U to the truth set of P, the negation "∃x ∈ U such that P(x) and ~Q(x)" is logically equivalent to its existentially quantified opposite "∃x ∈ P such that P(x)→~Q(x)." This opposite statement is indeed the negation.

Universal Quantification as a Distinction between Negation and Proving False

Carmen's unquantified negation, P and ~Q, was correct. But, when prompted to quantify it, Carmen chose universal quantification (and further elaborated in her SRI). She consistently made distinctions between negation and proving false. She explained that negation is an "alteration" of the original statement "that would make it false." Perhaps because the original statement was about all values in the universal set and the negation was an "alteration" of this statement, Carmen reasoned that the negation should also be universally quantified. While existential quantification is indeed an assertion about the state of the universe, because Carmen associates "there exists" with a specific x, she may have felt that existential quantification of the negation would not be a broad enough statement about the universe.

When the researcher asked Carmen to distinguish between opposite and negation in her SRI, she gave the example of "not hot," saying "you can negate it in many ways" (e.g. warm or cold). This language may indicate that Carmen sees negating as a collection of all the ways one might say a statement is false. If Carmen was viewing negation as the collection of *all* possible counterexamples, then this might also explain her choice of universal quantification.

Implications and Future Research

We have discussed how EOs related to attending to quantification, treating the vacuous case, and making distinctions between negation and proving false may significantly influence students' treatment of negation. Nevertheless, there are other potential factors that may contribute to students' conception of negation. Such factors include treating an LI as an action (Norton et al., 2023), students' difficulties with transforming LIs (e.g., Durand-Guerrier, 2003), understanding the Principle of Universal Generalization (Norton et al., 2022), and others. Future research should shed light on the effect of these factors on students' reasoning about negation.

Our findings suggest potential instructional interventions. Teachers might draw attention to the distinction between using an LI to draw conclusions (modus ponens) and determining the validity of an LI. In particular, they might emphasize that the latter requires considering all values in the universal set U, not just values in the truth set of P. To make explicit their tendency to narrow their focus to only those values in P, students might compare and contrast the logical difference between "∃x ∈ U such that P(x)→~Q(x)" and "∃x ∈ P, such that P(x)→~Q(x)."

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References

Antonides, J., Norton, A., & Arnold, R. (in review). Linking structures across logic and space: The role of Euler diagrams. Manuscript submitted for publication.

Arnold, R., & Norton, A. (2017). Mathematical actions, mathematical objects, and mathematical induction. *Proceedings of the 20th annual conference for Research in Undergraduate Mathematics Education*, San Diego, CA.

Balechard, G. (1938). La formation d'esprit scientifique: Contribution à une psychanalyse de la connaissance. Vrin.

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990*, 79–117.

Barnard, T. (1995). The impact of 'meaning' on students' ability to negate statements. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th International Conference on the Psychology of Mathematics Education*. (pp. 3-10). PME.

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education, 3*, 496-522.

Dawkins, P. C., & Hub, A. (2017). Explicating the Concept of Contrapositive Equivalence. *North American Chapter of the International Group for the Psychology of Mathematics Education*.

Dawkins, P. C., & Roh, K. H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education, 6*, 1-22.

Dawkins, P. C., Roh, K. H., & Eckman, D. (2023). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. *Journal of Mathematical Behavior, 70*, 101043.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics*, 8(2), 44–51.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective, *Educational Studies in Mathematics, 53*, 5–34

Epp, S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12, 1999 Yearbook* (pp. 188-197). NCTM.

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886–899.

Griggs, R. A., & Cox, J. R. (1983). The effects of problem content and negation on Wason's selection task. *Quarterly Journal of Experimental Psychology, 35*(3), 519–533.

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*, 193–223.

Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. *Proceedings of the 45th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 69-78). University of Nevada, Reno.

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, 1-27.

Sellers, M. E. (2020). Students' quantifications, interpretations, and negations of complex mathematical statements from calculus [Doctoral dissertation, Arizona State University].

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *Journal of Mathematical Behavior*, *61*, 100802.

Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and Its Applications, 35*, 41–49

Zaslavsky, O., & Ron, G. (1998). Students' understandings of the role of counterexamples. In A. Olivier & K. Newstead (Eds.) *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, ed. A.* (pp. 25-32.). PME.

Measuring Mathematical Knowledge for Teaching College Algebra at Community Colleges

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The VMQI Research Team[1]

We present preliminary findings regarding the dimensionality of a 34-item instrument designed to measure the mathematical knowledge used in teaching college algebra at community colleges and the performance of items within the instrument. The instrument assumed a six-dimension model with two organizers of knowledge, one related to knowledge needed to perform two specific tasks of teaching, choosing problems *and* understanding student work, *and the other related to knowledge of three function types,* linear, exponential, and rational functions*. Multidimensional item response theory models were applied to a sample of 416 community college mathematics instructors. A three-dimension model structured by function types better fitted the data than a unidimensional model. Two- and six-dimensional models, structured by the tasks of teaching or the combination of function types and tasks of teaching, did not converge so they are not discussed. We discuss implications and work to further validate the instrument.*

Keywords: mathematical knowledge for teaching, college algebra, community colleges

Our project (Mesa et al. 2020) sought to develop and validate an instrument to measure community college faculty knowledge of teaching college algebra (MKT-CCA). Theoretical work on teacher knowledge has hypothesized distinct components of such knowledge (e.g., pedagogical, curricular, and content, Shulman, 1986) but empirical support has been elusive (e.g., Copur-Genturk et al., 2019). While there is a consensus that such knowledge should be multidimensional, and multiple efforts that further specify content or pedagogical knowledge according to cognitive level, knowledge type, or mathematical topic have flourished (Ball et al., 2008 [MKT]; König et al., 2011 [COACTIV]; Krauss et al., 2008 [TEDS-M]; Saderholm et al., 2010 [DTAMS]), the question about multidimensionality remains open (Schilling, 2007).

Using an approach that focuses on content-specific teacher decision making (e.g., in high school algebra or geometry), Herbst and colleagues (e.g., Herbst & Chazan, 2012; Ko & Herbst, 2020; Herbst & Ko, 2019) have empirically shown that teachers' mathematical knowledge used in two tasks of teaching are distinguishable via two distinct (albeit correlated) measures for knowledge needed to choose givens for a problem and understanding students' work in geometry. They surmise that the main challenge previous studies had in developing multiple measures is related to their knowledge-type-based framework given that there is no clear boundary among the types (e.g., common and specialized content knowledge) and that teachers need to use multiple dimensions of knowledge simultaneously in the moments of teaching. Grounding their work on the notion of instructional exchanges, they seek to understand, from the teacher's perspective, how knowledge is "transacted" in classrooms. Herbst (2006) notes that in their role as teachers, they are expected to provide students with activities or assignments that

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would support the development of mathematical ideas and that as students produce work in response to those activities and assignments, teachers must interpret the work to ascertain whether the student has learned or not. These two activities, choosing problems and understanding students' work, require teachers to sort through their own mathematical knowledge to make decisions that are guided by the goal of the instructional situation at stake.

We took this perspective in the design of an instrument that would assess community college instructors' knowledge of college algebra topics for teaching (Mesa et al., 2023). The core tasks of teaching used in this study are choosing problems and understanding student work. We defined *choosing problems* as a task of teaching that requires instructors to select activities or assignments that represent mathematical ideas accurately and give students opportunity to work on a mathematical idea at stake. Further, we defined *understanding student work* as a task of teaching that requires instructors to make sense of the ideas that students produce, either orally or in writing, and evaluate their mathematical correctness. Each task demands knowledge specific to teaching mathematics. Choosing problems requires knowledge about mathematical ideas presented in the problem, and about how to assess if ideas in the problem align with an instructional goal. For understanding student work, an instructor needs to have knowledge about interpreting students' work, typical errors students make, various strategies for solving a problem and how they could be represented, and the mathematically correct answer. We grounded these two tasks of teaching in three different types of functions, linear, rational, and exponential functions, as they are foundational to college algebra and for further work in mathematics. In this paper, we document the process of validating the structure of the instrument, seeking to identify whether there are indeed six distinct dimensions of mathematical knowledge for teaching, through the hypothesized organization along two tasks of teaching and the three function types (see Figure 1).

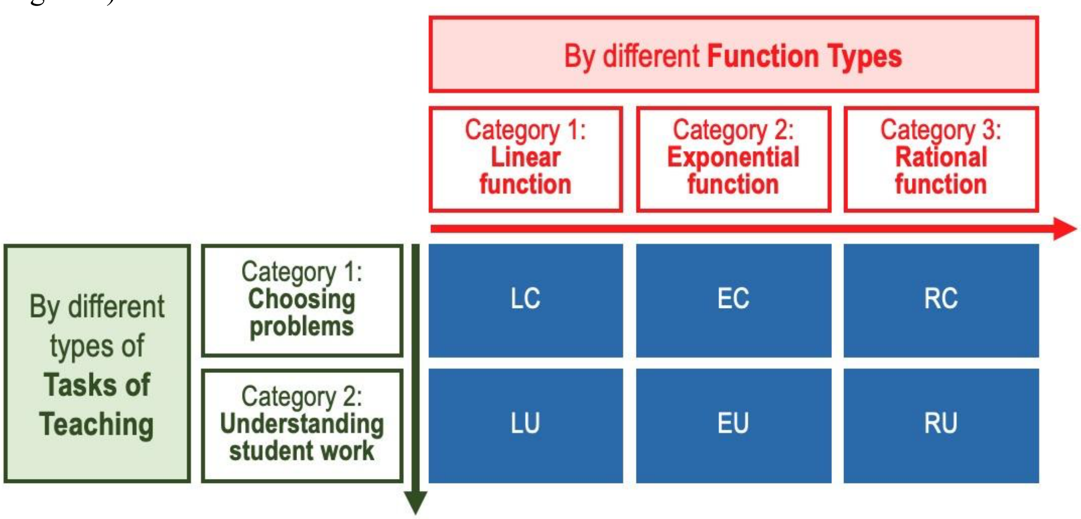

Figure 1. Blueprint for the development of the MKT-CCA instrument. (Mesa et al., 2023)

Methods

The instrument was developed over a two-year period that included an item development camp (13 practitioners, 11 researchers, 240 items), cognitive interviews to improve the items (12 instructors, 36 items), solicitation of situations and student work (15 instructors, 12 items), expert review of the mathematical content (2 mathematicians, 36 items), piloting (120 instructors, 60 items), and further revision of the items (see Figure 2).

Mr. Trevena is creating a lesson about applications of exponential functions. Which of the following news headlines would be the most likely to provoke a discussion about exponential growth patterns? **Choose one option only.**

A. Census data show that the population of the city tripled from 1970 to 1980 then doubled from 1980 to 2000.

B. Prices have increased 7% over the past 5 years, while usage of public transportation has decreased 5% over the same period.

C. Twice as many single people ages 20-25 are living with their parents today as they were 20 years ago and economists predict this trend will continue.

D. The water level of the local river has increased by 13% in the past decade and 35% during the past 30 years.

(a)

Ms. Ucci's class has just finished discussing the notation of $f(t)=a(b)^{rt}$ and the meaning of parameters and variables for exponential functions. To assess her students' understanding she gave them this task:

A radioactive substance decays according to the function $A(t)=100(1/2)^{(1/5)t}$ where t is measured in years and $A(t)$ denotes the amount of the unstable portion of the substance in micrograms at t. What is the meaning of ⅕ in the function?

Which of the following students' responses demonstrates a correct understanding for the meaning of ⅕? **Choose one option only.**

A. ⅕ means that the substance needs to decrease four more times to get to half of its original amount.

B. ⅕ means that ⅕ of the substance *A* is left after half of a year.

C. ⅕ indicates that it will take five years for the substance *A* to reach its half life.

D. ⅕ means that ⅕ of the substance A has decayed in half of a year.

(b)

Figure 2. Sample items in exponential function addressing (a) choosing problems and (b) understanding student work.

The final instrument included 34 items (27 multiple-choice items (MC) and seven testlets (T), resulting in 55 different questions) that were balanced across the six hypothesized dimensions (see Figure 3).

		Function type (#MC, #T)			Total
		Linear	Rational	Exponential	
Task of teaching	Choosing Problems	3, 2	5, 1	3, 3	11, 6
	Understanding Student Work	5, 1	5, 0	6, 0	16, 1
Total		8, 3	10, 1	9, 3	27, 7

Figure 3. Total number of multiple-choice items (MC) and testlets (T) per dimension in the final instrument.

We invited a sample of 4,350 instructors teaching from 752 different institutions to complete the instrument. The sample was constructed to ensure representation of the diversity of the colleges in terms of size, location, and diversity of the student population. The data were collected between October 2022 and March 2023. We obtained responses from 954 participants. Data were excluded if (a) the participant's position as an instructor at a community college could not be verified (n=185), (b) it was a duplicate response (n=85), (c) the participant entered their assigned ID but did not complete any portion of the survey (n=91), (d) the participant indicated that they had never taught college algebra (n=17), or (e) the participant responded to the 34 items in less than 1,200 seconds (*n*=20). There were 556 teachers who responded to some portion of the instrument after these data exclusions were applied. Final analyses focused on the 416 teachers who responded to all 55 questions on the MKT-CCA instrument. Accuracy on each question was submitted to a 2PL multiple item response theory (MIRT) model using maximum likelihood estimation in Mplus (Muthén & Muthén, 1998-2023). The dimensionality of the instrument was tested using likelihood ratio tests comparing alternative models. The six-dimensional model, where each factor represents one task of teaching and one function type, and two-dimensional models, where each factor represents one task of teaching, did not converge. The three-dimensional model included correlated factors, with each factor represented by a set of function-type items (linear, exponential, and rational). In the one-dimensional model all items were loaded onto one factor. Items were considered for deletion when the item did not provide much information and had a low discrimination estimate (less than 0.61, indicating low ability to differentiate teachers).

Results

The sample included faculty from 260 different community colleges in 43 states with 50% enrolling a majority of non-White students; 47% of the colleges were located in a city, about a quarter (26%) in a suburb, and the rest in rural (12%) or small towns (14%); most of the colleges were in the West (37%) or South (36%), with 19% located in the Midwest, and 9% in the Northeast region of the United States. Forty-eight percent of the participants identified as male, and 46% as female; in terms of race, 76% identified as White, 10% as Asian, 4% as Black, 2% as mixed, and 4% chose Other. Seventy-eight percent of the participants held full-time positions; 32% had a temporary appointment, and 9% were on tenure track. The average number of years of teaching experience was close to 17 years (mean=16.76, SD = 9 years; range: 1.5 to 47 years). The majority (63%) held a master's degree in mathematics, mathematics education or another mathematics related field, and 12% held PhDs (about 5% were in Mathematics Education.)

Discrimination and difficulty within the item response theory (IRT) paradigm are indices of item performance. Discrimination gives a measure of the differential capacity of an item, that is the ability of the item to differentiate participants by their ability level. The discrimination is the slope of the item response function graph (i.e., the item characteristic curve); the higher values suggest that the item has a high ability to differentiate examinees. Values of discrimination are reported as positive numbers (i.e., a negative value would be concerning because the probability of getting the answer correct should not decrease as ability increases). Difficulty is defined as the ability level at which we would expect examinees to have a probability of 0.50 (assuming no guessing) of selecting the correct response to the item. The higher the difficulty is, the higher the ability required from an examinee to have a 50% chance of answering an item correctly. Values of item difficulty typically range from –3 to +3; items with difficulty close to –3 are very easy items; items with difficulty close to +3 are very difficult for the pool of examinees. The discrimination values from the 3-dimensional model across the whole set of 55 items ranged from 0.183 to 2.555, whereas the level of difficulty of the full set of items ranged from –4.914 (very easy) to 2.140 (difficult).

The three-dimensional model including all 55 MKT-CCA items fit the data significantly better than the one-dimensional model, $\chi^2(3) = 12.37$, $p < .01$. The Akaike's Information Criterion (AIC) and sample size adjusted Bayesian Information Criterion (BIC) also favored the three-dimensional model. Four items contributing to the linear latent factor, seven items contributing to the exponential latent factor, and six items contributing to the rational latent factor were identified for deletion. The discrimination for all items considered for deletion was less than .61. After removing these 17 items, indicators representing the accuracy on the remaining 38 items were submitted to a final 2PL multidimensional IRT(MIRT) model using maximum likelihood estimation. As before, the three-dimensional model fit the data significantly better than the one-dimensional model, $\chi^2(3) = 9.93$, $p = .02$. The AIC and sample size adjusted BIC also favored the three-dimensional model. All item discrimination parameters were greater than .61 in this final model on the reduced set of items. Examination of the total information curve for each dimension (see Figure 4) revealed that the MKT-CCA instrument provided good coverage of a wide range of latent ability levels, but the test provided the most information about teachers with lower mathematical teaching knowledge. A total of 15 out of 16 of the linear items had negative difficulty parameters (below average MKT-CCA is needed to have a 0.5 probability of getting the item correct), 12 out of 14 of the exponential items had negative difficulty parameters, and all the rational items had negative difficulty parameters. The discrimination across the retained 38 items ranged from 0.612 to 2.472, whereas difficulty level ranged from -3.465 (very easy) to 2.181 (difficult).

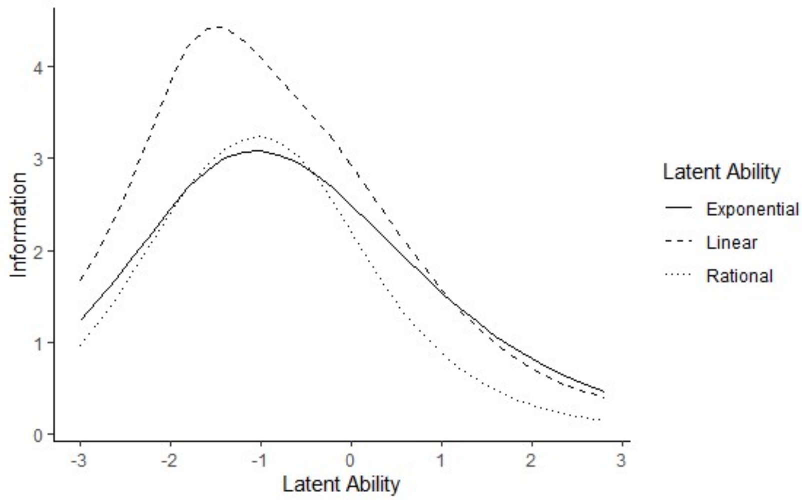

Figure 4: Total information curves for the three dimensions of the final model.

Discussion

The six-dimensional model did not converge, potentially because the sample size was insufficient. While the three-dimensional model fit the data better, we still need to further examine whether the three dimensions do really correspond to distinct function types, given that our hypothesized two-dimensional model was by tasks of teaching, and not about whether the types of functions were distinct. In other words, we need to rule out that the items that are purportedly designed to assess a specific function type, assess that function type and not others.

A two-dimensional model by tasks of teaching did not converge; in other words, it was not possible to differentiate knowledge along the two dimensions associated with choosing problems or understanding student work. We believe that this might be due to how we operationalized the choosing problems dimension, relative to the work by Ko and Herbst (2020) who focused exclusively on choosing givens (set of givens that had enough information for students to solve the problem) for problems in geometry. Such specificity might have contributed to make a clear distinction possible in Ko and Herbst's context. In our study, choosing problems is a task of teaching that requires instructors to select activities or assignments that include much more than specific givens, and thus our broader definition of a task may cause multidimensionality within the choosing problems dimension: that is in crafting items about choosing problems, we may have triggered knowledge related to understanding students' work.

As is apparent in Figure 4, we found that, overall, the instrument was not very difficult even though the instrument allows us to discriminate between high and low scorers. Because of resource limitations, and the need to reach faculty in multiple institutions in the United States, we administered the instrument online (instead of in person), which might have contributed to the test not being as difficult, as we believe that participants might have used resources such as textbooks or online applications to respond to the items, even though this was discouraged. Despite this, we believe that the instrument could be a promising tool to examine community college faculty knowledge for teaching these types of functions and design content-focused faculty professional development.

Next Steps

Our next steps in the validation process include conducting cognitive interviews with participants and testing the model with a population without teaching experience. With the cognitive interviews, we plan to validate our hypotheses about how instructors with or without the hypothesized knowledge would answer a randomly selected set of items. In doing so, we plan to corroborate that instructors who answer the item correctly are doing so because they have the needed knowledge, and that when an item is answered incorrectly, is because the instructor does not possess such knowledge. In randomly selecting the items, we will be able to establish the generalizability of the instrument. Moreover, cognitive interviews will allow us to shed light regarding possible ways in which the two tasks of teaching we hypothesized were intertwined in ways that instructors could not call on one or the other task of teaching independently of each other.

To determine whether there is something special about the instrument as related to community college mathematics faculty, we administered the test to a sample of 100 undergraduate students, who would be knowledgeable of the mathematics, but who did not have any teaching or tutoring experience. We are in the process of analyzing the data to identify whether the samples behave similarly or not. We anticipate major differences in the modeling, which would further confirm that the instrument is useful to assess knowledge for teaching these types of functions at community colleges.

As part of the data collection process, we gathered background information about faculty and their teaching experiences, their history of courses taught, their educational background, engagement in professional development, and their beliefs about mathematics, its teaching, and its learning. Further analysis of these data will shed light about how these characteristics relate to performance on the instrument by different groups of instructors, which would corroborate prior findings relating associations between courses community college faculty teach and performance on instruments assessing teacher knowledge (Ko et al., in press). Such analysis will provide further validation for the dimensionality, difficulty, and discrimination of the instrument.

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References

Ball, D. L., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Copur-Gencturk, Y., Tolar, T., Jacobson, E., & Fan, W. (2019). An empirical study of the dimensionality of the Mathematical Knowledge for Teaching construct. *Journal of Teacher Education, 70*(5), 485-497. https://doi.org/10.1177/0022487118761860

Herbst, P. (2006). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*, 176-203.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and the sources of justifications for the actions of the mathematics teacher. *ZDM Mathematics Education*, *44*, 601-612.

Herbst, P., & Ko, I. (2019, April). *Reconsidering the organization of mathematical knowledge for teaching*. Paper presented at the Annual conference of the American Educational Research Association, Toronto, Canada.

Ko, I., & Herbst, P. (2020). Subject matter knowledge of geometry needed in tasks of teaching: Relationship to prior geometry teaching experience. *Journal for Research in Mathematics Education*, *51*(5), 600-630.

Ko, I., Mesa, V., Duranczyk, I., Herbst, P., Kohli, N., Ström, A., & Watkins, L. (in press). Understanding the characteristics of mathematical content knowledge for teaching algebra in high schools and community colleges. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2021.2006348

König, J., Blömeke, S., Paine, L., Schmidt, W. H., & Hsieh, F.-J. (2011). General pedagogical knowledge of future middle school teachers: On the complex ecology of teacher education in the United States, Germany, and Taiwan. *Journal of Teacher Education, 62*(2), 188-201. https://doi.org/10.1177/0022487110388664

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM*, 40(5), 873-892.

Mesa, V., Ko, I., Boeck, C., Duranczyk, I., Akoto, B., Lim, D., Kimani, P., Watkins, L., Ström, A., & Beisiegel, M. (2023). *A blueprint for measuring mathematical knowledge for teaching college algebra at community colleges*. 13th Congress of the European Society for Research in Mathematics Education, Budapest, Hungary.

Mesa, V., Watkins, L., Duranczyk, I., Beisiegel, M., Kimani, P., & Ström, A. (2020). *Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction (AI@CC 2.0)* [Grant]. University of Michigan, Maricopa Community Colleges, University of Minnesota, Oregon State University.

Muthén, L. K., & Muthén, B. (1998-2023). *Mplus user's guide: Statistical analysis with latent variables, user's guide*. Muthén & Muthén.

Saderholm, J., Ronau, R., Brown, E. T., & Collins, G. (2010). Validation of the Diagnostic Teacher Assessment of Mathematics and Science (DTAMS) instrument. *School Science and Mathematics, 110*(4), 180-192. https://doi.org/10.1111/j.1949-8594.2010.00021.x

Schilling, S. G. (2007). The role of psychometric modeling in test validation: An application of multidimensional item response theory. *Measurement, 5*(2-3), 93-106.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Guided Reinvention of the Definitions of Reducibles and Irreducibles

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Informed by Realistic Mathematics Education, we designed a hypothetical learning trajectory on graduate students' guided reinvention of reducible and irreducible elements in unique factorization domains. We created experientially real tasks for use in a teaching experiment, in which students used algebra tiles as an emergent model of factoring integers and quadratics in $\mathbb{Z}[x]$. *In students' mathematical activity, this became a model for abstracting the shared structure of (ir)reducible elements in* $\mathbb{Z}$ *and* $\mathbb{Z}[x]$, *which students used as they formally defined (ir)reducibles.*

Keywords: Realistic mathematics education, abstract algebra, hypothetical learning trajectory

Abstract algebra courses that serve in-service and prospective teachers (IPSTs) should guide IPSTs to reason about the advanced content in a way that helps them reshape their understanding of secondary mathematics (Wasserman, 2018). Abstract algebra can be beneficial for helping teachers develop unified understandings of algebraic concepts that span K-16 curricula, such as the factorization of integers and polynomials (e.g., Murray et al., 2017; Novotná & Hoch, 2008; Usiskin, 1974). The Common Core State Standards of Mathematics (National Governors Association & Council of Chief State School Officers, 2010) suggested that teachers should teach students to "understand that polynomials form a system analogous to the integers" (p. 64). An abstract algebra course can support IPSTs in recognizing properties shared by the sets of integers ($\mathbb{Z}$) and polynomials with integer coefficients ($\mathbb{Z}[x]$). $\mathbb{Z}$ and $\mathbb{Z}[x]$ are *unique factorization domains* (UFD), defined as an integral domain R in which every nonzero element $r \in R$ that is not a unit has the following two properties: (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct), $r = p_1 p_2 \dots p_n$. (ii) The decomposition in (i) is unique up to associates: namely, if $r = q_1 q_2 \dots q_m$ is another factorization of r into irreducibles, then $m = n$, and there exists some renumbering of factors so that p_i is associate to q_i for $i = 1, \dots, n$ (Dummit & Foote, 2004). Reasoning about UFDs could help IPSTs better understand the properties related to factorization of elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$, e.g., the fundamental theorem of arithmetic and the fundamental theorem of algebra. Before one can understand this structure of UFDs, they need to understand the structure of irreducible and reducible elements. An *irreducible* is a nonzero, non-unit element r of an integral domain R such that if $r = ab$ with $a, b \in R$, then a or b must be a unit in R. A *reducible* is a nonzero, non-unit element r of an integral domain R such that we may write $r = bc$ for $b, c \in R$ and neither b nor c is a unit.

There has not yet been research done on how IPSTs can define the shared structure of factorability of the elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$. We address this by investigating how graduate student IPSTs reinvent the definitions of irreducibles and reducibles. We used the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1991; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) to design a hypothetical learning trajectory (Simon, 1995) of how IPSTs can reinvent the definitions of reducibles and irreducibles. This is part of a larger project in which we designed a local instructional theory (Gravemeijer, 2004) guiding IPSTs to reinvent UFDs in connection to their teaching. We address the following research questions: *Which aspects of the task sequence support IPSTs' reinvention of (ir)reducibles? Which ways of reasoning are productive to leverage to guide IPSTs toward reinventing (ir)reducibles?*

Literature Review

Reasoning about the properties of abstract algebraic structures can help IPSTs develop more robust understandings of the secondary algebra content they teach. Researchers have shown how reasoning about the properties of algebraic structures can help students or IPSTs develop unified understandings (Lee & Heid, 2018; Zandieh et al., 2017) of secondary mathematics content. Having a *unified understanding* of a concept involves recognizing the shared overarching structure of the concept among different instantiations of that concept in different contexts and being able to reason about that concept generally without needing to reference a specific context. Cook et al. (2023) identified students' unified ways of reasoning about the structure of inverses shared by different instantiations of that inverse concept. Serbin (2023) showed how reasoning about the group axioms supported IPSTs in recognizing the shared overarching identity structure shared by additive, multiplicative, and compositional identities. Several other researchers have explored students' unified understandings of algebraic concepts, including homomorphisms (Melhuish et al., 2020), linear transformations (Bagley et al., 2015; Zandieh et al., 2017), and binary operations (Wasserman, 2023). These studies illustrate how students and IPSTs reason about the shared structure of various algebraic concepts in different contexts.

We aimed to guide IPSTs to recognize the structure shared by certain types of integers and polynomials, those that are composite or factorable (reducibles) and those that are prime or not factorable (irreducibles). There has been minimal research done on students' understanding of reducible and irreducible elements. Two studies (Lee, 2018; Lee & Heid, 2018) explored the coherence of students' understandings of the factorization of integers and polynomials as instantiations of the decomposition of elements in integral domains. Lee (2018) investigated how students developed a unified understanding of factoring across elementary, secondary, and tertiary contexts. The students who perceived factoring whole numbers and factoring polynomials as disparate procedures could later recognize the shared structure of factorization of elements in integral domains. Juxtaposing integers and polynomials allowed the students to reflect on parallel structures within the sets, such as primes and irreducibles, composites and reducibles, and prime factorization and irreducible factorization. The students developed unified understandings of the shared structure of integers and polynomials as integral domains in which not every element had a multiplicative inverse. Lee and Heid's (2018) study informed our study design that guides IPSTs to identify the shared structure of primes and irreducible polynomials, as well as composites and reducible polynomials, to reinvent irreducibles and reducibles.

Theoretical Background

Design Research (Gravemeijer et al., 2003) involves a cycle of designing sequences of mathematical tasks to be used during instruction, implementing the task sequences, analyzing students' learning, and revising the task sequences based on the student learning outcomes. In designing the task sequence, design researchers develop a Hypothetical Learning Trajectory (HLT), which consists of the learning goal, the learning activities, and the hypothesized thinking and learning that students might engage in as they perform the activities (Simon, 1995). Through the process of designing, implementing, and revising HLTs, researchers can discover students' ways of thinking that anticipate formal mathematics and identify ways to leverage those ways of thinking to guide students' development of formal concepts. Our design of an HLT is guided by the instructional design theory of RME (Gravemeijer, 1999), which conceptualizes mathematics as a human activity. The HLT is designed based on the three RME heuristics: 1) experientially real tasks, 2) emergent model, and 3) guided reinvention. Experientially real tasks allow the students to connect with their informal knowledge, developing an intuitive understanding that will help them create an emergent model for use in that given context, defined as a *model-of* their situated activity. The emergent model shifts from being *model-of* a student's activity to a *model-for* formalizing mathematics as students mathematize their mathematical activity (Zandieh & Rasmussen, 2010). *Horizontal mathematization* involves "describing a context problem in mathematical terms to be able to solve it with mathematical means" (Gravemeijer & Doorman, 1999, p. 117), and *vertical mathematization* involves "mathematizing one's own mathematical activity" (p. 117). Design researchers create experientially real tasks that require horizontal and vertical mathematizing by which students, with guidance, can reinvent mathematical concepts.

Mathematizing occurs as students engage in mathematical activity, such as defining and conjecturing (Rasmussen et al., 2005). Defining is a process of formulating, negotiating, and revising a definition (Zandieh & Rasmussen, 2010). Freudenthal (1973) described two types of defining: descriptive and constructive. In *descriptive defining,* students build a definition by suggesting ideas about an object's properties, and the instructor helps organize and clarify such ideas. *Constructive defining* involves students modeling a new object by using prior examples or ideas. Rasmussen et al. (2005) deemed descriptive defining as horizontal mathematizing, in which the student organizes their ideas into a structured definition, and constructive defining as vertical mathematizing, where students abstract and generalize previously organized activities to create a new concept. Descriptive defining (horizontal mathematizing) grounds the context for students to progressively advance their activity to constructive defining (vertical mathematizing).

Methods

We designed an RME-based HLT with the goal of guiding students to reinvent the definitions of reducibles and irreducibles. We created "experientially real" tasks related to a high school/college algebra context that used algebra tile manipulatives. These tasks were designed to help the IPSTs develop an intuitive understanding of the mathematical concepts and engage in mathematical practices of defining and conjecturing, which would support them in the process of formalizing them. We made conjectures of students' reasoning for each of the designed tasks in our task sequence. We tested this HLT by administering tasks to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in a Hispanic-serving institution in the Southern US. Six mathematics graduate student IPSTs participated in this study. They all took Abstract Algebra courses at the undergraduate or graduate levels but have not learned UFDs in their coursework. They were grouped into three pairs (Group A with Josie and Raul, Group B with Javier and Roberto, and Group C with Kim and Taylor). Groups A and B participated in person, and Group C participated on Zoom. In-person groups were given physical algebra tiles, and the virtual group was provided with an online whiteboard and virtual algebra tile manipulatives.

The research team consisted of a teacher-researcher (TR), secondary instructor, and research assistant. The TR's role was to guide the session, provide students with prompts to work on, and ask follow-up questions to better understand and clarify the students' thinking. The research assistant operated the video camera and took field notes. The secondary instructor observed, took field notes, and asked questions to students when needed. We collected and transcribed the video recordings of fifteen 1.5-hour sessions of the teaching experiment (five sessions per group). The groups' written and virtual whiteboard work was collected. We used inductive coding (Miles et al., 2013) to analyze participants' reasoning, mathematizing, and the development of emergent models as they used algebra tiles in the task sequence. We focused our analysis on students' ways of reasoning that were leveraged by the TR to guide them to reinvent (ir)reducibles. In what follows, we describe the design of each task set in the HLT, how they were informed by RME, and the high-leverage student reasoning (italicized) evident in students' responses.

Results

Task Set 1: Contrasting Composite and Prime Integers

The experientially real tasks in Task Set 1 (see Figure 1a) were designed to help students develop an intuitive understanding of the factorability of integers as they use the emergent model of algebra tiles. The students created a *model of* factoring irreducibles and reducibles as they manipulated algebra tile representations of integers or quadratics in $\mathbb{Z}[x]$. Tasks guided students to see some integers, like 12 represented by 12 one-unit-squared tiles joined together, can be arranged into rectangular arrays where its factors are the side lengths of the rectangle. The first way of reasoning evoked was that *the factors of an integer correspond to the possible side lengths of the rectangular array that represents the integer*. This is evident in group C:

Kim: My thought process is just like, how are all the different ways that…you can factor 12, like with different, with smaller decompositions of numbers.

TR: … how does the factor translate to this rectangle concept here?

Taylor: It's length times width, right? That's what I would think of it. You know, 3 times 4 and 4 times 3, and 1 times 12, and 12 times 1, and 6 times 2, and 2 times 6.

When Kim and Taylor worked on problem 1, they recognized that 12 could be represented in different rectangular arrays that each had side lengths of the different pairs of factors of 12.

This led to their second high-leverage idea was that *the rectangle that was 12 units* $\times 1$ *unit was essentially the same as the rectangle with dimensions of 1 unit* $\times 12$ *units due to the commutativity of integer multiplication*. This is evident in this transcript from group A:

Josie: Do these count as two separate ones?

TR: That's a great question. So you're saying, 1 by 12, is that the same as 12 by 1?

Josie: …Well, I figured…multiplication is commutative, so they're exactly the same.

Josie and Raul thought that two rectangles with dimensions from the same factor pair were essentially the same, so they did not count as two different rectangles. This idea was used in the next task, 2a (see Figure 1), which was designed to help students recognize that prime numbers of square tiles can have only one arrangement that forms a rectangular array. Josie explained:

> Numbers that can only be written in one way would be the prime numbers. They don't have too many factors. They only have that one and itself. And if we're only counting like the- the 1 by 5 as the same as 5 by 1- that's really the only way you can do it.

Josie used the aforementioned way of reasoning that rectangular arrays with the same dimensions counted as the same representation to justify her claim that *numbers that could only be represented in one rectangular array must be prime*. This claim is the third way of reasoning that could be leveraged toward guiding the students to reinvent reducibles and irreducibles. Once the students recognized that those types of integers were prime, they used that way of reasoning in task 2c to identify the difference between the integers that could only be represented in one rectangular array and the integers that could be represented in many rectangular arrays. Josie responded, "The amount of factors that they have. Primes only have 1 and the number itself, and composites have at least 2." This elicited the way of reasoning that *the difference between the integers that could only be represented in one rectangular array and the integers that could be represented in many rectangular arrays was whether they were prime or composite*. This is the first instance where they use the emergent model (the algebra tiles) as a *model for* identifying the differences between irreducible and reducible elements in $\mathbb{Z}$. Overall, these tasks were designed to prompt the students to contrast the primes and composites using the emergent model. We posit that making this contrast is an essential waypoint in guiding students to reinvent (ir)reducibles.

a) Task Set 1	1. Find all the possible ways that the number 12 can be represented using rectangular arrays of squares. 2. Like the number 12, some numbers can be written in many different ways using rectangular arrays of squares. But some numbers can only be written in one way. a) Find several numbers that can only be written using one rectangular array of squares. b) Find several numbers that can written using different rectangular arrays of squares. c) What is the difference between these two kinds of numbers?
b) Task Set 2	1. Try to arrange these quadratics in a rectangular array using algebra tiles. Each side length of a rectangular array cannot be a constant number and must include the variable x. a) $3x^2 + 9x$ b) $x^2 + 3x + 3$ c) $2x^2 + 4x$ d) $x^2 + 4x + 3$ e) $x^2 + 5x + 7$ 2. What is the difference between the quadratics you **can** arrange into a rectangular array and the quadratics you **cannot** arrange into a rectangular array? Make a conjecture. 3. Give 3 examples of quadratics whose algebra tile representations could be arranged into rectangular arrays. 4. Give 3 examples of quadratics whose algebra tile representations could NOT be arranged into rectangular arrays. 5. Does your conjecture from Task 2 about the arrangement of the algebra tiles hold true for these examples? 6. You conjectured that: *A quadratic can be arranged into a rectangular array with algebra tiles if and only if the quadratic is factorable*. Let's test this conjecture with the following quadratics. a) $x^2 - 4$ b) $x^2 - 2$ c) $x^2 + 4$ 7. Refine your conjecture: *Quadratics can be arranged into a rectangular array with algebra tiles iff:*
c) Task Set 3	1. What similarities can you see between the integers that can be represented in different rectangular arrays and the quadratics that can be represented as a rectangular array with algebra tiles? 2. What similarities can you see between the integers that can only be represented in one rectangular array and the quadratics that cannot be represented as a rectangular array with algebra tiles? 3. We can name the first type of integers and quadratics that can be represented as a rectangular array as "**reducibles**." We can name the second type of integers that can only be represented in one rectangular array and quadratics that cannot be represented in any rectangular array as "**irreducibles**." We will formally define these terms soon. When irreducible integers can be arranged into a rectangle, one of its side lengths must be 1 or –1. These numbers are called "**units**." Definition: An element of a ring is called a unit if it has a multiplicative inverse. In a ring, an element is either 0, a unit, an irreducible, or a reducible. What are the units of $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients? 4. We can name the first type of **integers** and **quadratics** that can be represented as a rectangular array as "reducibles." Create a definition for a reducible. 5. We can name the second type of integers that can only be represented in one rectangular array and quadratics that cannot be represented in any rectangular array as "irreducibles." Define irreducible.

Figure 1. The three task sets in the HLT's task sequence.

Task Set 2: Contrasting Reducible and Irreducible Quadratics in $\mathbb{Z}[x]$

The tasks in Set 2 guided students to use the intended emergent model of algebra tiles as a *model of* factoring quadratics in $\mathbb{Z}[x]$ by attempting to arrange quadratics into rectangular arrays. The students recognized $3x^2 + 9x$, $2x^2 + 4x$, and $x^2 + 4x + 3$ can each be arranged into a rectangular array, where each side length of a given array is a factor of the quadratic, whereas $x^2 + 3x + 3$ and $x^2 + 5x + 7$ could not be arranged into a rectangular array (see Figure 2). The tasks had them find the difference between these quadratics and guided students to conjecture *(a) quadratics that can be factored can be arranged into a rectangular array where the factors of the quadratic are the side lengths of the rectangle and (b) the quadratics that cannot be factored cannot be arranged into a rectangular array.* The conjectures are high-leverage ways of reasoning that can be used to support students in reinventing (ir)reducibles. Kim used these:

> For the quadratics that you can arrange in a rectangular array, the quadratic is factorable, … if you have a quadratic you cannot arrange into a rectangular array, then…it is not factorable… if the quadratic is factorable, then you can arrange it in a rectangular array.

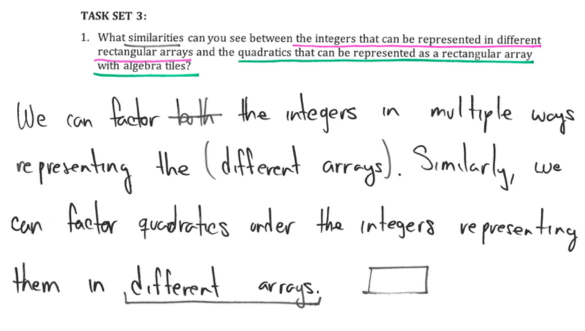
This discussion led group C to refine their conjecture to be "A quadratic can be arranged into a rectangular array with algebra tiles if and only if it is factorable." The tasks guided students to consider quadratics that are factorable over $\mathbb{R}$ but not over $\mathbb{Z}$, which cannot be arranged into a rectangle, which led them to revise their conjecture to specify the set over which the quadratic was factorable: *A quadratic can be arranged into a rectangular array with algebra tiles if and only if the quadratic is factorable over the integers*. These tasks guided the students to contrast irreducible and reducible quadratics in $\mathbb{Z}[x]$ to help them develop intuitive understandings of the difference between them in this set. Making this contrast is a high-leverage way of reasoning that is an essential waypoint in guiding students to reinvent the definitions of (ir)reducibles.

Task Set 3: Comparing and Defining Reducibles and Irreducibles in $\mathbb{Z}$ and $\mathbb{Z}[x]$

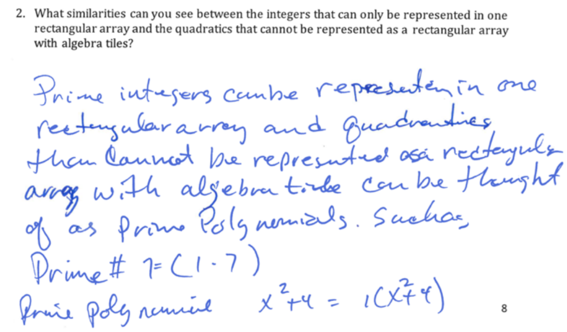
The tasks in Set 3 (see Figure 3) were designed to guide students to abstract and generalize the structure shared by reducible elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$, as well as the structure shared by irreducible elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$. The tasks prompted students to identify similarities among the modeled representations of the integers and quadratics in $\mathbb{Z}[x]$ that could be arranged into rectangular arrays with non-unit side lengths. Roberto responded, "They're factorable in the integers… one of them is composite, so it can be written in different ways, and the quadratics can be represented as a rectangular array." Prompting students to identify the similar structure of the modeled composite integers and quadratics allowed students to recognize that *integers and polynomials of this type can be factored or reduced into a product of integers or polynomials of lesser degree*. This way of reasoning about the reducibility property of these types of integers and quadratics can be leveraged to guide the students to reinvent the definition of reducibles.

In task 2, the students were asked to identify similarities among the modeled representations of the integers and quadratics in $\mathbb{Z}[x]$ that could not be arranged into rectangular arrays with non-unit side lengths. Javier responded, "Prime integers can be represented in [one] rectangular array. Like when 7 can be 1 times 7… quadratics that cannot be represented as a rectangular array with algebra tiles can be thought of as prime polynomials, such as $x^2 + 4$" (see written work in Figure 3). Prompting students to identify the similar structure of the modeled prime integers and quadratics led them to recognize that *prime integers and quadratics of this type have the same characteristic of not being factorable into a product of integers or polynomials of lesser degree*. This way of reasoning about the irreducibility of these types of integers and quadratics can be leveraged to guide students to reinvent the definition of irreducibles. These tasks intended to evoke a transition in the students' emergent model (i.e., the algebra tiles), provoking the students to use their *model of* factoring integers and quadratics in $\mathbb{Z}[x]$ as a *model for* abstracting the shared structure of reducibles and irreducibles, which could be used to define those concepts.

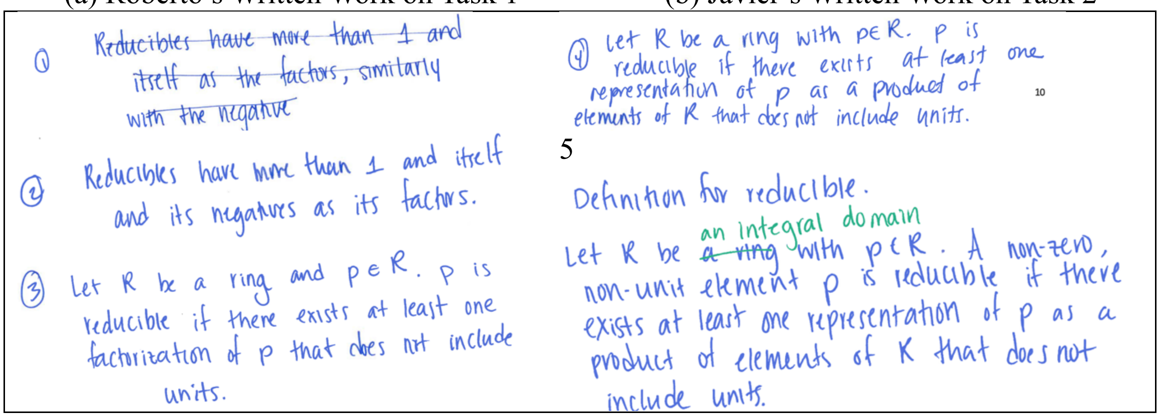
Before the tasks could guide students to formally define (ir)reducibles, the students needed to understand the concept of a unit in a ring because the definitions refer to a factor being either a unit or not. Task 3 gave the students the definition of a unit and prompted them to identify the units of $\mathbb{Z}$ and $\mathbb{Z}[x]$, which are 1 and -1 in both sets. Once the students had this terminology of "unit" they could use in their definition drafts, tasks 4 and 5 guided students to leverage their prior conjectures to give rough draft definitions of reducibles and irreducibles. The students iteratively refined their definition drafts with support from the TR's pedagogical moves of scaffolding, poking holes in arguments, posing counterexamples, pressing for precision, encouraging the definitions to be nonredundant and generalizable, and making sure any example of the concept satisfied the definition. Josie's five drafts of definitions of reducibles, which progress from using informal to more formal mathematical terminology, are shown in Figure 4. Overall, task set 3 were designed to engage students in constructive defining and vertical mathematizing, i.e., further mathematizing their previous conjectures about properties of reducibles and irreducibles in pursuit of reinventing and defining reducibles and irreducibles.



(a) Roberto's Written Work on Task 1



(b) Javier's Written Work on Task 2



(c) Josie's Drafts of Definitions of a Reducible Element

Figure 3. Students' written work in Task Set 3.

Discussion and Conclusion

This study shows how the task sequence and TR's pedagogical moves guided the students' reinvention of the definition of reducibles and irreducibles by facilitating their high-leverage reasoning about reducible and irreducible properties of integers and polynomials. The sequence of three task sets prompted students' reasoning in identifying and abstracting the reducibility/ irreducibility of integers (Task Set 1) and quadratics in $\mathbb{Z}[x]$ (Task Set 2). In particular, the experientially real tasks using algebra tiles evoked students' development of emergent models *of* factorizing integers and quadratics, which transitioned to become models *for* identifying and refining their conjectures about the shared structure of integers and quadratics (Task Set 3). In this process, TR's pedagogical moves played an essential role in developing the emergent models and facilitating students' mathematizations (Gravemeijer & Doorman, 1999) by helping them to make sense of their mathematical activities of creating and interpreting algebra tile models (horizontal mathematization) and by guiding them to formalize their conjectures of the shared structure of integer and quadratics in their definitions of reducibles and irreducibles in an integral domain (vertical mathematization). This finding suggests a hypothetical learning trajectory of students' reinvention of definitions in advanced mathematics by eliciting and leveraging students' reasoning in experientially real tasks. Future studies can extend the findings of this study by refining the task design and implementation and by examining different types of high-leverage reasoning that can also guide students in learning trajectories.

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References

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: The case of function and linear transformation. *The Journal of Mathematical Behavior*, *37*, 36–47.

Cook, J. P., Richardson, A., Reed, Z., & Lockwood, E. (2023). Using conceptual analyses to resolve the tension between advanced and secondary mathematics: the cases of equivalence and inverse. *ZDM–Mathematics Education*, 1–14.

Dummitt, D.S. and Foote, R.M. (2004). *Abstract algebra* (3rd ed.). John Wiley & Sons, Inc.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Netherlands.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, 6(2), 105–128.

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). Chapter 4: A hypothetical learning trajectory on measurement and flexible arithmetic. *Journal for Research in Mathematics Education. Monograph*, *12*, 51–66.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111–129.

Lee, Y., & Heid, M. K. (2018). Developing a structural perspective and its role in connecting school algebra and abstract algebra: A factorization example. In *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 291–318). Cham: Springer.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior, 60*, 100806.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Murray, E., Baldinger, E., Wasserman, N., Broderick, S., & White, D. (2017). Connecting advanced and secondary mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors. http://www.corestandards.org/Math/.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, 20(2), 93–104.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73.

Serbin, K. S. (2023). Prospective teachers' unified understandings of the structure of identities. *The Journal of Mathematical Behavior*, *70*, 101066.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education* (pp. 267-306).

Usiskin, Z. (1974). Some corresponding properties of real numbers and implications for teaching. *Educational Studies in Mathematics*, 279–290.

Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, *49*, 116–128.
Wasserman, N. H. (2023). Mathematical Challenge in Connecting Advanced and Secondary Mathematics: Recognizing Binary Operations as Functions. In *Mathematical challenges for all* (pp. 241-260). Cham: Springer International Publishing.
Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57–75.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, *95*(1), 21–38.

We're All Human, So Why Does Equity Matter?

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There is a need for mathematics instructors in higher education to have more equitable teaching practices. One way to address this need is through equity-centered professional development (PD). Using interviews across one year of professional development (PD), we describe how three immigrant mathematics instructors grapple with incorporating equity in their teaching practice using Gutiérrez *(2009) four dimensions of equity. We found that the instructors focused primarily on awareness and access with respect to equity but did not understand how to infuse equity in their practice. We also found that three themes, the discipline of mathematics, all students are human, and the identities of the instructors and beliefs seem to serve as barriers for instructors' implementation of equitable practices. Future research might consider how PD could be informed to account for the themes found in this study.*

Keywords: equity, identity, two-year college, community college

"It doesn't matter what my race is. Um, or my gender, because, you know, I'm teaching a subject that I feel is just it's universal. So it really does not matter." This quote came from a two-year college mathematics teacher. The statement is not surprising, and, in some cases might even be expected, as mathematics as a discipline has historically been positioned as universal, unbiased, and politically neutral. The implication being that mathematics is accessible to everyone. While this perspective is common, it is problematic as it removes any accountability or responsibility that the role mathematics as a discipline plays in *what* it means to do mathematics and *who* can do mathematics. This perspective is transmitted to individuals that attain advanced degrees in mathematics and to the students they teach. Having this perception makes it difficult to understand why equity is important in the teaching and learning of mathematics.

In this paper we focus on three two-year college mathematics instructors who are immigrants. We sought to understand how they negotiate their perspective of mathematics, which is closely aligned with the opening quote, when trying to understand and incorporate equity in their classrooms. The research question that guides our paper is: *How do instructors think about their teaching practices while grappling with tensions around self-identities, as well as personal experiences with and beliefs about mathematics as a discipline?*

Literature Review

Even as efforts are being made to advance STEM fields toward a more humanistic and equitable endeavor, mathematics tends to resist transformation more than other fields. One possible explanation can be traced back to how mathematics is perceived in society. Leyva and colleagues (2021) argue that mastery of mathematics is often seen as a precursor to one's social standing and frequently used as a measure for intelligence. Similarly, Moses and Cobb (2002) liken math literacy to civil rights, emphasizing its societal importance. Hence, when discussing mathematics as a discipline, we need to recognize the high esteem in which it is held in society and how it functions as a form of cultural capital.

In the pursuit of equity in mathematics education, it is crucial to examine four key characteristics: the universality of mathematics, the understanding of what it means to engage in mathematical practices, the historical exclusivity of the discipline, and its inherent political nature, especially as these factors relate to the often-understudied context of immigrant instructors in community colleges. First, mathematics is often conceptualized as universal. That is, mathematics is often portrayed as a universal discipline, suggesting its omnipresence and accessibility to all, irrespective of race, gender, or any other identity markers (Baber, 2015). The second characteristic to consider when thinking about equity in math education is understanding what it means to do mathematics. Understanding what it means to do mathematics is critical as there is often a misalignment between what it means to do mathematics and how mathematics is taught (Brown et al, 1989). The contrast between the two – school mathematics and the authentic practice of mathematics outside of the classroom – highlights the tension that exists between what it means to do mathematics and how it is taught in the classroom. The third characteristic is who can do mathematics. Historically the discipline of mathematics has been exclusive, highlighting that only certain individuals can do or are good at mathematics (Ernest, 1989; Martin, 2009; Hottinger, 2016; Nasir & Shah, 2011). The fourth characteristic is recognizing that mathematics is political (Gutierrez, 2013). The classroom is more than just a site for social reproduction and enculturation. Rather, it is a space where power dynamics, identity, and cultural constructs intersect. Indeed, mathematics inherently carries political aspects and power dynamics similar to other human activities. Collectively we argue these four characteristics inform instructor's teaching practices and how they conceptualize equity. Additionally, we build upon the literature by focusing on a population and context that is often understudied in mathematics education, immigrant instructors in mathematics and community colleges.

Theoretical Framing

The four dimensions of equity is a framework as described in Gutiérrez (2009) articulates considerations for equity in mathematics education. The framework highlights four components, access, achievement, identity, and power. The components are divided into two axes—the dominant axis and the critical axis. The dominant axis represents the prevalent emphasis for equity in mathematics education which centers on access and achievement in which access precedes achievement. Access refers to ways in which students can participate in mathematics. This includes students engaging with teachers, curriculum, and resources. Indeed, access may impact achievement. Achievement attends student outcomes such as grades and scores on standardized tests and can highlight ways in which students may demonstrate their mathematical knowledge. While the dominant axis is important, Gutiérrez acknowledges that it is only one half of equity in mathematics education. The second axis which is referred to as the critical axis, includes identity and power, where identity precedes power. Identity refers to the social markers of students and teachers, as well as their lived experiences, and cultural socialization. Understanding and incorporating students' identities is important to attend equity. Power, the last component, highlights the power and agency that both students and teachers have and how teachers can use their power to ensure that classroom instruction is inclusive and attends to the identities of students.

Methods

The Mathematics Persistence through Inquiry (MPIE) project is a five-year NSF-funded project that is using design-based research to build and implement a professional development (PD) program for two-year college mathematics instructors. The focus of the PD program is on supporting the development of inquiry- and equity-focused teaching practices. Goals of the project are to 1) study a two-year college's response to a state-mandated change in gateway mathematics courses (College Algebra, Precalculus, and Trigonometry), 2) use cycles of design research to build the capacity of math instructors in the two-year college to foster student success, and 3) investigate the effects of the capacity-building effort. In this paper we discuss participants' perspectives on equity, which are connected to their own personal experiences and relationships with mathematics, and, in turn, impact their teaching practices.

Setting and Context

The setting of this research project and PD efforts is a two-year Hispanic Serving Institution (HSI) in the Southwestern United States. In this paper we will refer to this institution as Southwestern HSI (SHSI). SHSI serves a student population that includes a majority of students from historically minoritized communities including Latinx students (68%) and students from low-income households (70%). PD participants include SHSI instructors who primarily teach gateway mathematics courses.

The MPIE PD program is a two-semester program running through the fall and spring terms. The participants discussed in this paper participated in the MPIE PD during the fall 2021 and spring 2022 semesters. All were math instructors at the SHSI, most of whom taught a variety of courses including gateway mathematics courses. The fall semester PD focused on inquiry-based teaching and learning with participants meeting six times, two hours each (for a total of 12 hours). In the spring, the PD shifted to an equity and inclusive emphasis, with participants meeting 4 times, 1 hour each (for a total of 4 hours). The reduced PD session time in the spring allowed for individual classroom observation and debrief sessions.

Data

For this paper we present three case studies: Paul, Nhung, and Savana. We selected them as cases because they are all immigrants with unique experiences that seemed to have shaped their beliefs about and ultimately their teaching of mathematics. The beliefs they shared highlighted a tension between their perspectives on what it means to do mathematics, what mathematics affords as a discipline, and the role of equity in disrupting their perspectives. Below is a brief description of each participant.

- Paul is a full-time mathematics instructor who grew up in Tijuana, Mexico and identifies as a Latino who has been perceived by others as white in some spaces and Latino in others, which has created a complex, dual identity for himself. He has taught a variety of courses at SHSI for more than two decades.
- Nhung is a full-time mathematics instructor who spent the first 11 years of his life in Vietnam before coming to the United States as a refugee. He has taught a variety of courses at SHSI for more than a decade.
- Savana is a part-time mathematics instructor who grew up in a large family in Tijuana, Mexico and was in the first generation in her family to attend college. She has taught a variety of courses at SHSI for more than two decades, as well as at other local colleges.

The data examined from the participants are three, one-hour interviews each across two semesters of PD. The first interview occurred after the fall PD, the second interview occurred before the start of the spring PD, and the third interview occurred after the conclusion of spring PD. Each interview was semi-structured and designed to capture information around the instructor's teaching background and style, mathematics experiences, equity beliefs and practice, and inquiry beliefs and practices.

Analysis

Each author on this paper read through a set of three interviews for at least one of the three participants, so that each set of interviews were analyzed by two different researchers. First round of coding involved members of the research team writing memos so that common themes regarding equity and identity could be identified, with Gutiérrez's four dimensions of equity as a guiding framework. After discussion, the themes of *self-identity, math as unbiased/universal, we are all human*, *awareness,* and *access* were identified as themes that appeared in each participant's data. Below we unpack how these themes address our research question.

Findings

These themes all arose across the three interviews from each participant that were used for data analysis. We present them in an order to help provide context for each participant, highlighting how their personal experiences and perspectives show up in their teaching practices.

Identity Impacting Practice

All three participants identified as immigrant teachers - each grew up in another country and then became a teacher in the United States. Their unique experiences growing up and becoming immigrant teachers has played a role in their teaching, impacting their practice and the way in which they interact with their students.

Paul. Paul grew up in Mexico and identifies as biracial. In Mexico he experienced racism and marginalization. Often referred to as "Gringo" or "El Blanco", he struggled to fit in or feel like he belonged in his high school classes. As a 15-year-old, he took a difficult physics class which he ended up failing. During a class period towards the end of the term his Physics teacher stated in front of the class that "The American white boy failed physics. So now he's going to learn what it is to be oppressed and what it is to be privileged." This experience was "hurtful" to Paul and has impacted his choices and values as a teacher. As a teacher at an HSI on the border with Mexico, he knows that there will be students in his class that will likely have an identity or experiences like his own. Having such students in his class is an opportunity for him to connect and provide the support students might need. He is fluent in Spanish and will share his identity with his students to let them know that he also has experienced challenges in his education.

Nhung. Nhung spent his childhood in Vietnam. He was 11 years old when his family moved to the United States (US). Growing up speaking Cantonese, he knew very little English at the time. Thus, as a student in US schools, he struggled in his classes and had multiple uncomfortable experiences when being called on during class. The language barrier made it hard to "verbalize" his thinking and these negative experiences impacted his confidence and learning processes. Mathematics classes usually did not pose the same level of difficulty for him, given the reduced amount of English vocabulary. However, he did struggle to read and interpret contexts presented in mathematical word problems, which made it difficult to apply the appropriate mathematics. These past experiences impact how Nhung teaches. For one, he never wants to put students in a position where they feel singled out and uncomfortable. And second, he wants to be available to help students if they are struggling and need additional support. He goes further to say "I feel like I can relate to students better because…the adjusting to classrooms in the United States is just something that I have a little bit more appreciation maybe for what they have to go through."

Savana. Savana was born and raised in Mexico. She was the youngest of 13 children and grew up in a traditional household. Neither of her parents were formally educated, so learning and writing were unfamiliar to them. Her father believed that the priority for women was to focus on family, children, and the home, but her mother viewed education as important. Savana embraced both values in her own life - family and education. Thus, Savana attended school through community college in her hometown of Tijuana, and then moved to the US to pursue her dream of becoming a teacher. This was not easy. She did not speak English, but she was motivated. She had a "dream…[to] teach math", and even though she did not know the language, noting that it "was really, really challenging", she strived to follow her dream. This aspect of Savana's identity is a big part of who she is and impacts how she interacts with students. She wants students to know and understand that pursuing an education (especially when not in your native language) is "not easy, but it can be done. If we work hard." Savana shares this story with her students, hoping it will motivate them to continue pursuing their education.

Mathematics is Universal and We're All Human

An interesting theme that arose across the interviews was the idea that anyone can do mathematics and that mathematics is a neutral subject. In fact, Nhung felt that mathematics is "unbiased", and that success is achieved with enough "time and effort". He went further to note that that was one of the motivating reasons for joining a PD that had a focus on equity - he wanted to understand how such a neutral subject could have inequities.

The notion that mathematics is a neutral subject was also intertwined with the idea that "we are all human". When asked about how she thought about race, Savana stated that she did not think about race and went further to say: "We are all the same. We're all human beings. We're all the same." While Paul had a similar sentiment, he situated it in the classroom setting about engaging with mathematics and promoting student success. Paul noted the sameness in which he wants to support students while still monitoring biases he may have:

> I'm trying to break the stereotype by, by trying to be okay now we can- no be the same with everybody. Be the example, communicate the same way with everybody, treat everybody with the same level of respect, communicate with the same level of acknowledgement of respect. Don't focus on only a handful of students or one gender. Everybody's the same, everybody's equal. We all belong to one to race the human race. If I, if I make sure that everybody to me is a human, that right there is helping me keep myself in check and avoid falling into those biases, those stereotypes. And if they see that I'm doing that and, and it's encouraging them to succeed.

Throughout PD, Nhung struggle with the idea of inequity existing in the mathematics classroom:

> I just have always seen myself as I'm just a math teacher. Right. It doesn't matter what my race is. Um, or my gender, because, you know, I'm teaching a subject that I feel is just it's universal. So it really does not matter.

Although all participants had this idea of mathematics being universal or we are all human, they each acknowledged the inequities that they themselves had experienced as students or inequities they recognized as being present in mathematics as a discipline. These inequities shaped their identity as mathematics teachers, and, in turn, impacted how they interact with students.

Getting Stuck: Awareness and Access

Throughout the interviews it was clear that Paul, Nhung, and Savana increased their understanding and awareness of classroom equity issues as they progressed through the PD.

Participants seemed to think about equity as access to opportunities. An instance that captures such awareness that was developed early on is when participants were provided with an image, which is now ubiquitous, showing two cartoon panels. In one panel, there are three children trying to look over a fence to see a baseball game. The children are of different heights, and each child is standing on the same type of box to try to look over the fence. Although each has a box to stand on, not all can see over the fence. In a second panel, each child is standing on a different size box that will allow for them each to see over the fence. Although this image does not capture equity as a concept in its entirety, it provides an entry point to a conversation about equity, especially for those who are new to having such discussions. It was this image that gave Paul clarity about equity in the classroom for the first time.

> And they showed the [cartoon description] and I thought to myself, by the way, please forgive me if I if I release a colorful metaphor, because I really get worked up about this. But I was thinking, "God damn w– we've been doing this all along! We've just been giving everybody the same box!" But hey look– there's tutoring, whoopie! Hey look– we have workshops, yee haw! But if students have different needs, how can we bring them up to the same level instead of just giving everybody the exact same opportunity? That's when it clicked, that's how it's done.

Paul realized that giving all students the same opportunities was not what it meant to attend to equity. He worked to meet the students where they are, and part of this process was allowing students to participate in ways that they feel comfortable. This notion of opportunity and comfort was also reflected in Nhung's and Savana's interviews.

Nhung felt that being equitable was giving "students the opportunity to learn" through group activities guided by "leading questions". He believed that each of his students were capable of the same achievement with enough practice, time, and effort. However, Nhung began to struggle with his definition of equitable teaching once he began to realize the varying levels of preparedness in his students, which impacted their ability to take advantage of such opportunities. He noted,

> Well, when you interact with students when you recognize that they are not as prepared to take advantage of those opportunities. Right? And now I see where some of it is falling short. But I don't really know how to address it yet.

While Nhung felt opportunities were important, he recognized that giving students the same opportunities did not address the gap that existed between students. Nhung struggled with this tension stating:

> Right. But it's not just equal opportunity. Right. Because if you start further up than someone else, and both of you are presented with the … same opportunities, you are going to rise or at least you have the opportunity to rise, um, at a greater rate than someone else who started way behind.

This tension caused Nhung to shift his perception of equitable teaching slightly, adding the caveat that students who entered his class with gaps in prerequisite knowledge just needed more chances to work one-on-one with him as the teacher. He further notes that if students "elect to not interact with" him, he will leave them alone and not "bother them". From Nhung's perspective one-on-one attention was a way to elevate students to a level commensurate with their classroom peers. Savana had similar sentiments stating that equity "means being able to provide every student what they need", which may be unique to each student. Ultimately Nhung viewed issues related to equity as access issues and students just needed opportunities to engage with mathematics. Savana, like Nhung, recognized that students were at different ability levels and felt that support needed to be customized based on a student's specific needs. However, while Nhung struggled to attend to students' needs because one-on-one help during class was difficult to accomplish for all students, Savana expressed not knowing how to begin to attend the issue.

> How do I help the student that, um, you know, uh, that has, I don't know, anxiety and, and, and, and just being asked to, to talk in front of the class, you know, makes just asking that person, a question, you know, uh, may block him and, and not be able to, you know, to, to, to follow, you know, what I'm doing. So it's, so it's so, uh, challenging

By the end of the first year of PD not much had changed with perceptions of equity in the classroom. Paul, Nhung, and Savana all discussed increased awareness about equity in their teaching, but little change could be seen in the data across their interviews. All continued to see increasing access to engaging through participation in mathematics as the key piece to attending to equity in the mathematics classroom.

Discussion

Paul, Nhung, and Savana provide an important description for what it means for immigrant mathematics instructors in higher education to grapple with incorporating equity in their teaching practices. Although each instructor had increased awareness with regard to what it meant to attend to equity in the classroom, at the end of their first year in PD that awareness was limited to focusing on increasing access to mathematics. This translated in their practice through their teaching by having more one-on-one interactions with students and encouraging more questions and discussion from students (as they felt comfortable).

There are several factors that might contribute to this limited view of equity in the classroom. Through themes identified across Paul, Nhung, and Savana, one potential barrier seems to be their own personal identities, beliefs, and experiences. The perspective that mathematics is unbiased or universal makes it difficult for the instructors to understand how shifting teaching practices beyond more "opportunities" would help attend to equity. This idea is further validated as Paul, Nghung, and Savana succeeded academically and professionally, in part because they took advantage of opportunities and were motivated. As a result, this view makes it difficult for these instructors to shift their practice beyond creating opportunities and encouraging participation.

Conclusion

Math is political and it is important for instructors to understand their own identity, power, and responsibility when implementing equitable teaching practices. In this paper, instructors highlight a tension that exists between their own identities, beliefs, and experiences, and the common narratives that exist within mathematics (e.g., mathematics is universal). Understanding these tensions and the deep-rooted impact that historical norms and beliefs about mathematics have given us a better understanding of how to reframe aspects of ongoing PD to try to get instructors to move beyond *awareness* and *access* and towards practice that also incorporates Gutiérrez's critical dimension (*identity* and *power*).

Acknowledgments

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References

Baber, L. D. (2015). Considering the interest-convergence dilemma in STEM education. *The Review of Higher Education, 38*(2), 251-270.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-42.

Ernest, P. (1989). *Mathematics teaching: The state of the art.* Falmer Press.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." *Teaching for Excellence and Equity in Mathematics, 1*(1), 4-7.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37-68.

Hottinger, S. N. (2016). *Inventing the mathematician: Gender, race, and our cultural understanding of mathematics*. State University of New York Press.

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It Seems like They Purposefully Try to Make as Many Kids Drop": An Analysis of Logics and Mechanisms of Racial-Gendered Inequality in Introductory Mathematics Instruction. *The Journal of Higher Education, 92*(5), 784–814. https://doi.org/10.1080/00221546.2021.1879586

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record, 111(*2), 295-338.

Moses, R. P., & Cobb, C. E. (2002). *Radical equations: Math literacy and civil rights*. Beacon Press.

Nasir, N. I. S., & Shah, N. (2011). On defense: African American males making sense of racialized narratives in mathematics education. *Journal of African American Males in Education (JAAME), 2*(1), 24-45.

Undergraduate Students' Use of Generative Artificial Intelligence in Proving

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In this paper, we explore how undergraduate students use and evaluate generative artificial intelligence (genAI) in proving. We view proving as a human activity and proof as a production of proving, and hence, we believe that students need to be the ones who evaluate the genAI-generated arguments and write their proofs. In the initial phase of research on genAI in proving, we conducted interviews with three undergraduate students to examine how they use genAI in proving and how they write their proofs after their use of genAI. This study revealed that there is a flow in students' use of ChatGPT in proving, and there are factors that impact their use of genAI. We think that the flowchart presented here can serve as a guide for both teaching and research on the use of genAI in the context of proving.

Keywords: Artificial Intelligence (AI), Undergraduate Students, Proving, ChatGPT

Since the end of 2022, when a generative Artificial Intelligence (genAI) tool became public, ongoing discussions about the role of AI in education have accelerated. People started viewing genAI as a tool that can both benefit and hinder students' learning. As the prevalence of the use of genAI in education is inevitable, it is important to consider how educators can help and guide students' responsible use of genAI in education. Based on its affordances (e.g., instant feedback or personalized learning) and limitations (e.g., wrong information or ethical issues), we believe that students should be able to be aware of the pitfalls of AI tools and critically evaluate responses generated by AI tools.

We chose proving as our research context because proving is a context where students need to persuade themselves and convince others (Bieda & Lepak, 2014; Hanna, 2000; Hersh, 1993). Using the feature of genAI that provides instant conversation, it can be used as a good tool to practice convincing each other. Another benefit of using a genAI as a tool in proving can be that it does not necessarily involve negative human interactions with the user. For example, when students ask questions to their instructors or fellow students, power relationships exist between them. Then, some students may take mathematical statements and proofs from the authorities, including instructors, peers, or textbooks, without questioning them (Weber & Mejia-Ramos, 2014). On the contrary, using AI might help students feel more agency and authority by being free from those power dynamics because there is no human behind the tool and their (intentional or unintentional) judgment about the student. Thus, communicating with genAI can become a place where students can have authority in evaluating the argument if used appropriately.

As there have been only a few, if any, studies regarding the use of AI in undergraduate mathematics-level proving tasks, this study initiates exploring this area. We investigated the following research question: *How would undergraduate students use ChatGPT when they are proving a mathematical statement?* Through the study, we call for more conversation about the use of genAI tools in proving by proposing a set of aspects to be considered when using genAI in proving and how those aspects are connected and interact with each other, by revising the IDEE framework by Su and Yang (2023).

Backgrounds and Theoretical Perspectives

In this section, we review 1) the affordances and limitations of genAI, 2) our perspectives on proving, and 3) a framework for using a genAI in proving by modifying the IDEE framework (Su & Yang, 2023).

Affordances and Limitations of GenAI in Education

As emerging of genAI tools was recent, at the end of 2022, which is about a year ago, not much research exists in education in general, and even less in mathematics education. The most common form of current research articles is examining genAI as a tool for education to find affordances and limitations. Researchers viewed that genAI could be of benefit to students' learning, including but not limited to the following: 1) provide personalized and interactive learning environment tailored to individual students' preferences and levels (Baidoo-Anu & Owusu Ansah, 2023; Qadir, 2023; Su & Yang, 2023); 2) provide generate formative assessment tools (Baidoo-Anu & Owusu Ansah, 2023; Qadir, 2023); and 3) provide instant feedback (Baidoo-Anu & Owusu Ansah, 2023). GenAI tools, however, have some limitations and challenges as well. First, several researchers pointed out that genAI can provide wrong information depending on the data set that it was trained in (Baidoo-Anu & Owusu Ansah, 2023; Qadir, 2023; Su & Yang, 2023). Researchers claimed that the accuracy of genAI responses would be lower in complex tasks (Su & Yang, 2023). Second, ethical concerns exist regarding using genAI, such as plagiarism and overreliance (Qadir, 2023), bias in the data set (Baidoo-Anu & Owusu Ansah, 2023), users' privacy (Baidoo-Anu & Owusu Ansah, 2023; Qadir, 2023; Su & Yang, 2023), and equity issues (Cooper, 2023; Qadir, 2023).

Based on this review, as mentioned previously, we view the critical use of genAI as a skill for students to take advantage of the tool, and hence, we aimed to provide a guide for the goal.

Proving and Proofs

In this study, we viewed proving as human activities with the goal of generating proofs. By proof, we adapt the definition of Stylianides (2007). Stylianides defined mathematical proof as

> … a mathematical argument, a connected sequence of assertions against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291).

The key point of this definition is that an argument can be referred to as proof when it is approved by the classroom community, which is their mathematical community. It aligns with other researchers' arguments that a mathematical argument becomes proof when a mathematical community agrees on it (Weber & Mejia-Ramos, 2014). From this perspective, what genAI produces is a mathematical argument, but it does not become proof until a mathematical community sanctions it. In other words, students who use genAI become the first person in the line who gets to decide if the generated argument can be qualified as a mathematical proof considering the three components mentioned in the definition of proof before they present this to other people. Thus, the subject who writes and presents mathematical proof should be and is a person, not genAI.

IDEE Framework

Su and Yang (2023) suggested a theoretical framework as a guide to using generative AI in education. The framework has four aspects:

Identify the Desired Outcomes. Identifying the objects of generative AI before using it.

Determine the Appropriate Level of Automation. Automating teaching or learning experiences using generative AI or using it as a supplement to traditional teaching methods, depending on the objectives.

Ensure Ethical Considerations. Considering ethical implications of using generative AI, including potential biases and their impact on teachers and students.

Evaluate the Effectiveness. Evaluating the effectiveness of generative AI in achieving the desired outcomes.

While the IDEE framework (Su & Yang, 2023) is for teaching and learning in general education, this study focuses on students' use of generative AI in the context of mathematical proof. We interpreted the framework by focusing on (1) mathematics education, more specifically for students' proving, and (2) students' use of generative AI for their learning. Thus, the following is our interpretation of the IDEE framework.

Identify the Desired Outcomes. Identifying the objects of generative AI before using it when proving mathematical statements

Determine the Appropriate Level of Automation. Deciding the ways to use generative AI and the extent to which they want to take from what generative AI generates depends on the objectives.

Ensure Ethical Considerations. Students' recognition of potential ethical issues, such as determining whether using generative AI for homework aligns with academic integrity.

Evaluate the Effectiveness. Evaluating the effectiveness of generative AI in achieving the desired outcomes when proving mathematical statements.

Methods

Data Collection

Participants. In July 2023, we conducted individual interviews with three participants, Yuna, Jiho, and Kitae (all pseudonyms), from two different universities in Korea. All of them were freshmen mathematics education majors. Yuna was a woman, and Jiho and Kitae were men. Yuna took a Linear Algebra course, which she described as a proof-based course, as well as Calculus I, and the other two took only Calculus I.

GenAI. We used ChatGPT 3.5 as a genAI in this study because it was the most commonly used genAI in Korea at the time of data collection, July 2023.

Interviews. At the beginning of the interview, students were given six mathematical statements, mostly from number theory and one from advanced calculus. The statements were written both in Korean and English. Students were asked to choose one or two statements they wanted to prove or disprove and allowed to use any resources. We did not ask them to use ChatGPT for this part to see if it occurs naturally for them. However, none of them used the ChatGPT in the first part, so we asked them if they were aware of the tool and had used it. All of them said they had used ChatGPT before, so we asked them to use ChatGPT for proving tasks, observed how they used ChatGPT, and evaluated the argument generated by ChatGPT. During the interviews, a main interviewer led the interview, and a secondary interviewer took notes and asked supplementary questions if needed. Each interview lasted 80 to 100 minutes and was transcribed into Korean after the collection.

Data Analysis

For data analysis, we used the modified IDEE framework. We first coded the transcripts using the four categories: 1) desired outcomes, 2) level of automation, 3) ethical consideration, and 4) effectiveness. In applying the initial categories, we viewed the level of automation as students' decision process to get an argument and effectiveness as students' evaluation of the argument generated by ChatGPT. If students' use of ChatGPT in proving does not fit into one of the four categories, we coded those as others. First, each participant's transcripts were coded by one of the team members; then, different team members did the second round for cross-checking purposes. When the team members detected differences, it was discussed until the team agreed on specific coding. Through this repetitive process, we revised and specified the codes (Table 1).

Table 1 Code Descriptions

Codes	Revised Codes
Desired outcomes	Objectives of using genAI.
Level of automation	Decisions on prompts to ask (differently) to genAI. Decisions on continuing conversation with genAI. Decisions on how much they want to take from the generated arguments by AI.
Ethical consideration	Opinions regarding ethical issues
Evaluation of effectiveness	Evaluations of genAI's argument
Others	GenAI belief Proving belief Proving knowledge

Findings

In this section, we present how Yuna used and understood ChatGPT as a learning resource and how she used ChatGPT in a proving task. We only present Yuna's case here because of the page constraints.

Yuna's Previous Experiences in Using ChatGPT

Yuna was the participant who often used ChatGPT for her undergraduate education in general. She used ChatGPT in mathematics when she had a problem that she did not know how to solve or did not have enough time to complete assignments. Yuna generally noted that ChatGPT was a useful tool that expanded her draft and met the quantity requirement for writing assignments in other contexts. She said ChatGPT was not that useful in studying mathematics because 1) it is not easy to type mathematical symbols into it, and 2) sometimes it generates different answers even if the same problem was typed in. Because of these reasons, she preferred searching online (e.g., Google or MathStackExchange) to ChatGPT for proving tasks. Lastly, despite her negative evaluations for ChatGPT in proving, that she thought using ChatGPT was not cheating and did not have an ethical problem. She thought using ChatGPT did not matter what she used as resources for assignments and said maybe it was even better than copying peers' assignments.

Yuna's Use of ChatGPT in a Proving Task

As she did not use ChatGPT without the interviewer's direction, we asked her to use ChatGPT. In Yuna's use of ChatGPT for a proving task, we found that she had a conversation with ChatGPT by entering prompts several times. We found a repetitive pattern of 1) making decisions on prompts, 2) executing the prompts in ChatGPT, and 3) evaluating the responses until Yuna felt that she was ready to write her proof, which is detailed below (Yuna's pattern of using ChatGPT in proving tasks in Figure 1).

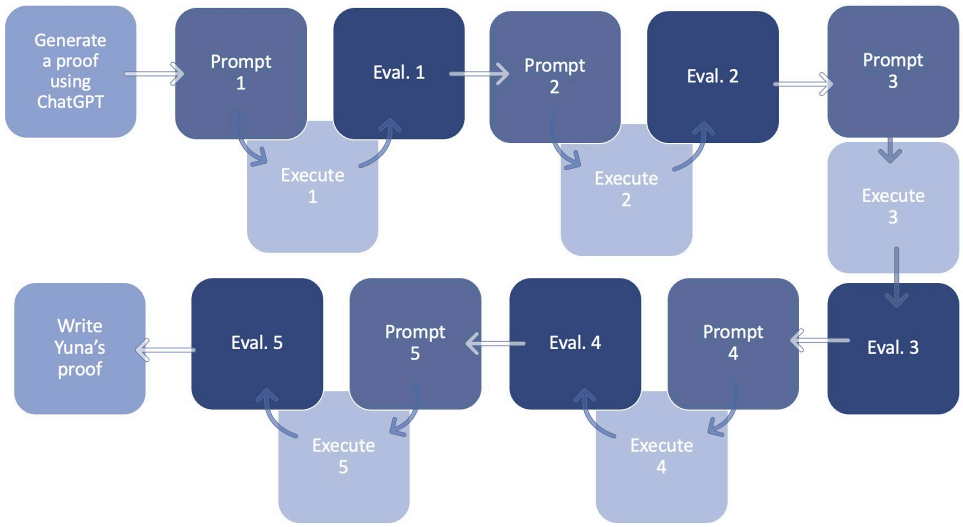

Figure 1 Yuna's pattern of using ChatGPT in proving tasks

Given a set of questions, Yuna selected the following statement to use ChatGPT for proving: *For all integers m and n, if the product mn is odd, then m is odd and n is odd.* As soon as she chose the problem, she copied and pasted the statement directly to prompt ChatGPT (Prompt 1 in Figure 2). ChatGPT provided an argument claiming that the statement was false. When Yuna read the ChatGPT-generated argument, she found the argument was conflicting with her conjecture that the statement was true. Then, she revised the prompt by adding "can you prove this statement:" in front of the previous prompt and typing "m*n" instead of "mn" to clarify that it is about multiplication (Prompt 2 in Figure 2). With Prompt 2, ChatGPT generated an argument using proof by contradiction that discussed the other three cases, except the case when both *m* and *n* are odd. Yuna initially felt that the argument seemed correct but soon questioned:

> It seems like it is saying the right thing… There exist four cases for *m* and *n,* and it used all the cases except when both *m* and *n* are odds, and claiming these cases led to a conclusion that *mn* is even, but I don't know where the remaining case is. [Translated from Korean]

Then, we asked if she knew what the word "proof by contradiction" meant and translated it into Korean. Then she typed Prompt 3 in Figure 2 in Korean, saying: "There are too many English words than mathematical expressions compared to what I am used to, so it is hard to recognize the argument in skimming." When ChatGPT produced the response for Prompt 3, she was not satisfied again and prompted Prompt 4 in Figure 2. Yuna noted that both responses did not include many mathematical expressions, and she did not feel ChatGPT's response was rigorous enough to be considered as proof.

When the interviewer asked if she wanted to prompt further, she typed Prompt 5 (Figure 2), adding "considering cases for m and n," into the prompt because she did not understand why the argument generated by ChatGPT used proof by contradiction. She felt this response was similar to what she was thinking and decided she would not use ChatGPT anymore as she knew what to do. Yuna noted that she probably would not use proof by contradiction to write her final answer. Even though Yuna said that she would not use "proof by contradiction" in her final answer, she used the format of proof by contradiction without the word "proof by contradiction" in her answer.

Prompt	Yuna's Prompts	Translation
1	For all integers m and n, if the product mn is odd, then m is odd and n is odd.	N/A
2	can you prove this statement: For all integers m and n, if the product m*n is odd, then m is odd and n is odd.	N/A
3	모든 정수 m 과 n 에 대하여, 만약 그 곱 mn 이 홀수이면, m 도 홀수이고 n 도 홀수이다.	For all integers m and n, if the product mn is odd, then m is odd and n is odd.
4	귀류법을 사용해서 모든 정수 m 과 n 에 대하여, 만약 그 곱 *mn* 이 홀수이면, *m* 도 홀수이고 *n* 도 홀수이다. 를 증명해봐.	By using proof by contradiction, prove. For all integers m and n, if the product mn is odd, then m is odd and n is odd.
5	귀류법을 사용해서 m 과 n 을 경우를 나누어서 아까의 명제를 증명해봐.	By using proof by contradiction and considering cases for m and n, prove the previous statement

Figure 2 Yuna's prompts used in ChatGPT

Discussions

By combining the analysis of Yuna's and the other students' use of ChatGPT in their proving tasks, we present a flow chart (Figure 3) to illustrate students' process of using ChatGPT in proving by revising IDEE framework (Su & Yang, 2023).

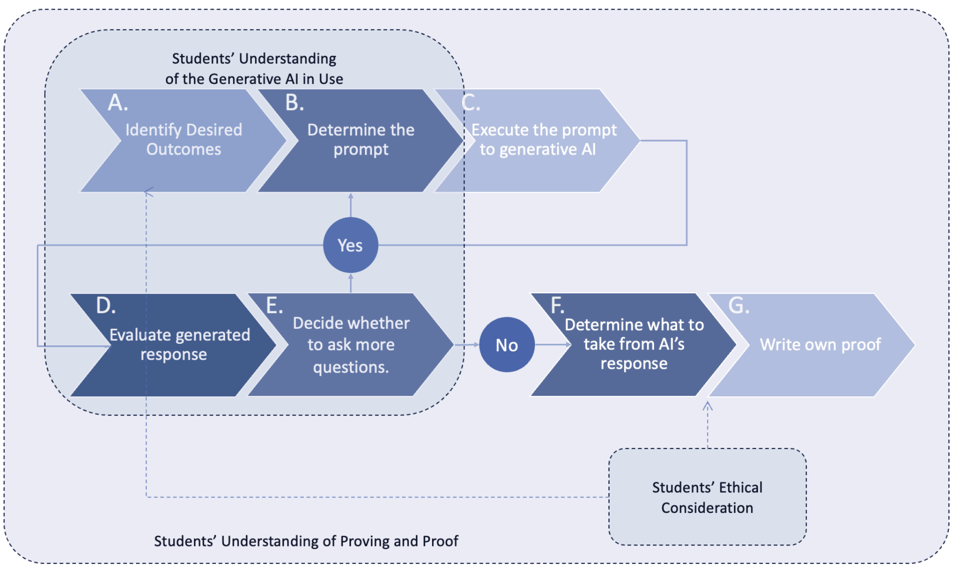

Figure 3 Flowchart of students' use of ChatGPT in proving

A. **Identify desired outcomes.** Students had their objectives by using genAI in proving. The outcomes could be determined by students themselves or given by other people such as an instructor or researcher.
B. **Determine the prompt.** Once students decided on their objectives for using genAI, they decided how they wanted to ask such questions. Yuna's various prompts are in Figure 2.
C. **Execute the prompt to genAI.** As students entered their determined prompts, they received the answers from the ChatGPT immediately.
D. **Evaluate the generated response.** When students evaluated generated responses, their evaluation varied based on the objectives identified in the first stage. As mentioned in the

example, Yuna made several evaluations of the arguments based on her understanding of proofs and mathematical knowledge.

E. **Decide whether to ask more questions.** At this point, students decided if they wanted to use genAI further to get another response based on their evaluation at D. Based on the answer, students could repeat the cycle between B and E as many as they wanted to. In Yuna's case, she repeated the cycle five times until she decided to ask no further questions.
F. **Determine what to take from AI's response.** When they decided to stop using genAI, students made a decision on which part of genAI's responses they would like to take and discard for their proof. It could include a set of accepted arguments, modes of argumentation, and modes of argument representations. For example, Yuna hesitated to use proof by contradiction in her proof because that was not the mode of argumentation she had seen in her mathematics classrooms.
G. **Write their own proof.** As a last step, students wrote their proof as the author of their proof. This stage emphasizes proof is human activity, and what counts as a proof is determined by humans, and in this case, the user of the genAI.

This process underscores that while automation tools like genAI are used, proving remains fundamentally a human activity. In her interactions with genAI, Yuna maintained the authority to critique genAI's arguments and determine their incorporation in her proofs. This finding resonates with the significance of students' agency and autonomous actions in proving (Castle et al., 2022) could critically engage with genAI (Cooper, 2023) by resisting passively accepting arguments from external resources, including gen AI, by giving them the authority (Castle et al., 2022).

In addition to the processes, we identified three other aspects (highlighted in shaded areas of proving Figure 3) considered in their use of genAI for proving:

1. **Students' understanding of proving and proof:** Influences the entire process, including evaluating genAI's suggestions and writing students' final proof. For instance, Yuna believed a mathematical proof should be more equation-heavy and less verbose. She thus modified ChatGPT's verbose argument to a more symbolic form in her proof.
2. **Students' understanding of genAI in use:** Concerns how students perceive the genAI tool they employ. This might include their knowledge about genAI's capabilities based on prior experiences or beliefs. Yuna, believing it cumbersome to type mathematical notations, often refrained from doing so.
3. **Students' ethical considerations:** Relates to students' awareness of potential ethical dilemmas in using genAI. This might be influenced by class norms or personal beliefs about leveraging resources. Yuna, for example, felt using ChatGPT was "better" than copying others' work as at least one tried to type questions and checked the argument by ChatGPT.

We believe these three factors would need to be considered in designing and implementing tasks using genAI. Also, the presented flowchart in this paper could provide a guideline to mathematics instructors to devise their proving activities using genAI in their teaching by suggesting some sequences that should be considered. Also, it means that instructors need to know their students' understanding of proving and proof and genAI and should develop and negotiate the classroom norm with their students. This flow chart could also serve as a tool for analyzing how students use genAI in their proving tasks and the extent to which students critically think about the responses generated by genAI.

References

Baidoo-Anu, D., & Owusu Ansah, L. (2023). Education in the era of generative Artificial Intelligence (AI): understanding the potential benefits of ChatGPT in promoting teaching and learning. SSRN Electronic Journal, 1–22. https://doi.org/10.2139/ssrn.4337484

Bieda, K. N., & Lepak, J. (2014). Are you convinced? Middle-grade students' evaluations of mathematical arguments. School Science and Mathematics, 114(4), 166–177. https://doi.org/10.1111/ssm.12066

Castle, S. D., Smith III, J. P., Levin, M., Hwang, J., Karunakaran, S. S., Küchle, V. A. B., & Elmore, R. (2022). Shifts in external authority and resources for sense-making in the transition to proof-intensive mathematics: the case of Amelia. In S. S. Karunakaran & A. Higgins (Eds.), Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education (pp. 100–107).

Cooper, G. (2023). Examining science education in ChatGPT: An exploratory study of generative Artificial Intelligence. Journal of Science Education and Technology, 32(3), 444–452. https://doi.org/10.1007/s10956-023-10039-y

Hanna, G. (2000). Proof, explanation and exploration: An overview. Educational Studies in Mathematics, 44(1–3), 5–23. https://doi.org/10.1023/A:1012737223465

Hersh, R. (1993). Proving is convincing and explaining. Educational Studies in Mathematics, 24(4), 389–399. https://doi.org/10.1007/BF01273372

Qadir, J. (2023). Engineering education in the era of ChatGPT: Promise and pitfalls of generative AI for education. In IEEE (Ed.), 2023 IEEE Global Engineering Education Conference (pp. 1–9). IEEE.

Stylianides, A. J. (2007). Proof and proving in mathematics education. Journal for Research in Mathematics Education Journal for Research in Mathematics Education, 38(3), 289–321. https://doi.org/10.1007/978-94-007-2129-6

Su, J., & Yang, W. (2023). Unlocking the power of ChatGPT: a framework for applying generative AI in education. ECNU Review of Education, 1–12. https://doi.org/10.1177/20965311231168423

Weber, K., & Mejia-ramos, J. P. (2014). Mathematics majors' beliefs about proof reading. International Journal of Mathematical Education in Science and Technology, 45(1), 89–103. https://doi.org/10.1080/0020739X.2013.790514

Towards Operationalizing What Is Learned During Modeling

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In this theoretical paper we leverage constructivist theories of learning to operationalize what is being learned during modeling. We do this by examining model construction through two theories: (i) schemes and concepts and (ii) abstracted quantitative structure. In addition to operationalizing what is being learned, we illustrate our operationalizations by providing examples from STEM undergraduates' model construction activities.

Keywords: mathematical modeling, learning theories, cognition

Scholars have embraced diverse perspectives on mathematical modeling (hereafter, modeling) to address a variety of research problems within the field (Kaiser, 2017). For example, within the cognitive perspective of modeling, researchers have investigated the ways in which to reduce or mitigate the cognitive obstacles students encounter during modeling. Irrespective of the perspective adopted, the researchers share a common goal: improving modelers' learning outcomes in modeling. In order to do this, it is crucial to understand what exactly is being learned during modeling. This knowledge will shed insight onto the instructional strategies and assessment criteria that educators can employ to guide modeling instruction. Two theoretical lenses have been proposed on how modeling enables the learning of mathematical concepts. The models and modeling perspective puts forward the idea that individuals learn through constructing conceptual systems (models) that are used to "construct, describe, or explain the behaviors of other systems that occur in the world' (Lesh et sl., 2003, p. 213). The emergent perspective describes how learning is occasioned as conceptual models are evolved from one context to the other (Gravemeijer, 1999). While these theories are useful to understand how modeling occasions learning, they do not explicate what exactly is being learned during modeling. In this paper, we draw on two distinct (but related) constructivist theories on knowledge construction: (a) schemes and concepts (von Glasersfeld, 1995), and (b) abstracted quantitative structure (Moore et al., 2022) to operationalize what is being learned during modeling. We support our operationalizations with illuminating data from two undergraduate STEM majors' work on modeling tasks.

Theoretical Framework #1: Applying Schemes and Concepts to Modeling

We take the stance that learning occurs through the change of schemes and formation of concepts. von Glasersfeld (1995) identifies three parts of a scheme: (1) An individual's recognition of a certain situation (S); (2) Specific activity associated with that situation (A); (3) and an anticipated result of that activity (R). When a learner sets a goal, it initiates a scheme (S-A-R) (Simon et al., 2004). That is, the goal triggers similar situations (S) where similar goals were made. This recognition is a result of assimilation. The assimilated situation then triggers the specific activity sequence associated with the situation. In effect, this activity leads to a result (say $R *$). If $R = R^*$ (i.e., the result of the activity turns out to be what the learner anticipated), the learner again assimilates. If $R \neq R^*$ (i.e., the result of the activity is not what the learner anticipated), the learner will experience a *perturbation*, also known as a state of disequilibrium. As individuals desire to stay in a state of equilibrium, the learner will attempt to overcome the perturbation through *accommodation*. An individual can accommodate through either (a) modifying the scheme, or (b) re-organizing the scheme. This results in a change in the scheme. (See Figure 1). Therefore, learning has happened, even if the specific change was unintended by the curriculum or teacher.

Von Glasersfeld (1982) describes concept as following:

> "concept" refers to any structure that has been abstracted from the process of experiential construction as recurrently usable, for instance, for the purpose of relating or classifying experiential situations. To be called "concept" these constructs must be stable enough to be re-presented in the absence of perceptual "input" (p. 194).

Hackenberg (2010) makes the case for concept to be "recurrently usable, for instance, for the purpose of relating or classifying experiential situations," the *abstraction* itself is implemented by the activities of an individual's schemes. Therefore, Hackenberg (2010) operationalizes concepts as "results of schemes that [individuals] have abstracted from their use of the schemes that the [individuals] can take as given prior to operating… [That is, concepts are] results of schemes that have been *interiorized*" (ibid, p.387). In other words, the results of a scheme, and the activities that produced those results can be taken as given during *assimilatio*n of a new situation. We adopt this view on concepts as well.

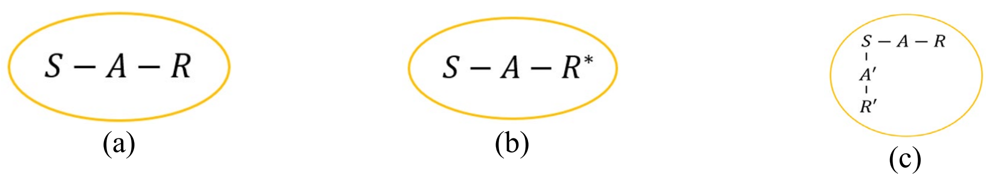


Figure 1. Our Representation for a scheme (a), modification to a scheme (b), and reorganization of a scheme(c)

Object of Validation as Schemes in Use and Standards of Validation as Concepts

Model validation is crucial for mathematical modeling (Czocher, 2018; Zbiek & Connor, 2006) because non-viable models are of little use for solving real-world problems. In Kandasamy & Czocher (2020), we argued that looking deeply into model validation would provide insight into how modeling enables learning because (a) the outcome of model validation may lead to the modification of the model, and (b) learners validate both their final model and their evolving models (p. 924).

When a learner engages in model validation, she holds two models in her mind: the one she is constructing (the object of validation) and the one she anticipates constructing (the standard of validation). The model she is constructing resulted from embracing a goal, which triggered a situation, which triggered a sequence of activities. Thus, the object of validation can be seen as a scheme under use. The model she anticipates is the result of an interiorized scheme and the activities that produced the result. Thus, the standard of validation can be seen as a concept. When the object of validation meets the requirements of the standards of validation, the learner assimilates and remains in a state of equilibrium. When the object of validation does not meet the requirements of the standards of validation, the learner is in a state of disequilibrium. To overcome the perturbation, the learner may choose to either (1) revise the object of validation or (2) accommodate the object of validation through modifying the standards of validation. In these terms, we claim that what modelers learn through modeling are the modifications to the standards of validation. We exemplify this claim using Eshonai's work on the Cancerous Mass task.

Eshonai and The Cancerous Mass Task

Eshonai was an undergraduate majoring in electrical engineering with a concentration in computer science. In this section, we present a portion of her work on the cancerous mass task, one of 5 modeling problems she solved in a modeling instructional sequence. The interviewer's goal was for Eshonai to build on her knowledge of absolute and relative change to develop a change equation as a model for exponential growth (rate of change is proportional to value of the function). The full task statement included information about the HeLa cell lines and a table of hourly measurements of the sample's mass.

> *The Cancerous Mass Task (abridged version):* Cancer cells can be grown in a lab, for study in their own right and also as a basis for further medical research. The HeLa cell line has a 24-hour propagation rate of 69% of its current mass. Create an expression that would model how quickly the sample is growing.

Using the table of values, Eshonai computed the hourly percent change in mass for the first, second, and third hourly intervals. She obtained 2.93% for each interval. Eshonai concluded that the hourly percent change in mass remained constant as time passes and had a value of 2.93%. She validated her model by computing the hourly percent change for the 22nd time interval. Next, The following conversation between Eshonai and the interviewer was exchanged.

Interviewer: What is the percent change [in mass] during 2-hour segments? So, if you were to look at a segment of time or an interval of time that ranged for 2 hours, lasts for 2 hours, what do you suspect the percent change in mass is?

Eshonai: My gut instinct is saying that it would just be double.

Interviewer: Okay. Double what?

Eshonai: Double the percent change of one hour.

Interviewer: Okay, and why do you suspect this? What is making you suspect that?

Eshonai: Well, the percent change is going to be fixed throughout each individual hour, so I assume that it would be fixed over 2 hours as well.

[Eshonai validated her answer by computing the percent change in mass during [1, 3] and confirmed its "roughly double," as in Figure 2(a)]

Interviewer: What about for a 3-hour increment? What does your gut say?

Eshonai: It would be 3 times. I'll do the math right now.

Eshonai confirmed her hypothesis by evaluating the percent change in mass during [1,4] as shown in Figure 2 (b). After computing the values, she realized that her hypothesis did not hold true and there was a discrepancy between the "actual' value and the "hypothesized" value (Figure 2). Eshonai nominalized the percent change in mass she evaluated through using the percent-change formula as "actual" and the percent change in mass she evaluated through her hypothesis—3 times the percent change in mass during an hour—as "hypothesized."

For 2-hour intervals:

$$\frac{5.29735-5}{5}=5.9463\%$$

$$0.02930224\cdot 2=0.05860$$

(a)

For 3-hour intervals:

$$\frac{5.452539-5}{5}=9.05088$$ "Actual"

$$0.02930224\cdot 3=0.087906$$ "Hypothesized"

(b)

Figure 2. Eshonai evaluates the percent change in mass during a 2-hour interval (a) and a 3-hour interval.

After computing the "actual" and "hypothesized" values, Eshonai was hesitant to conclude that the percent change in mass during a 3-hour interval was roughly 3 times the percent change in mass during a 1-hour interval. When the interviewer asked her why she thought there was a discrepancy, Eshonai replied, "I'm not sure. I mean, maybe there could be an error in my part. I feel like they should be exactly the same...I have no idea why there is a discrepancy." With the interviewer's guidance, Eshonai realized that as the length of the interval increases, her "hypothesized" values further depart from the "actual values". She modified her claim, asserting that her hypothesis would hold only for small intervals. The interviewer verified, asking if her hypothesis would hold for a half hour interval. Eshonai added, "I would say it is just half [of 2.93%], because if it is straining accuracy with more time [large intervals] then it should be more accurate with smaller of an interval we have."

In this scenario, Eshonai's object of validation was *the precent change in mass during the 3-hour interval that she evaluated using the percent change formula*. Eshonai's standard of validation, the result she was anticipating, was the *percent change in mass during a 3-hour interval is 3 times the percent change in mass during a 1-hour interval*. Her formulation of the standard of validation may have drawn on Eshonai's prior mathematical experiences with linear growth patterns and was re-confirmed when she computed the percent change in mass during a 2-hour interval (for which she got "roughly" 2 times the percent change in mass during a 1-hour interval). That is, she had interiorized that if the percent changes in mass are the same for each time interval of equal length Δt, then the percent change in mass during N such intervals is N times the percent change in mass during Δt. However, while evaluating the percent change in mass during a 3-hour interval using the percent change formula (the object of validation) she obtained a discrepant value, leading to a perturbation. As a result, she modified her standard of validation by limiting the values of Δt to be small. Therefore, we conjecture, Eshonai's modification to the standard of validation—her conjecture holds true for only small values of Δt—was what she learned during this particular modeling scenario.

Theoretical Perspective #2: Applying Abstracted Quantitative Structure to Modeling

Moore et al. (2022) reframed von Glasersfeld's (1982) definition of a concept using quantitative and covariational reasoning theories. Moore et al. (2022) define abstracted quantitative structure as a system of quantitative operations an individual has interiorized. The authors adopt Thomson's view on quantitative reasoning as conceiving a situation consisting of quantities and relationship among those quantities, where quantities are conceptualized as mental constructs of measurable attributes (Thompson, 2011). Quantitative relationships are a result quantitative operation. The authors operationalize quantitative operations as the mental operations involved in constructing new quantities (Thompson, 1990) and the mental operations involved in reasoning about varying quantities (Carlson et al., 2002). For example, within a disease transmission context, an individual may construct the quantity *how many more people got infected on day 2 than day 1* by additively comparing the quantities: *number of infected people on day 1* and *number of infected people on day 2*. Furthermore, an individual may reason "*as the number of infected people increase, the number of susceptible people decrease*," engaging in the directional variation of new quantities. Moore et al. (2022) posit that an abstracted quantitative structure has the following characteristics: (1) is recurrently usable beyond its initial experiential construction; (2) can be re-presented in the absence of available perceptual material including that in which it was initially constructed; (3) can be transformed to accommodate to novel contexts permitting the associated quantitative operations; (4) is anticipated as re-presentable in any figurative material that permits the associated quantitative operations (p.44). We argue that modeling activity constructs abstracted quantitative structures, and therefore they are the outcome of learning. We exemplify this claim below.

Pai's Abstracted Quantitative Structure Associated with Interactions

Using Pai's work on an instructional sequence of modeling tasks, we illustrate abstracted quantitative structures as a result of learning during modeling. Pai was an undergraduate senior majoring in economics and minoring in mathematics. The instructional sequence was built to examine participants' quantitative reasoning as they constructed models for real-world situations. The overarching goal of the instructional sequence was to examine how participants' reason quantitatively as they constructed models for real-world situations. Of the 8 tasks Pai worked on, we draw on Pai's work from the following tasks.

The Speed Networking Task (abridged): Suppose you and your friends attend a job fair organized by your department. Each student will have 5-minute "quickfire dates" with a representative from each firm. Create a mathematical model that would represent the total number of meetings between your friends and the representatives from the various firms.

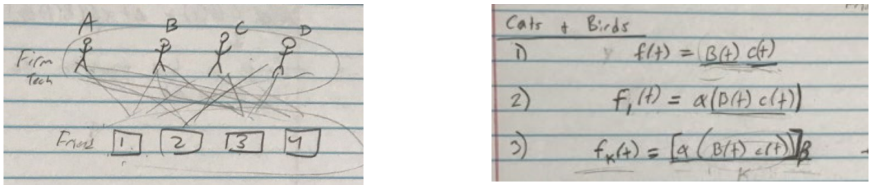
The Cats and Birds Task (abridged): (i) Consider a backyard habitat, where cats are the natural predators of birds. Let $B(t)$ be the number of birds and $C(t)$ be the number of cats at time t. How many potential cat-bird interactions are there at time t? (ii) Not every cat and bird encounter each other. Only some percentage α of potential cat-bird interactions are realized. How would you adapt your model above to account for that fact? (iii) Cats are very good hunters, but they aren't perfect. Sometimes the bird gets away. How would you adapt your model above to represent the number of interactions that results in a bird's death?

The Disease Transmission Task: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model for the dynamics of how the disease would spread through the population.

Below we explain how Pai constructed abstracted quantitative structures associated with interactions between species by providing examples of Pai's initial construction, re-presentation and construction, and finally accommodating to novel contexts.

Initial Construction: The speed networking task. Pai initially modelled this scenario as $y = 5 + 2.5x$, where y represented the total number of meetings between students and representatives and x represented the total number of questions the students may ask from the representative. Pai's model reflects his assumptions about the event's organization: each student would get 5 minutes with each firm, and then an additional 2.5 minutes per question. However, Pai was unsure whether this model satisfied the problem statement. He drew a diagram to represent the meetings between the group of friends and representative from firms (Figure 3a). He counted the number of lines that connected the representatives (A,B,C,D) to the friends (1,2,3,4). Pai then constructed $y = t \cdot f$ to model the number of meetings between friends (f) and representatives (t). Pai used his drawing to justify the model, arguing, "You would simply multiply the number of tech representatives by your number of friends." Pai expressed confidence with the combinatorial model, explicitly qualifying its scope: "every friend talks to every firm, which we cannot know for certain" and "what I have $y = t \cdot f$ is the maximum possible" given that "you cannot have two meetings with the same firm."

In Speed Networking, Pai first constructed an unsatisfactory model. Using his diagram, he was able to construct the quantity *total number of meetings between friends and representatives* as a multiplicative combination of the number of friends and the number of representatives. We argue his activity established a quantitative structure for the total number of possible interactions between two disjoint sets of objects. Next, we argue that he abstracted that structure to carry it to a new scenario.



(a) (b)

Figure 3. Pai's diagram in the Speed Networking task (a) and Pai's models for the Cats & Birds tasks

Re-presenting and constructing: The cats and birds task. After reading the task, Pai immediately wrote the expression $f(t) = B(t)C(t)$ where $f(t)$ represented the total number of potential cat-bird interactions at any time t. Pai stated that the situation was similar to the Speed Networking task. He next modelled the percentage of cat-bird interactions that realized as $f_1(t) = \alpha[B(t)C(t)]$. He justified his expression for the realized cat-bird encounters as "since $f(t)$ is the potential number of cats and birds interactions, assuming they all interact with each other, $f_1(t)$ will be a percentage of that $[B(t)C(t)]$, since not all of them actually interact." He saw a clear link between the subtasks, explaining "I'm assuming each of these models are based on the previous ones."

For the second subtask, Pai first constructed $f_k(t) = \frac{\alpha B(t)C(t)}{K(t)}$ to model the number of cat-bird interactions that resulted in a bird's death. He nominalized $f_k(t)$ as the number of bird-cat interactions that result in a bird's death and $K(t)$ as the number of birds killed at time t. However, he expressed uncertainty around whether $f_1(t)$ should be divided by $K(t)$ or if he should subtract $K(t)$ from $f_1(t)$. The interviewer probed Pai's image of how bird deaths occur, and Pai stated that cats and birds need to meet. He then realized that $f_k(t)$ and $K(t)$ represented the same measurable attribute, number of birds killed due to interaction with cats. Pai then changed his model to $f_k(t) = [\alpha B(t)C(t)]\beta$, where he nominalized β as the "percentage of these interactions [*referring to $\alpha B(t)C(t)$ as he was drawing square brackets, see Figure 3b]* that result in a kill for the cat." Pai justified his revision, arguing that the $B(t)C(t)$ corresponded to total, potential interactions, that $\alpha B(t)C(t)$ corresponded to the actual interactions, and that $[\alpha B(t)C(t)]\beta$ corresponded to the interactions resulting in a dead bird. He concluded, "this makes more sense to me."

Pai wrote $B(t)C(t)$ to model the total number of potential cat-bird interactions without having to draw a diagram, suggesting he re-presented the quantitative structure from Speed Networking in the absence of available perceptual material. The Cats & Birds task presented Pai the opportunity to refine his initial quantitative structure for the total number of interactions between two disjoint sets of objects through removing the assumption that all objects interact exactly once. Pai's model for the number of realized cat-bird interactions, $\alpha B(t)C(t)$, was constructed through shrinking the size of the whole, $B(t)C(t)$, via a scalar α. We interpret that he had constructed a quantitative structure for a subset of the number of interactions. We took Pai's placement of the square brackets in his models (Figure 3(b)) and the excerpt above as evidence that Pai was engaging in shrinking the size of the whole (quantity), in both instances. Next, we show the task that occasioned his accommodation of the quantitative structures.

Accommodating to novel contexts: the disease transmission task. To model the rate of disease spread, Pai introduced the variables $h(t)$ and $I(t)$ to represent the number of healthy people and the number of infected people, respectively. His immediate goal was to model the interaction between healthy and sick people that would result in a healthy person falling sick. He constructed $h(t)I(t)\alpha$, where he defined α to represent "[rate of] successful infections." He justified his model, explaining that "a healthy person interacts with an infected person [gesturing over $h(t)I(t)$], the disease doesn't necessarily spread because they interact. It's going to be some sort of rate, and I have that as α."

In comparison to Speed Networking and Cats & Birds, the Disease Transmission task was a different context, offered less information, and was less scaffolded. Still, Pai set a sub-goal to model the interaction between healthy people and sick people and the interaction that results in a healthy person getting sick. To accomplish this goal, Pai accommodated the quantitative structures he created in the previous task—total number of possible interactions and a subset of the total number of interactions—to this novel context. We take this as indication of Pai having constructed an abstracted quantitative structure associated with the number of interactions between two disjoint set of objects and subsets of those interactions, as it satisfies the characteristics proposed by Moore at al. (2022). Consequently, we claim that through engaging with the instructional sequence of modeling tasks, Pai learned the number of interactions between disjoint set of objects can be constructed via multiplicatively combining the number of each objects and that a subset of interactions can be constructed by scaling the size of the whole.

Discussion

In this paper, we operationalized what is being learned during modeling through (i) schemes and concepts and (ii) abstracted quantitative structures. Through the first lens, we argued that modifications to the standards of validation are what is being learned during modeling. Through the second lens we made the case that learners' construction of abstracted quantitative structures is what is learned during modeling. Our analysis sheds insights into concretizing the end goal of modeling, in terms of learning. This has implications: first, knowing what is learned during modeling will help educators to design modeling tasks that engender that learning. For example, when adopting the view that modifications to the standards are learned during modeling, tasks can be designed to challenge learners' existing standards of validation, providing learners the opportunity for accommodation. Second, the knowledge of what is being learned during modeling is instrumental for educators to be intentional with their instructional goals for modeling and learning environments that support those goals, mitigating some of the inadvertent cognitive obstacles learners may experience during modeling. Finally, it gives educators a sense of what to expect when assessing learners' modeling competencies. We close with two considerations: limitations and future directions. Using the schemes and concepts theoretical orientation, we neither analyzed what reorganizing a standard of validation may look like nor what is learned in the absence of perturbation. Additionally, as standards of validation reside in the mind of the individual, it is not always possible to observe or make predictions about learners' standards of validations nor delineate between the object and standard of validation. To attend to this, future interview protocols can include specific questions regarding learners' model validation such as, *what made you revise your model? Were you anticipating some other result? What was your goal in revising?* However, this theoretical paper is a significant contribution to the field as it initiates a discourse on leveraging established theories of learning to effectively operationalize the learning goal of modeling, and the instructional strategies for nurturing the learning of modeling both as a means and as an end.

Acknowledgments

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References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Czocher, J.A. (2018). How does validating activity contribute to the modeling process? *Educational Studies Mathematics, 99*, 137–159.

Gravemeijer, K. (1999). How Emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 267–291). The National Council of Teachers of Mathematics, Inc.

Kandasamy, S., Czocher, J.A., (2020). How does mathematical modeling enable learning? Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 922-930). Mexico

Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003). Beyond constructivism. *Mathematical Thinking and Learning*, *5*(2), 211–233.

Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., Paoletti, P. (2022). Abstracted quantitative structures: Using quantitative reasoning to define concept construction. In G. Karagöz Akar, I. O. Zembat, S. Arslan, & P. W. Thompson (Eds.). *Quantitative Reasoning in Mathematics and Science Education* (Vol. 21, pp 35–69). Springer International Publishing.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, *35*(5), 305–329.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra.* Unpublished manuscript, Center for Research in Mathematics & Science Education, San Diego State University, San Diego, CA.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming.

von Glasersfeld, E. (1982). Subitizing: The role of figural patterns in the development of numerical concepts. *Archives de Psychologie, 50*, 191–218.

von. Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning.* Washington, D.C: Falmer Press.

Pre-service Teachers Understanding of Commutativity, Associativity and the Challenge of Order

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Commutativity and associativity are recurring properties in school and university mathematics. In design experiments with pairs of secondary pre-service teachers (PSTs) we investigate what they are focusing when discussing commutativity and associativity, and how they deal with the notion of order. A qualitative discourse analysis shows for commutativity that the focus is on the order of two elements for most PSTs. Few focus on the order of several elements. Regarding associativity, bracketing is a main focus for many PSTs. The order in which operations are computed is sometimes discussed explicitly with also emphasizing the order of elements and sometimes more implicitly. The PSTs struggle with the notion of order so that we suggest coping strategies like "demanding precision in language" and "using alternative terms for order".

Keywords: pre-service teachers, commutativity, associativity, qualitative discourse analysis

Introduction

The arithmetic properties commutativity and associativity are recurring mathematical topics that can be found at various levels in school and university mathematics (Wasserman, 2016). One could assume that dealing with different binary operations and their properties in different settings should have eliminated any difficulties in dealing with the properties. However, according to empirical data, this does not appear to be the case (Melhuish & Fagan, 2018; Zaslavsky & Peled, 1996). When discussing commutativity and associativity, one of the biggest struggles for university students seems to be the notion of order (Larsen, 2010). Larsen (2010) suspects that a lack of precision in the informal language is one cause for this difficulty. Though it is not yet certain what language could be hindering or supportive, and what are possible coping strategies for dealing with the notion of order. This is why the reported study in this paper deals with the following research questions:

1. What do (German) secondary pre-service teachers near the end of their studies focus on when discussing commutativity and associativity?
2. How do secondary pre-service teachers deal with the notion of order concerning the properties commutativity and associativity?

Theoretical Background

In university mathematics, commutativity is defined for an operation $*: M \times M \rightarrow M$ on a set M referring to two elements $a * b = b * a, \forall\, a, b \in M$ and associativity referring to three elements $(a * b) * c = a * (b * c), \forall\, a, b \in M$ (Alcock, 2021). In (German) school textbooks, however, we sometimes find definitions of commutativity that are not based exclusively on two elements anymore, e.g. "The summands of a sum or the factors of a product can be changed in any way" (translated: Braun, 2019, p. 91). For commutativity, the school definition might evoke the idea that the order of several elements may be changed. Whereas the university definition has the idea that the order of two elements may be changed (Larsen, 2010). As a consequence, the understanding of commutativity in school and university mathematics may not match.

One understanding of associativity suggested by the formal definition is that, if the property is valid, brackets may be arbitrarily moved, inserted, or removed (Melhuish & Fagan, 2018). Behind this still rather superficial idea, if bracketing refers to two elements, is the understanding that elements can be grouped differently when computing the operations (Weber & Larsen, 2008). For this, the order in which the operations are computed does not matter (Larsen, 2010) One interesting aspect is, how Cayley formulated associativity as a condition for a group. In general, that commutativity does not hold would not actually need to be explicitly addressed. He postulated "symbols are not in general convertible [commutative], but are associative" (Kleiner, 1986, p. 208 cited in Larsen, 2010). Larsen (2010) observed exactly this phenomenon with his students. They operated with symmetry mappings and addressed (non)commutativity in their definitions for a group even though it is not necessary. To some students it seems inherently important to emphasize that the order of the elements from left to right is not allowed to be changed (Larsen, 2010). Nevertheless, there are also students who have difficulties with keeping the order of the elements when using associativity (Larsen, 2010; Zaslavsky & Peled, 1996). When checking operations for commutativity, one common difficulty is confusing the operation sign with the number sign when changing the order of the elements (Zaslavsky & Peled, 1996). For checking operations for associativity, difficulties can be caused if learners primarily have the understanding that associativity allows brackets to be moved (Melhuish & Fagan, 2018). Students are asked to verify that the operation of the mean $*$: $a * b = \frac{1}{2}(a + b)$ is associative. One given choice to justify that the operation is not associative is based on random shifting of brackets $\left(\frac{1}{2}a\right) + b \neq \frac{1}{2}(a + b)$. This option was chosen by 16.9% of PSTs (Melhuish & Fagan, 2018). More generally, it can be said that the understanding of and checking for commutativity seems to be easier for students than associativity (Findell, 2001).

In comparison of the two properties it becomes evident that the notion of order is apparent in the understanding of commutativity and associativity. However, in the case of commutativity, the order refers to the order in which the elements are arranged. While in the case of associativity, the order refers to the order in which the operations are computed. This similarity entails the risk that students could conflate the two properties when not paying attention to the differences (Larsen, 2010; Zaslavsky & Peled, 1996).

If both properties apply, they could be replaced by a single property (Pinto & Cooper, 2017). For example, if multiplication is considered "it is not clear why these two rules should not be replaced by a simpler multiply in any sequence rule" (ibid. p. 322). It would be sufficient to know that in a multiplication the order may be interchanged arbitrarily without differentiating what the order refers to. Empirically, it also appears that some students combine the two properties into one "order doesn't matter" (Findell, 2001, p. 148) property. This notion can lead students to make statements such as "because it's associative, you can move it all around" (Findell, 2001, p. 148) or "this property [meaning commutativity] allows us to switch around the elements in an expression so that it doesn't matter which elements will operate first" (Findell, 2001, p. 148). The difficulty with order and differentiating what the order refers to could also lead to the misconception that associativity also has something to do with swapping the order of elements (Ding et al., 2013). Furthermore, a common misconception that one property might imply the other property (Melhuish & Fagan, 2018; Tirosh et al., 1991).

Method

Data Collection

This study is part of a design-based research project at a German university with the aim of connecting university mathematics and teaching school mathematics for secondary pre-service teachers (PSTs) (Dellori & Wessel, in press). In this project, three teaching units following the sequencing instruction model (Wasserman et al., 2017) for PSTs algebra learning are developed and tested in laboratory design experiments. The second one addresses learning goals related to associativity and commutativity. At first (*building-up*), the PSTs work with school textbook excerpts about associativity and commutativity. In the second phase (*relearning*), the PSTs deal with commutativity and associativity in different algebraic structures. Finally (*stepping-down*), they reflect on the textbook excerpts with reference to the university mathematics. The data for this paper is situated at the beginning of the relearning phase of the teaching unit about commutativity and associativity. It concerns the following tasks:

1. Let M be a set with the binary operation $*: M \times M \to M$. Write down the definitions for commutativity and associativity.
2. In your own words, describe commutativity and associativity verbally.
3. What are differences and similarities between commutativity and associativity?

For the first two tasks, PSTs first discuss orally and then give a written answer. Task 3. is only discussed orally. Afterwards PSTs are prompted with the statement: "Commutativity implies associativity". Although, some PSTs already mention this claim during task 3.

The design experiments were conducted with pairs of PSTs (N=8x2) by the first author. The PSTs were in their first to third semester of their master's program at a German university and volunteered to participate in the study. At the time of data collection, 15 of the 16 PSTs were enrolled in an Abstract Algebra course. The design experiments were videographed and transcribed. The focus of the analysis is on tasks 2 and 3, as these encourage to reflect on the understanding of the properties.

Data Analysis

To answer the research question about what PSTs focus on when discussing commutativity and associativity, a qualitative content analysis is conducted (Kuckartz & Rädiker, 2022). The main categories are deductively formulated from previous research. For commutativity the two main categories are "order of two elements" and "order of elements" (Braun, 2019; Larsen, 2010). Inductively, the distinction is added as to whether a more general or example-based discussion of the order of the elements took place (examples see Figure 1).

Category	Definition	Example
Commutativity		
Order of elements	There is a focus on the fact that commutativity allows the order of elements to be rearranged.	General: The set under the operation is commutative, if we can say we can change the order around. […] And that for all the elements. Example-based: So, we know that this is commutative, then we can rearrange it as we want. [points to the "a*b*c" on sheet 1]
Order of two elements	There is a focus on the fact that commutativity allows the order of two elements to be rearranged.	General: That it doesn't matter in which order I combine two elements? Example-based: Whether I now combine a with b or combine b with a doesn't matter.
Associativity		
Order of operation	There is a focus on the fact that associativity allows the order in which operations are computed to be interchanged arbitrarily.	First computation: So, it just doesn't matter what we calculate first. Order: Here it is about the operation so to speak. It doesn't matter in which order we compute it. With regard to the operation. Example-based: And it doesn't matter for […] my result, if I, uh, first combine the elements a and b, the result from that with c. Or if I first combine b and c and then again combine what comes out of that with a.
Emphasis on order of elements	There is a focus on the fact that when the associative property is applied, the order of the elements is not changed	The order of the operation pair by pair doesn't matter. […] Whereas maybe … The order of the operation doesn't matter if the order of the elements is the same.
Bracketing	There is a focus on the fact that associativity allows brackets to be set arbitrarily.	When combining more than two elements of a set, brackets may be omitted or set as you like.

Figure 1: Categories for the focus on commutativity and associativity

For associativity the “order of operation”, “emphasis on order of elements” and “bracketing” are the three main categories (Larsen, 2010; Melhuish & Fagan, 2018) (see Figure 1). It has been observed that the order of operations could be addressed explicitly in using the term “order” or more implicitly in describing it on an “example-based” level or focusing on what is “computed first” (see Figure 1).

To answer the second research question (how do the PSTs deal with the notion of order), all passages in which the term “order” occurs were paraphrased and summarized for each pair. These summaries were contrasted and compared in order to identify typical difficulties and coping strategies for the notion of order.

Results

Pre-service teachers’ discussion of commutativity

To answer the first task and give a definition for commutativity and associativity, all PSTs give a symbolic based answer. The definition refers to *two elements* a and b for all pairs. Minor differences between the definitions are the notation of the operation ($*: M \times M \rightarrow M$) or the algebraic structure $(M,*)$. The pair #8 is the only pair that does not specify that $a * b = b * a$ must hold for all elements. The other pairs use the quantifier $\forall$ for this in their definitions. Furthermore, this pair also added $a * b * c = c * a * b$ later, when Sven (#8) talks about switching the elements back and forth. According to their understanding, commutativity is not just about the order of two elements but the *order of elements* in general.

Sven: If I can switch *a* and *b* and have a *c* there as well, then I can also, yes, … switch that back and forth if I want to.

In describing commutativity in their own words, six out of eight pairs clearly focus on the *order of two elements*. For example, pair #3 wrote: “The two combined elements of a set may be interchanged arbitrarily.”. The two pairs #1 and #5 start out with an example-based description of commutativity such as “It doesn't matter whether I combine *a* with *b* or combine *b* with *a*”. During their discussion, passages are also at a general level, for example like “It doesn't matter in which order I combine two elements”. Similarly, pair #8 has a mix of example-based and general passages in their discussion (see Figure 2). Pairs #2 and #4 tend to be almost exclusively at the general level while pairs #3 and #7 engage in discourse solely at this level (see Figure 2).

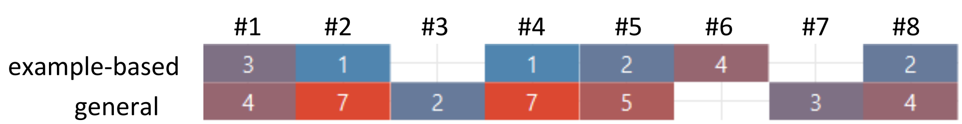

Figure 2: Passages focusing on the order of two elements

Pair #6 was the exception to talk about the *order of two elements* merely on an *example-based* level. On a more general level, this pair only talked about the *order of elements* such as “An operation is commutative if the sequence of elements is freely selectable and may be changed.”. It cannot be said with certainty that they are talking about multiple elements, but as they never explicitly talk about only two elements, it is assumed. The two pairs #6 and #8 are the only ones where the focus is on the *order of elements* rather just two elements. In the discourse of the other pairs we find some passages in which it is not clear from the language whether two or more elements are considered. Since, however, these pairs previously have multiple passages in which they explicitly only refer to two elements, one can assume that the missing explication is only due to an abbreviated form of expression.

Pre-service teachers' discussion of associativity

Like in their definitions of commutativity, all PSTs first state a mainly symbolic definition of associativity. Again, the only differences are in the notation of the operation and the algebraic structure. Pair #8 is the only pair that doesn't express that $(a * b) * c = a * (b * c)$ must hold for all elements.

The understanding that associativity allows *brackets* to be set arbitrarily is coded for six pairs. Pairs #2 and #4 are the only ones that don't discuss brackets in connection with associativity at all. For four pairs (#3, #5, #7, #8), their initial main idea for the verbal description of the property is associated with brackets. For example, in the verbal definition of pair #8: "The result is independent of the bracket placement." Later in the design experiment, these pairs are asked to give an alternative definition without using the idea of bracketing. The other pairs (#1, #6) discuss the notion of bracketing, e.g., see Pia and Nora (#6), but it does not seem to be the focus of their understanding of associativity. For Nora, focusing on the order seems more meaningful than the idea of bracketing.

Pia: Then/ Mhh. Yes, first you are allowed to place the brackets differently somehow. Then, uh, it doesn't matter how the brackets are set.

Nora: Mhh yes but that/ I think that doesn't say much. Because I would just write that/ here the order doesn't matter, right? So, whether I do *a* and *b* first or *b* and c first.

Looking at the discussions about associativity, it can be noted that an *emphasis on the order of the elements* is apparent in six of the pairs while pairs #3 and #8 didn't talk about it all. In some cases, students point out that the order of *a*, *b* and *c* from left to right is not allowed to be changed and explicitly contrast this with the property of commutativity. For two pairs (#4, #7) it was such an integral part of their understanding of associativity that they included it in their verbal definition: "Considering 3 elements from a set, the order of the pairwise composition does not matter if the order of the elements remains the same."

	#1	#2	#3	#4	#5	#6	#7	#8
first computation	5	1	2	1	6	1	1	2
order	2	3	3	5	1	4	8	3
example-based	2		2	1	2	1	6	

Figure 3: Passages focusing on the order of operations

In some way, all PSTs discuss the *order of operations*. Initially, three pairs (#3, #5, #7) consider the order of operations at an *example based-level*. Like Nico and Marcel (#5, see three passages below) these pairs use the specific elements a, b and c to implicitly take the order into consideration (first part). In further discussion, they move away from the three concrete elements and start contemplating that it doesn't matter what is *computed first* (second part). Afterwards, they go further and explicitly interpret "what you do first" as the *order of operations* (third part).

Nico: Yes, well. Um. You could say something like: First *a* with *b* and then with *c* or/

Marcel: First/

Nico: First *b* with *c* and then with *a*.

Marcel: Kind of like … Yeah, you can do it the same way or? Associativity means that it doesn't matter/

Nico: Which operation you do first.

Nico: Mhh. So here we have/ Here it is about the operation. There it doesn't matter in which order we compute it. Referring to the operation.

The other five pairs don't begin their discussion about associativity on an *example-based* level. With two pairs not referring to associativity on this level at all. Some mention the *order of operations* right at the beginning while some start with discussing which operation is *computed first*. Essentially, it can be observed that two pairs (#1, #5) have their biggest focus on which operation is computed first, and two pairs (#4, #6) explicitly on the order of operations. For the other pairs, it's more broadly distributed (see Figure 4).

Pre-service teachers dealing with the notion of order

While for some pairs (#1, #5, #6, #8) the notion of order is only provoked by the third task about similarities and differences between/ of the properties, some pairs (#2, #3, #4, #7) already discuss the notion of order when describing the properties.

The two pairs who manage particularly well to elaborate the similarities and differences always specify what the order refers to when they use the term. Adrian (#3) emphasizes that the order regarding commutativity refers to the elements and the order regarding associativity refers to the computation of the operations. Adrian (#3) has already shown the precise use of the term "order" in the verbal definitions in task 2. Pair #5, on the other hand, was the only one of all pairs that did not use the term order in task 2 at all. Nevertheless, most of the other pairs struggle more to specify what the order refers to. They often simply talk about the "order of operations" and not the "order in which the operations are computed".

Adrian: In both cases, it's about a kind of variable order. Just in one case I may switch the elements and in the other case I may/ It's maybe a bit fare fetched. But there I am also allowed to choose the order, but just which operation I compute first. So that would be a similarity and a difference. So, the similarity is that it's about an order that I'm free to choose. And the difference that it's not about elements, it's about, um, the operation.

The pairs that discuss order at the beginning have in common that they use or wanted to use the term "order" in both their verbal definitions of commutativity and associativity. This causes students to struggle with the notion of order like Anton and Lina (#1, see below). They have a problem using the term "order" for both properties and their way of dealing with it is deciding not to use the term "order" for both properties after all. For commutativity, they prefer to use the alternative term of "direction", which "fits better" to commutativity, according to Lina.

Anton: Because somehow order fits both. That's why I was a bit confused by the term. But somehow… But I mean, it fits more to this [means associativity].

Lina: Maybe it's better to use direction, which is what we said first. So, because you can combine from the left or from the right.

Anton: Yes. I mean/ So I would say both of them are correct. But/

Lina: No, direction fits better. Because that's really, you can look at it from the left and from the right. And with order, there the direction stays the same and then you have *a* and then the bracket or you have first the bracket and then your element.

This approach of resorting to alternative terms - such as direction, placement and position - can be observed in several pairs (#2, #6, #7). In all of these occasions, the alternative terms are used to refer to commutativity. Using alternative terms can lead to rejecting the notion of order for a property, as it is the case for Lina. However, when trying to explain the similarities and differences in task 3, the alternative terms can also be supportive in disentangling the notion of order concerning commutativity and associativity. For example, Anton later used the term "placement order" to clarify what the order is for him in commutativity.

In those pairs (#1, #6, #8) that use the term "order" for only one property in task 2, many of the students tend to associate the term more strongly with that one property even later on. Eva and Mirco (#1) use the term “order” only for commutativity in task 2. In task 3, they quickly give the answer that both properties are in fact about some sort of order. However, Eva is not particularly firm in this view. In the subsequent discussion she is tempted to reject the notion of order for associativity because of her understanding of bracketing. Andreas (#8) behaves like Eva. A similar pattern can be observed for Pia (#6) and Sven (#8). Only that in their case the term “order” originally was used for associativity, and they are inclined to reject order for commutativity.

Eva: Yes, strictly speaking, we don’t switch the order [means associativity]. We just rearrange the brackets.

Mirco: But that is an order in which you compute the operation.

Discussion

In summary, for the majority of the PSTs in this study, the focus of commutativity is on the order of two elements and consequently aligns with the formal definition in university mathematics. For almost all PSTs we found passages in which they just spoke about the “order of elements”. It is assumed that for most students, this is just due to inaccuracy in their language. For associativity, PSTs seem to have a strong focus on the idea of bracketing. Their focus on bracketing leads some students to reject the notion of order for associativity. Just like Larsen (2010), we also observe that the majority of the PSTs emphasize the order of elements when speaking about associativity. The task about finding similarities and differences between commutativity and associativity seems to be very productive in initiating discussions about the notion of order. Those PSTs who lack precision in their language concerning what the order refers to have difficulties with the notion of order. Struggling with using the term “order” for both properties, some PSTs (momentarily) reject the notion of order for one of the properties. To cope, some PSTs develop alternative terms for the term “order” such as “direction”.

The results indicate that a formal symbolic representation of the properties seem not to be particularly challenging for the PSTs. Difficulties arise when explicating the meaning of the properties is concerned. As shown in previous research, symbolic representations and formal vocabulary are not sufficient to explain meanings of mathematical concepts. Thus, meaning-related language is needed “especially for expressing and thinking about abstract relationships” (Post & Prediger, 2020, p. 112). Post and Prediger (2020) suggest providing meaning-related phrases in learning arrangement to support students in understanding the meaning of the part-whole relationship in probability (Post & Prediger, 2020). For understanding the meaning of order regarding commutativity and associativity, our findings suggest that phrases with alternative terms for order seem to be part of the meaning-related language. In line with the theory of scaffolding, we suggest providing students with phrases like that to support them in verbalizing the meaning of order. Besides using meaning-related language, demanding precision in language (in this case to clarify what the term “order” refers to) seems to be productive for explaining the meaning of order as Larsen (2010) already assumed.

The methodological limitations of this study, above all (a) the small sample size, (b) being tied to the specific teaching units, and (c) being situated in a German context (language and university), must be considered when interpreting the empirical findings. Concerning the latter, seeing how the identified language difficulties, coping strategies and proposed supportive means appear and may vary in different languages with other vocabulary could be an interesting aspect for further research.

References

Alcock, L. (2021). *How to think about abstract algebra*. Oxford University Press.

Braun, A. (Ed.). (2019). *Lambacher Schweizer 5 - Mathematik für Gymnasien* [Mathematics for secondary school](1. Vol.). Ernst Klett Verlag.

Dellori, A., & Wessel. (in press). Prospective teachers' mathematical knowledge: Relating representations as design principle for developing knowledge of non-local mathematics for teaching in abstract algebra. In *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education.*

Findell, B. R. (2001). *Learning and understanding in abstract algebra*. *Doctoroal Dissertations: Vol. 51*. https://scholars.unh.edu/dissertation/51

Kuckartz, U., & Rädiker, S. (2022). *Qualitative Inhaltsanalyse. Methoden, Praxis, Computerunterstützung: Grundlagentexte Methoden* (5. Vol.). *Grundlagentexte Methoden*. Beltz Juventa.

Larsen, S. (2010). Struggling to disentangle the associative and commutative properties. *For the Learning of Mathematics*, *30*(1), 37–42. http://www.jstor.org/stable/20749437

Melhuish, K., & Fagan, J. (2018). Connecting the Group Theory Concept Assessment to Core Concepts at the Secondary Level. In N. H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 19–45). Springer International Publishing. https://doi.org/10.1007/978-3-319-99214-3_2

Pinto, A., & Cooper, J. (2017). In the Pursuit of Relevance – Mathematicians Designing Tasks for Elementary School Teachers. *International Journal of Research in Undergraduate Mathematics Education*, *3*(2), 311–337. https://doi.org/10.1007/s40753-016-0040-3

Post, M., & Prediger, S. (2020). Decoding and discussion part-whole relationships in probability area models: The role of meaning-related language. In J. Ingram, K. Erath, F. Rønning, A. Schüler-Meyer, & A. Chesmais (Eds.), *Proceedings of the Seventh ERME Topic Conference on Language in the Mathematics Classroom* (pp. 105–112). hal-02970604

Tirosh, D., Hadass, R., & Movshovitz-Hadar, N. (1991). Overcoming overgeneralizations: The case of commutativity and associativity. In F. Furinghetti (Chair), *Proceedings of the Fifteenth Annual Conference of the International Group for the Psychology of Mathematics Education,* Assisi, Italy.

Wasserman, N. H. (2016). Abstract Algebra for Algebra Teaching: Influencing School Mathematics Instruction. *Canadian Journal of Science, Mathematics and Technology Education*, *16*(1), 28–47. https://doi.org/10.1080/14926156.2015.1093200

Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making Real Analysis Relevant to Secondary Teachers: Building Up from and Stepping Down to Practice. *PRIMUS*, *27*(6), 559–578. https://doi.org/10.1080/10511970.2016.1225874

Weber, K., & Larsen, S. (2008). Teaching and Learning Group Theory. In C. Rasmussen & M. P. Carlson (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 139–152). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.012

Zaslavsky, O., & Peled, I. (1996). Inhibiting Factors in Generating Examples by Mathematics Teachers and Student Teachers: The Case of Binary Operation. *Journal for Research in Mathematics Education*, *27*(1), 67. https://doi.org/10.2307/749198

Student Reasoning in Quantum Mechanics Examined Through Modeling and Sensemaking

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As part of an effort to examine student mathematical and physical reasoning in quantum mechanics, a paired, semi-structured, think-aloud interview was conducted. The students were asked to interpret a quantum mechanical operator expression in a functional (position) representation and to think about an analogous expression in Dirac notation. When the students ceased to make progress on this task, they were given a modeling task in which they constructed an eigenvalue equation for a different operator in quantum mechanics. After examining their newly constructed eigenvalue equation, students were able to determine more precisely the nature of the original expression given in a functional representation. These data were coded in accordance with mathematical modeling and mathematical sensemaking frameworks to examine the intersection of the two frameworks.

Keywords: Quantum Mechanics, Modeling, Sensemaking

Introduction

Due to its prevalence in modern research and applications in industry, quantum mechanics is an essential part of an undergraduate physics curriculum. Research on student understanding of quantum mechanics can be largely differentiated by the instructional methods used with the student populations. Undergraduate quantum mechanics is typically taught with either a functions-first or spins-first approach. In a functions-first approach, instruction first focuses on continuous systems represented in functional notations. This often requires solving differential equations to model (e.g, Griffiths, 2018). In spins-first instruction, students begin with discrete systems represented in Dirac\bra-ket notation (a vector-like notation developed for use in quantum mechanics) before moving onto continuous systems in functional notations (e.g., McIntyre et al., 2012).

One motivation for spins-first quantum mechanics courses is that they tend to fall more in line with graduate instruction. They allow for students to build intuition for and a qualitative understanding of quantum mechanical systems while working with two-state systems, described by relatively simple mathematics, before moving onto the continuous systems that require functional representations (McIntyre et al., 2012). Ideally, students would be able to apply the ideas learned in the context of spins to continuous systems. If this were the case, one could reasonably expect to see evidence of mathematical sensemaking during this transition. Sensemaking in physics is also closely related to modeling frameworks and activities in mathematics education research. The interplay between modeling and sensemaking will be discussed herein. We begin to address these issues by examining students' reasoning and sensemaking while working with multiple representations in the context of quantum mechanics.

Background

We first situate this study among the relevant literature associated with teaching and learning in quantum mechanics. This is followed by a more in-depth discussion of modeling and mathematical sensemaking, which are the two primary frameworks used in analysis.

Teaching and Learning in Quantum Mechanics

Much of the work on student understanding of upper-division quantum mechanics has focused on identifying difficulties students face in engaging with the material. Work has been done at varying levels of specificity ranging from specific concepts, contexts, or calculations to broad overviews of difficulties. Some of the specific concepts include time dependence (Emigh et al., 2015) and measurement (Gire & Manogue, 2008). Others have focused on students' use and understanding of notations (Gire & Price, 2015; Kohnle & Passante, 2017; Schermerhorn et al., 2019). Many of these have been used as the basis for curriculum development projects aimed directly at the identified difficulties (DeVore & Singh, 2015; Emigh et al., 2018; Kohnle & Passante, 2017; Singh, 2016). Others have examined student understanding from the perspective of resources in efforts to determine what may lead to certain lines of unproductive reasoning (Gire & Manogue, 2008, 2011). A variety of studies have addressed how students think about operator and eigenvalue equations in quantum mechanics (Her & Loverude, 2020; Singh, 2008; Singh & Marshman, 2015; Wawro et al., 2020). There is also a growing body of research around representations in quantum mechanics due to their variety and the need for both fluency with and fluidity between them. The three primary representations in quantum mechanics are Dirac, matrix-vector, and functional. The first two are closely related and well suited to modeling discrete systems, while functional representations are well suited to modeling continuous systems. The *resources framework* posits that people have a variety of cognitive resources that are activated in different contexts and utilized in reasoning (Hammer, 2000). Gire and Manogue identified student resources *quantum measurement as agent* and *operator as agent* that could be combined unproductively to lead students to the idea that an operator acting on a state is a representation of making a measurement of the corresponding observable (Gire and Manogue, 2008). This association between operators and measurements is consistent with other findings in the mathematics education community (Wawro et al., 2020).

Mathematical Modeling Frameworks

Modeling activities are commonly implemented in mathematics instruction to present students with opportunities to practice developing their skills in a real-world context, or to demonstrate how different mathematical concepts can manifest in the real world. This has led the mathematics education community to develop several different frameworks for modeling that typically depict cycles (e.g., Blum & Leiß, 2007). Given that modeling tasks often utilize contexts relevant to the natural sciences, the development of frameworks has not been exclusive to mathematics education (e.g., Modir et al., 2017; Redish & Kuo, 2015; Uhden et al., 2012).

Zbiek and Conner (2006) proposed a model consisting of a variety of processes and sub-processes both in the physical context ("real world") and the mathematical context (see Fig. 1). Their distinction between a mathematical entity and real world situation also proved productive for the analysis presented below. Some of the processes are ways one could engage with or think about a real world situation, some are ways one could engage with a mathematical entity, and others are ways that one could go back and forth between a real world situation and mathematical entity. *Exploring* is obtaining more information about the real world situation by questioning, clarifying, or paying special attention to a specific portion of the situation. By exploring one can *observe mathematically*, using mathematical ideas to describe aspects of the situation. Observing allows the interpretations of the physical system to be informed by formal mathematics, addressing a common sentiment that the mathematics and physics concepts are sometimes inextricable. *Specifying* is identifying conditions and assumptions (C&A) relevant to the real world situation. This is also a step where one is likely to come to some conclusion about the relevant conditions of the real world situation or make simplifying assumptions that will inform the *mathematization* of the problem. *Mathematization* in this case is generating a mathematical representation of the real-world situation. *Combining* is identifying that the mathematical entity one is working with has the appropriate properties and parameters (P&P). This checks that the mathematical entity is representative of the C&A. *Analyzing* includes many activities that can be used to manipulate or interpret the mathematical entity and derive one or more new properties. Included in analyzing is formal mathematical reasoning in which one utilizes their understanding of formal mathematics to derive new P&P of the mathematical entity. A subprocess of analyzing is *associating*, which is drawing on real world knowledge not necessarily associated with the real world situation being modeled to determine something about the mathematical entity. Like the way exploring mathematically provides an avenue for addressing how the interpretation of the real world situation is informed by mathematics, associating accounts for the ways in which the mathematics done are informed by physics concepts. *Highlighting* is the selection of specific properties to assist in reasoning that allow for the drawing of a mathematical conclusion. This conclusion can become one about the real world via *interpreting*. *Examining* is determining if the real world conclusions make sense given the real world situation. This could include things like validity checking or special case analysis. *Associating* and *observing mathematically* are significant features of the Zbiek and Conner model that make it well suited to analysis of physics problem solving. In physics problem solving there often processes engaged in that are not explicitly mathematical or physical. The focus on process is a primary reason this was chosen for analysis in this project.

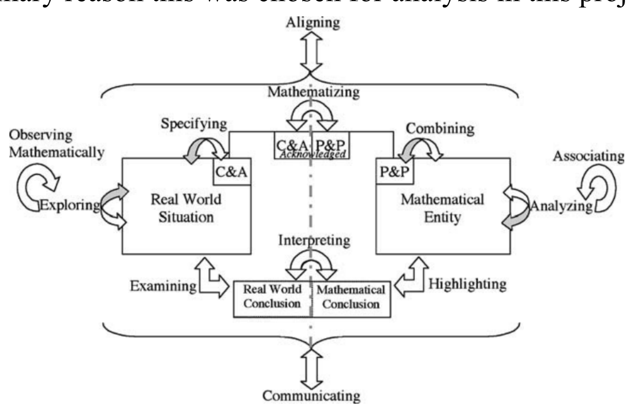
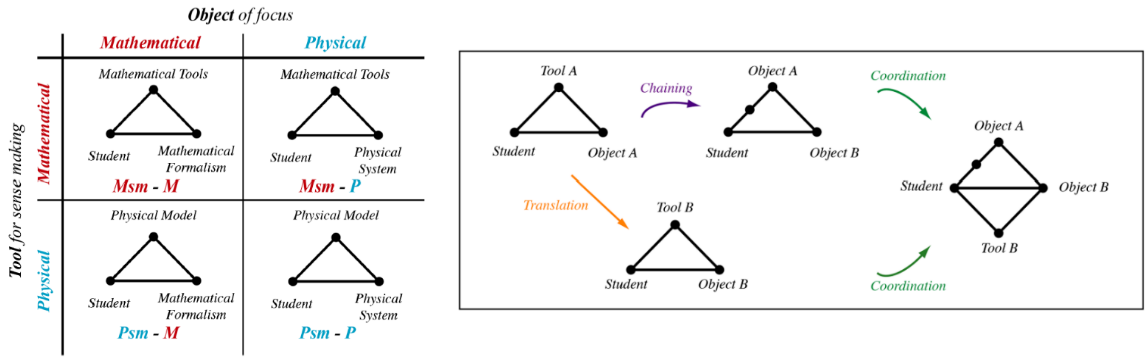


Figure 1. Framework for modeling developed by Zbiek and Conner (Zbiek & Conner, 2006).

Sensemaking

Broadly, mathematical sensemaking (MSM) can be considered a part of the larger activity of seeking coherence in which students, and even experts, seek connections between conceptual and mathematical understanding in physics (Schoenfeld, 2016). Odden and Russ worked toward building a more operational definition for sensemaking in the sciences (Odden & Russ, 2019). Their definition of sensemaking focuses on the goal – to figure something out – and a threshold, the recognition of an inconsistency. They characterize sensemaking as the work done to figure out or resolve that inconsistency.

With this definition in mind, Gifford and Finklestein developed a categorical framework for analyzing different modes of sensemaking, combining mediated cognition and activity theory (Gifford & Finkelstein, 2020). Mediated cognition is a model for cognition in which a mediator, or tool, is used to help the reasoner's understanding of an object and potentially produce some conclusion (Redish & Bing, 2009)(Vygotsky, 1978). The structure of the model is a result of activity theory, which positions the reasoner (as the subject), the tool being used, and the object being reasoned about amidst a larger context, resulting in an outcome (Engeström et al., 1999). Gifford and Finkelstein focus on the subject, tool, and object; these make up the basis for the model and for the nodes of their sensemaking triangles (see Fig. 2a).



(a) (b)

Figure 2. Categories in Gifford and Finkelstein's framework for (a) mathematical sensemaking in physics and (b) mechanisms for transitions between reasoning spaces (Gifford & Finkelstein, 2020).

For the purposes of analyzing mathematical sensemaking in physics, the tools are either mathematical tools or a physical model. These can be used to engage with either mathematical formalism or a physical system as the object. The different permutations of tools, models, formalism, and system result in four different modes of sensemaking that can be used to model reasoning. The researchers modeled transitions within and between these different modes of reasoning via three different mechanisms, summarized in Figure 2b. Students can "translate" between these modes, reasoning can be "chained" in a multi-step sequence, in which the object of one mode becomes the tool in the next mode, or two different ways of reasoning can be "coordinated" to provide two ways to make sense of the same idea (object).

Methods

The data being discussed in this paper are from a clinical, think-aloud interview with two students from a senior-level, spins-first quantum mechanics course at a mid-sized, research-focused institution in the northeast United States. The first portion of the interview was focused on how the students interpreted, both mathematically and physically, an equation for the momentum operator acting on an energy eigenstate of the infinite square well potential:

$$\hat{p}\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right) = -i\hbar\frac{3\pi}{L}\sqrt{\frac{2}{L}}\cos\left(\frac{3\pi x}{L}\right). \quad (1)$$

In a functional position representation, the momentum operator is expressed as a derivative with respect to position $(-i\hbar\frac{d}{dx})$, resulting in the additional constants and change in function, making it explicitly not an eigenvalue equation. While this operation is something that students could need to calculate in solving some problems in quantum mechanics, it is not one they are likely to have thought about in the context of eigentheory. When given to students in a previous survey however, it was common for them to label it as an eigenvalue equation, making it of interest and posing the question of the ways in which students determine whether a functional equation was an eigenvalue equation.

The students were then asked to generate an analog of Eq. 1 in Dirac notation. A correct response to this could take any form in which the kets on the left- and right-hand sides of the equation are different (e.g., $\hat{p}|\varphi\rangle = |\xi\rangle$). The students engaged with Eq. 1 until the interviewer deemed that no more progress could be made. In the next portion of the interview, students engaged in a modeling task involving a system consisting of a single particle constrained to exist in one dimension. They were asked to generate an eigenvalue equation for an operator representing the position of the system (e.g., $\hat{x}|x_n\rangle = x_n|x_n\rangle$), identify the different terms in their expression, explain their individual meaning, connections that exist between the terms, and connections between the terms and the physical system. After talking through the eigenvalue equation generated by the students, the original expression (Eq. 1) was revisited, and the students were asked if anything had changed about their interpretation of the equation.

The authors engaged in collaborative qualitative analysis (Richards & Hemphill, 2018) to code student activity into the processes and subprocesses described by Zbiek and Conner. Criteria included which entity students were primarily working with (real world situation or mathematical entity) and whether they were making identifiable conclusions; certain conclusions were associated with specific processes. In the process of interpreting the given expression, students engaged in mathematical sensemaking. The framework for mathematical sensemaking in physics (Gifford & Finkelstein, 2020) was used to analyze student activity when the data reflected students' recognition of an inconsistency. Similar to the modeling analysis, this involved identifying an object and tool utilized in sensemaking. Our aim in using both frameworks is to describe the processes the students go through in reasoning with the given equation more richly than with either framework alone, and to apply that description to student understanding of eigenvalue equations.

Episodes and Outcomes

In the interest of space, we are limiting our discussion to the episodes in which sensemaking occurred. Both frameworks were utilized in the analysis. These sections provide some insight into the interplay of modeling and sensemaking, which is addressed later, in the discussion.

Recognizing Inconsistencies: A Start to Sensemaking

After some initial analysis, the students began trying to rewrite Eq. 1. The explicit manipulation and rewriting of the equation were coded as *analyzing*. The result of their re-writing was, $\hat{p}\varphi_{E_3}(x) = p_{E_3}$, which is incorrect for two reasons: it ignores the fact that there is a function on the RHS of Eq. 1, and it also implies that the energy eigenstate in Eq. 1 is also an eigenstate of the momentum operator, which it is not. The pair continued working with this expression for some time, tweaking different aspects of it before pausing long enough for the interviewer to ask what was causing them to pause. Bob then articulated the inconsistency that led to their sensemaking saying, "Yeah. And my hesitation comes from this, where we say, okay, the thing is we don't have like the, the ket [motions to RHS of $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$]." The students had previously written the equation they referred to here. The equation is the eigenvalue equation for the Hamiltonian operator which represents energy of a system. It is a very well-known eigenvalue equation which is used repeatedly throughout the course, and so it is likely that the students are attempting to use it as a template in their reasoning.

Bob articulated that while the energy state appears on both sides of the energy eigenvalue equation, their expression did not have a state represented on the RHS. It could be argued that the students noticed that the form of the two equations did not match. If this is the case, then the use of another mathematical entity for the purposes of extracting properties and parameters of eigenvalue equations would be considered *combining*. Bob's leveraging his interpretation of the energy eigenvalue equation as a tool for making sense of Eq. 1 can be modeled as a sensemaking triangle (see Fig. 3). Their sensemaking attempt is unsuccessful since the students are not able to resolve the inconsistency between the forms of the two equations. In their reasoning, the students discuss whether the ket on the RHS of the energy eigenvalue equation is there as a formality.

> *Alice*: Well, you, I think you technically still do, but it, it just, it doesn't, this [motions to $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,] is just a representation of it. Right?
>
> *Bob*: A representation of what?
>
> *Alice*: Like when you actually mathematically do this [pointing to $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,] and you get your value for energy, like the ket, you wrote there. ... And that's really what you find, but when you write it mathematically and like, to be formal. You include that [points to ket on RHS of $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,]. Which might be like not the best way to think about it ...

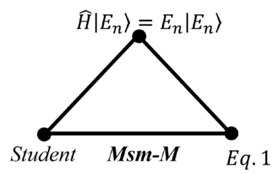


Figure 3. Model of students unsuccessful sensemaking.

Tying Modeling into Sensemaking

After the students' articulation of having essentially hit a wall, the interviewer had them move onto the modeling task, where they immediately wrote, $\hat{x}|\ \rangle = x|\ \rangle$. When asked to explain the terms in their expression, the students identified a position operator ($\hat{x}$), a measured value of position (x), and a state ($|\ \rangle$). As their discussion continued, the students made connections between elements of the expression and the real world situation. Similar to above, this was coded as *combining*, as the students were connecting their generated equation to features of the physical system. In their discussion of what label they would like to use for the kets in the expression, the students again compared it to the energy eigenvalue equation, which eventually leads them to both accept the equation $\hat{x}|L_n\rangle = x_n|L_n\rangle$ as an appropriate eigenvalue equation for an operator representing the position of a particle constrained to exist in one dimension.

The interviewer asked the students to think about what makes the equation they generated for a position operator an eigenvalue equation. Alice and Bob focused on the fact that when operating on the state with the position operator, the state does not change.

> *Bob*: … when we operate our operator, which we designate as *x* hat [$\hat{x}$], um, to represent the position on one of our states, which that could be anything, but since when we operate $\hat{x}$ on this state, we get a scalar value times the same state back. It means that we didn't, we didn't, we, it, […] - if we had changed our state, they wouldn't commute.
>
> *Alice*: By measuring, we don't change the state.
>
> *Bob*: Yeah. But by measuring, we don't change the state. We just find the scalar value that represents the state, I guess.

The students' justifications that their equation is indeed an eigenvalue equation was also coded as *combining*: they are ensuring that the mathematical entity they generated has the appropriate properties to describe the real world situation in the prompt. The primary property they reference in this justification is that the state does not change. When asked if their analysis had any impact on their interpretation, the students were able to fully articulate that the state is changed in Eq. 1 and therefore Eq. 1 is not an eigenvalue equation. The students used their general understanding of eigenvalue equations in quantum mechanics as a tool to determine the relevant features of the eigenvalue equation they generated. This reasoning was then chained to make sense of another object, the non-eigenvalue operator equation, resulting in the final conclusion. Figure 4 models the sensemaking process in its entirety including the students' final conclusion.

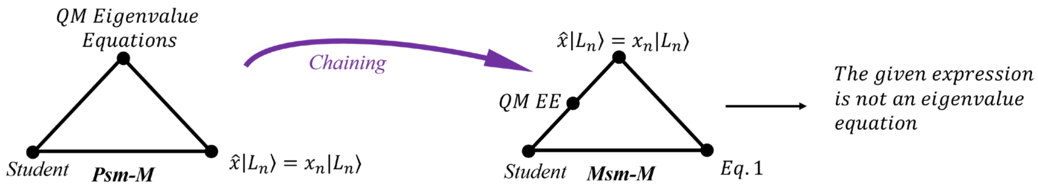


Figure 4. A model of the students sensemaking leading to their final conclusion about Eq. 1.

Discussion

In this study, Zbiek and Conner (2006)'s modeling framework is used to get a fuller picture of the processes students engage in as they work through the tasks, while Gifford and Finkelstein (2020)'s sensemaking framework is utilized to deepen the understanding of how the students resolve an inconsistency in their understanding while working in a modeling frame. When the students recognize an inconsistency in their analysis of Eq. 1, they shift into a sensemaking frame in order to resolve that inconsistency. That is not to say that a modeling frame is requisite for sensemaking, only that the two can coexist, as they do in this case study.

Odden and Russ (Odden & Russ, 2019) claim that sensemaking is often a part of modeling activities, and could potentially occur in parallel with modeling. Our study documents an example of sensemaking within modeling, and explicitly identifies behaviors and strategies students use to resolve an inconsistency. The modeling and sensemaking frameworks come together here to provide insight into the kinds of activities students engage in when given a task that places them in what is typically the middle of a modeling cycle, how they go about trying to resolve an inconsistency, and how analogous modeling tasks can help students refine their reasoning on a task with which they are struggling.

While productive, the Zbiek and Conner (2006) model may not be sufficient to model more advanced physics problem solving. In coding, the authors had several discussions in which they were unable to singularly code some of the students' work because it was difficult to determine whether the object of the students' thinking was a mathematical entity or physical system (e.g., real world situation). We suspect this is due to the blended nature of quantum mechanics content rather than a lack of student articulation thereof. This may suggest a need for modifications to existing frameworks, or a new framework entirely, that accounts for the ways mathematics and physics are blended in physics reasoning and modeling.

The primary limitation of this study is that the data are from a single pair of students so results may not be generalizable. However, these students were articulate and thoughtful, and the ideas they present corroborate previous findings in the literature. We look forward to further empirical and theoretical work in this and other physics topics to provide additional insight.

References

Blum, W., & Leiß, D. (2007). 5.1 - How do Students and Teachers Deal with Modelling Problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modelling* (pp. 222–231). Woodhead Publishing. https://doi.org/https://doi.org/10.1533/9780857099419.5.221

DeVore, S., & Singh, C. (2015). *Development of an Interactive Tutorial on Quantum Key Distribution*. 59–62. https://doi.org/10.1119/perc.2014.pr.011

Emigh, P. J., Passante, G., & Shaffer, P. S. (2015). Student understanding of time dependence in quantum mechanics. *Physical Review Special Topics - Physics Education Research*, *11*(2). https://doi.org/10.1103/PhysRevSTPER.11.020112

Emigh, P. J., Passante, G., & Shaffer, P. S. (2018). Developing and assessing tutorials for quantum mechanics: Time dependence and measurements. *Physical Review Physics Education Research*, *14*(2), 20128. https://doi.org/10.1103/PhysRevPhysEducRes.14.020128

Engeström, Y., Miettinen, R., & Punamäki, R.-L. (1999). Perspectives on Activity Theory. In *Learning in Doing: Social, Cognitive and Computational Perspectives*. Cambridge University Press. https://doi.org/DOI: 10.1017/CBO9780511812774

Gifford, J. D., & Finkelstein, N. D. (2020). Categorical framework for mathematical sense making in physics. *Physical Review Physics Education Research*, *16*(2), 20121. https://doi.org/10.1103/PhysRevPhysEducRes.16.020121

Gire, E., & Manogue, C. (2008). Resources students use to understand quantum mechanical operators. *AIP Conference Proceedings*, *1064*, 115–118. https://doi.org/10.1063/1.3021230

Gire, E., & Manogue, C. (2011). Making Sense of Quantum Operators , Eigenstates and Quantum Measurements. *AIP Conference Proceedings*, 195–198.

Gire, E., & Price, E. (2015). Structural features of algebraic quantum notations. *Physical Review Special Topics - Physics Education Research*, *11*(2), 1–11. https://doi.org/10.1103/PhysRevSTPER.11.020109

Griffiths, D., & Schroeter, D. (2018). *Introduction to Quantum Mechanics* (3rd ed.). Cambridge: Cambridge University Press. doi:10.1017/9781316995433

Her, P., & Loverude, M. (2020). Examining student understanding of matrix algebra and eigentheory. *Physics Education Research Conference Proceedings*, 210–215. https://doi.org/10.1119/perc.2020.pr.Her

Kohnle, A., & Passante, G. (2017). Characterizing representational learning: A combined simulation and tutorial on perturbation theory. *Physical Review Physics Education Research*, *13*(2), 1–13. https://doi.org/10.1103/PhysRevPhysEducRes.13.020131

McIntyre, D., Manogue, C. A., & Tate, J. (2012). *Quantum Mechanics* (1st ed., Vol. 59). Pearson.

Modir, B., Thompson, J. D., & Sayre, E. C. (2017). Students' epistemological framing in quantum mechanics problem solving. *Physical Review Physics Education Research*, *13*(2), 1–12. https://doi.org/10.1103/PhysRevPhysEducRes.13.020108

Odden, T. O. B., & Russ, R. S. (2019). Defining sensemaking: Bringing clarity to a fragmented theoretical construct. *Science Education*, *103*(1), 187–205. https://doi.org/10.1002/sce.21452

Passante, G., Emigh, P. J., & Shaffer, P. S. (2015a). Examining student ideas about energy measurements on quantum states across undergraduate and graduate levels. *Physical Review*

Special Topics - Physics Education Research, *11*(2), 1–10. https://doi.org/10.1103/PhysRevSTPER.11.020111
Passante, G., Emigh, P. J., & Shaffer, P. S. (2015b). Student ability to distinguish between superposition states and mixed states in quantum mechanics. *Physical Review Special Topics - Physics Education Research*, *11*(2), 1–8. https://doi.org/10.1103/PhysRevSTPER.11.020135
Passante, G., & Kohnle, A. (2019). Enhancing student visual understanding of the time evolution of quantum systems. *Physical Review Physics Education Research*, *15*(1), 10110. https://doi.org/10.1103/PhysRevPhysEducRes.15.010110
Passante, G., Schermerhorn, B. P., Pollock, S. J., & Sadaghiani, H. R. (2020). Time evolution in quantum systems: A closer look at student understanding. *European Journal of Physics*, *41*(1). https://doi.org/10.1088/1361-6404/ab58bf
Redish, E. F., & Kuo, E. (2015). Language of Physics, Language of Math: Disciplinary Culture and Dynamic Epistemology. *Science and Education*, *24*(5–6), 561–590. https://doi.org/10.1007/s11191-015-9749-7
Richards, A., & Hemphill, M. (2018). A practical guide to collaborative qualitative data analysis. Journal of Teaching in Physical Education. *Journal of Teaching in Physical Education*, 225–231. https://doi.org/doi.org/10.1123/jtpe.2017-0084
Schermerhorn, B. P., Passante, G., Sadaghiani, H., & Pollock, S. J. (2019). Exploring student preferences when calculating expectation values using a computational features framework. *Physical Review Physics Education Research*, *15*(2). https://doi.org/10.1103/PhysRevPhysEducRes.15.020144
Schoenfeld, A. H. (2016). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics (Reprint). *Journal of Education*, *196*(2), 1–38. https://doi.org/10.1177/002205741619600202
Singh, C. (2008). Student understanding of quantum mechanics at the beginning of graduate instruction. *American Journal of Physics*, *74*(3), 277–287.
Singh, C. (2016). Quantum Interactive Learning Tutorials. *American Journal of Physics*, *76*(4), 400–405. https://doi.org/10.1119/1.2837812
Singh, C., & Marshman, E. (2014). Investigating Student Difficulties with Dirac Notation. *Physics Education Research Conference*, 345–348. https://doi.org/10.1119/perc.2013.pr.074
Singh, C., & Marshman, E. (2015). Review of student difficulties in upper-level quantum mechanics. *Physical Review Special Topics - Physics Education Research*, *11*(2), 1–24. https://doi.org/10.1103/PhysRevSTPER.11.020117
Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G. (2012). Modelling Mathematical Reasoning in Physics Education. *Science and Education*, *21*(4), 485–506. https://doi.org/10.1007/s11191-011-9396-6
Vygotsky, L. S., & Cole, M. (1978). *Mind_in_Society* (M. Cole, Ed.; 2nd ed.). Harvard University Press.
Wawro, M., Thompson, J., & Watson, K. (2020). *Student Meanings for Eigenequations in Mathematics and in Quantum Mechanics Megan*.
Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, *63*(1), 89–112. https://doi.org/10.1007/s10649-005-9002-4

Believability in Mathematical Conditionals: Generating Items for a Conditional Inference Task

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This paper describes design issues for a conditional inference task with mathematical content. The task will mirror those used in cognitive psychology to study inferences from everyday causal conditionals: its items will present a conditional premise (if A then B) and a categorical premise (A, not-A, B, or not-B) and ask participants to evaluate whether a conclusion (respectively, B, not-B, A, not-A) necessarily follows. To assemble items, we asked six mathematics education researchers with expertise in conceptual understanding to generate conditionals covering a range of mathematical topics. To mirror the structure of tasks with everyday causal content, we asked that these conditionals should vary in believability. In this paper, we analyze the content and phrasing of the submitted conditionals in order to assess their suitability for use in a conditional inference task, and describe our planned use of this task to investigate the relationship between logical reasoning and mathematical expertise.

Keywords: conditional, inference, logic, reasoning, proof

Introduction

Logical reasoning is central to mathematics, overtly so at the transition to proof (David & Zazkis, 2019). Students must learn to validate inferences that they make during proof construction and inferences that they read during proof comprehension (Hodds, Alcock & Inglis, 2014). This does not mean that every inference must be explicitly justified – demands vary by context (Weber, 2008). It does mean that every inference could in principle be checked, in a process that in experts involves considerable back-and-forth reading (Inglis & Alcock, 2012).

Inference checking can fail for semantic reasons. Alcock and Weber (2005), for instance, reported that some students were willing to accept an inference from the premise that $(\sqrt{n})$ is an increasing sequence to the conclusion that $(\sqrt{n})$ tends to infinity. This inference is deductively invalid (Smith, 2020) because some increasing sequences do not tend to infinity. The argument, however, has the form below. If the conditional premise (interpreted as a generalized conditional per Durand-Guerrier, 2003) were true, then the inference would be valid.

If (a_n) is an increasing sequence, then (a_n) tends to infinity.
$(\sqrt{n})$ is an increasing sequence.
So $(\sqrt{n})$ tends to infinity.

Inference checking can also fail for syntactic reasons. Selden and Selden (2003), for instance, reported that some students were willing to accept a proof of the first claim below as a proof of the second (see also Hoyles & Kuchemann, 2002; Inglis & Alcock, 2012). This is logically invalid (Smith, 2020) because a conditional and its converse are not equivalent.

For any positive integer n, if n is a multiple of 3 then n^2 is a multiple of 3.
For any positive integer n, if n^2 is a multiple of 3 then n is a multiple of 3.

In the semantic case, failure to detect an invalid inference is readily explained by the fact that individuals' example spaces (Sinclair et al., 2011) or concept images (Tall & Vinner, 1981) do not fully 'match' the defined concepts, so that people might overlook counterexamples. In the syntactic case, failure to detect an invalid inference might occur for what amounts to the same reason at a more abstract level: individuals' example spaces or concept images *for conditionals* are unlikely to 'match' the material or truth-functional interpretation used in mathematics (Dawkins & Norton, 2022; Epp, 2003). This occurs because everyday conditionals lend themselves to multiple distinct interpretations: for 'If it is a dog, then it is an animal', a material conditional interpretation is sensible; for 'If you mow the lawn, I will give you $5', a biconditional interpretation is sensible (Cummins et al., 1991). Hence, mathematics students must learn to restrict their interpretations of conditionals just as they must learn to restrict their interpretations of words like 'limit' and 'group'. As they learn, we can expect to see errors in relation to normative validity. Indeed, we might expect errors to persist – maybe mathematical experts reason normatively across all contexts, but maybe they reason normatively in mathematics only, and maybe their reasoning is vulnerable to errors even there.

To investigate these possibilities and thereby to improve knowledge relevant to the teaching of logical reasoning, we plan to build on the extensive research on everyday conditional reasoning in cognitive psychology. Specifically, we plan to construct a mathematical *conditional inference* task and use this, together with a task with everyday causal content, to study whether and how reasoning differs across contents and levels of mathematical expertise. In the present paper, we explain the theoretical background of our work and report first steps in task design.

Theoretical Background

Cognitive psychology has a long history of studying reasoning with and about conditionals (e.g., Evans & Over, 2004; Oaksford & Chater, 2020). Here we focus on conditional inference tasks, as we believe their structure is clearly relevant to proof validation. A typical conditional inference task presents participants with a conditional premise and one of four categorical premises, as illustrated below with everyday causal content (De Neys et al., 2003); participants are asked to evaluate the conclusion in the third line. Normatively, MP and MT inferences are valid, and DA and AC inferences are invalid.

Modus Ponens (MP)
If John studies hard, then he does well on the test.
John studies hard.
John does well on the test.

Affirmation of the Consequent (AC)
If John studies hard, then he does well on the test.
John does well on the test.
John studied hard.

Denial of the Antecedent (DA)
If John studies hard, then he does well on the test.
John does not study hard.
John does not do well on the test.

Modus Tollens (MT)
If John studies hard, then he does well on the test.
John does not do well on the test.
John did not study hard.

The task instructions might emphasise logic, asking participants to assume that the premises are true and state whether the conclusion necessarily follows, or they might seek to elicit everyday reasoning, asking participants to assume that the premises hold and use a scale to express certainty that the conclusion can be drawn (e.g., Evans, Handley, Neilens & Over, 2010). Instructions that emphasise logic mirror the task of proof validation: participants are asked to assume that some premises are true and decide whether a conclusion necessarily follows, which involves recognizing normatively invalid DA and AC inferences.

However, everyday content introduces complexity, because a conditional like 'If John studies hard, then he does well on the test' is not simply true or false. It is a reasonable assertion about a causal relationship, but studying hard is neither necessary nor sufficient for doing well. *Disablers* (Byrne, 1989) might intervene to prevent the antecedent from causing the consequent: John might study hard but not do well if, for example, the test is very difficult or if he studies the wrong subject (De Neys et al., 2003). *Alternative antecedents* (Cummins et al., 1991) might account for the consequent without the stated antecedent: John might study little but still do well if the test is easy or if he is lucky (De Neys et al., 2003). For causal conditionals, disablers highlight the fact that the antecedent might not be sufficient for the consequent and are known to make people less likely to endorse MP and MT inferences; alternative antecedents highlight the fact that the stated antecedent might not be necessary for the consequent and make people less likely to endorse DA and AC inferences (Byrne, 1989; Cummins et al., 1991). Disablers and alternative antecedents can be more or less accessible – for instance, people typically generate few disablers but many alternatives for 'If Alvin read without his glasses, then he got a headache' (Cummins, 1995). Thus, conditionals are more or less believable and support the four inferences to different extents (De Neys et al., 2003). Also, there are individual differences in ability to prioritise logic over believability, though rates do not approach normative perfection. Evans, Handley, Neilens and Over (2010) reported that for participants of higher cognitive ability, acceptance rates under deductive instructions were 93% for MP inferences, 41% for DA, 41% for AC and 47% for MT; these are far from the normative 100%, 0%, 0% and 100%.

In order to study conditional inference across levels of mathematical expertise, we aim to design a mathematical conditional inference task with structure that parallels tasks with everyday content, using all four inference types and conditionals that vary in believability. However, this raises several issues in task design.

One issue is that to parallel everyday tasks, we would like to use simple phrasing as in 'If x is less than 2, then x is less than 5'. However, in mathematics, such a conditional is considered a predicate, not a proposition, so that formally it has no truth value (Durand-Guerrier, 2003). We consider this relatively unproblematic because sensible universal quantification is commonly assumed (Dawkins & Roh, 2022) and because tasks with everyday content have this feature too (Oaksford & Chater, 2007): participants are implicitly invited to consider 'If John studies hard, then he does well on the test' as a generalized conditional applying across multiple situations.

Another issue is that in mathematical proofs, universal instantiation is routine. In the Introduction, the conditional premise 'If (a_n) is an increasing sequence, then (a_n) tends to infinity' appears not with the general categorical premise '(a_n) is an increasing sequence' but with the instantiation '$(\sqrt{n})$ is an increasing sequence'. Premises of both types can appear in proofs, and instantiations have been used in the few studies to investigate conditional inference in mathematics (Case & Speer, 2021; Durand-Guerrier, 2003). However, premises of the general type are used in typical research on everyday causal tasks so, while noting that this would be an interesting variation for the future, we prioritise comparability across contexts.

A third issue is that with assumed universal quantification and no instantiation, mathematical conditionals are either true or false. For any true conditional (that is not a true biconditional), the antecedent is sufficient but not necessary for the consequent. Sufficiency means that and MP and MT inferences are always valid because standard logic is monotonic (Smith, 2020) so there can be no disablers – nothing could 'intervene' to prevent an x less than 2 from being less than 5. Lack of necessity means that DA and AC inferences are always invalid, though counterexamples differ from the alternative antecedents of everyday tasks because they are often singular ('zero') or in classes of a single type ('all negative numbers').

For these reasons, it is impractical in mathematics to follow the typical research approach (e.g., De Neys et al., 2003) of distinguishing more and less believable conditionals using pre-tests in which participants list possible disablers and alternatives. Nevertheless, we suggest that believability for mathematical conditionals does plausibly vary on a continuous scale. For instance, 'If X is a square then X is a parallelogram' is true but perhaps not immediately or wholly believable. While a square is a parallelogram, it would be conversationally uncooperative (Grice, 1989) to call X a parallelogram if one knew that it was a square. In contrast, 'If $x < 3$ then $1/x > 1/3$' is false but definitely not immediately or wholly unbelievable. Negative x values might easily be overlooked – Alcock and Attridge (2023) reported that about one fifth of mathematics undergraduates initially judged this conditional true. We might, in theory, assess believability by asking participants to score it directly (cf. Evans et al., 2010). However, we think that mathematically experienced people would likely balk at assigning a believability score that is not 1 or 0 when they know that a conditional must be true or false.

We therefore take an alternative approach, asking not about disablers/alternatives or believability scores, but instead about *relative* believability. In the work reported here, we asked experts in conceptual understanding to generate five mathematical conditionals each and to rank their conditionals for believability. If believability does vary continuously, this should ensure that we gather conditionals from across the scale. In work to follow, we will use comparative judgement (see, e.g., Davies et al., 2021) with both experts and undergraduates to assess more robustly whether believability is a reliably shared construct.

Method

Participants, Data Collection and Data Processing

We set out to collect conditionals covering a range of mathematical topics, to assess whether believability might vary as anticipated by the theory above, and to refine our understanding of how language around mathematics reflects or differs from that around everyday causal content. We therefore approached six mathematics education researchers with expertise in secondary-school-level conceptual understanding, who would be able to anticipate what undergraduate students would find more and less believable. These participants were informed about the purpose and theoretical background of the study and were each asked to generate five mathematical conditionals that would be familiar to typical students aged 13-14 and therefore basic for mathematics undergraduates, postgraduates, and experts. They were instructed that to parallel those used in tasks with everyday causal content, their conditionals should:

- Cover a range of mathematical topics;
- Have plausibly related antecedent and consequent;
- Not be obviously false;
- Not use additional connectives ('not', 'and', 'or') in the antecedent or consequent;
- Vary in believability (where the most believable could be clearly true).

They were also asked to:

- State whether each of their conditionals was technically true or false;
- Rank their conditionals from 1 (least believable) to 5 (most believable);
- Give a one-sentence explanation of their ranking for each conditional.

Prior to analysis, we requested amendments where participants had included conditionals with extra connectives (often in symbols, for instance '$\leq$' or '$\neq$'). We checked the stated truth values, inviting participants to explain/adjust if there was any doubt about, for example, the set over which universal quantification was assumed. We then ensured that all conditionals were written in the same way, with an explicit 'then' after a comma, and without extra words. For example, '…then it is also convex' became 'then it is convex', and '…then the median must be 7' became '…then the median is 7'. Overall, participants were able to comply with our instructions and generate conditionals with non-obvious truth values, though some observed that ranking was difficult. Naturally, we do not take individual rankings as inherently reliable; the planned comparative judgement will provide more robust measures.

Analysis: Removing Unsuitable Conditionals

Our analysis focused on suitability for use in a conditional inference task. Seven of the 30 submitted conditionals were not suitable because, although they met our criteria, their consequents could not readily be asserted as categorical premises, even with rephrasing (equivalently, the converses could not readily be stated). Not coincidentally, some were long and some invoked an agent 'you' or 'I' or described physical actions as well as abstract relationships. These conditionals, listed below, were therefore removed.

- If I flip a fair coin twice, then the probability of getting two heads is 1/3.
- If you sum the first *n* odd numbers, then you get n^2.
- If a circle is divided into regions by straight lines connecting *n* dots, then the maximum number of regions is 2^n.
- If coin lands on tails 42 times out of 100 flips, then there is not enough information to tell whether it is biased.
- If £12 is shared in the ratio 1:3, then the smaller share is 1/3 of the total.
- If two normal 6-sided dice are thrown and I tell you that one of the dice shows a 2, then the probability of both being a 2 is 1/6.
- If the perimeter of a square is the same as the circumference of a circle, then the area of the square is less than the area of the circle.

Along similar lines, we also removed the conditional 'If $a = -1$, then $|a| = -|1|$' because its consequent is false. We then removed a further four conditionals were not suitable because, although they met our criteria and were phrased in ways better fitted a conditional inference task, they had true converses. This would potentially create a confound because it would mean that DA and AC inferences would be valid for semantic reasons; it also makes these conditionals less 'like' everyday causal conditionals in which there is a clear cause in the antecedent and effect in the consequent. The following conditionals were therefore also removed.

- If $x^3 < x^2$ then $x < 0$.
- If $x = 24$ is the solution to $4x + 9 = 105$, then $x = 12$ is the solution to $8x + 9 = 105$.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a rectangle.
- If *n* is a positive integer, then it can be written as a product of prime factors in exactly one way.

Analysis: Content and Form in Suitable Conditionals

This left 18 conditionals potentially suitable for a conditional inference task. We list these in Table 1, ordered by participants' individual believability rankings (most to least believable) and together with their truth values and mathematical topics. The final column captures the way in which scope of quantification is handled, with codes (discussed further below):

- 'I' where the scope is implicit, usually where a variable is assumed to be a real number;
- 'E' where the scope is explicit, usually where a set is specified in the antecedent;
- 'S' where the antecedent involves a single specific number or situation.

Table 1. Conditionals potentially suitable for a conditional inference task

Conditional	Rank	Truth	Topic	Scope
If a quadrilateral is cyclic, then it is convex.	5	T	geometry	E
If a polygon is a square, then it is a rhombus.	5	T	geometry	E
If $x^2 = y^2$, then $xy = yx$.	5	T	algebra	I
If $x = -4$, then $x^2 + x - 12 = 0$.	5	T	algebra	S
If a quadrilateral has a reflex angle, then it will tesselate.	4	T	geometry	E
If x is positive, then $\tan x > \sin x$.	4	F	trig	E
If $x - 12{,}345 = .67$ then $x > -12{,}345.67$.	4	T	number	S
If a fraction has denominator 7, then it is equivalent to a non-terminating decimal.	4	F	number	E
If $\sin x > 0$ then $\cos x < 1$.	3	T	trig	I
If a number is a multiple of 13, then it has an even number of factors.	3	F	number	E
If a composite number ends in a 3, then it is a multiple of 3.	3	F	number	E
If a rectangle has area 10cm^2, then its perimeter is greater than 10cm.	2	T	geometry	E
If an equation is a quadratic, then it has exactly two roots.	2	F	algebra	E
If $a = 42$, then $a \times b > 42$.	2	F	number	S
If four consecutive numbers are added, then the result is a multiple of four.	2	F	number	E
If the mean of a dataset is 7, then the median is 7.	1	F	statistics	E
If $a > b$, then $ac > bc$.	1	F	number	I
If a rectangle is stretched so that its side lengths double, then its area doubles.	1	F	ratio	E

In terms of topics, truth values, and believability rankings, we find this initial collection satisfactory. Topics are spread across believability rankings. More believable conditionals are more likely to be true but, in line with our suggestion that truth and believability might not align perfectly, there is considerable overlap in the middle of the table. Our removal of unsuitable conditionals did not disproportionately remove those at specific ranks, so there is no evidence of a problematic interaction between believability and phrasing challenges or truth of the converse.

The scope codes highlight new issues in designing a task to parallel those involving everyday causal content, in terms of substance, phrasing and the interactions between the two.

Conditionals with scope code I are most straightforwardly like those used in everyday causal tasks. Their ‘making sense’ without explicit scope means that they lend themselves to presentation with all four corresponding categorical premises. For instance, for the conditional ‘If $x^2 = y^2$ then $xy = yx$’, an AC item would appear as follows.

If $x^2 = y^2$ then $xy = yx$.
$xy = yx$.
$x^2 = y^2$.

Conditionals with scope code E usually have a specified set in the antecedent and a deictic ‘it’ or ‘its’ in the consequent, as in ‘If a quadrilateral is cyclic, then it is convex’. In these cases, categorical premises cannot simply use the consequent as is ('It is convex' makes no sense in isolation). We do not want to lose these items, which make up the majority of those submitted and which are clearly natural in mathematics. As a sensible solution, we will likely mirror the phrasing typically used for everyday causal items, naming the object and using the name in both categorical premise and conclusion, as below. Mathematical conditionals are at least simpler in that they do not require manipulating tenses to accommodate the temporal aspects of causality.

If a quadrilateral Q is cyclic, then it is convex.	If John studies hard, then he does well on the test.
Quadrilateral Q is convex.	John does well on the test.
Q is cyclic.	John studied hard.

Conditionals with scope code S – whose antecedents focus on specific objects – have the interesting property that they do not 'feel' like generalized conditionals. We could avoid using such items, but we note that they have interesting properties around phrasing, necessity and sufficiency. For instance, the antecedent of the conditional 'If $a = 42$, then $a \times b > 42$' seems to be about a, when the conditional actually quantifies over b. It is false but invokes the common misconception 'multiplication makes things bigger', and the antecedent is so far from necessary for the consequent that accepting the AC inference from $a \times b > 42$ to $a = 42$ would seem nonsensical. This raises questions about how plausible false conditionals must be in order to work in our task, and we will discuss this further at the conference.

Discussion and Next Steps

The analysis above clarifies design issues for a mathematical conditional inference task. Despite the normative interpretation of conditionals in mathematics, 'natural' uses of if-then to express mathematical relationships vary considerably. Some uses do not lend themselves to a conditional inference task due to length and overall complexity in describing agents and/or actions. Some have true converses, and therefore do not match items used for causal conditionals, where the cause appears in the antecedent and the effect in the consequent (Cummins et al., 1991). For those that are in principle suitable, scope might be fully implicit – commonly in the case of quantification over sets of numbers – or explicit in the sense that a set is specified in the antecedent. In the latter case, rephrasing is necessary to construct an item.

We will use this analysis to inform our ongoing work. We aim to collect a total of approximately 40 suitable conditionals, with a view to constructing a 16-item task using eight conditionals that are more believable and eight that are less believable (should believability turn out not to be a reliable construct, we will simply use a spread of topics). We will therefore ask two more experts to generate five conditionals each, with the additional criterion that the converse of each should be readily articulated and false. We will then collect conditionals from the mathematics education literature and textbooks to complete a set that balances topics, truth and implicit/explicit quantification, and will seek to make phrasing as uniform as is practical. We will report on this work at the conference, together with the comparative judgement study of believability and a first test of the conditional inference task with undergraduates.

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References

Alcock, L.J. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior*, *24*, 125-134.

Byrne, R.M.J. (1989). Suppressing valid inferences with conditionals. *Cognition*, *31*, 61–83.

Case, J. & Speer, N. (2021). Calculus students' deductive reasoning and strategies when working with abstract propositions and calculus theorems. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *31*, 184–201.

Cummins, D. D., Lubart, T., Alksnis, O., & Rist, R. (1991). Conditional reasoning and causation. *Memory & Cognition*, *19*, 274–282.

David, E. J. & Zazkis, D. (2019). Characterizing introduction to proof courses: A survey of U.S. R1 and R2 course syllabi. *International Journal of Mathematical Education in Science and Technology*, *51*, 388–404.

Davies, B., Alcock, L., & Jones, I. (2021). What do mathematicians mean by proof? A comparative-judgement study of students' and mathematicians' views. *Journal of Mathematical Behavior*, *61*, 100824.

Dawkins, P. C. & Norton, A. (2022). Identifying mental actions for abstracting the logic of conditional statements. *Journal of Mathematical Behavior*, *66*, 100954.

De Neys, W., Schaeken, W., & d'Ydewalle, G. (2003). Inference suppression and semantic memory retrieval: Every counterexample counts. *Memory & Cognition*, *31*, 581–595.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, *53*, 5–34.

Epp, S. (2003). The role of logic in teaching proof. *American Mathematical Monthly*, *110*, 886-899.

Evans, J.St.B.T., Handley, S.J., Neilens, H., & Over, D. (2010). The influence of cognitive ability and instructional set on causal conditional inference. *Quarterly Journal of Experimental Psychology*, *63*, 892–909.

Evans, J.St.B.T. & Over, D.E. (2004). *If*. Oxford: Oxford University Press.

Grice, H. P. (1989). *Studies in the way of words*. Boston, MA: Harvard University Press.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*, 62-101.

Hoyles, C. & Küchemann, D. (2002). Students' understanding of logical implication. *Educational Studies in Mathematics*, *51*, 193–223.

Inglis, M. & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*, 358-390.

Oaksford, M. & Chater, N. (2020). New paradigms in the psychology of reasoning. *Annual Review of Psychology*, *71*, 305–330.

Oaksford, M. & Chater, N. (2007). *Bayesian Rationality: The Probabilistic Approach to Human Reasoning*. Oxford: Oxford University Press.

Selden, A. & Selden, J. (2003). Validations of proofs considered as texts: can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4–36.

Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *Journal of Mathematical Behavior*, *30*, 291–303.

Smith, P. (2020). *An introduction to formal logic: Second Edition*. Cambridge: Cambridge University Press.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*, 431–459.

The Price Does Not Appear the Same for Everyone: Racial Differences in Students' Perceptions of the Mathematics Graduate School Application Process

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Lack of racial diversity has been an ongoing issue in higher education. Recently, the Theory of Racialized Organizations has been used to help explain why, despite many calls for diversity, the demographics of higher education have not changed. Considering this framework, we seek to understand what aspects of the graduate school application process are viewed as barriers by minoritized students for applying. As part of a larger study of undergraduate student knowledge of the graduate school application process, we analyze 515 responses from undergraduate math majors using Mann-Whitney U tests to identify differences in what participants view as a barrier to apply to graduate school by race/ethnicity. We discuss two main results and recommend changes to graduate programs wishing to recruit more minoritized students.

Keywords: Graduate school application, undergraduate mathematics majors, Theory of Racialized Organizations, Minoritized students

On June 29, 2023, the U.S. Supreme Court struck down Affirmative Action on college admissions (Students for Fair Admissions Inc. v. President & Fellows of Harvard College, 2023). This decision has the potential to impact the ability of future minoritized[1] students to enter college both at the undergraduate and graduate levels. This will be especially problematic for the field of mathematics given its lack of racial diversity that becomes more pronounced at higher levels. While 31.9% of the U.S. population identify as Hispanic/Latinx or African American (U.S. Census, 2020), in recent years only 15.9% of mathematics and statistics (mathematics-only data unavailable) bachelor's degrees were earned by minoritized students (National Center for Science and Engineering Statistics, 2019). Finally, only 7.4% of new mathematics doctoral recipients were minoritized (Golbeck, et al., 2020).

Diversity in graduate admissions has become an important topic of research and conflict in the last decade. Recent higher education research has shed light on what faculty think about the role of diversity in final-round decisions in the graduate admissions process (Posselt, 2016). Often, diversity is discussed as a "goal" for institutions of higher education to achieve. Yet, in their admissions processes, the conditionality of diversity comes secondary to the perceived obligation of "protecting well-established standards of conventional achievement," such as high program rankings and competitive test scores (Posselt, 2016). However, it is well-documented that gaps in standardized test scores fall along lines of socio-economic identity and are not adequate indicators of intelligence (Posselt, 2016). If diversity is considered a criterion for

[1] Minoritized is an alternative way of referring to people who are often labeled as "Underrepresented Minorities" in STEM. This alternative phrasing makes it clear that it is power imbalances and systematic oppression that cause these groups to be less represented in STEM (Wingrove-Haugland & McLeod, 2021).

graduate admissions only secondary to traditional quantitative measures, then current admission practices are likely to "perpetuate enrollment inequities" (Posselt, 2016).

While this research provides great insight on the admissions process across multiple disciplines it loses context and insight that can be gained through Discipline-Based Education Research. Physics education researchers have conducted multiple studies of the graduate application process (Chari & Potvin, 2019a, 2019b; Potvin, et al., 2017; Scherr, et al., 2017; Young & Caballero, 2019). Physics departments "express a … demand for greater numbers of students from [minoritized] groups, but simultaneously report a lack of such applicants" (Potvin et al., 2017). This finding mirrors smaller scale research in the field of mathematics which found "low graduate mathematics application rates from historically underrepresented groups" (Gevertz & Wares, 2020). If the U.S. is to increase diversity in STEM graduate programs, we must examine whether minoritized students apply for graduate school at the same rate as their non-minoritized peers and if not, determine how to address the disparity.

Students who want to pursue graduate school in mathematics often face multiple financial barriers. These burdens can include working to support their family (instead of acquiring research experience or studying), rent, transportation, paying off undergraduate debt, GRE costs, and application fees (Cochran et al., 2018). Multiple studies have shown that application fees limit the number of applications from students from low-income backgrounds (Cadena et al., 2023; Cochran et al., 2018; Roberts et al., 2021; Wilson, et al., 2018). Some programs have implemented fee waivers for this reason. However, the effort required to gain fee waivers deters students from applying to graduate school. For example, students may need to complete their application in advance of the normal deadline or achieve a higher GPA (Cadena et al., 2023; Roberts et al., 2021). In some cases, fee waivers require U.S. citizenship, so undocumented or international students may not qualify for waivers. In addition, the application fee may sometimes cost a student an entire month's salary (Cadena et al., 2023). Thus, financial burdens, and application fees in particular, negatively, and significantly impact low-income students applying to graduate school.

The Undergraduate Knowledge of the Mathematics Graduate School Application Process (Knowledge-GAP) project was created to examine undergraduate mathematics majors' knowledge about the graduate school application process and to facilitate an understanding of perceived barriers to applying to graduate school across different demographic groups. This paper focuses on how minoritized students perceive the application process and seeks to answer the following subset of research questions from the Knowledge-GAP project:

1. *Do perceptions of barriers to applying to graduate school differ by race/ethnicity?*
2. *What factors are most important to minoritized students planning to apply to graduate school?*

Theoretical Background

The Theory of Racialized Organizations (TRO) was developed to help explain "consistency of racialized organizational inequality" (Ray, 2019). This framework calls for researchers interested in racial inequality to critically examine how an organization's policies and institutionalized practices (e.g., admissions procedures) uphold racial disparities (Ray, 2019). It has been applied to many fields and types of organizations since its inception including undergraduate mathematics education (Leyva et al., 2021). In a recent study, Poon et al. (2023) applied the TRO framework "to examine the totality of (undergraduate) admissions as racialized organizations". They found that even supposedly "race-neutral" admissions policies can increase racial inequality due to the existing racial wealth gap in America (Poon et al., 2023). To explain this gap, they call upon the concept of racial capitalism (Poon et al., 2023). Melamed (2015) explained racial capitalism by stating: "Racism enshrines the inequalities that capitalism requires . . . by displacing the uneven life chances that are inescapably part of capitalist social relations onto fictions of differing human capacities, historically" along racial lines (p. 77). We extend this framework to graduate admissions to understand differences in perceived barriers to the graduate application process between minoritized and non-minoritized students.

Methods

Instrument Development

The research team created a survey based in part on a survey used to determine undergraduate physics majors' interest in graduate school and how important they believed different aspects of the application process were (Chari & Potvin, 2019b). Nineteen survey items were adapted from that instrument. A notable difference between that survey and ours was that we provided an opportunity for participants to express their lack of knowledge about different parts of the application process. The final survey had 57 items separated into four categories: (a) knowledge about different aspects of the application process, (b) barriers to applying, (c) interest in graduate school and what students look for in programs they apply to, and (d) demographic questions. Most questions were Likert scale or multiple choice, though four were open-ended and some of the multiple-choice items allowed participants to type in a text response. The full survey is available at this link: https://researchrepository.wvu.edu/faculty_publications/3291/.

Data Collection

The research team sent an email with the survey to department chairs and undergraduate program directors at all U.S. undergraduate mathematics programs at colleges and universities with at least 1000 students total ($N = 985$). We requested the survey be sent to all undergraduate mathematics majors. Initial emails were sent Fall 2022 through Spring 2023, via Qualtrics, and follow-up emails were sent to encourage a greater response rate. In addition to direct emails, the survey was also posted on social media, listservs and in newsletters for several professional organizations in mathematics.

Data Analysis

We received 1090 responses from students at 181 colleges and universities, with 519 complete responses. Note that students could miss part of a question and still have their response marked as complete. Thus, the *N*s for different items are not always the same. Statistical tests were run in IBM SPSS.

To address these research questions, we analyzed data collected through two survey items: *To what extent are the following factors a potential barrier to your pursuit of graduate school?* and *How important are the following factors in choosing which schools you apply to?* Both questions were Likert scale items adapted for this study from Chari and Potvin (2019b). The first item had 17 sub-item topics (potential barriers), which students rated on a scale of 1 (not at all a barrier) to 5 (very significant barrier). The second item had 15 sub-item topics (potentially important factors for applying to graduate programs), which students rated on a scale of 1 (not at all important) to 5 (very important).

Results

Participant Demographics

Tables 1 and 2 show annual income for participants while growing up, and racial/ethnic demographics for participants with complete responses. Note that participants were able to select more than one category for racial/ethnic identification.

Table 1. Yearly Income for Participants When They Were Growing Up.

To the best of your knowledge, which category best describes your family's yearly household income while you were growing up?

Income	*N*	Percentage
Less than $60,000	119	22.9%
Between $60,000 and $100,000	128	24.7%
More than $100,000	208	40.1%
Do not know	44	8.5%
Prefer not to say	20	3.9%
Total	519	100%

Table 2. Race/ethnicity of Participants.*

With which racial and ethnic groups do you identify?

Race/Ethnicity	*N*	Percentage
American Indian or Alaskan Native	7	1.4%
Asian or Asian American	80	15.5%
Black or African American	21	4.1%
Hispanic, Latine/Latinx, or Spanish Origin	59	11.4%
South Western Asia and North African (Middle Eastern or North African)	8	1.6%
Native Hawaiian or Other Pacific Islander	5	1.0%
White	381	73.8%
Prefer not to say	10	1.9%
Total	516	

Perceptions of Potential Barriers & Important Factors

Participants were separated into two groups based on their response to the survey item asking for their race and ethnicity. Participants who said they belonged to at least one of the following groups were labeled as "minoritized" in the dataset: American Indian or Alaskan Native, Black or African American, Hispanic, Latine/Latinx, or Spanish origin or Native Hawaiian or Other Pacific Islander. While there are issues with combining different identities that have been historically and through modern times excluded in STEM disciplines, this method provides insight into factors potentially excluding these groups from graduate education. In addition, sample sizes in many of the individual groups were too small to run meaningful statistical analyses.

We report here only on a subset of the sub-item topics for both items, seven for the first item and five for the second item. A one-way analysis of variance (ANOVA) was not employed because for 8 of the 12 sub-item topics, the Homogeneity of Variance assumption was violated. Thus, for ease of comparison and consistency, Mann-Whitney U tests were performed using the minoritized/non-minoritized variables for all sub-item topics. Table 3 contains Mann-Whitney U test results for the minoritized/non-minoritized groups for the 515 participants who responded to the selected sub-item topics from the first survey item.

Table 3. Mann-Whitney U test results for selected items for the question, "To what extent are the following factors a potential barrier to your pursuit of graduate school?" using the minoritized/non-minoritized variable.

Item	Group	*N*	Mean	Mean Rank	*U*	*Z*	*p*	*r*
Graduate application fees	Minoritized	85	3.08	313.68	13542	-3.88	<.001	0.17
	Not	430	2.47	246.99				
Paying for the General GRE Test ($220)	Minoritized	85	3.40	317.02	13003.5	-4.22	<.001	0.19
	Not	427	2.69	244.45				
Paying for the GRE Mathematics Subject Test ($150)	Minoritized	85	3.38	320.90	12843.5	-4.41	<.001	0.19
	Not	429	2.62	244.94				
Sending GRE scores to programs ($30 per program)	Minoritized	85	3.16	311.82	13530.5	-3.83	<.001	0.17
	Not	428	2.50	246.11				
Availability of scholarships/funds or my ability to pay tuition	Minoritized	84	4.19	320.87	12653	-4.47	<.001	0.20
	Not	429	3.55	244.49				
Parenting or family responsibilities	Minoritized	84	2.35	316.92	12901	-4.60	<.001	0.20
	Not	428	1.68	244.64				
A lack of mathematicians/scientists that look like me	Minoritized	83	2.82	322.11	12358	-4.74	<.001	0.21
	Not	429	1.93	243.81				

The output of a Mann-Whitney U test is a Z value on a normal distribution. The Z values in Table 3 indicate that the minoritized group has greater means than the non-minoritized group. These results show there is a statistically significant difference (all p's < .05) between the minoritized/non-minoritized groups in the responses for all seven sub-item topics. In all cases the minoritized participants were more likely to view each sub-item topic as a potential barrier to their pursuit of graduate school than their peers. All of these results had a small effect size (all r's between 0.1 and 0.3).

For the second survey item, "How important are the following factors in choosing which schools you apply to?", it should be noted that not all participants saw this item. Prior to this, participants were asked to state their interest in graduate school in mathematics. Only participants who responded with anything other than "Not interested in graduate school in mathematics" saw this item. Table 4 contains Mann-Whitney U test results for the minoritized/non-minoritized groups for the 435 participants who responded to the selected sub-item topics from the second survey item.

Table 4. Mann-Whitney U test results for selected options for the question, "How important are the following factors in choosing which schools you apply to?" using the minoritized/non-minoritized variable.

Item	Group	N	Mean	Mean Rank	U	Z	p	r
Availability/Amount of assistantships or scholarships	Minoritized	74	4.61	259.74	10194	-3.51	<.001	0.17
	Not	360	4.20	208.82				
Cost of living	Minoritized	74	4.54	278.77	8860	-4.84	<.001	0.23
	Not	361	3.92	205.54				
No GRE General Test requirement or no minimum score requirement	Minoritized	74	3.09	257.91	10329.5	-3.12	.002	0.15
	Not	360	2.52	209.19				
Having peers who are the same race/ethnicity as myself	Minoritized	73	2.52	278.45	8654	-5.52	<.001	0.27
	Not	360	1.51	204.54				
Having a thesis advisor of the same race/ethnicity as myself	Minoritized	73	2.19	274.76	8850.5	-5.62	<.001	0.27
	Not	359	1.34	204.65				

**Note the total N for these tables are strictly less than the previous tables since participants not interested in graduate school did not get this question.*

The Z values in Table 4 indicate that the minoritized group has greater means than the non-minoritized group. These results show there is a statistically significant difference (all p's $< .05$) between the minoritized/non-minoritized groups in the responses for all five sub-item topics. In all cases the minoritized participants were more likely to view each sub-item topic as an important factor in choosing which school to apply to than their peers. All of these results had a small effect size (all r's between 0.1 and 0.3).

These Mann-Whitney U test results show that minoritized participants are more concerned about the cost of different aspects of the graduate school application process compared to their peers. They are also more concerned about being able to afford to attend graduate school. Finally, they are more concerned about having peers and advisors with the same race/ethnicity in the graduate programs to which they apply.

Annual Income

To examine the relationship between family income and minoritized status, participant responses to the item "To the best of your knowledge, which category best describes your family's yearly household income while you were growing up?" were analyzed. A Chi-squared test of association determined that there was an association between ethnicity status and income category $\chi 2(4, N = 512) = 46.44$, $p = <.001$, $V = .30$. This result had a medium effect size. Minoritized participants were more likely to come from lower income families than their peers: half of the minoritized participants were from families that made less than $60,000 dollars a year while only 17.6% of non-minoritized participants were from families that made less than $60,000 dollars a year.

Discussion

These test results show two main threads in which minoritized participants demonstrate different concerns from their peers: finances and a lack of racial diversity in graduate education. The financial concerns are most obvious in results for 7 of the 12 sub-item topics showing that minoritized participants were directly concerned about funding or finances. Results from another two sub-item topics, familial responsibilities and favoring programs with less stringent GRE requirements, may also be caused (at least in part) by financial concerns. We also found that significantly more minoritized participants reported that they were from low-income households than their peers. The Theory of Racialized Organization and racial capitalism explain how the apparently uniform cost of the application process serves as a contextual barrier and thus a gatekeeper, preventing minoritized students from entering mathematics programs at the graduate level. The results from the last three sub-item topics that do not fit under the topic of finances demonstrate minoritized participants' well-grounded concerns that they will be the only person who looks like them in their department. We know from the demographics of mathematics graduate programs that many mathematics departments are likely to have few, if any, minoritized students (Golbeck, et al., 2019). This can have multiple repercussions for the few minoritized students at these programs. The Theory of Racialized Organizations calls into question why programs have so few minoritized students and the impact that could have on students who are applying. Did these programs previously admit minoritized students who either left of their own volition or were forced out? Is the work of minoritized students, both inside and outside of the classroom, systematically undervalued at these programs? The lack of diversity in these programs negatively impacts minoritized students' decisions to apply.

Based on these results we recommend that programs hoping to recruit and support minoritized students seek ways to minimize the cost of applying to, and staying in, the program. For example, consider removing the GRE General and Subject test requirements (for additional reasons to exclude the GRE from admission requirements, see McEldowney et al., 2024, Miller et al., 2019, and Posselt, 2016). Consider allowing unofficial transcripts in the application and only require official transcripts for admitted students. Advocate for the financial well-being of current graduate students. Try to obtain more funding for graduate students either internally or externally. If faculty are eligible for university childcare programs or childcare subsidies, advocate for graduate students to be eligible for those programs.

In terms of research implications, there is still more data to be analyzed from the Knowledge-GAP survey. A clear next step for the project is to test for differences in knowledge of application fees by race or ethnicity. We also need to examine differences in perceptions of the graduate school application process by other demographic information like gender, income, etc. the Knowledge-GAP is only the first step in studying mathematics graduate student application and admissions processes. More work is necessary to fully understand which factors impact students, especially minoritized students, choice of graduate schools and what obstacles they face. We also look forward to seeing future qualitative research can be done in this area.

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References

Cadena, M. A., Amaya, C., Duan, D., Rico, C. A., García-Bayona, L., Blanco, A. T., Agreda, Y. S., Villegas Rodríguez, G. J., Ceja, A., Martinez, V. G., Goldman, O. V., & Fernandez, R. W. (2023). Insights and strategies for improving equity in graduate school admissions. *Cell, 186(17),* 3529-3547

Chari, D., & Potvin, G. (2019a). Admissions practices in terminal master's degree-granting physics departments: A comparative analysis. *Physical Review Physics Education Research*, *15*(1), 010104.

Chari, D., & Potvin, G. (2019b). Understanding the importance of graduate admissions criteria according to prospective graduate students. *Physical Review Physics Education Research*, *15*(2), 023101.

Cochran, G. L., Hodapp, T., & Brown, E. E. (2018). Identifying barriers to ethnic/racial minority students' participation in graduate physics. In *Physics Education Research Conference*.

Estrada, M., Burnett, M., Campbell, A. G., Campbell, P. B., Denetclaw, W. F., Gutiérrez, C. G., Hurtado, S., John, G. H., Matsui, J., McGee, R., Okpodu, C. M., Robinson, T. J., Summers, M. F.,Werner-Washburne, M., and Zavala, M. E. (2016). Improving underrepresented Minority student persistence in STEM. *CBE-Life Sciences Education, 15(3)*, es5.

Gevertz, J. L., & Wares, J. R. (2020). Fostering Diversity in Top-Rated Pure Mathematics Graduate Programs. *Notices of the American Mathematical Society*, *67*(2).

Golbeck, A. L., Barr, T. H., & Rose, C. A. (2019). Fall 2017 Departmental Profile Report. *Notices of the American Mathematical Society*, 66(10) 1721-1730.

Golbeck, A. L., Barr, T. H., & Rose, C. A. (2020). Report on the 2017-2018 New Doctoral Recipients. *Notices of the AMS*, 67(8) 1200-1213.

Harris P. E. and Winger A. (2020). *Asked and answered: Dialogues on advocating for students of color in mathematics*. Independently published.

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *The Journal of Higher Education, 92(5),* 784-814.

McEldowney, T., Townsend, E., Maldonado, D., Michaluk, L., & Deshler, J. (Accepted). Undergraduate Gender Differences in Knowledge of the GRE and Perception of the GRE as a Barrier to Applying to Graduate Mathematics Programs. *Proceedings of the 26th Annual Conference on Research in Undergraduate Mathematics Education*

Melamed, J. (2015). Racial capitalism. Critical Ethnic Studies, 1(1), 76–85. https://www.jstor.org/stable/10.5749/jcritethnstud.1.1.0076

Miller, C. W., Zwickl, B. M., Posselt, J. R., Silvestrini, R. T., & Hodapp, T. (2019). Typical physics Ph. D. admissions criteria limit access to underrepresented groups but fail to predict doctoral completion. *Science Advances, 5(1)* , eaat7550. https://doi.org/10.1126/sciadv.aat7550

National Center for Science and Engineering Statistics. (2019). *Women, Minorities, and Persons with Disabilities in Science and Engineering: 2019.* Special Report NSF 19-304. Alexandria, VA. Available at https://www.nsf.gov/statistics/wmpd

Poon, O., Lee, D. H., Galvez, E., Song Engler, J., Sérráno, B., Raza, A., Hurtado, J. M., & Chun, N. K. (2023). A Möbius Model of Racialized Organizations: Durability of Racial Inequalities in Admissions. *The Journal of Higher Education*, 1-26.

Posselt, J. R. (2016). *Inside graduate admissions: Merit, diversity, and faculty gatekeeping*. Harvard University Press.

Potvin, G., Chari, D., & Hodapp, T. (2017). Investigating approaches to diversity in a national survey of physics doctoral degree programs: The graduate admissions landscape. *Physical Review Physics Education Research*, *13*(2), 020142.

Ray, V. (2019). A theory of racialized organizations. *American Sociological Review, 84(1)*, 26-53.

Roberts, S. F., Pyfrom, E., Hoffman, J. A., Pai, C., Reagan, E. K., & Light, A. E. (2021). Review of racially equitable admissions practices in STEM doctoral programs. *Education Sciences*, *11*(6), 270.

Scherr, R. E., Plisch, M., Gray, K. E., Potvin, G., & Hodapp, T. (2017). Fixed and growth mindsets in physics graduate admissions. *Physical Review Physics Education Research*, *13*(2), 020133.

Students for Fair Admissions Inc. v. President & Fellows of Harvard College, 600 U.S. ___ (2023) https://supreme.justia.com/cases/federal/us/600/20-1199/

U.S. Census Bureau QuickFacts: United States. (2020). Retrieved September 29, 2020, from https://www.census.gov/quickfacts/fact/table/US/PST045219

Wilson, M. A., DePass, A. L., & Bean, A. J. (2018). Institutional interventions that remove barriers to recruit and retain diverse biomedical PhD students. *CBE—Life Sciences Education*, *17*(2), ar27.

Wingrove-Haugland, E., & McLeod, J. (2021). Not “Minority” but “Minoritized”. *Teaching Ethics*, *21*(1).

Young, N. T., & Caballero, M. D. (2019). Using machine learning to understand physics graduate school admissions. *arXiv preprint arXiv:1907.01570*.

Factors Influencing Undergraduate Students' Logical (in)Consistency (LinC) in Mathematical Contexts

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This study delves into undergraduate students' ability to self-monitor and organize their claims coherently to ensure logical consistency when evaluating mathematical statements and validating accompanying arguments. To assess the capacity of undergraduate students to maintain logical consistency in mathematical contexts, we designed an online instrument comprising twenty statement-argument pairs, each rooted in mathematical content. We administered this online instrument to 205 undergraduate students, encompassing various levels of proof experience, across three diverse U.S. higher education institutions, including two large public universities and one small liberal arts college. Our analysis reveals a substantial number of undergraduate students displayed logical inconsistencies. It also appears that students may exhibit different levels of logical inconsistencies when the accompanying argument is framed using proof by contradiction, as opposed to direct proof. This prevalence underscores a critical concern in mathematics education, particularly proof-oriented mathematics.

Keywords: logic, logical (in)consistency, statement evaluation, argument validation, proof by contradiction

Our primary aim of this study is to understand logical consistency as it pertains to students' mathematical reasoning and argumentation when evaluating mathematical statements and validating accompanying arguments. By logical consistency in an individual's mathematical thinking, we refer to a state of having no logical contradiction among their assertions. To uphold logical consistency, it is critical that individuals recognize and address any logical contradictions that might remain unnoticed within their assertions. Logical contradictions in mathematics arise when statements conflict, as in false statements such as "x is a multiple of 2 and x is not a multiple of 2." Fundamental to mathematical logic is the law of non-contradiction, which asserts that a statement and its negation cannot both be true simultaneously. For instance, if a student claims both "x is a multiple of 2" and "an argument properly proves x is not a multiple of 2," a logical contradiction arises since the second assertion negates the first, and vice versa. Overlooking such logical contradiction in their reasoning can lead to creating mathematical arguments that the community will not accept. To achieve our research aim, we pose several research questions that guide our investigation:

1. To what degree do students demonstrate logical consistency across a set of statements and accompanying arguments?
2. Is there a correlation between undergraduate students' logical consistency and their exposure to proof-oriented mathematics courses?
3. Do specific contexts or variables influence the likelihood of undergraduate students exhibiting logical inconsistencies (LinC)?

These questions will serve as the foundation for exploring logical consistency in students' mathematical reasoning.

Review of Literature & Theoretical Perspective

To comprehend the significance of logical consistency in mathematical reasoning, we turn to insights from cognitive psychology and educational theory. Cognitive psychologists have long posited the general tendency of people to maintain cognitive consistency and avoid cognitive dissonance in various contexts (e.g., Abelson et al., 1968; Festinger, 1957). The principle of maintaining cognitive consistency has been applied to interpersonal relations, beliefs, feelings, and actions (Cooper, 1988; Cvencek et al., 2014; Gawronski & Strack, 2008). However, the relevance to mathematical reasoning remains intriguing. Our study extends this notion into the realm of mathematical thinking, which we refer to as logical consistency. It is essential to recognize the unique challenges that mathematical reasoning presents, particularly concerning logical contradiction, while evaluating mathematical statements and validating accompanying arguments (Roh & Lee, 2018).

Our study is grounded in a theoretical framework that draws from cognitive psychology and constructivist learning theory. Cognitive psychologists have argued that individuals naturally strive to maintain cognitive consistency and resolve cognitive conflicts (e.g., Piaget, 1967). In learning, cognitive conflicts are considered essential for individuals to construct new knowledge or modify their existing knowledge structures (Glasersfeld, 1995). This perspective suggests that students actively seek logical consistency in their mathematical reasoning and argumentation. Research showcases that students can also enhance their reasoning by addressing logical inconsistencies in their mathematical arguments (Dawkins & Roh, 2016; Ely, 2010; Roh & Lee, 2011, 2017).

However, can students effectively identify logical contradictions within their mathematical assertions when they exist? Roh and Lee's study (2018) reported that more than 20% of undergraduate students ($n = 47$) who completed introductory proof courses displayed logical inconsistencies in their assertions. This suggests that even with substantial mathematical content knowledge and mathematical proofs, logical inconsistencies (LinC) may persist in students' mathematical reasoning. Building upon Roh and Lee's (2018) research, we extend our investigation to assess the frequency with which students exhibit logical inconsistencies and identify the prevalent types of such inconsistencies when students evaluate mathematical statements and validate accompanying arguments.

Methods

The LinC Instrument

We developed the LinC instrument to assess students' logical (in)consistency when evaluating a statement and validating an accompanying argument. The current version of the LinC instrument focuses on content areas in precalculus and calculus, employing two factors for statements (logical complexity and truth-value) and three factors for accompanying arguments (attempts, validity, and frames). Table 1 presents the characteristics of the final 20 statement-argument pairs derived from these factors.

Table 1 Summary of the LinC items (20 items in total)

Quantifiers in the Statement	Truth-Value of the Statement		An Attempt in the Argument		Validity of the Argument		A Frame of the Argument		
	True (10)	False (10)	Proof (10)	Disproof (10)	Valid (12)	Invalid (8)	Direct (4)	by Contradiction (7)	by Example (9)
∀ (4)	2	2	3	1	2	2	1	1	2
∃ (6)	3	3	2	4	4	2	1	2	3
∀∃ (4)	2	2	2	2	3	1	0	2	2

∃∀ (6)	3	3	3	3	3	3	2	2	2

Each LinC item starts with a statement-argument pair, followed by three questions: (1) evaluate if a given statement is true or false; (2) determine if a given argument is an attempt to prove or an attempt to disprove the given statement; and (3) evaluate if the given argument proves properly what it attempts to prove. Figure 1 illustrates an example of LinC items.

Item 1. Consider the statement and argument below:

Statement: For all $x > 0$, $x^3 - x^2 + x > 0$.
Argument: Suppose there exists $x > 0$ such that $x^3 - x^2 + x \leq 0$. Then, since $x > 0$ and $x^3 - x^2 + x \leq 0$, $x^2 - x + 1 \leq 0$. However,

$$x^2 - x + 1 = \left(x^2 - x + \frac{1}{4}\right) + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0.$$

1. Check the most appropriate one about the statement.
   (a) The statement is true.
   (b) The statement is false.
   (c) We cannot determine if the statement is true or false.
2. Check the most appropriate one to describe what this argument attempts to prove.
   (a) This argument attempts to prove the statement is true.
   (b) This argument attempts to prove the statement is false.
   (c) We cannot determine if this argument attempts to prove the statement is true or false.
3. Check the most appropriate one to describe the validity of this argument.
   (a) This argument proves properly what it attempts to prove.
   (b) This argument does not prove properly what it attempts to prove.
   (c) We cannot determine whether or not this argument proves properly what it attempts to prove.

Figure 1 An example of the LinC items

In each LinC item, we first asked students about the truth value of the given statement before asking them to evaluate the accompanying argument. However, students were allowed to change their answers to any of these three questions before moving to the next LinC item. As a result, students could determine or revise their determination of the truth value of the statement after engaging with the second and third questions about the accompanying argument. That is, students had opportunities to self-organize their thinking to have no logical contradiction among their responses to the three questions given in each LinC item. Thus, if there is a logical contradiction among a student's response to the three questions, we interpret the student's response as displaying logical inconsistency.

For example, a student might initially believe that the statement is true. Still, after reading the given argument about the statement, the student might construe and accept that the argument proves properly the statement is false. Suppose the student recognized the logical contradiction between their initial evaluation of the statement as true and their subsequent validation of the argument that proves the statement false. In that case, they may reconsider either their evaluation of the statement as false or their validation of the argument to maintain non-contradiction in their responses to the LinC item.

However, some students may not detect a logical contradiction even if it exists in their responses. Then, they would not experience psychological pressure to self-organize their thinking to have no logical contradiction, and consequently, they would not change their responses. In this case, the presence of logical contradictions in students' responses suggests that the student exhibits logical inconsistency. That is, the students cannot self-organize their thinking to have no logical contradiction. The three questions in each LinC item allow logical inconsistencies in students' responses to manifest if they exist. The LinC instrument's reliability is acceptable with Cronbach's α=0.686 (*n*=205).

Data Collection

In this study, we collected data via an online survey utilizing the LinC instrument. The survey was administered at two large public universities and one small liberal arts college in the United States from fall 2019 to spring 2022. We obtained responses from a total of 205 undergraduate students who dedicated sufficient time to complete the LinC instrument. Among them, 86 participants were recruited from in-person classes during fall 2019, fall 2021, and spring 2022. The rest of the participants (119) were from online classes during spring 2020 and fall 2020, coinciding with the COVID-19 pandemic, during which educational institutions experienced a rapid shift towards remote and online teaching modes.

Data Analysis

Table 2 provides a comprehensive list of all possible logical inconsistencies that could arise from evaluating a statement and validating an argument about the statement. This table serves as a valuable reference tool for analyzing logical inconsistencies in students' responses to the LinC instrument and identifying patterns in students' responses.

Table 2 All instances of responses identified as displaying logical inconsistencies to the LinC items

LinC Type	Response to Question (1)	Response to Question (2)	Response to Question (3)
aba	The statement is **True.**	The argument attempts to prove the statement is **False.**	The argument proves **properly** what it attempts to prove.
aca	The statement is **True.**	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.
baa	The statement is **False.**	The argument attempts to prove the statement is **True.**	The argument proves **properly** what it attempts to prove.
bca	The statement is **False.**	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.
caa	We **cannot determine** if the statement is true or false.	The argument attempts to prove the statement is **True.**	The argument proves **properly** what it attempts to prove.
cba	We **cannot determine** if the statement is true or false.	The argument attempts to prove the statement is **False**.	The argument proves **properly** what it attempts to prove.
cca	We **cannot determine** if the statement is true or false.	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.

In order to examine the first research question, we report descriptive statistics broken down at the item level. To examine the second research question, we considered a level of proof experience (in terms of the number of courses) as a predictor of the LinC overall score. We created a linear model and a hierarchical linear model (students nested within teachers), finding that the nesting added no explanatory power. Finally, for the third research question, our analysis focused on a subset of the data of 86 participants. This subset was drawn from the data collected during which classes were delivered through an in-person teaching mode. Note that the year 2020 coincided with the COVID-19 pandemic, and we acknowledge that the experience, perceptions, and academic performance of participants in online teaching mode during the pandemic may have been influenced by the unique circumstances and challenges posed by the pandemic. Focusing on the subset of participants who experienced in-person teaching, we aimed to mitigate the potential confounding variables influencing the third research question.

Results

RQ1: To what degree do students demonstrate logical consistency across a set of statements and accompanying arguments?

Of the participants ($n = 205$), 81% (187 students) displayed logical inconsistencies at least once while responding to the 20 items in the LinC instrument. Among those who showed logical inconsistencies, 73% displayed them at least twice, 64% at least three times, 53% at least four times, and 40% at least five times. These findings suggest that many undergraduate students in mathematics courses display logical inconsistencies in their thinking. These findings also highlight the need for further research and instructional interventions to address this issue.

RQ2: Is there a correlation between undergraduate students' logical consistency and their exposure to proof-oriented mathematics courses?

In order to explore this relationship, we developed a series of linear models. The first model included both in-person and online semester data. TeachingMode is 1 when online and 0 when in person. ProofExperience takes on a value of 0 for no-proof courses, 1 for one proof course, and 2 for two or more proof courses. We estimated:

$$LinC\ Score = 18.22 - 3.92 * TeachingMode + 0.13 * ProofExperience$$

finding that TeachingMode was significant with $p < .001$, but ProofExperience was not significant, $p = 0.58$. Since TeachingMode plays an unclear role (perhaps relating to modality or pandemic effect), we also ran a model on just the in-person data, finding similar results. The estimated coefficient for proof experience was 0.05 with a $p = 0.89$. These results were rather robust, with proof experience not being significant in any model our team ran. This indicates a surprising result: Logical inconsistency does not appear related to the amount of proof and formal representation system exposure.

RQ3: Do specific contexts or variables influence the likelihood of undergraduate students exhibiting logical inconsistencies (LinC)?

Because the prior analysis showed that teaching mode had a substantial impact on students' performance, we focus this early preliminary exploration of RQ3 on the in-person subset. Of the participants drawn from the in-person teaching mode ($n = 86$), 57% (49 students) displayed logical inconsistencies at least once while responding to the 20 items in the LinC instrument. Among those who showed logical inconsistencies, 40% displayed them at least twice, 30% at least three times, 15% at least four times, and 5% at least five times.

We examined if these students are more likely to display logical inconsistencies with certain types of statements or arguments. In particular, we analyzed the student responses considering the statements' complexity and the frames of the arguments as the factors that might affect students' logical inconsistencies. In Tables 3 and 4, the LinC frequency for each item represents the number of students displaying logical inconsistency in their responses to that specific item.

LinC and Statements' Logical Complexity. We may assume that students would have greater difficulty understanding mathematical statements with more complex logical structures, such as multiply quantified statements. Similarly, we might anticipate students encountering logical inconsistencies more like in statements with greater logical complexity. We might expect more students to display logical inconsistencies when engaging with multiply quantified statements in the LinC items 11-20 than statements involving a single quantifier in the LinC items 1-10. However, our analysis yielded unexpected results. Students' responses to LinC items contained logical inconsistencies, regardless of the logical complexity of the statements. Over 10% of the students displayed logical inconsistencies on each of the LinC items, including those with complex statements involving two quantifiers (items 11, 15, 17, and 20) and even those with relatively simple statements involving one quantifier (items 1, 2, 6, and 9) as detailed in Table 3. This finding suggests the frequency of logical inconsistencies of students does not significantly differ with respect to the logical complexity of the given statements.

LinC and Frames of Arguments. On the other hand, the type of argument frames points to potential differences in the frequency of logical inconsistencies. We observed a higher occurrence of logical inconsistencies when the argument was framed using proof by contradiction instead of direct proofs (see Table 4). To be specific, over 10 % of the students displayed logical inconsistencies on all but one LinC item with the seven arguments framed using proof by contradiction (e.g., LinC item 11), whereas none of the LinC items with the four arguments framed using direct proofs (e.g., LinC item 8). We can conjecture that the contradiction frame may relate to how students attend to logical consistency.

Table 3 LinC Frequency by Statement Types (n=86)

Item	LinC Frequency	Statement	
		T/F	Quantifier
1	10	T	$\forall$
2	13	F	$\forall$
3	0	T	$\forall$
4	5	F	$\forall$
5	1	T	$\exists$
6	9	T	$\exists$
7	4	T	$\exists$
8	4	F	$\exists$
9	10	F	$\exists$
10	5	F	$\exists$
11	15	T	$\forall\exists$
12	2	T	$\forall\exists$
13	5	F	$\forall\exists$
14	6	F	$\forall\exists$
15	10	T	$\exists\forall$
16	3	T	$\exists\forall$
17	10	T	$\exists\forall$
18	2	F	$\exists\forall$
19	8	F	$\exists\forall$
20	12	F	$\exists\forall$

Table 4 LinC Frequency by Argument Types (n=86)

Item	LinC Frequency	Argument		
		Attempt	Validity	Frame
1	10	Prove	Valid	Contradiction
6	9	Prove	Valid	Contradiction
11	15	Prove	Valid	Contradiction
15	10	Prove	Valid	Contradiction
9	10	Disprove	Valid	Contradiction
14	6	Disprove	Valid	Contradiction
20	12	Disprove	Valid	Contradiction
4	5	Prove	Invalid	Direct
16	3	Prove	Invalid	Direct
8	4	Disprove	Valid	Direct
19	8	Disprove	Valid	Direct
3	0	Prove	Invalid	Example
12	2	Prove	Invalid	Example
18	2	Prove	Invalid	Example
5	1	Prove	Valid	Example
7	4	Disprove	Invalid	Example
10	5	Disprove	Invalid	Example
17	10	Disprove	Invalid	Example
2	13	Disprove	Valid	Example
13	5	Disprove	Valid	Example

Conclusion & Discussion

Our study investigated the factors influencing logical inconsistencies among undergraduate students while evaluating mathematical statements and validating accompanying arguments. Initially, we expected students to struggle more with mathematical statements with greater logical complexity. However, our findings challenge this assumption. Surprisingly, students exhibited logical inconsistencies regardless of the logical complexity of the statements. On the other hand, students may exhibit different levels of logical inconsistencies when the accompanying argument is framed using proof by contradiction, in contrast to direct proof. The complexities involved in indirect proofs (e.g., Antonini & Mariotti, 2008) may be related to logical inconsistencies.

Mathematical reasoning relies fundamentally on avoiding logical inconsistencies, as even a single logical inconsistency can render an entire argument invalid. Our study reveals that a substantial number of undergraduate students displayed logical inconsistencies. This prevalence underscores a critical concern in mathematics education, particularly proof-oriented mathematics. We believe that identifying and addressing logical inconsistencies in students' thinking should be a top priority for promoting their learning of proof-oriented mathematics. Therefore, it is imperative for mathematics educators and researchers to carefully monitor students to see when logical inconsistencies exist in students' thinking and understand the factors that affect their occurrence.

Our study has limitations, including the specific context (precalculus and calculus) and sample size. Future research could explore logical consistency in diverse mathematical contexts and involve larger and more diverse student populations. Additionally, the high frequency of logical inconsistencies among undergraduate students highlights a potential gap in their mathematical reasoning skills. The frequency of these logical inconsistencies suggests a pressing need for further research and instructional interventions to address this issue. This is especially important as traditional proof courses do not seem to serve our students in developing their logical consistency. Investigating the effectiveness of specific instructional interventions in addressing logical inconsistencies is a promising avenue for future research.

References

Abelson, R. P., Aronson, E., McGuire, W. J., Newcomb, T. M., Rosenberg, M. J., & Tannenbaum, P. H. (1968). *Theories of cognitive consistency: a sourcebook.* Rand-McNally: Chicago.

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM*, *40*(3), 401-412. https://doi.org/10.1007/s11858-008-0091-2.

Cooper, J. (1998). Unlearning cognitive dissonance: Toward an understanding of the development of dissonance. *Journal of Experimental Social Psychology*, *34*, 562–575. https://doi.org/10.1006/jesp.1998.1365.

Cvencek, D., Meltzoff, A., & Kapur, M. (2014). Cognitive consistency and math-gender stereotypes in Singaporean children. *Journal of Experimental Child Psychology*, *117*, 73-91. https://doi.org/10.1016/j.jecp.2013.07.018.

Dawkins, P., & Roh, K. (2016). Promoting metalinguistic and metamathematical reasoning processes in undergraduate proof-oriented mathematics: A method and a framework. *International Journal of Undergraduate Mathematics Education*, *2*, 197–222. https://doi.org/10.1007/s40753-016-0027-0.

Ely, R. (2010). Nonstandard student conceptions about infinitesimals. *Journal for Research in Mathematics Education*, *41*, 117–146. https://doi.org/10.5951/jresematheduc.41.2.0117.

Festinger, L. (1957). *A theory of cognitive dissonance*. Evanston IL: Row Peterson.
Gawronski, B., & Strack, F. (2012). *Cognitive consistency: A fundamental principle in social cognition*. New York: Guilford.
Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.
Piaget, J. (1967). *The Psychology of Intelligence*. (M. Piercy, Trans.). London: Routledge & Kegam Paul Ltd.
Roh, K., & Lee, Y. (2011). The Mayan activity: A way of teaching multiple quantifications in logical contexts. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 21, 1-14. https://doi.org/10.1080/10511970.2010.485602
Roh, K., & Lee, Y. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: the case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education*, *3*, 34-68. https://doi.org/10.1007/s40753-016-0039-9.
Roh, K., & Lee, Y. (2018). Cognitive consistency and its relationships to knowledge of logical equivalence and mathematical validity. *Proceedings for 21st Annual Conference on Research in Undergraduate Mathematics Education,* San Diego, CA, 484-492.

Student Quantitative Understanding of Integration in Single Variable Calculus

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Student understanding of integration and key concepts from Single Variable Calculus and the role of infinitesimals was investigated. Analysis was done using a framework which highlights quantitative reasoning. Data showed rich understanding of integrals in multiple representations, with some gaps. Also revealed were connections between those concepts expressed over intervals in Algebra and expressed instantaneously in Calculus, and how conceptions of infinitesimals supported those connections. Implications for instruction are considered.

Keywords: Calculus, quantitative reasoning, infinitesimals, integral

Introduction & Literature Review

First semester Calculus is a critical course for all STEM majors, a required course containing foundational ideas for future study. Student success has been below desired levels for a long time, and efforts toward improving student success are extensive and ongoing (Bressoud, Mesa & Rasmussen, 2015). There has been much investigation into student understanding of Single Variable Calculus (SVC), exploring conceptions of limit (Oehrtman, 2009; Tall, 1992), derivative (Zandieh, 2000; Park, 2013), integral (Jones, 2015; Sealey, 2014), and the Fundamental Theorem of Calculus (Carlson, Smith & Persson, 2003; Radmehr & Drake, 2017). Some work has been done to connect those conceptions to a student's related prior conceptions (Thompson, 1994; Pustejovsky, 1999; Samuels, 2011). However, none of these studies investigated these questions for students learning Calculus using infinitesimals.

Recently there has been a growing call to revise Calculus instruction away from an Analysis-based approach using limits and toward a more quantitative approach (Augusto-Milner, Jimenez-Rodriguez, 2021). There have been a handful of documented attempts to do so with infinitesimals (Ely, 2021). Some research has theorized the underlying student thinking in such an approach (Ely & Ellis, 2018). None of these studies of infinitesimal-based Calculus instruction has examined how students conceive of important Calculus ideas. These gaps in the literature suggest the following research questions. (1) What is a characterization of student understanding of integration and key Calculus concepts after one semester of Calculus taught with infinitesimals? (2) How are understandings of Algebra concepts in discrete form used to generate understandings of related Calculus concepts in instantaneous form?

Framework

To analyze student understanding of Calculus, an approach oriented towards quantitative reasoning (Thompson & Carlson, 2017) was used. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017, p425), where a quantity is a measurable attribute combined with a way to measure that attribute. The quantities of SVC and the relationships between them can be described using the ACRA Framework (Samuels, 2022). The framework is summarized here briefly with a delineation of the quantities and some of their relationships. An *amount* is a real value, a *change* is a difference between two amounts, a *rate* is a ratio of two changes, and an *accumulation* is a sum of consecutive changes. One important relationship is that the product of a rate with a change (in the input variable) is a change (in the output variable), the *change equation*. Changes and rates over real intervals are encountered in a typical Algebra course;

changes and rates over arbitrarily small intervals first arise in SVC. A positive infinitesimal is a positive number which is smaller than all positive real numbers. Instead of using limits, one can interpret differentials as infinitesimals to describe the arbitrarily small quantities in Calculus. (For further details, see (Samuels, 2022).) This study was targeted specifically at student conceptions after taking Calculus I. In a previous study (Samuels, 2023), I focused on conceptions of instantaneous rate. In this study, I focused on conceptions of accumulation along with connected ideas and relationships.

Table 1. Part of the ACRA Framework for Quantities in Calculus (from (Samuels, 2022))

QUANTITIES	**Description**	**Real**	**Infinitesimal**
Amount	A magnitude or extensive quantity	x, y	ε
Change	A difference between two amounts	$\Delta y = y_2 - y_1$	dx, dy
Rate	A quotient of two changes (of different quantities)	$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$	$f'(x) = \frac{dy}{dx}$
Accumulation	A sum of consecutive changes	$f(b) - f(a) = \Sigma_i \, \Delta y_i$	$f(b) - f(a) = \int_{x=a}^{x=b} dy$
RELATIONSHIPS			
Change Equation	The product of a rate with a change in one quantity is a change in the other	$\Delta y = m \cdot \Delta x$	$dy = f'(x)\, dx$
CONVERSIONS	An equation of amounts converts to one of infinitesimal changes or rates, and those two convert back and forth	$y = f(x)$ ⇒ $dy = f'(x)dx$ ⇕ $\frac{dy}{dx} = f'(x)$	

Methodology

In 2023 Spring the author taught a Calculus I course at a northeastern college designed to use infinitesimals instead of limits. Two months after the end of the course, a clinical semi-structured interview (Hunting, 1997) was conducted with one of the students, who had volunteered. A script was prepared beforehand with questions on change, rate, accumulation, and the relationships between them, in both precalculus and calculus contexts, and in multiple representations (verbal, numerical, graphical, symbolic). Flexibility was allowed for follow-up questions which might be suggested in-the-moment by student responses. The interview was video recorded, transcribed, and coded using the framework.

Results

Travis [a pseudonym] was first asked "without Calculus, what's an example where we could talk about change?" He explained that "you could have two different amounts, and you basically subtract one from the other to get change". He then described an example involving toys produced, made a table of values, calculated various changes, and drew the graph in Figure 1a. When asked about rate, he expressed it as the ratio $\Delta y/\Delta x$ and calculated it over several different intervals (see Figure 1b).

When asked about accumulation, Travis said that "it's the sum of each individual change in x or y value." For the question, "If a plant grows 3in/week for 2 weeks, then 2in/week for 4 weeks, then 1in/week for 6 weeks, how much did it grow?", his calculations are in Figure 2a. He initially wrote the numerical calculations, and when prompted for "a more abstract notation" he

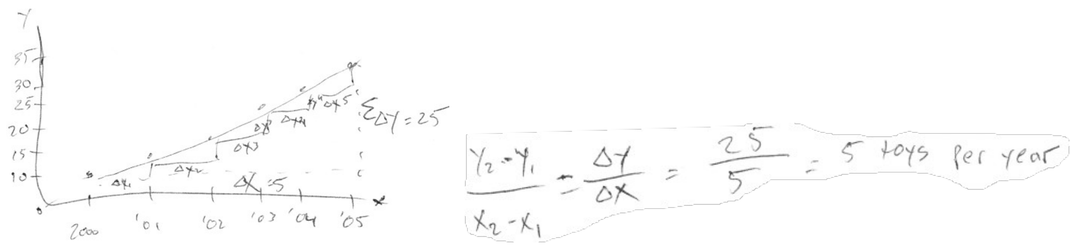

Figure 1. Travis' example to explain change and rate on a real interval (a) Travis' graph (b) Travis' calculations

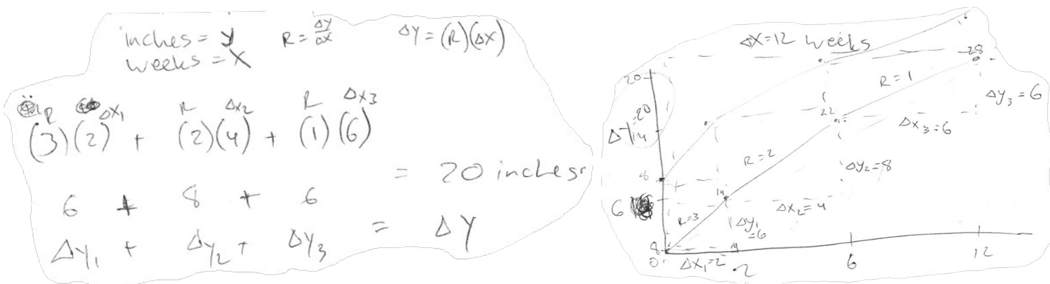

Figure 2. On a question to calculate an accumulation with a finite sum (a) Travis' calculation (b) Travis' graph

added more lines, expressing the accumulation as "$R{\cdot}\Delta x_1$ + $R{\cdot}\Delta x_2$ + $R{\cdot}\Delta x_3$" ("*R* for rate") and "$\Delta y_1 + \Delta y_2 + \Delta y_3 = \Delta y$".

Travis was asked to make a graphical representation of his calculations, and his work is in Figure 2b. It includes changes in *x* (e.g. Δx_1=2), changes in *y* (e.g. Δy_1=6), rates (e.g. R=3), and total changes Δx=12 and Δy=20.

Next, questions involving Calculus were asked. When asked "What is a situation where you would need calculus to talk about the slope or the rate?", Travis replied "You need calculus when you do not have a constant rate on a line, and you want to find the instantaneous rate at a certain point." He elaborated by creating the example $f(x) = x^2$ and then calculating $dy/dx = 2x$, which at the point (4,16) was 8 (see Figure 3b). In the following excerpt, he describes his reasoning.

Interviewer: Can you indicate on this graph [Figure 3a] what a *dx* is?
Travis: Sure. You would have to zoom in really closely on the point [draws a circle], and then you would have a little *dy* and a little *dx*. [draws and labels two line segments]
Int: And what do you mean by little?
T: Infinitely small. ...it's a little hard for me to describe it, but zoomed all the way in on that point, you get a line. The slope of that zoomed-in line is your infinitesimal rate.

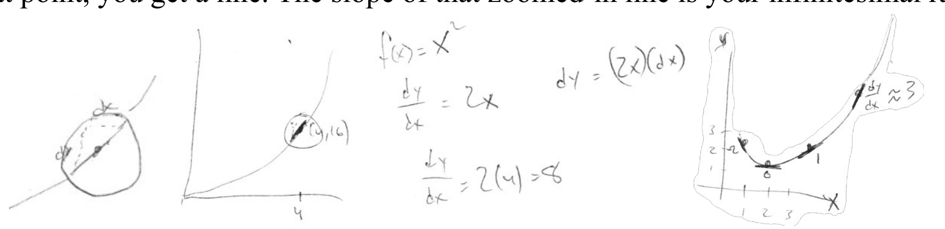

Figure 3.(a) Travis drew dx and dy (b) Travis created a function, graphed it, calculated the derivative and dy (c) Travis drew the graph, marked several points, and for each drew a short tangent line and a slope estimate

Given the graph of a function, Travis was asked to draw tangent lines and estimate the derivative at several points. He did this correctly, as shown in Figure 3c. When asked to solve for *dy* in his example, he wrote "$dy = 2x\ dx$". When asked what it means, he said "It means that the infinitesimal change of *y* equals the infinitesimal change in *x* multiplied by the rate."

When asked about infinitesimals, Travis said "you only have a point to work with, so you need to zoom in so far on that point in order to get a line... And the infinitesimal basically gives you a way to use real numbers to assign values to the changes going on at that infinitely small point." He said it is not possible to write down an infinitesimal number, so we use the symbol ε.

Travis was given an application in which the height of a ball was $h = 27t - 10t^2$. When asked for the rate, he calculated $dh/dt = 27\text{-}20t$; when asked for the units he said "meters per second". When asked to compare Δh and dh, he said "because you're dealing with *dh* and *dt*, these are infinitely... small changes in the ball's height, because you're only doing it in relation to an infinitely small change in the time. If it was going to be Δh and Δt, it would be a bigger change. So your Δt may be 2 seconds, in which case your Δh would be a bigger number."

When asked to compare *dh* versus *dh*/*dt*, he said that "*dh/dt* is a rate, and it is...the rate at which the height of the ball is changing in relation to time... And *dh* by itself is just the change in height of the ball, not the rate at which that height is changing."

When asked about integration, Travis described it as "calculating the sum [of changes] between two points". As to types of accumulation, when asked "is there a notation for doing" the sum in a finite accumulation, Travis responded "the sigma thing". When asked to compare with the "Calculus version of the same thing", Travis responded "this is going to be my integral".

When given a slope field and asked about its meaning, he said "these are all representations of, I guess, instantaneous rates of whatever the function is of these graphs" and each dash "tells you the direction that the function is going to be going at that point."

Travis was asked an application question. "A rope has a certain amount of mass *m* over its length *x*. Suppose it has $2x$ kg/ft at the *x*-foot mark. How much mass does it have between the 1-foot mark and the 4-foot mark?" Travis' correct calculations are in Figure 4a. He was also asked to sketch a graphical representation of the solution, and his inscriptions are in Figure 4b. It includes a slope field, a solution curve, with Δx=3ft and Δm=15kg marked. (He noted that "obviously my scale is all messed up".) He explained his work in the following excerpt.

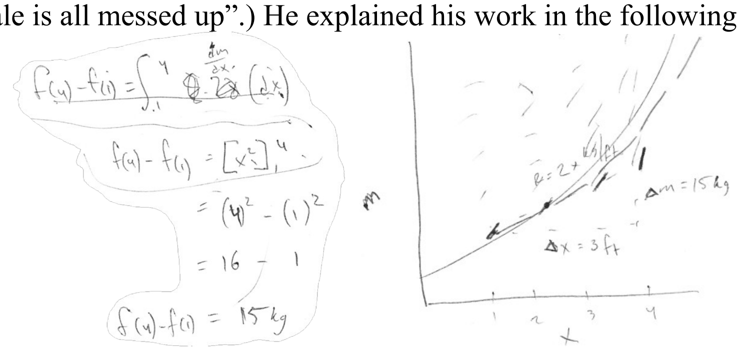

Figure 4. For an application to find mass given density (a) Travis' calculation (b) Travis' graph

Int: The first expression you wrote down, where did that come from?

T: You're asking from the 1 foot to the 4 foot mark. So those are my limits...

Int: ...in the original expression, it's the integral from 1 to 4, $2x\ dx$. So tell me how that's going to give you $f(4) - f(1)$ … Tell me what that means and how that's the thing that you need.

T: This [pointing to $f(4) - f(1)$] is my Δy, for lack of a better phrase. And then this [pointing to the right side] is my rate times my Δx...I know that by multiplying my rate and my *dx* will give me my *dy*... so the $2x$ times the *dx* that gives you a *dy*.

Int: ...what about the actual squiggle, the actual integral mark itself? Is that telling you something specific or is just something that you write down?

T: The squiggle is like the letter S stands for sum.
Int: And what is it that you're adding up? If it's a sum, what are you adding up?
T: The values of the mass between 1ft and 4ft. Yeah, the value of mass at 1ft plus the value of mass at 2ft, or I guess at every infinitesimal increment between 1 foot and 4 foot to give you the amount that it has between those two points.

While explaining his calculation, Travis noted that he was expecting an x to appear in the answer. He explained his confusion about it, eventually sorting it out and referencing the FTC.

> I'm thinking I'm... trying to figure out where this x mark on the rope is, but that's not what you're asking… There would be other times where you would be asking, how much mass does it have between the 1ft mark and the x foot mark? … I would even have been happier having the limitations from 1 to x and be solving for $f(x) - f(1)$ … This is very simple.

In the excerpt below, Travis described some infinitesimals and their quantitative characteristics.

Int: What does *dm* represent?
T: An infinitely small change in the mass of the rope.
Int: Okay. And in the course of doing this calculation or this problem, how many *dm*'s are you going to encounter?
T: Infinitely many.
Int: Okay. And *dm/dx*, what does that represent?
T: That would be my 2*x*. That would be my, that's my *m*'(*x*), my instantaneous rate.

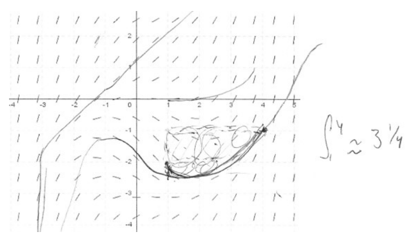

Figure 5. Travis represents the integral on a slope field.

When asked to use a slope field to find the integral from x=1 to x=4, Travis said "If you have an endpoint at four and endpoint of one, you have this little area here of your change in y from one to four and your change in x. I would shade in this and say that's your integral from one to four… your Δy accumulation would be one and a half-ish and your Δx would be three… Basically, you're going to find this area [counting boxes] one, two, three, and like a quarter." His inscriptions are in Figure 5.

In the excerpt below, he discussed two different graphical methods of representing integrals.

> When I was taking the class ... I was going to tutoring, and [he] explained [integration] to me using area…I was like, okay ... but slope field ... at the time I don't remember even linking it to integrals, which obviously now I see how it does. But all I remember was like, wow, this is easy. I just draw a line going along with the other lines... this is simple.

Discussion

Travis demonstrated a detailed understanding of the quantities of Calculus. He gave clear explanations of the concepts of change, rate, and accumulation, and made specific connections between them. For change, he represented a real change numerically as a difference between two real numbers and graphically as a line segment; he represented an infinitesimal change both symbolically (using *dx*) and graphically (using zooming). For rate, he represented discrete rate numerically and symbolically as a ratio of two changes and graphically as the slope of a straight line. For an infinitesimal rate, he calculated one using formulas, represented it graphically as a short tangent line (both on the original graph and after an infinite zoom, and subsequently as a slope field), and described it an instantaneous rate, using both prime and differential notation. (Although he did not explicitly refer to an infinitesimal rate as the ratio of infinitesimal changes, he frequently manipulated it as though it were.)

Travis was able to perform all 3 conversions, from an amount equation to an infinitesimal change or rate equation, and between the last two. From an equation with "y=", in addition to calculating the derivative, he wrote down the infinitesimal change equation with "dy=". Further, he was able to assign a quantitative meaning to each term - the infinitesimal change in x, the infinitesimal change in y, the infinitesimal rate - and interpret the equation as a relationship between those quantities, both in an abstract scenario and an application where he assigned correct units to each. Also, he distinguished between infinitesimal rate and change, in meaning and use. He converted between the instantaneous change and rate equations by multiplying or dividing by dx. In a standard Calculus course, students are told both that one cannot manipulate in that way and individual differentials have no meaning, as well as to manipulate that way and use isolated differentials in the procedure of integration by u-substitution. The inherent contradiction has been noted before (Ely, 2021). After Calculus instruction with infinitesimals, Travis had a coherent concept image (Tall & Vinner, 1981) of these equations, in meaning and in symbolic manipulation.

Travis was able to describe accumulation using the quantities of change and rate. In the finite case, he stated that it was the finite sum of changes; given a piecewise constant rate, he was able to calculate the accumulation using arithmetic, and graph it as a piecewise linear graph. He also stated that the accumulation was a sum of infinitely many infinitesimal changes; given a formula for instantaneous rate he calculated the accumulation with an integral, he graphed it as a solution curve in a slope field. He also (after some confusion) described the calculation both for a constant upper boundary to produce a constant answer and a variable upper boundary to produce a function answer (the FTC).

Travis expressed accumulation as a finite sum of changes in the output variable as well as a sum of rate times input variable change terms ($f(b) - f(a) = \Sigma_i \Delta y_i = \Sigma_i r_i \cdot \Delta x_i$). For integration, he made a parallel infinitesimal statement, $f(b) - f(a) = \int f'(x)\, dx$. Although he did not explicitly write the formula $\int dy$, in one application he did describe the integral which calculated the total mass as a sum of infinitesimal masses, and separately described how dy can be replaced with $f'(x)\, dx$. For integration, previous research has shown that applications are very difficult for students (Wagner, 2018; Jones, 2015), and that quantitative understanding is rare (Jones, 2015; Fisher & Samuels, 2016). Similar quantitative meanings for integration have been discussed in previous studies (Oehrtman & Simmons, 2023; Jones, 2015), but instruction relying on limits results in dx having no rigorous definition.

Travis made strong connections between the real and infinitesimal versions of the same concepts. For change, he indicated Δx as a horizontal segment on the original graph, and he indicated dx as a horizontal segment on the graph visible after you "zoom in really closely… [it's] infinitely small". In the ball problem, he referred to dh and dt as "infinitely small changes" whereas "Δh and Δt, it would be a bigger change". For rate, Travis drew (piecewise) lines for (piecewise) constant slopes, labeling slope values. For instantaneous rate, he described "zooming in" at a point to reveal a straight graph; he drew that straight tangent line on both the original graph and a zoomed-in graph, also marking the slope value at a point. For accumulation, he wrote the formula as a finite sum with Σ, and he used ∫ as an infinite sum of infinitesimals, and integration would be needed when the rate is non-constant. He noted that there would be "infinitely many" *dm*'s and *dx*'s in calculating the integral for mass. He made a direct comparison between Δy and dy in the context of rate, change, and accumulation. In various contexts, Travis made statements about real quantities and incorporated infinities and infinitesimals to express corresponding statements. Thus he exhibited *transfinite thinking* (Samuels, 2023) underpinning his Calculus conceptions. Learner connections between finite and infinite scenarios have been investigated previously (Oehrtman, 2009; Mamolo & Zazkis, 2008), but only recently was this term rigorously defined, or were the connections examined for their contributions to productive conceptions.

One initial point of confusion for Travis concerned whether or not the solution to an integration application should contain a variable. Previous research into the Fundamental Theorem of Calculus (FTC) has shown the extreme difficulty students have in recognizing the upper bound as the variable (Thompson, 1994). In this investigation, Travis eventually remembered the two relevant types of question, a fixed upper bound or a variable upper bound, even commenting that he would be "happier" with the variable problem, exhibiting knowledge of the FTC and the functional nature of its formula.

Another point of confusion was the graphical representation of the integral. Working from a slope field, Travis drew a solution curve, then shaded an area and stated that was the result of the integral. He conflated the two graphical representations, the solution curve in a slope field, and the area under the integrand function. This is a conception not previously documented, partly due to the dearth of research on Calculus instruction with infinitesimals. Travis discussed one possible cause, working with a tutor whose instruction did not align with the approach of the class. Previous research has shown that conflicting instruction can cause productive notions of infinitesimals to erode (Simmons et al., 2022). Having two separate graphical representations for the same situation or calculation is inherently confusing (the integrand plays different roles). One implication for teaching is the need to emphasize this distinction as much as possible. The optimal pedagogy is a topic for future inquiry. Other directions for future research include investigating quantitative conceptions of the rate equation, and the FTC.

It is important to point out that data in this study was drawn from a single student. I cannot claim that any observations generalize to the entire population. Rather, the analysis sheds light on how a student might form conceptions about Calculus and integration using infinitesimals. Due to the relative novelty of the context, this signifies a contribution to the literature.

Conclusion

In this study, I examined an understudied area of student understanding in Single Variable Calculus, conceptions of integration after one semester of Calculus I taught using infinitesimals instead of limits. The data showed rich connected student conceptions of integration, supported by conceptions of change and rate, in multiple representations. These conceptions were expressed both for real quantities and infinitesimal quantities, connected by the student via transfinite thinking.

Calculus instruction using infinitesimals is not typical practice. There is a growing body of evidence that it is a productive approach for students to make quantitative meaning in Calculus, and this study is a contribution in that direction. It also demonstrated how student knowledge of the quantities and relationships within Calculus, and their connection to prior knowledge, can be exhibited by students, and productively analyzed using the ACRA Framework.

References

Augusto-Milner, F. & Jimenez-Rodriguez, J. (2021). *Didactic contrasts between Calculus and Analysis.* Unpublished manuscript.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education III* (pp. 115-162). Providence, RI: American Mathematical Society.

Carlson, M., Smith, N. & Persson, J. (2003). Developing and connecting calculus students' notions of rate of change and accumulation: The Fundamental Theorem of Calculus. *Proceedings of the 2003 Joint Meeting of PME and PMENA*, 2, 165-172.

Ely, R. (2021). Teaching Calculus with infinitesimals and differentials. *ZDM*, 53, 591-604.

Ely, R. & Ellis, A. (2018). Scaling-continuous variation: A productive foundation for calculus reasoning. In S. Brown, G. Karakok, K. Roh & M. Oehrtman (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1180-1188), San Diego CA.

Fisher, B. & Samuels, J. (2016). Student conceptions of definite integration and accumulation functions. In T. Fukawa-Connelly, N. Infante, M. Wawro, S. Brown (Eds.) *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 761-767), Pittsburgh PA.

Hunting, R. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior,* 16(2), 145-165.

Jones, S. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behavior*, 38, 9-28.

Mamolo, A. & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education*, 10(2), 167-182.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, 44(5), 624–640.

Pustejovsky, S. (1999). Beginning Calculus students understanding of the derivative: Three case studies. Unpublished PhD dissertation, Marquette University.

Radmehr, F. & Drake, M. (2017). Exploring students' mathematical performance, metacognitive experiences and skills in relation to fundamental theorem of calculus. *International Journal of Mathematical Education in Science and Technology*, 48(7), 1043-1071.

Samuels, J. (2011). The relationship between learner characteristics and learning outcomes in a revised first-semester Calculus course. In L. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 666-674). Reno, NV.

Samuels, J. (2022). Calculus reconceptualized through quantities. In S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 876-884). Boston, MA.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *Journal of Mathematical Behavior*, 33, 230-245.

Simmons, C., Samuels, J., Reed, Z & Ely, R. (2022). A reemergence of a non-quantitative interpretation of the differential for definite integrals. In S Karunakaran & A Higgins (Eds.) *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 589-596). Boston.

Oehrtman, M. & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education*, 9, 36-61.
Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
Thompson, P. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, 26(2), 229-274.
Wagner, J. (2018). Students' obstacles to using Riemann sum interpretations of the definite integral. *International Journal of Research in Undergraduate Mathematics Education*, 4, 327-356.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (pp. 103–127). Providence, RI: American Mathematical Society.

Instructors' Use of Student Projects in Postsecondary Quantitative Reasoning Courses in Ohio

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Student projects in a postsecondary Quantitative Reasoning (QR) course can encourage students to think deeply about connections between a real-world situation and the corresponding mathematical or statistical model. These projects can help students collaborate and improve their 21st-century skills, such as critical thinking and oral and written communication. This paper reports how 13 QR instructors across two studies at 11 public postsecondary institutions in Ohio implemented student projects. We analyzed the course syllabus for each participating instructor, conducted at least one semi-structured interview, and observed their teaching using six Instructional Quality Assessment rubrics. Data revealed substantial variation in the implementation of student projects, resulting in varying opportunities for student learning. Our findings indicate that student projects are a critical variable in Quantitative Reasoning.

Keywords: Quantitative Reasoning, student projects, and project-based learning and assessment.

An entry-level postsecondary (gateway) QR course includes nontraditional mathematical goals for students (e.g., collaboration and oral and written communication using quantitative arguments) that aim to prepare them to solve problems in everyday and professional contexts (Mathematical Association of America [MAA], 1996; Stump, 2017). Policy documents and professional opinions call for methods of instruction and assessment that build the student QR competencies of interpretation, representation, calculation, analysis, assumptions, and communication (e.g., AAC&U, 2009; Boersma et al., 2011). They also emphasize the need for students to think deeply and critically about real-world phenomena and link them to relevant mathematics and statistics (Foley & Wachira, 2021).

QR instruction is relatively common to include projects requiring students to solve real-world problems and construct associated knowledge and skills (Lutsky, 2008; MAA, 1996). Learning via such projects provides constructivist and collaborative opportunities for students to grow academically and nonacademically (Harwood, 2018; Virtue & Hinnant-Crawford, 2019). Traditional mathematics assessments—quizzes, tests, and exams—do not address students' engagement in and learning through the projects and typically do not emphasize bona fide real-world problem-solving, collaboration, or communication. Professionals, including Virtue and Hinnant-Crawford, argue that students' learning in a nontraditional course that relies on projects can only be appropriately assessed through their engagement in similar opportunities, that is, project-based assessment.

Student Projects in Quantitative Reasoning

Project-based learning (PBL) and project-based assessment (PBA) address issues related to implementing several formative assessment strategies in mathematics courses, which may be challenging to implement in a typical mathematics course (Chanpet et al., 2020). PBL and PBA tasks include several nontraditional instructional and assessment tools, such as observation, performance tasks, portfolios, presentations, taxonomies of experts, reflection writing, and peer- and self-assessment in different contexts (Noonan & Duncan, 2005; Tal et al., 2000). These tools also allow students to enhance and demonstrate their understanding of mathematical relationships in several real-world phenomena.

Quantitative Reasoning courses and *project-based learning and assessment* (PBLA) share many things, including implementing real-world tasks to help students solve problems from their everyday contexts. Using rich tasks in projects for instruction and assessment in QR courses can help instructors understand a student's misconceptions and determine a student's learning gaps (Kish, 2017). QR and PBLA constitute a social constructivist approach requiring students' active engagement and critical inquiry into genuine phenomena. Students' coordination, collaboration, and communication are their essential components. Several recent studies have identified group projects as a critical variable for QR instruction and assessment (Budhathoki, 2022; Budhathoki et al., 2024; Foley et al., 2023). Therefore, PBLA can be instrumental in enhancing and assessing student learning in QR courses.

National and state-level recommendations (e.g., Leitzel, 2014; MAA, 1996) emphasize active student engagement and the critical inquiry of real-world phenomena. However, little or nothing is said about assessing student learning or using projects. Moreover, most gateway mathematics instructors lack experience using projects in their teaching. QR instructors in Ohio (the research contexts reported in this paper) and at the national level often say their unawareness about the types of projects to select, the number of projects to implement during a semester, ways to engage students in a project, and learning outcomes to focus on through such projects. Consequently, there are varied QR instructional and assessment practices in Ohio.

This paper reports the combined findings of two studies conducted to explore the formative assessment practices of QR instructors in Ohio public postsecondary institutions; the first was a pilot study, and the second a dissertation study. However, this paper focuses on reporting only QR instructors' implementation of student projects. The findings reported here may be significant to novice and experienced instructors who strive to implement projects in their QR teaching and similar other freshman-level mathematics courses.

Multiple Case Study and Cross-Case Analyses

The Ohio Department of Higher Education (ODHE) created common student learning outcomes and transferability across its 36 public community colleges and universities for all gateway mathematics courses, including QR (ODHE, 2015). Foley and Wachira (2021) and Leitzel (2014) have touted QR as the most appropriate gateway mathematics course for non-STEM majors. To earn transferability, the ODHE (2015) requires that critical thinking be the centerpiece of QR, that a QR course develops the competencies of interpretation, representation, calculation, analysis, assumptions, and communication, and that a QR course focuses on three main content areas: numeracy, probability and statistics, and mathematical modeling. However, the ODHE (2015) does not specify how to assess students' achievement of such learning outcomes. However, this is a part of the process for a given institution to have its QR course certified to be transferable across the state.

The researcher employed a multiple case study in both examinations. He defined a case in the pilot study as an instructor who taught at least one section of QR courses in public postsecondary institutions in Ohio during data collection, which he narrowed to the instructor teaching at least one section of Ohio Transfer 36-approved QR courses for the dissertation study. The researcher employed a purposeful selection of cases to recruit 6 instructors (3 male and 3 female) from 2 universities and 3 two-year colleges during the Spring of 2020 for the pilot study and 8 instructors (2 male and 6 female) from 3 universities and 5 two-year colleges during the Summer and Fall of 2021. However, 1 female instructor from a two-year college dropped off from continuing her participation, still giving consent to use the data thus far. Also, 1 male instructor from a two-year college participated in both studies, totaling 13 QR instructors from 11 public postsecondary institutions in Ohio. The instructors had a wide range of positions and experience; they ranged from graduate students to full professors. Their experiences in QR teaching ranged between one semester and six years. Also, some of them were working as course coordinators at their institutions.

The researcher conducted one qualitative interview with each instructor and analyzed their course syllabus in the pilot study. However, he conducted at least two virtual or in-person interviews for the dissertation study, analyzed course documents including course syllabus and teachers' artifacts, and observed two consecutive classes using 6 IQA rubrics (a) *Rigor of teachers' questions*, (b) *Accountable talk*, (c) *Clarity and detail of expectations*, (d) *Communications of expectations*, (e) *Teachers' press,* and (f) *Students providing* (Boston, 2019).

The data reported in this paper mainly include that obtained from the instructors' course syllabi analyses. The researcher categorized the instructors' project-related data into several categories, like the number of projects implemented during a semester, weights to the total course grade that instructors provided for projects and associated presentations, the content domains covered through the projects, whether the projects were individual or group assignments and the time they assigned the projects to students. The researcher transcribed the verbal data for the interview data and created codes using value coding focusing on the instructors' intrapersonal and interpersonal perceptions and experiences in implementing projects in this course (Saldana, 2016). Then, he developed themes by merging similar and related codes. He used the cross-case analysis technique to explore commonalities and differences in the instructors' actions, activities, and processes regarding project implementation (Stake, 2006).

Results and Discussions

The data analyses revealed exciting information about the instructors' use of student projects in teaching QR. Though all 13 instructors included projects in their course syllabi, they had mixed perceptions about using projects and associated presentations in this course. They also had a great deal of variation in their types, the number of projects, and how to implement them. Table 1 and Table 2 represent information obtained from the analysis of the course syllabi from the pilot study and the dissertation study, respectively. Both tables consist of the instructors' plans

Table 1: Use of Student Projects by Instructors in the Pilot Study

Instructors	Number of Projects	Weights for Projects	Content Domains	Individual or Group	Time of Implementation
Amanda	3	45%	Financial literacy Statistics Modeling	Group	One at each third of the semester
Cole	1	10%	Media literacy	Individual	The second half of the semester
Dani	3	40%	Business math Statistics Modeling	Group	One at each third of the semester
Evan	1	10%	Statistics	Individual	End of the semester
Fawn	1	4%	Comprehensive	Group	End of the semester
Zach	1	10%	Media literacy	Individual	The second half of the semester

for implementing projects, including the frequency of the projects; weights given to projects to determine student grades; content domains emphasized through the projects; nature of the projects; and the implementation time. Zach is included on both tables as he participated in both studies. Still, Zach had different project implementations during the two semesters of his participation; an increment in his number of projects and associated weights to the course grades may indicate QR instructors' increasing use of student projects in public postsecondary institutions in Ohio.

Wide Range of Number, Weights, and Time of Implementation

The instructors provided a wide range of importance to student projects and associated presentations in their course syllabi. The weights of student projects and presentations to the total course grades ranged between 4% and 75% through 1 to 8 projects a semester; the average number of projects a semester was 13, with an average weightage of 33%. However, as discussed above, this data may change based on their implementation; in his dissertation study, the researcher discovered that the average weight for student projects based on the 8 instructors' implementation decreased from 28.8% of total course grades to 27%, but with an increase in student presentation weights, from 12% to 15%. Out of 13 instructors reported in this study, 2 instructors (Evan & Fawn) provided a little weight to their projects to determine the course grades; they employed only one project during the semester and, like a traditional homework problem, required students to solve the given problems with the provided information. On the

Table 2: Use of Student Projects by the Instructors in the Dissertation Study

Instructors	Number of Projects	Weights for Projects	Content Domains	Individual or Group	Time of Implementation
Ashley	1	30%	Student choice	Group	Over the semester
Clara	3	40%	Personal finance, Statistical modeling	Group	One at each third of the semester
Gordon	2	30%	Financial Literacy, Statistical modeling	Group	One in each half of the semester
Julie	8	40%	Recent topics	Individual	Every other week.
Kande	6	75%	Recently topics	Individual	Every other week
Susan	4	50%	Numeracy, Statistics, Modeling, Comprehensive	Group	One in about a month.
Yara	4	60%	Financial literacy, Statistics, Media literacy, Modeling	Two individual, two group	One in about a month.
Zach	2	20%	Media literacy, Personal Finance	Individual	One in each half of the semester.

contrary, the other 2 instructors (Julie & Kande) used 8 and 6 projects in a semester, offering 40% and 75% weights to the total course grades. Still, these two instructors also implemented their projects like a traditional homework problem, emphasizing students' abilities to solve the problem, not the QR competencies like interpretation, representation, analysis, assumptions, and explanation. These 4 instructors did not have firm beliefs about using student projects in this course, but they had to put them in their syllabi as the Ohio Department of Higher Education required. However, some instructors heavily relied on student projects to determine course grades even though they did not have much weight on student projects. For example, Ashley and Susan from the dissertation study heavily relied on student projects. Though they had exams and quizzes in their course syllabi, they implemented such assignments as open, take-home group projects. They stated that they had to put exams and quizzes in their course syllabi as the mathematics departments in their institutions were traditionally required and required them to use exams and quizzes at least in some manner.

The instructors also had variations in the time they implemented the student projects, mainly depending on the number of projects during the semester. The instructors using multiple student projects generally divided the whole semester duration by their number of projects and used the results as the duration for each project. However, Evan and Fawn implemented their only one project at the end of the semester, like a test for a content domain or comprehensive exam. It is worth restating that these two instructors did not rely much on student projects. Likewise, though Julie and Kande implemented one project every other week, they used their projects as different names for their unit homework.

Content Domains

Even though the 13 instructors had variations in the number of, weights to, and time of implementation of their student projects, they had some commonalities in the content domains their projects covered. Notably, only 9 of the 13 instructors required students to work on any given quantitative situation, identify the available quantitative information, choose appropriate mathematical processes to solve the problems, make necessary assumptions, and derive conclusions. They provided open-ended quantitative situations, embracing the three content domains the Ohio Department of Higher Education suggested—*numeracy*, *probability and statistics*, and *mathematical modeling.* Their practices embrace all or any of the six QR competencies—*interpretation, representation, calculation, analysis, assumptions,* and *explanation* in the projects. The commonalities in content domains were more among the instructors who used more than one project during a semester. Notably, 7 instructors used more than one project during a semester, excluding the two who claimed to implement projects in each content domain. They all assigned a project in numeracy, budgeting, or personal finance. The statistical project was another popular choice; 6 of 7 instructors assigned multiple projects and 1 assigned only one project implemented statistical projects, requiring students to calculate and reason with statistics.

Similarly, 4 among the 13 instructors assigned mathematical modeling projects to students. In addition, 4 instructors included media literacy in their projects. In addition, 2 instructors, who had only one project, either let students choose any project they found interesting or provided a comprehensive worksheet; they also had chances to highlight some or all of the content domains discussed above.

The other 4 instructors, who implemented their projects like a traditional worksheet problem, generally emphasized students' ability to use recently learned mathematical knowledge and skills to solve problems. They usually provided all the required information, suggested appropriate mathematical steps, and required students to calculate answers. Their practices mainly emphasized 3 of 6 QR competencies—*interpretation, representation, and calculation.*

Collaboration and Communication

The majority of the 13 QR instructors emphasized student collaboration through their projects. Notably, 8 of the 13 instructors assigned some or all projects in groups and emphasized that students work together to understand quantitative situations, collect data, prepare reports, and present findings and conclusions. The instructors used various innovative approaches to ensure collaboration among students. Some instructors collected effort grade forms from each to determine how the team members contributed to the projects. Likewise, some required each member to submit individual reports, reserving rights to grade only one paper per group and assign grades to the whole group.

All QR instructors required students to submit a written report from their projects, but only 7 incorporated oral presentations. Depending on the nature of their projects, the instructors required students to prepare a detailed report individually or in a group or submit the worksheet. However, Cole and Zach needed students to prepare a report, develop QR activities out of their media literacy projects, and exchange with peers to seek mutual feedback.

The instructors' student presentation implementation was closely related to the nature of their projects, group or individual. Seven instructors implemented student presentations, and they all assigned group projects. Only the instructor who implemented group projects but not student presentations was the one who did not believe much in student projects. Still, there was a significant variation in the instructors' implementation of student presentations. While other instructors required students to present their project findings and conclusions, Gordan allowed students to present any interesting quantitative context. The four instructors who did not require students to present included the 4 instructors using their projects as worksheet problems and the 2 instructors from the same two-year college, Cole and Zach. Many of these 6 instructors argued that adding a presentation would add anxiety to students. They stated that students in QR courses often have low confidence in their mathematical abilities, and requiring them to present on a mathematical topic may adversely affect their learning.

Group Projects as a Critical Variable for Formative Assessment in QR

The dissertation study discovered group projects as a critical variable for formative assessment in Quantitative Reasoning courses. The 8 instructors in this study generally relied on group projects for students' learning with collaborative and collective endeavor; their average weight for group projects was 20.5%, which was 56.5% of their average collaborative assessments. They also had a strong positive association between collaborative assessments and group projects, $r = .87$. Five (5) of the 8 instructors used group projects, highlighting students' holistic development and ability to use the mathematical content in real-world problems through their group projects. They mostly used open-ended, ill-defined, and high cognitively demanding tasks in their group projects, emphasizing the reasoning competencies of QR, like analysis, assumption, and communication. They mostly gave less instruction and support and sought students' collaboration and communication to accomplish the projects. The group projects also had a close connection with student presentations. Only the instructors who assigned group projects required their students to present the findings and conclusions of the projects.

Group projects were also associated with the instructors' formative assessment strategies. Only the instructors using group projects asked a series of checking and advancing questions to check students' understanding, seek clarification, connect ideas, and further their learning and understanding, usually in a whole class setting. Likewise, only such instructors had sophisticated feedback practices; they provided formative feedback while students worked on their group projects and in-class activities, emphasizing clarifying students' misconceptions, furthering understanding, and fostering self-esteem and confidence. Likewise, the instructors generally orchestrated students' peer- and self-assessment only in the group projects and presentations. The peer- and self-assessment also highlighted the students' collaboration, communication, and reflective understanding abilities.

The group projects also connected to the instructors' scores in the IQA rubrics. Instructors who employed group projects scored higher than those who used only individual projects. For the 7 instructors in the dissertation study, the average score on the 6 IQA rubrics was 3.08 out of a possible 4, with at least 3 in each rubric, but the Accountable Talk. This study also suggested collaboration and communication as the premise and results for the group projects.

Conclusions

A Quantitative Reasoning course aims to develop students' quantitative and reasoning abilities to solve problems in their everyday and professional contexts. A typical QR instruction uses projects that include real-world tasks and performs as constructive and collaborative platforms where students work individually or in groups to foster their QR competencies. Therefore, many professional communities and policy documents explicitly or implicitly suggest instructors use projects as both instruction and assessment in QR courses.

Quantitative Reasoning instructors in public postsecondary institutions in Ohio greatly varied their plans for and implementation of student projects. They mainly varied in the number of projects they used in a semester, associated weights to determine course grades, the content domains covered by the projects, and the implementation, individual or in groups. There were significant differences among the instructors who used one project versus multiple projects in the semester. Likewise, in implementation, the instructors varied when they assigned student projects, the nature of such projects, opportunities for students to collaborate while working on projects, instructors and peer feedback practices, and the learning competencies emphasized through the projects. The instructors' student presentation and formative assessment practices mainly depend on the nature of their projects; only the instructors who employed group projects required students to present their findings and conclusions. The instructors' variations in their project implementation provided different learning opportunities for their students. Such variations may challenge the transferability of this course. Ten among 11 institutions represented here offered the *Ohio Transfer-36* approved QR courses. However, the instructors mostly used fermi projects, mainly covering the three content domains the state authority suggested–*numeracy, probability and statistics,* and *mathematical modeling.*

The discussion in this paper restated several findings of the first researcher's dissertation study: (i) a great deal of variation in QR instructors' use of student projects in public institutions in Ohio, (ii) group projects as a critical variable for formative assessment in QR. The group projects, supported by instructor autonomy, create opportunities for students to solve real-world problems, fostering their six QR competencies—*interpretation, representation, calculation, analysis, assumption,* and *communication* (AAC&U, 2009; Boersma et al. 2011), and (iii) collaboration and communication as the premise for and results from the group projects.

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References

Association of American Colleges and Universities. (2009). *Quantitative literacy VALUE rubric*. https://www.aacu.org/sites/default/files/files/VALUE/QuantitativeLiteracy.pdf

Boersma, S., Diefenderfer, C., Dingman, S. W., & Madison, B. L. (2011). Quantitative reasoning in the contemporary world, 3: Assessing student learning. *Numeracy: Advancing Education in Quantitative Literacy*, *4*(2), Article 8. http://scholarcommons.usf.edu/numeracy/vol4/iss2/art8

Budhathoki, D. (2022). *Formative assessment in postsecondary Quantitative Reasoning courses* (Doctoral Dissertation, Ohio University). http://rave.ohiolink.edu/etdc/view?acc_num=ohiou165903392100886

Budhathoki, D., Foley, G. D., & Shadik, S. (2024). Instructional decision making in a gateway Quantitative Reasoning course. *Numeracy, 17*(1), Article 2. DOI: https://doi.org/10.5038/1936-4660.17.1.1451

Chanpet, P., Chomsuwan, K., & Murphy, E. (2020). Online project-based learning and formative assessment. *Technology, Knowledge and Learning,* 1–21. https://doi.org/10.1007/s10758-018-9363-2

Foely, G. D., Budhathoki, D., Thapa, A. B., & Aryal, H. P. (2023). Instructor perspectives on quantitative reasoning for critical citizenship. *ZDM-Mathematics Education, 55*(5), 1009–1020. https://doi.org/10.1007/s11858-023-01520-4

Foley, G. D., & Wachira, P. W. (2021). From gatekeeper to gateway: The role of quantitative reasoning. *Ohio Journal of School Mathematics*, *87*, 29–36. https://library.osu.edu/ojs/index.php/OJSM/article/view/8339/6125

Harwood, R. C. (2018). Logistics of mathematical modeling-focused projects. *Primus, 28*(4), 360–385. https://doi.org/10.1080/10511970.2016.1277813

Kish, D. (2017). *Developing a Quantitative Reasoning course for college consumption* (Thesis). Ohio State University.

Leitzel, J. R. (Chair). (2014, March). *Rethinking postsecondary mathematics: Final report of the Ohio Mathematics Steering Committee.* Ohio Department of Higher Education. https://www.ohiohighered.org/sites/ohiohighered.org/files/uploads/math/MATH-REPORT_FINAL_4.22.14.pdf

Lutsky, N. (2008). Arguing with numbers: Teaching quantitative reasoning through argument and writing. *Calculation vs. context: Quantitative literacy and its implications for teacher education*, 59–74.

Mathematical Association of America. (1996). *Quantitative reasoning for college graduates: A complement to the standards*. L. R. Sons (Ed.). https://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/quantitative-literacy/quantitative-reasoning-college-graduates

Noonan, B., & Duncan, C. R. (2005). Peer and self-assessment in high schools. *Practical Assessment Research and Evaluation, 10*(17), 1–8. https://doi.org/10.7275/a166-vm41
https://doi.org/10.1016/j.sbspro.2014.03.196

Ohio Department of Higher Education. (2015). *TMM011–Quantitative Reasoning.* https://transfercredit.ohio.gov/static/files/transfer/documents/OTM/TMM011%20Quantitative%20Reasoning%20FINALIZED%20v2-%2012-21-2015.pdf

Saldana, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). Sage.

Stake, R. E. (2006). *Multiple case study analysis.* The Guilford Press.

Stump, S. (2017). Quantitative reasoning for teachers: Explorations in foundational ideas and pedagogy. *Numeracy: Advancing Education in Quantitative Literacy, 10*(2), Article 9. http://doi.org/10.5038/1936-4660.10.2.9

Tal, R. T., Dori, Y. J., & Lazarowitz, R. (2000). A project-based alternative assessment system.

Virtue, E. E., & Hinnant-Crawford, B. N. (2019). "We're doing things that are meaningful": Student perspectives of project-based learning across the disciplines. *International Journal of Problem-Based Learning, 13*(2), Article 9. https://doi.org/10.7771/1541-5015.1809

Do Instructors Value Examples in Proof?

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During the transition from procedure-based mathematics courses to proof-based mathematics courses, many students struggle to understand the purpose of examples. A productive use of examples can help both the generation and understanding of proofs. Although recent advances in research have argued the importance of productive uses of examples, there is little research investigating the alignment of research with current practices of instructors or of how the instructors view research on example-use. Findings from this study indicate instructors are aligned with mathematics education research on differing levels of example-use. However, many instructors may be hesitant to see examples be used by their students in written work. Despite this hesitation, instructors still encourage students to use examples to aid their understanding of proof.

Keywords: Proof, Examples, Instructors

Introduction

Student difficulties in understanding and producing proofs are well documented in mathematics education research (e.g., Ellis et al., 2019; Harel & Sowder, 2007; Stylianides, 2007). One such difficulty with proving comes from the effective use of examples to aid proof production and understanding (e.g., Aricha-Metzer & Zaslavsky, 2019; Epp, 2003; Zaslavsky & Knuth, 2019). Although there is much research in understanding how and why students use examples when proving, little research exists studying instruction in proof classes based around examples (Zaslavsky & Knuth, 2019). Zaslavsky and Knuth (2019) noted, "Very little research has focused on the nature and design of instructional practices that facilitate the development of students' abilities to strategically think about and productively use examples as they engage in proving-related activities" (p. 243).

This paper begins the process of addressing Zaslavsky and Knuth's (2019) call for research by investigating the perceptions of students' use of examples held by introduction to proof (ITP) instructors. To develop effective instructional interventions, it is imperative to understand what practitioners currently do in their classrooms (Desimone & Garet, 2015). This study aims to lay groundwork necessary for future research to implement instructional interventions by ascertaining the current perceptions and reactions held by ITP instructors across the United States. To accomplish this goal, I seek to answer the following research question: **How do instructors of ITP classes perceive students' understanding and use of examples when proving?**

Theoretical Framework

I draw upon the work of Aricha-Metzer and Zaslavsky (2019) in distinguishing between productive and unproductive use of examples when proving. Aricha-Metzer and Zaslavsky defined productive examples when the example helped the prover make progress or gain insight into key aspects of the proof. An unproductive use of examples in proof was when the prover used an example but no progress towards developing or understanding a proof was made.

One of the aspects of proof in my work is how examples are used and potentially included in the formal written work provided by students in an ITP course. Balacheff (1987) described categories of example-use in proof; these categories include: naïve empiricism, crucial experiment, and generic example. Table 1 summarizes these categories.

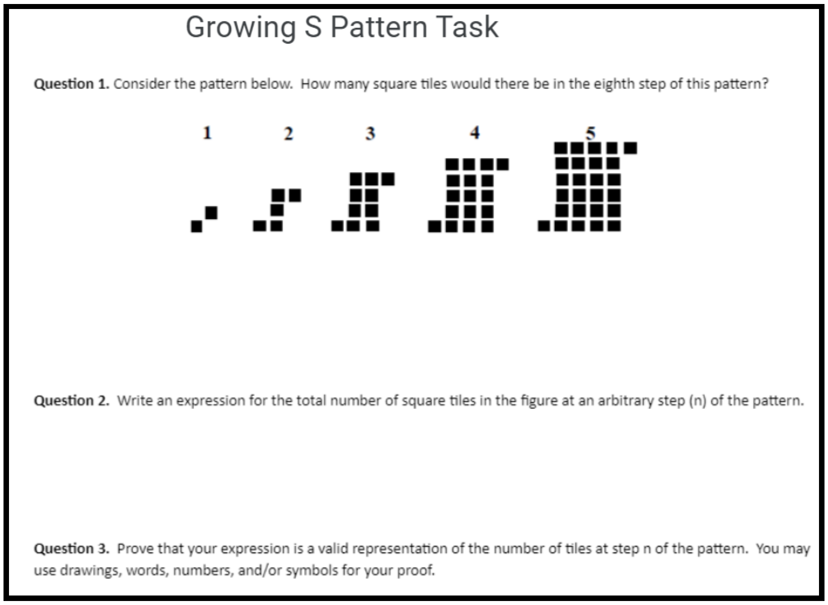
Table 1 Balacheff (1987) example-use categories

Name of Category	Definition (From Balacheff 1987, p. 19-20)
Naïve Empiricism	Drawing from the observation of a small number of cases the certainty of the truth of an assertion
Crucial Experiment	A process of validation of an assertion in which the individual explicitly poses the problem of generalization and solves it by betting on the realization of a case that he recognizes as being as unspecific as possible
Generic Example	Explanation of reasons for the validity of an assertion by carrying out operations or transformations on an object present not for itself, but as a characteristic representative of a class of individuals

Aricha-Metzer and Zaslavsky (2019) explain that when examples are used akin to the generic example level described by Balacheff (1987), the examples used by students were typically classified as productive. In defining generic example, I build on the work provided by Rø and Arnesen (2020) citing Reid and Vallejo Vargas' (2018) two criteria of a generic example: evidence of awareness of generality and mathematical evidence of reasoning. In short, the argument must conclude with a general claim about the original problem to be proven as well as contain reasoning directly linked from the example to be classified as a generic example.

Methodology

Participants of this study include university professors across the Southeastern United States actively teaching their university's version of an ITP course. Eleven professors agreed to participate in the study with varying demographic information such as Carnegie classification of the university, type of Ph.D. received, tenure at the university, and self-described teaching style. One-hour semi-structured interviews were conducted with all participants online through Zoom. Participants were emailed two questions before the study shown in Figure 1.



Question 1. Given the number of vertices in a polygon, find a method to calculate the number of diagonals of the polygon. Explain why your answer is correct.

Figure 1. Growing S pattern task and polygon diagonal task

Participants were not expected to solve these problems before the meeting. The problems were chosen to be easy enough to solve for most faculty members during the interview so time could be spent discussing student use of examples rather than proving the claim. Four participants mentioned they would use or have previously used one or both of the questions shown to them in the interview. Ten of the eleven participants expressed belief these two questions were appropriate for early semester in an ITP course.

During the interview, participants were shown three student work samples for each of the two questions provided. These student work samples were chosen to exemplify the three levels of Balacheff's (1987) framework on example-use. Figure 2 details one of these student work samples: Carla at the crucial experiment level solving the growing S pattern task.

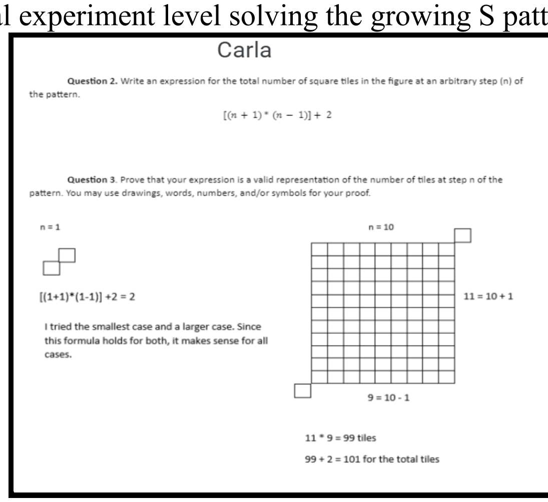


Figure 2. Carla solving the growing S pattern task at the crucial experiment level.

During the interviews, the order of the two questions was randomized as was the order the student work samples were shown. Student work samples were always kept clustered so a participant would either receive the three growing S pattern task solutions first or the three polygon diagonal task solutions first. When presented with student work, participants were asked to mention anything they noticed about the student work sample. Participants were informed this may include if they believe the student is correct, if they liked what the student did, if they had comments on any of the student work, if they had seen something similar to this solution in their classroom, or any questions they would have as instructors if a student in their class presented them with a solution like this as a homework assignment.

After responding to all six student solutions, participants were asked to group the student solutions into as many or as few categories as they deemed appropriate and name each category. Finally, participants were asked to rank the six student responses from 1 to 6 with a response of 1 signaling the best attempt at a proof and a response of a 6 signaling the most work needed to be done for a prof. After finalizing their rankings, participants were asked to provide instructional feedback to advance the students forward whom they ranked best and worst. This manuscript will focus on the ranking task and transcript from across the responses to student solutions.

Analysis

Video data from Zoom interviews with faculty were transcribed by talk turn. I define a talk turn as all utterances spoken by one participant until interrupted by the interviewer. This data was then open coded for common remarks about students' example use among the faculty participants. In order to be considered as a potential code, the code theme had to be included at least twice within the same participants' transcript and across three different participants. For instance, Dr. Amanda's transcript showed a potential code for *mathematical induction*. This was coded when Dr. Amanda remarked discussing Carla's work, "Okay, so here's some induction ideas. You've got the base case. And a small case for the largest." Similarly, Dr. Amanda remarked on Nancy's work (naïve empiricism, growing S pattern task), "[Nancy] tried a smallest case. And the larger case, is suggesting that there's maybe some more idea there of like, maybe I can interim check intermediate cases, which sort of feels like you're building to induction." Dr. Amanda continued to note similar remarks about two other participants' work. This was sufficient within Dr. Amanda's transcript to be a potential code. If this code showed up in another transcript from another participant, I included the code as a theme. For the ranking task, screen captures were taken and compiled to look for common rankings. The rankings were compiled into a table with the feedback given for each participant coded as described earlier.

Results

I will answer my research question in two parts. First, I will list how participants ranked the student responses. Then, I will discuss one of the common themes that emerged from transcribing the data.

Rankings

Participants were asked to rank the six student work samples from a 1, signaling the best attempt at a proof, to a 6, signaling the worst attempt at a proof. The six student work samples were split between the two questions listed in Figure 1 with one student work sample at each of the levels described by Balacheff (1987). Table 2 shows the rankings of each student argument across the 11 faculty participants.

Table 2. Participant rankings of student work samples from 1 to 6

	1st	2nd	3rd	4th	5th	6th
Gina's argument (generic example)	10	1				
Jacob's argument (generic example)	1	9	1			
Carla's argument (crucial experiment)		1	9	1		
Eric's argument (crucial experiment)					8	3
Aaron's argument (naïve empiricism)			1	5		5
Nancy's argument (naïve empiricism)				5	3	3

Table 2 shows how frequently each student was ranked in positions 1 through 6 in the 11 interviews. This table gives evidence to the participant's inclination to rank students' arguments in alignment with the levels of example-use described by Balacheff (1987). Ten of the 11 participants ranked Gina's argument as the best attempt at a proof. Many participants expressed although they similarly liked Jacob's argument, the design of the growing S pattern task led to a seemingly more appealing argument leading to Gina's ranking above Jacob's. Although Eric (crucial experiment – polygon diagonal task) was positioned to exhibit a more robust use of examples than students at the naïve empiricism level, some of the wording used by Eric caused concern for many participants. Eric included statements such as, "This led me to a pattern" and "If my pattern holds," without every explaining what his pattern was. Most participants expressed concern with these phrases when ranking Eric and decided to rank him in the bottom two for lack of clarity.

Theme of Acceptance of Example-Use

A common theme that emerged from the interviews was the theme of acceptance or rejection of example-use. Of the 11 interviews conducted, 5 participants expressed concern with seeing examples of any form included in the written work shown by students. The other six participants alternatively expressed interest and praise for the use of examples to aid proof production and understanding.

The theme of acceptance or rejection was common amongst all 11 participants. A participant was classified as either accepting or rejecting examples entirely based on unprovoked remarks of the first phase of the interview responding to student work samples. All participants expressed their preference of acceptance or rejection of examples at least twice within their interview. The acceptance category included comments promoting student reasoning and praise of the use of examples to aid the understanding or production of the proof. The rejection category instead expressed concern in every instance for the example included in the written work and felt the proof was more lacking with the inclusion of the example. There were no differences in demographic information between the acceptance and rejection groups.

One instance of an acceptance category comes from Dr. Tucker. When responding to Jacob (generic example – polygon diagonal task), Dr. Tucker remarked, "So, I do like the, I guess he kind of start out with, you know, thinking of a fixed number. Just to kind of get an idea of what to do with it." Later, Dr. Tucker similarly commented on both Gina (generic example – growing S pattern task) and Carla's (crucial experiment – growing S pattern task) work praise for the pictures drawn and an interest in having more students use this type of reasoning. These comments were sufficient to include Dr. Tucker in the acceptance of example-use category. Participants in the acceptance category did not accept all forms of examples as sufficient for proof. Five of the six participants in this category expressed concern for both Nancy (naïve empiricism – growing S pattern task) and Aaron's (naïve empiricism – polygon diagonal task) use of examples in their proof. Dr. Tucker commented on both Eric (crucial experiment – polygon diagonal task) and Aaron's work,

> I guess generally, a pitfall I see with students is trying to just do a couple of examples. And then that kind of just, I know this doesn't say prove (referring to the wording of the question given), but kind of thinking that counts as a proof. And yeah, it feels like they just kind of need to bring more justification of why would this work in general.

Although Dr. Tucker and others in the acceptance of example-use category appreciated the work shown by students when proving, this did not mean they accepted incomplete proofs. Participants in the acceptance of example-use category expressed appreciation for the productive uses of examples as described by Aricha-Metzer and Zaslavsky (2019) and concern for the unproductive use of examples.

Participants in the rejection category responded to student work regardless of Balacheff's (1987) associated level with explicit concern for the written inclusion of examples in the product produced. These comments were made for both productive uses of examples as well as unproductive uses of examples. For instance, Dr. Hubert remarked about Gina's work, "I mean, it's reasoning by example. Which is always rough. I would say the n minus one, n plus one part could have helped me earlier on." Dr. Hubert here refers to how Gina starts with a drawing of a specific case. From this case, Gina gathers information about the nature of the problem and correctly answers the problem producing a generic formula to solve the growing S pattern task along with a short algebraic proof. Dr. Hubert expressed concern that Gina left the reasoning with her example in the work instead of removing this first.

Later in the interview, Dr. Hubert again remarked about Gina's work, "So I'm struggling right now with if you can do this (referring to Gina's formal answer at the bottom of the page), how does it not stick? I mean, I ought to be able to just state that part earlier right?" Dr. Hubert was struggling in the interview to understand why if Gina knew the complete answer she could not remove the discussion about how she arrived at the answer and instead state her proof more formally. This idea of formality was raised by Dr. Sarah about Jacob's work with examples commenting, "The fact that Jacob does still have, like the thinking about his example. To me, I would not want to see evidence of it in the proof." The rejection of examples category was largely defined by the difference in how formality was viewed as a necessary condition in an ITP course.

Discussion and Conclusion

This research looks to make progress in answering the call to action from Zaslavsky and Knuth (2019) to include more instructional tools about examples in proving. I set out to answer my research question: How do instructors of ITP classes perceive students' understanding and use of examples? Answering this question will assist future researchers in developing instructional interventions for practitioners of ITP courses by considering the current state of the field. I answered this research question in two ways. I first answered my research question through the participants' rankings of the provided student work based on the completeness of a proof. Second, I showed a theme for common responses to student work involving example-use through the acceptance or rejection of examples in proving.

Through the ranking task, participants are largely aligned with the levels of example-use described by Balacheff (1987). Although there was a discrepancy with Eric, most of the participants still ranked the student work largely by type of example used by the student. This finding can help future instructional interventions find a starting point for helping instructors of ITP courses aid students in using examples more productively as instructors seem to be implicitly aware of the categories of example-use without any background information given. Participants of the study were asked in a short debrief after the interview if they were familiar with any research on generic examples or Balacheff's framework. None of the participants were familiar with any of this research.

Through the coding, a potential theme of acceptance or rejection of example-use came to light. This theme should be considered when designing instructional interventions in the future for example-use in proof. There were no differences in any demographic information collected between the acceptance of example-use group and the rejection of example-use group. For professional development to be most effective, practitioners need to be met at their current level (Desimone & Garet, 2015). This study gives some evidence that for some practitioners, work may need to be done by mathematics educators on convincing their future professional development groups of the effectiveness and usefulness of generic examples. There is a potential conflict with the message sent to students if practitioners value examples to aid a students' understanding while in class but then do not want to see examples in the written work. Finally, the issue of formality of written work in an ITP course was repeatedly seen. Participants in the acceptance of example-use group did not bring up issues with formality of the presentation of a proof. Participants in the rejection of example-use group frequently brought up issues of how including work in a proof detracted from the argument.

I encourage future researchers to investigate this disconnect of example-use and acceptance in other areas of mathematics, other types of proof, and in more detail. The comments taken in this study came unprovoked as the interview protocol did not include questions about acceptance or rejection of example-use. Directed questioning towards types of examples and what a proof should look like would benefit future work in developing instructional interventions for ITP classes. Mathematics education researchers and practitioners should consider the idea of formality of proof and determine how to best approach future research in the field of example-use in proof.

References

Aricha-Metzer, I., & Zaslavsky, O. (2019). The nature of students' productive and non-productive example-use for proving. *The Journal of Mathematical Behavior, 53*, 304–322. https://doi.org/10.1016/j.jmathb.2017.09.002

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics, 18*(2), 147–176. https://www.jstor.org/stable/3482413

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y.-Y. (2015). Transforming perceptions of proof: A four-part instructional sequence. *Mathematics Teacher Educator, 4*(1), 32–70. https://doi.org/10.5951/mathteaceduc.4.1.0032

Desimone, L. M., & Garet, M. S. (2015). Best practices in teachers' professional development in the United States. *Psychology, Society, and Education, 7*(3), 252-263

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (2019). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior, 53*(January 2017), 263–283. https://doi.org/10.1016/j.jmathb.2017.06.003

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886–899. https://doi.org/10.1080/00029890.2003.11920029

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805–842). Information Age Publishing.

Reid, D., & Vallejo Vargas, E. (2018). When is a generic argument a proof? In A. J. Stylianides, & G. Harel (Eds.), *Advances in Mathematics Education Research on Proof and Proving* (pp. 239-251). Springer International Publishing. https://doi.org/10.10007/978-3-319-70996-3_17

Rø, K., & Arnesen, K. K. (2020). The opaque nature of generic examples: The structure of student teachers' arguments in multiplicative reasoning. *Journal of Mathematical Behavior, 58*(2020), 100755. https://doi.org/10.1016/j.jmathb.2019.100755

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321. https://doi.org/10.2307/30034869

Zaslavsky, O., & Knuth, E. (2019). The complex interplay between examples and proving: Where are we and where should we head? *Journal of Mathematical Behavior, 53*(October 2018), 242–244. https://doi.org/10.1016/j.jmathb.2018.10.001

Playful Math, Problem Posing, and Discovered Complexity

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Problem posing is an important part of mathematical inquiry, but students can struggle to pose problems that are mathematically relevant. One route for supporting meaningful problem posing is through playful math tasks, which can emphasize agency and exploration. In this paper we report findings from a small-group teaching experiment with five pre-service secondary teachers who explored a variety of tasks, including open tasks, problem-solving tasks, and tasks designed to foster mathematical play. In investigating students' problem posing, we found that developing playful challenges for one another supported instances of discovered complexity, the experience of new conceptual challenges. We discuss examples of discovered complexity and the mathematical ideas students developed when grappling with novel ideas.

Keywords: Problem posing, mathematical play, covariation

Instructional approaches that engage students in experiences that are authentic to disciplinary inquiry should offer opportunities for problem posing, exploring, and conjecturing (Bonotto, 2013). Problem posing can be a particularly powerful activity to support students' mathematical inquiry (Cai et al., 2015), and classrooms that do not include problem posing can curtail students' mathematical exploration (Ellerton, 2013). Collective problem posing activities are one way to build students' autonomy in the development of the mathematics they learn, and they can provide a space for students to develop their creativity and problem-solving skills (Cai, 2012). There are, however, challenges with meaningfully fostering problem-posing skills. Students and teachers can pose a variety of problems when presented with a given set of information (Stickles, 2011), but may struggle to pose problems that are mathematically relevant, appropriate, and solvable (e.g., Cai & Hwang, 2003; Silver & Cai, 1996; Silver et al., 1996). As Cai et al. (2015) noted, more research is needed to understand not only which strategies are needed to pose mathematically interesting problems, but also how to effectively teach for problem posing.

We conjecture that *playful math* could be one way to foster productive problem posing. We have been running a series of studies investigating how to meaningfully incorporate playful elements into task design and instruction, and have found that playful mathematics tasks can emphasize student agency, exploration, and goal selection (e.g., Bloodworth et al., 2023; Horne et al., 2023; Ellis et al., 2022). When crafting playful mathematics tasks, we shift the design role to the student, enabling students to construct challenges for one another. This provides an opportunity to engage in problem posing that may support discovered complexity (Williams, 2001; 2002; 2003), which occurs when problem solvers "perceive *intellectual* and *conceptual* complexities not evidence at the commencement of the task" (p. 378, emphasis original). In this paper we address the following research questions: When does students' problem posing lead to discovered complexity? In particular, how does discovered complexity emerge in different task types, and what mathematical ideas are developed through discovered complexity?

Background Literature and Theoretical Perspectives: Problem Posing and Play

Cai and Hwang (2020) defined Mathematical Problem Posing (MPP) as "the process of formulating and expressing a problem within the domain of mathematics" (p. 2). While some emphasize problem posing in terms of the generation of a new problem from a given set of conditions (e.g., Stoyanova, 1998), others also include in problem posing the activity of reformulating existing problems (e.g., Silver, 1994). Problem-posing tasks, then, are ones that require students (or teachers) to generate new problems based on given situations, expressions, or diagrams (Cai et al., 2020). As an example, Stoyanova (1998) offered the following task: "Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on the previous ring. Ask as many questions as you can" (p. 66). Note that this task, which is typical in the literature, offers a constrained context with specific values, and then asks students to develop problems with the given information. That said, a good problem-posing task should leave room for different interpretations (Kontorovich et al., 2012; Silver & Cai, 1996).

Posing problems can deepen students' understanding of mathematics, as well as their understanding of problem-solving (Brown & Walter, 2004). Indeed, problem posing is an important part of problem-solving competence, and to become successful problem solvers, students need to encounter tasks that require both posing and solving problems (Cai et al., 2015; Kilpatrick, 1987; Niss & Højgaard, 2019). Problem posing can also capture students' interest, foster reasoning and communication abilities, and can help students understand that there is no one right way in mathematics (Cai et al., 2015; Silver, 1994).

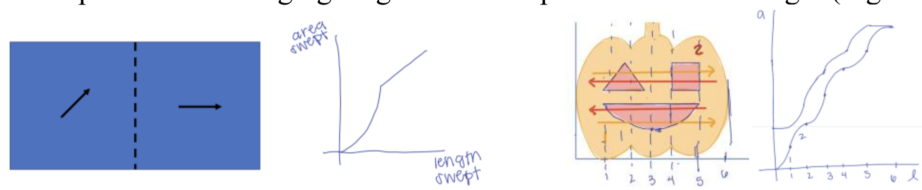
Research on people's abilities to pose problems is largely situated at the K-12 level, and the findings are mixed. Some studies show that both students and teachers can pose interesting and important problems (e.g., Silver & Cai, 1996; 2005; Cai & Hwang, 2020). However, more studies have identified challenges with successful problem posing (e.g., English, 1998). Stickles, for instance, found that secondary teachers could pose problems, but their success was partial and was related to experience and background (2011). Taken as a whole, the studies of students' and teachers' problem posing have identified issues such as a) posing nonmathematical problems, b) posing tasks that were not problems, and c) posing low-quality problems (Cai & Hwang, 2020; Leung & Silver, 1997; Silver et al., 1996).

In investigating strategies for supporting problem posing, researchers have noted the importance of providing opportunities for exploration (Koichu & Kontorovich, 2013). Crespo and Sinclair (2008), for instance, emphasized the need to allow students avenues to explore the limits of the mathematical situations they investigate. Cai and Cifarelli (2005) studied how undergraduate students posed problems, and identified two levels of reasoning strategies, hypothesis-driven and data-driven, that students incorporated into their problem-posing strategies. Brown and Walter (2004) proposed the "what-if-not" strategy, and others have suggested helping students learn to pose problems by extending or revising an existing problem (e.g., Abu-Elwan, 2002; Cai & Brook, 2006). However, as Cai et al. (2015) pointed out, "Much more research is needed to develop a broadly-applicable understanding of the fundamental processes and strategies of problem posing" (p. 17). As we discuss below, we hypothesize that some features of playful math could foster meaningful problem posing.

Mathematical Play and Playful Math. Drawing on five characteristics that recur throughout the literature, we define mathematical play in terms of a person's experiences, behaviors, and affective states during activity. First, mathematical play entails *freedom* and *agency*. It is player-centric, with the player in charge of the process (Holton et al., 2001; Gresalfi et al., 2018). Participation is voluntary (Williams-Pierce & Thevenow-Harrison, 2021) and ludic (Featherstone 2000). Second, mistakes have *low stakes*. When playing, a person is not afraid of failing (Williams-Pierce & Thevenow-Harrison, 2021) and may not even experience anything as "failure" (Barab et al., 2010; Su, 2017). Third, mathematical play entails *engagement and* *immersion* in one's activity (Gresalfi et al., 2018). Learners find pleasure in their experience (e.g., Burghardt, 2011; Sukstrienwong, 2018), and activity is imaginative and creative (e.g., Parks, 2015; Su, 2017). Fourth, mathematical play is *experimental and exploratory* (Mason, 2019). Play often leads to surprises (Su, 2017), and when one plays, one is in a state of openness to unexpected experiences and outcomes (Davis, 1996). Finally, mathematical play is *goal-driven* (Huizinga, 1955). Goals can emerge as play proceeds, and researchers have described instances of goal-directed behavior, such as learners solving challenges in a video game (Williams-Pierce, 2019). Consequently, we consider mathematical play to entail the following three traits: (a) exploration, (b) self-selection of goals, and (c) immersion, investment, and/or enjoyment.

Researchers have identified a number of learning benefits from mathematical play. It can support experimentation, reflection, and persistence (Barab et al., 2010; Gresalfi et al., 2018; Mason, 2019) and can provide a productive route for exploring and conjecturing (Jasien & Horn, 2018; Mason, 2019; Williams-Pierce, 2019). Beyond mathematics, classroom-based play offers a supportive environment for risk-taking and creativity (Barab et al., 2010; Brown, 2009; Radke & Ma, 2018; James & Nerantzi, 2019). It can help students connect theory and practice (Barnett, 2007), it can foster openness to new learning (Forbes, 2021), and it can support classroom engagement (James & Nerantzi, 2019; Whitton & Moseley, 2014). Many of these practices mirror mathematicians' activity, such as detecting patterns (Fernández-León et al., 2021; Melhuish et al., 2021), generalizing (Bass, 2008; Martín-Molina et al., 2018), conjecturing (Fernández-León et al.; 2021; Harel, 2008), and experimenting (Watson, 2008). These similarities have resulted in calls for instruction that encourages playful engagement. For instance, Gresalfi and colleagues (2018) argued that "much of what is considered to be sophisticated disciplinary engagement involves many of the same features as play" (p. 1335).

To study how to foster mathematical play, we used the term "playful math" to describe the elements of classroom activities and environments that can facilitate mathematical play. Drawing on the literature, we have identified five design principles for playful math tasks (Ellis et al., 2022): (1) enable free exploration within constraints; (2) engender anticipation within the task; (3) provide a method for intrinsic feedback; (4) offer meaningful challenge while still being feasible; and (5) allow the student to act as both designer and player. As an example, we created the *Guess My Shape* game, which draws on a set of research-based tasks to support student understanding of function through examining covarying quantities (Ellis et al., 2020; Matthews & Ellis, 2018). In these tasks, students investigate dynamically growing shapes, graphing a shape's area compared to its changing length as it sweeps out from left to right (Figure 1a).



(a) *(b)*

Figure 1. An instructor generated (a) and student generated (b) shape and associated area-length graph.

To playify the tasks, the *Guess My Shape* game prompts students to create shapes of their choice (design principles 1 and 5), construct graphs comparing length and area (principles 2, 4, and 5), then challenge each other to determine the shape or the graph based on what is provided (principles 2, 3, and 4). Figure 1b shows an example of a particularly creative shape and its associated graph created by our participants, which entailed sweeping a pumpkin from left to right, and then sweeping backwards right to left to remove area to create a jack-o'-lantern.

The playful math design principles are consistent with many features of problem-posing tasks. Problem-posing tasks require students to generate new problems (principle 5), and they offer opportunities for exploration (principle 1). However, our tasks shift the design role to students in a manner that is more open-ended than what is seen in the problem-posing literature. For instance, in contrast to Stoyanova's (1998) doorbell task, in which both the situation and the relevant numbers or mathematical expressions are given, we provide fewer constraints and encourage students to design novel challenges.

Discovered Complexity. We draw on Williams' (2001) construct of discovered complexity to examine the consequences of students' problem posing, particularly the spontaneous problem posing that occurred in the course of both designing and solving tasks. Williams (2002) wrote that when students discover a complexity, they spontaneously formulate a question that leads to intellectual and conceptual challenges: "the mathematical ideas in a discovered complexity are new to all students in the group and the teacher does not contribute new mathematical ideas during the interaction" (p. 403). The process of discovered complexity is autonomous, spontaneous, and creative, similar to the activity described by a research mathematician (Williams, 2002). Discovered complexity is associated with high positive affect (Williams, 2003) and meets the conditions for flow (Williams, 2001), as students can work just above their present skill level to meet a challenge almost beyond their reach.

Methods

We conducted a videoed teaching experiment (TE; Steffe & Thompson, 2000) with five secondary pre-service teachers, Phyllis, Meredith, Kelly, Ryan, and Toby (all pseudonyms). The students had all completed their first semester in a secondary mathematics education program, and they were familiar with non-routine covariation and graphing tasks. The students enjoyed a positive rapport with one another and were comfortable questioning each other and themselves.

The TE met for a total of 6 hours. We began with the function growth activities described above, which we call standard tasks, and which addressed polynomial, trigonometric, and piecewise functions. We also incorporated playified tasks in the form of the *Guess My Shape* game, as well as more traditional problem-solving and open tasks (e.g., tasks that were open-ended with many possible solutions, such as "create two growing shapes that could be represented by the same graph"). The problem-solving tasks were ones such as the handshake task: "If there are n people in a room, and they all shake hands with each other exactly once, how many handshakes occurred?" All of the standard, problem-solving, and open tasks were challenging and creative, but only the *Guess My Shape* game adhered to all five playful math design principles. Over the course of the TE we implemented six standard tasks, three problem-solving tasks, two open tasks, and three playified tasks.

Analysis. As part of a larger study investigating the characteristics of students' mathematical play, we developed emergent codes describing students' experiences and behavior. One code that became relevant for this paper was "wonderment", which represented instances of students spontaneously communicating curiosity or interest about a new mathematical idea, problem, or relationship. Wonderment is an instance of spontaneous problem posing. We initially found 23 cases of wonderment, and examined each case to determine: a) did it occur during a standard, open, problem-solving, or playified task; b) did it result in discovered complexity? We identified instances of discovered complexity by seeing whether each wonderment yielded a novel or more complex mathematics challenge, as opposed to simply trying to understand the given question, establish a convention or definition, or wondering how to create a challenging task for someone else (but not actually creating such a task). During this process, we abandoned two of the 23 wonderments as not actual wonderments. In one case, a student was trying to understand how another student's shape was being swept out, and in the second case, a student was curious about the logic another student was using for her solution. We analyzed the remaining 21 wonderments collaboratively, meeting weekly to refine and adjust our meanings and resolve discrepancies.

Results: When Wonderment Leads to Discovered Complexity

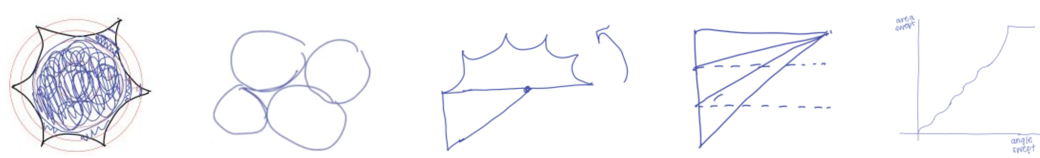
Table 1 shows the number of wonderments per task type and the number of wonderments within each type that led to discovered complexity. Five of the 21 wonderments occurred during the two open tasks, and the remaining 16 occurred during the three playified tasks. We did not find any instances of wonderment in the six standard tasks or the three problem-solving tasks.

Table 1. Number of wonderments and number of resulting discovered complexities by task type.

Task Type	Wonderments	Discovered Complexity
Standard	0	0
Problem Solving	0	0
Open	5	3
Playified	16	9

In examining how discovered complexity emerged, we first identified cases of wonderments that were not discovered complexities. As an example, Toby grappled with an open task in which he had to invent a shape with a monotonically increasing graph that changed concavity. He drew a trapezoid, and a corresponding piecewise area-length graph that had a concave up portion, a linear portion, and a concave down portion. Toby asked himself, "Does that count as a change in concavity? Like, what is the concavity of a straight constant line?" Toby's wonderment did not introduce a new mathematical idea or problem, and it did not count as problem posing. Instead, it was an attempt at a clarification of meaning.

In another example, during the *Guess My Shape* game, Phyllis and Meredith explored a novel way of sweeping. They created a figure that they called a police badge (Figure 2a) and imagined sweeping area from a point in the center moving outward in a circle. Phyllis wondered, "I'm just thinking about, like, now just the geometry stuff. When there's a bunch of circles connected (draws four circles, Figure 2b). You see what I mean, you see spikes here. Is there any relationship with that at all?" Later, she clarifies her wonderment: "How many circles could you fit around one circle of the same size?" Here Phyllis's wonderment posed a problem about circle packing, addressing mathematics that was novel to her. However, before she had an opportunity to attempt a solution, the students were interrupted and told to move back to the whole group, and thus she did not get a chance to engage with intellectual or conceptual complexity.



(a) (b) (c) (d) (e)

Figure 2. Phyllis and Meredith's shape (a), Phyllis's illustration of a circle packing problem (b), Meredith and Toby's shape (c), triangle divided into sub triangles (d), Meredith and Toby's graph (e)

In contrast, we present a case in which a wonderment did lead to discovered complexity. As seen in Table 1, discovered complexity only occurred a little over half the time during students' activity in the playified *Guess My Shape* game. It occurred more often when the students were creating a challenge for their classmates, a formal act of problem posing, than when solving a challenge (discovered complexity occurred during 5 out of 7 wonderments when creating a challenge, but only 4 out of 9 wonderments when solving a challenge). We found that discovered complexity emerged when students either developed curiosity about the nature of covariation, rates of change, or related graphs, or when students became curious about why a particular solution method or strategy was valid.

In the following example, Toby and Meredith posed a problem by creating a shape for their classmates, which they imagined sweeping radially from 0° to 360° (Figure 2c). The students wondered whether the associated graph comparing area and angle swept would be constant once the angle reached 180°:

Toby: So, what happens when we do that? I feel like you're getting more triangle each time you open up, right? Because this length gets longer (points to radial lines in Figure 2d; dotted lines are not yet present), but the base is the same and the height is the same, right? Wait, the base is definitely the same. I don't know about the height, it could also be increasing.

Meredith: I think it –

Toby: – Wait, wait, there's only base and height. So, the base is the same, the height's increasing, so it would get bigger.

Meredith: The base has to be the same.

Figure 2d divides the triangle into three smaller triangles, and Toby and Meredith referred to the vertical portion on the left of each smaller triangle as its base. They erroneously assumed that for each equal angle increment, the base of each small triangle would be the same. Toby and Meredith initially conceived of the height of each triangle differently. Toby assumed the heights would be the solid radial lines in Figure 2d, whereas Meredith realized that they would need to be the dotted horizontal lines, which she drew in: "I think the height has to be perpendicular to the base." The students agreed that the heights of each triangle must be equal, and therefore, they concluded that the amount of area added for each equal angle increment would be equal. Thus, the graph for that portion of the shape should be linear.

Toby remained, however, uncertain about this conclusion. As they worked on an initial graph, he returned to the issue of the triangle's growth, saying, "I still find that hard to believe, but I think it's true. How come this ends up being the same as if we had done that (draws another triangle with the base at a 45° angle, rather than vertical) instead?" Both students, however, believed that it must be linear, because they had justified it based on the belief that the base of each incremental triangle would be equal. Later, a teacher-researcher asked the students about their thinking, and Toby expresses doubt again, but this time with a different outcome:

TR1: How did that triangular part, what was your reasoning there?

Meredith: We were saying that for the bases, we're saying all have to be the same because that's how we're designing the –

Toby: Wait. Are they though? I'm questioning myself again. Just because the angle increases by the same amount, does that mean the base is the distance away, because…(pause). I'll draw a bigger one. There's this angle, and you open it up by the exact same amount. I feel like you're going to have *more* base over here (points to the bottom sector in Figure 2d) than you did up here (points to the top sector). You know what I mean?

Toby drew a much larger triangle, and relying on the grid feature of his notetaking app, then carefully subdivided it into triangles with equal increment angles. In this manner it became apparent that each new triangle had a larger base than the previous triangle. Consequently, the students revised their graph to reflect that the portion of the graph from 180° to 270° would not be constant, but rather would increase at an increasing rate (Figure 2e). Indeed, this can be determined by considering the smaller triangles as each having a swept angle θ, a height of 1, and a base *x*, in which *x* grows as area is swept. The area of each triangle will be $\left(\frac{1}{2}\right)x$. Because $\tan\theta = x$, the area function will be $A = \frac{1}{2}\tan\theta,\ 0 \leq \theta \leq \frac{\pi}{4}$, which yields a graph that increases at an increasing rate.

In posing the radial sweeping problem, Toby and Meredith encountered a discovered complexity. Toby's wonderment about whether the relevant portion of the graph would be linear led to the students perceiving a conceptual complexity, that of how to determine the growth of the triangle. That this was an intellectual challenge for the students was apparent in their discomfort with their initial graph, as well as the need to revisit the question more than once. In resolving the complexity, Toby and Meredith shifted from reasoning only about increasing or decreasing rates, without attending to a way to quantify area amounts, to reasoning geometrically with the triangle area formula to justify why the area had to grow at an increasing rate for each equal-increment angle sweep. In this manner, the problem Meredith and Toby posed for their classmates required them to grapple with a novel conceptual complexity.

Discussion

The *Guess My Shape* game formalizes problem posing by asking students to design problems, but in a manner that is less constrained than what is typically seen in the literature (e.g., Stoyanova, 1998). The primary constraint on the students was that they had to determine how to graph any shape they developed. Because the students were motivated to make their tasks "fun", "tricky", and "unexpected", they pushed themselves to create shape-graph pairs that included novel and creative elements, some that were particularly challenging to solve. However, wonderments emerged not just during formal problem posing; they also occurred when students were solving one another's challenges and exploring open tasks. Wonderments are spontaneous instances of problem posing, which may or may not be pursued to fruition. When they were pursued and resulted in discovered complexity, mathematical ideas emerged that addressed topics such as constant versus nonconstant rates of change in novel sweeping situations, the effects of the center of rotation on the change in area for equal changes in angle measure, and determining the areas of segments of circles formed by two non-radii chords.

Given the small numbers of each task type, we cannot make definitive claims about which types of tasks will necessarily lead to discovered complexity. That said, it was compelling that the students did not demonstrate any wonderments when solving standard or problem-solving tasks. It may be that the motivation to develop creative tasks and solutions during the playified and open tasks, combined with an emphasis on agency and exploration, afforded students opportunities to embrace risk taking, openness, and experimentation, mirroring results from studies on mathematical play (e.g., Gresalfi et al., 2018; Mason, 2019; Williams-Pierce, 2019). We also found that it was often the students' need for causality (Harel, 2013) – the need to explain or determine the cause of a phenomenon – that resulted in a wonderment becoming a discovered complexity. These findings suggest that playful math could potentially be one avenue meriting further exploration for supporting productive problem posing.

References

Abu-Elwan, R. (2002). Effectiveness of problem posing strategies on prospective mathematics teachers' problem solving performance. *Journal of Science and Mathematics Education in Southeast Asia, 25*(1), 56-69.

Barab, S.A., Gresalfi, M., & Ingram-Goble, A. (2010). Transformational play: Using games to position person, content, and context. *Educational Researcher, 39*(7), 525 – 536.

Barnett, L.A. (2007). The nature of playfulness in young adults. *Personality and Individual Differences, 43*(4), 949-958. http://doi:10.1016/j.paid.2007.02.018.

Bass, H. (2008, January). *Mathematical practices*. Paper presented at a Project NExT Session on Helping Students Develop Mathematical Habits of Mind, Joint Mathematics Meetings, San Diego, CA.

Bloodworth, A., Horne, D., & Ellis, A.B. (2023). Students' and mathematicians' playful engagement. In Cook, S., Katz, B. & Moore-Russo D. (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 108-117). Omaha, NE.

Bonotto, C. (2013). Artifacts as sources for problem-posing activities. *Educational Studies in Mathematics, 83*, 37-55.

Brown, S. (2009). *Play: How It Shapes the Brain, Opens the Imagination, and Invigorates the Soul.* Avery.

Brown, S. I., & Walter, I. M. (2004). *The art of problem posing* (3rd edition). Lawrence Erlbaum Associates Publishers.

Burghardt, G. M. (2011). Defining and recognizing play. In A. Pelligrini (Ed.), *The Oxford handbook of the development of play* (pp. 9–18). Oxford University Press.

Cai (2012). *Problem posing as lenses of improving students' learning in classroom*. University of Delaware.

Cai, J. & Brook, M. (2006). New perspectives of looking back in mathematical problem solving. *Mathematics Teaching, 196*, 42-45.

Cai, J., & Cifarelli, V. V. (2005). Exploring mathematical exploration: How two college students formulated and solved their own mathematical problems. *Focus on Learning Problems in Mathematics, 27*(3), 43-72.

Cai, J., & Hwang, S. (2003) A perspective for examining the link between problem posing and problem solving. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the Joint Annual Meeting of the PME and PME/NA*. University of Hawaii, July 13–18 (Vol. 3, pp. 103–110).

Cai, J., & Hwang, S. (2020). Learning to teach through mathematical problem posing: Theoretical considerations, methodology, and directions for future research. *International Journal of Educational Research, 102*, 101391. https://doi.org/10.1016/j.ijer.2019.01.001

Cai, J., Hwang, C., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. M. Singer, et al. (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 3–34). New York: Springer.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*, 395-415.

Davis, B. (1996.) *Teaching Mathematics: Toward a Sound Alternative*. Garland Publishing Inc.

Ellerton, N.F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: development of an active learning framework. *Educational Studies in Mathematics, 83*, 87-101.
Ellis, A.B., Ely, R., Tasova, H., & Singleton, B. (2020). Scaling continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87 – 103.
Ellis, A.B., Horne, D., Bloodworth, A., Nielsen, A., & Ely, R. (2022). Playful math: Modeling students' engagement in play-based algebra activities. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.), *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 771 – 780). Middle Tennessee State University.
English, L. D. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83–106.
Featherstone, H. (2000). "-Pat + Pat = 0": Intellectual play in elementary mathematics. *For the Learning of Mathematics, 20*(2), 14 – 23.
Fernández-León, A., Gavilán-Izquierdo, J. M., & Toscano, R. (2021). A case study of the practices of conjecturing and proving of research mathematicians. *International journal of mathematical education in science and technology*, *52*(5), 767-781.
Forbes, L. (2021). The process of play in learning in higher education: A phenomenological study. *Journal of Teaching and Learning, 15*(1), 57 – 73.
Gresalfi, M., Horn, I., Jasien, L., Wisittanawat, P., Ma, J.Y., Radke, S.C.,…,Sfard, A. (2018). Playful mathematics learning: Beyond early childhood and sugar-coating. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 1335 – 1342). London, UK: University College London.
Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, part 1: Focus on proving. *ZDM, 40*(3), 487–500.
Harel, G. (2013b). Intellectual need. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119–151). Springer.
Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology, 32*:3, 401-415. DOI: 10.1080/00207390118654.
Horne, D., Bloodworth, A., & Ellis, A.B. (2023). Modeling mathematicians' playful engagement in task-based clinical interviews. In Cook, S., Katz, B. & Moore-Russo D. (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 99-107). Omaha, NE.
Huizinga, J. (1955). *Homo Ludens: A Study of the Play Element in Culture*. Boston, MA, Beacon Press.
James, A., & Nerantzi, C. (2019). *The power of play in higher education: Creativity in tertiary learning*. Palgrave Macmillan.
Jasien, L., & Horn, I.S. (2018). "Ohhh, now I can do it!": School-age children's spontaneous mathematical sensemaking in construction play. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 624 - 631). London, UK: University College London.
Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Erlbaum.
Koichu, B., & Kontorovich, I. (2013). Dissecting success stories on mathematical problem posing: A case of the Billiard Task. *Educational Studies in Mathematics, 83*, 71–86.

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior, 31*(1), 149–161. https://doi.org/10.1016/j.jmathb.2011.11.002
Leung, S. S., & Silver, E. A. (1997). The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of prospective elementary school teachers. *Mathematics Education Research Journal, 9*(1), 5–24.
Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study. *International Journal of Mathematical Education in Science and Technology*, *49*(7), 1069-1082.
Mason, J. (2019). Pre-parative and post-parative play as key components of mathematical problem solving. In *Problem solving in mathematics instruction and teacher professional development* (pp. 89-110). Springer, Cham.
Matthews, P.M., & Ellis, A.B. (2018). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition, 4*(1), 19 – 58.
Melhuish, K. M., Vroom, K., Lew, K. M., & Ellis, B. (2021). Operationalizing authentic and disciplinary activity for the undergraduate context. In Olanoff, D., Johnson, K., & Spitzer, S.M. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 349–358).
Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics.* (online 18 June 2019) https://doi.org/10.1007/s10649-019-09903-9
Parks, A. N. (2015). *Exploring mathematics through play in the early childhood classroom.* National Council of Teachers of Mathematics.
Radke, S.C. & Ma, J.Y. (2018, June). Guiding Intent Participation at an Art Crating Company. Paper presented at the *International Conference of the Learning Sciences* (ICLS) 2018. London, England: International Society of the Learning Sciences.
Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.
Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521–539.
Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education, 27*, 293–309.
Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A., Kelly, & R., Lesh (Eds.), *Handbook of research design in mathematics and science education*. Lawrence Erlbaum Associates.
Stickles, P. (2011). An analysis of secondary and middle school teachers' mathematical problem posing. *Investigations in Mathematics Learning, 3*(2), 1-34.
Stoyanova, E. (1998). Problem posing in mathematics classrooms. In N. Ellerton & A. McIntosh (Eds.), *Research in mathematics education in Australia: A contemporary perspective* (pp. 164–185). Perth: Edith Cowan University.
Su, F. (2017). Mathematics for human flourishing. *The American Mathematical Monthly*, *124*(6), 483-493.
Sukstrienwong, A. (2018). Animo math: The role-playing game in mathematical learning for children. *Tem Journal*, *7*(1), 147.

Watson, A. (2008). School mathematics as a special kind of mathematics. *For the Learning of Mathematics, 28*(3), 3–7.

Whitton, N., & Moseley, A. (2014). Deconstructing engagement: Rethinking involvement in learning. *Simulation & Gaming, 45*(4-5), 433-449. http://doi:10.1177/1046878114554755.

Williams, G. (2001). Sustained overlap in zones of proximal development? If so, how did that happen? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th international conference for Psychology of Mathematics Education* (Vol. 1, p. 378). Utrecht, The Netherlands: PME.

Williams, G. (2002, July). Associations between mathematically insightful collaborative behaviour and positive affect. In *PME CONFERENCE* (Vol. 4, pp. 4-401).

Williams, G. (2003). Spaces of learning that promote insightful and creative mathematical behavior: A theoretical framework. Paper presented at the *Annual Meeting of the American Educational Research Association*, Chicago, April 21 – 25, 200.

Williams-Pierce, C. (2019). Designing for mathematical play: failure and feedback. *Information and Learning Sciences*.

Williams-Pierce, C., & Thevenow-Harrison, J. T. (2021). Zones of mathematical play. *Journal of the Learning Sciences*, *30*(3), 509-527.

Calculus I Students' Understanding of Implicit Differentiation

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Implicit differentiation is an important topic in first-semester Calculus, yet only recently have researchers in the RUME community turned their attention to it. Our study seeks to contribute to this growing body of knowledge. We examined students' performance on an instructional activity that emphasized the graphical representation of implicit curves alongside the symbolic competence of implicit differentiation. The students worked in small groups during recitations and then completed a similar type of question on an assessment. A researcher-designed Hypothetical Learning Trajectory was used to structure the instructional task and to analyze student written data. The results suggest that students' attainment of symbolic learning goals is higher than that of graphical ones, with most difficulties experienced with coordination between symbolic and graphic modalities. Implications for further research are discussed.

Keywords: Implicit differentiation, Calculus learning, Hypothetical learning trajectory

Implicit differentiation is one of the core topics of Calculus, yet only recently researchers have turned their attention to studying it (Speer and Kung, 2016). Implicit differentiation is a technique for finding derivatives of equations where y cannot be explicitly expressed as a function of x. Many popular textbooks on single-variable Calculus in the United States (e.g., Stewart, 2016), present implicit differentiation as a computation technique for finding derivatives, with a strong emphasis on symbolic manipulation. Little attention is paid to building the conceptual basis of implicit functions, justifying the legitimacy of differentiating both sides of the implicit equation (Mirin & Zazkis, 2019), or developing the graphical meaning of implicit functions. All these contribute to students' conceptual and technical difficulties with the topic.

Within the existing research on implicit differentiation, we identified two strands: survey studies that characterize students' difficulties with implicit differentiation (e.g., Chu, 2019; Kandeel, 2021), and small-scale intervention studies that closely examine the development of students' understanding of this topic (e.g., Borji & Martinez-Planell, 2019, 2020; Jeppson, 2019).

Our study occupies a middle ground. It was conducted in the first-semester single-variable Calculus I course, in an authentic classroom setting. After attending a lecture on implicit differentiation, students worked in groups, during a recitation on an instructional centered on graphical representation and the connections between symbolic and graphical meaning of implicit differentiation. In this context, we examined students' performance on the implicit differentiation task and the follow-up performance on a similar exam question. Due to space constraints, here we focus on the class work only.

Literature review

Calculus topics such as derivatives, the chain rule, and related rates have been previously researched (e.g., Cottrill, 1999; Infante, 2007; Martin, 2000, Zandieh, 2000). However, implicit differentiation came to the researchers' attention only recently. Several studies focused on students' difficulties with this topic. Chu (2019) surveyed 136 first-semester calculus students and found that only about 50% of solutions were correct; with most students' errors being related to calculus concepts and procedures, rather than prerequisite algebra skills. Kandeel (2021) classified common errors in an implicit differentiation survey completed by 117 Calculus students. Among common algebraic errors were isolating a common factor, simplifying expressions, dealing with exponents and radicals, and isolating y', of responses. Common calculus errors occurred in the application of the chain rule, product rule, and differentiating functions. These types of errors appeared in 25% - 50% of responses.

Researchers also attempted to unpack what it means to "understand" implicit differentiation. Mirin and Zazkis (2019) point to the conceptual difficulty of understanding the legitimacy of differentiating both sides of implicit equations. Borji and Martinez-Planell (2019) assert that first-year Calculus students have neither the background nor "mathematical tools to consider the statement, and even less the proof, of the Implicit Function Theorem, upon which a rigorous study of implicit differentiation would be based" (p. 15). The authors utilized the APOS (action-object-process-schema) theory (Arnon et al., 2014) to develop a genetic decomposition for implicit differentiation –a hypothetical sequence of mental constructs involved in learning this topic and five technology-assisted activities supporting the development of these mental constructs. The genetic decomposition and the activities emphasized a graphical understanding of implicit functions and symbolic-graphic connections. The intervention was carried out with 14 students who have completed a lecture-based Calculus I course. Ten out of 14 students showed an increased understanding of implicit differentiation, but others had no evidence of progress.

In another intervention study, Jeppson (2019) developed a hypothetical learning trajectory (HTL) for implicit differentiation. HLT is a sequence of learning goals, activities, and conjectured students' understanding of a particular content (Simon, 1995). Jeppson's HLT integrates the chain rule, implicit differentiation, and related rates under the overarching concept of nested multivariation - a covariation of multiple variables related to one another through function composition. Jeppson conducted single-subject teaching experiments with four Calculus students to study how they develop an understanding of implicit differentiation through meaningful context and careful sequencing of mathematical. Yet, Jeppson's HTL did not include graphical representations of implicit functions nor symbolic-graphic connections.

Theoretical Perspectives

Following Borji and Martinez-Planell (2019, 2020), we place a strong emphasis on symbolic-graphical connections in developing student understanding of implicit differentiation. However, the scale and methods of our study, detailed below, do not allow for examining mental constructs or cognitive processes underlying student understanding of implicit differentiation. Instead, we adopt Vygotsky's (1978) *sociocultural perspective* to examine what learners *say and do* while solving an implicit differentiation task in a collaborative group context, supported by more knowledgeable others – teaching and learning assistants (TAs and LAs). Within the sociocultural perspective, student activity is characterized by their discursive performances, which include vocabulary, visual mediators, actions, etc. (Lave & Wegner, 1990; Sfard, 2015)[1]. Student learning is cast in terms of increased, proficient participation in the discursive practices of a particular community and sifting from ritualistic to goal-oriented participation. In the Calculus course, students are enculturated into discursive practices of the mathematical community. We described these practices in a Hypothetical Learning Trajectory of implicit differentiation.

Hypothetical Learning Trajectory for Implicit Differentiation

An HLT is a theoretical model to design instruction supporting conceptual learning, which consists of learning goals, tasks, and hypothesized learning processes (Sion, 1995). Jeppson's

[1] See Sfard (2015) for the discussion of parallels between cognitive constructs and discursive perspective.

(2019) HTL was a main inspiration for our model. But our HTL (Table 1) had a strong focus on graphical representation and graphic-symbolic coordination (c.f., Borji & Martinez-Planell, 2019). The last column matches the HTL's goal to the item number in the activity (Figure 1).

Table 1. Hypothetical Learning Trajectory for Implicit Differentiation

Category	Description of Goal	Item #
Symbolic	**Develop symbolic meaning of implicit equations and correctly implement differentiation procedures**	
Symbolic Recognition (S_Rec)	Recognize an implicit equation cannot be written explicitly as $y = f(x)$.	1b
Symbolic implicit equation (S_Eq)	Given an equation with variables x and y, recognize y as an implicit function of x.	2
Symbolic Chain Rule (S_Ch)	Given implicit equations recognize the need for the chain rule in taking the derivative with respect to the implicit independent variable.	2
Symbolic differentiation (S_Diff)	Can correctly perform procedures for finding $\frac{dy}{dx}$ for an implicit equations.	2
Symbolic Evaluation (S_Eval)	Can correctly compute the derivative $\frac{dy}{dx}(x_0, y_0)$ at a point	3a, 3b
Symbolic Tangent Line (S_Tan)	Can correctly find the equation of a tangent line at a point.	3a
Graphic	**Interpret graphical properties of implicit functions**	
Graphic Recognition (G_Rec)	Recognize graphic representation of implicit equations as a curve in a cartesian place that does not pass the vertical line test.	1a
Graphic Tangent Slope (G_TS)	Interpret the meaning of $\frac{dy}{dx}(x_0, y_0)$ a slope of a tangent line at a point (x_0, y_0), and $\frac{dy}{dx}$ as slope of the tangent line or instantaneous rate of change at any point.	2a, 2b
Graphic Vertical Tangent (G_VT)	Recognize that tangent lines to a graph of implicit equations can be vertical, and at the points of tangency the derivative does not exist.	4
Graphic Constant Rate (G_CR)	Recognize in an explicit function, the lack of x in the derivative formula means the function has a constant rate of change, but a derivative formula of the implicit equation can contain no x, while the curve does not have a constant rate of change.	2b
Graphic Coordination (G_Cor)	Coordinates graphical and symbolic representations.	2b, 3a, 3b, 4

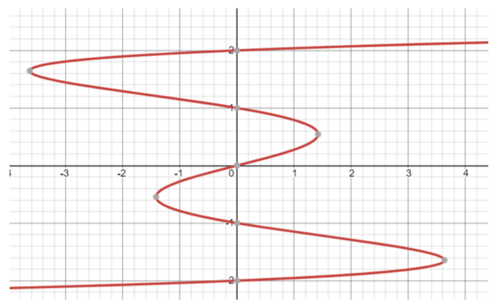
We are mindful that the construct of HTL belongs to the cognitive paradigm, hence our use of the term may not exactly align with the literature. We operationalize the HTL categories in Table 1 as *discursive competencies* that students can exhibit in their mathematical work. This allows us to characterize not just students' difficulties, but also their areas of strength. We examine the following research question: *What aspects of HTL can be observed in students' written responses to an implicit differentiation activity?*

Methods

This study is a part of the larger NSF-funded project for improving the teaching and learning of introductory STEM courses in a large public university in the northeast of the USA. Calculus I is taught in the format of large lectures (~120 students), taught by a faculty member three times a week; and recitations (~20 students), taught two times a week by graduate TAs. The reform efforts involved introducing active learning (CBMS, 2016) in recitations, the inclusion of LAs to support student work; and the use of conceptually rich activities that emphasize graphic representations of the calculus concepts. Figure 1 shows an activity on Implicit Differentiation.

Task 1: Consider the curve on the graph and the equation of the curve: $x = y^5 - 5y^3 + 4y$.

With your small group, answer the following questions using both the symbolic equation and the curve graph. Explain your reasoning.



1) Is it possible to express y as an explicit function of x?
   a) Explain your response using the graph.
   b) Explain using the symbolic equation for the curve: $x = y^5 - 5y^3 + 4y$.
2) Use implicit differentiation to find a formula for $\frac{dy}{dx}$.
   a) Explain the meaning of $\frac{dy}{dx}$. What does it represent?
   b) Sam noticed that his expression for $\frac{dy}{dx}$ does not contain x. Sam thinks that this means that the rate of change is constant for all points on the x-axes. Is Sam correct in his thinking or not? Explan.
3) a) Find the equation of the line tangent to the graph of $x = y^5 - 5y^3 + 4y$ at point (0,1). Add the tangent line to the sketch.
   b) Are there other points on the graph, where the slope of the tangent line is the same as you found in 3(a)? Respond using the graph, and then verify algebraically.
4) Determine all the points at which the graph of $x = y^5 - 5y^3 + 4y$ has a vertical tangent line. Explain how you know that. How many such points are there? Locate them on the graph.

Figure 1: Task 1 from the Implicit Differentiation Activity. Based on Boelkins et al., 2018 (p. 147).

Data Collection

Data collection occurred in Spring 2022. After a lecture on implicit differentiation, students worked on the Implicit Differentiation activity (ID-activity, hereafter) in the recitations. Students worked in groups of 3-4, but each person submitted their own worksheet, graded for completion. There was a standard textbook homework assignment, unrelated to graphical representation. The mid-term exam, about three weeks later, included an implicit differentiation close in style and content to the ID-activity. The data sources were 119 consenting students' written work on the ID-activity and the exam question. Here we only report on their ID-activity performance.

Data Analysis

The data were coded using the HLT (Table 1). Each response was coded as either exhibiting evidence of the student attaining a certain learning goal, not attaining it, or not enough evidence (Yes/No/NEE). For example, consider item 2a: "Explain the meaning of dy/dx. What does it represent?" A student's response "This represents the derivative, which is the slope of the tangent line. Rate of change" was coded as "Yes" for the evidence of attaining the goal *Graphic Tangent Slope* (G_TS). Only wrong or completely irrelevant answers were coded as "No" evidence of attaining an HLT goal. All other cases, including no response, were coded as NEE – not enough evidence. An example of NEE-coded response for question 2a is: "$\frac{dy}{dx} = y'$. Find derivative with respect to y." This response explains the notation rather than the meaning of dy/dx. Yet, it is not enough evidence to conclude the lack of attainment of the G_TS goal.

This method allowed us to identify the elements of HLT where students showed competence or lack thereof on a fine-grain scale. Figure 2 shows one student's work in response to question (2) "Use implicit differentiation to find the formula for dy/dx."

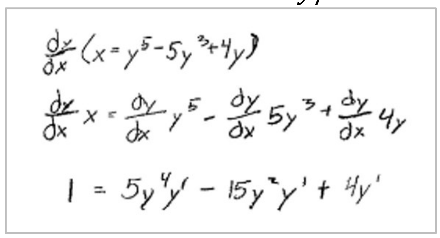


Figure 2: Coding Example

This work would score "Yes" for evidence of correctly using the chain rule to take the derivative with respect to the implicit independent variable (S_Ch) but would get NEE for symbolic differentiation (S_Diff), due to incomplete calculation of y'.

In our HLT (Table 1) the two codes S_Eval and G_TS were matched with two items, and the code G_Cor with four items. To get a single score per code we considered where most evidence is leaning, e.g., two or more "Yes" with all other NEE was coded as "Yes" (evidence of G_Cor attainment). One "Yes" with all other NEE was coded as NEE. Two or more "No" with all other NEE was coded as "No". Getting NEE in all four G_Cor items was coded as "No," since the student used none of the four opportunities to exhibit any evidence of coordinating graphic-symbolic representations. There were no "Yes/No" combinations in our data. The rest of the data was analyzed similarly, resulting in a total of 1190 codes (119 students x 10 codes/ HLT goals[2]).

Results

Table 2 shows the distribution of Yes/No/Nee codes in each of the HLT categories.

Table 2: Distribution of Yes/No/NEE codes by category, N=119

	Symbolic HLT goals					Graphical HLT goals				
	S_Rec	S_Ch	S_Diff	S_Eval	S_Tan	G_Rec	G_TS	G_VT	G_CR	G_Cor
Yes	67	100	109	107	95	109	92	35	102	20
No	0	0	0	1	0	0	2	0	1	66
NEE	52	19	10	11	24	10	25	84	16	33

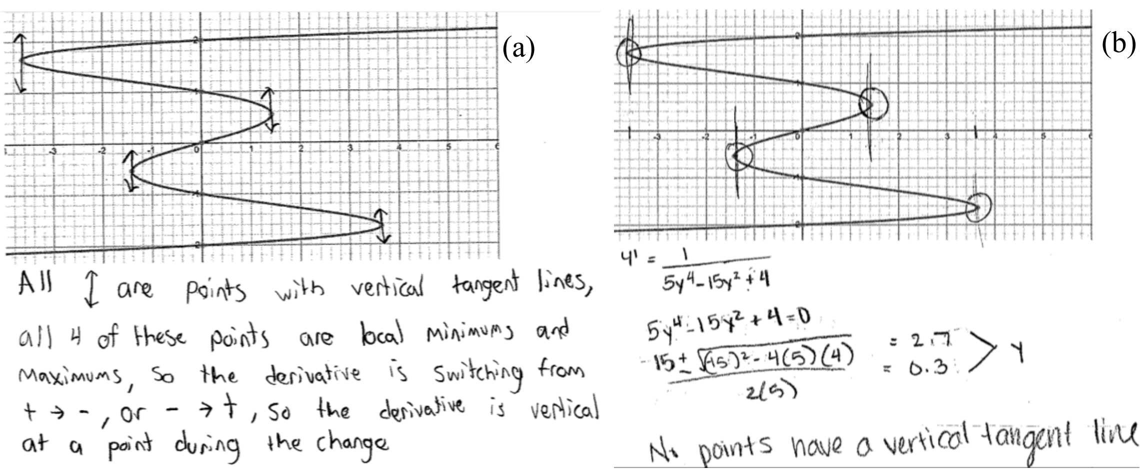
The modal categories with at least 90% show that students had the most success in calculating the derivative of the implicit function (S_Diff), evaluating the derivative at a given point (S_Eval), and recognizing that a graph of an implicit equation does not pass the vertical line test (G_Rec), hence cannot be considered as $y = f(x)$. We were encouraged by such a high percentage of "Yes" scores for the correct calculation of the implicit derivative (S_Diff), which requires correctly applying the chain rule (S_Ch) *and* correctly isolating y'. 100 students (84%) correctly applied the chain rule when taking the derivative with respect to an implicit variable.

The two goals posing the most difficulties to students were related to graphical representation: G_VT (vertical tangent) and G_Cor (coordination of symbolic and graphic modalities). The item related to G_VT was question 4 which called for finding all points where

[2] The codes S_Eq and S_Ch were merged since the worksheet explicitly asked for using implicit differentiation.

$x = y^5 - 5y^3 + 4y$ has vertical tangent line. Most students could correctly identify the four points on the graph, and state at these points y' is undefined. But only 35 students (29%) completed the calculation, and even fewer correctly interpreted the result.

We observed two types of difficulties in this item. One type is shown in Figure 3a. The student identified the points where the tangent line is vertical and marked them on the graph. But then proceeded to explain that "these points are local minima and maxima, so the derivative switches sign" from positive to negative and vice versa, hence the "derivative is vertical at a point during the change." The use of "vertical derivative" instead of "vertical tangent line" can be a mere carelessness, but it seems alarming that the student refers to these points as "local minimum and maximum." Moreover, although the derivative does change the sign at these points, it is unclear whether the student simply states known facts about local minima /maxima, or whether the student has some warrant for their claim, from the graph or from the derivative formula. This cannot be determined from a written text. Nevertheless, since this kind of response appeared repeatedly in our data, we assume that there is some underlying tendency at play, where students project their knowledge of critical points of explicit functions to implicit equations, but without recognizing how to adjust their prior knowledge to the new situation.



Figures 3 a & b: Sample student responses to item 4

Figure 3b shows another interesting response type. Here, the student marked the vertical tangent lines on the graph and correctly calculated the y-coordinate of the points where the derivative is undefined. But then the student made a seemingly contradicting conclusion, that there are "no points" with vertical tangent lines. Another student with a similar solution wrote that the answers "don't work." We conjecture the following explanation. The student solves the equation for y, obtaining the y-coordinates of the marked points on the graph. But then attempted to match these numerical values to the x-coordinates of these points. The x-coordinates being ± 3.63 and ± 1.91, they indeed do not match the calculations of 2.7 and 0.3, leading the student to a quite paradoxical conclusion that "no points have a vertical tangent line."

Students who correctly identified the points where the tangent line is vertical and calculated the y-coordinates by finding where the derivative is undefined scored a "Yes" for attaining the G_VT goal. However, correctly interpreting the meaning of the calculations corresponds to another HLT goal: coordinating graphic and symbolic representations of implicit equations (G_Cor). Four items contributed to a single G_Cor score: item 4, discussed above, and items 2b, 3a, 3b (Table 1). So, there were four opportunities for students to show some attainment of the

G_Cor goal, yet it appeared to be the most difficult one for the students. It was the only one with 66 students (55%) scoring "No" for that goal, and another 33 students (28%) scoring NEE.

The items that contributed most to these low scores were 3a and 3b, where students were asked to sketch the tangent lines, whose equations they found algebraically, on the graph. In item 3a, 95 students (80%) correctly found the equation of the tangent line at a point (0,1), but only 49 students (41%) could correctly draw this tangent line on the graph. The rest of the students either did not attempt this item (getting the NEE score) or drew an intersecting line instead of a tangent. In item 3b, 66 students (55%) justified algebraically that the tangent parallel to $y = -\frac{1}{6}x + 1$ will pass through (0,-1), but only 17 students (14%) drew the correct tangent line.

Overall, aggregating across all items and HLT goals, we obtained that the percentage of "Yes" scores was 80% for all symbolic HLT goals and only 60% for all graphic goals.

Discussion and Scientific Significance of the Study

The goal of the study was to examine students' performance on the implicit differentiation activity as they worked in small groups during a recitation in an otherwise traditional first-semester single-variable Calculus 1 course. Our study contributes to the existing literature in two ways. First, rather than using surveys or small-scale teaching experiments, our study was conducted in an authentic classroom setting and reflects what students can do collaboratively while being supported by GTA and LA. Second, our implicit differentiation instructional activity emphasized the graphical representation of implicit equations and coordination between the symbolic and graphic meanings of implicit differentiation.

Our results show that students performed much better on the symbolic HLT goals compared to graphical ones. Moreover, the percent of correct performance on utilizing the chain rule (S_Ch) and calculating y' (S_Diff) were much higher than what has been described in the literature 84% and 92% respectively, compared to about 50% in the studies by Chu (2019) and Kandeel (2021). The outcome may be attributed to the collective nature of student work, as opposed to individual survey data reported by Chu and Kandeel. In fact, the percentage of correct performance for S_Ch and S_Diff dropped to 66% and 62% respectively on individual exams, but the full discussion of these outcomes is beyond the scope of this paper.

The graphical interpretation of implicit curves, including graphing tangent lines and, specifically, coordinating between graphic and symbolic meanings of implicit functions presented the biggest challenges for the students. This outcome concurs with Borji and Martínez-Planell (2019). Students' challenges seem to be related to applying procedures and types of reasoning borrowed from their experience with explicit functions, e.g., attempting to interpret points with vertical tangent lines as "local minima and maxima", possibly due to the visual similarity, while ignoring the graph's orientation. Similarly, being used to finding the x-coordinates of critical points first, when obtaining y-coordinates from the calculation, students got confused and couldn't match them to the point on the graph. These challenges suggest that students require more exposure to and experiences with implicit curves to develop a better understanding of implicit functions. More research is needed to understand how students coordinate graphic-symbolic representations of implicit functions to support their learning.

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References

Boelkins, M., Austin, D., & Schlicker, S. (2018). *Active Calculus 2.1*.

Borji, V., & Martínez-Planell, R. (2019). What does 'y is defined as an implicit function of x'mean?: An application of APOS-ACE. *The Journal of Mathematical Behavior*, *56*, 100739.

Borji, V., & Martínez-Planell, R. (2020). On students' understanding of implicit differentiation based on APOS theory. *Educational Studies in Mathematics*, *105*(2), 163-179.

Conference Board of the Mathematical Sciences (CBMS). 2016 *Active learning in post-secondary mathematics education.* https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

Chu, C. G. (2019). *Investigation of student understanding of implicit differentiation*. The University of Maine.

Cottrill, J. F. (1999). *Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions.* (Unpublished doctoral dissertation). Purdue University, West Lafayette, Indiana.

Infante, N. M. E. (2007). *Students' understanding of related rates problems in calculus.* (Unpublished doctoral dissertation). Arizona State University, Tempe, AZ.

Jeppson, H. P. (2019). *Developing understanding of the chain rule, implicit differentiation, and related rates: Towards a hypothetical learning trajectory rooted in nested multivariation*. Unpublished Master's Thesis. Brigham Young University.

Kandeel, R. A. A. (2021). Learners' common errors in implicit differentiation: An analytical study. *Journal of Educational Research and Reviews, 9*(7), 198-207.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge University Press.

Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal, 12*, 74-91.

Mirin, A., & Zazkis, D. (2019). Making implicit differentiation explicit. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp.792–800). Oklahoma City, OK.

Sfard, A. (2015). Why all this talk about talking classrooms? Theorizing the relation between talking and learning. In Resnick, L. B., Asterhan, C. S. C., & Clarke, S. N. (Eds)., *Socializing intelligence through academic talk and dialogue*, (pp. 245-253). American Educational Research Association.

Speer, N., & Kung, D. (2016). The complement of RUME: What's missing from our research? In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp.1288–1295). Pittsburgh, PA.

Stewart, J. (2006). *Calculus: Early Transcendentals, 6E*. Belmont, CA: Thompson Brooks/Cole.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Harvard University Press.

Zandieh, M. (1997). *The evolution of student understanding of the derivative.* Oregon: Oregon State University Ph. D. Dissertation.

The Algebra Concept Inventory: Creation and Validation with Students Across a Range of Math Courses in College

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Even though algebraic conceptual understanding is recognized as a critical skill, existing larger-scale validated algebra assessments consist mostly of computational tasks, or only assess a very narrow range of conceptions in a smaller focused domain. Further, few instruments have been validated for use with college students. In this paper, we describe the creation and validation of an algebra concept inventory for college students. We describe how items were administered, revised, and tested for validity and reliability. Results suggest that algebraic conceptual understanding is a measurable construct, and that the instrument has reasonable validity and reliability. Revision and validation is ongoing; however, lessons learned thus far provide information about what conceptual understanding in algebra might look like and how it might be assessed.

Keywords: Algebra, conceptual understanding, concept inventory, assessment, validity

In college, needing to take algebra can be a barrier to degree completion (e.g., Adelman, 2006; Bailey et al., 2010), and extensive mathematics education research has documented K-12 students' difficulties with school algebra (e.g., Booth, 1988, 2011; Kieran, 1992). Struggles with basic algebra concepts learned in school also impact even those in higher-level college courses such as Calculus (e.g., Frank & Thompson, 2021; Stewart & Reeder, 2017). One reason students struggle with algebra is that algebra courses in college tend to focus on procedures disconnected from meaning-making (e.g., Goldrick-Rab, 2007; Hodara, 2011). While procedural fluency is important, it is critical to connect it with conceptual understanding (Kilpatrick, et al., 2001). Thus, there is a critical need to better understand and assess students' conceptions of algebra concepts. However, to date there are no widely-validated assessments that measure college students' conceptual understanding of algebra. Existing large-scale validated algebra assessments for K-12 tend towards computational skills, or focus on a narrow set of conceptions in a small subdomain. Tests of computational skills are often poor measures of understanding. Students may have robust conceptual understanding, but make smaller computational mistakes, especially if they have math or test anxiety (e.g., Ashcraft, 2002; Ashcraft & Kirk, 2001; Moran, 2016; Namkung et al., 2019). On the other hand, students may have little-or-no conceptual understanding, yet produce "correct" answers for "wrong" reasons (e.g., Aly, 2022; Erlwanger, 1973; Leatham & Winiecke, 2014).

We aim to address this gap by describing a first attempt to conceptualize, develop, and test college students' conceptual understanding in algebra using the *Algebra Concept Inventory* (*ACI*). This process continues, but we have chosen to write about results at this juncture with the hope they may be helpful for others interested in conceptualizing, measuring, and teaching conceptual understanding in algebra.

Literature Review

Several instruments have been created to test algebraic proficiency; however, none were designed to test a large body of algebraic concepts and conceptions. TIMMS and NAEP (Mullis, et al., 2020; National Center for Education Statistics, 2023) are widely validated at the international and national level, and contain some questions intended to assess conceptual understanding. There are also state-wide assessments that contain some questions intended to test conceptual understanding (e.g., Massachusetts Department of Elementary & Secondary Education, 2023; New York State Education Department, 2023). However, the main focus of all these instruments is computational skills.

There is one validated assessment that targets algebraic conceptual understanding in grades 1 to 5 (Ralston, et al., 2018), and one designed to assess a few specific algebraic concepts in middle school (Russell, 2019; Russell et al., 2009). Yet these instruments measure just a few conceptions, and were not designed for secondary or postsecondary students. As such, these often focus primarily on less complex or less abstract algebraic conceptions.

Some concept inventories have been developed that assess some student conceptions of algebraic concepts, but for students in more advanced courses only. For example, the Pre-calculus Concept Assessment (PCA) (Carlson, Oehrtman, & Engelke, 2010) and the Calculus Concept Readiness Instrument (CCRA) (Carlson, Madison, & West, 2010) explore some algebra concepts relevant to students in higher-level courses; while these have been tested through extensive cognitive interviews, larger-scale psychometric validation is still needed. Recently, researchers Hyland and O'Shea (2022) in Ireland generated a 31-item algebra concept inventory for college students, but it includes algebraic objects that would not be familiar to students in a first-year algebra course and has not yet been tested through cognitive interviews or psychometric analysis. Thus, an inventory that is valid for students starting in elementary algebra is needed, as well as more extensive large-scale psychometric testing of concept inventories more generally.

Method

A total of 402 unique items were developed and tested for the ACI. Items were administered to 18,234 students enrolled in all mathematics classes (except arithmetic) at a large urban community college campus. Data reported here were collected from spring 2019 to fall 2022, in eight separate waves. Data collection followed a common-item random groups equating design, which was selected because it allowed to investigate a large item pool while allowing a simultaneous calibration across multiple forms (de Ayala, 2009; Kolen & Brennan, 2004). For the first wave of testing, the last ten items on each form were anchor items, all taken from the National Assessment of Educational Progress (NAEP) grade 8 item bank. For subsequent waves, six anchor items were included: three of these were NAEP items and three were items that had performed well during the first wave of ACI testing. Each form had roughly 25 total items. Forms were randomly administered within in each class so there was no association between test form and class or instructor.

Just before answering inventory items, students were invited to participate in cognitive interviews, and paid for their time. In total, 135 interviews were conducted with students. Each was roughly 1-1.5 hours long and was structured as a "retrospective think-aloud" (Sudman et al., 1996). Research suggests that retrospective think-aloud protocols reveal comparable information to concurrent think-aloud protocols, and are less likely to have negative effects on task performance, particularly high-cognitive-load tasks (see e.g., van den Haak et al., 2003). Interviews were analyzed qualitatively to assess construct validity of the items, but there is insufficient space to report on that analysis here. Here we report only quantitative results.

We investigated each wave of the ACI through item-response theory analysis. First, items were dichotomized into pass-fail items using the response key. Then, two-parameter logistic models (Birnbaum, 1968) were estimated using marginal maximum likelihood (MML) on each wave, using the R package "mirt" (Chalmers, 2012). Because of planned missingness data collection design, the default number of model iterations was extended to allow for all models to converge successfully. Based on these models, we examined item parameters (difficulty and discrimination) and item information functions for item analysis, and computed person estimates using expected a posteriori (EAP) factor scores for convergent validity analysis. Reliability estimates were computed directly from IRT models. To investigate model fit, we computed item fit statistics, using the PV-Q1 statistic (and significance test) (Chalmers & Ng, 2017) for each item.

To investigate measurement invariance, we used multi-group IRT models and a model comparison approach. Because of the planned missingness design (and sometimes small observed subsample sizes), we used a piecewise DIF detection strategy (Thissen et al., 1993) that starts from a fully constrained model and drops constraints for each item separately. More specifically, with respect to each examinee characteristic considered, we first estimated a fully constrained model (where, across groups, item discriminations, difficulties, latent mean and variance are constrained to equality). Then, for each item, the same model was estimated, but with unconstrained item parameters (difficulty and discrimination), thus "temporarily" allowing differential item functioning (DIF) for the item. A likelihood ratio test was then performed to test if the model allowing DIF for the item had a better fit than the constrained model. This resulted in a series of tests of the significance of differential item functioning for all items. Because it is a multiple testing strategy, p-values were subsequently Bonferroni-corrected.

Validating the ACI

IRT Models: Item Discrimination and Difficulty

Results reported here were based on an item pool in which some items were dropped because they were deemed problematic (e.g., typographical errors; multiple correct answers); however, no items were dropped from analysis simply because of unsatisfactory IRT parameters. 2PL IRT models were run on all waves of data collection (Table 1). IRT models were run on all waves of data collection. While Rasch (or 1PL) models and 3PL models were also considered, 2PL models were chosen because unlike 1PL models, they allow discrimination to vary by item, and because they were considered more parsimonious, more useable for item selection (because item coefficients are more interpretable), and less prone to calibration errors than 3PL models due to their lower number of item parameters (San Martin et al., 2015).

Table 1. 2PL Model Coefficients Across all Eight Waves

Discrimination parameter	Proportion of Unique Items
>=0.65 "moderate"[a]	63.4%

>=1.35 "high"	31.3%
>=1.7 "very high"	18.5%
Difficulty parameter	Theta
mean	0.00
1st quartile	-0.85
median	-0.14
3rd quartile	0.63
Total number of unique items in 2PL models	399

[a] Characterizations of categories of discrimination parameters are taken from Baker (2001).

Discrimination is classified as "moderate" if it is ≥ 0.65, "high" if it is ≥ 1.35 and "very high" if it is ≥ 1.7 (Baker, 2001). Based on this, 63.4% of all items (253) have at least moderate, and roughly one-third have high or very high discrimination.

We also assessed item fit in the 2PL model for each wave using Chalmers' $PV - Q_1$ test, because it performs better than other fit statistics at controlling Type I error (Chalmers & Ng, 2017) (Table 2).

Table 2. Measures of Item Misfit in 2PL IRT Models

	Number of Items With Significant[a] Misfit[b]	Total Number of Items	Percentage of Items With Significant Misfit
Wave 1	1	33	3.0%
Wave 2	5	125	4.0%
Wave 3	4	66	6.1%
Wave 4	3	72	4.2%
Wave 5	8	100	8.0%
Wave 6	5	99	5.1%
Wave 7	2	39	5.1%
Wave 8	0	31	0.0%
Total	28	565	5.0%

[a] Significant at the $\alpha = 0.05$ level
[b] Misfit as measured by Chalmers' Chi-Square Statistic ($PV - Q_1$)

Only 5% of items were significantly misfitted by the 2PL models (for $\alpha = 0.05$), suggesting this is likely due to random variation.

Reliability

In IRT, the reliability of an item varies based on Theta, which represents the number of standard deviations above or below the mean an individual is on the measure of the latent trait. Table 3 shows various measure of reliability.

In Table 3 peak instrument values have excellent reliability ($R \geq 0.9$). There are also waves where excellent reliability ($R \geq 0.9$) can be obtained for values ranging from $\theta = [-2,7,2.2]$ (assuming a standard normal distribution of knowledge, this corresponds to satisfactory reliability for ~98% of examinees). Further, shorter tests can be constructed with only those items with the highest discrimination: for example, the 10 items with the best discrimination from Wave 1 yields a test with excellent reliability ($R \geq 0.9$) for $\theta = [-2,1]$.

Table 3. Reliability (R) for each wave of item administration of the ACI

	Theta at max info[a]	Info max[b]	R for info max[c]	theta w $R \geq$ 0.8	theta w $R \geq$ 0.9	Number of Items Tested
Wave 1	-1.4	26.4	0.96	[-2.8, 0.4]	[-2.4, -0.2]	33
Wave 2	-1.5	37.8	0.97	[-3.0, 2.1]	[-2.7, 0.9]	104
Wave 3	-0.6	24.3	0.96	[-2.3, 1.5]	[-1.8, 0.7]	57
Wave 4	-0.6	30.1	0.97	[-2.4, 2.1]	[-1.9, 1.2]	69
Wave 5	0.7	177.1	0.99	[-2.3, 2.9]	[-1.4, 1.8]	100
Wave 6	-0.6	105.3	0.99	[-1.7, 3.0]	[-1.0, 2.2]	99
Wave 7	-0.1	21.7	0.95	[-1.5, 1.8]	[-1.0, 1.1]	39
Wave 8	0.1	11.3	0.91	[-0.9, 1.2]	[-1.2, 0.3]	31

[a] info = 2PL IRT model information function
[bd] max = information function maximum for 2PL model
[e] $R = 1 - \frac{1}{Info}$
[c] expected reliability in Normal(0,1) ability distribution for 2PL models

Relationship Between ACI Score and Prior Algebra Course Completion: Convergent Validity

To explore convergent validity of the ACI, we explored the relationship between scores on the ACI (using theta scores from the 2PL model) to various measures of mathematics course level. For example, correlation of students' ACI scores with the level of algebra courses they have already successfully completed would be evidence of convergent validity. First, we consider linear regression models with level of student's course (where "level" is defined based on the algebra course pre-requisite requirements of the course) as the independent variable, and ACI score as the dependent variable (Table 4).

Table 4. Regression of course level (by algebra pre-requisite) in predicting theta scores from the 2PL model on the ACI, reference group: low

Course Level	Coefficient	SE	*p*-value (vs. low)	*p*-value (vs. high)
mid	0.347	0.014	0.000	0.000
high	0.700	0.017	0.000	

low = no algebra course prerequisite
mid = elementary algebra course prerequisite
high = intermediate algebra course prerequisite

In Table 4, differences in Theta score are significant ($p < 0.001$) for all pairwise comparisons. Students in "mid"-level courses scored on average 0.35 SD higher than those in "low"-level courses; and students in "high"-level courses scored on average 0.35 SD higher than those in "mid"-level courses (or 0.70 SD higher than in "low"-level courses). This provides strong evidence of convergent validity.

We also considered a more nuanced course sequence based on prerequisites (see Table 5).

Table 5. Sequence level of various courses in the sample, based on their prerequisites

Various elementary algebra courses	1
Various 100-level courses with an elementary algebra pre-requisite	2
Intermediate algebra courses	2
College algebra	2
Discrete math with intermediate algebra prerequisite	3
Precalculus	3
Math for elementary teachers with intermediate algebra prerequisite	3
Math for elementary teachers, second term	4
Advanced statistics with precalculus prerequisite	4
Introduction to geometry with precalculus prerequisite	4
Calculus I	4
Calculus II	5
Calculus III	6
Differential equations with Calculus II prerequisite	6
Linear algebra with Calculus II prerequisite	6
Abstract algebra	7

Rerunning linear regression models using this more refined set of levels again reveals a strong correlation between level and ACI score (Table 6).

Table 6. Regression of course position in longer mathematics curricular sequences (by classification given in Table 5) in predicting theta scores from the 2PL model on the ACI, reference group: sequence level 1

Course Position in Sequence	coeff	SE	*p*-value (vs. 1)	*p*-value (vs. 2)	*p*-value (vs. 3)	*p*-value (vs. 4)	*p*-value (vs. 5)	*p*-value (vs. 6)
2	0.504	0.017	0.000					
3	0.623	0.031	0.000	0.000				
4	0.888	0.023	0.000	0.000	0.000			
5	1.059	0.033	0.000	0.000	0.000	0.000		
6	1.232	0.041	0.000	0.000	0.000	0.000	0.000	
7	1.661	0.226	0.000	0.000	0.000	0.001	0.008	0.060

One of the largest gains (one half SD) was between sequence level 1 and 2 (see Table 6), which distinguishes between students who have or have not satisfied an elementary algebra prerequisite, providing further evidence of convergent validity, as the ACI is designed to focus on concepts relevant to elementary algebra specifically.

Differential Item Functioning: Measurement Invariance and Discriminant Validity

We also assessed potential differential item functioning (DIF) related to irrelevant examinee characteristics: race/ethnicity, gender, and English-language-learner status. This is an aspect of discriminant validity, as the ACI should measure algebraic conceptual understanding and not something else, like English literacy. Each wave was tested for DIF in three separate 2PL models: one for each characteristic. There was no consistent evidence of DIF on any of these factors. Only a negligible number of items had significant DIF for $\alpha = 0.05$ (using a Bonferroni correction for the number of tests within each model). Many items were tested in multiple waves, and none of these had significant DIF in more than one wave, suggesting that significant DIF in one wave for these items was likely due to random variation.

Limitations

The City University of New York, where this instrument was tested, is very diverse but not nationally-representative; however, this makes it useful for validation with marginalized students who have often been neglected in large-scale assessment validation. A current study is underway to validate the ACI on a larger national sample. The ACI has also only been validated with college students—further studies are necessary with younger students. The ACI has also been developed to make *diagnostic* judgements about *groups* of students—not high-stakes decisions for individuals—and thus we caution against that particular use of the ACI.

Discussion and Conclusion

Results from analysis suggest that algebraic conceptual understanding, as conceptualized by the items included on the ACI, is a measurable domain. IRT analysis indicated that a large proportion of items had good discrimination parameter estimates, suggesting that the final version of the ACI is likely to have an excellent ability to differentiate between students of various levels. Additionally, reliability was excellent for all waves, and results indicated that a shorter test could be constructed that would have excellent reliability for a large range of knowledge levels. The ACI also showed evidence of convergent validity, as students with higher algebra course prerequisites showed higher item success rates. Finally, only a negligible proportion of items showed differential item functioning with respect to race/ethnicity, gender, or English-language-learner status, indicating that the ACI had satisfactory measurement invariance with respect to these characteristics.

However, the ACI is only a first attempt at measuring algebraic conceptual understanding, and much more work needs to be done to map out in detail the various conceptions that students in different contexts hold of core algebra concepts, and determine how these can best be measured. The ACI provides only a single scale number; however, further work with cognitive diagnostic models on ACI items might provide more nuanced diagnostic information that could be particularly critical for instruction, by better modeling the complex layers of conceptions that students might have about various concepts in algebra. In reality, the kinds of knowledge that the ACI is trying to measure are quite complex, and capturing only a single score is, on its own, woefully inadequate if we hope to understand how students think algebraically and how various instruction and curriculum relate to this complex conceptual development. We see the ACI as just a first step in building out much more complex models of students' algebraic conceptions.

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References

Adelman, C. (2006). *The toolbox revisited: Paths to degree completion from high school through college* (ED490195). US Department of Education. http://www2.ed.gov/rschstat/research/pubs/toolboxrevisit/toolbox.pdf

Aly, G. (2022). Benny, Barbara, and the Ethics of EdTech. *Journal of Humanistic Mathematics*, *12*(2), 98–127.

Ashcraft, M. H. (2002). Math Anxiety: Personal, Educational, and Cognitive Consequences. *Current Directions in Psychological Science*, *11*(5), 181–185. https://doi.org/10.1111/1467-8721.00196

Ashcraft, M. H., & Kirk, E. P. (2001). The relationships among working memory, math anxiety, and performance. *Journal of Experimental Psychology: General*, *130*(2), 224–237. https://doi.org/10.1037/0096-3445.130.2.224

Bailey, T., Jeong, D. W., & Cho, S. (2010). Referral, Enrollment, and Completion in Developmental Education Sequences in Community Colleges. *Economics Of Education Review*, *29*(2), 255–270.

Baker, F. B. (2001). *The basics of item response theory*. ERIC.

Birnbaum, A. (1968). *Some latent trait models and their use in inferring an examinee's ability* (F. M. Lord & M. R. Novick, Eds.; pp. 395–479). Addison-Wesley.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In *The ideas of algebra, K-12: 1988 Yearbook* (pp. 20–32). NCTM. http://elementaryalgebra.cmswiki.wikispaces.net/file/view/Childrens+Difficulties+in+Beginning+Algebra.pdf/142535729/Childrens+Difficulties+in+Beginning+Algebra.pdf

Carlson, M., Madison, B., & West, R. (2010). The calculus concept readiness (CCR) instrument: Assessing student readiness for calculus. *Eprint arXiv:1010.2719.*

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruciton, 28*(2), 113-145. doi:https://doi.org/10.1080/07370001003676587

Chalmers, R. P. (2012). A Multidimensional Item Response Theory Package for the R Environment. *Journal of Statistical Software*, *48*(6). https://doi.org/10.18637/jss.v048.i06

Chalmers, R. P., & Ng, V. (2017). Plausible-Value Imputation Statistics for Detecting Item Misfit. *Applied Psychological Measurement*, *41*(5), 372–387. https://doi.org/10.1177/0146621617692079

de Ayala, R. J. (2009). *The Theory and Practice of Item Response Theory*. Guilford.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, *1*(2), 7–26.

Frank, K., & Thompson, P. W. (2021). School students' preparation for calculus in the United States. *ZDM – Mathematics Education*, *53*(3), 549–562. https://doi.org/10.1007/s11858-021-01231-8

Goldrick-Rab, S. (2007). What Higher Education Has to Say about the Transition to College. *Teachers College Record*, *109*(10), 2444–2481.

Hodara, M. (2011). *Reforming Mathematics Classroom Pedagogy: Evidence-Based Findings and Recommendations for the Developmental Math Classroom. CCRC Working Paper No. 27. Assessment of Evidence Series*. Community College Research Center, Columbia University; eric.

http://ezproxy.library.arizona.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=eric&AN=ED516147&site=ehost-live
Hyland, D., O'Shea, A. (2022). How Well Do High-Achieving Undergraduate Students Understand School Algebra?. *Can. J. Sci. Math. Techn. Educ.* 22, 818–834. https://doi.org/10.1007/s42330-022-00256-9
Kieran, C. (1992). *The learning and teaching of school algebra.*
Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In *Adding it up: Helping children learn mathematics* (pp. 115–118). National Academies Press. http://books.google.com/books?hl=en&lr=&id=df7ZX4a8fzAC&oi=fnd&pg=PA1&dq=Adding+it+up:+Helping+children+learn+mathematics.&ots=jtZBgTu9cF&sig=WWHYCFi9heYlUsZP0oIvw1wJWXw
Kolen, M. J., & Brennan, R. L. (2004). *Test equating, scaling, and linking*.
Leatham, K. R., & Winiecke, T. (2014). The case of the Case of Benny: Elucidating the influence of a landmark study in mathematics education. *The Journal of Mathematical Behavior*, *35*, 101–109. https://doi.org/10.1016/j.jmathb.2014.06.001
Massachusetts Department of Elementary & Secondary Education. (2017). Massachusetts comprehensive assessment system (MCAS). Retrieved from http://www.doe.mass.edu/mcas/
Moran, T. P. (2016). Anxiety and working memory capacity: A meta-analysis and narrative review. *Psychological Bulletin*, *142*(8), 831–864. https://doi.org/10.1037/bul0000051
Mullis, I. V., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). TIMSS 2019 international results in mathematics and science. *Retrieved from Boston College, TIMSS & PIRLS International Study Center website*: https://timssandpirls.bc.edu/timss2019/international-results.
Namkung, J. M., Peng, P., & Lin, X. (2019). The Relation Between Mathematics Anxiety and Mathematics Performance Among School-Aged Students: A Meta-Analysis. *Review of Educational Research*, *89*(3), 459–496. https://doi.org/10.3102/0034654319843494
National Center for Education Statistics. (2023). National assessment of educational progress (NAEP). Retrieved from https://www.nationsreportcard.gov/focus_on_naep/
New York State Education Department. (2017). Office of state assessment (OSA). Retrieved from http://www.nysedregents.org
Ralston, N. C., Li, M., & Taylor, C. (2018). The development and initial validation of an assessment of algebraic thinking for students in the elementary grades. *Educational Assessment*, *23*(3), 211–227.
Russell, M. (2019). Digital technologies: Supporting and advancing assessment practices in the classroom. In S. Brookhart & J. H. McMillan (Eds.), *Classroom assessment and educational measurement* (pp. 224–242). Routledge.
Russell, M., O'Dwyer, L. M., & Miranda, H. (2009). Diagnosing students' misconceptions in algebra: Results from an experimental pilot study. *Behavior Research Methods*, *41*(2), 414–424.
San Martín, E., González, J., & Tuerlinckx, F. (2015). On the unidentifiability of the fixed-effects 3PL model. *Psychometrika*, *80*, 450-467.
Stewart, S., & Reeder, S. (2017). Algebra Underperformances at College Level: What Are the Consequences? In S. Stewart (Ed.), *And the Rest is Just Algebra* (pp. 3–18). Springer International Publishing. https://doi.org/10.1007/978-3-319-45053-7_1

Sudman, S., Bradburn, N. M., & Schwarz, N. (1996). *Thinking About Answers: The Application of Cognitive Processes to Survey Methodology*. Jossey-Bass.
Thissen, D., Steinberg, L., & Wainer, H. (1993). Detection of differential item functioning using the parameters of item response models. In *Differential item functioning* (pp. 67–113). Routledge.
Van Den Haak, M., Jong, M. D., & Schellens, P. J. (2003). Retrospective vs. Concurrent think-aloud protocols: Testing the usability of an online library catalogue. *Behavior & Information Technology*, *22*(5), 339–351.

My Mom Supports My Teaching: One Mathematician's Journey in Learning to Teach

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In this contributed report, I tell a story surrounding one mathematician's learning to teach mathematics content courses designed for prospective elementary teachers. Namely, I describe the mathematician's background and the series of events that pushed this newer mathematics instructor to develop productive dispositions towards their students, leverage support from their mom, a former elementary teacher, and navigate challenging student questions in the classroom. This report draws on results from my dissertation study, a narrative inquiry of one mathematics instructor, involving a semester-long collection and analysis of classroom observations, observations of instructor meetings, weekly instructor reflection journal entries, three instructor interviews, and instructor-written mathematics teaching and learning autobiographies. A key implication is that mathematics instructors may benefit from early experiences developing productive mindsets towards teachers and teaching, and need access to individuals who value teachers and teaching and who avoid deficit discourses around teachers or students.

Keywords: narrative inquiry, post-secondary mathematics instructors, mathematics content courses, prospective elementary teachers, support for teaching

Introduction

Bruce was a prospective elementary teacher (elementary PT) in a mathematics content course designed for elementary PTs in a large midwestern university. During the second class of the semester, Bruce asked their instructor, Rowan, "in what grade would number lines be attainable?" Rowan responded:

> I'm not sure. I think counting numbers are Kindergarten and 1st grade. I think it would be attainable for 2nd or 3rd graders, but it would depend on the curriculum in the district. I'm not really the best person to ask because I know more about math--being from the math department--but if I was from the teaching department I might know more.

Bruce's question posed a challenge to Rowan, in part, because Rowan was teaching elementary PTs for the first time but also because Rowan did not have K-12 teaching experience. So how did Rowan learn to navigate challenging questions from their students, especially those that went beyond mathematics content and into intersections of mathematics content and knowledge/experience of K-6 students' mathematical thinking and/or K-6 schools?

As Bruce's question shows, learning to teach mathematics content courses designed for prospective elementary teachers (PTs) can be challenging. Mathematics content instructors must be prepared to not only teach mathematics content, but engage PTs in understanding children's mathematical thinking strategies, learning trajectories, and misconceptions (Carpenter et al., 1996; Carpenter & Moser, 1982; I et al., 2020). When considering how instructors might best be prepared and supported, however, one must recognize that there is significant variation in the backgrounds and expertise of mathematics content instructors (Masingila et al., 2012; Yow et al., 2016) and little is known about the preparation, knowledge, and experiences of mathematics content instructors (Even, 2008; Goos, 2009; Masingila et al., 2012; Oesterle, 2011; Zaslavsky & Leikin, 2004). Furthermore, most instructors who are newer to teaching PTs do not feel prepared and additionally report an absence of training, resources, and support at their institutions (Goodwin et al., 2014; Masingila et al., 2012; Quaisley, 2023; Yow et al., 2016). Mathematics content instructors, especially those newer to teaching PTs, need preparation and support that account for their background and teaching context.

In this contributed report, I draw on the findings of my dissertation study, a narrative inquiry connecting the background, preparation, and support of one instructor (Rowan) to their learning to teach mathematics content for PTs (Quaisley, 2023). Specifically, I detail the series of events that pushed Rowan to develop productive dispositions towards their students, leverage support from their mom, a former elementary teacher, and navigate challenging student questions, like Bruce's, in the classroom.

Literature Review

Not much research has been done with specific attention to newer mathematics content instructors' preparation and learning to teach PTs (Quaisley, 2023). What studies do exist typically focus on a few small slices of the knowledge or skills that may be important for instructors to develop (e.g., Masingila et al., 2018; Zopf, 2010) or discuss support structures that may benefit instructors (e.g., Castro Superfine & Li, 2014; Jackson et al., 2020; Shaughnessy et al., 2016; Suppa et al., 2020) but are not necessarily *about* the instructor and how they navigate this work outside of researcher intervention. Oesterle's (2011) summary of literature on post-secondary mathematics instructors holds relevance today:

> although the research is intended *for* instructors (or for course/program designers), there is very little research *about* the instructors… in this context there is a lack of documented research about what happens in ordinary mathematics content courses for preservice teachers, ones that are not undergoing studies for particular interventions. (Oesterle, p. 39, emphasis original)

What literature *about* instructors of PTs of mathematics do exist typically involve one experienced instructor reflecting on their experiences (e.g., Chauvot, 2009; Nicol, 1997; Tzur, 2001), or involve less than a handful of newer instructors (e.g., Oesterle, 2011; Van Zoest, 2006). Additionally, of the studies about newer instructors, most focus on mathematics *methods* instructors with experience teaching high school mathematics (e.g., Chauvot; Nicol, 1997; Tzur, 2001; Van Zoest, 2006). Hence, more perspectives need to be examined to thoroughly understand the complexities of learning to teach elementary PTs, specifically perspectives from mathematics content instructors who are not only newer to teaching elementary PTs, but newer to teaching more generally.

Studies about newer instructors of PTs of mathematics are important because of the opportunities they provide for insight into instructors' dilemmas, tensions, and challenges and to hypothesize about the connections among an instructor's background, teaching context and experiences. For instance, Nicol (1997) and Van Zoest (2006) both involved mathematics methods instructors with prior experience teaching high school mathematics and both studies revealed common tensions around teaching PTs to think critically about teaching. More studies that center instructor experience may reveal further commonalities among instructors with particular backgrounds and in particular teaching contexts. Such studies may also highlight key differences among instructors with similar backgrounds and teaching contexts, as well as previously undocumented challenges for those instructors.

Purpose Statement and Research Question

My research puzzle (Clandinin, 2013) was this: how might the backgrounds and teaching contexts of newer mathematics content instructors shape their experiences around learning to teach PTs? Chauvot's (2009) study calls for this kind of exploration: "different kinds of knowledge is needed to serve different roles" (Chauvot, p. 369). Connections between instructors' contexts and their learning to teach PTs do not appear to be well understood in the research literature yet are critical to making progress in understanding newer instructors' learning and growth, the challenges and successes they experience, the expertise they need to develop, and the support structures that might benefit them. Hence, the purpose of my study was to obtain a more nuanced understanding of a newer mathematics content instructor's experiences around learning to teach elementary PTs in their respective teaching context.

The central question that guided my study was: "How might a newer mathematics content instructor's background and preparation relate to the challenges and successes they experience around teaching and learning to teach mathematics content for elementary PTs?"

Theoretical Framing: Narrative Inquiry and 3-D Framework

To address my research question, I engaged in a narrative inquiry (Clandinin & Connelly, 2000) of one beginning instructor's experiences learning to teach mathematics for elementary PTs. Narrative inquiry centers on experience, both as the phenomenon being studied and the method used to understand experience (Creswell & Poth, 2018). I drew on Clandinin and Connelly's (2000) three-dimensional (interactional, temporal, and contextual) inquiry space to understand experience. First, the *interactional* dimension refers to both personal interactions, such as "an individual's feelings, hopes, reactions, and dispositions" (Creswell & Guetterman, 2019, p. 522), as well as social interactions and perspectives of other people. Second, the *temporal* dimension refers to time—past, present, and future—and means that one does not just consider the on-going experiences of the present, but also considers past events, anticipated future events, and how those might be relevant to someone's understanding of current events. Third, the *contextual* dimension refers to "the context, time, and place within a physical setting, with boundaries and characters' intentions, purposes, and different points of view" (Creswell & Guetterman, 2019, p. 522). When considered altogether, these three dimensions allow for a continuous and storied understanding of experience—not as an isolated event, but as a thread connected to the larger storyline of who someone was, is, and wants to become.

Context of the Study

The setting for my inquiry was Cardinal University (a pseudonym), a large mid-western university. Operations & Number Systems (O&NS; a pseudonym), the mathematics content course of interest for this study, is taken during the STEM semester, an integrated effort to connect the contents and pedagogies of mathematics, science, and technology. O&NS aims to develop elementary PTs' understanding of some of the earliest grades' mathematics content, such as the base-10 number system, the operations of addition, subtraction, multiplication, and division, fractions, and the properties of arithmetic. Because O&NS is taken alongside Elementary Mathematics Methods (EMM), as well as a practicum that meets at elementary schools for a full day twice a week, the hope is that elementary PTs will be able to see the value in what they are learning and apply it to teaching elementary mathematics.

I recruited one instructor, Rowan (she/they), at Cardinal University who was teaching O&NS and who identified as both a newer instructor of PTs and a newer instructor more generally. I am especially grateful to Rowan for giving me the precious gift of their time, energy, and thoughts, so that I could share their experiences. Other participants in my study included Rowan's students (18 total), as well as the other O&NS instructors (2 total) and EMM instructors (3 total).

Data Collection and Analysis

In spring 2022, I observed all of Rowan's classes and meetings with other instructors, conducted three interviews with Rowan (on their preparation, support, and overall learning), and collected Rowan's teaching and learning autobiographies, Rowan's weekly reflection journal responses, and Rowan's teaching and DEI statement. During each of my observations of Rowan's O&NS classes and instructor meetings, I composed detailed fieldnotes on what I saw and heard, as well as my interpretations. Following nearly every observation of a class or instructor meeting, I wrote descriptive and reflexive memos to assist me in interpreting situations and writing rich descriptions of themes (Emerson et al., 2011; Spradley, 2016) with specific attention to each dimension of Clandinin and Connelly's 3-D framework.

My goal for data analysis was to utilize the various data sources I collected to support me in restorying—a process of analyzing and reorganizing—Rowan's experiences. I analyzed my fieldnotes, along with the early (on preparation), mid-semester (on support), and end of semester interview (on overall learning) transcripts, Rowan's reflection journal entries, Rowan's mathematics learning and mathematics teaching autobiographies, and Rowan's teaching and DEI statement using cycles of open-coding (Saldaña, 2016) based on broad categories related to my research questions: (a) challenges, (b) successes, and (c) prior experiences, preparation, and support. As I re-read each and every piece of data and coded ideas, excerpts, or interactions within the above categories, I organized ideas and interactions from classroom fieldnotes and memos, instructor meeting fieldnotes and memos, and reflection journal responses into tables based on these categories chronologically and contextually. I also wrote analytic memos as I collected data, which provided a crucial support to me in hypothesizing about themes or narrative threads.

Results: Rowan's Learning to Navigate Challenging Student Questions

In Quaisley (2023), I described numerous concurrent challenges Rowan experienced in learning to teach mathematics content courses for elementary PTs, such as time constraints as a graduating Ph.D. candidate, the burden of unfinished grading assignments, and an overall sense of feeling "uneasy" and "unsteady," often with respect to learning "different" and "hard" content. At the same time, Rowan experienced successes around developing positive beliefs about PTs and teaching O&NS, navigating challenging student questions, learning to seek external support for teaching (and in general), having students pass exams, and surviving the semester without any "catastrophic failures." For this report, I detail the interconnected nature of Rowan's successes in developing productive dispositions towards their students, leveraging external support for teaching, and navigating challenging student questions in the classroom.

Rowan's Background & Prior Experiences

At the time of the study, Rowan (she/they) was a mathematics graduate student instructor in their fifth and final year in a mathematics Ph.D. program at Cardinal University (a pseudonym). Throughout those five years, Rowan taught calculus recitations and lecture and Intermediate Algebra, as well as assisted teaching a mathematics content course for practicing teachers during a two-week summer session (Math for Teachers). In some semesters, Rowan also taught these courses from the position of associate course convenor—a role involving additional responsibilities such as creating common exams and observing instructors' classes. In the semester in which this study occurred, Rowan was teaching O&NS for the first time.

An influential person in Rowan's story is their mom, a former elementary teacher, who made a significant impact on both Rowan's desire to teach and their perceptions of teaching.

> My mom impacts just my teaching generally, not even in relation to this course, because my mom was a grade-school teacher and then was an administrator of preschool. And she, she loved it. My mother loved teaching. And I think that that really has impacted the entire trajectory of my life... So, definitely my mom has had a big influence on like wanting to teach, and I think a lot of that comes from her passion for it, and also all the discussions we had as I was growing up about like what teaching is. (Rowan, Final Interview)

These early experiences with their mom further supported Rowan in developing an asset-oriented mindset towards practicing teachers in the Math for Teachers course.

> The students were a treat though. Since they teach, they have an empathetic understanding of the complexities of teaching. Several of the students also reminded me of my mom—my mother was a grade-school teacher, and they had similar spirits and mentalities around teaching and learning… we ended up having a lot of really productive conversations where I would give them advice on mathematics, and they would give me advice on teaching. (Rowan, Autobiography)

Specifically, Rowan positioned practicing teachers as knowledgeable experts, who not only know how to teach elementary students but have valuable knowledge relevant to an instructor of undergraduate mathematics courses.

Developing Positive Dispositions

Rowan began the semester excited to teach and voiced optimistic opinions about their O&NS students in their early reflection journals. For instance, Rowan wrote in their week one journal: "I think the students will be fun and easy to work with. Since they want to be teachers, they'll have an appreciation and understanding for both the teaching methods I use and the teaching methods for mathematics they're learning" (Week 1 Reflection Journal). By the end of the semester, some of Rowan's early semester beliefs about prospective teachers still resonated with Rowan. Rowan explained how their thinking was influenced by their mom and their previous experiences teaching Math for Teachers:

> [Rowan] says that it's really easy for them to think about what it's like to be a PT just because their mom was a grade-school teacher, and she shared with [Rowan] what it was like learning to teach. [Rowan] says, although that was 35 years ago, and things are different now. [Rowan] says that they also taught teachers when they were assisting with [Math for Teachers] and that was the greatest experience they ever had. [Rowan] says that the teachers were really great. (Researcher Fieldnotes, Week 15 Classroom Debrief)

Navigating Challenging Student Questions

From the beginning of the semester, one of the classroom events that Rowan came to rely on as "business-as-usual" was their students regularly asking Rowan questions not just about mathematics content, but questions at the intersection of mathematics content and knowledge of K-6 students' mathematical thinking and/or K-6 schools. Early on, Rowan found these questions especially challenging. Recall from the introduction that right from the second class at the beginning of week two, Bruce asked Rowan, "in what grade number lines would be attainable?" Bruce's question challenged Rowan to reflect on their ability to support their O&NS students' inquiries about K-6 students' mathematical thinking early on:

> I wish I had a better sense of the timeline on which elementary students would learn this material. A [O&NS] student asked when a [K-6] student would learn about or be able to conceptualize the idea of decimals in base-ten notation and I wasn't able to provide

anything useful, but I think it would help the [O&NS] students to be able to consider the age of their [K-6] students while they learn the material. (Week 2 Reflection Journal)

During the preparation interview (also in the second week), Rowan additionally reflected on the extent to which they "have the authority" to anticipate K-6 student thinking and worried about missing important ideas behind student thinking. Perhaps addressing questions involving knowledge of or experience with K-6 students' mathematical thinking might not have presented itself as a challenge for Rowan if they were asked these questions infrequently. However, Rowan was asked such questions in front of the whole class almost every week starting from the second week of the semester through at least week ten of the semester.

Why Not Seek Internal Support?

So how would Rowan navigate challenging questions from their O&NS students? One way in which Rowan might have chosen to navigate these questions is by seeking support internal to their institution. Rowan valued the knowledge and expertise of those with K-12 teaching experience, and as a result, they wrote about wishing to connect with practicing elementary teachers as a desired support in their reflection journals and considered reaching out to EMM instructors (all three of whom had K-12 teaching experience) as a support they did have access to. An unfortunate experience at the third and final joint O&NS-EMM instructor meeting influenced Rowan to not exercise this possibility, however. One of the EMM instructors who was looked upon to lead the meetings joined the Zoom call for the meeting and then promptly left after a few minutes. Rowan recounted feeling that O&NS instructors were "not worth their (EMM instructors') time" during the mid-semester interview about their support, and lost faith in the STEM semester as a collaborative experience between O&NS and EMM instructors.

Seeking External Support from Mom

As Rowan did not have access to practicing K-6 teachers or believe that EMM instructors could be relied upon for support, a crucial way in which Rowan navigated challenging questions from their O&NS students throughout the semester was by regularly relying on support from their mom. Following each class, Rowan would debrief with me, then call their mom and debrief with their mom on their drive home. Conversations with their mom supported Rowan in reflecting on their own K-6 mathematics learning experiences, as well as some of their mom's experiences teaching K-6 mathematics. These conversations also built Rowan's confidence:

> But definitely was impactful was thinking about, thinking about and being able to talk through what I had done (as a K-6 student) in comparison to what I am teaching. And it also helped because then I could more confidently tell my (O&NS) students, 'Oh, like, when I learned it, and possibly when you learned it, because you're only, like, a little bit younger than I am. Like, this is. I remember this being a thing'. My mom said that this was a thing. And I think that connection is useful to me. I wish I had more of it, but it would be so hard to track down because, like, I don't fully remember my experience, right? I did have to talk it through with my mom, and my mom was very involved in my schooling, which is why she remembers. (Rowan, Final Interview)

A Conversation with Kenny

An important way in which Rowan's positive dispositions towards students and their conversations with their mom may have supported their navigation of challenging questions was that Rowan demonstrated empathy for multiple perspectives, rather than asserting a superior perspective. Consider the following conversation[1] between Rowan and one of their O&NS students, Kenny, during week ten of the semester, in which Rowan offered a possible rationale for the actions of Kenny's cooperating teacher and directed the conversation away from judging those actions as right or wrong. Instead, Rowan discussed the cooperating teacher's actions as a possible approach to supporting students' learning:

Kenny: What is the most common way we do this today? My understanding is that [City] public school is moving towards partial products method. I've noticed that some students skip carrying a step, and that you say thirteen tens rather than a hundred and thirty. Does that make sense?

Rowan: That makes sense. I know from the outset that the 2 is in the tens place and allowing that mindfulness of place from the outset... [Rowan works an example with Kenny's input.]

Kenny: I understand it, but I want to know what's right for the process for kids to understand... [Rowan does another example with 38 x 6 and writes 228.] I think I understand why my cooperating teacher emphasizes it, but I also don't quite understand...

Rowan: Yeah, I think the partial products just emphasizes more of the steps. Have you ever seen kids who want to add up all the numbers from left to right and end up with too many digits? The partial products method is trying to prevent that. [Kenny nods their head up and down as Rowan explains.]

Kenny: Yeah, I think that makes sense. Sorry for taking up the time. [Rowan reassures Kenny that its fine and then asks if there are more questions.]

Discussion and Implications

Even though Rowan did not feel they initially had the preparation or the authority to address questions around K-6 students' experiences and mathematical thinking, their offering of mathematical perspectives through an empathetic lens in their conversation with Kenny provides one reason to view Rowan's navigation of this challenge as a success. Navigating students' questions and maintaining a classroom environment of openness, curiosity, and mutual respect for mathematical perspectives, but especially teaching and teachers, is not a given. Rather, this may be a skill Rowan developed through reflective practice and their desire to build productive dispositions towards teachers and teaching.

Rowan's experiences suggest that mathematics instructors may benefit from early experiences developing productive mindsets towards teachers and teaching. For professional developers, an explicit introduction to and discussion of asset-based frameworks or mindsets early in mathematics instructors' preparation for teaching may support instructors in fostering more productive beliefs in the long-run. Furthermore, not all mathematics instructors have significant relationships with individuals who value teachers and teaching and who avoid deficit discourses around teachers or students. Thus, it is important for departments to carefully consider the resources, including people, that instructors have access to in learning to teach mathematics content courses for PTs.

References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20. https://doi.org/10.1086/461846

[1] Conversation is shortened and approximated from my fieldnotes—close to participants' words, but not verbatim.

Carpenter, T. P. & Moser, J. M. (1982). The development of addition and subtraction problem-solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 9-24). Routledge. https://doi.org/10.4324/9781003046585

Castro Superfine, A. & Li, W. (2014). Developing mathematical knowledge for teaching teachers: A model for the professional development of teacher educators. *Issues in Teacher Education, 23*(1), 113-132.

Chauvot, J. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator-researcher. *Teaching and Teacher Education, 25*(2), 357-370. https://doi.org/10.1016/j.tate.2008.09.006

Clandinin, J. (2013). *Engaging in narrative inquiry: Mapping a methodology*. Sage Publications.

Clandinin, J. & Connelly, M. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass Publishers.

Creswell, J. W. & Guetterman, T. C. (2019). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (6th ed.). Pearson.

Creswell, J. W. & Poth, C. N. (2018). *Qualitative inquiry and research design: Choosing among five approaches* (4th ed.). Sage Publications.

Emerson, R. M., Fretz, R. I., & Shaw, L. L. (2011). *Writing ethnographic fieldnotes*. University of Chicago Press.

Even, R. (2008). Facing the challenge of educating educators to work with practicing mathematics teachers. In B. Jaworski & T. Woods (Eds.), *The international handbook on mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 57-73). Rotterdam, Netherlands: Sense Publishers.

Goodwin, A. L., Smith, L., Souto-Manning, M., Cheruvu, R., Tan, M. Y., Reed, R., & Taveras, L. (2014). What should teacher educators know and be able to do? Perspectives from practicing teacher educators. *Journal of Teacher Education, 65*(4), 284-302. https://doi.org/10.1177/0022487114535266

Goos, M. (2009). Investigating the professional learning and development of mathematics teacher educators: A theoretical discussion and research agenda. *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1), 209-216.

I, J., Martinez, R., & Dougherty, B. (2020). Misconceptions on part-part-whole proportional relationships using proportional division problems. *Investigations in Mathematics Learning, 12*(2), 67-81. https://doi.org/10.1080/19477503.2018.1548222

Jackson, B., Hauk, S., Tsay, J., & Ramirez, A. (2020). Professional development for mathematics teacher education faculty: Need and design. *The Mathematics Enthusiast, 17*(2&3), 537-582.

Masingila, J., Olanoff, D., & Kimani, P. (2018). Mathematical knowledge for teaching teachers: Knowledge used and developed by mathematics teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education, 21*(5), 429-450. https://doi.org/10.1007/s10857-017-9389-8

Masingila, J., Olanoff, D., & Kwaka, D. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education, 15*(5), 347-358. https://doi.org/10.1007/s10857-012-9215-2

Nicol, C. (1997). Learning to teach prospective teachers to teach mathematics: The struggles of a beginning teacher educator. In J. Loughran & T. Russell (Eds.), *Teaching about teaching: Purpose, passion and pedagogy in teacher education* (pp. 95-116). The Falmer Press.

Oesterle, S. M. (2011). *Diverse perspectives on teaching Math For Teachers: Living the tensions*. Unpublished doctoral dissertation. Retrieved from https://summit.sfu.ca/item/11259

Quaisley, K. (2023). *Learning to teach prospective elementary teachers: A narrative inquiry of a newer mathematics content instructor*. Unpublished doctoral dissertation. Retrieved from https://digitalcommons.unl.edu/teachlearnstudent/140/

Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). Sage.

Shaughnessy, M., Garcia, N., Selling, S., & Ball, D. (2016). What knowledge and skill do mathematics teacher educators need and (how) can we support its development? In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 813-820). The University of Arizona.

Spradley, J. (2016). *The ethnographic interview*. Waveland Press.

Suppa, S., DiNapoli, J., Thanheiser, E., Tobias, J., & Yeo, S. (2020). Supporting novice mathematics teacher educators teaching elementary mathematics content courses for the first time. *The Mathematics Enthusiast, 17*(2&3), 493-536.

Tzur, R. (2001). Becoming a mathematics teacher-educator: Conceptualizing the terrain through self-reflective analysis. *Journal of Mathematics Teacher Education, 4*(4), 259–283. https://doi.org/10.1023/A:1013314009952

Van Zoest, L., Moore, D., & Stockero, S. (2006). Transition to teacher educator: A collaborative effort. *AMTE monograph, 3*, 133-148.

Yow, J. A., Eli, J. A., Beisiegel, M., McCloskey, A., & Welder, R. M. (2016). Challenging transitions and crossing borders: Preparing novice mathematics teacher educators to support novice K-12 mathematics teachers. *Mathematics Teacher Education and Development, 18*(1), 52-69.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth in practice. *Journal of Mathematics Teacher Education, 7*(1), 5–23. https://doi.org/10.1023/B:JMTE.0000009971.13834.e1

Zopf, D. (2010). *Mathematical knowledge for teaching teachers: The mathematical work of and knowledge entailed by teacher education*. Unpublished doctoral dissertation. Retrieved from https://deepblue.lib.umich.edu/handle/2027.42/77702

College Students' Conceptualizations of Symbolic Algebraic Properties

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Here we explore how college students across a wide range of courses may conceptualize symbolic algebraic properties. We draw on the theory of Grundvorstellungen (GVs) to analyze how learner conceptions may or may not align with instructional goals. In analyzing interviews, several categories of conceptions (descriptive GVs) emerged that may help us to better understand how students conceptualize symbolic properties during instruction.

Keywords: Algebraic properties, Syntactic reasoning, Equivalence, Algebraic transformation.

Mathematical properties are critical to justifying symbolic transformation, especially in algebra and domains that rely on algebraic representation. However, learners often use properties in ways that are not mathematically valid (e.g., Hoch & Dreyfus, 2004; Mok, 2010) and instruction may not address the use of symbolic properties (and their role in justifying transformation) explicitly enough (e.g., Barnett & Ding, 2019). Here we focus on learners' ability to identify parallel syntactic structure between symbolic properties and symbolic algebraic representations, and we explore how this may relate to learners' conceptions.

Properties and Forms

Because here the focus is on how properties are used to transform symbolic representations, we define a *symbolic property* as any mathematical statement that can be used to transform a symbolic object into an equivalent one with a different form. Two examples are: 1) the definition of negative exponents, e.g.: $x^{-n} = \frac{1}{x^n}$ for $x \neq 0$; and 2) this statement about equivalent equations: $A \cdot B = C \leftrightarrow A = \frac{C}{B}$ (when $B \neq 0$). The key feature of this definition is that 1) could be used to replace an expression of the form x^{-n} with one of the form $\frac{1}{x^n}$ (or vice versa), and 2) could be used to replace an equation of the form $A \cdot B = C$ with one of the form $A = \frac{C}{B}$ whenever $B \neq 0$ (or vice versa). Symbolic properties are made up of two sides, each of which can be viewed as a separate object, which are often referred to colloquially during instruction as "*forms*" (e.g., the "forms" x^{-n}, $\frac{1}{x^n}$, $A \cdot B = C$, and $A = \frac{C}{B}$ above).

Relatively little is known about learners' conceptions of symbolic properties. Existing research has focused on classifying errors that learners make when using properties to compute or transform (e.g., Hoch & Dreyfus, 2004; Mok, 2010); on learners' justifications for why properties are true; or on learners' ability to derive properties from arithmetic patterns (e.g., Hunter et al., 2022). Schüler-Meyer (2017) has investigated learners' structure sense for the distributive property (e.g., Schüler-Meyer, 2017), there is a dearth of research looking at this for symbolic properties more generally. Given how critical using and understanding symbolic properties and forms is for transforming symbolic representations (Kieran, 2011), it is essential that more research investigate how students conceptualize symbolic properties more generally, to address this gap in the research literature.

Theoretical Framework

Grundvorstellungen

We use prescriptive and descriptive *Grundvorstellungen* (GVs) (or "fundamental conceptions") to frame this research. *Prescriptive* GVs describe conceptions that are the goal(s) of instruction (vom Hofe, 1995); *descriptive* GVs describe actual conceptions that learners hold, which may or may not reflect prescriptive GVs. Comparing these two types of GVs can then be used to guide curriculum or instruction (Greefrath et al., 2016). Both descriptive and prescriptive GVs are intended to evolve with research over time. Further, one concept may have multiple GVs and vice versa. First we describe two *prescriptive* GVs for symbolic properties (Figure 1). After analyzing student interviews, we will present some descriptive GVs in the Results section.

Equivalence-Preserving GV	Symbolic properties describe a method for replaceing one symbolic representation with another equivalent one, based on a context-dependent pre-existing definition of equivalence (e.g., insertion equivalence; Prediger & Zwetzschler, 2013)
Mapping GV	For equivalence to be preserved, the following criteria must be met: The form on one side of the symbolic property must be mapped bijectively to the symboic representation so that: 1) A unified subexpression is mapped to each variable in the form; 2) All other symbols are mapped to notation in the form with the same syntactic meaning (e.g., different notations for multiplication can be mapped to one another).

Figure 1: Two Related Prescriptive GVs for Symbolic Properties

Operational vs. Structural Conceptions and Extracted vs. Stipulated Definitions

In constructing models of learners' descriptive conceptions, we were also influenced by research on operational vs. structural conceptions (Sfard, 1992) and extracted vs. stipulated definitions (Edwards & Ward, 2004). A learner with an *operational* conception views properties as a process of computation, while a learner with a *structural* conception views them as abstract objects (e.g., canonical representations of particular algebraic structures). Sometimes learners treat something as an object that is not the reification of any process, and this is called a *pseudostructural* conception (Sfard, 1992, p. 75). The operational/structural distinction is related to the prescriptive Mapping GV of Symbolic Properties, which focuses on conceptualizing forms within a property structurally as objects.

Extracted definitions emerge organically from observed usage of a term (e.g., when a learner extracts meanings for a property based on how it was used during computation in instruction). In contrast, *stipulated* definitions are explicitly stated—to determine if something fits the definition, one must consult the definition directly (Edwards & Ward, 2004). The extracted/stipulated distinction is related to the Equivalence-Preserving GV of Symbolic Properties, as a core stipulated part of the properties definition is that they preserve equivalence (in addition, the type of equivalence that is preserved must be based on a stipulated definition).

Methods

This project is based on 102 cognitive interviews that were conducted with college students in the US in 18 courses ranging from elementary to linear algebra on items from the Algebra Concept Inventory (Wladis et al., 2018); courses included both STEM and non-STEM courses. Students interviewed were diverse in terms of gender, race/ethnicity, national origin, and English language learner status. Thematic analysis (Braun & Clarke, 2006) was combined with an initial theoretical stance focused on noticing both extracted vs. stipulated definitions (Edwards & Ward, 2004) and operational vs. structural (Sfard, 1992) conceptions, as well as the extent to which learners provided potential evidence of Equivalence-Preserving or Mapping GVs. This allowed for the resulting coding framework of learners' descriptive GVs of symbolic properties to contain both emergent and confirmatory aspects.

Results and Discussion

Analysis of cognitive interviews resulted in a framework of learners' *descriptive* GVs of symbolic properties (Figure 2). Here operational vs. structural conception categories relate to how closely learners' GVs align with a Mapping GV and extracted vs. stipulated definition categories relate to how closely they align with an Equivalence-Preserving GV (Figure 1).

	Extracted Definition	Stipulated Definition
Operational Conception of Properties	**Pseudo-process GV:** Learners see properties as a cue to a computational process, and their approaches are extracted from prior experience rather than based on stipulated definitions. They often draw on surface structure rather than syntactic meaning. For example, learners may conceptualize the distributive property as an instruction to "take what is on the outside of the parentheses and put it next to each thing on the inside", regardless of the specific operations involved.	**Process GV:** Learners see properties as a cue to a computational process, but attend to syntactical meanings and/or equivalence as a justification (e.g., checking for appropriate operations in the expression; checking that original and resulting expressions are insertionally equivalent). However, they may struggle to conceptualize properties as objects to which structures in the expression or equation can be mapped one-to-one, and thus may have difficulty generalizing the use of properties to more syntactically complex symbolic representations.
Structural Conception of Properties	**Pseudo-object GV:** Learners conceptualize a property as something that requires mapping to the specific forms in the property, but the mapping is still somewhat ill-defined and/or based on extracted notions, such as what "looks right".	**Object GV:** Learners conceptualize the property as an object, such as a canonical form, to which the specific mathematical object (i.e., expression, equation, etc.) must be mapped one-to-one, in such a way that preserves syntactic meaning. They recognize that it is these criteria that preserve equivalence.

Figure 2: Framework to Categorize Descriptive GVs for Symbolic Properties

We illustrate the framework by presenting some examples of student work.

Operational Conceptions

Many learners appeared to draw on operational conceptions of symbolic properties. First we consider a student, Iota, who was enrolled in an introductory statistics course (elementary algebra was a pre-requisite), who was given seven questions with the following form:

Q6: Which of the following could result from using the **distributive property** to rewrite the expression $(x + 2)(3x + 7)$?

a. $x + 2 \cdot 3x + 7$
b. $x \cdot 3x + 2 \cdot 7$
c. $x + 2 \cdot 3x + 2 \cdot 7$
d. $(x + 2) \cdot 3x + (x + 2) \cdot 7$
e. None of the above.
f. I don't know the distributive property.

Figure 3: Task discussed with Iota during the interview

Other versions of this item used the following expressions: Q1: $(2x + 1)2$; Q2: $x - (2x + 1)$; Q3: $2(2x \div 1)$; Q4: $2(x \cdot y)$; Q5: $(2x + 1)^2$; and Q7: $2(xy)$. Iota stated that the distributive property could be used to rewrite the expression for each of these. They correctly chose equivalent expressions that could be the result of the distributive property for Q1 ($2x \cdot 2 + 1 \cdot 2$), Q2 ($x - 2x - 1$), and Q6 $(x + 2)(3x + 7)$ (option D). However, they also incorrectly chose "results" of the distributive property for Q3 ($2 \cdot 2x \div 2 \cdot 1$), Q4 ($2x \cdot 2y$), Q5 ($(2x)^2 + 1^2$) and Q7 ($2x2y$) that suggest that they may conceptualize the distributive property as an instruction to do something like "take what is outside the brackets and apply it to each 'thing' inside the brackets". At the same time, Iota's is able to conceptualize $(x + 2)$ as a unified subexpression within $(x + 2)(3x + 7)$ that could be "distributed" to each term in $3x + 7$, which suggests that Iota is able to think structurally in key ways. Iota explains:

> Because obviously two can distribute [makes motion with fingers as though moving two from left to right twice] with the one in parentheses. So two in the front can distribute to $2x$ multiply by $2y$. So it's gonna be $2x$ multiply by $2y$ [repeats distributive motion with fingers]—that's the result.

Here Iota focuses solely on describing a computational process based on surface similarities, without considering the mathematical validity of that computation, consistent with a pseudo-process GV. We see further evidence of this later in the interview:

Interviewer: What is the distributive property?

Iota: Distribute property is like that you can use the main number or main groups to distribute to each of another number or another groups.

Interviewer: So is that like here [highlighting $(x + 2)$ in Q6], is $x + 2$ the main number?

Iota: It's a main group. Yes.

Interviewer: And then you apply that to each of the ones [motions to $3x$ and 7 in Q6]?

Iota: Yes.

Interviewer: Okay. So, I noticed that this one [highlights + in expression $(3x + 7)$ in Q6] has a plus sign in between them. Is the distributive property only for the plus sign or could it also be subtraction? Could it be multiplication or division?

Iota: So, yeah, it could be subtraction, multiplication ... Could be any sign, but when you calculate, when you are doing it, you have to do with that own sign.

Here Iota provides further evidence that they are viewing the distributive property as an extracted process here, where whatever is outside the brackets is multiplied by each "group" inside the brackets, preserving the original operation between the multiple "groups" inside the brackets. However, when Iota was interviewed about Q7 ($2(xy)$), they appear to shift to a process view, checking for mathematical validity of the transformation results by checking insertion equivalence through arithmetic computation:

Iota: Sometimes when I see these kind of questions, at first I may think its right answer is A ($2x2y$), but what I normally do is I double check the answer. So, I create some equations and I double check it, it's incorrect. So, for this case, I create like x is *3.* Okay, let me type it now, y is *2*.

x=3, y=2

2(3*2) = 2*6 =12

2*3*2*2 = 24

I think it's wrong. So, I say no... I don't know why, but this is very tricky question for me ... So, *x* and *y* multiply each other should be done before multiply[ing] the one outside.…I don't know, it's not look like a distributive property for me. It looks like the way to calculate is you do the xy first because in parentheses, and after you get the result of xy you do with the number *2.* So, I don't think this one is like a distributive property ...to be honest, I don't know why. I don't think it's A, but I just feel it's not.

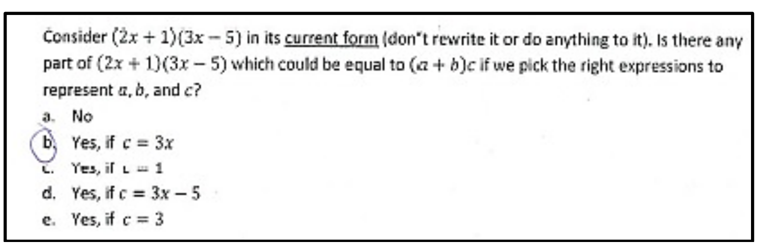
Interviewer: This strategy that you were doing, replacing x and y with numbers and seeing if they were the same: if you did that for number 6, for example, would get the same answers?

Iota: Oh, that's a good question. I don't ... Yeah. Right. I don't know ... I didn't ... I didn't try. But ... I mean, I'm just, I'm looking at it right now. Yeah, it should be the same. Because it should be only one value. Mm-hmm.

In this excerpt there is evidence of both process and pseudo-process GVs. Iota now shows evidence of the prescriptive Equivalence-Preserving GV, because they substitute numbers to check whether the result of their distributive property computation in Q7 is insertionally equivalent to the original expression, at least for one value. When it is not, they then question their use of the distributive property to replace $2(xy)$ with $2x2y$, providing evidence of a process GV. However, their explanation still draws on extracted meanings and some pseudo-process GVs: they several times mention “feeling” that the distributive property is not correct or whether an expression “looks like” the distributive property should be used. In the other six similar distributive property items, they do not use a process GV; however, when the interviewer asks them directly whether this checking process should work for those also, Iota then draws on their knowledge of the distributive property as an equivalence-preserving transformation to recognize that this is also relevant for the other expressions. Whether the pseudo-process or process GV was cued for Iota appears to be linked to the way that different expressions “look”, which may be important to keep in mind when designing curriculum and instruction. It may be that instruction and tasks that focus more on checking and justifying calculation as well as linking the equivalence-preserving GV to calculation procedures, especially for a diverse problem space with many different forms, may be critical for learners like Iota.

Pseudo-object GV

The next interview was conducted with an elementary algebra student, whom we call Eta. They were asked to interpret whether $(2x + 1)(3x - 5)$ could be viewed as equal to the form $(a + b)c$.


Consider $(2x+1)(3x-5)$ in its current form (don't rewrite it or do anything to it). Is there any part of $(2x+1)(3x-5)$ which could be equal to $(a+b)c$ if we pick the right expressions to represent a, b, and c?

a. No
b. Yes, if $c = 3x$
c. Yes, if $c = 1$
d. Yes, if $c = 3x - 5$
e. Yes, if $c = 3$

Figure 4: Eta considering whether $(2x + 1)(3x - 5)$ can be seen as having the form $(a + b)c$

Eta: $2x$ could be a then the one would be b, then the c would be $3x$… if c is equal to $3x$ then it would make sense…. I'm just doing it by order by the first number, second number, third number. Maybe that's not the best way, but that's what I was doing.

Interviewer: What’s being multiplied in each case [pointing to the expression]?

Eta: Two is being multiplied by three. Two is also being multiplied by the negative five. The same thing for the one, the one is being multiplied by three and then the one is also being multiplied by the negative five.

Here Eta focuses on mapping subexpressions to variables in the form “in order”, which reflects a pseudo-object GV: they map the “first subexpression” to the first variable, etc., without attending to grammatical meaning of the syntax. When considering $(2x + 5)(3x - 5)$ and mapping subexpressions to the form $(a + b)c$, Eta appears not to “see” the second set of brackets around $3x - 5$ initially (or not recognize them as a grouping symbol); but when further questioned by the interviewer, Eta explains that each term in $(3x - 5)$ will eventually be multiplied by each term in $(2x + 1)$. This provides evidence that Eta’s pseudo-object GV of symbolic properties is likely not caused by a failure to recognize the syntactic role of the second set of brackets. Rather, Eta appears not to focus on the existing syntactic meaning of expressions when mapping that expression to a form. Eta appears to conflate the existing syntactic meaning of $(2x + 1)(3x - 5)$ with the result of expansion, perhaps literally conceptualizing $(2x + 1)(3x - 5)$ as having the syntactic meaning $2 \cdot 3 \cdot x^2 + 2 \cdot -5 + 1 \cdot 3 \cdot x + 1 \cdot 5$. However, while these two expressions are equivalent, they do not have the same syntactic meaning, and conflating the syntactic meaning of the first expression with the second one appears to obscure the structure needed to map this expression to the form $(a + b)c$.Thus, Eta’s computational view of syntactic structure may be impacting their GV of symbolic properties. Instruction that more explicitly highlights the differences in syntactic structure of different expressions and links this explicitly to form mapping, may better prepare Eta (and learners like them) to draw on their existing knowledge of syntax, symbolic structures, and forms as objects. Future research is needed to explore this possibility.

Object GV

We now consider an interview with an elementary algebra student whom we call Theta, who was asked to interpret whether $\frac{2x^2(y-1)}{2}$ could be viewed as equal to the form $\frac{(ab)}{c}$ (where $c \neq 0$).

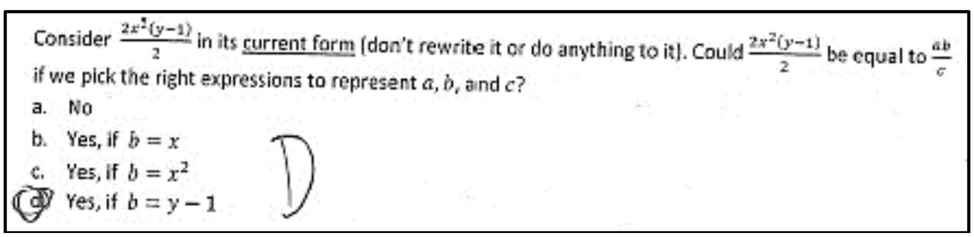


Figure 5: Theta mapping a multi-term expression to a variable in a form

Theta: I felt like D was the best option because looking at a and b over c the first equation fit that like a could be $2x^2$ squared and b could be $y - 1$ and c could be 2.

Interviewer: Did the parentheses impact your decision?

Theta: Yes.

Interviewer: How?

Theta: Because I saw that the $y - 1$, I saw it as separate from $2x^2$. And I know that looking at the second one that a and b in order for them to be multiplied they would most likely have to have parentheses around them. And I saw $y - 1$ in parentheses so I just ... looking at them all as substitutes, as soon as I saw a and b over c like I was just putting in my head okay, $2x^2$ squared is a, $y - 1$ is b, and the two is equal to c.

In this excerpt Theta provides evidence of an object GV. They identify mathematically valid subexpressions in $\frac{2x^2(y-1)}{2}$ and identify which of these should map to which variable in the form to preserve the structure. Later the interviewer asked Theta to identify different syntactic structures in the expression, and Theta was immediately able to do so correctly. Here Theta also appears to conceptualize brackets from an object view (as a grouping mechanism rather than a cue to a procedure [see Wladis et al, 2022a]) because they "separate" the $2x^2$ and $y-1$. Because Theta's object view of syntactic structure is critical to their identifying the subexpression structures that will create a syntactic-structure-preserving one-to-one mapping from $\frac{(2x^2)(y-1)}{2}$ to the form $\frac{ab}{c}$ illustrates how this object view of syntactic structure may be a critical precursor to having an object view of symbolic properties. Theta also specifically mentions substitution when describing how subexpressions relate to the properties form: thus, Theta's notions of substitution and substitution equivalence may be related to their symbolic properties conceptions (see Wladis et al., 2022b). Theta's explanations here are substantially more structural than most other students in the sample (including those in a wide variety of course levels). While this evidence is not causal, Theta's responses indicate that some elementary algebra students are capable of reasoning structurally about symbolic properties. Theta was part of an intervention that was designed to teach students the prescriptive GVs presented here (as well as others related to syntactic structure and equivalence). This may have influenced their GV formation; ongoing research is underway to explore this possibility. But regardless of whether this particular intervention played a role, Theta's responses show how elementary algebra students are capable of thinking structurally.

Conclusion

These different vignettes illustrate how conceptualizing student thinking around symbolic properties using the framework in Figure 2 may be productive for understanding the reasons that students work with symbolic properties in particular productive or non-productive ways. One interesting pattern across all three vignettes is that each of the learners shows evidence of potentially productive prior knowledge, however, the extent to which they were able to use this prior knowledge productively in the context of symbolic properties varied quite a bit. We also saw here how student conceptions of symbolic properties are also intricately related to their conceptions of symbolic structure and equivalence, and thus some conceptions of these related concepts may be essential precursors to student conceptions of symbolic properties (Wladis et al., 2023). More research is needed to explore the relationship among these various conceptions, as well as what factors enable or disable students from productively drawing on prior knowledge when working with symbolic properties. We continue to investigate these relationships in ongoing research, and hope that others will as well.

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References

Barnett, E., & Ding, M. (2019). Teaching of the associative property: A natural classroom investigation. *Investigations in Mathematics Learning*, *11*(2), 148–166. https://doi.org/10.1080/19477503.2018.1425592

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*. https://doi.org/10.1191/1478088706qp063oa

Edwards, B., & Ward, M. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411–424. https://doi.org/10.1080/00029890.2004.11920092

Greefrath, G., Oldenburg, R., Siller, H., Ulm, V., & Weigand, H. (2016). Aspects and "Grundvorstellungen" of the Concepts of Derivative and Integral: Subject Matter related Didactical Perspectives of Concept Formation. *Journal Für Mathematik-Didaktik*, *37*, 99-129. https://doi.org/10.1007/s13138-016-0100-x

Hoch, M., & Dreyfus, T. (2004). *Structure sense in high school algebra: The effect of brackets* (M. J. Høines & A. B. Fuglestad, Eds.; 3rd ed., pp. 49–56). PME.

Hunter, J., Miller, J., Bowmar, A., & Jones, I. (2022). " It Has the Same Numbers, Just in a Different Order": Middle School Students Noticing Algebraic Structures within Equivalent Equations. *Mathematics Education Research Group of Australasia*.

Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. *Early Algebraization: A Global Dialogue from Multiple Perspectives*, 579–593. https://doi.org/10.1007/978-3-642-17735-4_29

Mok, I. A. C. (2010). Students' algebra sense via their understanding of the distributive law. *Pedagogies: An International Journal*, *5*(3), 251–263. https://doi.org/10.1080/1554480X.2010.486156

Prediger, S., & Zwetzschler, L. (2013). *Topic-specific design research with a focus on learning processes: The case of understanding algebraic equivalence in grade 8* (T. Plomp & N. Nieveen, Eds.; pp. 407–424). SLO.

Schüler-Meyer, A. (2017). Students' development of structure sense for the distributive law. *Educational Studies in Mathematics*, *96*(1), 17–32. doi.org/10.1007/s10649-017-9765-4

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25, pp. 59–84). Mathematical Association of America.

Vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*.

Wladis, C., Offenholley, K., Lee, J. K, Licwinko, S., Dawes, D. (2018). Development of the elementary algebra concept inventory for the college context. In T. Fukawa-Connelly, N. Engelke Infante, M. Wawro, S. Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. (pp. 605-617). San Diego, CA.

Wladis, C., Sencindiver, B., & Offenholley, K. H. (2022a). An Exploration of How College Students Think About Parentheses in the Context of Algebraic Syntax. *Proceedings of the 44th Annual Psychology of Mathematics Education-North America (PME-NA) Conference*. (pp. 286-294), Middle Tennessee State University.

Wladis, C., Sencindiver, B. and Offenholley, K. (2022b) A model of symbolic structure sense: Meaning-making within abstract symbolic systems. *Manuscript under review*.

Wladis, C., Sencindiver, B., & Offenholley, K. (2023a). Reconceptualizing algebraic transformation as a process of substitution equivalence. In S. Cook, and N. Infante (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (RUME)*, Omaha, NE.

From an Inclination to Subtract to a Need to Divide: Exploring Student Understanding and Use of Division in Combinatorics

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In this report, we provide an initial exploration into a key but under-studied phenomenon in enumerative combinatorics – the use of division in solving counting problems. We present a case of one undergraduate student solving a combinatorics problem; this case is representative of a broader phenomenon in which students may intuitively desire to account for an overcount using subtraction, when division is a productive and useful approach. We highlight the conceptions a student demonstrated as she progressed from using subtraction to using division successfully. We frame our analysis in terms of a set-oriented perspective (Lockwood, 2014).

Keywords: Arithmetic operations, Combinatorics, Discrete mathematics, Sets, Division

When solving counting problems, we may find ourselves needing to remove some undesirable outcomes from a larger set, in order to find the cardinality of the set of outcomes in question. For example, in the problem "How many sequences of 5 digits contain at least one 9?", an efficient strategy is to count all possible 5-digit sequences and then to subtract those that do not contain a 9. This strategy is common, and can be viewed as a special case of the well-known Principle of Inclusion/Exclusion (e.g., Tucker, 2002); subtraction is a powerful tool for counting.

In some cases, however, the operation of division, and not subtraction, is most useful. In fact, division represents an important, often necessary, way to account for overcounting. Consider the Table Problem, which is the focus of our case study in this paper: "How many ways are there to arrange 10 people around a circular table?" A common efficient solution involves division – we first count the ways to arrange 10 people in a line (10!), and then we note that each of the desired circular arrangements is actually overcounted by a factor of 10 (since each linear arrangement can be rotated 10 times to yield equivalent circular arrangements). Thus, we can strategically divide the number of linear arrangements by 10 to arrive at the correct answer of 9!. In such problems, we have observed that students' initial approach tends to focus on subtraction as a way to account for overcounting. Our main goal in this paper is to consider and discuss ways in which students may progress from an initial intuitive desire to subtract to a combinatorial understanding of how and when to divide in appropriate circumstances.

There are many reasons why we might want students to develop robust, productive ways of thinking about division in combinatorics. Indeed, it occurs frequently in problems, and it is a fundamental aspect of why certain formulas (such as the binomial coefficients) work as they do. As important as division is, it has not commonly been addressed in the teaching and learning of combinatorics. We argue that better understanding division as it relates to counting could be beneficial for students. Our motivation and goal here is to provide evidence for – and to lay groundwork for – future studies to examine division in combinatorics.

In this report we focus on a case study of one undergraduate student, who solved the Table Problem and, in so doing, transitioned from an approach focused on subtraction to one that successfully leveraged division. We particularly want to highlight what conceptions about division emerged for the student that allowed her to successfully solve the counting problem (particularly when she was not able to use subtraction successfully). We attempt to answer the following research questions by examining this case: *What conceptions and ways of reasoning* *emerged for an undergraduate student as she progressed from using subtraction to using division to solve a counting problem that was designed to elicit division?*

Literature Review and Guiding Perspectives that Situate Our Work

A Set-Oriented Perspective

Lockwood (2014) introduced a set-oriented perspective as a way of thinking about counting "that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). Lockwood and others have since argued for the importance of having students connect counting to sets of outcomes in a variety of ways, connecting such a perspective to listing (Lockwood & Gibson, 2016), highlighting its importance in helping students understand and interpret counting formulas (e.g., Lockwood et al., 2015; Wasserman & Galarza, 2017; Wasserman, 2019), and emphasizing its centrality to being able to engage productively with combinatorial proof (e.g., Lockwood et al., 2020; Erickson & Lockwood, 2021a). Relatedly, Lockwood (2013) presented a model of students' combinatorial thinking that included three components: counting processes, formulas and expressions, and sets of outcomes. In this model, Lockwood emphasized the importance of sets of outcomes, suggesting affordances of having students think about ways in which their counting processes generate and organize sets of outcomes. Our findings in this paper help flesh out the relatively broad view of this set-oriented perspective initially presented by Lockwood (2014). We explore how certain ways of structuring sets of outcomes may serve to support students' reasoning about division in solving counting problems. The set-oriented perspective serves as a guiding theoretical principle, and a focus on sets of outcomes is central to how we conceive of counting.

Arithmetic Operations in Combinatorics

With the exception of multiplication, arithmetic operations have generally not been studied extensively in combinatorics education. Multiplication occurs so frequently in combinatorics that the community has developed a Multiplication Principle (MP) that describes conditions under which it is appropriate to multiply when solving a counting problem. Researchers have explored a number of ways in which the MP is presented in the teaching and learning of combinatorics, including its presentation in textbooks (e.g., Lockwood et al., 2017), students' reasoning about the MP (e.g., Lockwood & Purdy, 2020a, 2020b), and problems involving Cartesian products (e.g., Tillema, 2013). To this point, however, adequate attention has not been paid to other arithmetic operations in counting, especially subtraction and division. Lockwood and Reed (2020) describe an equivalence way of thinking in combinatorics, highlighting how equivalence relates to division in counting.

> Broadly, an equivalence way of thinking in combinatorics entails recognizing equivalence between particular outcomes, and then subsequently accounting for this equivalence. So, when employing an equivalence way of thinking, two things happen: a) one recognizes that in a given set of outcomes, there are certain outcomes that should be considered equivalent (or "the same," "duplicate," or "identical") for specified constraints in a situation or problem, and b) one understands that they can use the operation of division in order to account for the occurrence of such equivalent outcomes (Lockwood & Reed, 2020, p. 4).

Lockwood and Reed noted several places in which such equivalence and division naturally arise among topics in combinatorics. Our point here is that division in combinatorics relates to important underlying concepts, and it is worthwhile to pursue as a line of inquiry. Our data sheds light on students' conceptions of division (particularly as it relates to subtraction) that illuminate the kinds of productive meanings and ways of reasoning that have thus far been absent from the literature, and suggest a need for further exploration. We are thus motivated to think more broadly about ways in which other operations can support and augment students' combinatorial understanding beyond just multiplication.

Methods

We present an episode taken from a series of task-based clinical interviews (Hunting, 1997) exploring, among other things, students' engagement with division in counting. Nine students were recruited from a large university in the United States. The participants were a mix of six undergraduates, one early-stage graduate, and two late-stage graduate students who were enrolled in a math class and had taken (or were taking) a course that featured counting. There were no selection criteria aside from their coursework and willingness to participate. We present the work of Jillian (pseudonym), an undergraduate mathematics major, in her first interview.

Interview sessions were 90 minutes long. Because of student availability, some students participated in few sessions (1-3) while others participated in many sessions (6-9). Consistent with task-based clinical interviews (Hunting, 1997), the participants were asked to describe their work as they solved multiple counting problems, and were frequently asked hypothesis-confirming questions by the interviewer about their understandings of formulas, concepts, and strategies both during and after they solved the problems. Participants worked on an iPad to solve the problems, and their written work and gestures and utterances were recorded.

The interview protocol consisted of a diverse collection of counting problems, with many problems chosen as likely to elicit use of division as part of the problem-solving process. For instance, one might solve the *Table Problem* in at least three ways, though we hypothesized that many students would be successful by leveraging division as described in the Introduction. An alternative solution involves making an initial arbitrary choice that a single person sits first at the table. Following this, there are 9! arrangements of the remaining 9 persons around her.

Notably, one might also solve the *Table Problem* with subtraction. Beginning with the 10! linear arrangements of people, and recognizing the overcount, one might attempt to remove 9 extra outcomes for each 1 desired outcome. This leads to the insight that, if for each desired outcome there are 9 to be removed, then you solve the problem by the difference $10! - 9x$, where x is the desired number of outcomes. Since x is also the solution, you establish the equation $10! - 9x = x$ to yield the solution $\frac{10!}{10} = 9!$. As many counting problems involving division can be solved in multiple ways, we attempted to choose problems for which division would be a likely solution method. We also included other problems for which division was not a targeted solution method (e.g., problems involving sums of binomials) so students would not anticipate that division was the targeted operation of the study. Following our focus on a set-oriented perspective, other interventions were enacted to support student consideration of outcome sets during their problem solving.

The video records (iPad work and gestures and utterances) were spliced together so that we could view both the student and their work at the same time. Transcripts were made and enhanced with images, references, and comments. The research team analyzed the data for this report by searching the records for episodes where students utilized division in their solution, attending particularly to problems where subtraction was involved in an earlier solution attempt. We then reviewed the records of the episodes via conceptual analyses (Thompson, 2008) to build second-order models of the students' thinking, seeking viable explanations of the students' actions and utterances in the form of theoretical models (Steffe & Thompson, 2000; Thompson, 2008). The results that we present follow from our second-order models of Jillian's cognition.

Results

In this section, we briefly describe four episodes in Jillian's work on this problem, as we document her progress from her initial inclination to use subtraction to her use and justification of division. In each episode, we comment on her reasoning and connect it to what we think are broader important points related to student thinking on division in counting.

Episode 1: Starting from 10! with an Inclination to Subtract

Jillian had correctly solved a previous problem that asked for the number of ways to arrange 10 people in a line (the answer is 10!), and when answering the Table problem, Jillian began with that previous solution. She understood that 10! would give her too many outcomes, and she immediately demonstrated engagement with the sets of outcomes by identifying specific outcomes as "the same".

Jillian: There's going to be similar iterations, because if you have *a* through *j* [i.e. the sequence *(a,b,c,d,e,f,g,h,i,j)*], that's the same as *b* through *a* just around the circle goes *j* and then come back around [to] *a* [i.e. *(b,c,d,e,f,g,h,i,j,a)*],. So we're going to have another probably, I think, subtracting problem. So I think it would start similar to the line of 10 factorial [i.e. arranging 10 people into a line]. It'll give you all the possible ways to arrange them. Not accounting for possibilities being the same iteration around the table.

Jillian thus wanted to try to solve the problem by reducing that 10! in some way to account for duplicate rotations around the table. She went on to articulate correctly that there would be nine duplicates for every desirable arrangement, as seen in the following excerpt.

Jillian: So, for each order of 10 that you complete. There's going to be nine duplicates because each order of 10 can be shifted around the table, like there's 10 ways to express the same thing. And so, each of these 10 iterations is going to have nine duplicates.

Jillian's intuition was correct – in fact, she could even articulate what precisely would get overcounted and what she wanted to remove. However, she did not see how to use the subtraction to arrive at a solution and could not figure out what to subtract. The excerpt below shows her reasoning that because each desirable outcome had 9 duplicates, she wanted to subtract nine times the desirable (possible) outcomes (she wrote this in Figure 1).

Jillian: So there's 10 iterations. […] So each order [i.e. one desired outcome] has nine duplicates, but there's a lot of possible orders. [...] So number of possibilities [writing Figure 1a]. And then I want to take away or divide or somehow remove the nine duplicates of each of those possibilities. I think. And I believe this is a number of possibilities [circling 10!]. So how would you find the nine duplicates worth of each of those possible? […] minus 9 times "possibilities".

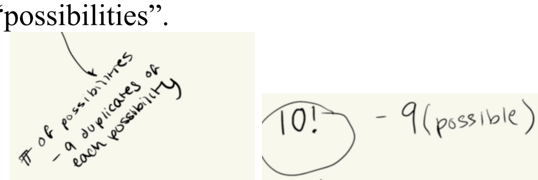

Figure 1a and 1b: Jillian's initial solution to the Table Problem, involving subtraction

She then clarified that she was using "possibilities" to mean two different things, and re-stated Figure 1a. as "# of possibilities and duplicates – 9 duplicates of each possibility". Jillian realized that the possible outcomes she sought was what she was trying to solve, which felt circular to her (note, she did not attempt to build on this towards the subtraction-based solution mentioned in the Methods Section). At this point, enough time passed to suggest an impasse, and the interviewer encouraged her to consider a smaller case, which we describe in Episode 2.

Episode 2: Investigating a Smaller Case, Still Focusing on Subtraction

Jillian wrote out a smaller case involving three people sitting around a table, and she wrote the outcomes in lexicographic order (Figure 2a). She noted, "So each of these [referring to *abc* and *acb*] had two duplicates, which makes sense because there's two iterations around the table."

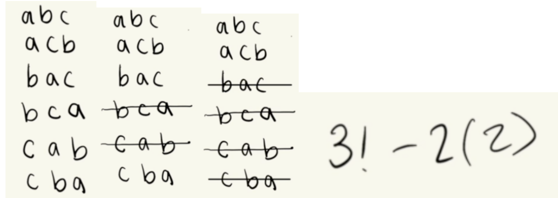

Figure 2a - 2d: Jillian's set of outcomes for the 3-person case

Jillian then proceeded to cross off some outcomes from her list. She noted that *bca* and *cab* were duplicates of *abc*, and so she crossed those off first in Figure 2b. Then, she noted that *cba* and *bac* were duplicates of *acb*, so she crossed them off next in Figure 2c. As she did this, she wrote the expression "3! – 2(2)" in Figure 2d, and she related that expression to her previous expression "10! – 9(possible)" in the larger case. Her language in the excerpt below shows her relating the expression "3! – 2(2)" to her process of crossing out the outcomes from her list; the bolded language summarizes her understanding of that equation. In this way, the smaller case and the set of outcomes helped her confirm that her expression was correct. We want to highlight that the expression here and the process by which she crossed out outcomes is reflected both in how the outcomes are listed, which aligns with Lockwood's (2013) model.

Jillian: Six original options minus […] and then instead of multiplying by nine [in the larger case] I will by two of each of those original options. By the previous logic, there would be three factorial, or six, so I'm happy with that of without not accounting for order, we're going to have the same number of options. And then instead of minus 9 times the actual amount of possibilities [i.e. desired possibilities and duplicates], I'd say minus 2 times the actual amount of possibilities, because **there's two duplicate iterations when you have three people and there's two legitimate possibilities that aren't duplicates. And so, the [expression] makes sense.**

Notably, Jillian realized that in this case she knew there were 2 possibilities (because she had actually written them out and counted them), but she wasn't sure how to get the answer in the bigger larger 10-person case. We infer that at this point the smaller case served to support her in confirming the formula could make sense if she knew what the number of possibilities were, but it did not help her actually solve the problem for the larger case. The interviewer let her wait and think, and we interpret that she had come to an impasse and was not sure how to proceed.

Episode 3: A Different Structure on the Set of Outcomes in the Smaller Case

We had hypothesized that an alternative way of organizing outcomes could help to motivate the use of division. So, once Jillian seemed unsure of how to proceed, the interviewer intervened by writing the outcomes in a different way (Figure 3a). He wrote the outcomes in two columns, with each equivalent rotation of an outcome in its same column.

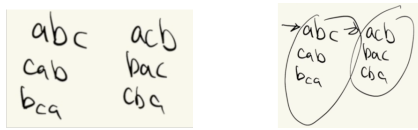

Figure 3a and 3b: The interviewer's, and Jillian's circling of two groups of 3

Jillian noted that she had two groups of three, and she realized (because she knew that the correct answer was 2) that to get the answer of 2 she would need to divide the 3! by 3. She circled the two groups and said, "Yeah, three factorial, you just needed to split it into groups. But it wouldn't be divided by two, by three." However, she continued to reason about the formula and the outcomes and to move back and forth between the smaller 3-person case and the original 10-person case. After some time, the interviewer re-stated her current strategy, as a means of confirming her current reasoning about the problem.

Interviewer: So, you're thinking we can make our two [solution to the 3-case] in this case out of three factorial divided by three. But you're noticing that your instinct is to - instead of divide - subtract by two, because there are two duplicates.

Jillian: Yes, yeah. Because you should be able to split it […] divide by. Oh. I guess in this case, it would be divided by ten [$\frac{10!}{10}$] and the other one too, divided by three [$\frac{3!}{3}$]. Because I'm dividing it into groups of three, the original and its two duplicates to see how many groups there are because that basically doesn't count the duplicates. It just counts how many original generators do we have, like for these different groups

We interpret that Jillian realized that although the desirable outcomes each had *two* duplicates, she could divide by *three* because the groups of three include the desirable outcome and its two duplicates. She also referred to generators, which the interviewer asked about and we discuss in the next episode. She then related this insight about two duplicates and a group of size three to the original 10-person case, and she said, "So wouldn't that work if we divided this [10!] by 10 because it would split it into […] 10, because we know each of these possibilities has 9 duplicates. So, if we split it into groups of 10, we should account for […] the 9 in addition to all of the original one."

Again, Jillian articulated that the division by 10 made sense because each possibility has 9 duplicates, making a group of size 10. The interviewer wanted to ensure he correctly interpreted her reasoning, and so he asked for additional justification, which we discuss in Episode 4.

Episode 4: Ultimately Justifying the Division

In this final episode, we document Jillian's understanding of why division made sense. The exchange below highlights how she came to talk about the groups of 3 (in the smaller case) or 10 (in the larger case) as including the desirable outcome and its duplicates.

Jillian: Yeah. So over here [smaller case] […] I'm counting *abc* and *acb* [draws arrows in Figure 3b] as our like "generators" [air quotes], any one of these [group members] could be a generator […]

Interviewer: And so, by generator there, you mean […] *abc* you said could be swapped out for any of the others. So, what makes that, like, a generator? Or what does generator mean.

Jillian: Yeah. I guess it's almost like if there was an operation that rotated them or something like it's operating on itself over and over again […]

Interviewer: [After confirming that Jillian was not formally referring to algebraic groups] So for you, maybe in layman's terms we would say you've got *abc* but you could generate any of the other duplicates from *abc*.

Jillian: Yes, yes, yes, yes, yes. It's like, it's kids or something. You have *abc* and then you have all of the family of *abc*. And I know that there's going to be in this [larger] case, there's going to be nine children of every parent. If we were to split it into groups of ten, even if the kids got all jumbled up and they weren't the right groups of ten, we still should account for the fact that each one is multiplied by ten different iterations.

Interviewer: So, we take the one generator, as you've been calling it, and you multiply by ten. That gives you like the entire collection of iterations.

Jillian: Yes. And so, if we divide by ten, we should get the number of generators.

Jillian could reason about the whole set of outcomes being split up into groups, and in this way she understood that the division was giving her the number of generators. We infer that she understood that each group would have one generator, and so the division would yield the number of desirable groups and, ultimately, of desirable outcomes.

Discussion and Conclusion

We highlight a couple of points of discussion here and articulate potential implications. In accounting for Jillian's transition from subtraction to division, we think that a key understanding was focusing not just on the number of duplicates (2 duplicates or 9 duplicates in the respective cases), but thinking of the entire sets of equivalent outcomes, including the desirable outcomes and the duplicates (a total of 3 or 10 in the respective cases); we call these the equivalence classes. Transitioning from a focus on a 1:2 or a 1:9 ratio and instead thinking of 3 or 10 seemed important in understanding the appropriate use of division. An implication then is that while subtraction of duplicates is useful, it is valuable to think of those duplicates not in a ratio to the desirable outcome but as part of a set (or equivalence class) with the desirable outcome itself. Jillian's notion of a generator was one useful way to think about this.

Another observation is that reasoning about sets of outcomes and a set-oriented perspective was useful, but certain ways of structuring the sets of outcomes might be associated with different solution strategies. The common lexicographic ordering of outcomes first accompanied Jillian's duplicate-removal strategy, whereas grouping the outcomes into equivalence classes provided a way for Jillian to focus on the 3 and the 10 (the sizes of the groups, rather than the sizes of the groups of duplicates). This re-orientation towards division gives insight into more nuances about *sets of outcomes* and how they might relate to *processes* and *formulas* that may be particularly suggestive of operations. Indeed, we feel that the listing and expression in Figures 2a-2d highlight Jillian's movement between components of Lockwood's (2013) model, and reinforce the currently underexplored idea that different lists of outcomes may be suggestive of different *processes* and *expressions*.

A final point and potential implication is that there are natural connections between counting problems and equivalence, and problems that focus on division in counting may offer rich opportunities and contexts in which to explore such connections. We believe that there is much more work to be done to investigate students' reasoning about division in combinatorics, and we hope that researchers will undertake systematic explorations into how students use and come to understand division in combinatorics.

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References

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior*, *16*(2), 145-165.

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior, 32,* 251-265. doi:10.1016/j.jmathb.2013.02.008

Lockwood, E. (2014a). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics, 34*(2), 31-37

Lockwood, E., Reed, Z., & Caughman, J. S. (2017). An analysis of statements of the multiplication principle in combinatorics, discrete, and finite mathematics textbooks. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 381-416. doi:10.1007/s40753-016-0045-y

Lockwood, E. & Purdy, B. (2019). Two undergraduate students' reinvention of the multiplication principle. *Journal for Research in Mathematics Education (50)*3, 225-267

Lockwood, E. & Reed, Z. (2020). Defining and demonstrating an equivalence way of thinking in enumerative combinatorics. *Journal of Mathematical Behavior, 58*. doi:10.1016/j.jmathb.2020.100780

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol 1, pp. 31-49). Morélia, Mexico: PME. Available at http://pat-thompson.net/PDFversions/2008ConceptualAnalysis.pdf.

Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior, 32*(3), 331-352. Doi: 10.1016/j.jmathb.2013.03.006.

Tillema, E.S. & Gatza, A. (2016). A quantitative and combinatorial approach to non-linear meanings of multiplication. *For the Learning of Mathematics*, *36*(2), 26-33.

Wasserman, N., Galarza, P. (2019). Conceptualizing and justifying sets of outcomes with combination problems. *Investigations in Mathematics Learning, 11*(2), 83-102.

Levers for Change: A Longitudinal Case Study of Two Departments

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University mathematics departments are making efforts to improve student success in STEM by implementing active learning in their introductory mathematics courses. These efforts are context-dependent, driving a need to understand the varying paths departments take in making these changes. In this paper we explore the change efforts of two mathematics departments that achieved varying levels of success. We take a longitudinal perspective to capture their efforts. We identify what levers these departments used, including how robustly the levers were implemented. We conclude with a discussion on how the levers identified contributed to each department's progress.

Keywords: Department change, calculus, student success

Research on departmental change, while prevalent for some time in higher education, has only recently been a focus of undergraduate mathematics education (e.g., Smith et al., 2021). In a recent research commentary, Reinholz et al. (2020) argue for an expanded research base that attends to the process of change in mathematics departments. Such research has the potential to provide both needed insights for others wanting to embark on a change initiative and to contribute to theories of change. Previous studies have focused on various aspects of departmental change including research on communities of practice, course coordination, and the intentionality needed to initiate desired change (Smith et al., 2021a), as well as research focusing on change efforts over a two-to-three-year period (Apkarian, 2018; Smith et al., 2021b). In this report we aim to contribute to this emerging research area by examining two different departmental change efforts over the span of a five-year period.

A primary reason mathematics departments in the United States (US) and elsewhere undertake transformational change efforts is poor student experience and success in the introductory mathematics courses required of most science, technology, engineering, and mathematics (STEM) majors (e.g., PCAST, 2012). In addition to traditional markers of success, such as grades, students completing calculus courses also report decreased confidence, enjoyment, and readiness (Bressoud, 2015). While the effects of these introductory courses apply to students in general, they disproportionately affect women and students of color (Bressoud, 2015; PCAST, 2012; Rocard et al., 2007, Seymour & Hewitt, 1997, Seymour et al., 2019).

High attrition and disproportionate outcomes have led to a myriad of calls to improve student experiences in introductory mathematics courses through the incorporation of active learning (CBMS, 2016; Abell et al., 2018). Analyses of efforts to improve student outcomes through active learning strategies have shown that students improve along traditional measures of success as well as affective gains, with a significant narrowing of achievement gaps for students from underrepresented groups (Freeman et al., 2014; Kogan & Laursen, 2014; Theobald et al., 2020).

Several ongoing efforts in undergraduate mathematics education are investigating departmental change efforts to better understand how mathematics departments are making these changes and what strategies might be successful (Laursen et al., 2019; Smith et al., 2021).

However, change is difficult, highly localized, and requires systemic strategies (Sabelli & Dede, 2001; Smith et al., 2021b). What works for one institution may not work for another. The two case studies reported here provide insights into various levers of change that are useful in different contexts over an extended period answering the research question: *How do two mathematics departments implement various levers of change in an effort to implement active learning in their Precalculus to Calculus 2 courses?*

Theoretical framework

We utilize Laursen et al.'s (2019) 13 levers of change to document the progress mathematics departments made with respect to changing instructional practices. Laursen et al.'s levers of change, which act on both individuals and academic units, describe a broader change theory. The levers themselves can also serve as a descriptive set of options that characterize change efforts. We selected this framework because of its flexibility in describing the factors of change at different institutions in which the levers are dependent on local context. One contribution of our work is the expansion of this framework to account for cultural and contextual factors that determine the generation and impact of site-specific levers. The theoretical framework includes three themes describing the change process.

The three themes are: (1) levers to motivate or provide rationale, (2) levers to prepare and enable, and (3) levers to stimulate and support action. Laursen et al. (2019) describe levers to motivate or provide rationale as, "Identifying the need, suggesting solutions, and prompting individual instructors or their departments and institutions to adopt [research-based instructional strategies]" (p.159). Levers to prepare and enable, as Laursen et al. (2019) state, "can best be described as offering mechanisms for change" (p. 166). That is, what structures are in place to support instructors to make changes? The third theme, levers to stimulate and support action, includes federal and private investment to support and sustain instructional change efforts - the only lever in the framework that impacts individuals and academic units together. Specifically, Laursen et al, (2019) state that, "Funding does not work directly on instructors, nor on academic units, but rather exerts secondary force on the other levers" (p. 175). We identified one additional lever, Administrative funding, that would fit under this broader definition of stimulating and supporting action. The 13 levers identified by Laursen et al. (2019) are provided in Table 1 as well as the additional change lever identified in this study shown in italics.

Table 1. Levers for change.

Lever Themes	Operating on Individuals	Operating on Academic Units
Levers to Motivate or Provide Rationale (LMPR)	1. Accreditation of certification by disciplines of states 2. Guiding documents from professional societies/leadership bodies 3. Demands from employers for specific competencies 4. Results from research	5. Models and exemplars from other institutions 6. Local data and evidence about student outcomes

Levers to Prepare and Enable (LPE)	7. Professional development 8. Resource collections or digital libraries 9. Educational technologies 10. Communities of practice	11. Collaboration with other disciplines or departments 12. Local leaders and internal change agents with a vision
Levers to Stimulate and Support Action (LSSA)	13. Federal and private funders' investments (operating on all levels and on the system as a whole) 14. *Administrative funding* (operating on all levels and on the system as a whole)	

Setting and Methods

This work is part of a larger project that studied change strategies at multiple institutions in the US to implement and sustain active learning in their Precalculus to Calculus 2 (P2C2) sequence. The research project was carried out in stages, each with its own goals and objectives. Phase 1 was a case study of six institutions that had successfully implemented active learning in their P2C2 sequence; Phase 2 was a longitudinal case study of nine departments that received nominal funding and participated in a networked improvement community (Smith et al., 2021b). Phase 3, which is the focus of this report, investigated the change strategies of departments who applied to be part of Phase 2 but were not selected. We investigate what levers influenced the progress these departments made in implementing their change initiatives. There were 38 departments that were not selected to join Phase 2, with 11 identified as having carried out at least part of their original proposal.

Of the 11 departments, six were deemed to have moderate to substantial change efforts and five were deemed to have made minimal progress towards their change efforts (LaTona-Tequida et al., 2022). Two departments from these 11 were selected for this report: One department, Mid State University (MSU), represents a modest to substantial successful change effort and one department, Large State University (LSU), represents a less successful change effort. MSU is a primarily Asian institution with very high research activity and an undergraduate student body of roughly 14,000 students. LSU has approximately 22,000 undergraduates, is a primarily white, and is a high research activity institution.

Data sources included proposals submitted by each department, audio-recorded and transcribed semi-structured interviews with key personnel in the mathematics departments, and final reports written by each department communicating their long-term progress toward their proposal goals. Our data sources span a period of five years allowing a longitudinal perspective, enabling us to evaluate change implementation over time, uncovering sustained changes and successes. Based on our knowledge of the literature, we took Laursen et al.'s (2019) 13 levers as an *a priori* coding scheme to analyze each data source but remained open to identifying additional levels (Miles & Huberman, 1994). Each department's data sources were coded individually by 2-3 of the authors who then met to discuss coding until a consensus was reached. We further organized the levers identified for each department and each data source into a table to capture changes in the use of levers over time.

Results

In Table 2 we present an overview of the levers utilized by the mathematics departments at each institution at three time points, Fall 2017, Fall 2020, and Fall 2022. Due to space constraints, we provide detail on the 2020 data and briefly discuss the 2017 and 2022 data. Levers in bold are those we found to be the most robustly implemented and likely the most effective catalysts for change.

Table 2. Levers utilized over five years (LMPR = Levers to Motivate or Provide Rationale; LPE = Levers to Prepare and Enable; LSSA = Levers to Stimulate and Support Action).

	Fall 2017	Fall 2020	Fall 2022
MSU	LMPR4, LMPR5, **LMPR6, LPE7**, LPE8, LPE11, LSSA.13	**LPE7**, LPE10, LPE11, **LPE12**, **LSSA13**	**LPE7**, LPE10, LPE11, LPE12, **LSSA13**, **LSSA14**
LSU	**LMPR6**, LPE7, **LPE8**, LPE10, **LPE11, LPE12**, LSSA13	LMPR6, **LPE7**, **LPE8**, **LPE12**	LMPR6, LPE7, **LPE8**, LPE12

Mid State University

MSU made significant progress toward their goals, which included: 1) implementing active learning in Calculus 1 with the support of learning assistants, 2) implementing course coordination in Precalculus and Calculus 1, and 3) increasing the enrolment and retention of Indigenous students in STEM by focusing on Calculus 1.

Fall 2017. At the time of their proposal, MSU used levers from each of the three themes, with plans to expand the use of active learning, strengthening existing levers, or including new ones. In their proposal they described two levers that provided motivation and rationale, and three levers that prepared and enabled, with both themes acting on individuals and academic units. The use of local data (LMPR6) and course coordination (as a form of professional development (LPE7)) were two areas where MSU focused their change efforts in Fall 2017.

Fall 2020. By the fall of 2020 MSU expanded the use of coordination and Learning Assistants (LAs), peer-to-peer instructional aids, making use of various levers, primarily focused on levers that prepare and enable. There was no progress in data use, and MSU was limited to temporary solutions for funding.

Levers to prepare and enable. Coordination expanded in both Precalculus and Calculus 1. A permanent position was created and filled for a Precalculus coordinator and co-coordinators were incentivized through course buyouts for Calculus 1. In line with the MSU vision for coordination, Precalculus coordination evolved to include common materials and instructional support (LPE7). Instructional support was facilitated through weekly meetings where LAs and coordinators discussed areas students were struggling and strategies for attending to students' understanding. Calculus 1 coordination grew to include common midterm exams in addition to a common final exam and common materials. Notably, MSU was able to resolve resistance to coordination in Calculus 1 through teaching assignments. Permanent faculty who were at odds with course coordination were shifted away from teaching coordinated courses to teaching upper division courses.

By the fall of 2020, the LA program expanded to Calculus 1 and Calculus 2 and was the cornerstone of active learning implementation at MSU. Components of the LA program included coordination, professional development, and strong leadership. LA coordinators were responsible for creating worksheets and meeting with LAs weekly in preparation for recitation and were also incentivized through course reduction. With respect to professional development, the new LA program director taught a semester-long pedagogy course for new LAs and provided optional workshops for instructors looking to make the best use of LAs in their classrooms. The program director also acted as a local change agent and helped promote the LA program's success to administrators.

Although coordination and the LA program previously made use of professional development, resource collections, and collaboration with other departments, these levers shifted and grew more robust as change efforts progressed. Course coordination now included communities of practice and professional development via weekly meetings with coordinators and LAs. The LA program continued to leverage collaboration with other departments but shifted from working with the Physics Department to the College of Education. Also, the creation of the LA program director position supported local leadership (LPE12) that could facilitate communication about the program's benefits to administration. In addition to these levers that prepare and enable, MSU was able to secure funding to support their active learning efforts.

Levers that stimulate and support action. The expansion of the LA program started with a round of one-time funding from the Vice Chancellor's office which supported a semester-long consultation with an LA program director from another university. An additional lever that stimulates and supports action, is *administrative funding* (LSSA14). At MSU, the Dean of the college was instrumental in securing the initial round of funding, supporting the genesis of the LA program. During this initial semester, the consultant ran an LA pedagogy course, while also training a MSU faculty member from the College of Education to run the course and fill the LA program director role locally. Additionally, the mathematics department provided funding that supported the Precalculus coordinator and Calculus 1 co-coordinator positions. Despite the positive growth in their efforts, MSU also reported in 2020 that funding for the LA program was an ongoing struggle, stating they often do not know where funding will come from until right before the semester begins.

Fall 2022. At the time of their final report, MSU described no changes to their previous coordination growth, showing a degree of sustainability and also described a robust and stable LA program. Despite previous issues securing funding, the program became a priority of the administration and funding was no longer an issue. MSU did not return to expanding their use of local data citing a need for access to "guiding principles for successful data collection, analysis, and presentation" and did not report any progress in addressing the needs of their Indigenous students, which was a need in 2017.

Overall, the change efforts at MSU included a range of levers. Analysis of local data functioned as an initial motivating lever, followed by levers included in course coordination and the LA program that prepared and enabled the department to reach their active learning goals. A significant turning point was their one-time funding through the Vice Chancellor's office, supported by the Dean of the college, which allowed them to implement the LA program, subsequently highlighting its success to administrators, and securing funding that facilitated the program's sustainability.

MSU relied on a variety of levers to different degrees to sustain their efforts until permanent funding was in place. They incentivized permanent positions for coordination and created a permanent position for the LA director. They also collaborated outside of the department. First with the Physics Department and then with the College of Education. Meanwhile all of these levers were maintained and supported by motivated local change agents. MSU's integrated approach, which adapted based on need, appears to have contributed to their success in sustaining their efforts to implement active learning.

Large State University

LSU is an example of a department that made minimal progress toward their proposal goals which included: 1) rewriting the Precalculus curriculum, 2) solidifying the use of active learning in Calculus 1, 3) creating a mentorship program for instructors new to active learning, and 4) expanding the use of local data to refine and redesign efforts.

Fall 2017. At the time of their proposal, LSU utilized levers from all three themes. Their proposal described one lever to motivate and provide rationale and three levers to prepare and enable. LSU's levers to motivate and provide rationale acted primarily on individuals, although their levers to prepare and enable acted on both individuals and academic units. During this time, LSU's change efforts were focused on the use of local data (LMPR6) and the dissemination of active learning materials (LPE8).

Fall 2020. In the fall of 2020, LSU reported mixed progress toward their active learning objectives. Areas of improvement included more consistency in the coordinated Precalculus course and more buy-in from instructors teaching these courses. However, despite relying on similar levers, active learning efforts in Calculus 1 were stalling. Relevant to these differences is the segregated nature of the Precalculus course. At LSU, the Precalculus course is run through the Mathematics Learning Centre (MLC), and the rest of the courses in the P2C2 sequence are housed in the mathematics department. This separation likely led to differences in the strength of similar levers used.

Levers to prepare and enable. The increased use of active learning in Precalculus relied on the use of three levers: material resources, professional development, and leadership of a local change agent. Revising active learning materials for Precalculus was a major component of the LSU proposal. Despite not making progress in this revision, materials remained a central lever in their active learning efforts (LPE8). Those teaching Precalculus, housed in the MLC, were strongly encouraged to use these materials. Their use was essentially compulsory as the MLC Director, the person leading active learning efforts in Precalculus, was also responsible for hiring instructors for the Precalculus course.

In addition to materials, the MLC hosted a formal professional development meeting at the beginning of each semester (LPE7), led by the MLC Director and attended by everyone teaching Precalculus. These meetings aimed to encourage active learning and included outlines for the implementation of active learning in the classroom. A major goal of these meetings, reported by the MLC Director, was to generate buy-in for the use of active learning, particularly among newer instructors. Professional development for Precalculus continued informally throughout the semester via course coordinators who advised instructors in implementing group work. Central to the use of materials and professional development was strong leadership by the MLC Director (LPE12).

Calculus 1, run through the Mathematics Department, also relied on the dissemination of materials. However, a change in leadership led to changes in the levers reported at the time of the proposal in the Fall of 2017. The previous course coordinator for Calculus 1 collaborated with the MLC Director (LPE11), using the structure of the Precalculus program. In addition to encouraging the use of active learning materials, the Calculus 1 coordinator also utilized the communities of practice lever, meeting regularly with Calculus 1 instructors to discuss and share materials. This progress was brief, the Calculus 1 coordinator working with the MLC Director left the position, becoming Associate Dean, and no longer had a role in change efforts. After his departure, LSU reported struggles maintaining the active learning curriculum, especially with respect to buy-in from those teaching the course.

Fall 2022. The final report from LSU reflected four levers in their change efforts: local data, professional development, resources collections or digital libraries, and local leaders or change agents. These levers were essentially unchanged from the time of their proposal, except for funding which they were not able to secure since their proposal submission. However, this consistency does not reflect the differences in progress between Precalculus and Calculus 1.

Overall, LSU used local data as an initial motivation for their change efforts. They had a brief period in which leaders in both the MLC and the Mathematics Department collaborated to promote active learning in Precalculus and Calculus 1. However, after a shift in leadership, the gains made in Calculus 1 were stifled. Precalculus continued to consistently use active learning, leveraging materials and professional development, both supported by their most critical lever, leadership.

Conclusion

As shown in Table 2, both MSU and LSU made use of multiple levers across all three points in time. Both institutions used local data to motivate and provide rationale (LMPR6). In Fall 2020 we saw that both institutions made robust use of professional development (LPE7) and that LSU also relied heavily on resources (LPE8) and local leaders (LPE12). The reliance on local leaders at LSU was problematic when the leader in the department, an ally of the leader in the Math Learning Center, left the department for the dean's office. This left both a gap in leadership in the department and left the leader of the Math Learning Center somewhat disconnected from the department. In contrast, MSU relied less on individuals and put structures in place that were sustainable. Additionally, they were actively seeking continued financial support in Fall 2020, ultimately leading to permanent funding in Fall 2022. In comparison, LSU was not able to secure continued funding beyond 2017.

At MSU there was a clear thread from 2017 through 2022 of local leadership (LPE12) and growing opportunities for professional development (LPE7), both through course coordination. MSU was also able to demonstrate how their change initiative aligned with the interests of upper administration (LSSA14). The identification of this lever for change makes a modest contribution to the change literature and levers for change detailed by Laursen et al. (2019). Another important difference that likely contributed to the differential success of the two change initiatives was structural in nature. At LSU, precalculus was run out of the Math Learning Centre, which potentially did not position the director to be a central figure in the mathematics department. In contrast, at MSU members of the mathematics departments were central and consistent change agents.

In addition to contributing to the change literature, we hope that these two stories of change provide guidance for others wishing to embark on their own departmental transformation efforts. Key takeaways from these data include caution on relying too heavily on individuals, garnering the support of upper administration (which can lead to resources for sustainability), and putting structures in place (e.g., professional development for LAs) that allow for continuity and sustainability.

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References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *Instructional practices guide*. Mathematical Association of America.

Apkarian, N. (2018). Transforming Precalculus to Calculus 2: A Longitudinal Study of Social and Structural Change in a University Mathematics Department. ProQuest Dissertations and Theses.

Bressoud, D. (2015). Insights from the MAA national study of college calculus. *The Mathematics Teacher*, *109*(3), 179–185. https://doi.org/10.5951/mathteacher.109.3.0178.

Conference Board of the Mathematical Sciences. (2016). *Active learning in post-secondary mathematics education.* Conference Board of the Mathematical Sciences.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America (PNAS), 111* (23). Retrieved from www.pnas.org/cgi/doi/10.1073/pnas.1319030111.

Kogan,M., & Laursen, S. L. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education, 39*, 183–199.

LaTona-Tequida, T., Ralston, K., Rasmussen, C., & Apkarian, N. (2022, February 24-26). An analysis of eleven department change initiatives [Poster presentation]. *Proceedings of the Twenty-Fourth Annual Conference on RUME*. Boston, MA, United States

Laursen, S., Andrews, T., Stains, M., Finelli, C. J., Borrego, M., McConnell, D., … Foote, K. (2019). *Levers for change: An assessment of progress on changing STEM instruction. (D. Smith, Ed.)*. American Association for the Advancement of Science.

Miles, M., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. Sage.

Presidential Council of Advisors on Science and Technology (PCAST). (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Office of the President.

Reinholz, D., Rasmussen, C., & Nardi, E. (2020). Time for (research) on change in mathematics departments. *International Journal of Research in Undergraduate Mathematics, 6*(2), 147-158.

Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walberg-Henriksson, H., & Hemmo, V. (2007). *L'enseignement scientifique aujourd'hui: Une pédagogie renouvelée pour l'avenir de l'Europe*. Bruxelles: Commission Européenne, Direction générale de la recherche, science, économie et société.

Seymour, E., & Hewitt, N. M. (1997). *Talking about leaving* (Vol. 34). Westview Press, Boulder, CO.

Seymour, E., Hunter, A. B., & Harper, R. P. (2019). Talking about leaving revisited. *Talking About Leaving Revisited: Persistence, Relocation, and Loss in Undergraduate STEM Education*.

Smith, W. M., Rasmussen, C., & Tubbs, R. (2021). Introduction to the Special Issue: Insights and Lessons Learned from Mathematics Departments in the Process of Change. *PRIMUS*, *31*(3–5), 239–251. https://doi.org/10.1080/10511970.2021.1886207

Smith, W. M., Voigt, M., Ström, A., Webb, D. C., & Martin, W. G. (Eds.). (2021). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning* (Vol. 138). American Mathematical Society.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., ... & Grummer, J. A. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476-6483.

Student Thinking with a Non-Traditional Linear Coordinate System

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This study explores different ways that linear algebra students reason with a non-traditional linear system, referred to as the Gulliver system, in a task-based clinical interview. Using the constructs of Naming and Locating developed in the conceptual framework, an a priori analysis outlines how students may engage in Locating and Naming tasks. The a priori analysis was used for data analysis as a basic framing. Students' engagement with the non-traditional linear system and the refined and extended a priori analysis will be presented. Students' adoption of their previous experience with the Cartesian coordinate system will be also discussed.

Keywords: Linear Algebra, Coordinate Systems, Conceptual Framework, RME, A priori analysis

Coordinate systems are widely used in secondary and collegiate mathematics. Students learn the Cartesian coordinate system in their early mathematics, where they associate an ordered pair (a, b) with a point and use it to graph equations in the plane. Later in Pre-Calculus and Calculus, many students progress to explore a new coordinate system, the Polar coordinate system. In Linear algebra, students encounter non-traditional linear coordinate systems that are similar to the Cartesian coordinate system but scaled and/or rotated. There are a decent number of studies which focus on student reasoning with the Polar coordinate system and how their understanding of the Cartesian coordinate system impacts their reasoning with the Polar system. (Montiel et al., 2008; Montiel et al., 2009; Montiel et al., 2012; Moore, Paoletti, & Musgrave, 2014; Sayre &Wittmann 2008) Despite the growing importance of linear algebra in STEM education (Tucker, 1993), there is a noticeable gap in study concerning student thinking of a non-traditional linear system and how students employ their understanding of the Cartesian coordinate system when engaging with a non-traditional linear system. This report foregrounds a non-traditional linear system that shares similarities with, yet is distinct from, the Cartesian coordinate system.

Literature

Some studies found that students' understanding of the Polar coordinate system is closely related to their understanding of the Cartesian coordinate system and sometimes students' familiarity with the Cartesian coordinate system delays the shift to other coordinate systems (Arcavi, 2003; Hillel & Sierpinska, 1993; Montiel et al., 2008; Montiel et al., 2009; Montiel et al., 2012; Sayre &Wittmann 2008). For example, Montiel and his colleagues found that students applied the *vertical line test* to a graph defined in the Polar coordinate system to check if it is a function over the Polar coordinate system even though the vertical line test is no longer useful. Similarly, Moore et al. (2014) found that the convention from the Cartesian coordinate system of using the ordered pair (input, output) may be problematic when constructing the Polar coordinate system which uses the reversed ordered pair (output, input).

In linear algebra, Wawro et al. (2013) created a lesson which includes a task that uses $y = x$ and $y = -3x$, as the new axes, to rename a location in a non-traditional linear system. Zandieh et al. (2017) found that students in the class using the task sequence symbolized locations in three different ways. (1) Some students renamed locations using geometry by identifying which new axis to treat as the x and y and the size and direction of a unit vector. (2) Other students used a matrix equation: setting up a matrix equation $A\boldsymbol{x}_{[a\ system]} = \boldsymbol{x}_{[another\ system]}$, solving for components in the two-by-two matrix A, and using A to convert names from a system to the other. (3) Another way that students renamed involves the idea of linear combination. Students found two vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\3\end{bmatrix}$ to match the two new axis directions of $y = x$ and $y = -3x$. Then, they determined c_1 and c_2, how much in each direction should travel along them in reaching a location in the plane. The sum of the scalar multiplications, $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\3\end{bmatrix}$, provided a coordinate pair with respect to the Cartesian system.

Other studies have discussed basis or other aspects of linear combinations but did not refer to these as coordinate systems. (Bernier & Zandieh, 2022; Bettersworth et al., 2022; Dogan, 2019; Dreyfus et al., 1999; Turgut et al., 2022; Wawro et al., 2012) Given that studies of non-standard linear coordinate systems are rare in the literature, this study intends to begin filling this absence.

Conceptual Framing

The conceptual framework was developed from the author's calculus and linear algebra textbook analysis (Author, year). It can serve as a useful framework when designing tasks that involve coordinate systems and analyzing students' mathematical activity. (Lee, year) The coordinate system framework includes two fundamental processes with representations: Naming and Locating. In Naming, a location in space is being measured following the convention imposed by a coordinate system and creates the measurement, a name. For example, a location in the 2D plane gets its name as (1,1) with the Cartesian coordinate system. On the other hand, in Locating, an existing name creates its location in space following the convention imposed by a coordinate system. An example of Locating is that the ordered pair (1,1) puts on a specific point in the Cartesian coordinate plane. Figure 1(left) illustrates that Naming and Locating are the reverse processes to each other. The two processes can be extended to represent an object with multiple coordinate systems: Re-Naming and Re-Locating. In Re-Naming, a location that has been measured by a coordinate system gets its new name measured in a new coordinate system that is laid atop the location. That is, the location previously paired with (1,1) is being renamed with $(\sqrt{2}, \frac{\pi}{4}) \approx (1.414, 0.785)$ in the same space using the Polar coordinate system (Figure 1, middle). On the other hand, in Re-Locating, an existing name creates two different locations in space depending on coordinate systems being used. The new location may appear different from the first, but they share the same name. For example, (1,1) corresponds to two locations: one defined by a horizontal and vertical distance of 1 each, and the other determined by a distance of 1 from the origin and an angle measure of 1 radian from the horizontal axis. (Figure1, right)

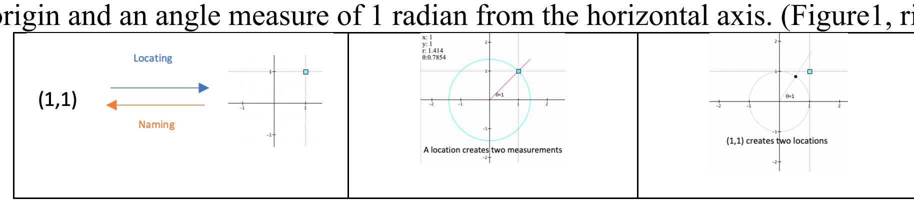


Figure 1. Naming and Locating (left), Re-Naming (middle), Re-Locating (right)

A priori analysis

Prior to conducting interviews with students, the author described an a priori analysis of how students might engage in Locating and Naming. Students' possible steps in Locating include (1) pairing known each component in an ordered pair with a proper axis, (2) identifying the location of each component by comparing it to the unit length imposed on each coordinate axis, (3) finding the intersection that comes from two locations on the axes. Reversely, in Naming, students would do (1) splitting the known location into two locations, one on the x-axis and one on the y-axis (if it is the Cartesian system), (2) measuring the length of each location by comparing it to the unit length imposed by the coordinate system, (3) expressing the measurements on the two axes symbolically. Figure 2 outlines the a priori analysis.

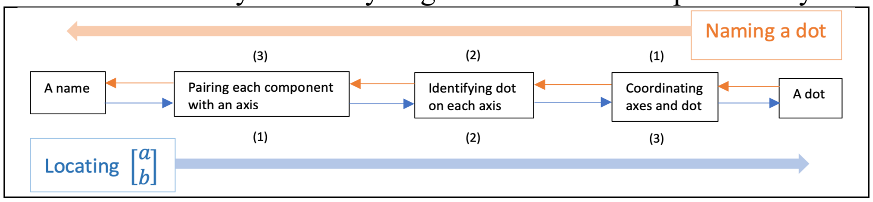


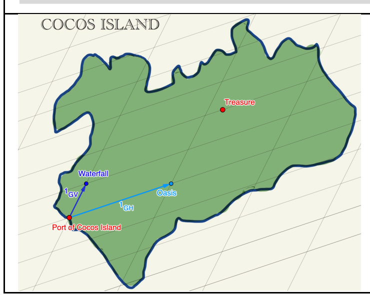
Figure 2. An a priori analysis for Locating and Naming

This study reports student reasoning with a non-traditional linear coordinate system in a task-based clinical interviews, designed to answer the research questions: (1) What are the different ways that linear algebra student reasons in a new linear coordinate system? (2) How do they employ their familiarity with the Cartesian coordinate system in working with the new system?

Methods

This proposal includes the first two tasks of a clinical interview, part of a longer dissertation study that consists of clinical interviews and teaching experiments. The clinical interview tasks were designed based on the central idea of Realistic Mathematics Education (RME), which emphasizes tasks to be experientially real starting point to students informed by Freudenthal (1991). A motive from the famous book "Gulliver's Travels" (Jonathan Swift, 1726) is combined with a treasure hunt. (Figure 3) Task 1 is a *Locating* task to place a dot with the number pair provided. Task 2 is a *Naming* task to name the treasure location. Both the tasks are built in the Gulliver system that is a new linear coordinate system different from the Cartesian system.

Gulliver discovered an ancient treasure map of Cocos Island in his hometown.
A grid and two arrows pointing to the location of an Oasis and a Waterfall were drawn on the map. Gulliver named the arrows $1_{GH} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $1_{GV} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Task 1:
Place a dot on the map indicating where $A=\begin{bmatrix} 1.3 \\ 0.5 \end{bmatrix}$ is located.

Task 2:
Describe the location of the Treasure.

Figure 3. Problem setting with Task 1(Locating) & Task 2(Naming)

Data Source and Analytic Method

The author conducted face-to-face clinical interviews with five students who have taken linear algebra at a large public university of the Southwestern United States. The students were STEM majors who had taken Calculus 1 or 2 as a prerequisite. Both their written work and interview conversations were recorded. The interview data were transcribed into spreadsheet and coded line by line, based on the author's a priori analysis. Whenever students engaged with the steps outlined in the a priori analysis, their quotes were noted and examined to characterize their reasoning. Additionally, the steps were refined and extended, resulting in the separation of one step into two distinct steps. All the names used in Results are pseudonyms reflecting their ethnicity.

Result

As shown in Figure 3, each task has its own goal aimed at examining student reasoning that corresponds to Locating and Naming. In this section, the different ways that linear algebra student reasons with the Gulliver coordinate system within steps outlined in the a priori analysis will be explored.

Locating: Determining a location in space

The students have demonstrated their ways of determining location in space. They coordinated the components in $\begin{bmatrix}1.3\\0.5\end{bmatrix}$ with the appropriate axes and identified the intersecting point corresponding to 1.3 and 0.5 on the axes.

Pairing each component with an axis

Given the components of A as 1.3 and 0.5, the participants positioned them on either the axes of the Cocos Island map or the axes separately created in a blank space. Positioning the number values requires the students' two-step commitments. First, they figure out which component of the first (1.3) or second (0.5) is paired with each direction of the horizontal or vertical axis. Second, they compare each number value to the unit length of 1. The following excerpts illustrate how they engaged with coordinating and locating on the axes.

Hann determined that the first coordinate should be in the direction of the Oasis and then it is positioned to the right of it. "*1.3, that's going to be in the Oasis direction, so that would be somewhere here* [points to the right of the 1]*"*. Wilson also attempted to mark the number values, 1.3 and 0.5, on each direction of axis, however his pairing of the first and second component with the horizontal and vertical axis went opposite. *"to go and establish where that 1.3 , 0.5 is, 1.3 roughly there* [marks on the vertical axis, above the 1] *and 0.5 there* [marks on the horizontal axis, to the left of 1]".

Jeraldo, Wanita, and Neeman marked on axes indicating where the size of the number values is located by comparing them to the size of units. Wanita said, "*I'm in going a little bit further* [than 1 for 1.3], *and then 0.5 is a little bit less from the 1*". Neeman also mentioned "*the y* [0.5] *is the midpoint here* [points on the vertical axis]…*and then so this is 1*[points to the Oasis location, the unit], *then 1.3*[marks on the horizontal axis the right of 1]."

Coordinating axes and dot; Intersecting two locations from axes. Once the students determined the location of each coordinate on the respective axis, some of them drew auxiliary lines. For instance, Jeraldo indicated a dotted line starting at 0.5 on the vertical axis and is parallel to the horizontal axis, then made some extended portion at 1.3 vertically. He finally placed a dot A by intersecting the auxiliary segments (Figure 4. Jeraldo). Neeman also indicated some auxiliary lines in yellow that pass the two locations representing 1.3 and 0.5 on the axes he marked earlier (Figure 4. Neeman). He constructed a dot for A where the two lines intersect. The written work of Jeraldo and Neeman shows that they think of the location of [1.3 0.5] as the intersection of the extended two locations from axes. Even though the other students did not draw auxiliary lines, they seemed to engage with the intersecting process in that the final dot was marked once they determine each coordinate on its corresponding axis.

I note that the auxiliary lines are not always parallel to the Gulliver coordinate system's grids; rather the vertical part of the auxiliary lines appears somewhat perpendicular to the horizontal axis. This demonstrates that the students tend to think of perpendicular grids even when it does not precisely match the appropriate projections of measurements from the axes. The students' final answers to this task are shown in Figure 4.

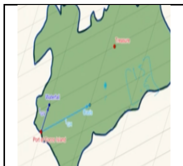
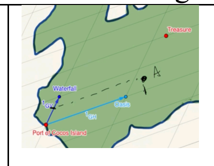
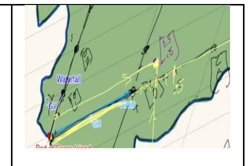
				
Hann	Wilson	Jeraldo	Wanita	Neeman

Figure 4. Location of $\begin{bmatrix}1.3\\0.5\end{bmatrix}$

Naming: Obtaining coordinates-pair

Similar to Task 1, the students utilized their understanding of the rectangular coordinate system to determine the coordinates of the Treasure location in Task 2 (Figure 3). One of the participants, Hann, seemed to first look for measurement on each axis and then coordinate them with their corresponding axis. "*The location of the treasure looks to be probably about* <u>*1.3 in that* [Oasis] *direction and then exactly 2 in the waterfall or y direction*</u>".

Some of the participants described how to obtain the coordinates-pair from the Treasure location given on the map. I present two distinct ways that students have engaged in obtaining the coordinates: Projecting the location onto axes and Over-and-up reaching the location.

Coordinating axes and dot; Projecting a given location onto axes. Jeraldo and Neeman indicated that they need some kind of projection to obtain measurements of the Treasure location. Jeraldo noted, *"If* <u>*we follow, like, the parallel line*</u>*, it's almost at the same exact place* [refers to the Task 1's horizontal component, 1.3]. *So, I'd probably say it's around the same as the previous at 1.3".* From his description, he seemed to be looking at the slanted projection from the Treasure location onto the Gulliver horizontal axis, recognizing that the slanty line passes through location A that he had placed earlier with 1.3 in Task 1. Similarly, Neeman's written work depicted reasoning with a slanted projection indicated from the auxiliary lines in black (Figure 5. left). He drew the lines to pass through the Treasure location and to be parallel to the Gulliver coordinate grids. The measurements that he obtained result from the slanted projection of the Treasure location onto the two axes. "*This* [where the slanted projection meets on the horizontal axis] *is a little bit before the 1.3, that is like 1.2. And it's* [where the slanted projection meets on the vertical axis] *on this one* [points the Waterfall]*, which is 2. So, let's say it's a 1.2 over 2".*

Figure 5. Written work of Neeman (left) and Wanita (right) in Naming

Coordinating axes and dot; Over-and-up reaching a location. Another student, Wanita, also demonstrated her reasoning with the rectangular coordinate system to obtain the coordinates of the Treasure. She illustrated the process of moving along the Gulliver horizontal axis and then turning up towards the Treasure location (Figure 5. right).

Wanita: It would be like 1.5 and then 2. I'm supposing the port is at the zero. It would be like 1.5, 1.6 or so. <u>It's going up [indicates the horizontally positive direction] and then 2.</u>

Interviewer: Why do you say 1.5 or 1.6?

Wanita: …2 is over here [points to (2, 0) location] 1.5 should be like, right in the middle, and then you just go up [draws an arrow from the middle to the Treasure] and that's where the treasure would be…T is equal to the 1.6 and then 2.

I note that the two distinct ways of finding the coordinates-pair, slanted projection onto axes and over-and-up process, are not exclusive to each other. For example, Jeraldo, one of the students who engaged in the slanted projection, initially answered that the horizontal coordinate has to be 1.5 by over-and-up process, similar to Wanita. He was looking at the locations of one unit, two units, and midpoint on the Gulliver horizontal axis, estimating the right location to be 1.5 in order to reach the Treasure. Soon after, he realized that the slanted projection is not precisely matching the upward movement from 1.5 and switched over to the projection onto axes way from the over-and-up process.

Types of names in Naming

Once the participants determine the measurements for the Treasure location, some students attempted to represent the coordinates-pair in different ways, whether in response to my request or without any prompting. There were three different ways of representing the same coordinate found in students written expression: Vector form $\begin{bmatrix} a \\ b \end{bmatrix}$, Linear combination with opaque symbolics of 1_{GH} and 1_{GV} provided in the problem, and Linear combination with actual vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hann's written answer includes all three representations. He first wrote $\begin{bmatrix} 1.3 \\ 2 \end{bmatrix}$ in a vector form and represented it using the linear combination format employing the provided opaque symbolics, 1_{GH} and 1_{GV}. (Figure 6) He commented that the symbolics could be substituted with the actual vectors consisting of 1's and 0's.

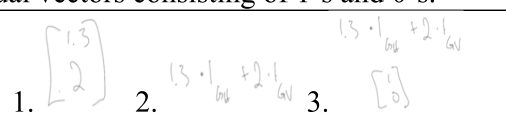
	1. Vector 2. Linear combination with opaques 3. Linear combination with actual vectors

Figure 6. Hann's symbolic representations for the same coordinates-pair

Wilson chose to represent the determined coordinates using the linear combination format as well in addition to the vector form. (Figure 7) The distinction between Wilson and Hann's representation is that Wilson made a slightly different modification to the opaque symbolics. That is, GH and GV have been used instead of 1GH and 1GV. This is an indication that Wilson conceives of the number '1' in the opaque symbolics of 1GH and 1GV as an actual measurement rather than as a symbol emphasizing a unit.

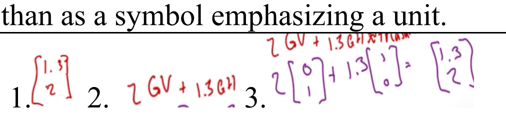
	1. Vector 2. Linear combination with opaques 3. Linear combination with actual vectors

Figure 7. Wilson's symbolic representations for the same coordinates

The other three participants, Jeraldo, Wanita, and Neeman represented the Treasure location as a vector form only using the measurements obtained from either slanted projection or over-and-up. I note that Neeman read the vector form of $\begin{bmatrix} 1.1 \\ 2 \end{bmatrix}$ "*1.1 over 2*". His treating the vector like a fraction sometimes comes a long later in the interview when he writes $\begin{bmatrix} \frac{14}{6} \end{bmatrix}$ to mean $\begin{bmatrix} 14 \\ 6 \end{bmatrix}$.

Discussion

The students have adopted their previous experience with the rectangular coordinate system to answer the first task, even though the Gulliver coordinate system is a non-Cartesian rectangular system. This aspect is well represented by one of the interview participants, Hann's comment "*Even though it's not rectangular, there's no reason not to act like it is*". The following lists different pieces of the rectangular coordinate system understanding that the participants brought to bear: identifying units as two directional line segments, labeling with 'x' and 'y', employing perpendicular axes, and pivoting a reference point. While some of them assisted the students in addressing the tasks, others were applied even though they were no longer useful.

Jeraldo called the 1_{GH} and 1_{GV} as matrices and noted that they represent one unit in each direction of x and y. The letters x and y, widely used in the mathematical community for coordinates in the rectangular coordinate system, have been used to denote the first coordinate as x and the second coordinate as y. *"I remember in back in the class that the matrices they obviously defined <u>that the first one represents the x coordinates, the second one equals the y.</u> So, x y x y* [writes x's and y's next to 1 0 and 0 1]". (Figure 8. Left) Wanita drew perpendicular axes on a blank space, and then placed two dots one on the horizontal axis and one on the vertical axis to coordinate them with $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. She indicated that these dots are pointing to the Oasis and Waterfall locations. (Figure 8. Middle) Neeman indicated the Port of Cocos Island on the map where the two axes intersect as the coordinate pair [0, 0]. That is, the reference point was identified as two components of null. He brought the letters x and y to indicate the first coordinate and the second coordinate, respectively and noted x to be corresponding to the horizontal axis and y to be the vertical axis. Additionally, he drew the perpendicular axes labeled with x and y. (Figure 8, right)

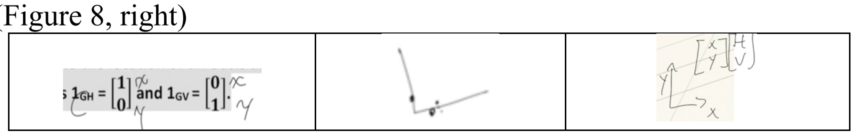


Figure 8. Employing rectangular coordinate system: Jeraldo(left), Wanita(middle), Neeman(right)

The Locating and Naming activities are reverse processes to each other. The processes were outlined in an a priori analysis in Figure 2, and it has been further refined in two ways: variations within a step and separation of one step into two distinct steps. Students have been engaged differently with the remaining steps when they progress to the subsequent set of tasks following Task 1 and 2.

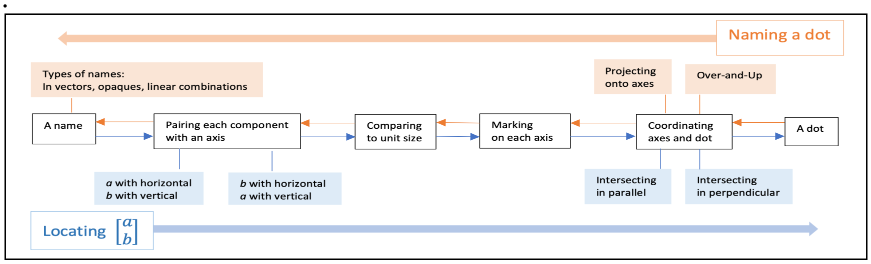


Figure 9. Locating and Naming in a linear coordinate system

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References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. 28. *Educational Studies in Mathematics* , 2003, Vol. 52, No. 3 (2003), pp. 215-241. https://www.jstor.org/stable/3483015

Bernier, J. & Zandieh, M. (2022). Comparing Student Strategies in Vector Unknown and the Magic Carpet Ride Task. Proceedings of the 2022 Conference on Research in Undergraduate Mathematics Education (pp. 46-53). Boston, MA: Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME).

Bettersworth, Z. & Smith, K. & Zandieh, M. (2022). Students' written homework responses using digital games in Inquiry-Oriented Linear Algebra. Proceedings of the 2022 Conference on Research in Undergraduate Mathematics Education. Boston, MA: Special Interest Group of the Mathematical Association of America on Research

Dogan, H. (2019). Some aspects of linear independence schemas. ZDM Mathematics Education 51, 1169–1181. https://doi.org/10.1007/s11858-019-01082-4

Dreyfus, T., Hillel, J., & Sierpinska, A. (1999). CABRI BASED LINEAR ALGEBRA: TRANSFORMATIONS. 13. European Research in Mathematics Education 1, 209-221. Inge Schwank

Freudenthal, H. (1991). Revisiting mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hillel, J. and Sierpinska, A. (1994). On One Persistent Mistake in Linear Algebra. In *The Proceedings PME 18*, pp. 65–72, University of Lisbon, Portugal.

Lee. (2023). A Coordinate System Framework. Proceedings of the 2023 Conference on Research in Undergraduate Mathematics Education. Boston, MA: Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.

Moore, K. C., Paoletti, T., & Musgrave, S. (2014). Complexities in students' construction of the polar coordinate system. *The Journal of Mathematical Behavior*, *36*, 135–149. https://doi.org/10.1016/j.jmathb.2014.10.001

Montiel, M., Vidakovic, D., & Kabael, T. (2008). Relationship between students' understanding of functions in Cartesian and polar coordinate systems. *Investigations in Mathematics Learning*, 1(2), 52–70.

Montiel, M., Wilhelmi, M. R., Vidakovic, D., & Elstak, I. (2009). Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context. *CERME 6– WORKING GROUP* 12, 2286.

Montiel, M., Wilhelmi, M. R., Vidakovic, D., & Elstak, I. (2012). Vectors, change of basis and matrix representation: onto-semiotic approach in the analysis of creating meaning. *International Journal of Mathematical Education in Science and Technology*, 43(1), 11–32. https://doi.org/10.1080/0020739X.2011.582173

Sayre, E. C., & Wittmann, M. C. (2008). Plasticity of intermediate mechanics students' coordinate system choice. *Physical Review Special Topics - Physics Education Research,* 4(2), 020105. https://doi.org/10.1103/PhysRevSTPER.4.020105.

Swift, J.(1950). Gulliver's travels. New York :Harper,

Tucker, A. (1993). The Growing Importance of Linear Algebra in Undergraduate Mathematics. *THE COLLEGE MATHEMATICS JOURNAL*, 24(1), 8.

Turgut, M., Smith, J. L., & Andrews-Larson, C. (2022). Symbolizing lines and planes as linear combinations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *66*, 100948. https://doi.org/10.1016/j.jmathb.2022.100948

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G., & Larson, M. (2012). An inquiry-oriented approach to span and linear independence: The case of the Magic Carpet Ride sequence. *PRIMUS*: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22(8), 577–599.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An Example of Inquiry in Linear Algebra: The Roles of Symbolizing and Brokering. *PRIMUS*, 27(1), 96–124. https://doi.org/10.1080/10511970.2016.1199618

Exploring How Use of a Shared Google Doc for Collective Proof Construction Supports Students' Mathematical Progress

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While the pandemic brought much hardship for mathematics education, it also brought opportunity for ingenuity and transformation. This study builds on results from prior work that found Google Docs to be rich tools that supported students in effectively engaging in collaborative proof activity in a remote introduction-to-proofs course. To study if Google Docs would have similar benefits for students' collective proof activity in a face-to-face setting, we investigate how two students in a face-to-face course leveraged a shared Google Doc to support their collaborative work on a proof construction task. We found that the students used the shared Google Doc to support their collective proof construction in similar ways as students in the remote class. In addition, we observed that the Google Doc was an inherent component of students engaging in an active proof writing process.

Keywords: Proof Construction, Technology, Google Docs, Collaborative Activity

For mathematics education the COVID-19 pandemic brought drastic changes in how courses were structured and how interactions between students and teachers took place (Tate & Warschauer, 2022). While these changes brought many challenges both educationally and emotionally (Engelbrecht et al., 2023), they also brought the opportunity for ingenuity and for reimagining what mathematics classrooms could look like. Specifically, in many ways the pandemic forced educators to find new means by which to use technology to engage students with mathematics and with each other, potentially, enriching students' learning experiences. Literature on educational technology has highlighted how we can create rich learning environments with the aid of technological tools (e.g., Clark et al., 2007; Öner, 2008; Stahl, 2006). As we transition (or have transitioned) back to face-to-face (F2F) courses, it becomes imperative to reflect back on what we have learned as educators from our experience teaching remotely. Specifically, we should consider what aspects of our remote mathematics courses can be (and perhaps should be) transferred to F2F. Studying remote instruction in an introduction to proof course, Alzaga Elizondo (2022) found valuable implications of using Google Docs for students' collective work in small groups. Most notably, she found that by coordinating the use of Google Docs with Zoom, students were able to actively engage with one another's thinking and produce truly collaborative work. Subsequently, we wondered if and how shared Google Docs would provide similar support to students in a F2F version of this course. In this paper, we present a case study of two students working on a task to develop a proof in a F2F class with access to a shared Google Doc. We explore the connection between the students' use of the Google Doc to support their collective proof activity and their proof construction progress.

Theoretical Grounding

To study the role that the Google Docs played in students' collective proof construction, this study builds on the work done by Alzaga Elizondo (2022). Alzaga Elizondo studied students' small group activity in a synchronous online Introduction to Proofs course where she investigated how students operationalized technological tools (e.g., Zoom, Google Docs) in the remote environment to engage in various collective proving activities (i.e., proof construction, defining, conjecturing). To analyze the students' tool use, Alzaga Elizondo leveraged the theory of Instrumental Genesis (Rabardel, 2002) which describes a reciprocal relationship between artifacts (or tools) and their users and the impact of that relationship on cognition. Any given tool can be transformed into an instrument by assigning it a *utilization scheme* (designated methods for using the tool) for a specific *goal directed activity* (Béguin & Rabardel, 2000; Carvalho et al., 2019). In her study, Alzaga Elizondo identified several collective instruments that students developed by coordinating the different technological tools available to them. These were collective in the sense that students coordinated their utilization schemes across different tools to develop collective schemes that were closely tied to their mathematical activity. For instance, students developed an instrument to *illustrate a new idea by writing a mathematical statement* where they would coordinate the use of Zoom and Google Docs to verbally communicate and simultaneously visually illustrate a new idea. Simpler instruments were created to support the development of more complex ones, these are identified in Table 1 with the supportive instruments italicized. We conjecture that students may develop similar instruments in a F2F class where, in this setting, students coordinate their use of the Google Doc with their verbal face-to-face interactions rather than through Zoom.

Table 1. Collective Instruments

Instrument	Observed Student Action
Engage in collective argumentation by coordinating visual mediators and verbal communication	Discuss and debate ideas by using visuals on the Google Doc to illustrate verbally spoken idea.
Illustrate new idea by writing a mathematical statement	Write mathematical statement on Google Doc to illustrate an idea to group before group has agreed on new idea.
Analyze shared idea by referencing written text	Reference written text by highlighting (or placing cursor next to) text on the Google Doc and/or by using indexical terms like “this” or “that”.
Develop shared understanding by introducing diagrams and examples	Adding examples or diagrams to Google Doc for the goal of developing shared understanding among group.
Co-construct a group solution by refining shared text	Create final solution by refining written text on the Google Doc.
Execute solution by repurposing illustrated idea	Using existing text in the Google Doc for the final solution that was not originally written for that purpose.
Execute solution by validating illustrated idea	Keeping text in the Google Doc that was added to final solution for the purpose of illustrating a new idea.

Further, Alzaga Elizondo conjectured that use of these instruments supported students in engaging in a *process approach* to writing, which describes an iterative collective writing process in which students plan, draft, and revise repeatedly by incorporating feedback from peers and the instructor (Graham & Perin, 2007; Sun & Feng, 2009). These kinds of activities can reflect more meaningful engagement with proof-based tasks, as they promote exploration and testing which have been noted to reflect more authentic mathematical practices (Larsen & Zandieh, 2008; Melhuish et al., 2022; Öner, 2008). As such, students, regardless of course modality (i.e., face-to-face or remote) should be encouraged to engage in a similar writing process. The work presented here builds on Alzaga Elizondo's (2022) work by investigating how and/or if the collective instruments she identified appeared in one group's collective proof construction when using Google Docs in a F2F course. Further, and most notably, it investigates how the students' Google Doc use supported them in engaging in a process approach to collective proof construction.

Methods

Data Collection and Episode Selection

The data for this study is from a university introduction-to-proof course that was part of a larger NSF-funded project (DUE #1916490) that developed introduction-to-proof curricula and accompanying instructor support materials. The inquiry-oriented curriculum included tasks designed for small group collaborative work. In this 10-week F2F course each student had access to an iPad which enabled each group of students to share a Google Doc to work together on tasks during small group work. A stationary camera and iPad screen recordings captured one focal group's interactions each day (26 days total).

This study focuses on two students' collective work on one task during one day of this course. Jake and Roger (pseudonyms), worked together on proving the sequence $a_n = \lfloor 10/n \rfloor$ is eventually constant. The class had previously established a definition for an eventually constant sequence: *a sequence a_n is eventually constant if there exists a natural number n such that for every natural number $k \geq n$, $a_n = a_k$*. In other words, a sequence is eventually constant if at some point it becomes and stays constant. The Google Doc that the students used included the task and this definition at the top of the page. This episode was selected because the students worked together in their small group to complete the task and there was substantial discourse, both written using the Google Doc and verbal, between the students.

Data Analysis

To begin our analysis, we created a multimodal transcript (Hoffman, 2018) that captured both the students' written work on their Google Doc and their spoken interactions. Since the students' iPads recorded both their written and spoken work, time stamps added to our transcript provided an accurate record of the temporal relationship between spoken and written statements. Then, using the observed student actions in Table 1 as operationalizations of the collective instruments, the first author identified for each of the spoken and written communications in the multimodal transcript if and when a collective instrument from Table 1 was used and described any mathematical progress reflected in the communication. The second author reviewed the first author's identifications and descriptions and any disagreements were resolved through discussion. Through the analysis the authors found that students developed a sequence of proof drafts. Together, the authors then identified six points in the students' work on the Google Doc that represented completion of interim drafts of their final solution (see Figure 1). We identified a point in their Google Doc as a draft if it represented a significant mathematical change from the previous draft and seemed to be the culmination of the students' argumentation at the time. We then were able to describe how the students' instrumentation process impacted their progress through development of the draft. We present the findings from this analysis in the following section.

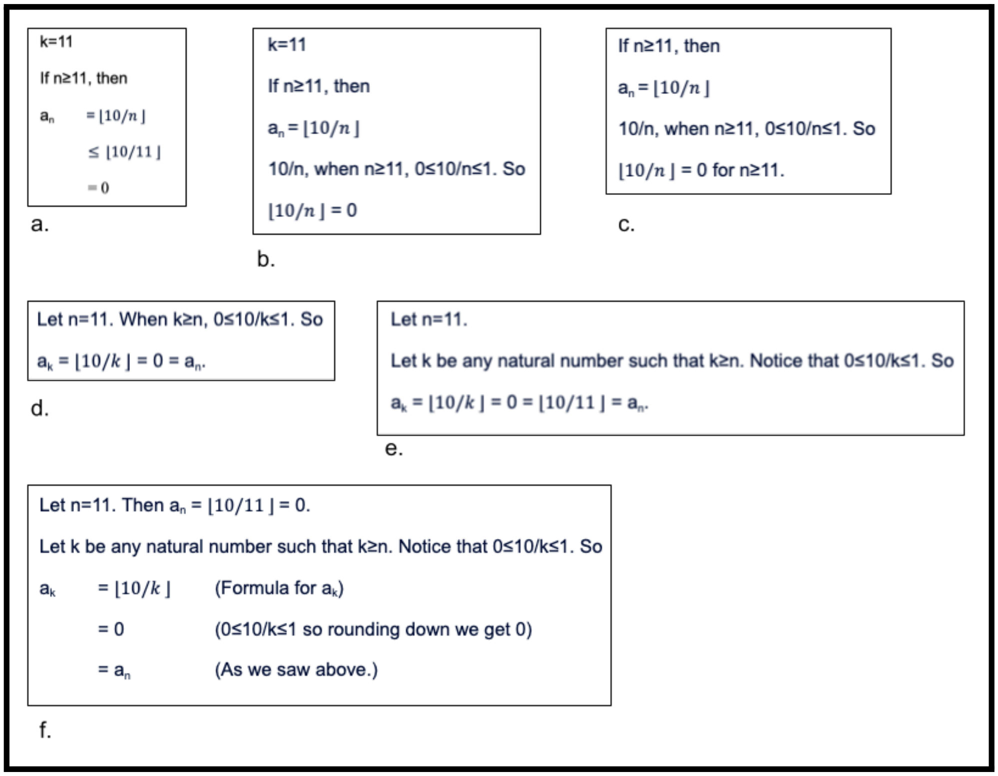

Figure 1. Students' drafts of their proof that the sequence $a_n = \lfloor 10/n \rfloor$ is Eventually Constant

Results

In the episode analyzed below we observed students using the Google Doc as a tool to engage in collaborative proof activity. Here we report the students' progress through the development of each draft, highlighting their use of collective instruments to achieve each draft.

Development of Draft 1

To start off their work, Jake suggested that they write a "few terms of the sequence and just see what it converges to". Without verbal confirmation, Roger responds by writing the first few values of the sequence in the Google Doc, writing "$a_n = 10$, 10/2=5, 3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0,...". Simultaneously, Roger verbalized his reasoning: "10 over 8, 1, 10 over 9, 1, 10 over 10, 1, and then 10 over 11, 0. I think at that point." Here Roger used the Google Doc to *develop a shared understanding by introducing examples* (i.e., the first terms of the sequence). As Roger typed, we saw evidence of Jake following along, verbally stating "and after *that* it becomes zero" at the same moment Roger says "I think at that point". In doing this, Jake is using the Google Doc to *analyze the shared idea* (the sequence terms) *by referencing text* when saying "and after *that*" where "that" referred to the 11th term in the sequence. At this point, Jake suggested that they construct a direct proof that follows the structure of the definition. Roger proceeds to *illustrate an idea by writing mathematical statements* on the Google Doc, writing

"$k = 11$, If $n \geq 11$, then $a_n = \lfloor 10/n \rfloor = \lfloor 10/n \rfloor$". As Roger is writing, Jake follows along and suggests they use inequalities instead which prompts refinement from Roger. Together, they produce the first draft of their proof (see Figure 1a).

Development of Draft 2

After a brief pause, Roger suggests "Maybe we should say something different. So like 10 over *n* when *n* is greater than 11." Simultaneously he adds to the Google Doc "10/*n*, when $n \geq 11$, 10/*n*" before the line "$\leq \lfloor 10/11 \rfloor$". This suggests Roger recognizes the inequality in Draft 1 is not correct and chooses to *illustrate a new idea by writing a mathematical statement* in the Google Doc to fix this error. After Roger writes this line in the Google Doc Jake responds by saying "is less than 1" (in other words, proposing "< 1" should follow the "$10/n$" Roger has just written). Here, Jake is not only suggesting the completion of this line, he is also communicating to Roger that he concurs with what Roger has written so far. As such, Jake is using the Google Doc to *execute a solution by validating an illustrated idea* (the line Roger has just added). In other words, Jake agrees with Roger's new addition and thinks it should be kept as part of their proof. Robert acknowledges Jake's idea and proposes a variation, saying "well yeah right sort of bound it like this. It's between zero and one." As he says this, Roger replaces "$10/n$" with "$0 \leq 10/n \leq 1$," a compound inequality. Again, using the Google Doc to *illustrate a new idea by writing a mathematical statement* that he only partially communicates verbally. Jake then again *executes a solution by validating an illustrated idea*, concurring with Roger's compound inequality by saying "Oh true. Yeah, it's not just less than one, it's [...] positive." With this validation, Roger then changes "$\lfloor 10/11 \rfloor = 0$" to "$\lfloor 10/n \rfloor = 0$." This change moves the last two lines of the proof closer to showing how the sequence formula generates 0 when $n \geq 11$ and results in Draft 2 (see Figure 1b).

Development of Draft 3:

Roger's addition to the last line in Draft 2, "$\lfloor 10/n \rfloor = 0$," prompts Jake to respond verbally "So in that case a sub n equals zero for *n* greater than or equal to 11." Jake is observing that the sequence formula does equal zero, as Roger has written in the Google Doc, but only when $n \geq 11$. Roger then *illustrates this new idea by writing the mathematical statement* "for $n \geq 11$" directly following "$\lfloor 10/n \rfloor = 0$." Roger *validates this illustrated idea* verbally when he follows up by saying "Yeah". This addition directly connects $\lfloor 10/n \rfloor = 0$ to the index value where the sequence becomes constant. This new addition prompts Roger to say "I guess we don't need *k*" and delete "$k = 11$" from the first line. Presumably, Roger suggests deleting *k* because it is now an unused variable in their proof (see Figure 1c).

Development of Draft 4

Prompted by Roger removing the variable *k* from the proof, Jake makes the case that this proof does require two variables. He says, "Yeah. I guess maybe just to make it match the definition as much as possible, what if you said at the beginning let *n* be 11. And then we used *k* after that [...] like let *n* be 11 and *k* is greater than *n*." Roger takes Jake's suggestion and adds "$Let\ n = 11.\ When\ k > n$" to the beginning of the proof, *illustrating a new idea* (Jake's) *by writing the mathematical statement* at the beginning of their proof draft. Further, Roger *executes the solution by* (implicitly) *validating the illustrated idea* when he keeps the new text and *illustrates his own new idea* by refining the rest of the proof to be consistent with this change saying, "I think I can actually cut some of this out. Let's see." He proceeds by deleting the first few lines of their proof and, changing "$0 \leq 10/n \leq 1$" to "$0 \leq 10/k \leq 1$," and changes "$\lfloor 10/n \rfloor = 0$" to "$a_k = \lfloor 10/k \rfloor = 0$" to align with the new use of "*k*" leading to Draft 4 (see Figure 1d.). Jake follows Roger's written work and eventually *validates the illustrated idea* by verbally rephrasing the argument being presented in their current draft. With the addition of these two variables in Draft 4 the students refined their proof to better match the structure of the definition of an eventually constant sequence.

Development of Draft 5

A few minutes later, with a hint from their instructor, the students have added "Let *k* be any natural number" to the beginning line of their proof. The addition of this line in the Google Doc prompts Jake to say "might need to combine *these* two statements here. I'm going to take out the "when" and say "and" or "such that". At this point, Jake is *analyzing a shared idea by referencing* lines in the proof when he says "*these* two statements here." He follows up by *illustrating his idea,* replacing "When" with "such that" so the text that defines the variable *k* now reads "Let *k* be any natural number such that $k \geq n$." This prompts Roger to follow a similar pattern when he says "I think *this* should be a new sentence. Notice that." Like Jake, Roger begins by *analyzing their shared text by referencing* a line in their proof when he says "*this* should be" and then *illustrates his new idea* when he then adds "Notice that" before "$0 \leq 10/k \leq 1$" which separates this statement from the definition of the variable and results in Draft 5 (see Figure 1e.). While minimal, the refinement in this draft supported the readability and logic flow of the proof.

Development of Draft 6

The students make two changes between Draft 5 and their final proof, Draft 6. Roger says, "[The instructor] might want *these* on new lines. He kind of likes writing justifications." Here Roger *analyzes their shared text by referencing* "these" lines in their draft. He then breaks the last line of Draft 5, "$a_k = \lfloor 10/k \rfloor = 0 = \lfloor 10/11 \rfloor = a_n$," into individual lines so that he can *illustrate his new idea by* adding justifications for each of those steps (see Figure 1f.). The second change they make is led by Jake when he says, "Maybe we should put in a part where it says then a sub n equals [mumbles to self]." As he's talking, he *illustrates his new idea by* adding "Then $a_n = \lfloor 10/11 \rfloor = 0$" following "Let $n = 11$" in the first line of the proof. This addition prompts Roger to delete "$= \lfloor 10/11 \rfloor$" from the second last line in the proof. This suggests Roger *validates* Jake's *illustrated idea* as Roger recognizes they do not need this at the end of the proof if they have already established it at the beginning. This change brings them to Draft 6, their final proof.

Discussion

These results show how a group of students used a Google Doc as a collective instrument in a F2F course to support their process of collectively developing a proof. By coordinating use of the Google Doc with their verbal F2F interactions, we saw similar collective instruments developed as Alzaga Elizondo (2022) found in a remote class, including *illustrating a new idea by writing a mathematical statement* in the Google Doc, *analyzing a shared idea by referencing written tex*t in the Google Doc, and *executing a solution by validating an idea illustrated* in the Google Doc. This suggests the productive use of Google Docs for collective activity extends beyond the online setting.

Further, throughout the students' work, we found that as a result of sharing and debating different ideas the students iteratively refined their proof, developing several drafts of this proof throughout their collective work. In other words, we found students engaging in a *process approach* to proof writing. Most notably, we found that the students' use of the Google Doc not only supported this refining activity, it was inherently tied to it. The results presented above, highlight *how* students used the Google Docs to engage in this process approach. In particular, there were multiple instances of a mathematical statement written in the Google Doc that served as a springboard for subsequent mathematical ideas. Frequently, when a student *illustrated a new idea by writing a mathematical statement*, it was immediately followed by a student *analyzing this shared idea by referencing this text in the Google Doc* with a verbal remark that reflected building on the referenced text. Often this cycle continued with the analysis comment leading to *executing a solution by validating an illustrated idea.* The validation was sometimes in the form of verbal agreement, but frequently the validation was implicit in that one student would just add to another's idea suggesting they accepted it. As an example, while developing Draft 4, Roger removing k from the Google Doc was the springboard that prompted Jake to suggest changing the variable definitions. Roger then executed and validated Jake's idea when he added k back into the Google Doc. This, in turn, served as a springboard when it prompted Roger to delete lines that were no longer needed.

We conjecture there may be several reasons the students' use of the Google Doc instruments supported their progress through this proof construction task. The Google Doc provides a written format and, unlike spoken ideas, mathematical statements placed in a Google Doc do not disappear into thin air. Relative to spoken ideas, Google Doc ideas give students something visual to consider, acting as a visual mediator (Sfard, 2008) for the students' communication. In addition, ideas in the Google Doc provide the writer with a way to express a thought they may not be able to adequately verbalize. As a written format, the Google Doc plays two roles. It is both the place where students record their interim written ideas as they work and the place where their final solution lives. As a result, when a student is considering an idea presented in a Google Doc, the format of the presentation bears some resemblance to a solution. The format itself suggests progress toward a solution. At the same time, an idea in the Google Doc is easily editable by all group members. This combination of a format that moves toward a solution and is easy to edit may well invite student participation and inspire ideas.

This case study explored how a shared Google Doc supported a small group of students working on developing a proof in a F2F class. Future research could explore the role Google Docs might play when students were faced with other types of mathematics tasks besides developing a proof. Also, while we highlight certain advantages of the Google Doc, we wonder how students' activity might compare if they were to use other non-technological tools like a whiteboard for instance which is also easily editable and shared by all members of the group. Overall, this study presents a case for using Google Docs in a face-to-face proof course as it can promote a more active writing process that naturally ties the students' collective discourse and sharing of ideas with a truly collaborative final solution.

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References

Alzaga Elizondo, T. (2022). *Collective proving activity in a synchronous online environment* (Publication No. 29993973) [Doctoral dissertation, Portland State University]. ProQuest Dissertations Publishing. https://www.proquest.com/docview/2758131153/abstract/6412722F939F4959PQ/1

Béguin, P., & Rabardel, P. (2000). Designing for instrument-mediated activity. *Scandinavian journal of information systems*, *12*(1), 1. https://aisel-aisnet-org.proxy.lib.pdx.edu/sjis/vol12/iss1/1

Carvalho, L., Martinez-Maldonado, R., & Goodyear, P. (2019). Instrumental genesis in the design studio. *International Journal of Computer-Supported Collaborative Learning*, *14*(1), 77–107. https://doi.org/10.1007/s11412-019-09294-2

Clark, D. B., Stegmann, K., Weinberger, A., Menekse, M., & Erkens, G. (2007). Technology-Enhanced Learning Environments to Support Students' Argumentation. In S. Erduran & M. P. Jiménez-Aleixandre (Eds.), *Argumentation in Science Education: Perspectives from Classroom-Based Research* (pp. 217–243). Springer Netherlands. https://doi.org/10.1007/978-1-4020-6670-2_11

Engelbrecht, J., Borba, M. C., & Kaiser, G. (2023). Will we ever teach mathematics again in the way we used to before the pandemic? *ZDM – Mathematics Education*, *55*(1), 1–16. https://doi.org/10.1007/s11858-022-01460-5

Graham, S., & Perin, D. (2007). A meta-analysis of writing instruction for adolescent students. *Journal of Educational Psychology, 99*(3), 445-476. https://doi-org.proxy.lib.pdx.edu/10.1037/0022-0663.99.3.445

Hoffman, E. B. (2018). Untangling the Talk: A New Multimodal Discourse Analysis Method to Investigate Synchronous Online Learning. *Journal of Digital Learning in Teacher Education*, *34*(3), 179–195. https://doi.org/10.1080/21532974.2018.1453895

Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, *67*(3), 205–216. https://doi.org/10.1007/s10649-007-9106-0

Melhuish, K., Vroom, K., Lew, K., & Ellis, B. (2022). Operationalizing authentic mathematical proof activity using disciplinary tools. *The Journal of Mathematical Behavior*, *68*, 101009. https://doi.org/10.1016/j.jmathb.2022.101009

Öner, D. (2008). Supporting students' participation in authentic proof activities in computer supported collaborative learning (CSCL) environments. *International Journal of Computer-Supported Collaborative Learning*, *3*(3), 343–359. https://doi.org/10.1007/s11412-008-9043-7

Rabardel, P. (2002). *People and technology*. université paris 8. https://hal.science/hal-01020705

Stahl, G. (2006). Supporting Group Cognition in an Online Math Community: A Cognitive Tool for Small-Group Referencing in Text Chat. *Journal of Educational Computing Research*, *35*(2), 103–122. https://doi.org/10.2190/Q435-7611-2561-720P

Sun, C., & Feng, G. (2009). Process Approach to Teaching Writing Applied in Different Teaching Models. *English Language Teaching*, *2*(1), 150–155. https://eric.ed.gov/?id=EJ1082330

Tate, T., & Warschauer, M. (2022). Equity in online learning. *Educational Psychologist*, *57*(3), 192–206. https://doi.org/10.1080/00461520.2022.2062597

Uncovering Mathematics Capital: Using Structural Equation Modeling to Investigate Interrelations with Traditional Measures of Mathematics Assets

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Students enter undergraduate mathematics classrooms with a variety of mathematical backgrounds. We view these past mathematical experiences as assets that support students in their future mathematical learning. In this paper, we seek to characterize these assets by introducing a new concept in mathematics education: mathematics capital. This concept emerges from a framework connecting Archer et al.'s (2015) conceptualization of science capital and Bourdieu's seminal work on the forms of capital (1986). Survey data were gathered in Spring 2023 from a total of 219 students in undergraduate mathematics courses at an urban midwestern university. Path analyses in structural equation modeling showed a strong association between the variables of mathematics capital and self-efficacy. Other associations are also discussed. Our results indicate that mathematics capital is a measurable, quantifiable variable, independent from others. Future research is ongoing to better understand the nature of mathematics capital and how it is related to other variables.

Keywords: Mathematics capital, Self-efficacy, Mathematics mindset, Structural equation modeling, Path analysis

The current state of undergraduate mathematics education involves a multitude of variables related to student demographics and success. An important variable to student success is equity, which addresses purposeful accommodations and adaptations regarding student socioeconomic status, gender, and cultural differences (National Council of Teachers of Mathematics [NCTM], 2014). Students bring their individual characteristics and identities into the classroom, which ultimately impacts their learning experiences and success in mathematics (Gutiérrez, 2009, NCTM, 2014). In this study, we seek to identify student backgrounds, history, and characteristics that potentially impact their success in mathematics education. We view these student characteristics through a Bourdieusian style lens, parallel to the previously researched theory of science capital espoused by Archer and colleagues (2015). Herein, we describe our approach of using structural equation modeling to investigate a novel theoretical construct, which we define as *mathematics capital.* We posit that mathematics capital is a variable that can be measured and is related to other important variables in mathematics education.

Conceptual Framing

Bourdieu (1986) rationalized that each field has inherent capital veiled in three ways: economic, cultural, and social. He fashioned the idea that patterns are not coincidental but rather are researchable and employable trends in the social world. Previous research from Archer et al. (2015) widened the Bourdieusian lens from art-based concepts to include the theory of science capital in society. Through their work, Archer et al. developed a three-part model in order to quantify students' science capital. Our research extends this work to the field of mathematics.

Using the three-part model from Archer et al. (2015), we introduce the concept of mathematics capital. We build on the following three categories for our variables, analogous to Archer et al. (2015), that we posit are related to mathematics capital. The three variables, which we define below, are mathematics teaching and learning experiences, mathematics self-efficacy, and cultural capital. Additionally, we include a fourth variable: mathematics mindset. We include mathematics mindset to incorporate research from Boaler (2015) and Dweck (2006) that highlights the importance of how students' view themselves as mathematical learners and developing a growth mindset towards learning mathematics.

Mathematics teaching and learning experiences are defined as exposures to mathematics as process of doing (Cuoco et al., 1996), such that a student develops flexibility and fluidity within multiple mathematical representations (NCTM, 2014), and a student's responses to these experiences. Mathematics self-efficacy is rooted by Bandura's (1997) seminal work on self-efficacy where self-efficacy is defined as a person's perception of their ability to complete tasks. Specific to mathematics self-efficacy, developing agency and identity in mathematics is integral to the development of mathematics self-efficacy and outcomes in mathematics (Boaler & Greeno, 2000). Finally, cultural capital described by Bourdieu refers to a person's family background, educational attainment, and perhaps social refinement (1977, 1986). Although this is not inclusive, these short phrases can act as descriptors to understand Bourdieu's vision of cultural capital.

Mathematics capital is theorized using Bordieu's (1986) definition of social capital where social capital is a not only a person's social relations, but also the power that grows as a result of those social relationships. In other words, social capital is not just having relationships with people, but having these relationships and using them to harness forward movement in a social network. A nonexample of social capital is having positive social relationships, but not accessing these relationships for any form of person gain. In this study, we used structural equation modeling to investigate the relationship between mathematics capital, cultural capital, and the collection of one's mathematics mindset, self-efficacy and mathematics teaching and learning experiences. Most importantly, our model exhibits the independent variable that we are interested in characterizing: mathematics capital, where mathematics capital is defined as mathematical assets that actively influence a student's capability or power in the field of mathematics education and how the student utilizes these assets to exert power in the field of mathematics education. Identification of mathematics capital, and how it has historically been acquired and used by successful mathematics students, extends current mathematics education research as it relates to equity. Defining mathematics capital will help researchers and educators better understand how to take proactive steps to support students in more equitable ways. Thus, the purpose of our study is to characterize mathematics capital and understand its relationship with other variables related to student success in mathematics. To this end, we investigate the following research questions:

1. *Is mathematics capital measurable?*
2. *What relationships exist among mathematics capital, cultural capital, mathematics mindset, mathematics self-efficacy, and mathematics teaching and learning experiences?*
3. *Is mathematics capital distinctly different than the other variables in the study?*

Methods

Participants

The participants in this study were students enrolled in undergraduate mathematics courses at an urban midwestern university. Survey data were collected through Qualtrics from 219 participants where 104 participants were enrolled in foundational mathematics courses (e.g., Pre-calculus, Trigonometry, Quantitative Literacy), 79 participants were enrolled in either Calculus I, II or III, 8 participants were enrolled in Mathematics for Elementary Teachers I or II, and 28 participants did not respond to this question. No other descriptive data were collected in this study including gender, socio-economic status, age, or ethnicity. Participants were recruited through their mathematics instructor's request and were not offered any compensation. Participation was voluntary and collected anonymously.

Variables

Data were collected for the following variables: mathematics mindset (MIND), mathematics self-efficacy (SE), mathematics teaching and learning experiences (TEAC), cultural capital (CC), and mathematics capital (MC). Participants were asked to respond to survey questions using a 5-point Likert scale where options varied from strongly agree to strongly disagree. We used existing scales for both MIND and SE (Cribbs et al., 2021) where reliability had already been established. Our Chronbach's alpha values for the existing scales used for MIND and SE were .914 and .960, respectively. Thus, we proceeded with the validated scales as anticipated. For both TEAC and CC we used existing scales as templates to build items that mirrored those that had been previously validated but better fit our research. The TEAC items were based on research from Ottmar et al. (2014), while the items for CC were developed from existing research from Dumais and Ward (2010), Noble and Davies (2009), and Zabihi and Prodel (2011). Finally, to address our research question, we developed 15 new survey items that were used for MC (see Table 1).

Table 1. Survey items for TEAC, CC, and MC.

TEAC

While growing up:

1. I discussed solutions to math problems with my peers.
2. I worked and discussed math problems that reflect real life situations.
3. I solved math problems in small groups or with a partner.
4. I wrote a few sentences about how to solve a math problem.
5. I used visual representations (e.g. diagrams, tables, models).
6. I learned how to communicate ideas in mathematics effectively.
7. I worked with manipulatives (e.g. geometric shapes).

CC

While growing up:

1. I read.
2. I was read to.
3. I attended museums.
4. I was taught that education was important.
5. I was encouraged to go to college.

6. I participated in my education.
7. I participated in extracurricular activities.
8. I was taught about historic events.
9. I felt supported.
10. I did not have a lot of screen time.
11. I read music.

MC

While growing up:

1) I earned good grades in math.
2) Math was generally easy for me.
3) I used my math skills outside school.
4) I noticed patterns in math.
5) Experiences in my life helped me make sense of math problems.
6) I advanced in STEM because of my math skills.
7) I received opportunities, awards, or recognition because I was good at math.
8) Math made sense to me.
9) I worked through math problems even if they took longer than expected.
10) I worked through math tests quickly.
11) I usually knew when I had solved a math problem correctly.
12) I knew where math formulas came from.
13) I knew how to use math formulas.
14) I knew why math formulas worked.
15) I enjoyed math.

To attempt to measure TEAC and CC, we utilized 7 and 11 items, respectively, derived from existing scales. Our initial recorded alpha values were .805 for TEAC and 0.738 for CC. Through the process of investigation, we attempted to eliminate items but could not come up with a higher Chronbach's alpha value for either variable. Thus, we retained all 7 items for TEAC and all 11 items for CC as they developed the most parsimonious scale with acceptable reliability.

To attempt to measure MC, we utilized the 15 items listed above in Table 1. Our initial recorded alpha value was 0.943. Then, through the process of investigation, we reduced down to 6 items (retaining items (1), (2), (6), (7), (8), and (15) from above) which developed the most parsimonious scale with acceptable reliability. The recorded alpha value for this variable reduced to 6 items was 0.946.

Analysis

Data were collected using Qualtrics then exported from Qualtrics to SPSS (Ver. 29). In SPSS, initial analyses were performed to preliminarily assess the reliability of the survey measures. To assess the shared associations between the variables of interest, we used path analyses in structural equation modeling (M-Plus Ver. 7, Muthén & Muthén, 2012). The scale means for each construct were modeled simultaneously and the interrelations were explored.

Results

The constructed scores of mathematics capital spanned the entire range of the scale. However, most of the responses (57.20%) were between "Somewhat agree" and "Somewhat disagree" with the remaining 14.4% below towards "Strongly disagree" and 28.40% above towards "Strongly agree". Overall, the distribution of scores was less normal and more platykurtic than ideal. However, there was nothing else to suggest that the measure was not a valid reflection of the overall distribution of mathematics capital.

Path analyses in structural equation modeling (M-Plus Ver. 7, Muthén & Muthén, 2012) were used to examine the associations in the variables of interest. Figure 1 shows the correlations between each pair of variables that were interrelated. Self-efficacy was positively related to mathematics mindset ($r = .36$, $p < .05$) and teaching and learning experiences ($r = .27$, $p < .05$). Higher reported self-efficacy was linked with a more positive mathematics mindset in addition to more positive teaching and learning experiences, with 12.96% and 7.29% shared variance, respectively.

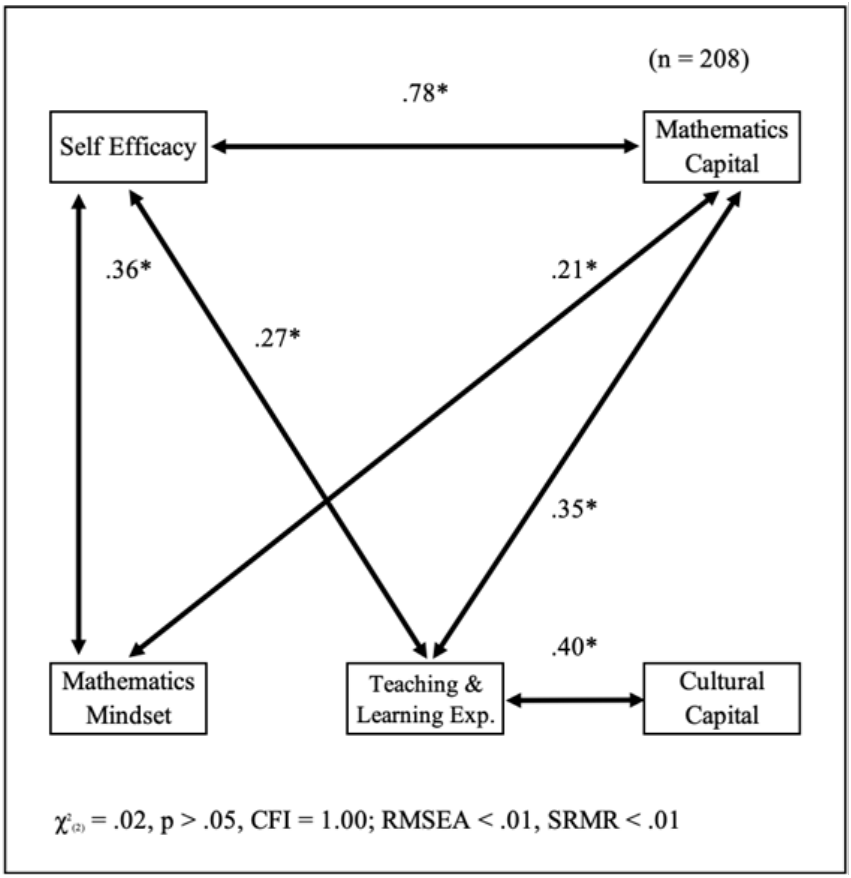

Figure 1. The shared association between the variables of interest.

Meanwhile, cultural capital was only significantly related to teaching and learning experiences ($r = .40$, $p < .05$). In other words, more cultural capital was related to more positive teaching and learning experiences with 16.00% of the variance shared between the two. Notably, however, cultural capital was not meaningfully associated with any of the other variables.

Finally, the measure of mathematics capital was positively related not only to mathematics mindset ($r = .21$, $p < .05$) but also to teaching and learning experiences as well ($r = .35$, $p < .05$). Here too, higher reported mathematics capital was linked with a more positive mathematics mindset in addition to more positive teaching and learning experiences, with 4.41% and 12.25%

shared variance, respectively. Perhaps most interestingly, mathematics capital's strongest association was with self-efficacy ($r = .78$, $p < .05$). In other words, the participants who reported more mathematics capital were also more likely to have high self-efficacy with 60.84% shared overlap in the variance. Lastly, mathematics capital and cultural capital were not related to each other in the current study. The resulting model ($n = 208$) was a good fit to the data ($\chi^2_{(2)} = .02$, $p > .05$, CFI = 1.00; RMSEA < .01, SRMR < .01).

Discussion

Our most important finding was that mathematics capital was directly and significantly related to almost all other variables in the model: mathematics teaching and learning experiences, mathematics self-efficacy, and mathematics mindset. This result suggests that increased mathematics capital is correlated to higher levels of mathematics self-efficacy, more positive mathematics teaching and learning experiences, and a growth-oriented mathematics mindset. This result also indicates that mathematics capital might be a candidate for exploring the potential mediating role in explaining the other associations we observed (e.g. the relationship between self-efficacy and teaching and learning experiences). While we did not measure the directionality between relationships in this analysis, work is ongoing to continue to understand and define the relationship between mathematics capital and these variables.

The strongest association in our model was between mathematics capital and self-efficacy. While there is research highlighting mathematics self-efficacy as a predictor of students' mathematics outcomes (e.g., Ayotola & Adedeji, 2009; Bonne & Johnston, 2015; Fast et al., 2010), our results contribute to the literature by highlighting the importance of the relationship between mathematics self-efficacy and mathematics capital. However, the nature of the relationship between these two variables must be further teased apart. Specifically, future research should investigate the items used to measure mathematics capital and mathematics self-efficacy to ensure that these two variables are psychometrically distinct. Given the strong correlation between mathematics capital and self-efficacy, we found that there was a 60.84% shared overlap in the variance, which was not high enough to claim these variables were the same. As variable validation continues for mathematics capital, we expect future research to help delineate between mathematics capital and mathematics self-efficacy.

We were surprised to find that our model showed no significant relationship between mathematics capital and cultural capital. As we continue to confirm the existence of mathematics capital, it is important to differentiate mathematics capital from other forms of capital. This distinction reiterates that mathematics capital is its own entity and does not overlap with cultural capital. This result also suggests that mathematics capital cannot be defined within cultural capital, and instead, requires its own definition. Said differently, mathematics capital is not the intersection of cultural capital and other mathematics variables such as self-efficacy, mathematics mindset, or mathematics teaching and learning experiences: mathematics capital is its own measurable, quantifiable variable.

While our analysis showed that mathematics capital and cultural capital are distinct forms of capital, it brings up the question of whether capital is transmutable. We expected to find some relationship between the two forms of capital due to the inherent transmutability of capital (Bourdieu, 1986). Thus, the lack of relationship between mathematics capital and cultural capital, raises the question: why is there no statistically significant direct relationship between these two forms of capital? Notably, mathematics capital and cultural capital are both statistically related to mathematics teaching and learning experiences. Therefore, any association between mathematics capital and cultural capital only exists inasmuch as they are both related to mathematics teaching and learning experiences. In addition, there may be other variables that were not included in our model, such as financial capital, that are associated to both mathematics capital and cultural capital. Thus, more research is needed to determine the relationships between these variables and different forms of capital.

Conclusion

A primary limitation of our study was the sample size. With 219 responses, we were able to investigate interrelations between the variables we were measuring. However, collecting a minimum of 300 responses would allow us to do latent variable modeling of the associations between the variables as latent constructs (i.e., to measure the error free associations between them). Data collection is currently ongoing to reach this sample size so that our analysis can be extended. We also plan to extend our analysis to investigate potential differences in student subgroups. Of particular interest are the differences between students who are enrolled in a calculus sequence course pursuing STEM majors and students who are non-STEM intending.

In addition to our quantitative analyses, our current survey allows students to respond to two free-response questions to gather qualitative data. In this portion of survey, we are looking to compare and contrast the measured variables, as quantitatively reported by the participants to the descriptions in their free-response questions. We are curious how students report their self-efficacy, mathematics teaching and learning experiences, and their mathematics mindset on a scale and how these compare to their experiential descriptions of these variables.

Our ultimate goal is to understand mathematics capital so that we can learn how it is acquired and better support students in attaining this capital. We view mathematics education as a holistic and comprehensive undertaking, as Bourdieu saw many capital-related concepts (1986). Developing students' mathematics capital requires understanding and describing how mathematics capital intersects with students' teaching and learning experiences in mathematics, their mathematics self-efficacy, their mathematics mindset, and their cultural capital. Having this holistic approach to mathematics education will help us as we continue to uncover more relationships between these variables and discover new ones.

References

Archer, L. Dawson, E., DeWitt, J. Seakins, A. & Wong, B. (2015). "Science capital": A conceptual, methodological, and empirical argument for extending Bourdieusian notions of capital beyond the arts. *Journal of Research in Science Teaching,* 52(7), 922-948.

Ayotola, A. & Adedeji, T. (2009). The relationship between mathematics self-efficacy and achievement in mathematics. *Procedia Social and Behavioral Sciences,* 1, 953-957.

Bandura, A. (1997). *Self-efficacy: The exercise of control.* W.H. Freeman and Company.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. John Wiley & Sons.

Boaler, J., & Greeno, J.G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple Perspectives on Mathematics Teaching and Learning, 1*, 171-200.

Bonne, L., & Johnston, M. (2016). Students' beliefs about themselves as mathematics learners. *Thinking Skills and Creativity*, *20*, 17-28.

Bourdieu, P. (1977). *Outline of a theory of practice.* Cambridge University Press.

Bourdieu, P. (1986). The Forms of Capital. *Handbook of Theory and Research for the Sociology of Education,* 241-258.

Cribbs, J., Huang, X., & Piatek-Jimenez, K. (2021). Relations of mathematics mindset, mathematics anxiety, mathematics identity, and mathematics self-efficacy to STEM career choice: A structural equation modeling approach. *School Science and Mathematics*, *121*(5), 275-287.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15,* 375-402.

Dumais, S. A., & Ward, A. (2010). Cultural capital and first-generation college success. *Poetics*, *38*(3), 245-265.

Dweck, C. S. (2006). *Mindset: The new psychology of success*. Random house.

Fast, L. A., Lewis, J. L., Bryant, M. J., Bocian, K. A., Cardullo, R. A., Rettig, M., & Hammond, K. A. (2010). Does math self-efficacy mediate the effect of the perceived classroom environment on standardized math test performance? *Journal of Educational Psychology*, *102*(3), 729.

Gutiérrez, R. (2009). Embracing the inherent tensions in teaching mathematics from an equity stance. *Democracy & Education*, *18*(3), 9-16.

Muthén, L.K., & Muthén, B.O. (2012). *Mplus Version 7 user's guide*. Los Angeles, CA: Muthén & Muthén, 10.

National Council of Teachers of Mathematics. (2014). *Principles to action.* National Council of Teachers of Mathematics.

Noble, J., & Davies, P. (2009). Cultural capital as an explanation of variation in participation in higher education. *British Journal of Sociology of Education*, *30*(5), 591-605.

Ottmar, E., Konold, T., Berry, R., Grissmer, D., & Cameron, C. (2014). Evaluating the structure of the Mathematics Teacher Questionnaire: A measure of exposure to mathematics instructional practices and content. *Learning Environments Research, 17,* 127–138.

Zabihi, R., & Pordel, M. (2011). Design, application, and factor structure of a cultural capital questionnaire: Predicting foreign language attributions and achievement. *English Language and Literature Studies*, *1*(1), 67.

"Acknowledging Students as Humans First and Students Second": Mathematics Graduate Teaching Assistants' Conceptions of Equity

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This paper explores the intersection of two critical areas in undergraduate mathematics education: the pursuit of diversity, equity, inclusion, and accessibility (DEIA) in mathematics classrooms and the understanding of equity by mathematics graduate teaching assistants (MGTAs). MGTAs play a pivotal role in teaching undergraduate mathematics courses, yet their grasp of equitable teaching practices remains underexplored. In this study, we investigate 21 MGTAs' conceptions of equitable teaching. The results indicate the importance of considering the critical axis of Gutiérrez's (2009) equity framework. Ultimately, the findings offer insights into MGTAs' conceptions of equitable teaching, providing a foundation for developing effective professional development programs and advancing DEIA efforts in mathematics.

Keywords: Equity, Mathematics Graduate Teaching Assistants, Student Thinking

Background

Researchers have emphasized the need for teaching practices that support Diversity, provide Equitable and Inclusive learning spaces for students, and improve Access to mathematical content, ideas, and learning resources, referred to as DEIA (AMS, 2019; MAA, 2019; Voigt et al., 2023). Mathematics classrooms that support DEIA are often those that are active, engaging learners in meaningful mathematical activity, such as inquiry-based learning (IBL) and inquiry-oriented instruction (IOI) (Hassi et al., 2011; Kuster et al., 2017; Laursen et al., 2014). However, Brown (2018) noted that not all active-learning classrooms are equitable and provided a vision for equity-oriented IBL (E-IBL). In a recent review and synthesis of the literature on these types of classroom practices, Laursen and Rasmussen (2019) posited that mathematics classrooms should incorporate the activities outlined by four pillars: "students engage deeply with coherent and meaningful tasks, students collaboratively process mathematical ideas, instructors inquire into student thinking, instructors foster equity in their design and facilitation choices" (p. 138). For our paper, we focus our attention on the last pillar - equity. More specifically, we focus on mathematics graduate teaching assistants' (MGTA) conceptions of equity.

Mathematics departments often rely heavily on MGTAs for teaching or supporting 100- and 200-level courses. MGTAs often lead multiple recitation or lab sections for large lecture courses or teach courses as the lead instructor. At some doctorate-granting institutions, MGTAs teach up to 68% of Calculus I courses (Selinski & Milbourne, 2015). This means that during their graduate programs, MGTAs likely contribute to the learning experiences of hundreds and possibly thousands of undergraduate students. Despite their contact with and impact on undergraduate students, few studies have investigated MGTAs' understanding of teaching practices (Miller et al., 2018; Speer et al., 2010) and whether/how they comprehend instructional practices that support DEIA. We found one recent study (White et al., 2023) in which mathematics faculty members were asked how they would define equitable and inclusive teaching practices after an equity-focused professional development program. Among the 13 participants, White and colleagues (2023) found three major themes: “awareness and consideration of students, student engagement, and reflective practice” (p. 451).

Due to their impact on undergraduate learners, we investigate MGTAs’ understanding of DEIA because their understanding will influence their instructional decision-making and abilities (Schoenfeld, 2011, 2014; Speer, 2008; Speer & Hald, 2008). Additionally, studies that reveal what MGTAs associate with equitable teaching can inform the field on what is needed for an effective professional development program. Thus, to add to this literature and focus on MGTAs, we aim to explore the research question: What are MGTAs’ conceptions of equitable teaching?

Theoretical Framework

Gutiérrez (2002, 2009, 2011) defines equity as “the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language” (Gutiérrez, 2002, p. 153). Within these definitions, Gutiérrez (2011) describes dominant and critical axes (Figure 1), each of which has two dimensions. Along the dominant axis are *access* (“the tangible resources students have available to them to participate in mathematics,” p. 19) and *achievement* (“measured by tangible results,” includes “participation in class as the mathematics pipelines,” p. 19). On the critical axis are *power* (“voice in the classroom, opportunity to use mathematics as a tool to critique society, rethinking mathematics as a humanistic enterprise,” p. 20) and *identity* (“a balance between self and others”; “opportunities to see themselves in the curriculum as well as have a view on the broader world,” p. 19-20).

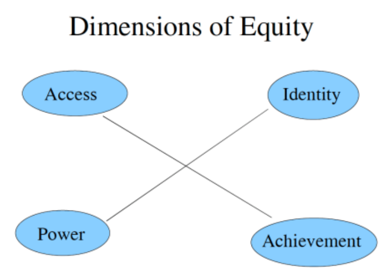

Figure 1. Gutierréz’s (2009) dimensions of equity.

While Gutiérrez (2009) noted that all dimensions are necessary for true equity, she problematizes the dominant axis through the lens of “playing the game” (p. 5) as participating in the status quo. In particular, she notes that access is “necessary but not sufficient” (Gutiérrez, 2009, p. 5) and that “equal access assumes sameness” (Gutiérrez, 2011, p. 19). In addition, she argues that focusing on providing all students with access or closing achievement gaps does not rectify past injustices that students may have experienced. In contrast, the dimensions of the critical axis help teachers and students to “change the game” (Gutiérrez, 2009, p. 5) by leveraging students’ cultural and linguistic backgrounds. Moreover, the power dimension provides students with opportunities to “rethink the field of mathematics” and to consider mathematics as a “more humanistic enterprise” (Gutiérrez, 2009, p. 6).

Researchers have used Gutiérrez’s (2002, 2009, 2011) framing of equity to redefine the actions of teachers and students in classrooms as well as describe people’s conceptions of equity. For example, Tang and colleagues (2017) used Gutiérrez’s framing to describe how inquiry can support equity. Similarly, by studying mathematics classrooms through the lens of Gutiérrez’s work, Brown (2018) revised the notion of Inquiry-Based Learning to include an equity-oriented (E-IBL) component. White et al. (2023) applied Gutiérrez’s framing to participants’ answers to interview questions to understand how people talk about equity more deeply. We continue this trend of leveraging Gutiérrez's framework to forward mathematics education.

Methodology

For this study, we conducted clinical interviews (Clement, 2000) with 21 MGTAs across three universities in the United States. MGTAs were recruited at the beginning of their graduate program (either Masters or PhD). These interviews were semi-structured (Zazkis & Hazzan, 1998) in that they were planned in advance, but the interviewer asked different follow-up questions based on the interviewee's responses. The interview took place at the start of the school year and consisted of 16 questions focused on investigating the MGTAs' conceptions of teaching. A substantial portion of these questions involved ideas of equitable teaching, such as; "How would you define equity and inclusivity?", "How would you describe equitable teaching?", "How would you describe a classroom that is not equitable?", and "Do you foresee equity and/or inclusivity impacting your plans for group or student interactions?"

Data Analysis

The analysis involved Open and Axial Coding (Strauss & Corbin, 1990) for moment-by-moment coding of students' responses and interpretations. This analysis was inductive in that the analysis was driven by what was in the data and that no pre-existing theoretical frameworks were used when creating the codes. However, as researchers aware of the literature, our knowledge of existing frameworks likely influenced what we noticed and how we decided on codes. Each interview was individually coded by at least 2 of the authors and then subsequently reconciled. During the reconciliation of codes, the authors created and refined the codes to capture patterns across students' responses. The authors repeated this process until no new codes were created, and there was agreement on using the codes in each interview transcript. This process was done to improve intercoder reliability to ensure that each code was used in the same way. We present a portion of our findings highlighting the major themes of these 21 MGTAs associated with equity.

Table 1. Summary of Code Counts Organized by Level of Association.

Code	Major	Minor	Other	Total
Access	2	3	8	13
Active Learning		3	12	15
Addressing Biases	1	1	5	7
Awareness of Diversity	3	6	9	18
Barriers		5	9	14
Content is Relevant		1	2	3
Equity and Inclusivity are Similar		1	8	9
Equity as Equality	1	1	6	8
Everyone Participating	1	3	9	13
Gauging Students		5	8	13
Individualized Approach	4	6	9	19
Lecturing is not Equitable		1	3	4
Making Things Equal	4	7	7	18
Multiple Teaching Methods		1	7	8
Not Leaving Students Behind	6	3	4	13
Students are Comfortable		2	11	13

Students are Heard		3	8	11
Valuing Different Types of Thinking	1		4	5
Valuing Students			5	5

Table 1 summarizes the most prominent themes among the graduate students, where each individual's codes were organized into *Major*, *Minor*, and *Other Association* levels of equity. The counts in each column indicate how many times a particular code was classified in each respective association level. *Major Associations* were determined by the code that emerged most frequently for each graduate student. *Minor Association*s were determined by the codes that came up less frequently than those counted in the *Major Association* category but still more than once or twice for the graduate student. All other codes were determined as *Other Association*, representing themes that did come up for the graduate students but not as much as the major or minor ones. Two graduate students had two codes come up significantly more frequently than the rest of the codes and, therefore, had two codes counted in the *Major Association* category, while the rest had only one. Most MGTAs had several minor codes, and several were counted in the *Other* association. Table 1 highlights each column's most commonly referred to codes.

Results

In this section, we highlight the three most prominent codes amongst the 21 graduate students. To do so, we explicate the meaning of these codes with excerpts that align with that code. It should be noted that each excerpt can be composed of multiple codes and does not necessarily fall under only that one code.

Individualized Approach

Excerpt 1

"I would try and focus on making what I'm teaching ***specific to each student****. I know it is hard to do that, but getting almost like* ***a personal lesson plan****, or like* ***a personal path that each student can take****."*

Excerpt 2

"So, in equity I assert that ***some people need more than others****. For example, if I have office hours, everybody is welcome to. But* ***if one student has more gaps or problems understanding something they might need more time with me.****"*

Excerpt 3

"There might be small things like, ***'Hey this student is sick, can they do makeup?'*** *or like "Can we change this policy* ***so if they miss an exam they can have their final exam score replace it?****' like little things like that"*

Figure 2: Sample Excerpts Coded with "Individualized Approach"

The most prominent theme in 19 of the 21 MGTAs was their idea that equitable teaching entailed an *Individualized Approach*. We used this code whenever an MGTA mentioned wanting to provide differentiated ways of instruction for their students, which was typically grounded in their desire to support students based on their individual needs [Excerpts 1 and 2]. Additionally, we used this code when they verbalized their desire to make accommodations for students depending on their respective situations [Excerpt 3]. In terms of individualizing education, the majority of these MGTAs discussed this in terms of a student's content understanding. In particular with Excerpt 2, this MGTA wanted to differentiate their time with students based on whether they had "more gaps or problems understanding" the material. This was a common sentiment among the 19 students where they wanted to help those who were behind by spending more time with them or by providing alternative explanations [Excerpt 1]. Overall, what is mutual amongst all the excerpts coded with *Individualized Approach* was that the MGTA expressed a desire not to treat everyone with the same standard. Instead, each MGTA explicated that depending on the individual's way of learning [Excerpt 1], understanding of mathematical content [Excerpt 2], personal situation [Excerpt 3], or other aspects, the MGTA wanted to address these and consider what would be best for each individual.

As a significant observation, only a few passages that we coded with *Individualized Approach* included a discussion on students' identities (e.g., race, gender, sexuality, ability/disability, etc.). The only exception was the few passages that considered a student's social class and the MGTA providing accommodations for them (such as flexible deadlines for students with jobs or materials for those without the necessary income). We note this because the literature clearly shows that people do not experience mathematics equally, and negative experiences are disproportionately felt by marginalized populations (Tatum, 1992; Yosso et al., 2009). For example, many women still face sexist narratives from their male peers (Yang & Carroll, 2018; Ernest et al., 2019), and therefore, one way to combat this is to take deliberate steps to ensure women have safe spaces in mathematics. In this case, educators can take an individualized approach in recognizing how to support marginalized populations, yet none of the MGTAs explicitly mentioned such actions. To be clear, we do not claim that the MGTAs are unaware of racism, sexism, etc. Instead, we assert that there is no direct evidence that they think of them as being concerned with "equity" or fully realize how a student's identity significantly impacts their experiences in the mathematics classroom.

Making Things Equal

Excerpt 4

"Like some students come without a pencil. Some students just had problems at home and that made me realize that sometimes students deserve extra attention ***in order to put them on a fair footing****."*

Excerpt 5

"I would say equitable teaching would be, not necessarily that every student gets the same. I would say it's more about ***making sure that everyone is at the same level of help or has the same opportunities as everybody else****."*

Excerpt 6

"I want to say like teaching so that it's fair. Like ***making sure that you're taking steps*** *to be fair and not giving preferential treatment.* ***Making sure that anybody should have the same ability to understand…****like everyone should be able to continue."*

Figure 3: Sample Excerpts Coded with "Making Things Equal"

Another popular theme was that equitable teaching involved *Making Things Equal* for all students. This code was used whenever an MGTA indicated the intent to ensure that all students had the same opportunities or ability to succeed. It is important to note that this code does not capture the idea of treating all students the same way. Instead, excerpts tagged with this code involved descriptions of something the teacher or university needed to do to provide equal opportunities for students. For example, in Excerpt 4, the MGTA described their awareness of students' access to resources and their background as impacting their ability to succeed. Thus, in order to provide an equal learning opportunity, a teacher should provide "extra attention in order to put them on a fair footing." Excerpt 5 also hints at this diversity of backgrounds by mentioning that not every student needs to get the same and that equitable teaching involves "making sure" everyone has the same opportunities. While Excerpt 6 may initially read as "treating everyone the same," the MGTA's emphasis on "making sure" of fairness indicates their assumption that, by default, things are likely not fair in the classroom and that the teacher needs to "make sure that anybody should have the same ability to understand."

18 of the 21 interviewed MGTAs shared this sentiment that equitable teaching involved deliberate actions from the teacher or university to ensure that students have an equal opportunity in the classroom. Underlying this idea was the awareness that students may come into the classroom with different backgrounds, associations with mathematics, and the possibility of being treated differently than other students. This code was often coded alongside "Awareness of Diversity," which makes sense since if an MGTA assumes everyone is equal, they would not articulate the need to ensure that things are equal. Instead, these MGTAs noticed the diverse populations in their classroom and the impact of those differences in mathematics education.

Awareness of Diversity

Excerpt 7

"A lesson that is not equitable? Maybe a lesson in which the assumption is that everybody knows all of the terms, or ***assuming that students have the same background, that there is no non-traditional students or foreign students.****"*

Excerpt 8

"I think that ***people come from different backgrounds have different ways they communicate and that could be a result of their environment and culture****. Those sum up who we are and how we think and to include all these different ways of thinking is to, I guess give everybody a voice that's equally relevant.*

Excerpt 9

"For equitable teaching, you get people who work like 40 hours a week and they're taking a summer class and all the material is really crammed together. So equitable teaching, I would describe it as not necessarily giving people a free pass but definitely trying to accommodate situations."

Figure 4: Sample Excerpts Coded with "Awareness of Diversity"

18 MGTAs mentioned an "Awareness of Diversity" as being associated with equitable teaching. We used this code whenever an MGTA made specific reference to people of different backgrounds when answering equity and equitable teaching questions. In all three of these excerpts, each student responded to a question by explicating an awareness of how their students come from various backgrounds. These kinds of differences that the MGTAs mentioned included the types of students [Excerpt 7], ways a student in a classroom interacts [Excerpt 8], things a student might deal with outside of the classroom [Excerpt 9], mathematical background, different learning styles, and levels of mathematical understandings. Of importance to note is that none of the students specifically mentioned that "you need to be aware of diverse backgrounds to teach equitably." Instead, we infer from their aggregate responses that their realization of students with different backgrounds was the impetus for how they envisioned teaching equitably. For example, in Excerpt 8, the MGTA described that students have different ways of communicating, and because of that, equitable teaching entails giving "everybody a voice that's equally relevant." Similarly, in Excerpt 9, the MGTA initially addresses a student's situation as the primary reason for equitable teaching involving providing situational accommodations for students.

Discussion

Overall, between the 21 MGTAs, the common theme of equitable teaching entailed supporting student success by individualizing education, making things equal, and being aware of the diversity of student backgrounds. One hypothesis for why these three codes appeared frequently is that the most prominent image associated with equity is the baseball field analogy

(Figure 5). In this widely used metaphor, equity is enacted by distributing the boxes according to each individual's needs. It is not too far of a reach to claim that the three most salient aspects of this metaphor involve "people have different needs" (*Awareness of Diversity*), "give according to those needs" (*Individualized Approach*), and that the overarching goal is to "*make things equal.*" In fact, one MGTA directly mentioned this metaphor when explaining equity: "It's the classic photo of equity versus equality. You don't give everyone a crate, you give crates according to

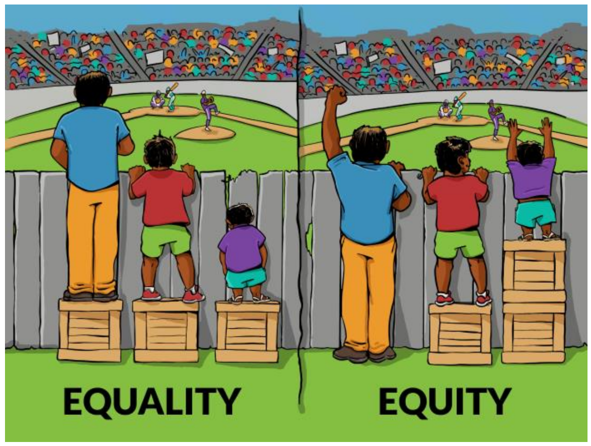

Figure 5: A Widely Used Metaphor for Equality Versus Equity

who needs to be able to see (over) the fence."

In considering Gutiérrez's framework for equity (2009), we note that the MGTAs mostly attended to the dominant axis by considering ideas of access and achievement. Most of the MGTAs focused on supporting academic success and closing achievement gaps. However, as Gutiérrez explains, attention to the critical axis is crucial for students to have true equity in the classroom. We note that we do not want to ironically gap-gaze at what the MGTAs lack. Instead, we make this comparison to highlight what professional development focused on equity for MGTAs might include. In particular, the results indicate that many MGTAs have admirable intentions and want to support students from various backgrounds. What they would likely need then, is a deeper understanding of how issues of racism, sexism, ableism, and other forms of oppression play a significant role in mathematics education. Therefore, it stands to reason that a productive professional development program for MGTAs would include aspects that address issues of inequity, specifically targeting the critical axis of Gutiérrez's framework.

We titled our paper with a quotation from one of the MGTAs who explained equitable teaching as "acknowledging my students as humans first and students second" to highlight that the MGTAs mostly have the intent of treating mathematics education as what Gutiérrez calls a "more humanistic enterprise." We believe the findings of this study indicate a better future for mathematics education and that providing training to graduate students on teaching and equity can be a worthwhile endeavor toward improving student experiences in mathematics classrooms.

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References

American Mathematical Society (AMS). (2019, April). *Statement on equity, diversity, and inclusion*. Retrieved from http://www.ams.org.

Brown, S. (2018). E-IBL: An Exploration of Theoretical Relationships Between Equity-Oriented Instruction and Inquiry-based Learning. In (Eds.), A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1-15), San Diego, California.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, *547*, 341-385.

Ernest, J. B., Reinholz, D. L., & Shah, N. (2019). Hidden competence: Women's mathematical participation in public and private classroom spaces. *Educational Studies in Mathematics*, *102*, 153-172.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, *4*(2-3), 145-187.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." *Teaching for Excellence and Equity in Mathematics*, *1*(1), 4-8.

Gutiérrez, R. (2011). Context matters: How should we conceptualize equity in mathematics education?. In *Equity in discourse for mathematics education: Theories, practices, and policies* (pp. 17-33). Dordrecht: Springer Netherlands.

Hassi, M.-L., Kogan, M., & Laursen, S. L. (2011). Student outcomes from inquiry-based college mathematics courses: Benefits of IBL for students from under-served groups. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 73-77). Portland, Oregon.

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2017). Inquiry-oriented instruction: A conceptualization of instructional principles. *PRIMUS*, http://dx.doi.org/10.1080/10511970.2017.1338807

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406-418.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 129-146.

Mathematical Association of America (MAA). (2018). *Instructional Practices Guide*. Retrieved from https://www.maa.org/sites/default/files/InstructPracGuide_web.pdf

Miller, E., Uhing, K., Hass, M., Zigterman, R., Quigley, K., Wakefield, N., & Lai, Y. (2018). Graduate student instructors' growth as teachers: A review of the literature. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 185-197), San Diego, California.

Schoenfeld, A. H. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, *43*(8), 404-412.

Selinski, N. E., & Milbourne, H. (2015). The institutional context. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of College Calculus* (pp. 31-44). Washington, DC: MAA Press.

Speer, N. M. (2008). Connecting beliefs and practices: A fine-grained analysis of a college mathematics teacher's collections of beliefs and their relationship to his instructional practices. *Cognition and Instruction*, *26*(2), 218-267.

Speer, N., & Hald, O. (2008). How do mathematicians learn to teach? Implications from research on teachers and teaching for graduate student professional development. *Making the connection: Research and practice in undergraduate mathematics education*, 305-318.

Speer, N. M., Smith, J. P., & Horvath, A. (2010). College mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*, 99-114.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Sage publications.

Tang, G., El Turkey, H., Cilli-Turner, E., Savic, M., Karakok, G., & Plaxco, D. (2017). Inquiry as an entry point to equity in the classroom. *International Journal of Mathematical Education in Science and Technology*, 48(sup1), S4-S15.

Tatum, B. (1992). Talking about race, learning about racism: The application of racial identity development theory in the classroom. *Harvard educational review*, *62*(1), 1-25.

Voigt, M., Hagman, J. E., Gehrtz, J., Ratliff, B., Alexander, N., & Levy, R. (2023). *Justice through the lens of calculus: Framing new possibilities for diversity, equity, and inclusion*. Mathematics Association of America.

White, D., Williams, M., Triplitt, C., Funk, R., Uhing, K., & Machen, R. (2023). Mathematicians characterizations of equitable and inclusive teaching. In S. Cook., B. Katz, & D. Moore-Russo. (Eds.) *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 449-458). Omaha, NE.

Yang, Y., & Carroll, D. W. (2018). Gendered Microaggressions in Science, Technology, Engineering, and Mathematics. *Leadership and research in Education*, *4*, 28-45.

Yosso, T., Smith, W., Ceja, M., & Solórzano, D. (2009). Critical race theory, racial microaggressions, and campus racial climate for Latina/o undergraduates. *Harvard Educational Review*, *79*(4), 659-691.

Zazkis, R., & Hazzan, O. (1998). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, *17*(4), 429-439.

The Cognitive Obstacles Associated with Structuring for Mathematization Undergraduates Encountered During Dynamic Modeling Tasks

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Studies have described a number of blockages to mathematizing, the process of transforming a real-world situation into a mathematical model. Recently, researchers have documented four cognitive obstacles associated with those blockages in physics-based modeling tasks. We attempted to extend these findings to biological-based modeling tasks. However, we found it difficult to use the theoretical lens as described in the previous literature. This difficulty arose because the previous theoretical lens required the delineation of real-world objects and mathematical objects, a pursuit recently shown to be unattainable in some contexts. By examining students mathematization under a quantitative reasoning and symbolic forms lens, we were able to find two cognitive obstacles analogous to the ones found in previous research, confirming their existence with a different demographic of students and different task scenarios.

Keywords: modeling, quantitative reasoning, differential equations

The idea of using mathematical modeling to enrich STEM education is well established. It is an idea upheld by the Common Core State Standards (CCSSI, 2010), and researchers alike (Blum & Niss, 1991). One line of inquiry into how students learn to model focuses on identifying and improving the competencies students need to make progress in a modeling problem (Schukajlow et al., 2018). The competency which has received most attention is *mathematizing* – constructing a mathematical representation of the mental model of the scenario. It is particularly difficult for students (Brahmia, 2014; Galbraith & Stillman, 2006; Stillman & Brown, 2014). Because each phase of modeling builds on decisions the modeler made earlier, some of the difficulties students encounter carrying out mathematizing come from how they structured the real-world scenario. In order to understand why some students experience blockages when they work on modeling problems, researchers have begun to catalog the cognitive obstacles associated with structuring for mathematizing that arise when students work on modeling problems.

Literature Review

Mathematical models are mathematical systems that represent real-world systems. They are conceptual systems "consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)…[A model] focuses on structural characteristics of the relevant system" (Lesh & Doer 2003 p. 10). Historically, scholars have conceptualized the process of generating the representational system as an idiosyncratic cycle that can be disrupted by inadequately performing key activities to transition from one competency to another. Studies have described a number of blockages to mathematizing such as representing elements mathematically so known formulae can be applied (Galbraith & Stillman, 2006) and failing to define variables or failing to realize interdependences among variables (Klock & Siller, 2020). These blockages should be understood as arising from structural choices about which elements, relations, and operations are relevant. Recent studies affirm that lack of content knowledge was not the root of students' difficulties during modeling, but rather a two-part set of skills: structuring the scenario and then translating the structure into a mathematical representation (Jankvist & Niss, 2020; Niss, 2017). Niss (2017) observed four primary obstacles to mathematizing that could be attributed to cognitive processes. He framed mathematizing as making simplifying assumptions about objects in the real world that are specified by the modeler and generating relationships between those objects that are relevant to solving the real-world problem. From this perspective, student blockages could be categorized as (a) not identifying a mathematical object to impose on the scenario (b) not identifying a key real-world object to impose onto the scenario (c) imposing unhelpful mathematical objects, and (d) making structural choices that led to a mathematical representation that students were unable to work on (e.g., choices that would require too-advanced mathematics). In concluding remarks, Niss called for other researchers to corroborate and expand upon this list of cognitive obstacles in other content areas. A call we set out to respond by extending Niss's framework to undergraduate STEM majors' work on canonical modeling problems from biology, including disease transmission and predator-prey dynamics. We found that Niss's framework depends on being able to reliably identify and distinguish between a mathematical object and a real-world object, which was challenging to do empirically in the biological contexts since it was not obvious what the relevant mathematical or real-world objects would be. We were unable to understand what the cognitive obstacles associated with structuring for mathematizing from this perspective. Recently, Zimmerman et al., (2023) argued that there is no clear distinction between mathematical objects and real-world objects, shedding light onto limitations of Niss's framework for settings outside of physics problems. Thus, a promising approach to understanding the difficulties students face while mathematizing would need to absolve the researcher from assuming a distinction between real-world object and mathematical object.

To address this theoretical and methodological need, we adopt constructs from quantitative reasoning (QR) and symbolic forms. Quantitative reasoning is the conceptualization of a situation into a network of quantities and quantitative relationships (Thompson, 1993). A quantity is a triple of an object, attribute, and quantification (Thompson, 2011). Quantification means to conceptualize an object with a measurable attribute so that the measure is proportional to its unit (Thompson, 2011). A quantity is made from an individual's conceptions of objects meaning a quantity is idiosyncratic to the individual (Ellis, 2007). An example of a quantity in a disease transmission context could be the number of sick people at time t. The object is the sick population, the attribute is amount, and evidence of quantification could be a student explaining they could feasibly measure the number of people on one day by counting them. A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to already created quantities (Ellis, 2007; Thompson, 2011). Symbolic forms are a type of cognitive resource comprising of two components: a symbol template and a conceptual schema (Sherin, 2001). For example, a common conceptual schema for the symbol template $[\,] \times [\,]$ is iterating one quantity by the magnitude of another (i.e., repeated addition). In this way, a mathematical model is a conceptual system encompassing a modeler's ideas they find relevant about a real-world system (e.g., a fish tank, or an island habitat). The elements of the model are quantities the modeler imposes onto the scenario. The relationships between elements, operations, and rules governing interactions are determined by the modeler's QR. A modeler externalizes quantitative operations by selecting the symbol template whose conceptual schema comports with the quantitative operation. For example, a modeler could multiplicatively combine number of sick people at time t and number of healthy people at time t to create the quantity number of interactions between sick and healthy people at time t. The modeler might externalize that quantitative operation via the symbol template [ ] × [ ], because repeated addition comports with how they multiplicatively combined the two quantities. In Niss' (2017) framing, cognitive obstacles arose from incongruencies in identifying real-world objects, and assigning the appropriate mathematical objects to those real-world objects. In our framing, cognitive obstacles arose from imposing quantities onto the scenario, creating new quantities by combining other quantities, and expressing that combination externally. From this perspective, we pose the question: What cognitive obstacles to mathematizing, arising from structuring, do students encounter when working on dynamics modeling problems?

Methods

This qualitative study draws data from a larger design study of facilitator scaffolding moves that foster undergraduates' modelling competencies. In this paper, we report on 12 participants' work on three canonical modelling tasks: a predator-prey dynamics scenario, a contaminated tank scenario, and a disease transmission scenario. Participants were volunteers from STEM courses who stated they had some familiarity with mathematical concepts like instantaneous rate of change and reported getting at least C's in their mathematics course work. The task design and task-based interview protocols attended to participants' QR and intentionally focused on similarities of conceptual schema across task contexts. In all three tasks, participants aimed to write a (system of) differential equation(s) modeling the dynamics of the scenario. In the predator-prey task, students were asked to create two differential equations that modeled the rate of change of cats and birds accounting for their interactions. In the contaminated tank task, a buffering agent is mixed with water to create a buffering solution. Students were asked to model how quickly buffering agent in the tank changed given the buffering solution entered the tank at 5 liters per minute at a concentration of $1 - e^{-\frac{t}{20}}$ grams per liter. In the disease transmission task, participants were asked to model how quickly a disease was spreading through a community of susceptible, infected, and removed populations.

Analysis of recordings of and written work from the interviews proceeded in four stages. First, we generated a list of quantities each participant imposed onto the task scenario by describing the object, attribute, and how the participant exhibited quantification for that attribute according to the quantification criteria developed by Czocher and Hardison (2021). Second, we created a chronological narrative documenting how participants combined quantities and externalized that combination using symbolic forms. Third, we identified instances where a participant's progress in the task halted. Some key indicators were: the participant said they were stuck, the participant wrote down two mathematical expressions for the same quantity (or quantitative relationship) and said one of the expressions should be correct, or the interviewer noticed that what the participant wrote down did not mathematically mean what the participant was explaining. In each instance, we inferred the cognitive obstacle that impeded the participant's progress based on the quantities and quantitative operations previously documented. Finally, we looked across instances for commonalities which we report as types of cognitive obstacles below.

Results

Across the three canonical modeling problems and participants, we observed many blockages to mathematization that could be attributed to cognitive processes and report on two with origins in structuring: an unhelpful quantitative relationship took precedent, and a key quantity was not salient for the participant. A third type of cognitive obstacle (a participant's specific quantification of a quantity did not permit them to sensibly combine it with other already-present quantities using well-known formulas) was previously reported in Roan and Czocher (under review). We illustrate each kind of cognitive obstacle to evidence its existence and its role in connecting structuring to mathematizing. To evidence the two types of cognitive obstacles existence, we selected instances that best exemplified a specific type of cognitive obstacle. To do that will show work from two students, Ivory (Physics) and Niali (Electrical Engineering).

An unhelpful quantitative relationship took precedent.

This cognitive obstacle occurred when a participant focused on externalizing a quantitative relationship that was not helpful (from the interviewer/researcher perspective) in creating a differential equation that modeled the scenario. To evidence this type of cognitive obstacle, we show work from Ivory (Physics) on the disease transmission task. To start, we show the quantities, quantitative relationships and the corresponding symbols and expressions she had created up to the point of interest in Table 1.

Table 1. Ivory's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Amount of susceptible people at time t	$S(t)$
Amount of infected people at time t	$I(t)$
Amount of removed people at time t	$R(t)$
Mortality rate	d
Accumulated amount of dead people at time t	$m(t)$
Amount of recovered people at time t	$r(t)$
Amount of dead people at time t and amount of recovered people are two disjoint subsets of amount of removed people at time t	$r(t) = 1 - m(t)$

Ivory was working on creating an expression for *amount of dead people at time t*. She noted some pertinent information she needed to consider, namely, that "there's a two-week delay" between symptom onset and death "…because if you haven't died within two weeks, you've recovered, is what we're assuming here." Ivory assumed that (A) the *amount of dead people at time t* depended upon *amount of infected people at time t, 14 days ago*. We identified that Ivory had stopped progressing towards creating a differential equation for the scenario because she wrote down multiple mathematical expressions for the same quantitative relationship, as if searching for the correct mathematical expression. Table 2 shows the different expressions Ivory tried and then rejected to depict quantitative relationship (A).

Table 2. Ivory's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
amount of dead people at time t depended upon amount of infected people at time t, 14 days ago.	$m(t) = I(t+14) \times d$
	$m(t) = \frac{dS}{d(t+14)}$
amount of removed people at time t depended upon amount of infected people at time t, 14 days ago.	$\frac{dR}{dt} = \frac{dI}{d(t-14)}$
	$\frac{dR}{dt} = \frac{dS}{d(t-14)}$
	$\frac{dR}{dt} = I(t-14)$

Ivory experienced a cognitive obstacle related to her attempts to depict a quantitative relationship she did not know how to represent adequately. In Ivory's work, the quantitative relationship (A) took precedence over more helpful relationship like (B) positive flux in the removed population is proportional per unit time to the current infected population. She could not adapt her models for relationship (B) because they did not account for relationship (A) in an explicit way. Neither did she think her models accounting for (A) were adequate. She reflected she was "still caught up on the 14-day lag thing." We assert that this was a cognitive obstacle for Ivory because she was not able to meet her goal (to create a system of differential equations that modeled the disease spread) until this quantitative relationship lost precedence.

A key quantity was not quantified by the modeler.

This type of cognitive obstacle occurred when a key quantity was missing from the participant's structure so other key quantities could not be constructed. Across our participants and tasks, this cognitive obstacle seemed to occur when participants thought they could not use a quantity in an expression because they thought they needed to find an equation to determine it (a way to measure its value) first. To evidence this type of cognitive obstacle, we show work from Niali (Electrical Engineer) on the contaminated tank task. To start, we show the quantities, quantitative relationships and the corresponding symbols and expressions he had created up to the point of interest in Table 3. Niali stated that he needed to find a mathematical expression for the quantity concentration of buffering exiting, as evidenced in the exchange below:

Niali: OK, we have a-- let's see. Have a fluid in the tank. We have a tank. We have a pump which adds water with concentration. The water starts full. Uh, buffer enters through here, and out comes the water. The water exits at a rate of 5 liters per minute. It enters 5 liters per minute at a rate of-- that-- oh, what is this, grams of buffer per liter. So what does it exit at?

Interviewer: And by exit at, you're talking-- you're thinking specifically about the concentration, the true concentration of the liquid exiting, right?

Niali: Yes. So how many grams per liter of the chemical is exiting?

Table 3. Nilai's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Rate of buffering solution entering in liters per minute	$water\ in = 5$
Concentration of buffering solution entering in grams per liter	$1 - e^{-\frac{t}{20}}$
Rate of buffering agent entering in grams per min	$5 \times (1 - e^{-\frac{t}{20}})$
Rate of buffering solution exiting is the same as the rate of buffering solution entering	$water\ out = 5$
Maximum volume of the tank in liters	300

We identified that Niali had stopped making progress towards creating a differential equation for the scenario because he wrote down multiple mathematical expressions for the same quantity, evidencing an ongoing search for an adequate one. In Table 4, we showcase all the different expressions Niali tried and rejected to depict the quantity *concentration of buffering exiting.*

Table 4. Nilai's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Concentration of buffering solution entering is the same as concentration of buffering solution exiting	$\%\ concentration\ in = 1 - e^{-\frac{t}{20}} = \%\ concentration\ out$
Concentration of buffering solution exiting	$\frac{5 \times (1 - e^{-\frac{t}{20}})}{300}$
	$\left(\frac{5 \times \left(1 - e^{-\frac{t_{Final}}{20}}\right)}{300} - \frac{5 \times \left(1 - e^{-\frac{t_{iniital}}{20}}\right)}{300}\right) \times \frac{1}{2}$

Niali experienced a cognitive obstacle related to his attempts to create a quantity when he did not have the quantities available to him to do so. In Nilai's work, the quantity *concentration of buffering exiting* could not be constructed until the quantity *amount of buffering agent in the tank* was quantified. We assert that this was a cognitive obstacle for Niali because he was not able to meet his goal (to create a differential equation that modeled how quickly the buffering agent in the tank was changing) until the quantity *amount of buffering agent in the tank* was quantified. Further, other students in the study who did explicitly quantify *the amount of buffering agent in the tank* did not encounter this obstacle.

Discussion

In this paper, we found two cognitive obstacles associated with structuring for mathematizing students encountered in three canonical modelling tasks: a predator-prey scenario, and a contaminated tank scenario. These were (1) an unhelpful quantitative relationship took precedent, and (2) a key quantity was not quantified by the modeler. While not all participants experienced each type of cognitive obstacle, no one obstacle was specific to one task. This implies that the cognitive obstacles we showcase here are not a consequence of the task scenario but are rooted in how students' reason quantitatively while mathematizing. We set out in this work to confirm the cognitive obstacles found by Niss (2017) existed in these different task scenarios with a different demographic of student. However, due to the nature of the tasks we selected, we found it difficult to delineate real-world objects and mathematical objects. This difficulty has been empirically shown to occur in physics as well (Zimmerman et al., 2023). To continue our investigation, we elected to investigate these cognitive obstacles through a quantitative reasoning and symbolic form lens. We were successful in finding analogous cognitive obstacles found by Niss (2017). The cognitive obstacle an unhelpful quantitative relationship took precedent is analogous to Niss (2017) cognitive obstacle "making structural choices that lead to a mathematical representation that students were unable to work on". However, here Ivory made structural choices that she could not translate into a mathematical representation. Further, the cognitive obstacle a key quantity was quantified by the modeler is analogous to Niss (2017) cognitive obstacle "not identifying a key real-world object to impose onto the scenario". Our theoretical perspective offers more nuance. It is not that Niali just "didn't think" to impose the quantify *amount of buffering agent in the tank,* onto the scenario. It is that Niali thought that he couldn't. Later in the interview, when the interviewer suggested that Niali use the symbol $B(t)$ to represent *amount of buffering agent in the tank* he said "We're going to solve for $B(t)$ or something, or is that something we can do?" We infer from this question that Niali anticipated solving for $B(t)$ and so he did not think he could use $B(t)$ in his equation. This comports with the findings that it is specifically skills for structuring and mathematizing that produce cognitive obstacles not mathematical nor real-world knowledge (Niss, 2017; Jankvist & Niss, 2020). However, we did not find cognitive obstacles analogous to the other two cognitive obstacles by Niss (i.e., not identifying a mathematical object and imposing unhelpful mathematical objects). Our framing fails to pick up on those kinds of cognitive obstacles because our framing has no analogue for mathematical object. Further research is needed to determine if this is a shortcoming in the quantitative reasoning-symbolic form framing, if this kind of cognitive obstacle can be detected when looking at cognitive obstacles associated with other parts of mathematizing, specifically when externalizing the structure of the scenario via symbolic forms, or even whether they could be replicated in this student population.

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References

Blum, W., & Niss, M. (1991). Applied Mathematics Problem Solving, Modelling, Applications, and Links to Other Subjects: State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics*, *22*(1), 37-68.

Borromeo Ferri, R. (2007). Modelling Problems from a Cognitive Perspective. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modelling* (pp. 260-270). Woodhead Publishing. https://doi.org/https://doi.org/10.1533/9780857099419.5.260

Brahmia, S. M. (2014). *Mathematization in Introductory Physics* The State University of New Jersey].

Common Core State Standards Initiative (2010), National Governors Association Center for Best Practices and Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Czocher, J., & Hardison, H. L. (2021). Attending to Quantities through the Modelling Space. In F. Leung, G. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West* (pp. 263-272). Springer.

Ellis, A. (2007). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition and Instruction*, *25*(4), 439-478.

Galbraith, P. L., & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM: Mathematics Education*, *38*(2), 143-161.

Jankvist, U. T., & Niss, M. (2020). Upper secondary school students' difficulties with mathematical modelling.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In *Compendium* (Vol. 267, pp. 267-291).

Klock, H., & Siller, H.-S. (2020). A Time-Based Measurement of the Intensity of Difficulties in the Modelling Process. In G. A. Stillman, G. Kaiser, & C. E. Lampen (Eds.), *Mathematical Modelling Education and Sense-making* (pp. 163-173). Springer International Publishing. https://doi.org/10.1007/978-3-030-37673-4_15

Lesh, R., & Doerr, H. M. (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Taylor & Francis Group.

Niss, M. (2017). Obstacles Related to Structuring for Mathematization Encountered by Students when Solving Physics Problems. *International Journal of Science and Math Education*, *15*, 1441-1462.

Roan, E. Czocher, J.A. (under review). *Reasoning students employed when mathematizing during a predator-prey modeling task*. International Conference on the Teaching of Mathematical Modeling and Applications, Wurzburg, Bavaria, Germany: 2022.

Schukajlow, S., Kaiser, G., & Stillman, G. (2018). Empirical research on teaching and learning of mathematical modelling: a survey on the current state-of-the-art. *ZDM: Mathematics Education*, *50*(5), 5-18.

Sherin, B. L. (2001). How Students Understand Physics Education. *Cognition and Instruction*, *19*(4), 479-541.

Stillman, G., & Brown, J. P. (2014). Evidence of implemented anticipation in mathematising by beginning modellers. *Mathematics Education Research Journal*, *26*, 763-789.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25*(3), 165-208.

Thompson, P. W. (2011). Quantitative Reasoning and Mathematical Modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 3357). University of Wyoming.
Zimmerman, C., Olsho, A., Loverude, M., & Brahmia, S. W. (2023). Empirical evidence of the inseparability of mathematics and physics in expert reasoning about novel graphing tasks. *ArXiv*. /abs/2308.01465

National Picture of Providers of Collegiate Professional Development for Teaching Mathematics: Formats, Topics, And Activities

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Teaching professional development (TPD) in collegiate mathematics has expanded over the last few decades. Providers of TPD, people who organize and facilitate professional learning about teaching, are at the center of this growth. Yet, little is known about who Providers are and what they do. To better understand the national landscape of Providers of TPD within university mathematics departments, this report shares data from a national survey where respondents were Providers. The focus here is on findings from survey questions asking about characteristics of Providers and the "providees" with whom they work, along with formats, topics, and activities used in TPD. Results suggest that Providers value active, learner-centered instructional methods promoted by research and policy. However, in the TPD itself, formats, topics, and activities commonly used by Providers may preach but not regularly practice activity-based methods.

Keywords: Professional Development Provider, Graduate Student Instructor, Teaching Assistant

Decades of research on undergraduate mathematics teaching, learning, and curriculum development have created an evidence-based foundation of resources for equitable teaching and effective learning. These resources include instructional materials, assessment tools, and practice guides for instructors (e.g., MAA, 2018; Carver et al., 2016; Garfunkel & Montgomery, 2019). However, getting implications from research into college mathematics classrooms remains a challenge (Archie et al., 2022; Pengelley & Sinha, 2019). Entry-level college mathematics classes are often taught by graduate students (about 35% [Blair et al., 2018, p.17]). In fact, 94% of graduate students will teach at some point. Graduate students can have a variety of instructional roles, from teaching assistant responsible for a lab or recitation session associated with a primary course where the instructor-of-record is a faculty member (LabTA) to a graduate student who is an instructor-of-record (GSI). Research indicates that college mathematics instructors (CMIs) in effective undergraduate programs, particularly novices, benefit from well-structured teaching-focused professional development (TPD) (Connolly et al., 2016; Friedlaender et al., 2014; Gehrtz et al., 2022). Those responsible for offering TPD to novice CMIs, including those who lead workshops, courses, and seminars as well as those who facilitate TPD as course coordinators, have come to be known as Providers of TPD (Braley & Bookman, 2022; Braley et al., 2023). Providers have a critical role within departments. However, little is known about who Providers are, what they provide, and how they provide it.

Context of the Study

In the U.S. there are 418 institutions granting a doctorate or master's as the highest mathematics degree and more than half have at least one Provider (some have three or more; American Mathematical Society, 2023; Braley & Bookman, 2022). This survey study was part of a larger project creating support for mathematics departments for designing teaching preparation programs for graduate students. One step in the project was gathering baseline data for understanding the current status of post-secondary mathematics TPD. Among the 200 people who responded to the survey, 95 completed all aspects of it. This report uses analysis of those 95 full responses from 56 institutions. The focus is addressing two questions:

RQ1: (Who) Who are the Providers and who does their TPD target?

RQ2: (What) What formats, topics, and activities do Providers use in their TPD?

Related Literature

Understanding what Providers provide is critical at this time in undergraduate mathematics education because the field has reached a critical mass of policy and resources for TPD. In 2018, the Mathematical Association of America (MAA) released the *Instructional Practices Guide* (IPG)*,* a report on recommended teaching practices in undergraduate mathematics. The guide summarizes the evidence-base on effective methods of instruction and promotes student-centered classroom-active methods as an expectation of and for the field. Moreover, multiple calls to include inquiry-based mathematics education (Laursen & Rasmussen, 2019) and active learning methods (Braun, et al., 2017) have been on the forefront of suggestions for TPD. Concurrently, a central resource for Providers has been developed by the College Mathematics Instructor Development Source (CoMInDS) project with hundreds of documents from and for Providers in an online repository and well-established networks of people to share resources (MAA, 2020). Also, recent projects such as *Student Engagement in Mathematics through an Institutional Network for Active Learning* (SEMINAL; Gobstein, 2016) have expanded use of collaborative learning methods through active-learning for Providers.

A study by the CoMInDS project found there is a great desire, but a struggle by department leaders to make sense of resources for their institutions (Bookman & Murphy, 2019). There is a clear need for guidelines for preparing LabTAs and GSIs. However, there is a reasonable concern: institutions vary in structure and resources in ways that are consequential for graduate student TPD. Thus, localized support–as through the work of Providers–is needed to demonstrate the impact and viability of such evidence-based research.

Prior research has identified the value of TPD for graduate students and its impact on teaching and learning, from course pass rates to level and frequency of complaints by undergraduates, to improvements in responsiveness and inclusivity (Hauk & Speer, 2023, Yee et al., 2023; Yee & Rogers 2022 and references therein). There is a gap in the research about both what and how TPD is provided (Hauk, et al., 2017). To address the gap, this survey study leveraged and revised some approaches used to examine secondary teacher preparation (Yee, Otten, & Taylor, 2018) to gather baseline data on current collegiate mathematics TPD.

Methods

Quantitative methods were used to analyze the survey data to determine who Providers are, what they provide, and how they provide it. Using multiple national listservs to contact Providers (e.g., CoMInDS, SEMINAL, RUME), a Qualtrics survey (vetted by an Advisory Board and Expert Providers) was sent out to more than 500 potential Providers across the United States. Among the 95 who responded to all the items (participants), 80% were from mathematics doctoral-granting institutions and 20% from masters-granting institutions. To answer the research questions, responses to five survey items were analyzed: (1) Who was the Provider, (2) Who was the main group the Provider worked with, (3) Which formats were used for TPD, (4) Which topics were discussed in the TPD, and (5) Which activities were used in the TPD. The

"Who" survey questions required a single selection. The "Formats," "Topics," and "Activities" questions were checkboxes, where more than one could be selected. Options offered on the survey were based on prior research on formats, topics, and activities prevalent across the United States (Bookman & Braley, 2014; Braley et al., 2023; MAA 2018, 2020) and interviews with Expert Providers conducted as part of the larger project. In what follows, descriptive statistics are used to describe the Providers and providees, while matrix-like "Upset" plots are used for combinations of checkboxes.

Findings

As indicated in Table 1 and Yee and colleagues (2023), 80% reported being either non-tenure-line Teaching Faculty (42%) or tenure-track faculty (38%). Thus, non-tenure-line teaching faculty have about as much responsibility for supporting graduate student's learning about teaching as do tenure-line faculty. The majority of Providers being teaching faculty suggests care is needed in future work examining questions around a Provider's role in the department. Also, more than 80% reported a focus on graduate students – including graduate students who are teaching assistants (42%), instructor-of-record (34%), or both (9%).

Table 1. Who: Providers and providees

The **faculty position** you hold		The **main group** served by professional development about teaching in your department	
Teaching Faculty (not tenure-line)	42%	LabTA	42%
Tenure-line Research Faculty	35%	GSI	34%
Tenure-line Teaching Faculty	3%	Novice Faculty	8%
Part-time or Adjunct Faculty	3%	Undergrad. Learning Assistant	1%
Other: chair (7%), time-limited full-time/post-doc position (10%)	17%	*Other*: both LabTA & GSI (10%), post-doc (3%), faculty (2%)	15%

Format

Providers indicated their use of seven different formats (six were described, the seventh was "other" and had room for the respondent to describe it). As indicated in Figure 1 (next page), a majority of participants selected some combination of three of the six formats:

(1) pre-semester orientation (bottom horizontal bar, 79 responses),
(2) meeting with a course coordinator (67 responses), and
(3) offering a single course about teaching (45 responses).

The right side of the "Upset diagram" in Figure 1 illustrates the connections among format co-selections. Most respondents selected two or more of the three most frequently selected formats. Among the 15 respondents choosing "other" as a format, mentoring and coaching (3) and weekly meetings (2) were the most common responses.

It is worthwhile to note that very few Providers selected only "one seminar or workshop" with most selecting other formats as well. Moreover, only one respondent had only pre-semester orientation and only one respondent had only meetings with coordinators. This echoes results from the CoMInDS surveys that the field as a whole is using multiple formats for TPD over longer periods of time with an increasing frequency of course-like structures (Bookman & Braley, 2014; Braley & Bookman, 2022; Braley et al., 2023). Indeed, any combination selected by five or more respondents included meetings with coordinators, a course, multiple seminars, or multiple courses to complement a pre-semester orientation.

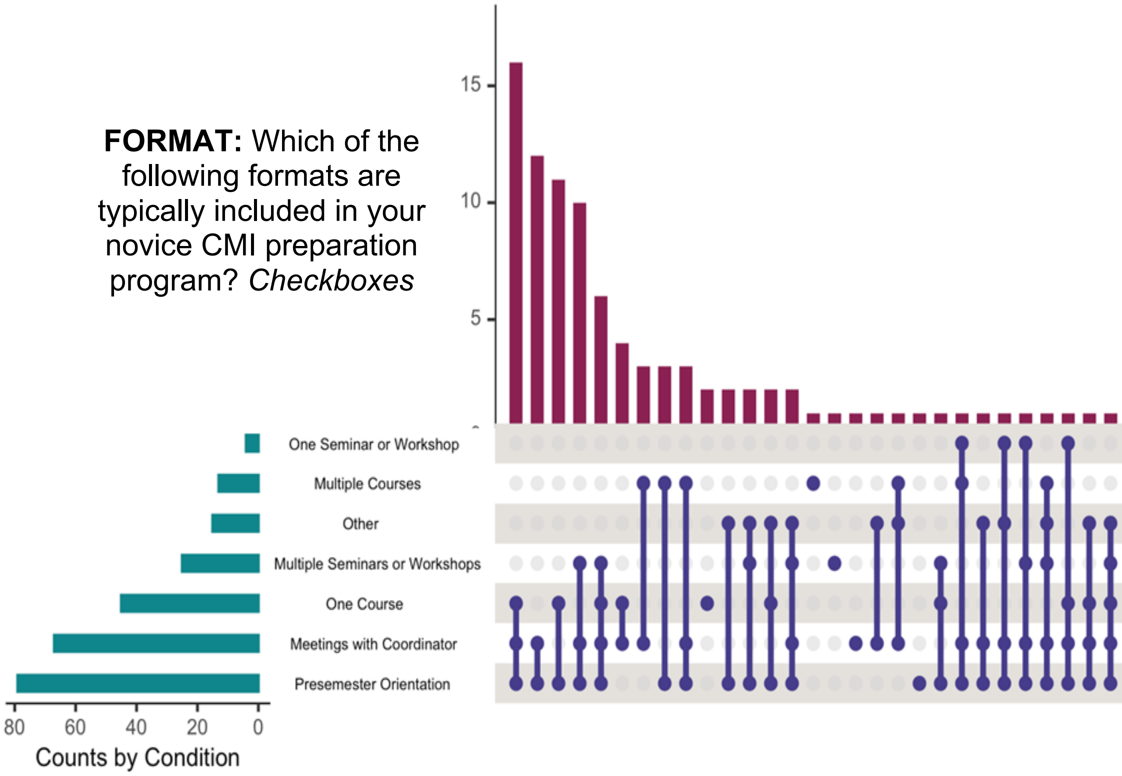

Figure 1. Formats of TPD reported by Providers.

Topics

Providers indicated their use of eight different topics (seven were described, the eighth was "other" and had room for the respondent to describe it). As illustrated in Figure 2 (next page), in order of frequency these were:

(1) university and departmental policies (90) chosen by nearly all participants,
(2) active learning (78),
(3) learning management systems such as Canvas (63),
(4) grading strategies (58),
(5) formative assessment (53).

Twenty respondents described other topics, including particular teaching techniques, (e.g., lecturing was written in by (4), student-centered teaching (3), and equitable practices (2)).

A majority selected topics aligned with the topics suggested in the IPG (MAA, 2018). Combinations of topics among more than three respondents had many of the five most popular topics (policies, active learning, learning management systems, grading strategies, and formative assessment). Notable is the pattern of exclusion for two topics: generally, those who did not select active learning also did not select formative assessment. A majority selected five of the seven possible choices, with IBL the least commonly selected.

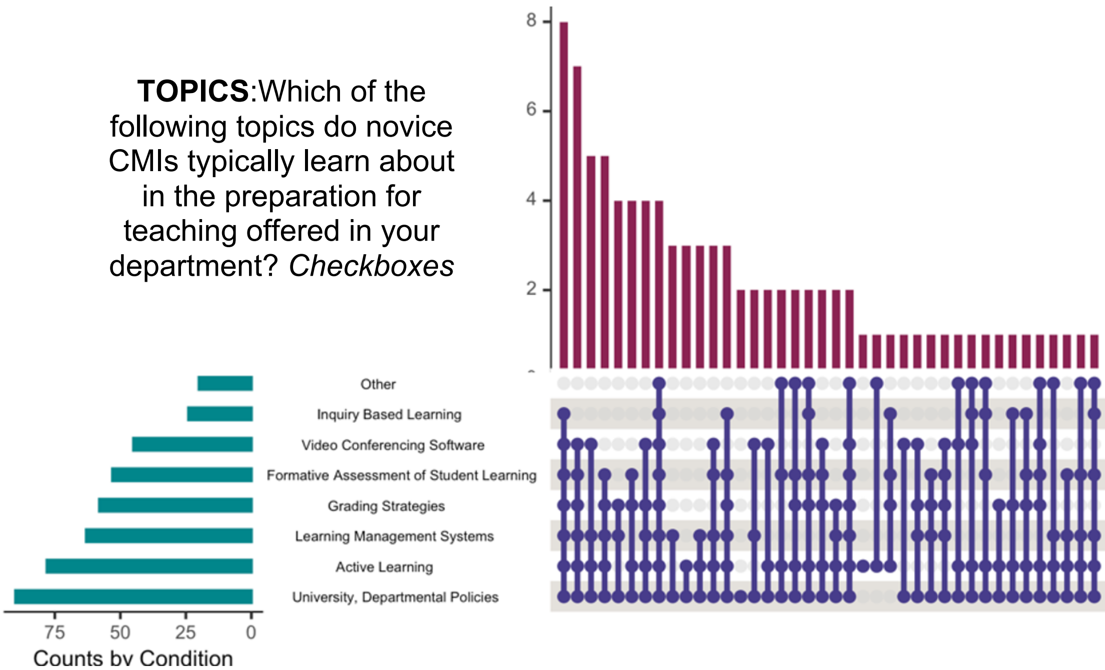

Figure 2. Topics in TPD reported by Providers.

Activities

As seen in Figure 3, (next page) the most frequently selected activities were:

(1) listen to presenters (74),
(2) discuss example cases of teaching and learning (69),
(3) read provided articles or other information (64),
(4) present a practice lecture (59), and
(5) practice assessing a student assignment (52).

Despite the knowledge in the field of the value of student-centered instruction, "listen to presenter" was chosen alongside "discussion of example cases" by more than half of respondents (52). The 30 responses about "other activities" included peer and instructor observations (6), collaborative learning (e.g., about facilitating discussions and writing lesson plans; 6), practice with teaching (e.g. "mini-lecture," "co-teaching"; 5). Six respondents only included "other" activities and did not select any of the seven given types of activity suggested as effective by the literature. Finally, it is worth noting that the most common combinations *excluded* developing a course website or delivering a non-lecture-based practice lesson.

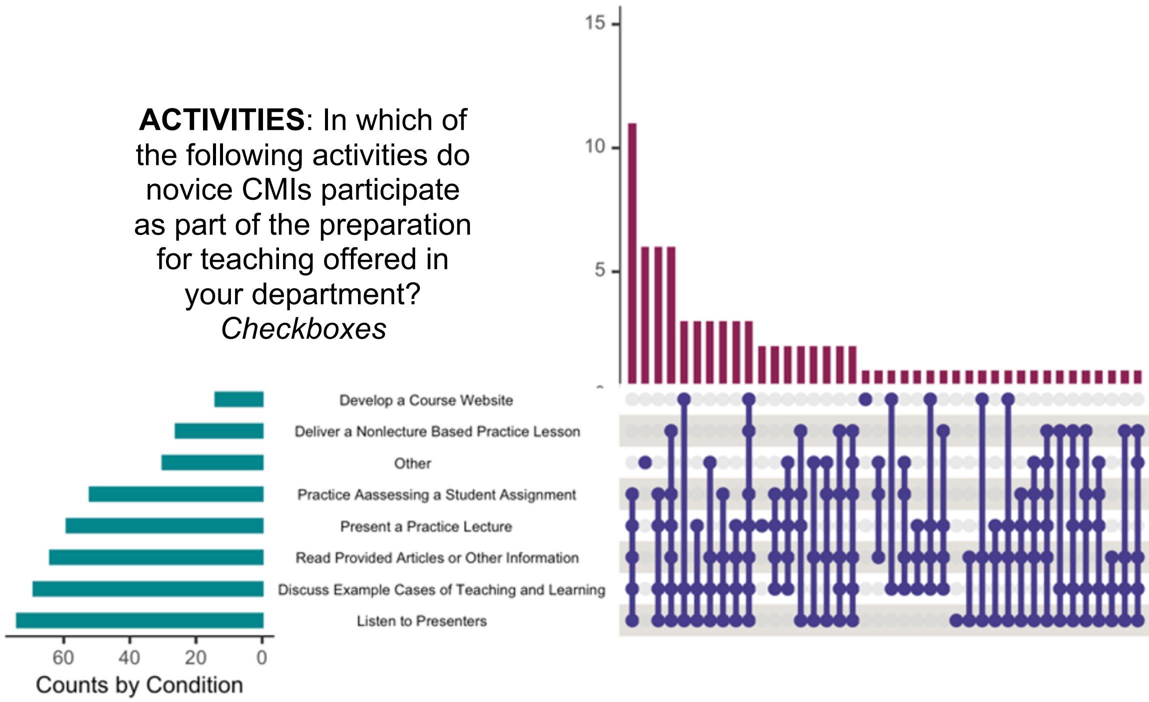

Figure 3. Activities in TPD reported by Providers

Comparative Analysis with Providers and Providees

Given that Providers were teaching faculty (42%) or research faculty (35%) and providees were split between GSIs and LabTAs, analysis included examination of variation in Formats, Topics, and Activities depending on faculty role and whether providees were GSIs or LabTAs. Analysis of Variance indicated there were no significant or practical differences in formats among Provider type (teaching versus research faculty). Additionally, there were no differences larger than 5% among activities, and only one difference among topics around video conferencing software. The existing slight differences follow from preferences of meeting the needs of particular providees. For example, pre-semester orientation was 10 percentage points higher as a format for Providers focusing on LabTAs instead of GSIs (37% vs. 27%). This coincided with one course being more commonly used when providees were GSIs (23% versus 15%). GSI-focused Providers chose coordinator meetings more than LabTA-focused (29% vs. 22%).

Conclusion and Discussion

The purpose of this study was to learn more about (1) Who Providers and their target TPD audiences are (2) What formats, topics, and activities Providers use in their TPD. Information on the first question suggests most Providers were as likely to self-identify as teaching faculty as they are as research faculty (~45% and 35% respectively) with the balance of Providers reporting roles as adjunct or time-limited faculty (e.g., postdoc). The target audiences (the "providees") for most respondents (~86%) were LabTAs and/or GSIs, with novice faculty indicated by fewer than 10% of respondents and undergraduate learning assistants by only 1%. The results from this study suggest that graduate students are the primary focus for most Providers of TPD in doctoral- and masters-granting departments.

What Providers provide in TPD was explored in terms of the formats, topics, and activities reported. Three *formats* of TPD were the most popular with the combination of these three also being the most commonly indicated by respondents: pre-semester orientation, course coordination, and a single course about teaching college mathematics. This supports Hauk and Speer's (2023) work that most doctoral and master's programs have multiple structures available to help novice instructors. The most frequently selected *topics* for TPD were university and departmental policies, active learning, learning management systems, and grading policies. A majority of respondents asserted these four topics were part of their department's TPD offerings. Inquiry-based learning was the least frequently chosen. The *activities* Providers reported using in TPD were most frequently listening to presenters and discussions of teaching examples, followed by reading/discussing articles. The Upset graph in Figure 3 showed how a majority of respondents had a combination of at least two of these three activities.

Implications and Impact

The finding that active learning was reported as a dominant topic, but not a part of TPD activity itself is worth exploring in further research. Do Providers practice what they preach? For example, activity-based learning about teaching such as "practice assessing a student assignment" or "practice a lecture" or "practice a non-lecture activity" might be expected to be more frequently reported than they were if active learning *in TPD* was highly valued. The field of college mathematics TPD is still striving to implement evidence-based teaching practices in a way that is meaningful for instructors and meaningful for TPD itself. Just as student-centered classrooms focus on student involvement and engagement, TPD can productively focus on novice instructor/LabTA/GSI involvement and engagement (in addition to listening and reading). More broadly, this indicates a need for assessment of the effectiveness of TPD. Such assessment would provide indicators and standards for the ways in which TPD is accomplishing intended goals for instructional development (Hauk & Speer, 2023; MAA 2020).

Limitations and Future Studies

Limitations of this study include the fact that this study's sample provides a national picture rather than a more granular understanding of the program. For example, this study collected survey data where respondents chose what they did according to our pre-defined categories. Although this gave us a broad national picture, it limits the understanding of how specific activities are enacted for each topic. For example, for the selection "reading articles" within the survey did not gather information on what occurred after the reading of the articles. Current projects are underway to better understand exemplar programs (Yee et al., 2022), and future studies could use the results of this study to further detail how U.S. mathematics departments implement activities within topics and within formats of TPD.

Acknowledgements

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References

Archie, T., Hayward, C. N., Yoshinobu, S., Laursen, S. L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *PLoS ONE, 17*(4):e0267097.

American Mathematical Society (2023). *The Mathematical and Statistical Sciences Annual Survey*. AMS. Available online: http://www.ams.org/groupings

Blair, R. C., Kirkman, E., & Maxwell, J. (2018). *Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States: Fall 2015 CBMS Survey*. Conference Board for the Mathematical Sciences.

Bookman, J. & Murphy, T. J. (2019). *MAA Invited Paper Session: Using research about teaching and learning to inform the preparation of graduate students to teach*. Joint Mathematics Meetings (Baltimore, MD).

Braley, E., & Bookman, J. (2022, March). A survey of programs for preparing graduate students to teach undergraduate mathematics [presentation]. In M. Jacobson, Chair, *AMS Special Session on Rethinking the Preparation of Mathematics GTAs for Future Faculty Positions.* AMS Spring Western Virtual Sectional Meeting. https://meetings.ams.org/math/spring2022w/meetingapp.cgi/Session/4489

Braley, E., Bookman, J., Gehrtz, J. & Speer, N. (2023). A survey of programs preparing graduate students to teach undergraduate mathematics [paper]. In S. Cook, B. Katz, and D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, Omaha, NE.

Braun, B., Bremser, P., Duval, A. M., Lockwood, E., & White, D. (2017). What does active learning mean for mathematicians? *Notices of the AMS, 64*(2), 124-129.

Carver, R. H. F., Everson M., Gabrosek, J., Horton, N. J., Lock, R. H., Mocko, M., Rossman, A., Roswell, G., Velleman, P. F., Witmer, J. A., & Wood, B. L. (2016). *Guidelines for assessment and instruction in statistics education (GAISE) – College Report.* American Statistical Association.

Connolly, M. R., Savoy, J. N., Lee, Y. G., & Hill, L. B. (2016). *Building a better future STEM faculty: How teaching development programs can improve undergraduate education*. Wisconsin Center for Education Research, University of Wisconsin-Madison. https://links.imagerelay.com/cdn/2210/ql/bb8ba5ac40984f4fa1121cbe01223940/Building_a_Better_Future_STEM_Faculty.pdf

Friedlaender, D., Burns, D., Lewis-Charp, H., Cook-Harvey, C. M., & Darling-Hammond, L. (2014) *Student-Centered Schools: Closing the Opportunity Gap* (tech. report). Stanford Center for Opportunity Policy in Education, CA.

Garfunkel, S., & Montgomery, M. (Eds.) (2019). *GAIMME: Guidelines for assessment and instruction in mathematical modeling education*, 2nd edition. COMAP and SIAM.

Gehrtz, J., Vroom, K., LaTona-Tequida, T., Martinez, A., and Rasmussen, C. (2022). Professional development and course coordination: One-time and ongoing supports. In E. Johnson, N. Apkarian, K. Vroom, A. Martinez, C. Rasmussen, & D. Bressoud (Eds.), *Addressing challenges to the precalculus II sequence through case studies* (pp. 45–54). MAA Press.

Gobstein, H. (2016). *Collaborative Research: Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL)*. Improving Undergraduate STEM Education (IUSE), National Science Foundation (NSF).

https://www.nsf.gov/awardsearch/showAward?AWD_ID=1624610&HistoricalAwards=false
Hauk, S., & Speer, N. (2023). Decentering and interconnecting as professional skills in the preparation of new college mathematics instructors. In S. Cook B., Katz, and D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 895-905), Omaha, Nebraska.
Hauk, S., Jackson, B., & Tsay, J. J. (2017). Those who teach the teachers: Knowledge growth in teaching for mathematics teacher educators. In *Proceedings of the 20th Conference on Research in Undergraduate Mathematics Educatio*n.
Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 129-146.
Mathematical Association of America (2020). *CoMInDS Resource Suite.* Mathematical Association of America. Online at: www.connect.maa.org
Mathematical Association of America (2018). *Instructional practices guide.* Mathematical Association of America.
Pengelley, D., & Sinha, D. (2019) Evidence-based teaching: How do we all get there? *MAA Focus, 39*(4), 20-24.
Yee, S., & Rogers, K. (2022). Active learning and STEM education: Who is active? Who is learning? *School Science and Mathematics*, 122(2), 71-73.
Yee, S. P., Hauk, S., LopezGonzalez, T., Wang, H. (2022, June). College Mathematics Instructor Preparation Design Tool: Helping Departments Grow their Teaching. *American Association for the Academy of Science (AAAS) Improving Undergraduate STEM Education (IUSE) Summit*. Washington, D.C.
Yee, S. P., Otten, S., & Taylor, M. W. (2018). What do we value in secondary mathematics teaching methods? *Investigations in Mathematics Learning*, *10*(4), 187-201.
Yee, S. P., Papalia, N., Deshler, J., Rogers, K. C., Lamarche, A., & Petrulis, R. (2023). Graduate Student Instructor Peer-Mentoring: Design and Impact. *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS)*, 1-19.

Experiencing Tensions of Nepantla While Working Toward Critical Transformations From Within

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The ACT UP Math project is studying the role and impact of research-practice partnerships between mathematics education experts and mathematics department faculty to critically and systematically initiate transformative efforts to improve the experiences of students with marginalized identities in introductory mathematics programs. This paper explores the shared experiences we encountered while working toward critical transformations of three departments from within those departments, focusing on three tensions that surfaced and framing these tensions as expected and necessary components of the process of critical change. These three tensions explore the role of identity as neutral or central, the enactment of power as power over or power with, and the role of students as experts or novices. These expected and necessary tensions are evidence of the transformations from dominance toward criticality happening within the departments.

Keywords: critical transformation, equity, department change, research practice partnership

Motivation and Overview

Barriers to equitable learning outcomes and inclusive learning experiences in mathematics education have been widely researched and include issues of implicit bias (Greenwald & Banaji, 1995), microaggressions (Leyva et al., 2021), instructors' minimization of existence and impact of microaggressions (McNeill et al., 2022), negative instructor relationships (Battey et al., 2018; Hill et al., 2010), stereotype threat (Steele et al., 2002), and sense of isolation (Good et al., 2012). Math stakeholders are aware of the need to attend to equity within their math programs, but they are often not familiar with how to accomplish this, and they feel disengaged from these conversations due to a lack of training (Apkarian, et al., 2021). Stakeholders need support to develop critical understandings of the factors contributing to differences in outcomes and experiences for marginalized students and to translate their understanding into action. To accomplish this, the ACT UP Math project is studying the role and impact of research-practice partnerships between critical mathematics education researchers and mathematics department faculty to critically and systematically initiate transformative efforts to improve the experiences of students who are members of marginalized identity groups in introductory mathematics programs.

In Fall 2022, the leaders from the three mathematics departments in collaboration with critical mathematics education researchers began the formation of a Networked Improvement Community (NIC) to initiate improvement cycles to critically transform their introductory mathematics programs. Each of the mathematics departments formed local NICs composed of various stakeholders (instructors, administrators, and students). The local NICs collected quantitative and qualitative data to observe the equity issues needing to be addressed; then they met to reflect and discuss what they saw in the data and developed a plan for programmatic changes to implement in Fall 2023-Spring 2024. Throughout this process, the mathematics education researchers worked with the local NICs to support their understanding of critical transformations and develop action plans and collected data to better understand the experiences of the NICs. This paper explores the shared experiences we encountered while fostering critical transformations, focusing on the tensions that have surfaced and framing these tensions as expected and necessary components of the process of critical change.

We are guided by the following research question: *What tensions are experienced by research-practice partnerships, as mathematics department stakeholders work together to critically transform their introductory mathematics programs?*

Framing Critical Transformations and Tensions

Critical transformations require that we critique and challenge the existing structures that shape students' mathematical experiences, and we seek to make improvements that extend beyond the confines of these systems. These systems include the mathematical content we teach, the way we teach it, the support programs for students, the departments that house these programs, the advising processes placing students into these courses, and much more. As we work in partnership to change introductory mathematics programs from within these programs, we draw on Gutiérrez's (2002) distinction between the dominant and critical axes of equity within mathematics education. The dominant axis attends to achievement and access, and is the primary - and sometimes only - component of equity that gets attention when undergraduate mathematics stakeholders discuss the importance of diversity, equity, and inclusion (DEI) (Tremaine et al., 2022). Focusing equity efforts on the dominant axis positions these changes within the existing system. While programs seeking to improve access and achievement can be important in supporting students from marginalized identities to succeed (Palid et al., 2023), such programs do nothing to change the system that these students are entering and can actually function to maintain systems of oppression rather than reform them (Martin, 2019). To discuss equity fully, we must also attend to the critical axis, which encompasses issues related to identity and power. Attention to the critical axis emphasizes not only valuing the representation and success of a diverse population within a current system, but also valuing the changes that population may require or make to the system. Therefore, attending to the critical axis necessitates working toward changes to the existing system.

Gutiérrez draws on the indigenous Nahuatl word *nepantla* to emphasize the tensions that exist within the intersection of the dominant and critical axes. Nepantla describes the in-between, liminal space between worlds. Anzaldúa (2002) emphasizes that "transformations occur in this in-between space, an unstable, unpredictable, precarious, always-in-transition space lacking clear boundaries," and that nepantla is associated with "being in a constant state of displacement - an uncomfortable, even alarming feeling" (Anzaldúa, 2002, p. 243). Gutiérrez (2015) emphasizes the power of these tensions; within this transformative and liminal space of nepantla we will experience tensions, but these tensions are necessary to transform mathematics education.

In this study, we explore the tensions observed as the NIC members work within the systems of introductory college mathematics programs to change these very programs. We specifically view the NIC members as walking with one foot in each of the worlds of the dominant and critical discourses in undergraduate mathematics. By operating from within the existing systems, one must have experienced some degree of success within that system: all of the NIC members are working to complete a mathematics degree or already hold graduate degrees in mathematics or mathematics education and are employed within or work frequently with mathematics departments. Recognizing the success the NIC members have experienced in these systems does not ignore the systemic barriers and struggles the NIC members themselves experience, but it does place them in a precarious position to see these systems and critique them. This analysis identifies shared tensions expressed among the three NICs and tensions that the research team observed and expressed. Drawing on the framing of nepantla, we emphasize that these expressions of tensions are evidence of the transformations occurring within these departments as they move within one world (dominant discourses) to create another (critical discourses), and we hold these expressions of tensions up as positive enactments of working to change the systems from within. We deeply respect and value the NIC members for engaging in this work and expressing their tensions and vulnerability.

Methods

Networked Improvement Communities

A consensus is emerging that "the department" is a prime locus for change (e.g., Austin, 2011; Larnell, 2023; Lee et al., 2007; Reinholz & Apkarian, 2018; Smith et al., 2021; Voigt et al., 2023). ACT UP Math seeks to support critical changes through Networked Improvement Communities (NICs; Penuel, 2020). In a NIC, a group of stakeholders convenes around a common aim, conducts a problem analysis, engages in continuous improvement cycles, and shares information across the network to contribute to collective progress. Each of the three local ACT UP Math NICs are positioned as *change agents* working to instigate critical transformations within their departments. The broader ACT UP Math NIC is made up of the members of the local NICs and the research team.

Alpha University is a public master's degree-granting university with moderate research activity. Alpha University is both a Hispanic-Serving Institution and an Asian American Native American Pacific Islander Serving Institution. Introductory math courses are taught primarily by faculty members in small classes using active learning strategies. The NIC is composed of two faculty leaders, two faculty members, one lecturer, one graduate student instructor, and two undergraduate transfer students. The NIC met every two weeks for two hours during Spring 2023. After a first iteration of their data exploration, the NIC developed plans to disrupt the placement system for lower division mathematics courses through self-placement and is driven by a goal to create positive relationships between students and mathematics.

Kappa University is a private, not-for-profit, highly selective doctoral degree-granting university that is research-intensive. Introductory math courses at Kappa had previously been taught primarily by graduate students in small, coordinated courses with a group-work focused recitation once a week, but very recently have shifted to having all calculus I and II courses taught by a teaching faculty member in large classes with weekly recitations taught by graduate students. The Kappa NIC includes eight members, six of whom regularly attended meetings. Of these, four are administrators outside of the mathematics department and four are mathematics instructors and/or coordinators, including the two co-leaders. The Kappa NIC meets monthly via Zoom. Many NIC members had personal relationships with the co-leaders and joined because of those relationships and their trust in the co-leaders. Although it was not an explicit intention of the NIC leaders to only recruit women, all regularly attending NIC members identify as women.

Tau University is a public, doctoral degree-granting, comprehensive university with a *high research activity* Carnegie classification and two campus locations. Most math courses are taught by faculty in small classes with some instructors using active learning strategies. The Tau NIC includes 11 members and is led by three faculty members. Nine members are instructors in the mathematics department and two members are administrators at the college level. The NIC meets monthly via Zoom, partly due to their institution being spread across two campuses, and the leaders meet monthly to plan each NIC meeting. One of the leaders recruited the other two co-leaders, and together they encouraged other faculty members to join with a mass email describing the project. After a first iteration of their data exploration, the NIC has decided to focus on improving individual instructor pedagogy given the varied nature of the data they observed and cultural values about instructional autonomy.

The **Research Team** is composed of eleven people from five different, research-intensive universities. Our varied identities and experiences shape how we enter this work, the power dynamics at play within our team and how we relate to the local NIC members, our individual motivations for engaging in this research, and how we interpret our work. We practice reflexive journaling as researchers and often engage in discussions about how our own identities may be leading to biases and blindspots in our approach to this work. The **ACT UP Math NIC** is made up of the research team and the members of the local NICs, and we meet once a month. As a research practice partnership, the research team is considered part of the broader NIC, and as such we view the tensions the research team experiences as part of the data for this study.

Data and Analysis: Critical Ethnography

Our analysis draws on approaches from critical ethnography, an ethnographic research method that seeks to explicitly critique and transform systems of oppression and inequitable power relations (Palmer & Caldas, 2015). This study takes a participatory research approach using research-practice partnerships such that the researchers and the local NIC participants are in ongoing dialogue and engagement, operating together as a research-practice NIC to make changes to departments and cultures (Bryk et al., 2015; Coburn et al., 2021; Martin et al., 2020).

The data from this analysis comes from structured field notes of NIC meetings, semi-structured interviews, and reflexive journals completed by NIC members during Spring 2023. The structured field notes were completed by a researcher who attended each of the NIC meetings and documented our observations, reflections, successes, challenges, noteworthy language, and observed power dynamics. The semi-structured interviews asked NIC members about their experiences over the last semester, how they became a part of the NIC, how data were explored, and the role of equitable decision making. The interviews were conducted via Zoom and were audio recorded and transcribed. NIC members engaged in reflexive journaling throughout the semester, completing 2-5 journal entries, which allow emancipatory dialogue with the data and can help mitigate power relations between participants (Malacrida, 2007).

The research team conducted an analysis of each NIC's experiences over the past year, seeking to identify stories that were emerging from the data. This included reading all reflexive journal entries, field notes from the NIC meetings, and interview transcripts and notes from the end of the semester. We then identified storylines focused on tensions that were either directly expressed by the NIC members or leaders (including the local NICs and the broader ACT UP Math NIC) or observed by the research team. In the results, we identify these tensions as specific instantiations of moving between the two worlds of dominant and critical discourses within undergraduate mathematics.

Results

Through our experiences, observations, and analyses, we recognized ways these tensions were experienced by all local NICs and the broader NIC. In telling the story of these tensions, we chose to center the ways each tension was experienced by one of the three local NICs.

Role of Identity: Neutral and Central

The NICs are positioned in nepantla between two worlds: the critical world of centering students with marginalized identities in their change efforts and the dominant world of not naming the specific populations and instead seeking improvement changes for all students. As we set out to support mathematics departments to critically transform their introductory mathematics programs from within, we purposefully chose to not specify any populations of students that we wanted the NICs to center in their action plans. Specifically, we did not encourage the NICs to explicitly attend to race, gender, sexuality, or other social identities. We recognize that this choice may perpetuate color evasiveness and other identity neutral stances which can serve to reinforce power structures (Goldin & Khasnabis, 2022).

Our choice to not prescribe a student population for each site came from wanting to support the equity needs specific to each institution through exploring disaggregated data from their institution in comparison to a national data set. We created data dashboards (Bolick & Voigt, 2023) drawing from extant data from 21 institutions (including each of the local NIC institutions) about students' experiences in introductory college mathematics. These dashboards, and the data exploration reflection activities created for them, encouraged the NIC members to explore the data by filtering various identity markers, including race and ethnicity, gender, first generation status, sexuality, and institutional comparisons. Some of the research team assumed the NICs would use these data explorations to identify populations of students who expressed exclusionary experiences and to use this to inform their action plans in their unique context, while others did not expect the NICs to center an identity group. The research team reflected on how we did not communicate these assumptions or expectations to the local NICs, because some of the research team believed their engagement with data dashboards would naturally lead to identifying populations of students at their institution who were especially in need of critical transformations to their introductory mathematics program.

Though the NICs did attend to student identity in their data explorations, they did not attend to a specific identity in their action plans. The Tau NIC's goal was to motivate all students to engage/succeed in mathematics, with little attention brought to any specific population. The Alpha NIC have been exclusively attending to lower division courses and the students who take those courses. The Kappa NIC members expressed interest in focusing on students who persist in the calculus sequence and those who do not and are interested in students who did not have access to calculus. Both of these populations can be proxies for other populations, given that research shows women are more likely to not persist in the calculus sequence (Ellis et al., 2016) and that low-income students have less access to advanced mathematics courses (Battey, 2011; Solorzano & Ornelas, 2002), but the Kappa NIC did not explicitly name any student population.

Role of Power: Power Over and Power With

The NICs are positioned in nepantla between two worlds: the critical world of collectively leading from within their departments to make changes to them, and the dominant world of hierarchies and privileged groups exerting power over others. While striving to increase equity within mathematics departments, we see a tension between discomfort with an image of power that we are most experienced with—*power over*—while working collectively toward changing systems built on this version of power. To work within this space of tension, we turn toward a different understanding of power. Allen (1998) defines *power with* as "the ability of a collectivity to act together for the attainment of a common or shared end or series of ends" (p. 35), contrasted with *power over*, defined as "the ability of an actor or set of actors to constrain the choices available to another actor or set of actors in a non-trivial way" (p. 33). The tension we see related to power describes the difference between enacting leadership through power over others via hierarchies and top-down decision making versus enacting leadership through power with others, which emphasizes democratic decision making and a rejection of hierarchies.

Tensions related to power dynamics were present among all three local NICs and among the researchers and the broader NIC. The Kappa leaders expressed not wanting to center their own "random ideas" about how to improve the department, discounting their own informed perspectives. The Alpha NIC leaders similarly expressed hesitancy toward enacting power over their NIC members and were "reluctant leaders," as noted in a researcher's summary of their NIC dynamics. Power dynamics were especially apparent within the Tau NIC. The researchers who observed the monthly Tau NIC meeting documented in the fieldnotes a sense of "power avoidance" or hesitancy to make decisions among the three co-leaders of the Tau NIC (Olivia, Natalie, and Garrett) throughout the spring semester. These co-leaders also reflected on the power dynamics at play in their leadership, noting individual experiences of these tensions. Olivia is a math education researcher and administrator at the college level at Tau with primary responsibilities to lead instructor professional development; this formal leadership role may have led to implicit assumptions she would lead the Tau NIC. She opted for distributed leadership by selecting Garrett and Natalie as co-leaders of the NIC, purposefully choosing instructors within the introductory mathematics courses to support more buy-in to any initiatives proposed by the NIC. Olivia reflected on the challenge of simultaneously valuing their voices and recognizing that they have not "gone through some sort of training ahead of time," so she supports them and tries "to push them out of those areas where they're comfortable to get them to do things that they might not otherwise do if they weren't pushed." After co-leading the Tau NIC for the semester, Natalie and Garrett expressed confusion and feelings of imposter syndrome in leading a group to enact changes while learning more about equity in higher education themselves. By putting the onus to enact equity-centered changes on mathematics department members, rather than equity experts from outside the mathematics department, such discomforts and feelings of inadequacy are to be expected and supported.

Role of Students: As Novices and as Experts

The NICs are positioned in nepantla between two worlds: the critical world of viewing students as experts of their own experiences and the dominant world of believing students are novices. In their work toward adopting more critical views of teaching mathematics, all three NICs recognized the importance of students' voices in telling their own stories but met tension when implementing policies that reflect this belief due to dominant structures. This tension arose for the Alpha NIC while navigating the inclusion of students within their NIC, for the Kappa NIC while coming to the realization that they wish to not only gather student feedback as data but also to incorporate students into the structure of their NIC, and the Tau NIC while transitioning from thinking of student voices as aggregated and biased opinions on course evaluations to meaningful perspectives that can be actively sought as qualitative data.

Here we highlight how the inclusion of students in the Alpha NIC surfaced tensions. NIC members were cognizant of the power imbalances arising within the NIC, and the research team noted this dynamic at the onset of the NIC formation. One faculty NIC member, Caroline, called out the power dynamics within the NIC, acknowledging in an interview that there is a "power dynamic between faculty (of different ranks/positions) and students in the NIC." The NIC faculty attributed these power imbalances due to "natural" hierarchies (student/instructor), logistical aspects (who leads the meeting), or the result of student characteristics (shyness).

NIC leaders attempted to mitigate this power differential by creating inclusive spaces for students to hold equal weight in the conversations through intentional pairings of instructors and students, small group discussions and share-outs, and a democratic voting system to decide next steps. While the democratic process of voting within the NIC introduced a measure of equality in decision-making, it did not counter the authority of faculty perspectives in decisions since faculty members outnumbered the students in the NIC. Students did not explicitly name a power dynamic occurring across students and faculty, but shared experiencing discomfort while in NIC meetings. One student NIC member shared that although they are not shy, settings "like [the NIC] make me a little bit nervous," and "it is a bit intimidating having professors in the group, especially since three are current professors of mine." Students identifying their own discomfort without recognizing the power imbalance placed the blame on themselves. The Alpha NIC was simultaneously grappling with the role of student perspectives to inform action plans. The Alpha NIC created two action plans focused on suggestions disclosed during a student focus group: restructuring the placement system for first year mathematics courses for students to self-place and engaging in interdisciplinary mathematics seminars. By valuing the voices of students in the focus group, the NIC saw students as experts on their own experience. Juxtaposing the students in the focus group with the status of student NIC members illustrates the tension well. The students outside of the NIC are positioned as experts, yet the students within the NIC are positioned as novices.

Conclusion, Implications, and Next Steps

Our research team felt empowered in naming the tensions around the roles of identity, power, and students and sharing them with our local NIC partners. By framing these tensions through the lens of nepantla, we feel more validated in our experiences and empowered to live with them instead of trying to rush away from them. When we shared these tensions with our local NIC partners, they expressed also valuing the tensions being named and shared, that these tensions "resonate," and that they "appreciate the space to think about the tensions." When responding to each specific tension anonymously on a Jamboard, most participants reflected on how they experienced these tensions, however one commenter shared that they did not resonate with the power tensions: "Perhaps I'm oblivious, but I haven't noticed power-over tensions, despite the fact that we have college-level and department leadership working alongside junior faculty."

These tensions also bring to the forefront the benefits and challenges of working within a research-practice partnership (such as through NICs) and how to do so equitably (Denner et al., 2019). Ryoo et al.'s (2015) definition of equitable research-practice partnerships includes "challeng[ing] power dynamics and hierarchies; regularly clarify[ing] and surfac[ing] needs, wants, and expectations relating to the partnership and partners' professional contexts; and attend[ing] to the changing needs of the collaboration" (p. 1). We position tensions as part of our work of being an equitable partnership, and by framing these tensions as existing within nepantla we view them as evidence of the progress the local NICs and the broader NIC are making to shift from a dominant to a critical framing. Our next steps within this work include asking the local NICs and the researchers to regularly reflectively journal about these tensions as they engage in action planning and implementation and use these insights to guide our attention during our observations of the NIC meetings.

References

Allen, A. (1998). Rethinking power. Hypatia, 13(1), 21-40.

Anzaldúa, G. (2002). Preface:(Un) natural bridges,(un) safe spaces. This bridge we call home: Radical visions for transformation, 1-5.

Apkarian, N., Kirin, D., Gehrtz, J., & Vroom. K. (2021). Connecting the stakeholders: Departments, policy, and research in undergraduate mathematics education. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 30*(1), 17–36. https://doi.org/10.1080/10511970.2019.1629135

Austin, A. E. (2011, March). Promoting evidence-based change in undergraduate science education. In the Fourth committee meeting on status, contributions, and future directions of discipline-based education research (Vol. 1).

Battey, D. (2013). Access to mathematics:"A possessive investment in whiteness". Curriculum Inquiry, 43(3), 332-359.

Battey, D., Leyva, L. A., Williams, I., Belizario, V. A., Greco, R., & Shah, R. (2018). Racial (mis) match in middle school mathematics classrooms: Relational interactions as a racialized mechanism. Harvard Educational Review, 88(4), 455-482.

Bolick, M. & Voigt, M. (2023). Using Interactive Data Dashboards to Address Diversity, Equity, and Inclusion in Introductory Math Programs. MAA Focus, 43(6), http://digitaleditions.walsworthprintgroup.com/publication/?m=7656&i=809747&view=articleBrowser&article_id=4690178&ver=html5

Bryk, A. S., Gomez, L., Grunow, A., & LeMahieu, P. (2015). *Learning to improve: How America's schools can get better at getting better.* Harvard Education Publishing.

Coburn, C.E., Penuel, W.R., & Farrell, C.C. (2021). Fostering educational improvement with research-practice partnerships. *Phi Delta Kappan, 102* (7), 14-19.

Denner, J., Bean, S., Campe, S., Martinez, J., & Torres, D. (2019). Negotiating trust, power, and culture in a research–practice partnership. AERA open, 5(2), 2332858419858635.

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. PloS one, 11(7), e0157447.

Goldin, S., & Khasnabis, D. (2021, December). In the pursuit of justice: Moving past color-evasive efforts. In The Educational Forum (Vol. 86, No. 1, pp. 1-4). Routledge.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. Journal of personality and social psychology, 102(4), 700.

Greenwald, A. G., & Banaji, M. R. (1995). Implicit social cognition: attitudes, self-esteem, and stereotypes. Psychological review, 102(1), 4.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. Mathematical Thinking and Learning, 4(2-3), 145-187.

Gutiérrez, R. (2015). Nesting in Nepantla: The importance of maintaining tensions in our work. Interrogating Whiteness and relinquishing power: White faculty's commitment to racial consciousness in STEM classrooms, 253.

Hill, C., Corbett, C., & St Rose, A. (2010). Why so few? Women in science, technology, engineering, and mathematics. American Association of University Women. 1111 Sixteenth Street NW, Washington, DC 20036.

Larnell, G. V. (2023). Building an Evolving Framework: A Clarion Call/Manifesto. Justice through the lens of calculus: Framing new Possibilities for diversity, equity, and inclusion. The Mathematical Association of America, p. 1.

Lee, V. S., Hyman, M. R., & Luginbuhl, G. (2007). The concept of readiness in the academic department: A case study of undergraduate education reform. *Innovative Higher Education, 32*(1). 3-18. https://doi.org/10.1007/s10755-006-9032-6

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. Cognition and instruction, 39(1), 1-34.

Malacrida, C. (2007). Reflexive journaling on emotional research topics: Ethical issues for team researchers. Qualitative health research, 17(10), 1329-1339.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. Race Ethnicity and Education, 22(4), 459-478.

Martin, W. G., Lischka, A., Smith, W. M., & Lawler, B. R. (Eds.) (2020). *The Mathematics Teacher Education Partnership: The power of a networked improvement community to transform secondary mathematics teacher preparation.* Volume 4 in B. Benken (Ed.), Association of Mathematics Teacher Educators Professional Book Series. Information Age Publishing.

McNeill, R. T., Leyva, L. A., & Marshall, B. (2022). “They’re just students. There’s no clear distinction”: A critical discourse analysis of color-evasive, gender-neutral faculty discourses in undergraduate calculus instruction. Journal of the Learning Sciences, 31(4-5), 630-672.

Palid, O., Cashdollar, S., Deangelo, S. et al. Inclusion in practice: a systematic review of diversity-focused STEM programming in the United States. IJ STEM Ed 10, 2 (2023). https://doi.org/10.1186/s40594-022-00387-3

Palmer, D., & Caldas, B. (2015). Critical ethnography. Research methods in language and education: Encyclopedia of language and education, 1-12.

Penuel, W. R. (2020). Promoting equitable and just learning across settings: Organizational forms for educational change. In Handbook of the cultural foundations of learning (pp. 348-364). Routledge.

Reinholz, D. L., & Apkarian, N. (2018). Four frames for systemic change in STEM departments. International Journal of STEM Education, 5(1), 1–10.

Ryoo, J.J., Choi, M., & McLeod, E. (2015), Building equity in research practice partnerships, Retrieved from http://researchandpractice.org/wpcontent/uploads/2015/10/BuildingEquity_Oct2015.pdf, 2015.

Smith, W. M., Voigt, M., Ström, A., Webb, D. C., & Martin, W. G., (Eds.) (2021). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning*. American Mathematical Society and Conference Board of Mathematical Sciences.

Solorzano, D. G., & Ornelas, A. (2002). A critical race analysis of advanced placement classes: A case of educational inequality. Journal of Latinos and Education, 1(4), 215-229.

Steele, C. M., Spencer, S. J., & Aronson, J. (2002). Contending with group image: The psychology of stereotype and social identity threat. In Advances in experimental social psychology (Vol. 34, pp. 379-440). Academic Press.

Tremaine, R., Hagman, J. E., Voigt, M., Damas, S., & Gehrtz, J. (2022). You don't want to come into a broken system: Perspectives for increasing diversity in STEM among undergraduate calculus program stakeholders. International Journal of Research in Undergraduate Mathematics Education, 8(2), 365-388.

Voigt, M., Hagman, J. E., Gehrtz, J., Ratliff, B., Alexander, N.& Levy, R. (Eds.). (2023). *Justice through the lens of calculus: Framing new Possibilities for diversity, equity, and inclusion.* The Mathematical Association of America.

Investigating How Students Try to Solve Real Analysis Problems

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This study observed real analysis students attempting the homework assigned by their instructor to identify how they used resources, specifically example proofs, to complete their homework. The data consisted of seven students in total over three instructors. Overall, the study found that students tended to adopt a recurrent style of how they used their notes, or did not use their notes, to advance their solutions. However, there were some instances where the students would deviate from their style which could be linked to the type of problem as well as how the instructor taught that topic. This study contributes insight into how real analysis students may use their resources for the tasks assigned by the instructor.

Keywords: Problem situation image, Tentative solution start, Resource framework

Introduction

In Real Analysis, instructors introduce definitions and theorems which are part of the material that the student needs to know. Theorems and other statements come tied to proofs that are both material the students need to know as well as something the students must learn to write. The proofs that the instructor demonstrates in-class, thus, serve a dual purpose. One role of the proof is to prove the theorem/statement the instructor has introduced. The second role of the proof is to serve as a demonstration of the process of proving to the students. Research has already suggested that lecturers tend to recommend their notes as the primary resource for students (Ni She, Bhaird, Fhloinn, & O'Shea, 2017). It stands to reason that a skill the students should develop is to be able to reference the proofs they do in class as examples to inform themselves on how to write the proofs that they are asked to do for homework.

Studies have shown that students do tend to use curriculum materials and class notes as a primary resource when working on class material (Anastasakis, Robinson, & Lerman, 2017; Ni She, Bhaird, Fhloinn, & O'Shea, 2017; Pepin & Kock, 2021). Albeit, the students may leverage certain resources more heavily when with certain goals such as high grades or simply passing in mind (Anastasakis, Robinson, & Lerman, 2017). This study wanted to dive deeper into *how* students used their resources. As such, this study documented students working on homework tasks and found that students tended to have a style on how to use their class notes to complete homework tasks. Interestingly, certain homework tasks tended to deviate students away from their usual style. This seemed to occur due to certain features of the task and how the instructor presented related proofs. Building on an existing framework, this study extends the literature on how students use resources and worked examples in the context of proof-based mathematics courses.

Research Questions

1) When and how do real analysis students leverage the examples presented in-class to complete their homework tasks?
2) What factors influence the reason that real analysis students use their notes to complete their homework tasks?

Resource use has been investigated by a number of researchers (Galbraith et al., 2000; Guedet & Pepin, 2018; Muir, 2014; Radovic & Passey, 2016). However, research on resource use remains limited for a proof-based mathematics context. This study focuses on that context in an effort to broaden our understanding of how students use their instructor-provided resources for the problems they are assigned for homework.

Firstly, it is important to describe how a student comes about making a solution attempt for a homework problem. While working on homework, students may have different problem situation images associated with the topics in the question as well as the task assigned to them (Selden et al., 1999). Students typically make an initial judgement of the difficulty of the task and assesses whether they need to reference their notes or if they think they know how to do it. This initial assessment leads the student to their initial tentative solution start (Selden et al., 1999). For this study, it was important to note the instances of when students did not feel the need to reference their notes or note what they referenced when they did reference their notes. This helped define *when* students would use their notes by looking both at the task itself as well as how a student initially responded to the task.

As for *how* students used their resources, I adapted the technology framework developed by Galbraith, Goos, Renshaw, & Geigher (2000) into a Resource Use Framework. I conjectured the use between resources and technology can be similarly classified since in both cases the object being used by the student is still a tool for completing a task. As such, there are four categories that students could be classified for how they used their resource:

Table 1: Resource Framework

Resource as Master	A student copies a proof or makes every decision based on their notes without a way to check the adequacy of their decisions
Resource as Partner	A student has a tentative strategy but searches their notes to either find a new strategy or justify their own, or the student finds a relevant example but needs assistance to adapt it into a solution
Resource as Servant	A student either has a strategy and uses their notes to complete their strategy, or a student finds a relevant example and is able to fully adapt it into a solution without assistance
Resource as Extension of Self	A student knows exactly what they need from their notes so the notes act more as an extension of memory rather than a tool

Data Collection

Data was collected from three Real Analysis classes over two semesters each of which had different instructors from a large, public university in the United States. I observed the instruction and took field notes. The field notes consisted of what was written on the board, the comments the instructor added, and questions or remarks the students made. The field notes provided a detailed reference for the information the students received during class and what their resource of written notes could possibly resemble when they are doing their homework.

I recruited students from the aforementioned classes. In total, seven students participated. The student interviews consisted of asking students to attempt problems from their assigned homework. Priority was given to problems that I could connect to parts of the instruction that occurred in class. This provided an opportunity to observe whether the student needed the resource from class or could identify the same resources I had if they did look into their notes. Students were videoed as they attempted to solve each problem. The video provided both verbal and visual evidence of what the student was doing at the time. The overall structure of the interview had three parts per question. Firstly, I let the student attempt the problem independently, noting whether or not they used any resources. If they formed a solution independently, I moved on to the next task. If not, I would ask them if they knew of anything from class that could help. Sometimes the student would find something and other times they would not find anything. If the student did not find anything or they did not yet look at what I thought was most relevant, then I would suggest for them to consider what I thought was relevant. Sometimes the student could use what I suggested and other times they could not. If the student was struggling with using the resource I referenced, I would continue to work through why the example could be useful to see if they could ever connect why I would use that part of their notes. Overall, this provided insight into the problems that a student may not need notes for and what problems a student may want notes for. If the student wanted notes, there was an opportunity to observe what the student deemed possibly relevant to the problem. If they found something they thought was relevant, then there was a chance to see if they could evaluate whether or not it was relevant. Lastly, if provided something relevant according to the interviewer, then it could be observed whether the student could make connections from what was provided to the problem.

Data Analysis

For each student, I made an initial profile based on the experience of the interviews. These profiles were made by considering each problem the student did and how they used resources during each one. Each interview consisted of about 5 problems and most students had 5 interviews providing me with approximately 25 questions to use as my unit of analysis per student. This provided a frequency on the ways they used their resources which provided insight into the tendencies of each student.

Students' resource use was broken down per task and I coded each student attempt at a task using the following categories: (a) Did not use any resources, (b) referenced class notes, (c) made their own example, (d) looked for an example or tried to make their own but failed, and (e) could or could not use an example provided by the interviewer. Of these actions, they also have varying degrees of success as well as different ways of being performed. However, these cover the actions around resources that students could perform. A frequency table was generated showing how frequently each student performed any of those actions. With the frequencies, I sought to identify whether each student had a recurrent style of resource use, addressing Research Question 1. As a note, the students were not asked directly to describe their own resource use style.

After this, I examined students' resource use by tasks. Though not all students were assigned the same problems, the topics were largely consistent among all the students. The topics mostly spanned set theory, like supremum and infimum, and sequences. Thus, the data gathered covers similar problems for each student. This uncovered that some problems tended to have the students perform similar actions to one another even for students that did not typically behave in that way. This suggests that the task itself has influence on how the student will decide to tackle the problem regardless of their usual tendency. The characteristics of the task itself as well as how similar tasks may have been presented in class both serve as factors towards addressing Research Question 2.

Results

The following are a few exemplary student cases that show how students can adopt different styles on their usual resource use when attempting homework tasks.

Jordan

Jordan frequently looked through their notes for help on their homework tasks. On several occasions, Jordan was able to adapt proofs efficiently and effectively from class into solutions for their homework tasks. Here is an example: Given a set $E \subset \mathbb{R}$ and $r > 0$, define the set $rE = \{rx: x \in E\}$. In Jordan's notes, they found where the instructor proved that: $\sup(E + r) = \sup E + r$ for the set $E + r = \{x + r: x \in E\}$. The following table provides the proof that the instructor gave during class and how Jordan adapted it to solve the problem.

Table 2: Instructor vs Student Proof for a Similar Task

Instructor's Work	Jordan's Work
Assume that $a \in E + r$ Then, $a = x + r$ for some $x \in E$. Since $\sup E$ is an upper bound of E, $x \leq \sup E$ so $a = x + r \leq \sup E + r$. Thus, $\sup E + r$ is an upper bound of $E + r$. Now assume N is an upper bound of $E + r$. Let $x \in E$. So $x + r \in E + r$ so $x + r \leq N$. Then $x \leq N - r$. Since $N - r$ is an upper bound of E, $\sup E \leq N - r \Rightarrow \sup E + r \leq N$ Then $\sup E + r$ is the supremum of $E + r$.	Assume $a \in rE$. Then, $a = rx$ for some $x \in E$. Since $\sup E$ is an upper bound of E, $x \leq \sup E$ so $rx \leq rsupE$. Thus, $rsupE$ is an upper bound of rE. Now assume N is an upper bound of rE. Let $x \in E$. So $rx \in rE$ so $rx \leq N$. Then $x \leq \frac{N}{r}$. Since $\frac{N}{r}$ is an upper bound of E, $\sup E \leq \frac{N}{r} \Rightarrow rsupE \leq N$. Then $rsupE$ is the supremum of rE.

When working on this problem, Jordan read their notes and interchanged the parts as they read. Jordan was able to write the proof without assistance from the interviewer. This illustrates how Jordan was able to both recognize a relevant example to adapt as well as an ability to recognize what parts of that proof were interchangeable with different characteristics. I coded this as Resource as Servant. Jordan both identified the relevant proof as well as efficiently identified all the relevant parts that should be changed based on the context. The notes served as an outline, but Jordan had the knowledge on how to fill in the blanks or change the instructor's work.

Peyton

Peyton did not frequently use their notes. However, they did reference them a few times. More frequently, Peyton thought up their own *conceptual examples* or recall *conceptual examples* from class. Conceptual examples refer to specific cases from which Peyton would try to generalize from. Such as using the interval [2,3] to reason why a closed, bounded interval must contain its supremum. They would try to use those to reason through what they needed to do with their homework tasks. Occasionally, this strategy showed a lot of power for certain tasks. However, at other times this technique was not effective enough for Peyton to produce a proof.

One instance of Peyton using their notes was when asked to prove that $x_n = 2n + 6$ is a strictly increasing sequence. Peyton initially looked up where the instructor talked about monotonicity, but they did not find an example they judged to be similar. During the second phase of this task, I suggested to look at a proposition where the instructor proved that $\{x_n\}$ is non-decreasing if $x_n \leq x_m$ for all $n \leq m$. Peyton noted that the instructor used induction, so they tried to implement induction for this problem. With this approach, Peyton noticed that: $x_n = 2n + 6 \leq 2(n+1) + 6 = x_{n+1}$.

Peyton finished up their argument by establishing that $P(1) < P(2)$ and, by the logic before, $P(n+1) < P(n+2)$. After completing this, Peyton decided that a direct proof just stating that $x_n = 2n + 6 < 2n + 2 + 6 = 2(n+1) + 6 = x_{n+1}$ was a sufficient proof.

I coded this proof construction as Resource as Partner. Peyton initially could not find any examples they thought would help to solve this problem. With my suggestion, Peyton was able to find an instance where they did something sort of similar to this problem. Peyton noted that the instructor used induction and decides to take that as a suggestion to do that themself. With this suggestion, they were able to make a satisfactory solution. However, afterwards, they noted that they do not think induction is necessary. This suggestion at the end is what distinguishes this resource use as Resource as Partner rather than Resource as Master. Peyton took the suggestion from their notes, but they keep a critical eye on what we're doing.

Charlie

Charlie frequently did not use their notes nor construct their own examples. The frequency analysis of their task solutions revealed that they rarely used their notes and often attempted to reason using the definitions and theorems presented in class. Occasionally, Charlie would construct their own examples, but this usually did not yield a valid solution to the task. When they ran into obstacles, I would provide Charlie with examples from the class or create a unique example to reason with, but they frequently did not use these to produce a valid task solution. Charlie usually mentioned that they wanted to go talk to the instructor during office hours to get help when they got stuck. This overall disinterest in using their notes or reasoning with examples contrasts with all the other students. However, in many cases, Charlie was successful at making strong reasoning connections between the definitions and theorems from class and the problem. With these connections, Charlie demonstrated throughout the interviews that they could construct proofs in real analysis without using their resources for many problems. However, there were several times that Charlie got stuck, but rather than using their available resources at the time instead opted to visit the instructor. For the purposes of this investigation of example use, I found that this student showed individual preference against using examples. However, there were a few occasions where they did opt to use examples, which I will discuss next.

There were only two instances where Charlie directly used notes. One of these instances was for the first tasks asking to prove a limit of a sequence by definition. The other time was to answer the following question: Suppose $S \subset \mathbb{R}$ is nonempty and bounded below and let $v = inf\ (S)$. Prove the following statement: For every $y > v$, there exists $s \in S$ such that $s < y$.

Charlie noted that the instructor proved something similar for the supremum. They changed necessary components to achieve a correct argument. This time the instructor's proof was the analogous supremum problem. As such, the student had to change very little, but Charlie did both identify the relevant proof and adapted it effectively for this context. Similar to the first example, I coded this as Resource as Servant.

Summary of Student Cases

For the students in this study, there were no particular moments where the students seemed to demonstrate Resource as Extension of Self. As such, there is no example given here. Resource as Extension of Self would look like a student bringing up a specific instance where a process was showcased by the instructor, but the student did not necessarily need it to complete the problem. Rather, the student would like to have the resource available in case they need just a reminder. In some sense, the resource would act as an extension of memory rather than a tool needed to help actively.

Additionally, it is important to end off by noting that all the students could change their position to their resources from problem to problem. For instance, every student faced with their first sequential limit problem to solve using the definition of convergence used their Resources as Master. Every student flipped to an instance where the instructor proved the limit of a sequence by definition and used that as a line-to-line inspiration of their approach. I conjecture that this is due to how the instructors emphasized the routine of this type of proof. When doing these proofs in class, the instructors tackled the problems using the same routine and would always mention that this is how the definitions proof work. While this can sound similar to the Resource as Servant examples from above, these students approached the problem differently in these cases. They acknowledged that they were relying on the instructor's notes to complete the problem. Before, they knew those problems held everything necessary to prove the problem, so they strategically used that structure to make sure their proof was complete. Here, they did not know how to write the proof, so they needed to rely on everything from the notes to complete their proof. This shows that students can renegotiate their position with their resources as they realize different limitations in their preparedness for certain tasks as well.

Discussion

For this study, the class notes were the standard that the student resource use was considered against. The class notes could be divided into providing Conceptual Examples and Proof Examples. Conceptual being the specific examples such as $(1,3)$ as an open interval, and Proof Examples being any instance of a complete proof provided in class. When considering these types of resources, the uses of them can be restrictive. From the observations made, the four categories that the framework considers are sufficient to categorize the resource uses. Largely, Proof Examples are the main resource used by students for their homework. As such, for proof-based mathematics, the resource use framework here could be a useful tool to describe how students are interacting with their available resource in a certain context since it was adapted to fit this situation.

The analysis brings forth a few details about how students use their resources while working on homework tasks. Namely, real analysis students adopt typical styles on how frequently they use their direct class note resource. Some students use their notes at a much higher frequency than others. This frequency might be tied to how confident a student is in their capability to adapt what they read in their notes. This can be partially substantiated by the fact that when I would bring up aspects from class, students that would frequently not reference their notes tended to have a harder time adapting the resources brought forth to them. For instance, Charlie would not commonly use the resource I mentioned to them. Instead, Charlie would frequently continue thinking about the problem using just the definitions and theorems they were already considering. On the other hand, Jordan frequently used their notes to complete their homework and could commonly take at least one useful part from any resource that I offered to them.

A second result of this data is that some types of questions tend to influence students to seek out help from their resources regardless of their previously established style. For instance, when students were asked to prove a limit of a sequence using the ϵ-k definition, every single student looked up an example from class and attempted to copy that structure for the homework task's sequence. I conjecture that this results from how the instructor routinized the problem during the instruction, or the very specific technique the instructor required the problem to be completed using. With this, the students might have decided that their best strategy would be to attempt an adaptation of what the instructor was doing in-class. In turn, this leads them to return to their resources and attempt to use them as closely as possible. This could influence them approaching the Resource as Master.

Conclusion

In conclusion, students can use their resources in various ways. The way that students use their resources can even vary from problem to problem and context to context. This suggests that students may develop an overall style to how they approach their work, but the way that questions are worded or what content they are covering may change how the student decides to approach the problem than they usually would. One of the strong ways that an instructor may be able to influence resource use of the students is by emphasizing a routinization of certain problems like sequential limits. This can be beneficial when the instructor wants students to solve problems in certain ways. By both verbally reinforcing the routine being used as well as repeated demonstration of the routine in use can help bolster a connection for the students between the routine and the types of problems it is relevant for in a student's problem situation image. For students, knowing how to adapt a proof is a key competence for being successful in a class like real analysis. This is not something emphasized in the literature, but it may be an important part of learning at this stage in a student's apprenticeship in proving.

Future research into the link between routinization and resource use by students could provide interesting results into how instruction could be meaningfully designed. Also, this could provide more insight into how problems that share structure may not be seen as routinized by the students.

References

Anastasakis, M., Robinson, C. L., & Lerman, S. (2017). Links between students' goals and their choice of educational resources in undergraduate mathematics. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *36*(2), 67-80.

Galbraith, P., Goos, M., Renshaw, P., & Geiger, V. (2001). Integrating technology in mathematics learning: What some students say. In J. Bobis, B. Perry, & M. Michael Mitchelmore (Eds.), *Numeracy and Beyond. Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia*, 225-232.

Gueudet, G., & Pepin, B. (2018). Didactic contract at the beginning of university: A focus on resources and their use. *International Journal of Research in Undergraduate Mathematics Education*, *4*, 56-73.

Muir, T. (2014). Google, Mathletics and Khan Academy: students' self-initiated use of online mathematical resources. *Mathematics Education Research Journal*, *26*, 833-852.

Ní Shé, C., Mac an Bhaird, C., Ní Fhloinn, E., & O'Shea, A. (2017). Students' and lecturers' views on mathematics resources. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *36*(4), 183-199.

Pepin, B., & Kock, Z. J. (2021). Students' use of resources in a challenge-based learning context involving mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *7*(2), 306-327.

Radović, S., & Passey, D. (2016). Digital resource developments for mathematics education involving homework across formal, non-formal and informal settings. *The Curriculum Journal*, *27*(4), 538-559.

Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). *Do Calculus Students Eventually Learn to Solve Non-Routine Problems?* Department of Mathematics, Tennessee Technological University.

Representing Learning in Advanced Mathematics Courses for Secondary Mathematics Teachers

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It is essential to provide opportunities for prospective secondary mathematics teachers to connect advanced mathematics content to secondary mathematics teaching practice, if advanced mathematics courses are to be useful to these teachers. We argue that for teachers to frame teaching and learning in equitable ways, connections to teaching must represent learning in anti-deficit ways. We review a selection of curricular materials satisfying two criteria: first, they are written for use in advanced mathematics courses that prospective teachers may take; and second, they feature explicit connections to secondary teaching practice. We find that representations of learning in curricular materials tend to take binary views of mathematical legitimacy, and frame students as either being correct or incorrect. We conclude with implications for mathematics teacher educators in RUME.

Keywords: content knowledge for teaching, secondary mathematics teacher education, advanced mathematics courses

Many prospective secondary mathematics teachers (PSMTs) complete multiple courses in advanced mathematics (e.g., Ferrini-Mundy & Findell, 2004; Tatto & Bankov, 2018). As recent empirical work demonstrates, it is essential to provide opportunities for PSMTs to connect the content of these courses to the *practice of teaching*, if these courses are to be useful to prospective teachers (Álvarez et al., 2020a; Lai et al., 2023; Wasserman & McGuffey, 2021). Multiple scholars have followed this approach in creating curricular materials for secondary mathematics teacher education (e.g., Hauk et al., 2018; Heid et al., 2015; Lischka et al., 2020; Mathematical Education of Teachers as an Application of Undergraduate Mathematics [META Math], 2020; Wasserman et al., 2022). In this approach, scholars represent teaching through descriptions of student thinking and teaching scenarios. This approach to secondary mathematics teacher education raises the question of how secondary students' learning is represented in curricular materials. This is a critical issue, as instruction in teacher education, including in mathematics classes, is part of the socialization that shapes teachers' images of teaching practice (e.g., Lai et al., 2023).

Our purpose is to review how secondary students' learning is portrayed in curricular materials for advanced mathematics courses that enroll PSMTs. To do so, we reviewed a selection of curricular materials satisfying the following criteria: they were intended for use in advanced mathematics courses that PSMTs may take; and they featured explicit connections to secondary teaching practice. Our review was guided by the research questions: *(1) What approaches have been taken by curriculum writers to connect advanced mathematics* *coursework to secondary teaching practice? (2) How do these connections portray students' learning of mathematics?* We suggest that approaches taken are consistent with Grossman et al.'s (2009) pedagogies of practice. We then analyzed portrayal of students' learning of mathematics in line with Louie's (2017) exclusive and inclusive frames of mathematics and mathematical ability.

Background

Advanced mathematics knowledge can potentially strengthen secondary mathematics knowledge (Conference Board of the Mathematical Sciences [CBMS], 2012; Murray et al., 2017). A typical approach has been to follow Felix Klein's (1924/1932) prescription of considering "elementary mathematics from an advanced standpoint" (p. 1). The hope here is that with a more sophisticated understanding of secondary concepts, teachers will take more productive actions when teaching.

Yet secondary teachers lament their mathematical preparation was irrelevant to secondary teaching (e.g., Goulding et al., 2003; Wasserman & Galarza, 2018; Zazkis & Leikin, 2010). Even when prospective teachers develop a richer understanding of the secondary mathematical concepts that they will teach, they may still exit advanced mathematics believing that they may have become better mathematicians, but not better mathematics teachers (Wasserman & Ham, 2013).

Mathematics teacher educators have historically connected advanced mathematics to secondary teaching along two dimensions: *mathematics content at the secondary level* and *mathematical practice*. The concept of a capstone course for teachers, which is offered by many institutions in the U.S. (Cox et al., 2013), exemplifies the first dimension. In their review of then-present-day capstone coursework for secondary teachers, Murray and Star (2013) found that connections took the form of generalizations or abstractions of secondary mathematical ideas (e.g., geometric transformations and group theory, factoring polynomials and Galois theory), or specific uses of advanced mathematics to define concepts that secondary students may encounter (e.g., limits and irrational numbers). As for mathematical practice, scholars such as Cuoco et al. (1996) called attention to the importance of mathematical habits of mind in teaching and learning mathematics. CBMS (2001) theorized that mathematics teachers must experience mathematical practices in their own learning, so that they can cultivate experiences of mathematical practices when teaching secondary mathematics. This argument is compatible with research on the perspective of teachers. Some scholars have found that when teachers found coursework relevant, it was due to understanding the nature of mathematics practice, and how learning can feel (Baldinger, 2018; Even, 2011; Hoffman & Even, 2019).

The potential of these approaches is unmet. Begle (1979) found that secondary students' performance was associated with neither the number of tertiary mathematics courses taken by teachers nor the average grade received by teachers in these courses. Goulding et al.'s (2003) and Zazkis and Leikin's (2010) surveys found that many teachers report their mathematical preparation is disconnected from teaching. In a study of an abstract algebra course, Ticknor (2012) found that even when secondary teachers wanted to do well in the course, and the instructor saw connections between abstract algebra and secondary mathematics, the teachers still perceived the course as irrelevant to teaching. Wasserman et al. (2018) articulated concrete reasons, given by prospective and in-service teachers, for why their knowledge of real analysis did not inform their teaching, even when they understood the material.

There is a promise of advanced mathematics to shape teachers' practice and an observed inefficacy of advanced mathematics courses to do so. In response, secondary mathematics teacher educators have pushed for explicit connections from advanced mathematics to secondary teaching practice and enacted such connections in written curriculum materials (Álvarez et al., 2020a; Hauk et al., 2018; Heid et al., 2015; Lai, 2019; Lischka et al., 2020; Wasserman et al., 2022). Many of these authors are in the RUME community; indeed members of the above cited teams behind curricula and research have all presented at RUME in the past 5 years.

Another factor in the ecology of secondary mathematics teacher education, particularly when it comes to courses taught by mathematics faculty for PSMTs, is a shift in the teacher education community's focus. For a long time, the sentiment that 'you can't teach math and have students learn without knowing the math yourself' drove funding, research, and course design; now, there is the urgent problem of increasing equitable access to mathematics. A corollary of this observation is that when assessing connections of advanced mathematics to secondary teaching practice, one must not only assess the mathematics, but also the images of equity and inclusiveness in teaching in connections.

To our knowledge, there has been no systematic review of materials taking the approach of connecting advanced mathematics to teaching, including the ways that these curricular materials represent student learning and teaching. In view of the posited connection to teaching practice, and the abundance of materials aligned with this approach, now is the right moment to analyze how exactly teaching and learning are portrayed. The results of this analysis, particularly in terms of equitable and inclusive educational principles, is essential to future work by the RUME community in improving teacher education.

Conceptual Perspective

Mathematics *teaching practice* at any level entails relational work with students, routines for disciplinary discourse, and norms for establishing the legitimacy of solutions. In this sense, the term *practice* is about what practitioners do rather than think or know (Lampert, 2010). Moving up a level from teaching to the teaching of teaching, Grossman et al. (2009) proposed the concept of *pedagogies of (teaching) practice*. By this term, Grossman and colleagues refer to the intentional design of experiences for future teachers that develop their future teaching practice.

Representations and approximations of practice

To articulate potential pedagogies of practice, Grossman et al. (2009) analyzed the education of future teachers, pastors, and therapists. These professions were chosen as comparisons due to their commonality of requiring relational work in combination with the expectation of particular routines and harnessing technical knowledge. Through this work, Grossman and colleagues identified pedagogies of practice that spanned the fields of teacher education, liturgical education, and clinical education. These pedagogies of practice included representations and approximations of practice. In the descriptions below, we follow Grossman et al. (2009).

Representations of practice allow novices to observe aspects of practice. When teachers engage with representations, they can develop ways of noticing and understanding teaching practice. Representations can vary in comprehensiveness and authenticity, and can take a variety of forms, including short narratives of teaching scenarios, videos, animations, and case studies. We consider curriculum materials (for advanced mathematics courses for PSMTs) to feature a representation of practice if they feature a description of a teaching scenario that for instance engages PSMTs in evaluating, describing, or reflecting upon features of the presented scenario.

Approximations of practice allow novices to simulate practice so they can attend to particular aspects of practice, rather than all aspects of the complex, relational practice that is teaching. Approximations may vary in authenticity and complexity. We consider curriculum materials to feature an approximation of practice if teachers are asked to simulate responding to students' mathematical work, evaluate students' work, explain content, or teach a full lesson. Note that approximations of practice may embed representations of practice, as the prompt for an approximation of practice may describe a teaching scenario.

Exclusionary and inclusive frames for the nature of mathematics and mathematical ability

We posit that inherent in any representation of practice of mathematics teaching is an at least implicit frame for mathematics and mathematical ability. Louie (2017) conceptualized exclusionary and inclusive frames for the nature of mathematical activity and the nature of mathematical ability. Within the exclusionary frame are the "rote practice frame", where "mathematics is a fixed body of knowledge" involving closed questions and a focus on answers; and the "hierarchical ability frame", where speed and correctness are valorized, and some students are positioned as helpers and others as in need of help. Within the inclusive frame are the "sense-making frame", where "mathematics is about making sense of ideas and understanding connections" and "the multidimensional math frame"; and the "multidimensional ability frame", where a variety of students are named as resources for peers' learning and skills outside of speed and correctness are valorized (p. 496).

Data & Method

We sought to review curricular materials satisfying two criteria: (1) they were intended for use in advanced mathematics courses that secondary teachers may take, and (2) they sought to make explicit connections to secondary mathematics teaching practice. We also restricted this review to materials developed in the US. In searching for curricular materials, we looked for both the materials themselves as well as reports of their enactment. We considered textbooks commonly used in capstone courses (as reviewed by Cox et al., 2013), those published by professional organizations of mathematicians, and the literature reporting the use of these materials and textbooks in advanced mathematics courses. As well, we conducted a Fastlane search for materials created with the support of the U.S. National Science Foundation.

The authors of this manuscript are developers of curricular materials, namely the MODULE(S^2) materials (Lai, Strayer, Casey, Lischka) and ULTRA materials (Wasserman, Weber, Fukawa-Connelly). We contacted all curriculum writers and investigators that came up in our search (other than ourselves) for copies of materials. Not all authors and investigators responded to requests. Table 1 (on the next page) displays materials and reports we obtained. Note that in this table, "algebra" refers to the study of number systems, functions, or relations, whereas "abstract algebra" refers to the study of mathematical groups, rings, fields, or related constructs.

We analyzed descriptions of enactments and the text of curricular materials in three stages. First, we identified instances of presenting secondary teaching context to PSMTs. By context we mean the contextual elements of teaching practice such as hypothetical or actual secondary student talk or pieces of hypothetical or actual secondary curriculum materials (Lai & Jacobson, 2018). Second, we coded these instances for whether they satisfied conceptualizations of representation of practice and approximation of practice (consistent with the descriptions given in the section on Conceptual Perspective) and then inductively coded categories of representations of practice and approximations of practice. Third, we analyzed these instances holistically for how their prompts and descriptions of students and teachers in the teaching context aligned or not with various aspects of Louie's (2017) framework. For instance, if student work was described as "correct" or "incorrect", this is indicative of an exclusionary frame for mathematical activity. Or, if teachers were asked to name strengths of student work beyond correctness, this would align with an inclusive frame for mathematical ability.

Table 1. Reviewed materials and reports.

Citations reviewed, grouped by project	Topics addressed
1. Álvarez et al. (2021)	Algebra
2. Bremigan, Bremigan & Lorch (2011)	Algebra
3. Buchbinder & McCrone (2018, 2020); Buchbinder (2018)	Number Theory, Geometry, Use of conditionals
4. Hauk, Hsu, & Speer (2017, 2018)	Algebra, Abstract Algebra, Geometry
5. Heid, Wilson, & Blume (2015)	Algebra, Abstract Algebra, Geometry, Statistics, Proof by induction
6. Lai & Hart (2021); Hart & Lai (2021), Aubrey et al. (2021); Casey et al. (2021a, 2021b, 2021c); Alibegović & Lischka (2021a, 2021b, 2021c); Anhalt et al. (2021a, 2021b, 2021c)	Abstract Algebra, Algebra, Geometry, Mathematical Modeling, Statistics
7. MAA META Math (2020a, 2020b, 2020c, 2020d, 2020e, 2020f, 2020g, 2020h, 2020i)	Abstract Algebra, Calculus, Discrete Mathematics, Proof, Statistics
8. Sultan & Artzt (2011); Artzt et al. (2011)	Algebra, Geometry, Statistics
9. Usiskin et al. (2003); implementation of using this textbook in Winsor (2009)	Algebra, Geometry, Abstract algebra
10. Wasserman & McGuffey (2021); Weber et al. (2020); Fukawa-Connelly et al. (2020), Wasserman et al. (2022); McGuffey et al. (2019); Wasserman et al. (2019)	Real analysis

Results

In all 10 projects, we found evidence of representations of practice and approximations of practice. Within representations of practice, we found 5 aspects of teaching context constituting the representation: student contributions (that is, depictions of student talk, work, or thinking, where teachers are not explicitly described); teacher-student interactions; written curriculum and assessments (such as actual or hypothetical textbook or lesson plans); personal experience with secondary-level tasks (where teachers are asked to notice features of their own experiences solving a secondary-level task); and an educator's modeling of a teaching practice (where the experience of the task becomes an object to notice). Within approximations of practice, we found 4 teaching practices that PSMTs were asked to simulate: responding to students' mathematical contributions; evaluating students' work; explaining content; and teaching a full lesson (where PSMTs were either teaching a full lesson of the advanced mathematics course in which teachers were enrolled [Artzt et al., 2011; Winsor, 2009], or teaching a lesson to secondary students [Buchbinder & McCrone, 2020]).

We now turn to frames within representations and approximations of practice. Evaluating student contributions was by far the most prevalent structure for engaging with representations of practice, and evaluating teacher contributions (such as appraising a described secondary teacher's teaching moves) was the second most common. A standard format for evaluating student contributions was showing teachers a sample of student work or dialogue, and then asking teachers to identify strengths and weaknesses of the mathematics shown (e.g., Bremigan et al., 2011; Casey et al., 2021a; Hauk et al., 2017, 2018; META Math 2020a; Sultan & Artzt, 2011; Wasserman et al., 2022). See Figure 1 for an example. Across the multiple secondary education projects reviewed, the majority of approximations featured the practices of responding to students' mathematical contributions, evaluating students' work, and explaining content (e.g., Álvarez et al., 2020b; Alibegović & Lischka, 2021b; Aubrey et al., 2021; Bremigan et al., 2011; Buchbinder & McCrone, 2020; Hauk et al., 2017; MAA META Math, 2020b; Sultan and Artzt, 2011; Wasserman et al., 2022). For instance, the Capstone Math materials provided teachers with a set of student work samples and asked teachers to work in groups to "sort the work into categories that represent different ways of thinking and/or difficulties" (Hauk et al., 2017, p. 9).

Two parts of [a student's] work on a task are shown.
(i) Name things [that this student] has done well when analyzing the data.
(ii) Name things [that this student] needs to work towards understanding about analyzing univariate quantitative data.

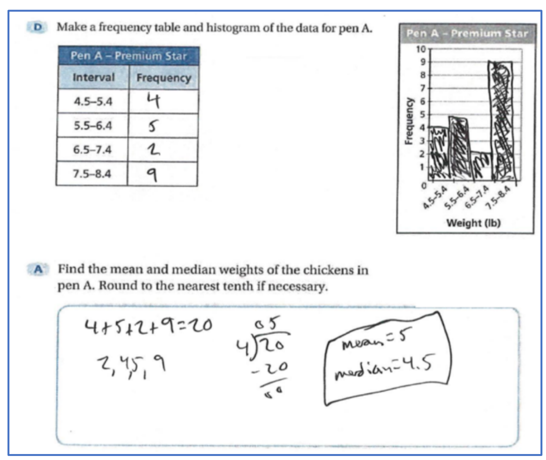

Figure 1. Example approximation of practice with embedded teaching context (Casey et al., 2021a, p. 175).

Across the reviewed materials, we saw a dominance of the inclusive sense-making frame, where mathematics is about making connections across ideas. However, within representations and approximations of practice, the secondary-level tasks tended to be closed rather than open. While this may a practical consideration for the curriculum developers, it also means that when PSMTs are asked to simulate teaching practice, they engage with an exclusionary frame. In multiple approximations of practice, an incorrect response was juxtaposed with a correct response. This juxtaposition may feed into an exclusionary hierarchical ability frame where some students are positioned as helpers and others as in need of help.

For brevity, we discuss only one example here in depth, that shown in Figure 1. The approximation of practice shown potentially supports an inclusive multidimensional frame of mathematical ability mathematics. In this example, PSMTs are asked to name strengths of student work. However, PSMTs may also gravitate toward correctness or incorrectness of the student work, which aligns with the exclusionary frame of mathematics as a fixed body of knowledge. On the other hand, within the materials reviewed, we saw multiple instances of representations and approximations which explicitly asked PSMTs to evaluate student work in terms of selecting which was the best work, or being presented with work that was incorrect and being asked to articulate what is incorrect (without asking PSMTs to articulate what may be worth building on). One set of materials contained prompts that asked PSMTs to rate solutions as "high, medium, or low". While assessing mathematical work for mathematical correctness and incorrectness is a skill that teachers need to have, only turning to this skill in engaging with student work leads to an exclusionary frame of mathematics and mathematical ability.

Conclusion

How secondary teaching is represented impacts the opportunities that PSMTs have to engage with inclusive frames of mathematics and mathematical ability. Representations and approximations may also reinforce exclusionary frames of mathematics and mathematics ability within PSMTs. In our review, we found examples of representations and approximations of practice that may foster an inclusive frame of mathematics. At the same time, these representations and approximations of practice did not explicitly support reframing away from exclusionary frames. Supporting PSMTs in inclusive framing of mathematics and mathematical ability is ongoing work that will require potential revision of existing materials (including our own) and broadening constructions of what counts as representations and approximations of secondary teaching practice in advanced mathematics courses that PSMTs experience.

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References

* denotes reviewed citations

Álvarez, J. A., Arnold, E. G., Burroughs, E. A., Fulton, E. W., & Kercher, A. (2020a). The design of tasks that address applications to teaching secondary mathematics for use in undergraduate mathematics courses. *The Journal of Mathematical Behavior, 60*, online first.

*Álvarez, J. A., Jorgensen, T., & Beach, J. (2020b). *Using multiple scripting tasks to probe preservice secondary mathematics teachers' understanding*. Paper presented at the 14th International Congress on Mathematics Education, Shanghai, China.

Artzt, A. F., Sultan, A., Curcio, F. R., & Gurl, T. (2011). A capstone mathematics course for prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *15*(3), 251-262.

*Aubrey, J., Patterson, C., Hart, J., & Tuttle, J. (2021). *MODULE(S^2): Algebra for secondary mathematics teaching, Module 3: Fields and number systems.* Mathematics Teacher Education-Partnership.
Baldinger, E. E. (2018). Learning mathematical practices to connect abstract algebra to high school algebra. In N. Wasserman (Ed.), *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers* (pp. 211-239). Springer.
*Casey, S., Ross, A., Maddox, S., & Wilson, M. (2021b). *MODULE(S^2): Statistical knowledge for teaching, Module 2: Statistical inference.* Mathematics Teacher Education-Partnership.
*Casey, S., Ross, A., Maddox, S., & Wilson, M. (2021c). *MODULE(S^2): Statistical knowledge for teaching, Module 2: Statistical association.* Mathematics Teacher Education-Partnership.
Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. American Mathematical Society and Mathematical Association of America.
Cox, D. C., Chesler, J., Beisiegel, M., Kenney, R., Newton, J., & Stone, J. (2013). The status of capstone courses for pre-service secondary mathematics teachers. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, *4*, 1-10.
Even, R. (2011). The relevance of advanced mathematics studies to expertise in secondary school mathematics teaching: Practitioners' views. *ZDM Mathematics Education, 43*, 941-950.
*Fukawa-Connelly, T., Mejia-Ramos, J. P., Wasserman, N., & Weber, K. (2020). An evaluation of ULTRA: An experimental real analysis course built on a transformative theoretical model. *International Journal of Research in Undergraduate Mathematics Education, 6*(2), 159-185.
Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *6*(4), 361-393.
Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055-2100.
*Hart, J. & Lai, Y. (2021). *MODULE(S^2): Algebra for secondary mathematics teaching, Module 2: From numbers, to powers, to logarithms.* Mathematics Teacher Education-Partnership.
*Hauk, S., Hsu, E., & Speer, N. (2017). *Transformational geometry teaching notes.* Capstone Math Project.
*Hauk, S., Hsu, E., & Speer, N. (2018). *Covariation and rate: Flag and bike teaching notes.* Capstone Math Project.
*Heid, M. K., Wilson, P. & Blume, G. W. (Eds). (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations*. Information Age Publishing.
Hoffman, A. & Even, R. (2019). The contribution of academic mathematics to teacher learning about the essence of mathematics. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 360-367). Pretoria, South Africa: PME.

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education, 48*(5), 488-519.
Lai, Y. (2019). Accounting for mathematicians' priorities in mathematics courses for secondary teachers. *The Journal of Mathematical Behavior*, *53*(4), 164-178.
*Lai, Y. & Hart, J. (2021). *MODULE(S^2): Algebra for secondary mathematics teaching, Module 1: Concepts of relations and functions.* Mathematics Teacher Education-Partnership.
Lai, Y., & Jacobson, E. (2018). Implications of pedagogical context for eliciting pedagogical content knowledge. *For the Learning of Mathematics*, *38*(2), 28-33.
Lai, Y., Strayer, J.F., Ross, A., Adamoah, K., Anhalt, C.O., Bonnesen, C., Casey, S., Kohler, B., & Lischka, A.E. (2023). Enhancing prospective secondary teachers' potential competence for enacting core teaching practices---through experiences in university mathematics and statistics courses. *ZDM-Mathematics Education*, available online. https://doi.org/10.1007/s11858-023-01485-4
Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean?. *Journal of Teacher Education*, *61*(1-2), 21-34.
Lischka, A., Lai, Y., Strayer, & J., Anhalt, C. (2020). MODULE(S^2): Developing mathematical knowledge for teaching in content courses. In A. E. Lischka, B. R. Lawler, W. G. Martin, & W. M. Smith (Eds.), *The mathematics teacher education partnership: The power of a networked improvement community to transform secondary mathematics Teacher Preparation* (AMTE Professional Book Series, Volume 4, pp. 119-141). Information Age Publishing.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020a). *Groups of transformations: Abstract (Modern) Algebra I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020b). *Logarithms and isomorphisms: Abstract (Modern) Algebra I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020c). *Solving equations in* $\boldsymbol{Z}_n$*: Abstract (Modern) Algebra I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020d). *Finding inverse functions and their derivatives: Calculus I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020d). *Finding inverse functions and their derivatives: Calculus I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020e). *Newton's Method: Calculus I.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020f). *Foundations of divisibility: Introduction to proof or discrete mathematics.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020g). *The binomial theorem: Introduction to proof or discrete mathematics.* The MAA META Math Annotated Lesson Plans.

*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020h). *Developing a margin of error through simulation: Statistics.* The MAA META Math Annotated Lesson Plans.
*Mathematical Education of Teachers as an Application of Undergraduate Mathematics. (2020i). *Variability: Statistics.* The MAA META Math Annotated Lesson Plans.
*McGuffey, W., Quea, R., Weber, K., Wasserman, N., Fukawa-Connelly, T., & Mejia-Ramos, J. P. (2019). Pre- and in-service teachers' perceived value of an experimental real analysis course for teachers. *International Journal of Mathematical Education in Science and Technology, 50*(8), 1166-1190.
Miles, H., Huberman, A. M., & Saldaña, J. (2018). *Qualitative data analysis: A methods sourcebook, 3rd edition.* Sage Publications.
Murray, E., Baldinger, E.E., Wasserman, N., Broderick, S., White, D. (2017). Connecting advanced and secondary mathematics. Issues in the Undergraduate Mathematics Preparation of School Teachers, Vol. 1. https://eric.ed.gov/?id=EJ1151024
*Sultan, A., & Artzt, A. F. (2010). *The mathematics that every secondary school math teacher needs to know*. Routledge.
*Usiskin, Z., Peressini, E., Marchisotto, A., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective*. Prentice-Hall
*Wasserman, N., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS*, *27*(6), 559-578.
*Wasserman, N., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' ''attention to scope.'' *Journal of Mathematics Teacher Education, 22*(4), 379-406.
Wasserman, N., Fukawa-Connelly, T., Weber, K., Mejia-Ramos, J. P., & Abbott, S. (2022). *Understanding analysis: Connections for secondary mathematics teachers*. Springer.
Wasserman, N., & McGuffey, W. (2021). Opportunities to learn from (advanced) mathematical coursework: A teacher perspective on observed classroom practice. *Journal for Research in Mathematics Education, 52*(4), 370-406.
*Weber, K., Mejia-Ramos, J. P., Fukawa-Connelly, T., & Wasserman, N. (2020). Connecting the learning of advanced mathematics with the teaching of secondary mathematics: Inverse functions, domain restrictions, and the arcsine function. *Journal of Mathematical Behavior, 57(1)*, 100752.
*Winsor, M. S. (2009). One model of a capstone course for preservice high school mathematics teachers. *PRIMUS*, *19*(6), 510-518.

To Do Calculus, We Need to Specify R: Lecture Meta-stories & their Underlying Assumptions.

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This papers examines the stories two instructors tell about why formally defining real numbers in the context of a Real Analysis course is a desirable objective. While the two stories exhibit some variations, the analysis shows that both narratives relied on the following constitutive elements: (1) calculus is a desirable activity, (2) students' current understanding of R is lacking, and (3) to use R for calculus, there is a need to specify it precisely. I discuss how these assumptions can be misrepresenting of practice and experienced as problematic impositions on students.

Keywords: real analysis, real numbers, meta-mathematics, narratives, values.

The stories we tell about the nature and purpose of mathematical practice matter. They frame mathematical activity in and out of the classroom, render it sensible (or not), and help delineate what actions are deemed appropriate, valuable or even feasible in a given context (Schoenfeld, 1989). Stories about math can be problematic. Many mainstream stories, such as "there is only one math" (Hersh, 1991) or "math is a young man's game" (Barany, 2021), both misrepresent disciplinary practice and alienate students by constructing a world that few can see themselves in. One context for meta-mathematical story-telling is introductory proof-based university courses, such as Real Analysis (RA). In such courses, instructors may feel the need to explicitly explain and justify the course and its new way of doing math.

RA is a challenging course in the undergraduate curriculum, largely due to the novelty of its proof-oriented epistemic game. Dawkins and Weber (2017) suggested that the norms and practices of this epistemic game are difficult to learn in part because the underlying values and purposes these norms uphold are not made explicit in instruction. However, we have little empirical evidence about whether and to what extent instructors address such meta-issues in lectures. This paper addresses this issue, by examining how instructors describe and justify real analysis in the first lectures of their RA courses. In particular, I address the following research question: What meta-stories about the nature and purpose of RA do instructors tell in lectures?

This study is informed by sociocultural theories (Wertsch, 2012). Mathematics is conceptualized as a discourse that involves telling stories (Sfard, 2008). Here, I focus on meta-stories, which I define as stories *about* mathematical practice. I refer to these as stories, rather than instructors' beliefs, to highlight their contingent and situated nature.

Methods

This study is part of a larger video-based micro-ethnography (Derry et al., 2010; Erickson, 1992) of undergraduate RA lectures taught in Fall 2020 at a large public research university in the US. All lectures were delivered online, over Zoom, due to covid-19. Primary data for the larger study include video recordings of all lectures, and auxiliary documents such as lecture notes. Four instructors (Alex, Cai, David and Emmett) participated in the study. All four worked as research mathematicians at the time of data collection. Alex, Cai and David were early career scholars and Emmett was a senior faculty member in the department. For this study on meta-stories, I consider only the first lecture of each instructor (Alex, David & Emmett, 50 min; Cai, 90 min).

Data Analysis

Data analysis proceeded in two steps. First, I generated secondary data through transcription, segmentation and descriptive coding of episodes. Then, I coded episodes for meta-talk, consolidated into story themes, and iteratively constructed summary descriptions for each theme.

Transcription, segmentation & descriptive coding. Each of the introductory lecture videos was transcribed for talk and salient gestures. Transcripts were segmented into short episodes, of 1 min length or less. Segmentation was based on "natural" transitions. These included linguistic markers (e.g. "So" and "Okay"), and shifts in topic or inscriptional focus. Each 1 min episode was given a title that served as its descriptive summary, using a mixture of content and in-vivo coding (Saldaña, 2021). I then grouped the short 1 min episodes into larger coherent wholes, according to the overall type of classroom activity (e.g. if episodes 3-8 were all part of a single proof, they were grouped together under the heading "proof of …"). This multi-step process resulted in a hierarchically structured outline of each lecture's transcript (Erickson, 1992). The structured transcripts were the data sources used for subsequent analysis.

Coding meta-talk & meta-story summaries. Meta talk appeared in lectures in two ways. There were entire episodes devoted to meta talk, and there were shorter instances of meta-talk (e.g. a single sentence) that occurred in the midst of an episode with a different focus. The coding process was inductive and proceeded as follows. First, I flagged episodes of meta-talk that addressed either what the course or the sub-field of RA are, or why they are worth studying or doing. To determine whether an episode or utterance counted as meta-talk, I relied on explicit linguistic indicators such as "our goal in this course" and "Real Analysis is." I then went through each episode that was flagged for meta talk, and created sub-codes to capture different story types. Shifts to different stories were often marked by the instructors themselves (e.g. "another goal of this course is…"). Through an iterative process of refinement, I delineated distinct categories of meta-stories about RA that together account for all meta-talk episodes and instances. With these meta-story categories, I went through the structured transcripts again, looking for confirming and disconfirming evidence and more instances of meta-talk. This process led to further refinement of the codes and consolidation in the form of story summaries.

Findings

All four instructors devoted class time to meta-talk about the nature and purpose of RA during their first lecture. Across the data set, I identified the following five meta-stories about RA:

1. To do calculus, we need to specify R (Real numbers). Our current understanding of R is vague. In RA, we define R precisely and build calculus from it.
2. RA is calculus with proof. Since calculus is known, RA is good for learning to prove.
3. Calculus is a tool that sometimes breaks. RA is a theory of how the tool work. It is good to learn the theory for future tool use and new tool development.
4. Mathematicians made mistakes in calculus and found solution in rigor. We follow them.
5. RA is a theory of connections between calculus and more fundamental topics. It is good because it makes calculus simpler and more elegant.

The table below (figure 1) shows the presence or absence of the five narratives in each of the instructor's lectures. In this paper, due to space limitations, I elaborate only on the first story: "to do calculus, we need to specify R." In the remainder of the findings section, I will describe how Emmett and David each constructed a version of this story in their first lecture, and then compare their version across key underlying assumptions: (1) calculus is a desirable activity, (2) students' current understanding of R is lacking, and (3) to do calculus, there is a need to specify R.

	Alex	Cai	David	Emmett
S1: precise R			precise R	precise R
S2: learn to prove	learn to prove	learn to prove	learn to prove	
S3: fix calculus	fix calculus	fix calculus		
S4: follow community		follw community		
S5: connections	connections			

Figure 1: Appearance of meta-stories in each instructor's first lecture.

Meta-story 1: To Do Calculus, We Need Precise R.

Emmett's meta-story. Emmett discussed the two constitutive terms of the course title, 'real' and 'analysis,' separately. Starting with 'analysis,' he offered the following definition:

> … an informal and incomplete definition of analysis is that analysis is that part of mathematics in which limits are used to solve problems. So, the key concept in analysis is that of a limit. I would say maybe the second concept is that of an inequality. But maybe the most fundamental concept is the limit. And that's really what this course is about. Limits and closely related concepts. (episode 10)

In the above quote, Emmett defined analysis as a subfield of math, characterized by its focal concepts (limits, inequality) that are framed as tools for solving problems.

Next, he shifted attention to the other part of the title, namely, the adjective 'real.' He posed the question "so, what about real numbers?" and shared a slide with the following famous quote from the German mathematician Leopold Kronecker: "God made the whole numbers; humans made the rest." Emmett interpreted the distinction Kronecker's quote makes between god-made and man-made numbers as implying that the real numbers are "not as fundamental as the whole numbers," and declared that the course will take that "point of view."

On the next slide, he elaborated on what makes real numbers less fundamental. He asked "what is a real number?" rhetorically and provided three "informal descriptions" as answers: real numbers are (1) points on the number line ("when you teach a calculus class, you draw a line on the board and you say here are the real numbers"), (2) expressible using decimals ("Any number you can write like that, positive or negative, is a real number"), or (3) possible values of a variable x in a calculus course. Emmett then explicitly framed these descriptions as inadequate:

> So that's obviously not a very rigorous notion, but that captures what we're talking about. I hope that you'll all agree with me that none of these answers are anywhere near as clear and unambiguous as the positive whole numbers. We all are confident that we can start with one, add one to it, add one to it, add one to it, and keep going. And we more or less understand what sort of objects we're getting that way. The real numbers are a bit more mysterious. (episode 12)

In the perspective offered above, familiar descriptions of real numbers (standard in calculus courses) are declared problematic because they are "informal" and not "very rigorous." The ambiguity and mystery of real numbers is justified through a comparison to the presumed clarity with which whole numbers are comprehended. According to Emmett, whole numbers as mathematical objects are clear and unambiguous; by imagining repeated addition, "we can more or less understand what sort of objects we're getting." Emmett uses this appeal to a presumably shared experience of different degrees of ontological clarity to justify assigning real and whole numbers a different epistemic status: “And in this course, we will not take basic properties of the real numbers for granted. Unlike these more basic kinds of numbers.”

In Emmett’s story, filling this ontological and epistemological void is a key problem the course aims to solve. Accordingly, constructing the real numbers in a clear an unambiguous way played a central role in “the program” Emmett laid out next:

> Instead, we're going to spend some time and we're going to use the rationals, which we've agreed, we do understand that we can assume things about. We're going to use the rationals to construct some objects. And this collection of objects will have all the properties that the real numbers ought to have. And then we'll just call these objects the real numbers. And as the course goes on, we'll use them. We'll use them, we will develop the concept of a limit. And we'll use them with, together with limits to develop the theorems that underpin calculus and to solve various other kinds subproblems. Ok, so that's the program. (episode 13)

The “program” Emmett articulated describes RA as sequential coverage of mathematical content (R, limits, calculus), where each step is ‘built’ on top of the previous one: rational numbers are used to construct real numbers, real numbers are used to develop limits, limit are used to develop calculus theorems. Analysis is situated later than real numbers in the sequential development of content. But, how are these two connected? Why are real numbers needed for doing analysis, assuming the latter is an agreed upon goal? Later in the lecture, Emmett made additional meta-comments, that can be seen as filling this gap in the story:

> … the rationals are not complete. And that, in a nutshell, is why this course is real analysis instead of rational analysis, the rationals are defective from our point of view. We want to be able to take limits, completeness, the rationals are not complete and there will be instances when we can't take limits when we want to. So the rationals don't work for us and we need the reals. (episode 34)

Here, the course’s focus on real numbers is justified by their necessity for doing limits. In this argument, we don’t care about real numbers *just* because they are unclear and complex. We care about them because we want “to take limits,” and rational numbers do not always allow it. The problem is that rational numbers, unlike real numbers, “are not complete” which makes them “defective from our point of view.”

In a later meta-comments, Emmett repeated the course’s program description, adding detail:

> … we're going to construct a set of objects that has all the properties that the real numbers ought to have. We'll just call that set the real numbers, and then we'll use it together with limits to solve problems and to justify the foundations of calculus. OK. And the objects that we construct are called cuts, or they are called Dedekind cuts after, after a mathematician named Dedekind. (episode 43)

In this last comment, Emmett provided additional information. Namely, that the to-be constructed real numbers are “objects” called Dedekind cuts. He immediately followed this programmatic outline with a comment about the resulting ontological status of real numbers:

> So notice that this approach sidesteps the basic question of what the real numbers actually are. And, you know, that's some sort of metaphysical question that doesn't really have a mathematical meaning. So, we're simply not going to go there. We're simply going to construct something and we'll call it the real numbers. Because of the theorem I told you at the end of the last set of slides, that's a reasonable approach. Any fully mathematical question you can ask will have the same answer for our real numbers, as for anybody else's real numbers. (episode 43)

In this last quote, Emmett clarified that the course's "program" will not provide an answer to the ontological question of "what the real numbers actually are." Despite the fact that he used the (presumed) ontological ambiguity of real numbers to motivate the program at the beginning of the lecture, in this later comment, he framed the ontological question as "metaphysical" and devoid of "mathematical meaning." To justify this dismissal of the original question, he referenced the theorem that any two complete, Archimedean, ordered fields are isomorphic.

Based on this meta-talk, I summarized Emmett's narrative as follows: *We are interested in doing analysis, which is using limits to solve problems. To do limits, we need real numbers. But, we don't know what real numbers are in a clear and unambiguous way. We do, however, clearly understand natural, whole and rational numbers. So, the course starts with constructing [something we call] real numbers from rational numbers, and develops calculus from that.*

David's meta-story. David began his meta-story by rhetorically posing the question: "what's the goal of this course?", and immediately providing the following answer:

> This is introduction to analysis or real analysis. So what we're <u>really</u> trying to do is come up with a set of tools to rigorously study what happens on… what the real numbers are and how the real numbers behave. (episode 9)

The above description features some of the themes we saw in Emmett's presentation. David claimed that the course's objective ("what we're really trying to do") is to develop "tools." The tool metaphor is similar to Emmett's framing of limits as something that can be used to solve problems. However, unlike in Emmett's story where the nature of the problems remains underdetermined, in David's description the objective is characterized as the exploration of a mathematical reality in which real numbers are the landscape and focal characters: "what happens on [the number line]" "what real numbers are," "how they behave." At this point in David's story, understanding phenomena pertaining to real numbers is the goal, whereas in Emmett's story, it was a necessary step in a 'program' whose ultimate objective was calculus, i.e. 'using limits to solve problems'. But, this was just the start of David's story. He continued:

> … the core ideas in this course, and the core themes we are going to explore are. Well, I guess one core theme is essentially approximation. And the real question we have, right? You know you have a vague notion, probably sort of what a real number is, but if you <u>actually</u> want to specify what a real number is, it can be a little tricky. (episode 9)

David mentioned approximation as a theme but did not connect it to the declared focus on real numbers. It seemed as though he was about to list several core ideas, not just approximation, but he quickly shifted to what he called "the real question," which is that of specifying real numbers. Similarly to Emmett, David problematized real numbers by invoking a binary opposition between two epistemic stances: having a "vague notion" versus "actually" specifying. This binary allows seeing the specification of "what a real number is" as a problem to be solved.

David proceeded to justify why specifying real numbers is useful. Recall that in Emmett's story the only justification is the purported experience of ambiguity (for David, "vagueness") with familiar descriptions of real numbers. David's narrative features more rationales. Specifically, he claimed that specifying real numbers is necessary for more advanced calculus:

> … if we want to start doing complicated things like calculus or analysis of things in higher dimensions or working on- or doing calculus on a surface that isn't actually the real line. If you want to do calculus on the surface of a sphere, for example, it gets a little bit tricky and it becomes important to know precisely what sort of thing you're talking about. And so the goal of this class is to sort of lay that groundwork that dealing with the real numbers are, so that we can then in the future expand it to talk about more interesting, more interesting topics. (episode 10)

Similarly to Emmett, David invoked a building-foundation metaphor ("lay the groundwork") to interpret the significance of 'specifying real numbers' as a base step in the development of a hierarchically organized system of mathematical knowledge. However, unlike in Emmett's story where the end point was 'regular calculus,' here the motivating horizon is more advanced versions of calculus – calculus in higher dimensions or on surfaces such as a sphere – that are "interesting," yet "tricky" and "complicated." Thus far in David's story it is the complexity of these more advanced topics, rather than any difficulty with regular calculus, that necessitates an ontological specification of R ("it *becomes* important to know precisely what sort of thing you're talking about"). He developed this idea further in relation to infinite dimensional calculus:

> You might be curious. Can you do something that looks like calculus in infinitely many dimension? I mean, you've probably taken- I'm sure one of the courses you've taken as a prerequisite to this is a multivariable calculus class that probably did two or three, maybe n-dimensional calculus. But can you have meaningful kinds of calculus in infinitely many dimensions? And the answer is yes. But developing that, there are many, many subtleties to get through to develop that. So we're going to start somewhere to have something to build on. And so we really want to make these ideas of <u>what</u> makes the real numbers, what they are, precise. (episode 10)

We see here a repetition of the same theme. Doing more advanced calculus (in this case, infinite dimensional) is subtle and complex. Mathematical knowledge is a hierarchical structure. To be able to (in the future) cope with this complexity of more advanced levels, it is necessary to do preparatory work at the base ("start somewhere to have something to build on"). In particular, the prep work involves specifying real numbers. Unlike Emmett, David did not provide an explanation for why real numbers, specifically, is the "somewhere" where one "starts."

I summarized David's narrative as follows: *We are interested in doing advanced calculus, such as calculus on surfaces, and calculus in multiple and infinite dimensions . To do advanced calculus, we need precise real numbers. Our current understanding of real numbers is vague. So, the course focuses on specifying real numbers, to have something to build on.*

Summary. I compare Emmett and David's meta-stories across constitutive assumptions.

The first assumption is (1) students are interested in doing calculus. Emmett treated interest in calculus as a taken-for-granted goal, both in defining analysis and in positioning calculus theorems and limits as the end goals of content development. In David's meta-story, the objective is not regular calculus, but rather more advanced versions, such as calculus on surfaces and in infinite dimensions. But, like for Emmett, the interest in such topics is assumed and positioned as an ultimate goal. A RA instructor may reasonably assume that a motivation for calculus – What kinds of problems do limits help solve? Why should we care? – was addressed in previous courses. However, David's framing of interest in infinite dimensional calculus as something one is naturally curious about after doing finite dimensional calculus can be problematic. It is historically inaccurate and potentially alienating, as few students are likely to see themselves reflected in this positionality of wondering about artificial extensions of theory.

The second assumption is that (2) students' current understandings of real numbers are lacking. Emmett conveyed this by listing descriptions of real numbers students are familiar with and framing them as "not very rigorous," unclear, and ambiguous. David, similarly, claimed that students only "have a vague notion … of what a real number is." In both stories, the negative epistemic stance is not fully justified. Emmett's use of phrases such as "obviously" and "I hope that you'll all agree," actually highlights the absence of an explanation. Indeed, a negative epistemic and ontological stance toward real numbers is not self-evident. Students (and various professionals, including mathematicians) have been successfully using 'informal descriptions' of real numbers for a long time (e.g. in calculus) without feeling the need to specify them in the way Emmett and David suggested. To date, in many (if not most) contexts of doing calculus, people unproblematically rely on representations of real numbers which Emmett and David labeled as ambiguous and vague. Thus, the offered stance can be experienced as an imposition.

The final assumption is that (3) to do calculus, R needs to be specified. Emmett did not provide explicit justification for this claim, but the programmatic description he offered relies heavily on building-foundation metaphors for mathematical knowledge, and within such a metaphorical conceptualization of math it may seem as self-evidently true that concepts need to be 'built on solid ground' to be viable. David's story incorporated building-foundation metaphors too (e.g. "lay the ground work"). However, David offered an additional rationale by positioning the specification of R, not as something done for its own sake, or as something needed for doing regular calculus (that may directly contradict students' experiences of doing calculus without specifying R), but rather as something necessary for doing more advanced calculus, one that students have not yet seen. This story defers a compelling motivation for specifying R to a future context (e.g. infinite dimensional calculus). The fact that students are tasked with 'taking David's word for it' further highlights the absence of an accessible rationale.

Discussion

Mainstream stories about math are rife with idealizations of the discipline that both alienate many learners and do not tell the full story of what the practice is like and what, and for whom, it is good for (Hersh, 1991; Wagner, 2022). This paper aims to contribute to our understanding of how such idealizations are constructed in the gatekeeping educational context of RA lectures. Critically examining the assumptions instructors' meta-stories rely on and how these assumptions relate to past and current professional practice and students' past curricular experiences, can help us craft meta-stories that are both realistic and more compelling. The analysis also illustrates that instructors *do* address axiological issues in lectures (Dawkins & Weber, 2017), though the effect of their stories on students requires further examination.

References

Barany, M. J. (2021). "A Young Man's Game": Youth, gender, play, and power in the personae of midtwentieth century global mathematics. In K. Niskanen & M. J. Barany (Eds.), *Gender, Embodiment, and the History of the Scholarly Persona: Incarnations and Contestations* (pp. 21–53). https://doi.org/https://doi.org/10.1007/978-3-030-49606-7_2

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123–142. https://doi.org/10.1007/s10649-016-9740-5

Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Erickson, F., Goldman, R., … Sherin, B. L. (2010). Conducting video research in the learning sciences: Guidance on selection, analysis, technology, and ethics. *Journal of the Learning Sciences*, *19*(1), 3–53. https://doi.org/10.1080/10508400903452884

Erickson, F. (1992). Ethnographic microanalysis of interaction. In M. D. LeCompte, W. Millroy, & J. Preissle (Eds.), *The Handbook of Qualitative Research in Education* (pp. 201–225). New York: Academic Press.

Hersh, R. (1991). Mathematics has a Front and a Back. *Synthese*, *88*, 127–133. https://doi.org/10.1007/BF00567741

Saldaña, J. (2021). *The coding manual for qualitative researchers*. SAGE Publications Ltd.

Schoenfeld, A. H. (1989). Explorations of Students ' Mathematical Beliefs and Behavior. *Journal for Research in Mathematics Education*, *20*(4), 338–355.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge University Press.

Wagner, R. (2022). Mathematical consensus: a research program. *Axiomathes, 32*, 1185–1204. https://doi.org/10.1007/s10516-022-09634-2

Wertsch, J. V. (2012). *Mind as Action*. Oxford university press.

Lesson Planning Practices of Undergraduate Mathematics Instructors: What Do We Know?

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We surveyed university mathematics instructors across the United States about their planning practices. We sought to better understand the complexities involved in the work that goes into preparing for instruction at the university level. Research indicates that the quality of instructors' lesson plans can be linked to the quality of their instruction (Akyuz et al., 2012). Instructors self-reported as either using inquiry-based practices, lecture-based practices, or an even mix of both. Instructors that indicated using both inquiry-based and lecture-based practices spend more time lesson planning, but often feel less supported by the lesson plans they create. These instructors attempt to do practices that both the inquiry-based instructors do (i.e., spend time preparing for and accounting for student thinking) and lecture-based instructors do (i.e., ensuring a clear understanding of the mathematics for themselves). We discuss implications for instructors and the undergraduate mathematics education community.

Keywords: lesson planning, university mathematics instructors, teaching practices

For decades researchers have investigated the teaching and learning of mathematics. Much of the focus has been on understanding how students think and learn particular topics or ideas, and what in-the-moment instructional practices support student learning. This work is crucial, and more recently some have pointed to the importance of carefully considering this type of information during the instructional planning process (e.g., Akyuz et al., 2012; Stephan et. al., 2017). Much of this work lives in the context of K-12 education, and though it is largely relevant to university education, there are some added complexities. More specifically, pedagogical freedom at the university level means not all instructors teach the same topics or even teach them the same way. This freedom also means individual instructoprs can significantly change pedagogies from semester to semester. Little is known, however, about how instructors build their lesson plans, what they consider when doing so, and how their pedagogical decisions impact their planning practices. This research paper describes an investigative journey into the planning practices of undergraduate mathematics instructors, aiming to better understand the complexities involved in the work that goes into preparing for instruction at the university level. We also explore some similarities and differences in these practices across broadly defined instructional categories: Inquiry-Based, Lecture-Based, and an Even Mix of Both.

Literature Review

While there is some literature about planning practices for K-12 teachers as either pre-service or in-service teachers, there is a lack of literature regarding the lesson planning practices of university mathematics instructors. For example, O'Donnell and Taylor (2007) investigated the relationship between preservice mathematics teachers' focus on student thinking in lesson analysis and lesson planning tasks. Their results highlighted a pedagogical lesson planning format consistent with Shulman's (1987) model of pedagogical reasoning and action which includes comprehension, transformation, instruction, evaluation, reflection, and new comprehensions. This model is used to theorize the knowledge required to develop the knowledge of teaching. Additionally, Amador (2016) investigated the role of teacher noticing during lesson design. Amador's findings emphasize the importance of supporting teachers through professional development opportunities, specifically focusing on student outcomes, student mathematical understanding, and teacher noticing. Further, Amador (2016) raises questions about how to better support these teachers but also how to further support teachers more broadly as they work in the space between lesson planning and enactment. Additionally, this thinking can also be applied to middle grades educators. For instance, Akyuz et al. (2012) aimed to extract the planning practices middle school mathematics teachers used to create adequate assessments, introduce important ideas, and choose tasks that are appropriate given the objectives of lesson materials. The results mention five core practices that are the foundation of the development of effective planning: preparation, reflection, anticipation, assessment, and revision. These practices align with Shulman's (1987) dimensions. These five practices also align with Stephan et. al.'s discussion of Lesson Imaging (2017) - where they build on Schoenfeld's (1998) work on how teachers expect their plans to be enacted in the classroom. Stephan et. al. outline a visualization process through which K-12 STEM teachers can develop effective lessons by unpacking the goals, imaging the launch of the lesson, anticipating student thinking and reasoning, and then imagining how they will use that thinking and reasoning to support meaningful discussions. Though the previously mentioned literature focuses primarily on K-12 planning, we argue that the same notions may apply to researching the lesson planning practices of university instructors. However, literature about university instructor's lesson planning practices is limited, as the field continues to grow.

Undergraduate instructors' pedagogical decisions in the classroom are represented in literature. Uniquely, undergraduate mathematics educators have the choice to teach in certain ways, often in a lecture format (Fukawa-Connelly et al., 2016). This freedom to choose could impact lesson planning practices. Johnson et al. (2018) stated that the "distinction between lecture and non-lecture pedagogy is not clear cut, especially when looking at how instructors self-identify" (p. 256). In an attempt to investigate why instructors choose to lecture, Johnson (2018) detailed a survey of national abstract algebra instructors at masters and doctorate-granting institutions. Ultimately, Johnson provided insight into factors that influence pedagogical decisions in American abstract algebra classes as well as avenues for approaching and providing better support for those interested in implementing non-lecture teaching approaches (Johnson, 2018). Johnson et al. (2013) indicated an opportunity for the research field of undergraduate mathematics education to "gain insight into the factors and considerations that matter to mathematicians," (p. 2) specifically in teacher identity and stance towards instructional materials. Lastly, research indicates that the quality of instructors' lesson plans can be linked to the quality of their instruction (Akyuz et al., 2012). Thus, understanding undergraduate instructors' lesson planning practices can also shed some light on their instructional practices.

Methods

Our study investigated the lesson planning practices of university instructors through a survey with both quantitative and open-ended items.

Data Collection

Survey creation. We designed a survey using Akyuz et al. (2012)'s lesson planning practices as the overall dimensions that characterize lesson planning activity: preparation, reflection, anticipation, assessment, and revision. In addition to these dimensions we asked questions about different classroom environments, teaching styles, and university characteristics. In light of the discussion by Johnson and colleagues (2017) concerning reasons for making certain pedagogical choices, we were particularly interested in how instructors defined success in terms of their lesson planning. In this work, we will focus primarily on the preparation and reflection dimensions. Preparation outlines any work prior to the start of instruction. Generally, it includes activities such as creating a learning trajectory, designing an instructional sequence, studying relevant literature, and considering the big ideas of the unit (Akyuz et al., 2012). Reflection involves taking a look into past teaching experiences and classroom interactions to assess quality instruction as well as evaluation of unexpected situations, misconceptions of students, and improvement of instruction (Akyuz et al., 2012).

Survey description and questions. The survey included 11 demographic questions such as instructor gender identity, university type, average class size, and how long they had been teaching. We also asked quantitative and qualitative questions about teaching styles. For example, we asked, "Which of the following best typifies your teaching style?" (with choices of inquiry-based, lectured-based, or a mix) and "Briefly describe why you teach this way. Please be detailed." Then each of the five main dimensions had both quantitative and qualitative items. For preparation, we asked questions such as "On a scale of often, sometimes, rarely, never, please indicate how frequently the following activities are part of how you prepare for a lesson. - Reading relevant education research on the topic (e.g., student learning of the topic)." The anticipation stem contained questions such as "Consider preparing for lessons. On a scale of strongly agree to strongly disagree, please select if you agree with the following statements. - I consider what might happen during a lesson." Questions such as "When teaching, approximately how often do the following occur on a scale of often, sometimes, rarely, never? - I stopped to consider how the students were thinking about the content" was used to assess the reflection stem. The assessment stem asked questions such as "Consider times when you used informal assessment. On a scale of strongly agree to strongly disagree or not applicable (N/A), please select if you agree with the following statements. - I change future lesson plans if grades are poor." Finally, questions including "Consider preparing for lessons. On a scale of strongly agree to strongly disagree, please select if you agree with the following statements. - I often change the order of the content from the textbook as a means to support student learning" were used to assess the revision stem. For quantitative questions, a scale was used for all 5 stems in order to get a clear response, while still offering a range of answer options.

Qualitative questions on the survey included inquiries such as "How does the teaching style you use impact your lesson planning practices?" These questions were designed to differentiate the planning practices of university instructors with different teaching styles and provide us with more data to compare to our quantitative results.

Survey population. The sample for our survey utilized several email listservs as well as directly emailing mathematics department chairs. Listservs included Research in Undergraduate Mathematics Education (RUME), Associate of Mathematics Teacher Educators' (AMTE) Service, Teaching, and Research (STaR), and the American Mathematical Society (AMS) directory of institutions. Potential participants were told that they had to be full-time instructors and that they had to teach mathematics content classes (i.e., general education, mathematics content classes for future teachers, or mathematics majors classes). We received 65 complete survey responses.

Data Analysis

We broke the respondents into three groups based on their responses to a question asking which teaching style they best fit into. We then conducted a series of Analyses of Variance

(ANOVA) to examine the relationships between their chosen instructional style and the responses to the various survey items. Norman (2010) and Carifio and Perla (2008) note that when examining multiple likert-scale items, using a method that compares means and standard deviations (such as ANOVA) is appropriate and yields valid results. We found many statistically significant differences in the way the respondents in the three groups tended to answer the survey questions. All open ended survey items were coded by three of the researchers using thematic analysis (Braun & Clarke, 2006). All three researchers did a first pass of coding then met to discuss codes and repeat the process until a codebook was developed. Then we all coded the data to be able to describe themes from the open-ended items. All disagreements were discussed by all three coders. Lastly, the qualitative data gave us a deeper understanding of what we found from results from the quantitative data and it supported some quantitative findings.

Results

Demographics & Teaching Style

The average length a respondent had taught was 16 years. 80% of respondents worked at 4-year public universities. Likewise, 80% worked at universities that granted PhDs or Masters. 36% of respondents taught K-12 prior to their current teaching position. 25% had degrees in mathematics education and the rest in some field of mathematics. 32 respondents identified as female, 26 as male, 1 as nonbinary, and 6 that preferred not to answer.

Participants were asked “which of the following best typifies your teaching style?” 18 said inquiry-based (i.e., student centered), 15 said lecture-based (i.e., teacher centered), and 32 said an even mix of lecture and inquiry activities. We also asked respondents the following question: “When teaching, approximately how often do the following occur on a scale of often, sometimes, rarely, never? - I stop to consider how the students are thinking about the content.” Inquiry-based instructors stop significantly more often to think about student thinking than even-mixed ($p=0.026$) and lecture-based ($p<0.001$), and even-mixed stop significantly more often to think about student thinking than lecture-based ($p=0.048$). This, to us, aligns with a broad interpretation of these teaching styles. In the most basic sense, the role that student thinking plays in these teaching styles is crucial. Instructors that identify as lecture-based indicated they stopped significantly less frequently than even-mixed or inquiry-based instructors.

Lesson Planning

Respondents who identified as inquiry-based do not spend as much time making or reviewing their lesson plans; however, they feel more supported by them. When asked Q21_4: “On a scale of strongly agree to strongly disagree, rate the following statement: I use a textbook to determine how I teach material,” respondents who identified as an even mix of teaching styles said that they agree significantly more with this statement than either lecture-based or inquiry-based ($p=0.01$).

Similarly, when asked Q20_8: “On a scale of often, sometimes, rarely, never, please indicate how frequently the following activities are part of how you prepare for a lesson - Reviewing from course materials (e.g., textbook, notes)”, respondents who identified as an even mix of teaching styles reviewed course materials significantly more than inquiry-based respondents ($p=0.003$), but there was no significant difference between even mixed and lecture-based and inquiry-based and lecture-based.

However, on Q21_7 despite indicating that they do not create detailed lessons, inquiry-based respondents indicate that they feel supported by their lesson plans, significantly more so than respondents who identified as an even mix of teaching styles (*p=0.014*). There was no significant difference between other groups.

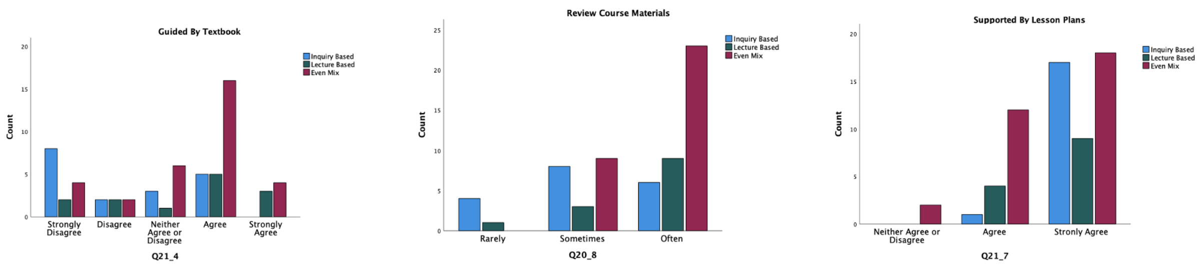

Figure 1. Bar charts showing all responses to Q21_4, Q20_8, and Q21_7.

These results also appeared in the qualitative data. Respondents were asked "How does the teaching style you use impact your lesson planning practices?" and "Please fill in the blank, and be detailed: I feel like I am prepared for a lesson when I have ____." Respondents who identified as lecture-based indicated overwhelmingly that their lesson plans are good because they know what they are going to talk about and the lesson plans are their notes (although those notes varied in format of in their head, on paper, in a PowerPoint, etc.). One participant said, "My planning process is mostly deciding which examples I want to include in class. I usually work through a few examples for each topic. The topic may be explained as I work through the first example." Inquiry-based respondents indicated on open-ended survey items that because they are sensitive to student thinking, their lesson plans are intentionally fluid to account for this. One participant said, "Because I teach in an inquiry-based style, it is important to carefully structure student interaction with each other and the course material to set the stage for their mathematical connections and discoveries." Respondents who indicated using an even mix of teaching styles said that their lesson planning notes are guided by the development of the mathematics, but they know they need to be sensitive to how students are reacting to it. One participant said,

> Rather than a lesson plan "template" that uses the same template or outline for every lesson, at the college level, once I consider the content and the students' needs for that content, then I prepare lesson materials that are most appropriate for that lesson in that moment, whether it be lecture notes, guided notes, activity/inquiry/discovery instructions for the students to engage, problems for the students to explore in problem-based learning, etc. Regardless, I typically will have in my notes (increasingly on Google Docs) a timeline with short notes about process, and more detailed notes where I see fit.

Reading research plays a role in lesson planning. Unsurprisingly, those that engage in inquiry-based teaching or an even mix of teaching styles are more interested in performing research on teaching and learning ($p<0.001$, $p<0.001$) and are more interested in reading literature on teaching and learning ($p=0.009$, $p=0.018$). However, while even mixed respondents are interested in research and anticipating student thinking, they are not doing it as often as inquiry-based respondents ($p=0.002$).

Lesson Reflection

Respondents were asked Q31_4: "On a scale of strongly agree to strongly disagree or not applicable (N/A), please select if you agree with the following statements. - I change future lesson plans if students indicate the content is not clear to them." Even-mixed instructors are more likely to change their lesson plans if students are struggling to understand than both inquiry-based and lecture-based ($p=0.015$). One even-mixed participant said they are prepared when they have:

> ...A goal in mind for that lesson. I am prepared if I have activities that will support that goal and have anticipated questions that the students may ask. I also like to have a means of measuring learning by providing closing questions or homework that I can monitor to see how the material is being understood. This measure allows me to reflect on how the next lesson will begin.

When asked Q24_4: "On a scale of often, sometimes, rarely, never, please indicate how frequently you discuss how a lesson went with colleagues" inquiry-based instructors do this significantly more than either other group ($p=0.005$, $p=0.028$).

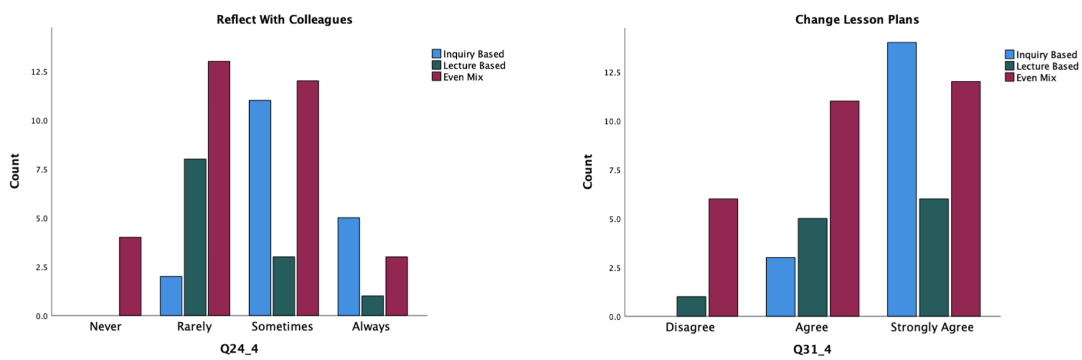

Figure 2. Bar charts showing all responses to Q24_4 and Q31_4.

Limitations

Before discussing our results, we want to acknowledge important limitations of this work. First, 65 responses, while large, is still small in terms of the power of our study. Second, as the data are self-reported, all reference to self-identifying their preferred teaching style was based on their selection of those three categories. Importantly, teaching practices are more than three categories, it is a continuum and it's not easy to distinguish between the ends of that continuum (Johnson et al., 2018). Additionally, we acknowledge that just because someone self-selected into a category, we do not have actual data from their classroom to confirm that their teaching practices in fact do align with these categories. Nevertheless, we believe our study sheds important light on the lesson planning practices of university mathematics faculty.

Discussion, Conclusion, and Next Steps

Our data indicates that different teaching styles do indeed relate to different planning practices. Respondents who identified that they use both inquiry-based and lecture-based teaching styles plan more than either inquiry-based or lecture-based instructors. However, even-mixed instructors do not feel as supported by their lesson plans compared to both inquiry-based and lecture-based instructors. This begs the question, what are the three groups including in their lesson plans to cause these differences and how do they relate to their goals?

Inquiry-based instructors do not report prioritizing doing the mathematics for themselves prior to a lesson as part of their lesson planning. Lecture-based instructors report having a strong sense of the mathematics for themselves. Whereas, even-mixed instructors seem to be pulled in both directions. They reported that they wanted to ensure they knew the mathematical progression of the content for themselves but still provided enough space to let student thinking drive the lesson. This aligns with the fact that their lesson plans and lesson planning are more involved; they appear to indicate a need to be prepared for everything (i.e., supporting the students' thinking and the mathematics).

Even-mixed instructors also indicated that they are most willing to change their lessons and lesson plans based on student reactions and their perception of how lessons are going. This further indicates the potential that even-mixed faculty are pulled in different directions. Both inquiry-based and lecture-based instructors stay their course, reporting they do not as frequently change lesson plans. Despite inquiry-based instructors reporting they reflect on their lessons more frequently than the other groups, even-mixed faculty make changes to their lessons more often. This further provides insight into why their lesson plans and lesson planning practices are more involved. Perhaps changing their lesson plans so frequently leads to less certainty in the success of lessons.

Collectively, our results indicate that the instructional goals and preferred teaching styles of instructors impacts their lesson planning practices. But more specifically, while most of our respondents indicated they used both inquiry-based and lecture-based teaching practices, that same group also indicated being the least comfortable doing so. Research on teaching practices tells us that instructors use both inquiry-based and lecture-based practices often (Johnson et al., 2018). In fact, as indicated earlier, teaching practices are on a continuum and are not dicotomous. However, regularly shifting along that continuum appears to be the most difficult in terms of preparing for those lessons.

Next Steps

In this work, we only discussed the lesson preparation and reflection (Akyuz et al., 2012) dimensions concerning how university mathematics instructors plan lessons. There is a clear indication that some faculty who try to use both inquiry-based and lecture-based teaching practices need support. Considering our results with those of Johnson et al. (2018), even-mixed instructors often feel overwhelmed by their lesson planning processes and they often change their plans as they go; more so than instructors who self-identify as lecture-based or inquiry-based. Thus, we wonder what support do these faculty need? The undergraduate mathematics education community has provided many professional supports (e.g., TIMES NSF#1431641, SEMINAL NSF#1624643), but it appears that institutional support is just as needed. The average years of teaching was 16 years (which was roughly the same for each teaching category as well) and our data did not find significant differences in terms of how supported instructors feel from their institutions. However, even-mixed instructors did indicate feeling the least supported by their institutions and departments. And these instructors are the ones doing more lesson planning.

References

Akyuz, D., Dixon, J. K., & Stephan, M. (2013). Improving the quality of mathematics teaching with effective planning practices. *Teacher Development*, *17*(1), 92–106. https://doi.org/10.1080/13664530.2012.753939

Amador, J. M. (2016). Teachers' considerations of students' thinking during mathematics lesson design. *School Science and Mathematics*, *116*(5), 239–252. https://doi.org/10.1111/ssm.12175

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, *3*(2), 77-101.

Carifio J & Perla R. Resolving the 50-year debate around using and misusing Likert scales. Med Educ. 2008 Dec;42(12):1150-2. doi: 10.1111/j.1365-2923.2008.03172.x. PMID: 19120943.

Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can math education research improve the teaching of abstract algebra? *Notices of the AMS*, *63*(3), 276-281.

Johnson, E., Caughman, J., Fredericks, J., & Gibson, L. (2013). Implementing inquiry-oriented curriculum: From the mathematicians' perspective. *The Journal of Mathematical Behavior*, *32*(4), 743–760. https://doi.org/10.1016/j.jmathb.2013.03.003

Johnson, E., Keller, R., & Fukawa-Connelly, T. (2017). Results from a survey of abstract algebra instructors across the United States: Understanding the choice to (not) lecture. *International Journal of Research in Undergraduate Mathematics Education*, *4*(2), 254–285. https://doi.org/10.1007/s40753-017-0058-1

Norman, Geoffrey. (2010). LIkert scales, levels of measurement and the "laws" of statistics. Advances in health sciences education : theory and practice. 15. 625-32. 10.1007/s10459-010-9222-y.

O'Donnell, B. & Taylor, A. (2007). Research, reflection, practice: A lesson plan as professional development? You've got to be kidding! *Teaching Children Mathematics*, *13*(5), 272–278. https://doi.org/10.5951/tcm.13.5.0272

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in education*, *4*(1), 1-94.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard educational review*, *57*(1), 1-23.

Stephan, M., Pugalee, D., Cline, J., & Cline, C. (2016). *Lesson imaging in math and science: Anticipating student ideas and questions for deeper STEM learning*. ASCD.

Taylan, R. D. (2016). The relationship between pre-service mathematics teachers' focus on student thinking in lesson analysis and lesson planning tasks. *International Journal of Science and Mathematics Education*, *16*(2), 337–356. https://doi.org/10.1007/s10763-016-9778-y

Exploration of TAMI-OP as a Professional Development Tool for Mathematics Instructors

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The VIP-Math project aimed to understand whether and how teaching practices of mathematics instructors are influenced by discussions of their classroom data visualized using the TAMI-OP classroom observation tool. We found that TAMI-OP data visualizations supported discussions that led the participating mathematics instructors to set both short- and long-term teaching goals, and to experiment with teaching practices related to at least one of their goals. Given the limitations of the participant pool, further research is recommended with a greater diversity of instructors and coaches.

Keywords: classroom observation tool, professional development, college instruction

Introduction

Research in STEM higher education has investigated a number of tools for measuring teaching, in order to study teaching practices in real time. Alongside surveys, interviews, and classroom artifacts, classroom observation is a valued tool in the education researcher's toolbox because it provides direct evidence of what teachers and students are doing and can be used to measure change in effective teaching over time (AAAS, 2013; Weston et al., 2021). Observations can also be used for evaluating teachers for retention or advancement, for providing feedback to teachers about their work, and for evaluating professional development offerings and their impacts.

For any of these distinct purposes, the observation protocol and sampling must be carefully chosen to align with the goals of the measurement and the purposes for which the data will be used (Esson et al., 2022; Hora & Ferrare, 2013; Weston et al., 2021). Classroom observation tools tend to be either segmented or holistic. Segmented tools typically describe teacher and student classroom activities by taking data across short time intervals. Holistic tools tend to use Likert-scaled items to arrive at ratings. Holistic tools are more inferential, requiring more expertise to use reliably (Weston et al., 2021), though both types require attention to features that affect the tool's validity and interrater reliability (Weston et al., 2023).

Many tools have been developed and tailored for specific kinds of classrooms, including for example the RTOP (holistic, applied to K-12 and college contexts; Sawada et al., 2022), the TDOP (segmented, for college instructors; Hora et al., 2013), the COPUS (segmented, designed for college STEM classrooms; Smith et al., 2013), the GSIOP (segmented, designed for use with graduate student teaching assistants; Yee et al., 2019), and the CRIOP (holistic, for middle school instructors; Powell et al., 2013).

One observation tool developed for college mathematics classrooms is the TAMI-OP, a segmented, descriptive tool (Hayward et al., 2017, 2018; Weston et al., 2021, 2023). The classroom observer records what both the instructor and the students are doing across each two-minute interval of class time and can record notes. The tool includes codes for modalities of active learning often seen in mathematics classrooms—students working in groups or individually, answering or asking questions, giving presentations—as well as listening to lectures (Laursen et al., 2014). Instructor codes include asking and answering questions, reviewing student thinking, moving and guiding student work, lecturing, and working problems. Thus TAMI-OP can capture teaching activities that involve students actively or more passively, and the mix of these. Designed to track changes in teaching behaviors such as may result from professional development, the TAMI-OP is deliberately behavior-focused and non-evaluative.

The TAMI-OP tool automatically generates two visualizations that display 1) a timeline of what students and teachers were doing across the class period, and 2) the proportion of two-minute periods of class time in which certain codes appear (Hayward et al., 2018). The visualizations highlight how class time is used, enabling instructors to quickly see class features such as how much class time is devoted to activities other than lecture, the use and timing of questions and answers, and the balance between instructor questions and student answers.

With a few exceptions (Dillon et al., 2020; Reinholz et al., 2020; Yee et al., 2022), the use of observation data has not been well explored as a formative feedback mechanism for improving college instruction (in contrast to education research). Interest in this application is growing, however, and follows a long tradition of employing observation for feedback (and evaluation) in K-12 education (Martinez et al., 2016). Recent work exploring the use of observation tools in the college arena for feedback purposes have emphasized their application to spur instructor self-reflection (Dillon et al., 2020), and to assess the impact of feedback on graduate student teaching assistants' teaching practices (Yee et al., 2022). Observed instructors have reported that observation-grounded feedback opportunities are helpful (Dillon et al., 2020). Graduate students were found to improve instruction when the feedback was well contextualized (Yee et al., 2022). It has been suggested that interactions between expert teaching faculty and research-oriented faculty (including through observation-grounded feedback) could support the adoption of student-centered instructional practices (Rozhenkova et al., 2023).

Previously, the TAMI-OP had been used for research and evaluation purposes, but its descriptive emphasis and simplicity led us to explore how the TAMI-OP tool could be used in a professional development context focused on supporting observation-grounded feedback and instructor reflection. The theory of planned behavior (Ajzen, 1991; Archie et al., 2022) shapes our understanding of how teaching arises from planning. The theory led us to design a cyclical process of gathering data, reflecting, and conversing about teaching can help teachers to assess and change their practices in self-driven ways. Thus, we did not use the theory to make predictions or model quantitative data, but it informed the design of the professional development process.

The process specifically involved two coaches facilitating a series of goal-oriented discussions with instructors about TAMI-OP visualizations. We wanted to know if instructors would use the data to experiment with their teaching or change some teaching practices in ways aligned with their goals and, ideally, also aligned with evidence-based teaching. We nicknamed this project VIP-Math (Visualization Instructional Practices). The coaches engaged each VIP-Math participant as follows: they met with instructors to learn their initial set of teaching goals and orient them to the TAMI-OP instrument. Instructors then video-recorded a session of their mathematics class. The coaches scored the teaching video using TAMI-OP and met with the participant to discuss patterns in the data and visualizations. Participants were prompted to reflect on progress around their goals and set future intentions. This cycle was repeated two more times over the academic term, roughly 4 weeks apart.

Here "VIP-Math process" refers to this process of repeated gathering, sharing and discussing the TAMI-OP data and visualizations. We asked the following research questions:

- RQ1: Which elements of the TAMI-OP data and visualizations did VIP-Math participants tend to focus on?
- RQ2: How did the VIP-Math process influence participants' teaching practices?

Using the findings, we consider the affordances and limitations of the TAMI-OP as a tool for coaching, and the implications of the results for observation-based professional development.

Methods

Development of the VIP-Math process. Two of the authors were the coaches for this project, an early-career research assistant with classroom and professional development observation experience and a mid-career researcher with teaching and research experience in K12 and higher education. The coaches co-developed the VIP-Math process in consultation with the third author and conducted all participant-facing activities. Participants were sent a digital video camera and instructions for recording, and they uploaded each recording to a secure cloud storage folder.

Participants. Three mathematics instructors participated in this study, recruited from a list of participants in teaching professional development workshops. All three were white women with over 5 years of college-level instructional experience, and each stated a commitment to creating student-centered and active learning classrooms. Each also described herself as influential to colleagues, through formal or informal means. Two were full-time tenured faculty and one was a full-time, non-tenure-track lecturer. All worked at primarily undergraduate institutions, two at community colleges and one at a small liberal arts college.

Coding and analysis. The two coaches coded several sample videos to arrive at a common understanding of TAMI-OP code application. Both coaches coded all participant videos, coming to consensus on codes prior to discussing them with participants. Coaches tracked qualitative features of the classroom using the notes feature of the TAMI-OP. During each discussion, coaches prompted instructors to reflect on their video-recorded class, patterns in the TAMI-OP visualizations, their initial teaching goals and ways they are addressing them, and to revise their goals and intentions for future sessions or classes. Coaches responded to such reflections with a mix of follow-up questions, encouraging comments, and ideas for the instructor to consider in planning. Discussions were conducted and recorded on Zoom, and coaches also documented the flow of ideas in a research journal. Both the research journals and discussions were analyzed qualitatively for evidence related to the research questions, and the TAMI-OP data provided quantitative evidence about teaching.

Results

We present brief summaries of each participant's engagement with VIP-Math, referencing each by their first initial (Table 1). These summaries focus on the goals the instructors set and related patterns in TAMI-OP data we observed across their three recorded sessions, which provide evidence of change in their teaching practices.

Table 1. Characteristics of courses, goals, and key change made by the three participants.

Participant	J	S	M
Course	Calculus 1	Precalculus	Finite Math
No. students	20	20	15
Class length	90 min.	60 min.	90 min.
Initial goals	Assess use of class time	Active learning; wait time	Student math talk; equity
Key change	Increased use of reasoning questions; Created plan to adjust materials	Introduced structured group work	Alternated questions between Zoom & room students

Participant S

Coaches noted that S's first videotaped class was highly interactive, with over 55% of time spent on activities other than lecture. She asked informational questions that students typically answered (Figure 1, Q&A). Students spent significant class time working on problems individually and used white boards to show individual work to S (Figure 1, IND).

The coaches suggested that S could expand the variety of her active learning techniques by incorporating structured group work and inviting students to present their work, and S set related goals. With the second recorded class, S incorporated structured group work for the first time. By the third class, she improved the written directions provided for group work, to increase their ability to work independently, and had one student present their group's work on the board.

S set long-term intentions to continue using structured group work and student presentations in both the Precalculus class that we observed and other classes, even though she anticipated some challenges:

> I am teaching precalc again in the fall so this is great, because I can implement it again. [I] want to start the student presentations earlier in the semester, and [I am] hoping that motivates them to be on time. My next goal: How do I add that level of structure in the group to ensure that everyone in the group participates and has a role?… I haven't tried [this] with the higher-level classes yet, but I definitely think I would try them. The actual physical classroom is logistically challenging because the class is really full and the room is really small. Could definitely do pairs though.

Participant M

In M's hybrid class for non-STEM majors, students could choose each day to attend online or in person. Her initial recording featured 90% of class time spent on activities other than lecture. In accordance with M's goal, coaches tracked the numbers of questions that M posed to students in the Zoom room and overall. The fraction of questions to Zoom students (typically about ⅓ of the class) rose from 20% to 35% across the three sessions. For M, making this change didn't feel entirely comfortable, but she had developed a system that seemed to work:

Intentionally directing Qs to Zoom vs room without knowing whether they are actually there – that feels tricky. I'm doing it alternating now, and if there is no pick-up in Zoom, I pivot to room.

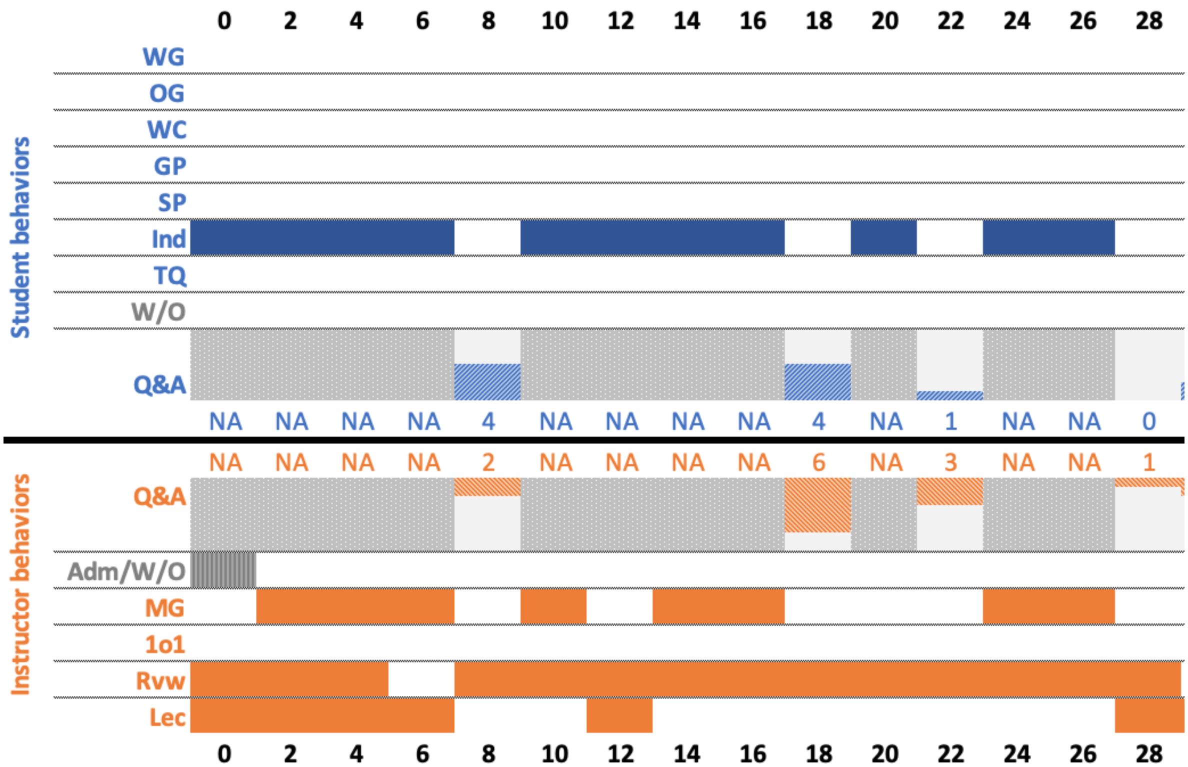

Figure 1. Excerpt of timeline visualization of S's first class, generated from TAMI-OP codes. Student codes (blue): WG, working in groups; OG; other group work; WC, whole class discussion; GP, group presentation; SP, student presentation; Ind, students working individually; TQ, taking a test or quiz; W/O, waiting/other. Instructor codes (orange): Adm/W/O, administrative/waiting/other; MG, moving and guiding; 1o1, teacher interacting with one student; Rvw, reviewing student thinking; Lec, lecturing. Horizontal axis represents elapsed time in class session, in minutes. Q&A numbers represent questions and answers from students (blue) and teacher (orange).

M devoted significant class time (25-90%) to student group work, in ways aligned with evidence-based teaching. Coaches noticed several ways in which class time was spent inefficiently, limiting opportunities for students to communicate mathematical understandings, but M did not engage with this observation.

During her final discussion, M continued to wrestle with the problem of engaging Zoom room students, in line with her initial goals. The coaches encouraged her to look at the norms she was setting around Zoom participation during her introduction to the course, and she came up with an idea to encourage Zoomers to use emojis to signal their presence and willingness to answer questions. She additionally generated an idea to create an introductory video of how Zoomers appear and sound in her hybrid-flexible classroom, to help Zoomers know how they appear to the in-person class.

Participant J

We observed three of J's Calculus I sessions. J's initial goals were to assess the balance of student group work time to other teaching activities, and to increase student voice in her classes. She wanted to devote 50% of class time to group work; in her initial class, she spent >60% of class time on activities other than lecture.

Across the three class sessions, TAMI-OP data revealed that J allocated 25-35% of class time specifically to group work. Discussion of J's second class centered around this balance:

> I have been thinking about restructuring the worksheets for this day. This is confirmation of that, that there is something missing in the balance of how it is set up.

J realized she needed to carefully choose example problems to illustrate new material, so there is "conceptual meat" yet without distracting details such as "dividing fractions." By our final discussion, J expressed a related long-term goal: "I need… to redevelop so I can get around them being bad at algebra." This was related to J's goal of increasing active learning: by restructuring class materials to maintain focus on new concepts, she could shift the balance of class time away from explanations and toward group work. Further, J described how TAMI-OP data led her to alter her expectations for use of class time:

> I have mentally adjusted by benchmark to realize that 50% group work may not be realistic… It will change how I approach others in my department. [I] don't want to set unrealistic expectations for other people.

Coaches also noticed that J asked 70-95 questions per class session, mostly informational. In discussions, J did not respond directly to the coaches' prompts around reasoning questions, but she did begin to ask more reasoning questions, increasing their share from 0% to 25% across the three class sessions.

RQ1: Which elements of the TAMI-OP data and visualizations did VIP-Math participants tend to focus on?

All three participants oriented quite quickly to the instrument. They had few questions about the meaning of different code categories and were able to interpret sample visualizations readily. Most frequently discussed were instructor codes LEC (lecture), REV (review), IQ and RQ (informational and reasoning questions), and MG (moving and guiding); and student codes SA (student answers), WG (working in groups), and SP (student presentations). Conversations focusing on TAMI-OP data tended to be brief. None of the participants tended to pore over the data, looking for patterns that were not obvious.

Participants tended to focus on TAMI-OP elements that aligned well with their initial goals. M tended to focus on the share of questions that she was asking to students in the Zoom room, while S and J tended to focus on the amount of class time devoted to student active learning. Participants also often commented on TAMI-OP data that they perceived as validating their current teaching practices.

RQ2: How did the VIP-Math process influence participants' teaching practices?

We found that the VIP-Math process influenced participants to change specific teaching practices in the short term. Each participant altered some teaching practices across their recorded classes, and mentioned altering some practices in classes that were not recorded. Some teaching practices changed as recommended by coaches, even when goals were not explicitly set around them. Changes were observed in how participants allocated class time to student group work and presentations, and how participants asked questions. At final discussions, all three participants also set long-term teaching goals, some of which could be traced to patterns in their TAMI-OP data. S set goals for increasing student presentations and structured group work, while M set goals about increasing the engagement of her online students, and J set goals for altering class materials so student group work can proceed more efficiently.

Discussion

Patterns across the three VIP-Math participants suggest some preliminary answers to our research questions. Participants readily engaged with the data visualizations generated about their class sessions and referenced them in reflecting on how their uses of class time related to their teaching goals (RQ1). Participants also made specific goal-related changes to their teaching, and they set long-term goals for future teaching changes (RQ2).

The VIP-Math project also allowed us to gather insights about the affordances and limitations of TAMI-OP as a professional development tool. As affordances, we found uploaded recordings of class sessions easy to score remotely, and we could accomplish an observation-feedback cycle with less than a week of lag between recording and discussion. Participants oriented quickly to the codes and visualizations, maintained a focus on their teaching goals, made related changes across the course of their sessions, and reacted positively to the VIP-Math process overall.

However, the VIP-Math process was time-intensive for coaches, and two coaches were needed to manage the logistics across three sessions. Coaches found some aspects of high-quality teaching challenging to address, particularly evaluative elements not captured by the protocol, and elements not well aligned to participants' teaching goals. For instance, coaches' concerns about M's efficiency in using class time were never discussed in depth, in part because these concerns did not align with M's goals.

This study is also limited by the small sample of participants with limited diversity. There were no instructors of color, novice or early-career instructors, instructors who expressed ambivalence around incorporating active learning into their teaching, or instructors who were teaching large classes. This limits our ability to generalize about the tool's benefits in professional development contexts.

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References

Ajzen, I. (1991). The theory of planned behavior. *Organizational Behavior and Human Decision Processes*, *50*(2), 179-211.

American Association for the Advancement of Science [AAAS] (2013). *Describing and measuring undergraduate STEM teaching practices*. Washington, DC: AAAS.

Archie, T., Hayward, C. N., Yoshinobu, S., & Laursen, S. L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *PLOS One*, *17*(4), e0267097.

Dillon, H., James, C., Prestholdt, T., Peterson, V., Salomone, S., & Anctil, E. (2020). Development of a formative peer observation protocol for STEM faculty reflection. *Assessment & Evaluation in Higher Education*, *45*(3), 387-400.

Esson, J., Wendel, P., Young, A., Frey, M., & Plank, K. (2022). Recent developments in classroom observation protocols for undergraduate STEM. *Journal of College Science Teaching*, *52*(1), 72-82.

Hayward, C. N., Laursen, S. L., & Weston, T. J. (2017). *TAMI-OP: Toolkit for Assessing Mathematics Instruction – Observation Protocol*. Ethnography & Evaluation Research, University of Colorado Boulder. https://www.colorado.edu/eer/content/hayward-tami-op-description-updated-jul21

Hayward, C., Weston, T., & Laursen, S. L. (2018). First results from a validation study of TAMI: Toolkit for Assessing Mathematics Instruction. In *21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 727-735).

Hora, M. T., & Ferrare, J. J. (2013). A review of classroom observation techniques in postsecondary settings. Wisconsin Center for Education Research.

Hora, M. T., Oleson, A., & Ferrare, J. J. (2013). Teaching dimensions observation protocol (TDOP) user's manual. Wisconsin Center for Education Research. http://tdop.wceruw.org

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406-418.

Martinez, F., Taut, S., & Schaaf, K. (2016). Classroom observation for evaluating and improving teaching: An international perspective. *Studies in Educational Evaluation*, 49, 15-29.

Powell, R., Cantrell, S. C., & Rightmyer, E. (2013). Teaching and reaching all students: An instructional model for closing the gap. *Middle School Journal*, *44*(5), 22-30. https://doi.org/10.1080/00940771.2013.11461869

Reinholz, D. L., Stone-Johnstone, A., & Shah, N. (2020). Walking the walk: Using classroom analytics to support instructors to address implicit bias in teaching. *International Journal for Academic Development*, *25*(3), 259-272.

Rozhenkova, V., Snow, L., Sato, B. K., Lo, S. M., & Buswell, N. T. (2023). Limited or complete? Teaching and learning conceptions and instructional environments fostered by STEM teaching versus research faculty. *International Journal of STEM Education, 10*(1), 1-20.

Sawada, D., Piburn, M. D., Judson, E., Turley, J., Falconer, K., Benford, R., & Bloom, I. (2002). Measuring reform practices in science and mathematics classrooms: The reformed teaching observation protocol. *School Science and Mathematics, 102*(6), 245-253.

Smith, M. K., Jones, F. H., Gilbert, S. L., & Wieman, C. E. (2013). The Classroom Observation Protocol for Undergraduate STEM (COPUS): A new instrument to characterize university STEM classroom practices. *CBE—Life Sciences Education, 12*(4), 618-627.

Weston, T. J., Hayward, C. N., & Laursen, S. L. (2021). When seeing is believing: Generalizability and decision studies for observational data in evaluation and research on teaching. *American Journal of Evaluation*, *42*(3), 377-398.

Weston, T. J., Laursen, S. L., & Hayward, C. N. (2023). Measures of success: characterizing teaching and teaching change with segmented and holistic observation data. *International Journal of STEM Education*, *10*(1), 24.

Yee, S., Deshler, J., Rogers, K. C., Petrulis, R., Potvin, C. D., & Sweeney, J. (2019, February). Bridging the gap: From graduate student instructor observation protocol to actionable post-observation feedback. In *Annual Conference on Research in Undergraduate Mathematics Education*.

Yee, S., Deshler, J., Rogers, K. C., Petrulis, R., Potvin, C. D., & Sweeney, J. (2022). Bridging the gap between observation protocols and formative feedback. *Journal of Mathematics Teacher Education, 25*(2), 217-245.

Are These Vectors Linearly Independent? Conceptions of Linear (In)Dependence in a Linear Algebra Textbook and Student Responses

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Using Balacheff's (2013) model of conceptions we analyzed textbook examples in the linear independence section of an interactive linear algebra textbook and 61 student responses to a similar reading question. Reading questions seek to entice students to read the textbook before attending the lesson when the ideas will be discussed; the responses are immediately available to instructors. We found ten additional parts of conceptions (control structures) used independently or in combination with the ones promoted in the textbook. We discuss the implications of our findings and our plans for future research.

Keywords: linear independence, conception, linear algebra, cK¢ model of conceptions

We contribute to investigations of students' understanding of fundamental notions in linear algebra by bringing a different theoretical approach to such investigation with the goal of exploring its usability in large-scale analysis. Linear algebra is a course that is nearly universally taught at four- and two-year institutions across the United States (Blair et al., 2018). Extensive research documents the difficulties students experience with the high abstract level of the notions, which also demands substantial computational dexterity. While much work has been done to address student difficulties (e.g., using curricula that emphasize building mathematical understanding through concrete experiences and visualizations that are progressively mathematized; Wawro et al., 2012), managing these difficulties in the classrooms is not straightforward. As they learn the material, independently of the curriculum that is being used, students face difficulties in transitioning from concrete to more abstract and from managing visual representations to working with spaces that can not be visualized. Our position is that independently of the textbooks or the mode of teaching, such difficulties will emerge; thus, finding ways in which these difficulties can be readily identified could promote better teaching and learning.

Literature Background

The research on the learning of linear (in)dependence has focused on identifying students' conceptual difficulties with the notion and its formal/abstract definition (Dorier, 2017; Hannah et al., 2013; Stewart & Thomas, 2007, 2009) and characterizing students' ways of thinking about linear (in)dependence. Regarding conceptual difficulties, researchers have shown that, although students make mistakes when solving routine problems, their procedural understanding is sounder and less varied than their conceptual understanding of this notion (Celik, 2015; Parker, 2010). To investigate ways of thinking, researchers have relied on a handful theorizations, and although they are named differently, the categorizations they produce are similar: abstract, algebraic, geometric (Ertekin et al., 2010; using Hillel's [2000] descriptive modes); travel, geometric, vector algebraic, matrix algebraic (Plaxco & Wawro, 2015; using Tall and Vinner's [1981] notion of concept image); embodied, symbolic, formal (Stewart & Thomas, 2007; using

Tall's [2004] three worlds of mathematics); or synthetic-geometric, analytic-arithmetic, analytic-structural (Dogan-Dunlap, 2010; Sierpinska's [2000] three worlds of mathematics). Across these studies, there is consensus that visual, embodied, or geometric forms of thinking about linear (in)dependence can help students overcome their difficulties with its algebraic and formal modes.

We focus on research addressing the problem of determining whether a set of vectors is linearly independent. Celik (2015) organized 186 student responses to this problem into 18 categories. The most common student approach was finding scalars that satisfied the definition of linear (in)dependence. Bouhjar et al. (2021) categorized the reasoning of 255 about the problem into six strategies: comparing Reduced Row Echelon Form (RREF) to the identity matrix, identifying the number of pivots, writing one vector as a linear combination of others, solving $Ax = 0$, comparing the number of rows and columns, and establishing the existence of free variables. Because linear independence was defined in terms of the solution to the homogeneous vector equation corresponding to $Ax = 0$, solving $Ax=0$ and comparing the RREF were considered the more appropriate approaches. Analyzing student responses to the same problem, Dogan (2010, 2018) showed that when not using geometric ways of thinking, most students use one of the following: (1) find a linear combination among the vectors, (2) find that the only solution to a linear combination is the trivial solution, (3) find whether scalar multiple and/or adding the vectors gives the zero vector, or (4) find RREF.

We complement this work in three significant ways. First, in addition to identifying student reasoning when learning how to solve the problem of determining whether a set of vectors is linearly independent, we study how their linear algebra textbook addresses it, which helps frame students' work. Second, while researchers have investigated students' various modes of thinking or types of reasoning, we only focus on one semiotic register: the symbolic vector-matrix. This allows us to fully investigate the difficulties of this representation, which is preferred for formalization to higher-order spaces. Third, although it is common to draw a line between procedural and conceptual understanding, we assume that these approaches are intertwined and can significantly inform one another. Therefore, we use a theoretical framework that bypasses analysis of conceptual and procedural understanding and instead focuses on the actions and the decision-making processes as students determine whether a set of vectors is linearly independent.

Theoretical Framework

Balacheff (2013; Balacheff & Gaudin, 2009) posits that meaning derives from the system that encompasses a milieu and a cognizant subject rather than being inscribed in either of them alone; meaning is created through the interactions between the milieu and the subject through the actions of the subject on the milieu and the feedback the milieu provides to the subject. His work responds to three needs related to investigating student thinking (1) explaining why particular held conceptions of a notion that appear contradictory to an observer are not to the holder of the conceptions; (2) describing the array of conceptions that can be held by learners, which are the result of historical and pedagogical process of concept development; and (3) understanding the connection between behaviors and knowing. Following this tradition, *savoir* (knowledge), the subject matter developed by a community, is differentiated from *connoisseur* (knowing, as a noun), the knowledge held by an individual, which can be incomplete or even mathematically invalid. For this reason, knowledge and knowings are tied to the problems in which they emerge. Thus, understanding the practices in which specific mathematical ideas are called for is the first step in understanding conceptions.

The cK¢ model is a heuristic of sorts for identifying conceptions. It has four components: problems, operators, semiotic and representation systems, and control structures. *Problems* correspond to the "class of the disequilibria the considered conception is able to recover from" (Balacheff & Gaudin, 2009, p. 190); they emerge from the sets of practices in which individual concepts are called for. *Operators* refer to "actions on the milieu" including those needed "to transform and manipulate linguistic, symbolic or graphical representations" (p. 190). The learner receives feedback from the milieu as a result of their actions (action-feedback loop). The *semiotic and representation systems* are defined as the "linguistic, graphical or symbolic means which support the interaction between the subject and the milieu" (p. 190). Finally, the *control structures* refer to the "components supporting the monitoring of the equilibrium of the [Subject - Milieu] system" (p . 190), or said differently, the strategies that the cognizant subject relies on to decide whether they had solved the problem and that they had done so correctly. This model has been used to identify potential conceptions that could emerge as students work with different components in textbooks, such as the problems, the examples, or the textbook presentation with different concepts (functions, Mesa, 2004; differential equations, Mesa, 2010; angles, Mesa & Goldstein, 2016). We use this theorization to identify: (1) What conceptions of linear independence are promoted by examples in an interactive undergraduate linear algebra textbook? and (2) What control structures do students use when answering reading questions about linear independence embedded in these textbooks?

Methods

The data for this study come from a larger study that investigates undergraduate mathematics students' and instructors' use of interactive textbooks in calculus, linear algebra, and abstract algebra courses (Beezer et al., 2018). The linear algebra textbook, *A First Course of Linear Algebra* (Beezer, 2021), is designed to be a bridge-to-proof linear algebra course and it follows the definition-theorem-proof presentation style (Love & Pimm, 1996). The textbook is authored in PreText (https://pretextbook.org/), which allows the inclusion of interactive features. *Reading Questions* are one such feature designed to entice students into reading material before coming to class; students are supposed to answer a few of these short answer questions in each section directly in their textbooks before coming to class. Their responses are then collected and made available for their instructor in real-time so that instructors can learn about their students' thinking before a lesson and alter their plans as needed.

To investigate how the textbook content related to the chosen reading question in this section (Figure 1), we identified one example in the Linear Independence and Spanning Sets (LISS) section directly related to it (Figure 2). The example, named Linear Independence in M_{32} (LIM32) is about the vector space of all 3x2 matrices and illustrates the work needed to decide whether two different sets of matrix vectors are linearly independent or dependent. We analyzed the example using Balachef's (2013) cK¢ model.

LISS Reading Questions

1. Is the set of matrices below linearly independent or linearly dependent in the vector space M_{22}? Why or why not?

$$\left\{\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 9 \\ -1 & 3 \end{bmatrix}\right\}$$

Figure 1: The first reading question in LISS

Example LIM32 Linear independence in M_{32}
Consider the two sets of vectors R and S from the vector space of all 3×2 matrices, M_{32} (Example VSM)

$$R = \left\{ \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix}, \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} \right\}$$

One set is linearly independent, the other is not. Which is which? Let us examine R first. Build a generic relation of linear dependence (Definition RLD),

$$\alpha_1 \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in M_{32} (Example VSM) we obtain,

$$\begin{bmatrix} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 & -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 & 4\alpha_1 - 3\alpha_2 + -5\alpha_4 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 & -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 &= 0 \\ -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 &= 0 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 &= 0 \\ 4\alpha_1 - 3\alpha_2 + -5\alpha_4 &= 0 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 &= 0 \\ -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this matrix we are led to conclude that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. This means there is *only* a trivial relation of linear dependence on the vectors of R and so we call R a linearly independent set (Definition LI).

So it must be that S is linearly dependent. Let us see if we can find a nontrivial relation of linear dependence on S. We will begin as with R, by constructing a relation of linear dependence (Definition RLD) with unknown scalars,

$$\alpha_1 \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + \alpha_4 \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in M_{32} (Example VSM) we obtain,

$$\begin{bmatrix} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 & \alpha_3 + 3\alpha_4 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 & -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 & 3\alpha_1 - 6\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 &= 0 \\ \alpha_3 + 3\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 &= 0 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &= 0 \\ 3\alpha_1 - 6\alpha_2 + 4\alpha_3 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & -2 & 0 & -4 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this we see that the system is consistent (we expected this since the system is homogeneous, Theorem HSC) and has $n - r = 4 - 2 = 2$ free variables, namely α_2 and α_4. This means there are infinitely many solutions, and in particular, we can find a nontrivial solution, so long as we do not pick all of our free variables to be zero. The mere presence of a nontrivial solution for these scalars is enough to conclude that S is a linearly dependent set (Definition LI). But let us go ahead and explicitly construct a nontrivial relation of linear dependence.

Choose $\alpha_2 = 1$ and $\alpha_4 = -1$. There is nothing special about this choice, there are infinitely many possibilities, some "easier" than this one, just avoid picking both variables to be zero. (Why not?) Then we find the dependent variables to have values $\alpha_1 = -2$ and $\alpha_3 = 3$. So the relation of linear dependence,

$$(-2) \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + (1) \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + (3) \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + (-1) \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is an iron-clad demonstration that S is linearly dependent. Can you construct another such demonstration? △

Figure 2: Example LIM32 in LISS

Next, we analyzed 61 responses to the reading question in LISS from 61 unique students from four different sections during Fall 19, Fall 20, Spring 21, and Fall 21. All four instructors used the textbook and assigned the reading questions to their students. During analysis, we noticed that the operators were implied in most of the student responses and the steps taken were not outlined. However, the justification for their conclusion was usually explicit. Therefore, we solely focused on the control structures stated in the student responses. After coding all the responses individually, we met to compare the coding and resolve disagreements. Six of the 61 students' responses to the reading question were uninterpretable or did not have a stated control structure. We also looked for patterns when students used more than one control structure.

Findings

Inferred conceptions from Example LIM32 in the textbook

Example LIM32 addressed the problem "where a given set of vectors is linearly independent" (see Figure 2). Relying on a 3x2 matrix, this example uses symbolic representations. We identified a total of five operators (OP) in the solution and two control structures (CS).

The author starts by constructing the homogeneous system by equating an arbitrary linear combination of all vector matrices with a zero matrix (OP1: *Construct homogeneous system*). He then creates an augmented matrix for the system (OP2: *Construct matrix*) and performs a row-reduced echelon transformation on the matrix (OP3: *Row reduction*). He proceeds to inspect the resulting RREF matrix. He first starts by asserting that "the system is consistent (we expected this since the system is homogeneous, Theorem HSC)", then he states the observation that the RREF "has $n - r = 4 - 2 = 2$ free variables, namely α_2 and α_4" (OP4.1: *Free variables*, feedback = "there are free variables"). This observation implies that "there are infinitely many solutions" (OP4.2: *Number of solutions*) which leads to the criterion "we can find a nontrivial solution." This enables him to use the control structure *Nontrivial* (CS1) that alludes to the existence of nontrivial solutions to the homogeneous system to justify the claim that the given set is linearly dependent. Conversely, if the homogeneous system only has a trivial solution (which is implied by having one unique solution, as it can not have no solution because the system is consistent by default), then the given set is linearly independent. Even though "the mere presence of a nontrivial solution (…) is enough to conclude that *S* is a linearly dependent set," the author also provides an optional operator, *Find nontrivial scalars* (OP5) by explicitly constructing a nontrivial relation of linear dependence. This suggests another control structure in solving this problem, namely *Scalars* (CS2) that justify the linear dependence of the set by providing a set of nonzero scalars that satisfy the homogeneous system.

Control structures in student responses to LISS reading question

Nontrivial (CS1) was observed in 14 responses (out of 55 total; e.g., "Linear independent, because it only contains the trivial solution", #50). Despite OP7: *Find nontrivial scalars* being described as optional, *Scalars* (CS2) was observed in 10 responses (e.g., "This set is linearly dependent as there is the nontrivial dependence relation \`a_1=-2, a_2=-1, a_3=1\`." #3). Here, because the student provided nontrivial scalars, we only coded the responses for *Scalars* and not *Nontrivial*.

We identified 10 additional control structures in student responses, four of which were identified in at least six responses: *FreeVar* (CS3), *NumSol* (CS4), *LinearCombo* (CS5), and *Pivots* (CS6). *FreeVar* (CS3, 10 responses) and *NumSol* (CS4, 6 responses) could derive from the solution path demonstrated in the textbook. *FreeVar* alludes to the number or existence of free variables in the justification (e.g., "Set is linearly dependent. These can be modeled by a system of 4 equations which can be represented in a matrix that, when row-reduced, yields a free variable." #17) whereas *NumSol* alludes to the number of solutions to the homogeneous system in their justification. For example,

> Following Example 2 of this section, if we form a general relation of linear dependence, find the corresponding system of homogeneous equations of each of the matrix entries, create the coefficient matrix and row reduce, we see that the resulting RREF of the coefficient matrix gives us a 4x3 matrix that has one free variable. This says that there are infinitely many solutions for the scalars in the relation of linear dependence, which further means (by Def LI) that the given set of matrices is linearly dependent. (#15)[1]

The other two control structures—*LinearCombo* (CS5, 14 responses) and *Pivots* (CS6, 8 responses)—were not directly related to the textbook examples. *LinearCombo* (CS5) alludes to the existence of linear combinations within the set or to finding the exact linear combination by expressing one vector matrix in terms of others (e.g., "The set of matrices are linearly dependent because 2[[1,3],[-2,4]]+[[-2,3],[3,-5]] = [[0,9],[-1,3]] thus making one vector a linear combination of the others." #1) whereas *Pivots* (CS6) is assigned when the students' responses suggest a comparison of the number of pivots with the number of vectors in their justification (e.g., "The set of matrices are linearly independent because there are less pivots than vectors.").

Other control structures were observed in at most three responses ("Others" in Figure 3). For example, students gave justifications: using the consistency of the homogeneous system (CS7:

[1] Note that this response also used *FreeVar* (CS3, "... has one free variable"). We could not find responses using only *NumSol* (CS4).

Consistency), using singularity of the coefficient matrix (CS8: *Singularity*), by reducing the vector matrix to identity (CS9: *VectorMatrix=I*), by reducing the coefficient matrix to identity (CS10: *CoefMatrix=I*), mentioning the shape of the coefficient matrix (e.g., number of rows or columns, number of vectors, number of equations; CS11: *MatrixShape*), and using a theorem, (Theorem Row-Equivalent Matrices represent Equivalent Systems, CS12: *REMES*).

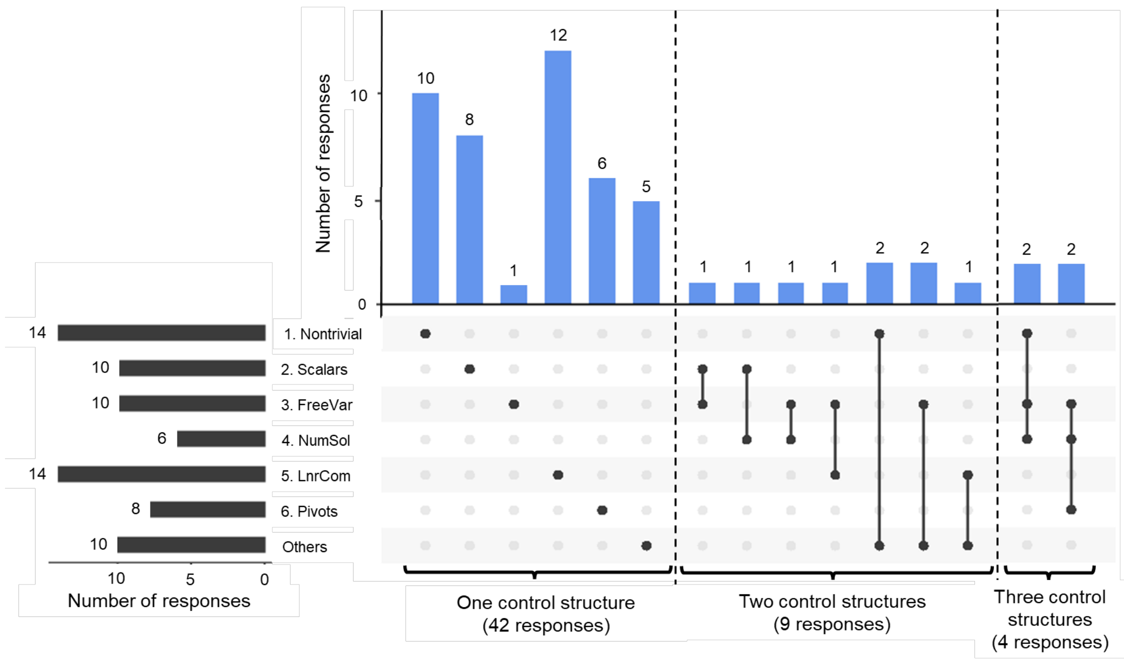

Figure 3: Distribution of control structures used in LISS-RQ1 (n = 55)

Figure 3 is an UpSet plot (Conway et al., 2017; Lex et al., 2012) that summarizes the distribution of control structures in student responses. Among the 55 interpretable responses, 42 used one control structure; nine used two control structures; and four used three control structures. *FreeVar* (CS3) and *NumSol* (CS4) was the most common combination, recorded in five of 13 responses with multiple control structures.

Discussion and Conclusion

Analyzing the control structures in student responses to reading questions allowed us to learn more about how students interpret the content in the textbook and potentially rely on other resources to tackle the problem of deciding whether a set of vectors is linearly independent. In the textbook example, the connection between the existence of free variables, the number of solutions to the homogeneous system, and the existence of nontrivial solutions are demonstrated sequentially, as if one needs to go through all these steps to answer the question. It is possible that the author wants to stay consistent with the definition of linear independence in the textbook (Figure 4) by ending the action-feedback loop with the criteria "whether there are non trivial solutions to the homogeneous system." However, given that the conditions are equivalent mathematically (e.g., free variable(s) exist ⇔ infinitely many solutions to the homogeneous system ⇔ existence of a nontrivial solutions), students may have realized that is possible to justify linear dependence after carrying out either or both operations—*Free variables* (OP4.1) and *Number of solutions* (OP4.2)—and inspecting the feedback. This suggests that students may see these three conditions as equivalent, rather than as a uni-directional line of reasoning as described by the author (free variable(s) exist ⇒ infinitely many solutions to the homogeneous system ⇒ existence of non-trivial solutions), which is an important learning objective and must be emphasized throughout the course.

> **Definition LI. Linear Independence.** Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ from V is ***linearly dependent*** if there is a relation of linear dependence on S that is not trivial. In the case where the ***only*** relation of linear dependence on S is the trivial one, then S is a ***linearly independent*** set of vectors.

Figure 4: Definition of linear independence in section LISS.

LinearCombo (CS5, 14 responses) and *Pivots* (CS6, 8 responses) were not mentioned in the textbook example. Independently, CS5 was more frequently observed than *Nontrivial* (CS1, 10 responses), which was demonstrated in the example. We suspect that students may be relying on the textbook's definition of the dependence relation (Figure 5) and not necessarily on the operator paths or the control structures presented in the textbook example. Likewise using *Pivots* might be justified by students making connections across sections as the connection between pivots and number of solutions is explained in section Types of Solution Sets (TSS), 17 sections before LISS. This implies a potential avenue for future research, which entails conducting a longitudinal examination of students' responses to reading questions across different sections in order to observe how their conceptions evolve over time.

> **Definition RLD. Relation of Linear Dependence.** Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, an equation of the form
>
> $$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$
>
> is a ***relation of linear dependence*** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is a ***trivial relation of linear dependence*** on S.

Figure 5: Definition of linear dependence relation in section LISS.

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References

Balacheff, N. (2013). cK¢, a model to reason on learners' conceptions. In M. Martinez, & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. University of Illinois at Chicago.

Balacheff, N., & Gaudin, N. (2009). Modeling students' conceptions: The case of function. *Research in Collegiate Mathematics Education, 16*, 183-211. https://doi.org/10.1090/cbmath/016/08

Beezer., R. (2021). *A first course in linear algebra*. http://linear.ups.edu/
Beezer, R., Judson, T., Farmer, D., Morrison, K., Mesa, V. ,& Lynds, S. (2018). *Undergraduate Teaching and Learning in Mathematics with Open Software and Textbooks (UTMOST)* (National Science Foundation, DUE 1821706,1821329,1821509,1821114) [Grant].
Blair, R., Kirkman, E. E., & Maxwell, J. W. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States. Fall 2015 CBMS Survey. American Mathematical Society.* www.ams.org/cbms/cbms-survey.
Bouhjar, K., Andrews-Larson, C., & Haider, M. Q. (2021). An analytical comparison of students' reasoning in the context of Inquiry-Oriented Instruction: The case of span and linear independence. *The Journal of Mathematical Behavior, 64*, 100908.
Çelik, D. (2015). Investigating students' modes of thinking in linear algebra: The case of linear independence. *International Journal for Mathematics Teaching and Learning, 16*(1), 2141-2159.
Conway, J. R., Lex, A., & Gehlenborg, N. (2017). UpSetR: An R package for the visualization of intersecting sets and their properties. *Bioinformatics, 33*(18), 2938-2940. https://doi.org/10.1093/bioinformatics/btx364
Dogan, H. (2018). Differing instructional modalities and cognitive structures: Linear algebra. *Linear Algebra and its Applications, 542*, 464-483.
Dogan-Dunlap, H. (2010). Linear algebra students' modes of reasoning: Geometric representations. *Linear Algebra and Its Applications, 432*(8), 2141–2159.
Dorier, J. L. (2017). Duality between formalism and meaning in the learning of linear algebra. In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of mathematics in higher education as a scientific discipline conference proceedings* (pp. 101-105). Universitätsbibliothek Kassel. urn:nbn:de:hebis:34-2016041950121
Ertekin, E., Solak, S., & Yazici, E. (2010). The effects of formalism on teacher trainees' algebraic and geometric interpretation of the notions of linear dependency/independency. *International Journal of Mathematical Education in Science and Technology, 41*(8), 1015–1035.
Hannah, J., Stewart, S., & Thomas, M. (2013). Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematical Education in Science and Technology, 44*(4), 475-489. https://doi.org/10.1080/0020739X.2012.756545
Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191-207). Kluwer Academic Publishers.
Lex, A., Gehlenborg, N., Strobelt, H., Vuillemot, R., & Pfister, H. (2014). UpSet: Visualization of intersecting sets. *IEEE Transactions on Visualization and Computer Graphics, 20*(12), 1983-1992. 10.1109/TVCG.2014.2346248
Love, E., & Pimm, D. (1996). 'This is so': A text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Vol. 1, pp. 371-409). Kluwer.
Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics, 56*, 255 - 286. https://doi.org/10.1023/B:EDUC.0000040409.63571.56
Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *Research in Collegiate Mathematics Education, 16*, 235-265.

Mesa, V., & Goldstein, B. (2016). Conceptions of angles, trigonometric functions, and inverse trigonometric functions in college textbooks. *International Journal of Research in Undergraduate Mathematics Education, 3*, 338-354. https://doi.org/doi:10.1007/s40753-016-0042-1

Parker, C. F. (2010). *How intuition and language use relate to students' understanding of span and linear independence in an elementary linear algebra class* [Unpublished doctoral dissertation]. University of Northern Colorado.

Plaxco, D., & Wawro, M. (2015). Analyzing student understanding in linear algebra through mathematical activity. *The Journal of Mathematical Behavior, 38*, 87-100. https://doi.org/10.1016/j.jmathb.2015.03.002

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). Kluwer.

Stewart, S., & Thomas, M. O. (2007). Embodied, symbolic and formal thinking in linear algebra. *International Journal of Mathematical Education in Science and Technology, 38*(7), 927-937. https://doi.org/10.1080/00207390701573335

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961. https://doi.org/10.1080/00207390903200984

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169. https://doi.org/10.1007/BF00305619

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS, 22*(8), 577-599. https://doi.org/10.1080/10511970.2012.667516

More Relatable? Students' Positioning of Undergraduate Teaching and Learning Assistants in Mathematics Courses Taught by Graduate Students

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Undergraduate teaching and learning assistants (UTLAs) can support learning and students' sense of belonging in undergraduate, active learning STEM classrooms. One main goal of incorporating UTLAs into a classroom is to connect students to a peer-like instructional figure whom, it is assumed, students find more relatable. However, little is known about how UTLAs change the social dynamic of undergraduate mathematics classrooms, particularly those taught by an instructor of record who is a graduate student (GSI). I used positioning theory as a lens to understand how students locate UTLAs in instruction. I analyzed 411 undergraduate precalculus student responses to the survey question: "Do you interact differently with [the GSI] than [the UTLA]? If so, please explain." From a qualitative analysis of responses, I found that students positioned UTLAs along three storylines that have different implications for the distribution of instructional duties between UTLAs and GSIs in an active learning environment.

Keywords: undergraduate teaching and learning assistants, graduate student instructors, active learning, student perceptions

For several decades, institutions have been hiring advanced undergraduates to provide academic and social support for their peers, with positive outcomes (Barrasso & Spilios, 2021; Dawson et al., 2014; Gafney & Varma-Nelson, 2008; Whitman & Fife, 1988). These efforts are grounded in the view that knowledge is socially constructed, peer-to-peer interactions are important, and that undergraduates, as "near-peers" (Whitman & Fife, 1988, p. 5) have a unique ability to support students (Gafney & Varma-Nelson, 2008; Otero et al., 2006). These near-peer teaching models can be particularly effective in acting as change levers (Laursen, 2019) to support the institutionalization of active learning in postsecondary education.

Undergraduate teaching and learning assistants (UTLA; Jardine, 2020) are undergraduates specifically hired to facilitate active learning during lectures or recitations. The learning assistant model is perhaps the most prominent peer teaching model that specifically integrates near-peers into regular class time (Barrasso & Spilios, 2021; Otero et al., 2006). Research on this model suggests that it supports increases in students' conceptual understanding, higher-order cognitive skills, satisfaction, and sense of belonging and decreases in failure rates (Alzen et al., 2018; Clements et al., 2022; Goertzen et al., 2011; Otero et al., 2010; Sellami et al, 2017; Talbot et al., 2015). But the field is still emergent. With few exceptions (e.g., Stringer, 2023; Webb et al., 2014) most research has been in physics (Barrasso & Spilios, 2021) and to date, focuses primarily on positive outcomes. This also holds true for other models that use UTLAs. There is a specific gap in research on UTLAs in classrooms led by graduate student instructors of record (GSIs). Indeed, most research focuses on classrooms taught by faculty. Given that a key assumed benefit of UTLAs is their positioning as a near-peer, and thus as a more relatable teaching figure, it is worth interrogating this assumption in a context in which the instructor is a graduate student.

The purpose of this study is to explore how students perceive UTLAs in relation to GSIs within active learning mathematics classrooms, and what influences that perception. Toward this aim, I focus on the following two research questions:

RQ1: How did students in active learning precalculus classrooms position UTLAs versus GSIs in interactions with students?

RQ2: Why might students have preferred to interact with either the UTLA or the GSI?

Theoretical Framework

Positioning theory has been used as a lens to understand the multiple ways that faculty and UTLA interact with one another. I argue that it can also illuminate the nature of student and UTLA interactions (Jardine, 2020). Positioning theory focuses on social interactions, specifically on "the ways in which people use action and speech to arrange social structures" (Wagner & Herbel-Eisenmann, 2009, p. 2). It specifically is useful in highlighting the assumed rights and duties of people in interactions. There are three mutually determining components of positioning: positions, speech acts - or communication acts as Herbel-Eisenmann et al. (2015) suggest - and storylines (Van Langenhove & Harré, 1999). Positions are a cluster of personal attributes which determine the distribution of rights (what one is owed) and duties (what one owes others) within a certain social milieu. These positions contribute to and are influenced by an unfolding storyline, which is likely also influenced by historically, personally, and culturally significant stories (Davies & Harré, 1999). Positions and storylines imbue meaning to the ways that people communicate in an interaction. Researchers interested in the positions taken up by participants may study communication acts for patterns to identify what storylines participants may be assuming, thereby illuminating how rights and duties are being distributed in an interaction.

Although positioning theory often focuses on immanent, moment-by-moment communication acts, others have used positioning theory to analyze how people position themselves and others through narratives (Bamberg, 1997; Deppermann, 2013). Kayi-Aydar (2021) presents an analytic framework to support the identification of positioning in narratives. They suggest focusing on four components: a.) attributes and biological dimensions, b.) categorical membership, c.) storyline structure, and d) emotions. Attributes and biological dimensions include character traits and dispositions (e.g., creative, helpful). Categorical memberships refer to an individual's membership in a cultural or identity group. Storyline structures can be identified through multiple means, including word choice, sentence structures, abrupt shifts in topic, and the introduction of new people. Emotion words include emotional states (proud) as well as words or sentences that express or imply emotion (e.g., "gross!", "I can do it"). In the methods section I discussed how I used this analytic framework from Kayi-Aydar (2021) to help identify the storylines students implicitly followed when positioning UTLAs and GSIs through the narratives they shared in survey data.

Methods

This study is part of a larger case study of the UTLA role[1] at a predominately white Midwestern University and focuses on data collected in fall 2021. UTLAs supported two precalculus courses: College Algebra and a combined College Algebra and Trigonometry course. All students in the sections of these courses were invited to participate in a survey for extra credit. In this survey, students were asked to directly position UTLAs in relation to GSIs by answering the question: "Do you interact differently with [the GSI] than [the UTLA]? If so,

[1] Locally, the UTLAs were called "LAs"; however, a defining feature of the LA model is that students receive pedagogical training throughout the semester (Otero et al., 2006). At Midwestern, UTLAs received pedagogical training at the beginning of the semester, but in fall 2021 a pedagogy course was not offered. Thus, Midwestern University did not implement the LA model as intended in fall 2021.

please explain." There were 411 responses to this question. I coded each of these responses as "Difference," "No Difference" or "N/A" to capture perceived differences between student-GSI and student-UTLA interactions. In total, I classified 390 responses as either "Difference" (n=194) or "No Difference" (n=196) and included these responses in subsequent analysis. I read through these 390 responses and took analytic memos to capture factors influencing differences or similarities in student-GSI and student-UTLA interactions. Several responses were general (e.g. "we generally ask him [the UTLA] more questions than [GSI]"); however, 209 responses provided sufficient detail to be included in a secondary analysis. Criteria for inclusion were broad to include all responses that could illuminate why students perceived interactions with their GSI to be the same or different from interactions with the UTLA. For example, the response "No, I respect them both equally as an instructor for this course" was included because, although not overtly specific, this response suggests that the student positions the UTLA and the GSI in similar ways (positioning both "equally" as instructors), because they respect them as teachers.

To identify how these 209 students positioned UTLAs and GSIs in their interactions with them, I developed codes informed by my memos about factors and Kayi-Aydar's (2021) recommended strategies for identifying positioning in texts. These codes were: dispositions, labels/categorical membership, emotions, and perceived teaching quality. See Table 1 for a summary of these codes. I then analyzed each of the 209 responses with these codes, before engaging in an axial coding process to identify related themes on how students interacted with their GSI and UTLA. I structured these themes around storylines to highlight how students positioned UTLAs in interactions. I also coded these responses for student preference, as in, their expressed preference to interact with the GSI or the UTLA (GSI, UTLA, or Other).

Table 1 Codebook for Identifying Storylines in SPIPS-M Data

Code	Definition	Example
Dispositions	Attributes personality traits and dispositions to a UTLA or GSI	"Yes, because [UTLA] seems more personable and available to help. I also like the way he explains things better."
Labels/ categorical membership	Mentions membership in particular cultural groups or uses specific labels to describe the UTLA or GSI. This includes designations such as student, peer, woman, etc. as well as titles (e.g., professor).	"Yes, [UTLA] is closer to our age so we talk to him more often."
Emotions	Mentions emotions arising from or during interactions with a UTLA or GSI.	"I can more easily say when I don't understand something to her and stop her while she's explaining - I feel a little rude to do that to Professor [GSI]."
Teaching Quality	Mentions a perceived teaching quality of the UTLA or the GSI.	[GSI] is better at answering questions.

Preference	Use to capture explicit preferences for the UTLA or GSI (or to indicate no explicit preference) Sub-codes: UTLA, GSI, Other	Other: “No, [UTLA] has helped me more since he's closer to our table, but we all interact the same.”

Findings

In the findings section, I use positioning theory to share how students in active learning classrooms positioned UTLAs and GSIs in interactions with students. I present three major storylines students followed to position UTLAs and GSIs, particularly related to one another. As I introduce these storylines, I share quotations from students as well as the number of responses that the storyline draws from, as a proxy for understanding how prevalent the storyline was in the data.

Storyline 1: UTLA ≅ GSI

In this section I present evidence of one of three storylines: *UTLAs and GSIs only differ superficially, and thus have the same duties in interactions with students*. Evidence for this storyline came from responses in both the “Difference” and “No Difference” categories. Seventeen of the 194 “Difference” responses indicated that the only difference between their interactions with the UTLA versus the GSI was due to incidental factors, such as proximity or the initiative of the instructional figure. One student said “usually [UTLA] helps answer our questions first.” Another student said that they did interact differently with the UTLA, “just because [UTLA] sits closer to our table. [GSI] helps people on his side.” Although this student did perceive there to be a difference in how they interacted with the GSI versus the UTLA, their reasoning for this difference was tied explicitly to their location in the classroom, rather than an explicit preference for interacting with the UTLA or the GSI. Several “No Difference” responses specifically labeled the GSI and UTLA similarly, which in effect positioned the GSI and UTLA as occupying the same position (e.g., "No, I see them as both of my teachers who teach me the course material in and out of class, "No, I treat them both as my professor..."). Teacher, instructor, and professor were the most common labels in this group, but one student also referred to their GSI and UTLA as “guides” through “the journey of math class," and another student labeled them as “mentors” to help them learn. Overall, this storyline calls attention to a group of students that did not perceive there to be any meaningful differences between the UTLA or GSI. However, these students did not provide enough information in their responses to glean the perceived, shared, duties of the UTLA and the GSI. The following storyline elaborates on these duties.

Storyline 2: UTLA ≅ GSI: Emotions, Dispositions, and Teaching Quality Matter

When students shared a preference for interacting with either the UTLA or GSI, they often explained their preference along three themes: (a) emotions felt during interactions (b) perceived teaching quality and (c) UTLA and GSI dispositions. Student responses related to these themes suggest that students judged UTLAs and GSIs based on a similar set of expectations, which effectively positioned UTLAs and GSIs similarly. Thus, the second storyline I present is an elaboration of the first: *UTLAs and GSIs hold similar positions, in which both have a duty to make interactions comfortable for students by having an inviting disposition and teaching content well*. Below I elaborate on each of the themes that contribute to this storyline.

Forty students discussed the emotions they felt in interactions with UTLAs or GSIs. Typically, students brought up emotions to explain with whom they preferred to interact. For example, one student said:

> *[GSI] makes me feel more comfortable because she is engaging and friendly. She explains the answer and why the problem works the way it does. She is encouraging and makes me feel like I can actually do the problem. [UTLA] walks around and occasionally gives the answer. However she is not very clear, and does not explain her thought process. She is reserved and does not make me feel confident in my answer.*

This student positioned their GSI as someone who makes them feel and capable of solving mathematics problems. In contrast, the student does not feel confident when working with the UTLA. Other students had similar sentiments, but about their UTLA. One student shared that they felt "more comfortable asking [UTLA] questions…because he makes sure I understand the problem." In contrast, they found interactions with the GSI "confusing" and time-consuming. Most emotions centered on the level of comfort and confidence felt by students. It was also common for students to simultaneously evaluate the teaching ability of their GSIs and UTLAs when comparing how they interact with these teaching figures[2].

In sum, 101 students described the teaching quality of their GSI or UTLA. Out of these responses, 61 responses focused on the ability of the GSI or UTLA to explain content or answer questions to support their understanding. One student shared that their GSI's approach to student questions (answering questions by asking questions) was ineffective. In contrast, they perceived their UTLA to be more adaptable to their learning needs:

> *Yes, I believe that both are good instructors however, my interactions with [UTLA] are different because he takes my learning disability into account and bases his methods of teaching or explaining off of it. No offense to [GSI], I think he is a good teacher and works very hard but I cannot learn efficiently or effectively with his style of teaching.*

Often, students positioned UTLAs as someone who provided more thorough explanations, which they valued. However, some students shared a preference for their GSI's explanations, usually for similar reasons as those cited by students who preferred interacting with their UTLA. For example, one student described a preference for interacting with their GSI because of their focus on understanding "why the answer is the answer it is," in-depth explanations, and scaffolding. Nine students also described how differences in the knowledge of the UTLA or the GSI impacted their perceptions of interactions with them. One student expressed a preference for interacting with the GSI because the "[GSI] knows what he is talking about usually and can help with homework and [UTLA] cannot." Interestingly, a different student in the same section said "[UTLA] is usually more helpful with homework or workbook questions, so I ask him rather than [GSI]." Students focused primarily on how the UTLA or GSI explained content and answered questions, but some students did discuss the opportunities given to students to figure things out on their own and receive feedback.

Fifty-two students attributed dispositions that a GSI or UTLA may have to describe their interactions with UTLAs and GSIs. Most of these students described preferring to interact with someone that had dispositions which promote interactions (e.g., nice, personable, trustworthy, relatable, talkative, friendly, approachable). For example, one student described their GSI as "more personable so it is easier to have a conversation with him." However, students also shared

[2] These evaluations reflect student perceptions of their interactions with instructors and UTLAs; however, I purposefully am not making any claims about the quality of the teaching by UTLAs or instructors involved in this study.

dispositions that inhibited their interactions with either the UTLA or the GSI. These included dispositions likely to result in less communication with students (e.g., quiet, reserved) as well as traits that were likely to result in negative interactions (e.g., rude, complainer). Often students used antithetical dispositions to highlight differences between the GSI and UTLA:

> *If I have a question about a problem that I really do not understand, I will ask [UTLA]. He takes the time to explain things and is super helpful and patient. [GSI] always seems like they are in a rush to get to the next table or does not understand/ listen to what you are trying to ask.*

Table 2 provides a complete list of the dispositions used to describe UTLAs and GSIs when students also expressed a preference for interacting with either the UTLA or the GSI. The dispositions students ascribed to UTLAs and GSIs, and the connection of these dispositions to their preference for whom they interact, suggest that students view it as the duty of the UTLA and the GSI to be inviting.

Table 2. Dispositions Students Attribute to UTLAs and GSIs

Preference	LA Disposition	GSI Disposition
LA	approachable (2); calm; communicator; easy-going; engaging; friendly (3); helpful (5)*; kind; patient; less intimidating; chill; outgoing; passionate; personable; relatable (3)	difficult to approach; quiet; rude; rushed; straightforward; kind*; helpful*; passionate*
GSI	complainer; down to business; less engaging; reserved; nice*	easy-going; encouraging (4); engaging (3); friendly; helpful (2); nice (2); not personal but knowledgeable; personable (4); talkative; relatable; trustworthy; understanding (2)

Note. Each disposition was mentioned by one student unless otherwise noted (n)
*Three students described both their UTLA and their GSI using similar dispositions (kind and helpful; passionate; and nice) while still expressing a preference for the other teacher figure. Preference was dictated by other factors (e.g., perceived teaching quality).

Storyline 3: UTLA ≇ GSIs

The prior two storylines focused on duties held in common between UTLAs and GSIs. However, several students described interactions with the UTLA and GSI that suggest they perceive their roles to be different, and at times, complementary. This is distinguished from the storyline above, in which students often expressed a preference for working with either the UTLA or the GSI based on a set of common criteria. I capture this idea as the storyline: *UTLAs and GSIs have different duties in instruction*. This storyline was the least prevalent; responses from 17 students form the basis of this storyline.

Sometimes, but not always, students felt that interactions were more comfortable, easier, or less intimidating with UTLAs than with GSIs. Further, some of these students attributed this to belonging to the same community as the UTLA. For example, one student shared:

I don't act differently between [GSI] or [UTLA] but I feel slightly more comfortable with [UTLA] just because he's closer to my age. I feel that I can crack jokes and make silly mistakes when [UTLA] helps me and I'm not afraid to not know an answer.

This student positioned the UTLA as someone that allows them to be mathematically vulnerable due to their closeness in age. Another student said it was "slightly easier" to talk to the UTLA because they were a "student," whereas they viewed the GSI as a "professor." Thus, some students preferred to interact with the UTLA because of the UTLA's status as a peer, student, or because of their age. For others, gender played a role in how they perceived interactions with their GSI versus their UTLA. As an example, one student said:

I'm a college girl and so is [UTLA], it's just easier for us to interact I think. And I can more easily say when I don't understand something to her and stop her while she's explaining - I feel a little rude to do that to Professor [GSI].

Another student said they interact "better with women than men, and with students vs. teachers" and that she felt she had "more in common" with the UTLA than the GSI. Both students preferred to interact with their UTLA because they shared assumed commonalities in gender, but also academic status as a student. Some students acknowledged similarities between themselves and the UTLA but did not connect this to a preference for interacting with the UTLA versus the GSI. As an example, one student said "I think I treat [UTLA] more as a peer because we are closer in age" but did not say that they preferred one over the other.

Some students described intentionally interacting with GSIs and UTLAs for different purposes. One student explained that the "[GSI] is my professor so I come to her with more important or difficult questions. I also act more professional with her" whereas with their UTLA they are "much more relaxed" and ask "mundane questions." Other students discussed approaching the UTLA for help on particular components of the course (e.g., homework) while approaching the GSI for different help (e.g., clarification on the notes).

Discussion

There were likely multiple storylines students drew from to make meaning of their interactions with the UTLAs and the GSI. However, these findings suggest that most students viewed, at one point or another, the UTLA as having the same position as the GSI in the classroom, and further judged the value of the UTLA according to their perceptions of what good teaching is, the dispositions of the UTLAs and GSIs, and how they felt in those interactions. Furthermore, although only a relatively small number of students did position UTLAs as providing a unique perspective or difference in instruction, for those students the impact of having someone they could relate to was impactful to their experience. Many of these students described similarities between the UTLA and themselves (i.e., categorical membership in the same communities) as supporting their desire to engage with the UTLA. Other students valued the different perspectives and expertise brought by the GSI and UTLA and used these differences to guide who they interacted with and for what purposes.

A major storyline in the literature about UTLAs is that the UTLA are near-peers, and thus more relatable. Yet, several students positioned UTLAs in ways that run counter to this storyline. Further research could investigate more deeply how the identities of UTLAs and instructors influence students' storylines about UTLAs, particularly regarding the near-peer assumption. Research could also investigate the tacit and explicit ways that mathematics instructors and departments position UTLAs in active learning classrooms and connect that to the storylines evoked by students. Such research would expand our understanding of the roles UTLAs can play in supporting active learning mathematics classrooms.

References

Alzen, J. L., Langdon, L., & Otero, V. K. (2018). A logistic regression investigation of the relationship between the Learning Assistant model and failure rates in introductory STEM courses. *International Journal of STEM Education, 5*(1), 1–12. https://doi.org/10.1186/s40594-018-0152-1

Bamberg, M. G. (1997). Positioning between structure and performance. *Journal Of Narrative And Life History, 7*(1–4), 335–342. https://doi.org/10.1075/jnlh.7.42pos

Barrasso, A. P., Spilios, K. E. (2021). A scoping review of literature assessing the impact of the learning assistant model. *International Journal of STEM Education, 8*(12). https://doi.org/10.1186/s40594-020-00267-8

Clements, T. P., Friedman, K. L., Johnson, H. J., Meier, C. J., Watkins, J., Brockman, A. J., & Brame, C. J. (2022). "It made me feel like a bigger part of the STEM community": Incorporation of learning assistants enhances students' sense of belonging in a large introductory biology course. *CBE-Life Sciences Education, 21*(2), 1-13. https://doi.org/10.1187/cbe.21-09-0287

Davies, B., & Harré, R. (1999). Positioning and personhood. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 32-52). Blackwell Publishers.

Dawson, P., van der Meer, J., Skalicky, J., & Cowley, K. (2014). On the effectiveness of supplemental instruction: A systematic review of supplemental instruction and peer-assisted study sessions literature between 2001 and 2010. *Review of Educational Research, 84*(4), 609-639. https://doi.org/10.3102/0034654314540007

Deppermann, A. (2013). How to get a grip on identities-in-interaction: (What) does 'positioning' offer more than 'membership categorization'? Evidence from a mock story. *Narrative Inquiry, 23*(1), 62–88. https://doi.org/10.1075/ni.23.1.04dep

Gafney, L., & Varma-Nelson, P. (2008). Peer-led team learning: Evaluation, dissemination, and institutionalization of a college level initiative. Springer.

Goertzen, R. M., Brewe, E., Kramer, L. H., Wells, L., & Jones, D. (2011) Moving toward change: Institutionalizing reform through implementation of the Learning Assistant model and Open Source Tutorial. *Physical Review Physics Education Research, 7*(2). https://doi.org/10.1103/PhysRevSTPER.7.020105

Herbel-Eisenmann, B., Wagner, D., Johnson, K., Suh, H. & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics, 89*(2), 185-204. https://doi.org/10.1007/s10649-014-9588-5

Jardine, H. E. (2020). Positioning undergraduate teaching and learning assistants as instructional partners. *International Journal for Students As Partners, 4*(1), 48–65. https://doi.org/10.15173/ijsap.v4i1.4032

Kayi-Aydar, H. (2021). A framework for positioning analysis: From identifying to analyzing (pre)positions in narrated story lines. *System, 102*(102600). https://doi.org/10.1016/j.system.2021.102600

Laursen, S. (2019). *Levers for change: An assessment of progress on changing STEM instruction.* American Association for the Advancement of Science. https://www.aaas.org/sites/default/files/2019-07/levers-for-change-WEB100_2019.pdf

Otero, V., Finkelstein, N., McCray, R., & Pollock, S. (2006). Who is responsible for preparing science teachers? *Science, 313*(5786), 445–446. https://doi.org/10.1126/science.1129648

Otero, V, Pollock, & Finkelstein, N. (2010). A physics department's role in preparing physics teachers: The Colorado learning assistant model. *American Journal of Physics*, *78*(11), 1218–1224. https://doi.org/10.1119/1.3471291

Sellami, N, Shaked, S, Laski, F. A., Eagan, K. M., & Sanders, E. R. (2017). Implementation of a learning assistant program improves student performance on higher-order assessments. *CBE Life Sciences Education*, *16*(4). https://doi.org/10.1187/cbe.16-12-0341

Stringer, B. P. (2023). Learning assistants: The assets of relationships in mathematics classrooms. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (pp. 966-971). Omaha, NE.

Talbot, R. M., Hartley, L. M., Marzetta, K., Wee, B. S. (2015). Transforming undergraduate science education with learning assistants: student satisfaction in large-enrollment courses. *Journal of College Science Teaching*, *44*(5), 24–30.

van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 14-31). Blackwell Publishers.

Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention to classroom positioning. *Educational Studies in Mathematics, 72*(1), 1-15. https://doi.org/10.1007/s10649-008-9178-5

Webb, D. C., Stade, E., & Grover, R. (2014). Rousing students' minds in postsecondary mathematics: The undergraduate learning assistant model. *Journal of Mathematics Education at Teachers College, 5*(2), 39-47. https://doi.org/10.7916/jmetc.v5i2.653

Whitman, N. A. & Fife, J. D. (1988). *Peer teaching: to teach is to learn twice.* ASHE-ERIC Higher Education Report No. 4. Association for the Study of Higher Education.

Analyzing an Instructor's Oral Follow-Up Assessment Strategies in an Introductory Proof Course

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This study investigates the characteristics of an instructor's communication in one-on-one oral assessments for an introduction-to-proof course and how they impact student learning. It examines the instructor's strategies for evaluating the depth of students' knowledge. Using thematic analysis, the study examines patterns in question types and their relation to revision quality and presentation fluency. Findings reveal that well-revised proofs prompted questions about comprehension and logical structure. For revisions needing improvement, questions guided students to establish correct proofs before addressing comprehension. The study also explores teaching elements resulting from the instructor's assessment practice using commognitive theory. These findings contribute to assessment practices, particularly in revision and oral assessment for introduction-to-proof courses. By highlighting communication's role in student learning, this research advances alternative assessment understanding in higher education mathematics.

Keywords: revision, oral assessment, objective-based grading, commognition, proof feedback

One crucial element of teaching practice revolves around the creation and execution of assessment methods. Gaining insight into how an instructor carries out his/her teaching practices can offer valuable insights into the implementation of a course. Given the significant role mathematical proofs play in the field of mathematics, evaluating proofs forms an integral aspect of teaching practice. Prior research has delved into how mathematics instructors assess students' written proof submissions using conventional point-based systems (Miller et al., 2018; Moore, 2016). This study aims to extend our understanding by exploring oral assessment methods, specifically focusing on objective-based grading (OBG) in an introduction-to-proof course.

The course that is being studied in this paper implemented oral assessments to evaluate proof revisions under the implementation of OBG. In OBG, student performance is typically assessed as pass or fail, based on near-perfect or perfect solutions, with minor errors sometimes overlooked. However, making a judgment about how serious a mistake is can be a challenging task when evaluating mathematical proofs (Miller et al., 2018; Moore, 2016). Oral assessment, however, has the advantage of clarifying whether a small mistake was indeed insignificant to the overall mastery of the material through follow-up questions.

Studies highlight the advantages of oral assessments for evaluating students' proof skills (Soto-Johnson & Fuller, 2012; Stylianides, 2019) and reveal that when students orally present their proofs, they often offer more comprehensive explanations, resulting in stronger proof outcomes (Soto-Johnson & Fuller, 2012; Stylianides, 2019). Other studies have discussed the implementation and outcomes of the use of OBG (e.g., Cooper, 2020; Iannone & Vondrová, 2023; Prasad, 2020; Zimmerman, 2020), and have discussed the benefits of oral assessment (e.g., Iannone et al., 2020; Iannone & Simpson, 2012, 2015). However, there has been limited research on oral assessment with OBG and this study will inform readers about the instructor's communicative approach to the implementation of oral assessment in an introduction-to-proof class using OBG. The main research goals of this study are: (1) What are the discursive characteristics of the instructor's oral communication during oral follow-ups when grading students' proof productions? and (2) What is the significance of the instructor's communication practices in the oral follow-up assessment?

Literature Review

Objective-based Grading

The traditional grading system still dominates higher education mathematics in the U.S., employing partial credit for each problem submitted (Nilson, 2015). There is, however, an alternative grading approach known as OBG, also referred to as mastery-based grading or standard-based grading. Under this system, the course is organized around a set of specific objectives that students must successfully achieve. Students are provided with multiple chances to demonstrate their comprehension for each objective and any initial failure to pass an objective does not impact their overall grade; only the final outcome is counted (Campbell et al., 2020).

Studies consistently demonstrate that OBG offers numerous advantages for both students and instructors (Knight & Cooper, 2019; Linhart, 2020; Zimmerman, 2020). Students benefit from reduced testing anxiety through multiple attempts, learning from mistakes, fostering autonomy, and prioritizing learning over points. This approach can significantly impact the depth of their understanding and intrinsic motivation (Linhart, 2020). Instructors, on the other hand, find satisfaction in streamlined grading processes without the need for complex rubrics, appreciate their students' increased effort and improved quality of work, gain insight into students' capabilities and understanding through objective-based assessment, and experience enhanced purposefulness in their teaching (Knight & Cooper, 2019; Prasad, 2020).

Proof Evaluation Practices

Teaching goes beyond classroom instruction; as Moore (2016) noted that when professors assess students' proof productions, they effectively engage in teaching by making corrections, providing comments, and discussing students' thought processes to enhance their proof-writing skills. Evaluating proof productions is a challenging aspect of teaching, influenced by individual judgments of the seriousness of proof errors, student comprehension, and perceived abilities, leading to considerable scoring variability among professors, especially in traditional point-based grading (Miller et al., 2018). However, OBG minimizes this variation, resulting in more consistent grading outcomes, except in cases when the seriousness of proof errors is disputed.

Mathematicians grade students' proofs based on several aspects, such as logic, clarity, fluency, understanding (Moore, 2016), and perceived ability (Miller et al., 2018). Some professors also focus on linguistic and academic rules, which students often overlook (Lew & Mejía-Ramos, 2019). Surprisingly, students and professors have contrasting views about language conventions on proofs. For instance, students, regardless of language skills, do not realize the importance of correct academic English in proofs. Mathematicians emphasize complete sentences, with most deducting points for incomplete ones, while only a few students find this unconventional. In addition, professors and students have varying opinions about overusing variables in proofs (Lew & Mejía-Ramos, 2019).

Oral assessments may have distinct evaluation criteria compared to written assessments, especially when students prepare written solutions in advance without supervision. In oral assessments, follow-up questions are crucial for evaluating a student's understanding, making the grading criteria more complex (Iannone & Simpson, 2012). This study aims to contribute to mathematics education literature by shedding light on the evaluation of students' proof productions in oral assessments, addressing a gap in the existing literature. It will provide insights for both researchers and professors in higher education mathematics on the practice of oral assessments and proof feedback in higher education mathematics.

Theoretical Framework

From a commognitive perspective, human actions fall within specific discourses, each defined by its objects, communication methods, and participant rules, thereby establishing distinct communities of communicators (Sfard, 2008). For example, discussions centered on mathematics form a mathematical discourse. Within mathematics, a discourse is built upon structured activities known as mathematical routines, which Sfard (2008) defines as "a set of metarules that describe a repetitive action" (p. 208). These routines serve the purpose of generating narratives about mathematical concepts and objects.

Kontorovich (2021) introduced the concept of the didactical discourse on proof (DDP) to encompass the discourse of the teaching and learning of mathematical proofs. DDP has two key aspects: the pedagogical aspect and the mathematical aspect (Kontorovich, 2021). For the pedagogical role, DDP guides educators on what and how to teach in a proof course. For the mathematical aspect, participants must adhere to community-accepted rules for mathematics when discussing mathematical proofs. The primary goal of this research is to uncover patterns of DDP within the context of assessing mathematical proof productions through oral assessments.

He also examined routines in feedback practices using DDP and the commognitive framework to analyze proof feedback in a graduate topology course, focusing on comments and point deductions on graded assignments. He found that fully-scored proofs received comments related to mathematical concepts (mathematizing), while reduced-score proofs often received comments addressing the student personally (subjectifying). Point deductions were always linked to issues concerning the proof's idea, but not its representation (Kontorovich, 2021).

Viirman (2014, 2015) explored routines among mathematical instructors during lectures. Viirman (2014) discovered variations in lecture routines, including construction routines (common to all instructors, involving concept definitions and diverse examples) and substantiation routines (related to theorem proving, with varying emphasis). Viirman (2015) listed three key didactical routines: explanation routines for clarifying concepts, motivation routines to engage students, and question posing routines (featuring rhetorical questions).

The didactic practices of mathematics instructors remain relatively unexamined. There is very little (or no) existing research that delves into the routines involved in the execution of oral assessments within higher education mathematics courses. This study aims to contribute to the body of literature in mathematics education by exploring the routines employed by mathematics instructors when conducting oral assessments to evaluate mathematical proofs.

Methods

Context

The research was conducted in an introduction to proof course, involving four students and an instructor with pseudonyms of Amy, Clara, Dana, Gina, and Dr. Jones. The study focused on evaluating proof productions on quizzes, following a specific process. Initially, students completed a closed-book, in-person written quiz in which each learning objective was graded as a success, growing, or not assessable. Only a grade of success counted as passing and contributed to the final grade. For students who did not obtain success on certain learning objectives on the quiz, they had the opportunity to revise their solutions and present them in a one-on-one virtual oral follow-up assessment within a week of receiving feedback. During this oral assessment, Dr. Jones could ask additional questions related to the presented proofs.

Data Corpus

The data for this study was comprised of: (a) original proof productions by students, (b) feedback from the instructor, Dr. Jones, (b) students' revised written proof productions, (d) recordings of the oral follow-up meetings, and (e) interviews with Dr. Jones. The study concentrated on proof constructions, assessed on Quizzes 2 through 5. Four participants (Amy, Clara, Gina, and Dana) presented 4, 10, 10, and 3 revised proofs, respectively, in some of the oral follow-up assessment meetings.

The interviews with Dr. Jones were conducted twice through Zoom using a semi-structured format, covering various themes. The main topics discussed during the interviews included Dr. Jones' teaching background, course structure, quiz assignment procedures, oral follow-ups, written feedback practices (both general and specific), and the implementation of mastery grading. The first interview with Dr. Jones took place at the end of the semester, and the second interview occurred approximately three months after the course had concluded, focusing more on the implementation and outcomes of mastery grading in the course.

Analysis

The types of follow-up questions were analyzed in relation to the quality of revised proof productions. The revised productions were categorized into one of the following: successful revised proof production with no improvement or correction suggested by Dr. Jones - *coded as P1*; revised proof production that is correct but could be improved by adding more information, such as additional details or explanations, or by refining some phrases - *coded as P2*; revised proof production that has one or more mathematical errors – *coded as P3*.

The oral follow-up recordings were the main data for this study. The thematical analysis coding was conducted using MAXQDA software for qualitative analysis. Because of our interest in depicting Dr. Jones' communication routines, the codes were developed to emphasize the aspects of communication in Dr. Jones' phrases. We particularly focused on capturing on the type of communication, rather than the specific content of the communication itself. For instance, when Dr. Jones inquired, "Why did you change the quantifier?" the code used is "question on changes" rather something related to quantifier. This approach allows us to center the analysis on the communication aspects rather than the mathematical aspects.

Results

The analysis revealed that Dr. Jones employed two distinct approaches when interacting with students based on the quality of the revised proof productions (P1, P2, or P3): the routines when encountered with correct narratives and reconstruction of an alternate narrative.

Routines when Encountered with Correct Narratives

There are two possible cases of Dr. Jones' actions when encountered with correct proof productions: the proof was instantly endorsed without any follow-up questions or there was an examination of the students' comprehension of the proof presented. When questions were posed, they were usually centered around logical aspects of the proof or the rationale for changes in the proof.

From a total of 27 proof presentations in the oral follow-up sessions, 11 of them were categorized as P1, which means the revised proofs had no issues. This categorization was based on the nonexistence of Dr. Jones' comments for disapproval or comments on improvement. Among these 11, there were five performances that received successes instantly. During the interview with her, Dr. Jones pointed out two important aspects describing the circumstances that do not require follow-up questions. The first aspect is the authenticity of writing. This is important because the nature of the assessment allows the students to access any resources available to them. However, they are expected to be able to write it in their own words. The second aspect is the fluency of their verbal communication in which the presentation of the proof should sound very natural. Therefore, a good quality of mathematical verbal communication is valued not only in terms of its validity, but also the fluency of the explanation part.

In most cases, error-free revised proof productions were subject to scrutiny by Dr. Jones. This was done to ascertain whether students had indeed learned and understood the material that they presented. Dr. Jones mentioned that "the oral follow-up gave me that option to figure out if they just copied it from somewhere or if they had really learned something." The routines of Dr. Jones implemented when she encountered an error-free proof productions can be distinguished into two types of inquiry: Question/prompts related to the logical structure of the proofs and questions/prompts related to the changes from the originals to the revised versions of the proofs.

Question/prompts related to the logical structure of the proofs. Dr. Jones collected information about the students' logical structures through implicit and explicit methods. In the explicit approach, Dr. Jones would request students to explain the reasoning behind their proof structures. Here's an illustration of how this explicit examination of the proof's logic works.

Gina revised her proof for the following objective: "I can construct a correct proof of a conditional statement involving sets." The statement to prove was as follow: Let A, B, V, and W be sets. Prove that if $V \subseteq A$ and $W \subseteq B$, then $V \cap W \subseteq A \cap B$. After Gina presented her revised proof for this objective, Dr. Jones asked her the following question, "So, what's the key word that tells us ultimately that definition of subset? That key word that's allowing us to conclude V intersect W is a subset of A intersect B?" Gina replied, "The key word?", to let Dr. Jones know she did not understand the question. Dr. Jones then clarified by stating, "So, somewhere in your proof you're saying by definition of subset. So, what is that definition of subset? Where's that in your proof that it's allowing you to make that conclusion?"

Dr. Jones questioned Gina's understanding of how the definition of "subset" influenced her proof's logic. Gina struggled to identify this crucial element in her proof. After some discussion, Dr. Jones clarified further, "that's the thing [circling the word *implies*] that actually allows you to make this conclusion [underlying $V \cap W \subseteq A \cap B$] at the end. Because if you had written that as an and then you will be talking about intersection […]. So, this is [pointing out the word *implies*], this word is what's telling us that you're using this definition [the definition of subset]."

In the implicit approach, Dr. Jones questioned the student about a statement's definition without directly connecting it to the proof. This allowed Dr. Jones to assess the consistency between the applied definition and the student's narrative. In the following discussion, Dana presented her revised proof for the statement: "If $A \subseteq B$, then $A - C \subseteq B - C$". After Dana presented the proof, Dr. Jones asked, "Alright. Here's two statements [wrote "$x \in A$ and $x \in B$" and "if $x \in A$, then $x \in B$"]. Are those two statements the same or are they different?" Although Dana accurately revised a proof that relies on the concept of a subset, she struggled to correctly identify the definition of a subset. Dr. Jones then provided the correct information in which she explained, "So, this one is the definition of subset [pointing to the second one]. We're talking about we've got a set A and it's inside of some larger set B. But the first one, we're talking about our two sets don't necessarily have to be nested and things are just in between them. So, this is the definition of intersection [pointing out to the first one]." This exchanged suggests that Dana might not have had a complete grasp of the logical structure underlying her written proof.

Dr. Jones did not immediately allow Dana to falter; instead, she offered her another opportunity through a series of different questions. Dr. Jones then tried to delve into Dana's logical structure more directly on another proof by asking, “How do you know to start with x is an element of A minus C?” During the second round of questioning, Dr. Jones found Dana's response satisfactory and deemed it a success and choose to provide further clarification after acknowledging the student's success, emphasizing the answers to the questions.

Questions/Prompts Related to Changes Made. On many occasions, success in the follow-up meeting requires students to not only construct a correct proof but also explain why the initial proof was rejected. This ability is crucial for meeting the learning target. Failing to articulate the errors in the original proof, even if the revised one is correct, can still result in failure.

This example illustrates how Dr. Jones probed a student’s understanding on the changes she made from the original to the revised proof. Gina presented her revised proof for the following problem: “Let A, B, and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$." In Gina's initial proof, she had notational problems, like omitting parentheses when denoting an element in the Cartesian product. However, she successfully rectified this issue in her revised proof, prompting Dr. Jones to pose the question, “Also, things like parentheses around that x, y, why is that important?” A student who presented a revised proof correctly may still not achieve success if she cannot provide a clear explanation for why the alterations were required.

Reconstruction of an Alternative Narrative. When a student presents a correct proof production, Dr. Jones may make a point to highlight an alternative version. The alternative version can involve a minor modification or substantial changes to student’s version of the proof. Among the data collected, there were three instances when Dr. Jones constructed alternative proof productions. The construction process was conducted through a dialogue with the student in which Dr. Jones guided the process. For example, Dr. Jones illustrated that a proof by cases (with even and odd integers cases) could be simplified to a straight-forward direct proof.

Routines when Encountered with a Flawed Narrative

When a student presented a revised proof that had an error, Dr. Jones provided her with prompts and questions to rectify the issue. The observed flaws in students' revised proofs fell into two categories: those needing additional information or details (P2) and those that were mathematically or logically incorrect (P3).

Eliciting Missing Information. When gaps in the proof arose from missing information, Dr. Jones prompted students to fill in that missing information. In one of Clara's revised proofs, there was a lack of crucial details regarding the requirement for a valid fraction, specifically that the denominator should not be zero. The excerpt below shows how Dr. Jones guided Clara to provide more details about the object's properties used in her proof. Dr. Jones began with the question, “So, those integers a and b that you’re using to make up as a rational [number], is there any other condition that should be there?” Clara replied, “Yeah, that they are co-prime, right?” Dr. Jones then stated, “Co-prime, I’m less concerned about that” and Clara exclaimed, “Okay, well, b is not zero.” Dr. Jones replied encouraging, “Yeah, that one is kind of important.”

Exploration of Errors. When a student presented a revised proof that had some mathematical errors, Dr. Jones would invite the student to discuss the revision. An example of this was shown in Amy’s revised proof presentation of the following statement: “Prove: For all integers x and y, if $x + y$ is odd, then $x \neq y$." In her revised proof, Amy initially wrote the contrapositive statement as, "There exist some integers x and y such that $x = y$.” In response to this alteration, Dr. Jones posed the question, "Why did you change the quantifier?" This inquiry prompted Amy to reconsider her choice regarding the quantifier. After careful thought about the quantifier and statement, Amy concluded that altering the quantifier was not necessary when applying the contrapositive. Dr. Jones confirmed the validity of Amy's explanation regarding the equivalence of the contrapositive statement. In summary, when a student struggled fixing a proof error, Dr. Jones would start the proof reconstruction process, which might include verbal guidance or an explicit conversation-based reconstruction.

Summary and Discussion

Dr. Jones' approach to assessing oral proof presentations combined flexibility and thoroughness. Dr. Jones occasionally granted success without follow-up questions for authentic and fluently presented valid proofs. However, she frequently asked follow-up questions to confirm understanding, even when proofs appeared correct These queries focused on the logical structure or the reasoning behind revisions. These inquiries helped gauge student comprehension, considering any resources were allowed. When alternative solutions are available, Dr. Jones would highlight this and occasionally provide explanations or walk students through the proof's reproduction. She also gave students opportunities to correct errors through questions or prompts to extract missing information or initiated discussions to encourage self-identification and correction of the proof errors.

Oral follow-up questions revealed a key insight: constructing a correct proof does not necessarily imply a deep grasp of the foundational concepts it relied on. These thought-provoking questions played a pivotal role in preparing students to actively participate in discussing the "why" behind the proof, delving into the rationale and reasoning that support their revisions and the validity of their proofs. This active engagement, in which students defended their arguments, fostered an increased awareness of the meta-discursive rules inherent in constructing a mathematical proof. Encouraging students to articulate and justify their thought processes helped them understand the fundamental principles and logical connections in mathematical proofs, improving their overall communication skills in this context.

Dr. Jones' assessment method aligns well with the conceptualized commognitive process "of becoming able to have mathematical communication" (Sfard, 2007, p. 573). Through revisions and oral presentations, students not only share their proofs but also actively engage in the written and oral communication to deepen their understanding of proofs. This study highlights the effectiveness of oral follow-up assessments in improving communication skills and shifting the focus from acquiring knowledge to expressing mathematical concepts sufficiently.

The act of verbally explaining ideas during the oral follow-up sessions allows students to become involved in an authentic mathematical conversation which promotes mathematical discourse and think critically when responding to questions related to their proof productions. Moreover, through follow-up questions, the students will learn that they need to be responsible to defend every statement presented in their proof narrative. This promotes the metalevel learning on substantiation in their mathematical proofs.

As a final remark, we observed that this assessment method also offers a valuable teaching and learning platform, allowing Dr. Jones to guide students in error identification and proof reconstruction on a one-to-one basis, which is highly effective for learning. Beyond assessing proof productions, the oral assessment encourages mathematical discourse and discussion, promoting deeper understanding and knowledge enrichment. Furthermore, it prioritizes proof comprehension over construction, with discussions shifting from correcting construction errors to probing comprehension when students present correct proofs, thereby facilitating a more comprehensive assessment of students' mathematical understanding.

References

Campbell, R., Clark, D., & OShaughnessy, J. (2020). Introduction to the Special Issue on Implementing Mastery Grading in the Undergraduate Mathematics Classroom. In: Taylor & Francis.

Cooper, A. A. (2020). Techniques grading: Mastery grading for proofs courses. *PRIMUS*, *30*(8-10), 1071-1086.

Iannone, P., Czichowsky, C., & Ruf, J. (2020). The impact of high stakes oral performance assessment on students' approaches to learning: a case study. *Educational Studies in Mathematics*, *103*(3), 313-337.

Iannone, P., & Simpson, A. (2012). Oral assessment in mathematics: implementation and outcomes. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *31*(4), 179-190.

Iannone, P., & Simpson, A. (2015). Students' views of oral performance assessment in mathematics: straddling the 'assessment of'and 'assessment for'learning divide. *Assessment & Evaluation in Higher Education*, *40*(7), 971-987.

Iannone, P., & Vondrová, N. (2023). The novelty effect on assessment interventions: a qualitative replication study of oral performance assessment in undergraduate mathematics. *International Journal of Science and Mathematics Education*.

Knight, M., & Cooper, R. (2019). Taking on a New Grading System: The Interconnected Effects of Standards-Based Grading on Teaching, Learning, Assessment, and Student Behavior [research-article]. *https://doi.org/10.1177/0192636519826709*. https://doi.org/10.1177_0192636519826709

Kontorovich, I. (2021). Minding mathematicians' discourses in investigations of their feedback on students' proofs: a case study. *Educational Studies in Mathematics*, 1-50.

Linhart, J. M. (2020). Mastery-based testing to promote learning: Experiences with discrete mathematics. *PRIMUS*, *30*(8-10), 1087-1109.

Miller, D., Infante, N., & Weber, K. (2018). How mathematicians assign points to student proofs. *The Journal of Mathematical Behavior*, *49*, 24-34.

Moore, R. C. (2016). Mathematics professors' evaluation of students' proofs: A complex teaching practice. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 246-278.

Nilson, L. B. (2015). *Specifications grading: Restoring rigor, motivating students, and saving faculty time*. Stylus Publishing, LLC.

Prasad, P. V. (2020). Using revision and specifications grading to develop students' mathematical habits of mind. *PRIMUS*, *30*(8-10), 908-925.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The journal of the learning sciences*, *16*(4), 565-613.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Soto-Johnson, H., & Fuller, E. (2012). Assessing proofs via oral interviews. *Investigations in Mathematics Learning*, *4*(3), 1-14.

Stylianides, A. J. (2019). Secondary students' proof constructions in mathematics: The role of written versus oral mode of argument representation. *Review of Education*, *7*(1), 156-182.

Viirman, O. (2014). The functions of function discourse–university mathematics teaching from a commognitive standpoint. *International journal of mathematical education in science and technology*, *45*(4), 512-527.
Viirman, O. (2015). Explanation, motivation and question posing routines in university mathematics teachers' pedagogical discourse: a commognitive analysis. *International Journal of Mathematical Education in Science and Technology*, *46*(8), 1165-1181.
Zimmerman, J. K. (2020). Implementing standards-based grading in large courses across multiple sections. *Primus*, *30*(8-10), 1040-1053.

The Problems with “Any” and “All”: How Language Impacts Student Meanings for Quantified Variables

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This study investigates undergraduate students’ meanings for quantified variables, and how the language of a mathematical statement impacts their meanings. I conducted 3-day exploratory teaching interviews with eight students from transition-to-proof and advanced calculus courses (four from each course). Over the course of the three days, I presented students with a variety of statements from calculus in order to test how both and language and mathematical content might impact students’ meanings for quantified variables. I found that students’ meanings did often change across mathematical contexts that contained different language. In particular, the words “any” and “all” were found to lead some students to use different meanings for quantified variables in calculus statements.

Keywords: quantification, quantifiers, meaning, calculus, mathematical language

Recently, many have called others to focus on students’ logic in elementary calculus courses (Case, 2015; Dawkins & Cook, 2017; Sellers et al., 2021). This is because many calculus statements are *complex mathematical statements* (Sellers, 2018), i.e., involving quantified variables and logical connectives. Yet, when investigating student quantifications of individual quantified variables, some students, in particular calculus students, use different quantifications for individual quantified variables, if they quantify at all (Sellers et al., 2021).

Furthermore, other researchers have called for more investigation into students’ meanings for quantified variables as related to the specific language and grammar given in statements across different mathematical domains (Shipman, 2013; Vroom, 2022). As such, students’ quantifications of the variables in these statements as related to the specific language given in the statements is important to investigate if we want to help them understand calculus ideas.

Others have noted that students’ logic might vary from one mathematical context to the next (Dawkins & Roh, 2020; Dubinksy & Yiparaki, 2000; Epp, 1999; Shipman, 2013). Thus, there is a need to compare students’ meanings for quantified variables across different mathematical contexts which involve different language and grammar.

This study is part of a larger study investigating both students’ interpretations for quantified statements from calculus and their negations of these statements (Sellers, 2020). In this paper, I focus on additional information gleaned regarding students’ meanings for quantified variables as related to the language given across different statements. Specifically, I focus on findings that extend the work of Sellers et al.’s (2021) framework for students’ meanings for quantified variables and continue to address the following research question: *How does language impact student quantifications across different mathematical contexts in calculus?*

Literature Review

Several researchers have noted the complexities of interpreting quantifiers specifically in complex mathematical statements (David et al., 2020; Dubinsky et al., 1988; Selden & Selden, 1995; Sellers et al., 2021). Furthermore, some have posited that interpreting quantified statements may be inherently difficult for students due to the different purposes for quantifier words and logical connectives in mathematics and colloquial English (Epp, 1999; Dawkins &

Cook, 2017; Grice, 1975; Roh, Lee, & Tanner, 2016). For example, if I state the following universally quantified statement, "Every book on the shelf is French," I colloquially infer there is at least one book on the shelf, whereas the statement could be vacuously true in propositional logic (Epp, 1999; Johnson-Laird, et al., 1989). Even more striking, there may be cases where the meaning of quantifier may be completely different in everyday language than in mathematics. Shipman (2013) concluded that students thought "unique" meant "unequaled" instead of "sole," and conjectured that her students' use of the word may have been attributed to the colloquial use of the word. Roh, Lee, and Tanner (2016) found that students assumed the article "a," attached to the existential quantifier "there is," referred to *one* single quantity.

While these previous works do suggest that colloquial language impacts students' reasoning in the mathematics classroom, I do not mean to suggest that instructors should utilize formal semantics or symbolism in the presentation of logical rules exclusively. Stylianides, et al. (2004) found that elementary majors' ability to reason about the truth of statement was higher in verbal cases than symbolic ones. Thus, verbal representations of mathematical ideas are not necessarily to be avoided; rather, an awareness of these colloquialisms may help teachers address these inconsistencies in natural language and mathematics as the need arises in their classrooms.

Theoretical Perspective

The findings from this study use and extend Sellers et al.'s (2021) framework for students' meanings for quantified variables. Similar to Sellers et al. (ibid), I approach this study from a constructivist (Glasersfeld, 1995) perspective. Thus, my focus is on *individual* students' meanings for quantified variables in a variety of calculus statements. A meaning is a scheme, and resides in a student's mind (Piaget, 1977; Thompson, et al., 2014).

Sellers et al.'s framework for student quantifications. In Sellers et al.'s (2021) framework for students' meanings for quantified variables (i.e., quantifications), the authors previously identified five different quantifications. Each of meanings for quantified variables (i.e., quantifications), abbreviated MQ1-MQ4 and NQ, are detailed below. MQ1-MQ4 are *not* intended to be developmental; they are only numbered for convenience of discussion.

MQ1 and MQ2 are indicative of meanings similar to conventional meanings for "there exists and x" and "there exists a unique x," respectively. Students using MQ3 in a specified moment chose a value for x from their chosen domain of discourse and determined whether or not their chosen value for x satisfies the predicate. They then repeated (or imagined repeating) this mental action by checking the predicate for *multiple* values of x until they exhausted *all* the elements in the domain of discourse.

While MQ1-MQ3 are akin to mathematical meanings for the existential, existential unique, and universal quantifiers, MQ4 does not align with mathematical convention. Students using MQ4 chose or looked at a value of x and then checked the predicate only for this selected value. These students sometimes chose another value of x and checked the predicate for this other value, but these values were not strategically selected, but *spontaneously* selected.

Finally, there were other moments where students did not quantify at all, and thus were described as using no quantification (NQ). These students appeared to search for the domain of discourse for a variable, but they did not appear to check the predicate for any values of x. Students that used NQ did not state that they are searching for a specific number of elements of x to satisfy the predicate, but instead interpreted the statement as if it were true. Furthermore, these students simply confirmed what they think the predicate means to them, rather than checking whether or not the values satisfy the predicate.

Methods

In order to further investigate students' meanings for quantified variables in calculus statements, I interviewed four students currently in transition-to-proof courses and four students from advanced calculus or above as part of a larger study.[1] I chose these students because they had *already completed* an entire calculus sequence, and had thus, been exposed to all of the necessary concepts in the statements. These students were selected from a pool of 24 students based on a survey used for theoretical sampling (Patton, 2001). Selected students participated in three exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020; Thompson et al., 2014).

Each exploratory teaching interview (ETI) lasted approximately 1-3 hours each, for a total of 5-8 hours, depending upon student responses. During these interviews, students were presented with 6 open statements and graphs of 4 functions and then asked to justify why the statements were true or false for each individual function. The six open statements are presented in Table 1, and the four graphs which were given are shown in Figure 1.

Table 1. Statements given in ETI 1 & ETI 2.

Statements	Statement Type
Statement 1 There exists a real number c in [$^-$1, 8.5] such that for all real numbers r in [$^-$1, 8.5], $f(r) \leq f(c)$.	$\exists x_1 \forall x_2$
Statement 2 For all real numbers w in [$^-$1, 8.5], there exists a real number k in [$^-$1, 8.5] such that $f(w) \leq f(k)$.	$\forall x_1 \exists x_2$
Statement 3 There exists a real number m such that for all real numbers p in [$^-$1, 8.5], $f(p) \leq m$.	$\exists y \forall x$
Statement 4 For all real numbers d in [$^-$1, 8.5], there exists a real number z such that $f(d) \leq z$.	$\forall x \exists y$
Statement 5 There exists a real number t in [$^-$1, 8.5] such that for all real numbers v, $f(t) \leq v$.	$\exists x \forall y$
Statement 6 For all real numbers j, there exists a real number q in [$^-$1, 8.5] such that $f(q) \leq j$.	$\forall y \exists x$

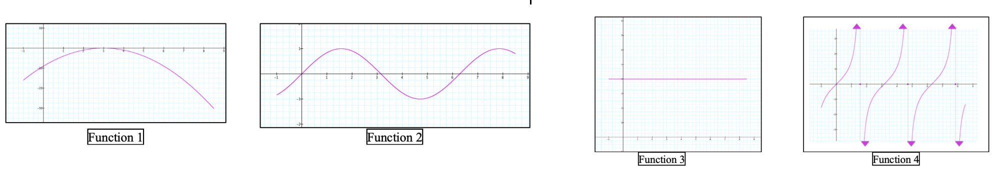


Figure 1. Graphs presented in first two exploratory teaching interviews.

The graphs of functions shown in Figure 1 provided opportunities for me to ask students to explain their thinking about the truth values of the given statements in more detail. Furthermore, these graphs were vetted along with Statements 1-6 in pilot studies and were found to be beneficial tasks for eliciting distinctions in student quantifications. Consider the Extreme Value Theorem (whose conclusion is shown in Statement 1). Since the sinusoidal graph (Function 2) has two values of the input that correspond to the same maximum value, some students in pilot studies stated that Statement 1 is false for Function 2 if they quantified "there exists" as "there exists a unique" (i.e., used MQ2 instead of MQ1).

In the final interview, I presented students with several alternative statements to test how well their quantifications transferred to different mathematical contexts. Additionally, I would

[1] The larger study focused on both students' interpretations and their negations of complex mathematical statements.

often change the quantifier language of these statements and ask interviewees if this language impacted their meanings for the statement. One of these statements is shown below:

Statement A: For all differentiable functions *f* on [*a*, *b*], there exists a *c* in [*a*, *b*] such that $f'(c)=0$. Statement A (Alternative): Suppose *f* is any differentiable function on [*a*, *b*]. Then there exists a *c* in [*a*, *b*] such that $f'(c)=0$.

Data analysis. While interviews were being conducted, I made initial models for students' quantifications and interpretations for each statement using Sellers et al.'s (2021) framework and also recorded students' evaluations for each statement across different moments. After each student's data was collected, I completed both a transcription and content log for each student interview. Initial open coding involved determining how to split my data into *relevant moments* (Sellers et al., 2021) where a student explained their interpretation of a statement or explained why a statement was satisfied. I determined that a new moment began whenever a student was presented with a new question or task, the student changed their evaluation of a statement, or if the student provided a new interpretation of a statement.

Once relevant moments were identified, I conducted iterative coding using Sellers et al.'s (2021) framework, as I could, for each student one at time, moment-by-moment, until I had codes that explained student quantifications across all moments for all students. However, a new code emerged in this process of trying to best explain student quantifications. I then re-observed students' words, gestures, and markings on graphs in order to characterize their mental actions that are associated with their quantifications until all the codes I made had explanatory power across all moments for all students in the study.

Finally, I used the evaluations that I coded along with the raw student data in order to determine why students evaluated statements in particular ways. Codes emerged in this phase for meanings that comprised students' interpretations of complex mathematical statements that also explained students' evaluations. For purposes of this paper, I only focus on one code or theme that emerged that related to students' interpretations of complex mathematical statements: students' meanings for quantifier language.

Results

In this study, three of the eight students in my study exhibited different quantifications *consistently* depending upon the statements at hand. Two of these students were in Transition-to-Proof at the time of the study, while one student was in advanced calculus. For purposes of this paper, I focus my attention on one student, Allison, who was currently in Advanced Calculus at the time of the study. I use Allison as a way in which to highlight some of the implications of quantifier language upon mathematical meanings.

The set-wise collection meaning and its connection to "all." The three students mentioned above each claimed that the word "all" led them to think of some variables as sets, and described how they interpreted and evaluated other statements with different quantifier words. Statements 1-6 all have universally-quantified variables and each of those variables are attached to the word "all." When students described these variables, I noticed that they often used additional words such as "all *at once*" to convey that they imagined grouping the values of the universally-quantified variables. In this subsection, I describe how student words for the values of the universally-quantified variable such as "all at once," "whole interval," and "entire set" helped me identify and describe this quantification.

I utilize the following moment with Allison to highlight how the word "for all" impacted her interpretation of Statement 2 ($\forall w(\exists k\, f(w) \leq f(k))$) along with Graph 4 (the modified tangent graph). Before this moment, Allison had determined that Statement 2 was false for the modified tangent function. During this moment, she was in the process of analyzing her evaluation for graphs against other student arguments. I asked Allison to examine several alternative student responses. I referred to an alternative student, Emma, and her argument for Function 4. Below is the argument I presented to Allison:

Alternate Student Argument: Emma said, "If I put w at 0, then $f(w)=0$, and if I put k at 8, $f(k)$ is approximately 1. In this case, the statement is true because $f(w)$ is less than or equal to $f(k)$. However, if I put w at 1.5, $f(w)$ would be approximately 1, and if I put k at 0, then $f(k)$ would be 0. In this case, the statement is false for this graph because $f(w)$ would be greater than $f(k)$. Therefore, I think that Statement 2 is sometimes true for this graph."

I presented Emma's argument for Function 4 in order to examine Allison's quantifications for each variable and why these quantifications led her to a false evaluation of Statement 2 for this function. I designed Emma's argument to represent the use of MQ4 for the variable w, since Emma checked the predicate for values of w individually, and then concluded that the statement is sometimes true. Allison rejected Emma's argument in the moment below based upon her own quantification for w.

Interviewer: [...] Do you think Emma's argument is the same as your argument?
Allison: She [Emma] was still just considering it one piece at a time, not all cumulatively (gestures as shown).

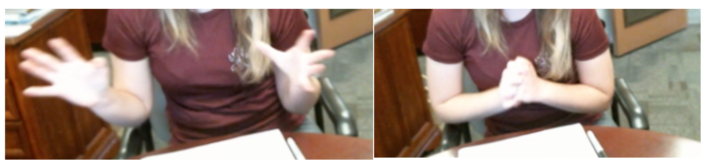


Figure 2. Allison's gestures when discussing w in Statement 2.

Interviewer: When you say all cumulatively, what are you thinking about in your head [...]?
Allison: I think if [...] you put k at 8, then you can't just check the case that w is 0 and say that that's the case. Because the statement is asking about all $f(w)$'s, I kind of feel like you have to like check them all. […] (still explaining Graph 4 from her perspective) Since this [function] is going to infinity, if I choose one [function output] that's higher up, and I say this [new point] is now w, then that's going to be false because you didn't choose a k for all of these numbers [$f(w)$] [...]
Interviewer: So, do you have to think about all of the values of w and all of their output values together at once?
Allison: Yes.
Interviewer: Okay, can you explain that a little bit more?
Allison: Well, you definitely have to think about all of the w's, because it says "for all." And if we're thinking about all the w's, then we're thinking about all their outputs being less than or equal to this one k and $f(k)$.

In determining Allison's meaning for the variable w, I used her descriptions of the statement, the values of w in relation to Function 4, as well as her gestures. First, I noted that Allison used the words "all cumulatively" which suggests that she was mentally grouping the values of w together. Additionally, when I asked if she was considering the values of w and $f(w)$ "all... at once," she confirmed that this is how she was thinking about the values as well. Beyond her description of her own thinking, she also stated that if she changed the values of w, that this would not be an accurate interpretation of the statement, in her view, because then she could not choose a k for *all* of the values of w, and that thinking about all values made her think that all of the values of w together had to be less than a singular $f(k)$ value. Additionally, Allison's gesture in Figure 2 that accompanied her words "all cumulatively," further indicates that she is collecting all the values of w into one set. I coded Allison's meaning for the variable w in this moment as a "set-wise collection" meaning for a quantified variable. (In light of Sellers et al.'s (2021) framework, I extended the framework and will refer to this meaning as MQ5.)

There are two different ways that a student with a normative meaning for "for all… there exists…" (∀∃) and "there exists… for all…" (∃∀) statements might map elements of one set of values to another set. First, for a statement of the form ∀x∃y, *each x* may be mapped to a different y. Yet, for a statement of the form ∃y∀x, the same y-value is mapped to *each x*. Yet, for the student who uses a set-wise collection meaning views "For all x, $P(x)$" as indicating that he needs to check the predicate for the *entire set*, X, at *once*, and grouping the set of x's into X before the predicate is checked. For a quantified statement of the form ∀x∃y, the student who uses a set-wise collection meaning for x will group the entirety of X at once *before* mapping and searching for the existence of a y-value. As a result, the student needs a *singular y-value* to satisfy the predicate for every x-value. Thus, for both ∀∃ and ∃∀ statements, the student will refer to an individual value of y that must satisfy the predicate. This set of cognitive steps is shown as a mapping in Figure 3.

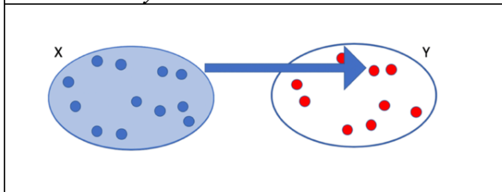
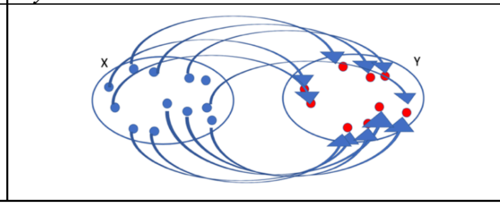
Set-wise Collection Meaning for x in "For all x… there exists a y…"	Conventional Meaning for "For all x… there exists a y…"
	

Figure 3. Illustration for student using set-wise collection meaning for universally- quantified variable.

Six out of the eight students (three Transition-to-Proof students and three Advanced Calculus students) I interviewed had some moments where they used MQ5 for quantified variables. MQ5 was used by these students only for universally-quantified variables, and typically when the quantifier word "all" was given in a statement.

How "any" leads to alternative quantifications. There were some cases in this study where ∀∃ statements using the words "all," "any" or "every" led to different cognitive processes than normative interpretations. In the next example, I present a moment with Allison to explain how students' quantifications for the word "any" can also deviate from convention.

On Day 3, I also posed both Statement A and an alternative version of Statement A, using the word "any" instead of "all" (see methods). While Allison did interpret Statement A as an ∀∃ statement, she did not interpret "for any... there exists..." as an ∀∃ statement. Below, I present Allison's description of how this language change impacted her interpretations of the two different statements in the moment below:

> *Interviewer*: So suppose I change this word here "all" [in Statement A] to the word "any." Would that change the meaning of Statement A?
> *Allison*: Seems like it kind of does.
> *Interviewer*: Ok, how so?
> *Allison*: Because, if you say "for any," I feel like that means like you have all of these choices of differentiable functions, and you pick one of them, and you happen to get lucky and you pick one that satisfies that this condition, that it's true.
> *Interviewer*: Okay, alright, so the word "any"-how does that… how is that different than the word "all?"
> *Allison*: So "for all" I feel like you have to […] go through all of them and make sure they all satisfy this condition. Whereas "for any," you have a choice of which one, which differentiable function you pick.
> *Interviewer*: So, then, how would you evaluate that statement if it had the word "any" there instead of "all?"
> *Allison*: I think it would be true, because when I hear the word "any," I just think of like […] any one you want.

Allison was classified by her use of MQ1 (that at least one value of f satisfies the predicate) for the variable f in the alternative version of Statement A that contains the word "any." I classified her meaning as MQ1 because she said that "any" made her think that she could select *one* function out of all of them that would make the statement *true*. Allison interpreted “any” as “any one you want,” and she explained that she wanted one that would make the statement true. This means that for Allison, “any” indicates that you can *choose one* from *an entire set*. As a result, she interpreted the alternative version for Statement A like a “there exists... there exists...” statement, rather than as an ∀∃ statement.

The example with Allison above reveals that the change of the word “all” to the word “any” in an ∀∃ statement should not be considered an automatic fix that will help all students interpret the statement as an ∀∃ statement. Additionally, one should note, changing “all” to “every” is not an automatic fix either, as another student used MQ5 when he was given the word “every.”

Conclusion & Discussion

One of the main findings of this study was a new theoretical addition to Sellers et al.’s (2021) framework for student meanings for quantified variables. I call this meaning MQ5, or the set-wise meaning for quantified variables. In Table 2 below, I show how the framework has been extended to include this new meaning.

Table 2. Characteristics of MQ5, the set-wise collection meaning for quantified variables.

Observable Behaviors	Mental Actions
• Identifies a domain of discourse, X. • Explains or illustrates whether or not X, the collection of all x's, satisfies the predicate. (May use gestures such as marking off boundary or use a grouping gesture.) • Uses words such as “whole,” “the set of,” or “entire” to refer to the collection of values of x in X; Uses words such as “all” or “all at once” to refer to X’s satisfaction of the predicate.	1. Identifies a domain of discourse, X. 2. Determines individual values (x_0) that comprise X. 3. Considers the collection of values of x from X as a singular unit. 4. Checks if the predicate is satisfied by the collective set X at once (i.e., checking $P(X)$, not $P(x)$).

In addition to this new meaning for quantified variables, this study also contributes to our knowledge on how language choice in mathematical statements impacts student quantifications. The word “for any” tends to lend itself to having students use MQ1, because they think “any” implies “any one that makes a statement true,” whereas “for all” might tend to lead students to the use of MQ5, or a set-wise collection meaning for a quantified variable.

Additionally, this study connects how the language of mathematical statements might relate to students’ interpretations of complex mathematical statements holistically. The word “for all” led students to using the set-wise collections meaning for quantification, which then led them to treat ∀∃ statements like ∃∀ statements, similar to Vroom’s (2022) finding. The theoretical construct of MQ5 allowed me to explain how students equate ∃∀ and ∀∃ statements in their minds by using set-theoretic interpretations of these meanings. On the other hand, the word “any” led the same student to treat a different ∀∃ statement as an ∃∃ statement.

This study suggests that in teaching and in curriculum development, certain quantifier language may be more advantageous in certain mathematical contexts than others (e.g., in presenting definition of a function, “each” or “every” might be more productive than using “all”). However, a change from one quantifier word to another is not a “cure all” and may bring other unconventional quantifications to light rather than highlighting the intended meaning of the statement for all students.

Acknowledgments

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References

Case, J. (2015). *Calculus students' understanding of logical implication and its relationship to their understanding of calculus theorems.* Retrieved from http://umaine.edu/center/files/2014/07/JoshuaCase_Thesis_DefenseDraft2015.pdf

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth.* (Doctoral dissertation, Arizona State University, Tempe, AZ, USA). Retrieved from http://patthompson.net/PDFversions/Theses/2010CastilloGarsow.pdf

David, E. J., Roh, K. H., & Sellers, M. E. (2020). Teaching the representations of concepts in Calculus: The case of the Intermediate Value Theorem. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 30*(2), 191-210. https://doi.org/10.1080/10511970.2018.1540023

Dawkins, P. C. & Cook, J. P. (2017). Guiding reinvention of conventional tools of mathematical logic: students' reasoning about mathematical disjunctions. *Educational Studies in Mathematics, 94*(3), 241-256. https://doi.org/10.1007/s10649-016-9722-7

Dawkins, P. C., & Roh, K. H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education, 6*, 1-22. https://doi.org/10.1007/s40753-019-00097-2

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics*, *8*(2), 44–51.

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CMBS issues in mathematics education* (pp. 239-289). Providence, RI: American Mathematical Society.

Epp, S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12, 1999 Yearbook* (pp. 188-197). Reston, VA: National Council of Teachers of Mathematics.

Glasersfeld, E. v. (1995). Constructing agents: The self and others. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism: A way of knowing and learning* (pp. 113 – 128). New York, NY: RoutledgeFalmer.

Grice, H. P. (1975). Logic and conversation. In P. Cole & J. Morgan (Eds.), *Syntax and semantics: speech acts* (Vol. 3). London: Academic.

Johnson-Laird, P. N., Byrne, R. M. J., & Tabossi, P. (1989). Reasoning by model: The case of multiple quantification. *Psychological Review, 96*(4), 658-673. https://doi.org/10.1037/0033-295X.96.4.658

Moore, K. (2010). *The role of quantitative reasoning in Precalculus students learning central concepts of trigonometry.* (Doctoral dissertation, Arizona State University, Tempe, AZ, USA). Retrieved from https://search-proquest-com.ezproxy1.lib.asu.edu/ docview/757614251?pq-origsite=primo

Patton, M.Q. (2001). Qualitative evaluation and research methods. 2nd edition. Thousand Oaks, CA: Sage.

Piaget, J. (1977). *Research on reflective abstraction*. Paris: P.U.F.

Roh, K. H., Lee, Y. H., & Tanner, A. (2016). The king and prisoner puzzle: A way of introducing the components of logical structures. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 26*(5), 424-436. https://doi.org/10.1080/10511970.2015.1107159

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151. https://doi.org/10.1007/BF01274210

Sellers, M. E. (2018). When 'negation' impedes argumentation: The case of Dawn. In *Proceedings of the 21st annual conference on Research in Undergraduate Mathematics Education* (pp. 242–256).

Sellers, M. (2020). *Students' quantifications, interpretations, and negations of complex mathematical statements from calculus.* (Doctoral dissertation, Arizona State University, Tempe, AZ, USA). Retrieved from https://keep.lib.asu.edu/system/files/c7/224733/Sellers_asu_0010E_20022.pdf

Sellers, M.E., Roh, K. H., David, E. J. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *Journal of Mathematical Behavior, 61,* Article 100802. https://doi.org/10.1016/j.jmathb.2020.100802.

Shipman, B. A. (2013). On the meaning of uniqueness. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 23*(3), 224-233. https://doi.org/10.1080/10511970.2012.718730

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*, 133–162. https://doi.org/10.1023/B.EDUC.0000017671.47700.0b

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic Algebra Students: Emerging Models of Students' Algebraic Knowing,* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.

Vroom, K. (2022). A functional perspective on student thinking about the grammar of multiply quantified statements. *Journal of Mathematical Behavior, 67*(100992). https://doi.org/10.1016/j.jmathb.2022.100992

An Exploration of Human Factors That Influence the Acceptance of Technology in Calculus Students

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Studies on the use of technology in calculus courses have focused on the use of different types of technology as learning tools for visualization, representations, and understanding. However, research on students' perspectives about the use of technology in calculus classrooms needs to be expanded. In this article, we interviewed eight calculus students in a US research institution to understand their perspectives about the use of technology. To analyze the data set, we used thematic analysis informed by the technology acceptance model focused on external variables related to human factors. As a result, students identified human factors that may influence their perspectives about using technology in calculus classes such as: their attitude and skills, selection of mathematical problems, feedback, classroom environment, and resources. These findings inform calculus instructors and coordinators about possible considerations before implementing the use of technology in their classrooms.

Keywords: calculus education, students' perspectives, technology to teach

In the last decades, many researchers have explored the use of technology to enhance students' learning experiences in calculus courses (e.g., Ellis Jr et al., 2000; Ferrara et al., 2006; Heid, 1988; Thompson et al., 2013). The use of technology is one of many approaches to target the retention problem of science, technology, engineering, and mathematics (STEM) programs in US institutions, which has been shown to rely on the effective teaching of calculus courses (Rasmussen & Ellis, 2013).

The effective use of technology can promote understanding of calculus concepts by facilitating visualization of structures, representations, and exploration of concepts such as limits, derivatives, and integrals (e.g. Bressoud et al., 2016; Ferrara et al., 2006). Tall (2002) explains that technology facilitates the understanding of fundamental ideas of calculus such as the approximation process and infinitesimals. For example, using numerical and graphical approaches to explore the concept of limits. In addition, the use of technology may build a stronger calculus conceptual understanding by allowing students to explore different ways of reasoning while solving problems (Ferrara et al., 2006).

Many studies have mainly focused on technologies that are used to do mathematics such as graphing or geometric software. While these technologies facilitate the learning of calculus concepts, it is imperative to understand that technology by itself will not ensure an effective learning experience. For example, Takači et al. (2015) found that students' learning about functions and graphs is better when using Geogebra while working in collaborative groups.

In our study, we explore external variables to the technology itself. Specifically we investigate students' perspectives about social aspects that may be controllable by humans when using technology. The research question that guided our study is: *How do students describe the social aspects controlled by humans that may influence their acceptance of the use of technology in calculus classes?* The understanding of these external variables complements research about how specific types of technology can enhance the learning of calculus concepts by providing possible considerations before implementing technology in the calculus classes as a coordinated effort in an R2 institution.

Theoretical Perspective

The Technology Acceptance Model (TAM) provides guidance on how students accept or reject the use of technology in calculus classes (Davis, 1989; Venkatesh and Davis, 2000)

According to this model, understanding a person's acceptance to use technology depends on two key variables: perceived ease of use and perceived usefulness. These determine if a person will accept the use of technology (Davis, 1989). Specifically, if someone thinks a technology is both easy to use and useful, then that person would like to use that technology tool. Conversely, if a person perceives it is not easy or not useful, then will not like to use the technology.

Since 1989, several researchers (e.g., Venkatesh and Davis, 2000) have proposed extensions to these two variables for understanding the complexities of using technology. For example, if a person intends to use the technology and if the person is using the technology, which are commonly known as behavioral intention and actual use. In addition, Venkatesh and Davis (2000) consider external variables that can influence the relationship between the perceived ease of use, perceived usefulness, behavioral intention, and actual use. These variables include social norms, availability of resources, and personal skills.

When considering the teaching and learning of calculus, we consider that these external variables are controllable by either students, instructors, coordinators, or administrators. Thus, they are also considered as human factors. Alomari et. al. (2020) categorize human factors into three main indicators: technological, psychological, and student-instructor interaction characteristics. However, their research identifies positive human factors attributes like attitude, enjoyment, experience, self-efficacy, and promptness that significantly impact user satisfaction and subsequently contribute to the success of learning using technology. Understanding these external variables provide a clearer picture to explore technology acceptance by an individual.

The TAM informs our study as we aim to understand the complexity of students' perspectives toward using technology for calculus classes. Considering the human factors of using technology will facilitate an informed decision-making process for instructors and coordinators to provide students with an effective introduction to new technologies while learning calculus. Specifically, this may help students embrace these technologies as important tools for their calculus learning experience by helping them see these technologies as both useful and easy to use.

Research Methods

To answer the research question, we collected audio-recording interviews from eight calculus students who volunteered to participate in this research study. Previous to the interview each participant completed demographic information.

Participants

Participants were sophomores, juniors, or seniors enrolled in Calculus I, II, or III in a US research university in the Midwest.

The university serves approximately 16,000 students from diverse backgrounds with more than half of the student population being first generation undergraduate students. About 46% of the population is White, 20% Latinx, 17% Black, and 6% Asian.

While reporting on demographic information, one student self-identified as Asian, one as Black, three as Hispanic or Latinx, and three as White. Regarding gender, there were four cis-men and four cis-women. Five participants reported being first generation students and one a non-traditional student. None of the students reported being a veteran or having a documented disability. Seven of them were in a STEM program and one of the students was a non-STEM major pursuing a middle school teaching license in STEM.

Instrument and Data Analysis

We conducted one hour interviews during the summer 2023 to students who were enrolled in a calculus section in the Spring 2023 semester. Questions of the interview were semi structured and focused on relevancy of the course, policies, resources, and technology. In this paper, we analyzed students' responses related to the use of technology in calculus classes.

The interviews were audio-recorded and transcribed for analysis. Data was de-identified by assigning pseudonyms to participants.

The three authors are all junior faculty who self-identify as Hispanic or Latinx. One is a mathematics researcher with focus on analysis, one is a computer science researcher, and the other is a mathematics education researcher.

To analyze data, the three authors conducted thematic analysis (Braun et al., 2019). The three authors initially coded four interviews guided by the TAM framework and met to agree on a codebook. After creating a codebook, two of the authors coded the eight interviews and met until consensus was reached.

Results of the Research

The categories of the code for identifying the human factors are the students' attitudes or skills, the selection of mathematical problems online, the learning feedback, the classroom environment, and the resources.

Table 1. Human factors when using technology for calculus classes identified by participants.

Human Factors	
Factor	Participants
Students attitudes or skills	Audrey, Ben, Diego, Elena, Josephine, Juan, Kate
Selection of online mathematical problems	Josephine, Lin
Learning feedback	Josephine, Kate
Classroom environment	Ben, Josephine
Resources	Audrey, Diego

Students' Attitudes or Skills

Students' attitudes or skills capture the ways students liked or disliked technology. For example, the level of difficulty with using technology to learn calculus, their familiarity, experiences, and perspectives against or in favor of using technology.

Seven students reported some of their attitudes and skills related to the use of technology in calculus classes. The most popular attitude was related to the use of online homework in calculus classes. For example, Kate thinks online homework is beneficial for the learning experience.

Interviewer: Do you think using technology is beneficial for your learning experience?

Kate: I feel like it has been useful just because chances are that's where the world's going is with technology being more useful. So having those lessons with it is helpful. Though I do, I still always end up like even if the assignment is online, I'll write it out on a paper before I do it because it's hard for me to do it all on the computer.

Interviewer: Okay, yeah, and what kind of technology would you like to use in the calculus classes? I don't know. It can be anything. It can be like a web page, an app, or software.

Kate: It can be useful if we use like graphing apps and stuff. So we can plug in like the equation and get an idea of what it's supposed to look like. Because when there's all those different things in it, it's really hard to visualize the graph and like do it and because there's so many different parts to the equation, it's really hard to graph by yourself. But and then I also I like being able to look up similar examples online that are like I don't even know like some sort of like especially with like trig stuff that's always the trouble with me, figuring out how people, if there's like a sine over a cosine or whatever how people treat that in certain problems and different things.

Kate, sees the use of technology as beneficial for the learning experience when perceiving technology as part of keeping up with the uses of technology in the world and as a tool for learning mathematics. It is interesting to see how Kate connects the calculus learning experience with a more global perspective on how people are increasing the use of technology "in the world". However, it is interesting how Kate still sees a need to work on math problems using paper and pencil before sharing the answers in online homework systems. In addition to the online homework, Kate thinks that the use of graphing apps is important to visualize graphs and to see different strategies when solving mathematical problems.

Students also reported on their familiarity with different types of technology. For example, Juan mentioned using several types of technologies in both high school and college.

Interviewer: Okay, so you have experience with scientific calculators. You also have some experience with online homework. Is there something else I'm missing about your experience with technology in math and calculus classes?

Juan: Umm, sometimes in my calc class as a teacher will like pull up a functions calculator or like a plot system to kind of just see things visually. I think those are nice. But for the most part, yeah, that's pretty much accurate. This is my first time ever having online homework for a math class. For my calc 3 class that I'm taking online right now. [...] WebAssign I think it's called Yeah, it's definitely a Cengage Webassign, yes.

Interviewer: You said something about functions and online graphing. Is that also inside WebAssign?

Juan: No, there are some applications in WebAssign where they can use those for certain problems. But I know like the one I'm using right now, because the one prior I can't use it anymore because a lot of my stuff is like in 3D, so I use 3D plot, personally, but that's not like through the instructor. I forget, there's another popular one. That a lot of instructors have used in my calc 1 and calc 2 experience.

Interviewer: So you do use them for your own learning. Is that correct?

Juan: Yes, I do

Interviewer: Okay, is there anything else that you use? Any other technology that you like to use for your own learning outside of class?

Juan: Lecture videos like sometimes if I need extra help on a certain topic. I'll watch a YouTube video. That kind of goes over it again. I use the online homework, which I like. Actually, I never knew if I would like online homework. But, since I've taken a lot of physics classes that have used, even in my high school, that have used WebAssign Cengage, and since I'm really familiar with it, I do like it now. Umm, let me think and then. It's nice to have a calculator. No, I think that's all the technology.

In this interview excerpt, Juan mentions using scientific calculators, mathematical software programs and online homework. He has a positive attitude toward instructors using visual representations in what he describes as function calculators that "are nice". This may be a graphing software that instructors use to visualize functions while teaching calculus classes. While he does not explicitly mention familiarity with these technologies, he may be familiar with using graphing softwares as he likes using other three dimensional graphing software programs outside of the classroom that helps him learn calculus. Regarding online homework, Juan mentions that he likes it because he is familiar with the technology as he used it in a physics high school class.

When asking students about using augmented reality in calculus classes most students mentioned unfamiliarity with how this may be used in a calculus class. They admitted they did not have enough information to have an opinion on how it may or may not be beneficial. For example, when asking Diego if using augmented reality would be beneficial, he said "I would say no [...] but I also don't have any experience with it, so I feel like my opinion just isn't too great just 'cause I don't have that knowledge." Later in the interview Diego explains having an open mind to maybe use it if someone explains how it would be useful with some specific examples.

Selection of mathematical problems online

Another human factor mentioned by some students was the way instructors or coordinators select the mathematical problems online. For example, Josephine finds mathematical problems in ALEKS as tedious.

Josephine: [...] Oh, I thought my Mathlab was horrible, but there is another like program like software, it's called ALEKS. Have you heard of ALEKS?

Interviewer: Yes.

Josephine: I hate it, absolutely hate it. I think I was using that app in appreciation for My Mathlab after using ALEKS because I took a trig summer class. But the difference between My Mathlab and ALEKS is that ALEKS will have you do like 30 different problems on a math subject lower than your math subject before you could even start practicing on your math subject if that makes some sense. So I feel like using the computer is just a little bit too tedious and just you know cause math is complicated, yes, but I feel like the computer makes it even more complicated than it needs to be. That's just my opinion as far as technology and math, I don't think math should have nothing to do with it. Well, math [inaudible] with technology, but as far as I'm learning it, I don't think it's necessary. Technology is not necessary for it though. Yeah.

In her interview, Josephine expresses a dislike of using a specific online homework system, however, the reason is related to the problems selected by the instructor. Specifically, she thought it was tedious to work on problems that were (1) not addressing the concept that was discussed and (2) too many. This presents two factors that can be controlled by an instructor: the alignment with the curriculum and the quantity of problems selected. What Josephine is experiencing with the online homework system may be reflecting the drill and train practice about mathematical procedures.

The lack of alignment when selecting mathematical problems online, was also mentioned by Lin when we asked about ideal technologies for calculus classrooms. Lin said not to use online homework because, in his experience, online homework "wasn't exactly tailored to what they learned for that day. It was more generic for the topic."

Learning Feedback

Students were also concerned about needing more assessment feedback in their calculus classes. Two students mentioned learning feedback as an issue when using technology.

Specifically, Josephine and Kate explained it was difficult for them to know where exactly they had a mistake. For example, when talking about online homework, Kate noted: "[...] it's hard for me to go through and know what I did wrong when it's such a big problem and there's so many little steps that I could have messed up on." While Kate is receiving immediate online feedback on whether a problem is correct or incorrect, she perceives this is not enough feedback. Here, Kate would expect feedback to point out mistakes in her rationale or computations while solving calculus problems.

Classroom Environment

Students reported on the classroom environment while using technology, for example, using a collaborative learning environment, working in small groups, or mimicking other non-math courses' classroom settings.

A student, Ben, noted that he has had a great experience in the college of engineering with collaborative learning when using technology. When asking about recommendations for types of technology we could use in calculus, Ben referred to this experience.

> Suppose potentially having some kind of like a screen that's more of like a table. I don't know what that would be but something that we could all kind of like look down at and kinda talk about like, like I said before, kind of in like that circle, you all kind of look down and kind of have more forward and open discussions versus the way classrooms kind of are, whether you're just kind of staring at the board which is understandably easier but yeah.

Ben perceives that the technological tools available in this engineering class would potentially be beneficial to include in calculus classes. Here, Ben perceives that technology could be used to foster more open discussions in the classroom by using screens and arranging classrooms in circles around those screens. It is worth mentioning that Ben uses the opportunity not only to introduce this as an option to improve mathematical discussions in the calculus classrooms but also to critique that the current classroom setting is traditional and is limiting discussions.

Resources

Participants mentioned that, when using technology, it is essential to consider the institution's available and needed resources. For example, having a computer at home to complete online homework and having access to computer labs in the institution.

For example, Diego agreed to implement technology in calculus classes and referred to resources available at the institution. He mentioned that the institutional access to the internet was pretty good, especially in computer labs. He explained that the institution had computer labs in case "a student did not have those types of resources [personal computer or laptop]."

Significance and Further Research

This study contributes to understanding students' perspectives about the use of technology in calculus classes, including human (students, instructors, coordinators, and administrators) factors. Specifically, students' skills and attitudes, selection of mathematical problems, learning feedback, classroom environment, and resources.

Students' skills and attitudes toward the use of technology would identify the human factor of the student as a learner. Skills and attitudes may result from their experiences or lack of experience with using technology in calculus classes. Understanding these factors may provide information about possible students' issues when mixing technology and mathematics, which could offer calculus instructors and coordinators targeting tools to overcome them. For example, a possible need to train students and help them feel more familiar with the technology before implementing it.

The selection of problems, learning feedback, classroom environment, and resources are primarily human factors related to instructors, coordinators, or institutions. These factors were mostly perceived as contributors to negative experiences from the students when using technology. Specifically, (1) students perceived that the selection of problems had alignment issues and became tedious because of the number of problems, (2) students perceived a need for more feedback than the correct (or incorrect) feedback in online homework, (3) students perceived a need for restructuring classroom environments, and (4) students discussed issues of accessibility when using technology. Some of these have been shown to be important for students' learning of calculus, such as considering the classroom environment when using technology (e.g., Takači et al., 2015) and their attitudes toward the use of online homework (e.g., Lampe and White, 2023).

Instructors may control the selection of problems, learning feedback, and classroom environment. Some of these issues may suggest a need to provide instructors with technology professional development activities before officially implementing online homework as a coordinated effort, as well as effective active learning practices and curriculum alignment. We must see these findings about learning feedback with caution as these vary in different online homework systems and it would not be appropriate to conclude that all online homework systems lack the feedback students perceive they need. For example, there are initiatives to use artificial intelligence to give more meaningful feedback to students when working on online homework. Future research may investigate students' perspectives about using artificial intelligence to provide feedback on assignments.

Administrators may also control some of the classroom environment and resources needed for effectively implementing the use of technology. While instructors have control over classroom settings, some of this effort relies on classroom conditions that facilitate active learning. In addition, when applying technology, it is important to consider if the type of technology would be available to all students.

Instructors, coordinators, and administrators must engage in open discussions to ensure equitable access to technology for all students, fostering an inclusive and technologically enhanced learning environment in calculus courses.

References

Alomari, M. M., El-Kanj, H., Alshdaifat, N. I., & Topal, A (2020). A framework for the impact of human factors on the effectiveness of learning management systems. *IEEE Access, 8*, 23542-23558. doi: 10.1109/ACCESS.2020.2970278.

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and learning of calculus*. Springer Nature. https://link.springer.com/chapter/10.1007/978-3-319-32975-8_1

Davis, F. D. (1989). Perceived usefulness, perceived ease of use, and user acceptance of information technology. *MIS Quarterly,* 319-340.

Ellis Jr, W. (2000). Technology and calculus. In S.L. Ganter (Ed.), *Calculus renewal: Issues for undergraduate mathematics education in the next decade* (pp. 53-68). Boston, MA: Springer US.
Ferrara, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus: Ideas discussed at PME over the last 30 years. In A. Gutierrez and P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 237-273). Brill.
Heid, K. M. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3–25.
Lampe, K., & White, K. (2023). The effects of online mathematics homework on learning, attitude, and efficacy in a college calculus class: A case study. *PRIMUS*, 1-15.
Rasmussen, C., & Ellis, J. (2013, July). Who is switching out of calculus and why. In *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80).
Takači, D., Stankov, G., & Milanovic, I. (2015). Efficiency of learning environment using GeoGebra when calculus contents are learned in collaborative groups. *Computers & Education, 82*, 421-431. https://www.sciencedirect.com/science/article/abs/pii/S0360131514002796?via%3Dihub
Tall, D. (2003). Using technology to support an embodied approach to learning concepts in mathematics. In L. M. Carvalho and L.C. Guimarães, *Historia e tecnologia no Ensino da Matemática, 1*, 1-28.
Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*, 124–147. doi:10.1080/07380569.2013.768941.
Venkatesh, V., & Davis, F. D. (2000). A theoretical extension of the technology acceptance model: Four longitudinal field studies. *Management Science, 46*(2), 186-204.

On the Unique Benefits and Challenges of Mathematics
Graduate Student Instructors Providing Teaching Feedback to their Peers

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In this paper, we compare the types of teaching feedback that graduate student instructors provide their peers in comparison to more senior faculty at a large research-oriented university. Additionally, we consider the challenges and benefits that graduate student instructors report concerning providing teaching feedback to a peer. Our results reveal that graduate student instructors and faculty contribute distinct perspectives on teacher growth and together can form a strong support system for first-time graduate student instructors. Additionally, while observing a peer does pose real challenges, we found that graduate student instructors develop strategies to overcome these and report more benefits than difficulties.

Keywords: graduate student instructors, feedback, teaching observation, peer mentoring

At many research universities, graduate students in the Mathematical Sciences become instructors of record for one or more undergraduate courses (Eller, 2017; Justice, Zieffler, & Garfield, 2017); we will call these educators graduate student instructors (GSIs). Oftentimes, GSIs have little to no assistance on the front-end in preparing to teach; however, departments across the country have begun to implement training programs for GSIs with the aim of both improving GSIs teaching abilities and improving the learning experience for their undergraduate students (Speer, et al., 2005; Ellis, 2014). One promising training component that can support GSIs in developing student-centered teaching practices is providing them with feedback from a teaching observation conducted by a more experienced educator (Yee et al., 2022). Within a mathematics department, this feedback can come from a faculty member or a more experienced graduate student instructor. In this paper, we explore the following research questions:

Research Questions

1) What types of teaching feedback do peers give to other graduate student instructors and how does it compare to the feedback provided by more senior faculty?

2) What are the challenges or benefits that graduate student instructors report concerning providing teaching feedback to a peer?

Theoretical Perspectives

Concerning feedback. We frame feedback using Kluger and DeNisi's (1996) Feedback Intervention Theory (FIT). This framework has previously been leveraged in the K-12 setting (Khachatryan, 2015) as well as in the setting of graduate student instructor professional development (Yee et al., 2022). We define feedback as action taken by an observer to provide information concerning aspects of the graduate student instructor's teaching performance. We categorize the components of feedback using three levels or scopes: (Whole) comments on the task of teaching as a whole, (Part) focused details within the task of teaching, and (Individual) comments concerning the affect, motivation, or skill of the instructor.

Because we are discussing feedback for novice instructors, who may not have particular knowledge of aspects of teaching, we expand our framework to categorize whole and part scoped components as description, feedback, or feedforward. While feedback focuses on what is effective or not in the present, feedforward focuses on the future by offering particular suggestions. It is important to track the scope of feedback because Yee et al. (2022) found that feedback that includes specific context and focal events (what we call part-type feedback) is more effective at promoting change in graduate student instructor's teaching behaviors than feedback that lacked this contextualization. Additionally, FIT suggests that comments focused on the individual are the least effective in motivating a change in performance (Kluger & DeNisi, 1996). Accordingly, we track individual scoped comments and categorize them as positive (a compliment) or negative (a criticism).

Concerning dimensions of mathematical teaching. To facilitate classification of the content of teaching feedback, we used the topics covered in the Mathematical Association of America's Instructional Practices Guide (Abell et al., 2017). This comprehensive resource operationalizes leading research concerning learning in the undergraduate mathematics classroom.

Methods

Context & Data

All data collection was done in the Department of Mathematics and Statistics at a large land-grant university in the United States. In a typical semester, 30 to 40 faculty members, the majority of which are research-oriented, participate in conducting a teaching observation of each graduate student instructor of record.In addition to this faculty-provided teaching observation, a peer conducts a teaching observation for each first time graduate student instructor. Each graduate student instructor who observed a first-time instructor also completed a reflection about the process. We analyzed 35 of these peer observations and the observer's corresponding reflection, from Fall 2020 — which was a planned remote semester due to the COVID-19 pandemic — ranging to Fall 2022 — which was a standard in-person semester.

For comparison purposes, we will share our analysis of 170 faculty observations of graduate student instructors conducted in the same time range. Please note that our reported analysis of faculty-generated observation data is a subset of an analysis conducted for a larger project focused on how mathematics faculty engage with teaching observation protocols; our analysis of the graduate student instructor-generated observation data and the corresponding reflections is novel and only appears in this report.

Observation protocols. The observation protocol used by graduate student instructors is heavily-adapted and pared-down from Rogers and Yee's (2018) Graduate Student Instructor Observation Protocol (GSIOP). We originally aimed to use the GSIOP, but struggled to effectively implement the required training with our graduate student instructors during the COVID-19 emergency. Our protocol opens with 5 likert-type items concerning student engagement followed by the student-centered techniques chart from the GSIOP. To account for the Zoom setting, we added Small Groups/Breakout Rooms, Polling, Chat feature to engage students with content, and use of Google Docs or Spreadsheets for activities/group work to the chart. Our protocol concludes with essay prompts to summarize the observed class, to discuss strengths, and to suggest areas for improvement. The observation protocol used by faculty begins with twelve Likert prompts asking the observer to rank components of the appropriateness of the mathematics, the quality of the instructor's communication, and the level of perceived student engagement. Next, the observer is provided two essay prompts, one requesting details of the observed session and the second requesting suggestions to help the graduate student instructor improve. The primary difference between the two protocols is that the one used by graduate student instructors has clearly defined notions of student engagement with very specific active learning techniques given. Our faculty, on the other hand, have quite varied notions of effective teaching. Consequently, the observation tool they use is necessarily less concrete.

Pedagogical & mentoring training for graduate student instructors. The graduate student instructors who served as peer mentors received both pedagogical and mentor training prior to conducting observations. The pedagogical training Promoting Success in Undergraduate Mathematics through Graduate Teacher Training (PSUM-GTT) included a teaching seminar during the spring of their first year and fall of their second which focused on modeling active learning strategies and reflecting on journal articles related to teaching and learning. Ongoing professional development was also provided through a critical issues in STEM education seminar facilitated 2-4 times per semester. Mentor training was provided to help clarify the role of a mentor and provide training around communicating effectively, building relationships, setting goals, and conducting classroom observations using an established observation protocol (Manzanares et al., 2023).

Results of First Research Question

Our first question considers what types of teaching feedback do peers give other graduate student instructors and how does this compare to the feedback provided by more senior faculty. We analyzed both the graduate student and faculty observations utilizing *a priori* coding (Saldaña, 2016) with two different code books, one focusing on the types of feedback and the other on the content of the feedback.

Feedback Intervention Theory

Our feedback codes are those types discussed above concerning Feedback Intervention Theory. Table 1 shows the frequency of scopes of feedback in both the graduate student peer observations and the faculty observations. Note that the gray-highlighted rows are the components of the most effective formative feedback, as discussed in theoretical perspectives.

Table 1
Frequency of feedback intervention theory codes in peer vs faculty observations

Scope & Category	Graduate Student Peer Frequency	Faculty Frequency
Whole - Description	0 %	< 1 %
Whole - Feedback	0 %	36 %
Whole - Feedforward	0 %	0 %
Part - Description	100 %	34 %
Part - Feedback	100 %	78 %
Part - Feedforward	100 %	61 %
Individual - Compliment	13 %	38 %
Individual - Criticism	0 %	1 %

Content of Feedback

Table 2 shows the frequency of content codes from the graduate student peer observations and the faculty observations. As briefly mentioned above, our codes concerning dimensions of mathematics teaching were developed from the Instructional Practices Guide.

Classroom community. This code encapsulates students' connections with their GSI, classmates, and university resources. Additionally, it covers classroom norms or atmosphere. For example, one peer observer wrote *"the class was kept at a pace the students can feel comfortable asking questions and I liked the idea of giving thumbs up when something is clear... In general, the class atmosphere was very learner - friendly"*. An example of classroom norms is the feedback:

> *I can tell that you are aware that your classroom is kind of a weird shape. It seems like it makes it hard to make sure everyone can see and participate. Since your room is so big, you could ask your students to all sit closer to the center or pull their tables over towards you. I know that's not the norm for your class since we're so far into the semester, but maybe the next time you teach in a room like that you can anticipate and make them sit closer together.*

Table 2

Frequency of content codes in peer vs faculty observations

Content Code	**Graduate Student Peer Frequency**	**Faculty Frequency**
Classroom Community	63 %	12 %
Student Engagement	69 %	41 %
Student Communication	25 %	2 %
Student Questions	38 %	31 %
Instructor Questions	81 %	28 %
Instructor Communication	44 %	35 %
Collaborative Learning	44 %	6 %
Tasks – Intrinsic Appropriateness	94 %	80 %
Tasks – Extrinsic Appropriateness	6 %	31 %
Technology	38 %	1 %

Student engagement or communication. Engagement refers broadly to the observer's perceptions of students actively participating in their learning. The student communication code captures the various modalities for students to communicate their ideas with one another and their instructor. For example, the quoted comment above concerning the use of the thumbs up button in Zoom is also student communication.

Student or instructor questions. Comments about questions posed by students were coded as student questions; Comments about questions posed by the instructor were coded as instructor questions. Many peer observers discussed wait time, or giving students adequate time to think about a question before expecting an answer. For example, one graduate student reflected *"He has a slight tendency when the answer isn't said almost immediately to give them the answers"* and wrote the following feedback on the observation form: *"waiting after asking questions. Gives students some time to internalize and some are shy."*

Mathematical tasks – intrinsic or extrinsic appropriateness. Intrinsic appropriateness concerns the characteristics intrinsic to a task and whether those characteristics support student learning while extrinsic appropriateness takes into account external factors such as classroom architecture or students' motivation. The discussion above of classroom shape making participation hard is an example of extrinsic appropriateness. The extrinsic appropriateness code only occurred when there was a concern.

Others. Instructor communication skills focused on legibility of handwriting or instructor's speaking volume. Collaborative learning is structured group work managed by the instructor. Technology primarily was comments concerning Zoom during the planned emergency remote semesters, but could be related to any technology in the classroom.

Comparing Peer Feedback to Faculty Feedback

Primarily, if a faculty member discussed an engaged-student learning strategy, it was group work. Our graduate students, on the other hand, leveraged a vast array of techniques coming from the GSIOP, suggesting when and where a strategy could be used; on average, each graduate student peer observer discussed 3.4 strategies. Graduate students 100% of the time employed the components of effective feedback: part description, part feedback, and part feedforward. Many faculty did not provide these components in their observation comments. Additionally, there was a wide difference in the types of topics discussed between graduate student instructors who were observing their peers who were teaching for the first time in the department and the topics covered by faculty observing all graduate student instructors. One might think this is because the first time instructors needed more help, however, we found this to not be the case. Even when a peer observer felt that the instructor they watched was effective, they still described what they noticed and explained why they found it effective. On the other hand, in a similar situation, where the faculty member found the session that they observed to be effective, they were more likely to give whole-type feedback such as "The class went quite well" with no further information about what they noticed.

One area in which faculty greatly outperformed the graduate student observers is their rich and nuanced discussions of mathematical task - intrinsic appropriateness. Our graduate student instructors generally mentioned the mathematical topic of the day and then turned their focus to the other dimensions of classroom instruction. On the other hand, faculty who discussed the mathematical content demonstrated a deep mathematical understanding, for example discussing necessary levels of correctness, alignment of that day's learning outcomes with the course and curriculum more broadly, or presenting the material at an appropriate level for the student population. We believe the differences we observed are primarily due (1) to the graduate student instructors having a tightly formed community of practice while the faculty have no common notions of effective teaching, (2) to graduate student instructors receiving specific training on providing teaching feedback while faculty received none, and (3) to our faculty primarily serving in research-intensive positions with experience teaching a broader range of courses and at an upper level.

Results of Second Research Question

We utilized emergent coding from a grounded perspective (Saldaña, 2016) when working with the 35 reflections. Our aim was to uncover the challenges and benefits that graduate student instructors report after providing teaching feedback to a peer. After individually gaining familiarity with the data, we met as a team to discuss and refine our codes. We met a second time to resolve any discrepancies in our final coding of the data. Importantly, we want to note that in our initial round of coding, we discovered a distinct difference in our perspective as coordinators of teaching professional development versus the perspective of the graduate student instructors. In particular, they found certain aspects of the experience to be a challenge, while we thought of that struggle as a benefit. Because of this tension in perspectives, we will be very purposeful in indicating *who* is perceiving the benefit or challenge.

Awkward or nervous because it is a peer. Four graduate student instructors felt nervous or awkward giving feedback to a peer, but also expressed a strategy to overcome this challenge. One reflected:

> *I thought our follow-up meeting went well, too. Even though I was still (and probably always will be) a little nervous to make suggestions for improvement, it wasn't awkward and I think we are gaining more rapport with one another to talk through those things.*

Another instructor also felt that developing a relationship over time would make this awkwardness not *"much of an issue"*. The source of one GSIs discomfort was their perfecption that their peer actually had more teaching experience than them, writing, "*I feel a little bit of imposter syndrome about giving her suggestions for strengthening/improving.*" The last instructor who felt awkward developed a strategy where *"instead of telling him he was wrong,"* they tried "*asking about certain things and letting him talk through to reach his own conclusion about how effective/ineffective certain things were.*"

Developed a feedback strategy. Eleven graduate student instructors discussed the strategies that they had developed for giving feedback to a peer. Three strategies focused on how to approach the feedback conversation, with one GSI explaining "*I approached it just as a conversation where we point out and maybe debate some aspects of each other's teaching style and approaches*", and the other two explaining that they wanted to come across as "*supportive instead of demanding*" and "*there to help and not out of a place of putting him on trial*", respectively.

One GSI decided to ask their peer if "*there was anything she wanted me to focus on during her observation*", which they felt made the other person "*more open to constructive criticism*". Two GSIs considered how their peer might be more open to suggestions and respectively decided on "*telling him more strengths than weaknesses and always interposing them*" and "*include more positive comments instead of only focusing on where my mentee can improve*". The last GSI also decided to specifically ask their peer what they "*wanted me to keep an eye out for, which allowed her to ask me things she wanted to know about, rather than just what I would say*"

Wrestling with different opinions of "good" teaching. This focusing on another person's perspective and recognizing that there may different ideas of "good" teaching presented a challenge to 3 GSIs. One wrestled with the ways in which different class settings may necessitate different teaching styles, reflecting "*If you have someone teaching finite math and it is active learning versus a calculus class which is lecture based, you do not do things the same way.*" One GSI was concerned that they did not know what the department values, writing:

> *There is not one method or style of teaching that seems to be the "ideal" for the department. It's not clear if the department wants the most effective lecture-style environment, or if they want more active learning, or if they want more in-class assessments, and so on.*

The final GSI wrestled with confronting their own biases and opinions, sharing, "*The thing I find most difficult is the actual assigning of values to her teaching. It is based off my own bias of what I think makes a good teacher and the qualities I identify as important.*" We do want to note that we value the fact that our GSIs are considering and reflecting on this issue to be a benefit of our program, even if the students themselves feel that this is a challenge.

Uncertainty in role. Every graduate student instructor reported feeling prepared to conduct a teaching observation and to provide feedback except for two students who expressed uncertainty in this new role. The first explained, "*I did not feel prepared going into our follow up meeting. I was unsure of what questions I should ask my mentee especially ones that would really make them examine specifics of my teaching style.*" The other expressed uncertainty in the purpose of the observation in the broader context of the department and the GSI training program explaining, "*I wasn't really sure what my mentee would have to do with regards to observation follow up*" This reveals an opportunity for us to provide additional support for GSIs prior to them conducting their teaching observations.

Providing self-reflection. One GSI valued having this dedicated time to discuss teaching with a peer, explaining, "*I was eagerly looking forward to it.*" Six GSIs explained that this process was an effective self-reflection tool, with one even explaining "*I learned more about my teaching*" and *"This was a good process in general because I feel like it encourages both the person being observed and the observer to reflect on their own teaching and seek how to improve.*"

Two reported a boost in confidence after self-reflecting, "*It was really encouraging to realize that my three semesters of experience made me feel completely confident and qualified to offer advice on my mentee's class."* and "*It was an encouraging reminder to me that I have been teaching for a couple of years now, and that experience has taught me a lot!*" The last GSI felt the process provided some accountability for their teaching, reflecting, "*I try to bring up her progress in these areas every time we meet to keep her accountable for working on this. Incorporating more active learning is something I also need to work on, so this keeps me accountable as well.*"

One GSI reported understanding her own teaching in relation to others' teaching practices after discussing feedback with a peer, writing, "*I knew that I had a laid-back attitude in my classroom ... but my mentee pointing out how informal my class was made me realize that that is really not the norm for everyone."* Another GSI also expressed understanding their own teaching better: "*My mentee is very open to getting help and being evaluated, so it made this process easy and eye-opening to things that I could improve on too!"*

Re-conceptualizing teacher growth. Two GSIs thought of growth as a teacher in a new way, with one expressing that this experience made them realize that improving their own teaching will be an ongoing and "*continuous process*" and the other reporting "*I think oftentimes (even in my own teaching) this is something slower to implement. I have tried to emphasize with her that changing teaching style/implementation can and probably should happen slowly. You can't change everything you do overnight.*"

Conclusion

We found that faculty were more effective at giving feedback on mathematical content knowledge while our graduate student instructors were more more effective at giving feedback on pedagogical approaches. This demonstrates that both populations contribute uniquely to teacher growth and together can form a strong support system for first-time graduate student instructors. Although asking graduate student instructors to observe a peer and provide feedback does pose real challenges, such as awkwardness and uncertainty, we found that GSIs develop strategies to overcome these and report many more benefits, such as personal accountability, teaching confidence, and a greater understanding of their own and others' teaching.

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References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., Soto, H., & Project Leadership Team (2017). Instructional Practices Guide. The Mathematical Association of America.

Eller, R. (2017). Training and mentoring graduate teaching assistants: A review of the literature. *AU eJournal of Interdisciplinary Research* (ISSN: 2408-1906), 2(1). p. 54 - 64.

Ellis, J. (2014). Preparing future professors: Highlighting the importance of graduate student professional development programs in calculus instruction. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), Proceedings of the joint meeting of PME38 and PME-NA36 (Vol. 3, pp. 9–16). Vancouver, Canada: PME.

Manzanares, R., Webb, J., Harrell-Williams, L. M., Olson, G. A., Houston, S., Lanius, M., Funkhouser, K., Shannon, D., Gomez, J. O., Jacobson, M., & Merchant, L. (2023). A Case Study of a Multi-Institution Replication of a Comprehensive GTA Teacher Training Program. PRIMUS, 1–17. https://doi.org/10.1080/10511970.2023.2214909

Speer, N., Gutmann, T., & Murphy, T. (2005). Mathematics teaching assistant preparation and development. College Teaching, 53. https://doi.org/10.3200/CTCH.53.2.75-80

Justice, N., Zieffler, A., & Garfield, J. (2017). Statistics Graduate Student Teaching beliefs, Practices and Preparation for Teaching Introductory Statistics. Statistics Education Research Journal, 16(1), 294-319.

Kluger, A. N., & DeNisi, A. (1996). The effects of feedback interventions on performance: a historical review, a meta-analysis, and a preliminary feedback intervention theory. Psychological bulletin, 119(2), 254.

Rogers, K. C., & Yee, S. (2018). GSIOP: Graduate Student Instructor Observation Protocol: Retrieved from https://www.bgsu.edu/PeerMentoringProgram

Saldaña, J. (2016). The Coding Manual for Qualitative Researchers. Thousand Oaks, CA: SAGE.

Yee, S., Deshler, J., Rogers, K. C., Petrulis, R., Potvin, C. D., & Sweeney, J. (2022). Bridging the gap between observation protocols and formative feedback. Journal of Mathematics Teacher Education, 25(2), 217-245.

Teaching Practices Addressing Multiple Uses of a Word in Mathematics – Case for Derivative

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We investigated teaching practices aiming to communicate multiple uses of a common term in mathematics with students focusing on the use of a single word "derivative" for both objects "the derivative at a point" and "the derivative function" using commognitive approach and intellectual needs. Our analysis of a teacher's teaching explicitly attending to this feature revealed several teaching practices with this aim: discussing discursive rules for distinguishing the two uses and using another mathematical object for connecting the uses (e.g., slope); discussing a colloquial object that has a similar feature and is familiar to students, which provided a familiar discourse that students could rely on; and avoiding multiple uses of a common word in one context, Those teaching practices are impacted by multiple uses of a common signifier in communication and mathematical relations among the two objects objectified by a common word, but also impacted how the objects became related.

Keywords: commognition; teaching practice; derivative; words with multiple uses

Multiple uses of same signifiers (words or symbols) have been known to cause potential difficulties in teaching and learning of mathematics (e.g., Biza & Zachariades, 2010; Thompson & Rubenstein, 2000). For example, various calculus terms are used as both process and object or as both function and a specific value (e.g., College Board, 2023; Güçler, 2013). This study focuses on the discourse about the derivative to examine the teaching practices that aim to help students learn about multiple uses of a common term in mathematics. In introductory calculus, derivatives can be separated into the derivative at a point and the derivative function (we use this term for a function obtained by differentiating another function). The observation that the word "derivative" is included in "the derivative at a point" and "the derivative function," the word "derivative" alone is often used for these objects (e.g., College Board, 2023; Stewart, 2016), and students may face challenges distinguishing or relating these objects (Font & Contreras, 2008; Park, 2013), motivated our study investigating potential teaching practices aiming to promote students' learning of the use of this common term with the following research question:

What can be the teaching practices that aim to promote students' learning of multiple uses of a common term for two mathematical objects in formal mathematical discourse?

We adopted the commognitive view of *learning* mathematics as "the process in which students extend their discursive repertoire by individualizing the historically established discourse called mathematics" which we refer to as canonic discourse (Sfard, 2018, p. 222). Such extensions are viewed as meta-level learning when they involve changes in rules of discourse (e.g., learning how to choose one of multiple uses of a new word like the 'derivative'). In contrast, object-level learning only involves changes or expansions of properties of objects that are already introduced in the discourse (Valenta & Enge, 2022). Individualizing a discourse means developing one's ability to communicate with others and oneself according to the rules of the discourse. Given the commognitive view of teaching, "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p. 303), we assume that there are teaching practices aiming to help students learn about how to apply those rules. Our goal is to reveal such practices in the context of the derivative based on a case study of a teacher explicitly addressing this discursive feature in class.

Theoretical Background

This study builds up on the existing literature about teaching and learning of multiple uses of a signifier in mathematics (Biza et al., 2008; Lamb et al., 2012; Rubenstein & Thompson, 2001; Thompson & Rubenstein, 2000; Zazkis & Kontorovich, 2016). We adopted the commognitive view of teaching and teaching practices (Cooper & Lavie, 2021; Nachlieli & Elbaum-Cohen, 2021; Tabach & Nachlieli, 2016) and intellectual needs (Harel, 2008, 2013) to identify teaching practices aiming to help students learn about multiple uses of 'derivative' and tasks they address.

Teaching and Teaching Practice Promoting Metalevel Learning in Commognition

The commognitive approach is a conceptual and analytic framework that combines cognition and communication and views mathematics as a type of discourse. A discourse is defined as a type of communication characterized by its distinctive use of words and visuals, and what they endorse as narratives. A feature of discourse – routines – are defined as a task paired with a procedure, where a task is "any setting in which a person considers herself bound to act" based on her interpretation of the task and a procedure is the prescription for her actions in the task situation "that fits both the present performance and those on which it was modeled" from her past experience (Lavie et al., 2019, pp. 160–161). Routines are regulated by object-level rules (e.g., rules about addition regulates the routine of adding of fractions) or meta-level rules (e.g., rules that determine the use of "derivative" regulates the endorsement of statements about it).

The commognitive view of learning is changes in one's discourse, and teaching is considered as communicational activities with the motive of bringing students' discourse closer to what is considered canonic discourse (Sfard, 2008). Recent commognition studies considered teaching practice as "the task as seen by the performing teacher together with the procedure she executed to perform that task" (Nachlieli & Elbaum-Cohen, 2021, p. 7). They examined the teaching practices that promoted meta-level learning and called for more studies on this topic. Meta-level learning usually happens when students are engaged in activities regulated by meta-level rules they are unfamiliar with (e.g., proving or defining) (Park et al., 2023; Schüler-Meyer, 2020; Valenta & Enge, 2022) or when they encounter a new mathematical object to which old discursive rules do not apply (e.g., from whole numbers to fractions) (Cooper & Lavie, 2021). Teaching practices aiming to promote meta-level learning often use characteristics of students' old discourses as a starting point towards a new discourse to create discomfort with old discourse (Nachlieli & Elbaum-Cohen, 2021) or to allow students to use old routines in emerging discourse in a way appropriate in the teacher's eyes (Cooper & Lavie, 2021). We contribute to this area by examining teaching practices aiming to promote students' meta-level learning about new mathematical objects involving meta-level rules about multiple uses of a word "derivative".

Intellectual Needs

This study views teaching practice as teachers' actions to address students' intellectual needs, which emerge in a problematic situation that is "incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" and the intellectual need is "the need to reach equilibrium by learning a new piece of knowledge" (Harel, 2013, p. 122). In commognitive terms, 'a problematic situation' can be seen as a task situation where students' existing routines do not work (i.e., do not lead to desirable results) or there is no existing routine they can apply. In this study, we considered task situations broadly in which students need to connect new words with their familiar words/symbols, realizations, narratives, and routines and to learn about how to communicate with the new words. Intellectual needs have been referenced in prior literature about teaching approaches to multiple uses of a mathematical word/symbol (e.g., Zazkis &

Kontorovich, 2016). Similar to those literature, we focused on three intellectual needs:

- Need for Causality: "one's desire to explain to determine a cause of the phenomenon"
- Need for Communication: the need to "externalize…an idea, or a concept, or the logical basis" for other to communicate with them
- Need for Structure: the need to organize "knowledge one has learned into a logical structure" (Harel, 2013, p. 126, p. 137, p. 140, respectively)

Since these needs are inextricably linked (Harel, 2013), we used them collectively to explain teaching practices aiming to address students' needs in their learning of multiple uses of a term.

Multiple Use of a Mathematical Word or Symbol and Teaching Approaches

As multiple uses of a signifier are prevalent in mathematics and cause potential difficulties for students, existing studies have investigated and suggested teaching approaches aiming to help students learn about such uses. Thompson and Rubenstein (2000) provided a list of such words including those that have more than one mathematical meaning (e.g., "square" in algebra and geometry). Rubenstein and Thompson (2001) provided a similar list of symbols with different meanings (e.g., "the raised -1 symbol" for "an inverse function" and a "reciprocal") (p. 268). Thompson & Rubenstein (2000) suggested teaching approaches to help students learn about such words/symbols for example by bringing their attention to different uses (e.g., a writing prompt, "I thought that a function was __. Now I know that a function is __.") and relations among multiple uses (e.g., "Square and cube have geometric meanings and are also used for second and third powers, respectively. How are the geometry and powers related?", p. 517).

Other studies that navigated teaching approaches specific to a word or symbol with multiple uses also focused on differences or relations among them. Lamb et al. (2012) pointed out difficulties that students have with multiple uses of the minus sign as "subtraction", "a symbol for a negative number" and "a unary operation of the opposite of" (p. 5) and suggested asking students to explain different uses of the minus sign in solution processes and providing numerical tasks that can expand to symbolic expressions (e.g., compare – –6 and –6, and then x and $-x$).

Biza et al. (2008) focused on multiple uses of "tangent line" that students encounter from Euclidian Geometry, Analytic Geometry, to Analysis and showed that many Analysis students (108 out of 196) still hold thinking developed in Geometry, "the tangent line is the one that has only one common point with the curve and leaves the curve in the same semi-plane" (p. 64) which do not generally apply to graphs of functions in Analysis. Biza et al. (2009) showed the similar results for teachers. The authors suggested providing students a strong image of "tangent line" from geometry and dominant but incorrect visual claims, and asking to verify algebraically.

Zazkis and Kontorovich (2016) analyzed secondary preservice teachers' (PSTs') responses to a hypothetical student questioning about the two uses of the exponent (-1) as the reciprocal and the inverse function. The PSTs addressed intellectual needs for structures for distinctive uses based on different contexts, terminologies, and locations of the symbol and related uses based on a common word ("inverse") that implies common actions (e.g., "undo") and also address the need for causality for why the same symbol is used in different contexts. They addressed intellectual need for communication using analogies to other symbols with multiple uses.

Two Uses of the Derivative

Our study deals with the two uses of the derivative, for each of the two mathematical objects, the derivative at a point and the derivative function, focusing on how they are different and related in the discourse about the derivative. For example, once the "the derivative at a point" is part of the discourse, it can be used to construct "the derivative function". There are many ways to do this, but one way is called *encapsulating* in Sfard's (2008) terminology. Specifically, once "the derivative at a point" is in the discourse, one can consider, for any number a, the pair $(a, f'(a))$. All the pairs can be collected into a set and encapsulated into a function called "the derivative function". Once the derivative function has been constructed, one can find its value at a specific input by substituting that value into the function. For example, the value of $f'(x)$ at $x = 1$ is $f'(1)$. We call this transition evaluating. Evaluating a function at a point is a process, typically done in calculus by substituting a value into a formula or reading a coordinate from a graph. In calculus, one of the important steps is appreciating the derivative at a point with the result of applying the evaluation process at a point to the derivative function as the same. Previous studies (Font et al., 2007; Park, 2013) have shown that the connection between evaluating the derivative function and the derivative at a point can be challenging for students.

Methods

Data Collection

The participating teacher, William (pseudonym) was educated in the U.S. He has B.S. and M.S. in Mathematics with over 10 years of teaching experience with 7 years of calculus teaching. We observed an Advanced Placement Calculus AB class with 34 students in a U.S. public high school. William used SMART board and generally organized his class starting with explaining key words and solving a few examples on the board, followed by students' individual or small group work on problems, and then whole class discussion about those problems. We video-recorded six 90-minute lessons at the beginning of the derivative unit, before the class began concentrating more on computation, where we could potentially observe the two uses of "derivative", how they were defined and connected to each other, and other realizations and real-life phenomena were discussed. We transcribed the videos including non-verbal communication.

Analysis

In our analysis, we treated the totality of the teacher's talk collected over multiple days as our unit of analysis (Sfard, 2008). However, we separated lessons into several episodes to see how words were used in different contexts. Episodes were defined by William's different teaching activities – e.g., defining a new mathematical term, providing a story involving a real-life object prior to defining a mathematical term, making connections between a newly defined term and previously defined terms, or making connections between a newly defined term and a real-life object – because changes in teaching activities correspond to changes in the context of discourse and, therefore, to potential changes in usage of words whose meaning is context-dependent (Biza & Zachariades, 2010; Thompson & Rubenstein, 2000), like "derivative".

Because we were interested in investigating potential teaching practices that aim to help students learn about multiple uses of the word "derivative" based on how William dealt with the feature, we first identified the mathematical objects that the common signifier signified in William's discourse. In class, William made an explicit distinction between two uses of the single word "derivative," namely as "a number" and as a "function", and we categorized William's use of the word "derivative" alone into two usages, namely as "the derivative at a point" or as "the derivative function," based on whether the surrounding context suggested it was being used as a number or as a function. We also recorded other terms and visual mediators (e.g., slope) used for those objects, and grouped episodes according to whether they included both objects or only one of them. We then recorded connections made between those terms and visual mediators. We operationalized teaching practice as a task–procedure pair (i.e., "the task as seen by the teacher together with the procedure she executed to perform that task") (Nachlieli & Elbaum-Cohen, 2021, p. 110872). Similar to other commognitive literature (e.g., Valenta & Enge, 2022), we first focused on William's actions related to the two uses of the derivative in the data, and wrote general instructional procedures based on these actions. We then identified the tasks William was trying to accomplish through these procedures based on intellectual needs.

Results

Teaching practices aiming to help students learn about the dual use of "derivative" based on our analysis of William's class are organized according to the intellectual needs they addressed.

Teaching Practice Addressing Intellectual Need for Structure and Causality

At the beginning of the derivative unit, William explicitly discussed multiple times how to distinguish the two uses of the signifier "derivative," as shown in the following excerpt observed on Day 2 after he constructed the derivative function graphically:

> This idea of the derivative can be thought of as a number, which represents the slope of the tangent line at a point. But, notice that I can talk about slope of the function at every point along this curve … And the slope is always changing. What we get then, is a function that represents the slope of the tangent line at any point as a function of x. That's called the derivative too. From the context, you understand what I'm talking about. …just remember now we're talking about numbers and we're talking about functions. We have to keep them straight. Then we get into the other half of Calc 1 and talk about integrals. There will be the same thing. There will be some integrals that are numbers, some integrals that are functions…we use the same words because the concepts are related.

Note that after using "derivative" to refer to both the derivative at a point and the derivative function, he provided a rule for distinguishing the two uses: if context shows the object signified by "derivative" is a number, then "derivative" signifies the derivative at a point and if context shows the object signified by "derivative" is a function, then "derivative" signifies the derivative function. This is a metarule because determining which object "derivative" signifies is something that participants in canonic discourse must do repeatedly even if often it is done tacitly. He also mentioned an equivalent metarule regarding how to distinguish the two uses of "integrals," another object with a similar feature that they were to cover soon. This addressed the students' intellectual need for structure for distinguishing the uses when the common word is used.

In the excerpt above, William also addressed intellectual need for connecting the two uses and causality of using one term for both. Specifically, using another object "slope" for both objects, he provided a connection between the two uses of the derivative – the derivative as a number and the derivative as a function – the latter comes into being when the former is expanded to several points over an interval. This connects the two uses via another mathematical object that are used for both objects and also shows how they are connected. He also provided the reason for using one word for two objects by stating "because the concepts are related."

Teaching Practice Addressing Intellectual Needs for Communication

Teaching practices addressing intellectual needs for communication were observed implicitly through the comparison among episodes. We identified the first teaching practice – connecting the dual uses of the derivative to a colloquial object with a similar feature – when William gave a story about a colloquial word with dual use by comparing an episode about "speed" and an episode about “derivative”. Specifically, he used a colloquial story about reading changing numbers on a speedometer to motivate the construction of the derivative function and followed a similar pattern as the colloquial story when constructing the derivative function. He used “the slope of the tangent line” at a point for both speed and the derivative and then considered them on an interval using numbers on a speedometer “changing all the time” similar to the slope “always changing” (see the excerpt in the previous section). Using the same structure and objects to discuss both speed and the derivative and talking about speed as a motivation to define the derivative seemed to address intellectual needs for communication because it allows students to use a new word “derivative” like they use a familiar word “speed” with the similar feature.

Another teaching practice addressed the need for communication. Specifically, the practice of avoiding the common term in one episode after the second object (the derivative function) is defined was observed with a shift in which object the common term “derivative” was used for, between the derivative at a point and the derivative function. In the beginning of the derivative unit, before the derivative function was defined, William used “derivative” dominantly for “the derivative at a point compared to other equivalent terms such as “slope of the tangent line,” or the symbol $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. However, after defining “the derivative” as a function, he almost exclusively used the word “derivative” for the derivative function in episodes that involve both objects, except in the two episodes where the other object became included due to students’ questions or participation. Even in those cases, William responded to students using their wording or expressions, and went back to his previous use of the “derivative”.

It should be noted that once this shift in his use of “derivative” was made, he dominantly used “slope of the tangent line” to signify the result of evaluating the derivative function at a point. Specially, on Day 1, William said, “it [the derivative] is a function of any point I give you here, I can tell you what the slope of the tangent line is there”. Then, on Days 3 and 4 he solved five problems about this connection by computing “the slope of the tangent line” at a point given the equation for $f(x)$ by computing $f'(x)$ and evaluated it at a point. In this computation, only the expression $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ was used for the derivative function and only the term “the slope of the tangent line” and a symbol f' with a number (e.g., $f'(3)$) were used for the derivative at a point. In other words, in his class, the derivative function evaluated at a point became equivalent to the phrase “slope of tangent line” and the notation (e.g., “$f'(3)$”), but not to “the derivative at a point” in the way he initially defined it (e.g., “ $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$”).

Discussion and Conclusion

Our analysis of William’s class suggested several teaching practices aiming to help students learn about multiple uses of a mathematical term that addressed intellectual needs for:

A. Structure – Distinction: Discussing the metarules that distinguish the two uses of “derivative” – a number and a function
B. Structure – Connection: Using another mathematical object (slope) for both objects and showing how one use – the derivative as a function—can be engendered by the other use – the derivative as a number.
C. Causality: Using the connection between the two objects as a reason for using one term
D. Communication: Using the same structure to discuss a word for two mathematical objects and colloquial objects
E. Communication: Avoiding one word derivative in one episode once the second object (the derivative function) is defined

In general, those teaching practices are aligned with the teaching practices identified in the existing literature addressing multiple uses of a mathematical signifier, but also different from those due to the distinctive feature of the two uses of the derivative; the two objects for which "derivative" is used are usually used in the same context and one object is built up on the other object, which we elaborate on with teaching practices A and E only due to the limited space. Similar to the teaching practices addressed in other studies, teaching practice A provided a way to distinguish the two uses of the term "derivative" (e.g., the PSTs in Zazkis & Kontorovich, 2016 suggested what the raised (-1) is next to in order to distinguish between its use between "inverse" or "reciprocal"). However, the teaching practice we identified was making canonic discourse about the "derivative" explicit by stating when it is used as a number, it is "the derivative at a point," and when it is used as a function, it is "the derivative function". Some instructors may not explicitly discuss this metarule about how to distinguish multiple uses or may explicitly use the unabbreviated terms "the derivative at a point" and "the derivative of a function". However, given that students are learning about multiple uses of a common term when they are learning about a new concept, explicitly discussing the metarules of how to distinguish those two uses seems particularly important because one use is built up on the other and they are later related to each other. Being explicit about metarules about a new discourse (about the derivative, in our case) seems aligned with the teaching practices found in the literature for making the boundary between students' old discourse and new discourse explicit with students (e.g., rules about real numbers and rules about complex numbers, Nachlieli & Elbaum-Cohen, 2021). Communicating this could be further aided by providing students opportunity to directly attend to and reflect on the multiple uses of the derivative (e.g., writing prompts, "I thought that the derivative was __. Now I think that the derivative is ___," similar to Thompson & Rubenstein, 2000). The teaching practice E provides a way to communicate when multiple uses of the same term could in the same context. This was observed with a discursive shift in our data, where William initially used the "derivative" as the main signifier for "the derivative at a point", but once he defined the "derivative" as a function, he exclusively used the term for "the derivative function" whenever he talked about the two objects in one context. We note that such a shift could be seen as William's didactic choice, and there is an obvious appeal to avoiding the potential for confusion, ambiguity, and circularity inherent in using one term "derivative" for different objects at the same time. This appeal is magnified by the commognitive observation that student discourse is often initially a (potentially imperfect) imitation of instructor discourse (Sfard, 2014), which suggests that, even if the teacher's discourse carefully avoids such problems, students may not. Notably, given that this shift could be made implicitly by the instructor (as seen in our data), students may not even explicitly notice that the shift occurred in the teacher's discourse. This implicit nature of using "derivative" observed in the shift seems aligned with other literature reporting teachers' implicit shifts between different uses of the same word (Güçler, 2013 for "limit"). These discursive shifts would be interesting to further investigate because the literature has shown that the connection between the derivative at a point and the derivative function can be challenging for students (Font & Contreras, 2008; Park, 2013) and implicit shifts in instructor discourse have been tied to student difficulties (Güçler, 2013). Although derivatives have been intensely studied in the literature, we have not seen shifts like the one we discuss documented before. Note, however, that the instructors studied in (Park, 2015) also dominantly connected "derivative function" to "slope of the of the tangent line" and this leads us to conjecture that this shift may be widespread, but not documented because previous analyses were not looking for such large-scale discursive patterns.

References

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education*, *10*(1), 53–70. https://doi.org/10.1080/14794800801916457

Biza, I., Nardi, E., & Zachariades, T. (2009). Teacher Beliefs and the Didactic Contract on Visualisation. *For the Learning of Mathematics*, *29*(3), 31–36.

Biza, I., & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. *The Journal of Mathematical Behavior*, *29*(4), 218–229. https://doi.org/10.1016/j.jmathb.2010.11.001

College Board. (2023). *AP Calculus AB Course and Exam Description*. chrome-extension://efaidnbmnnnibpcajpcglclefindmkaj/https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf

Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses – A commognitive look at instructional design in the zone of proximal development. *The Journal of Mathematical Behavior*, *61*, 100822. https://doi.org/10.1016/j.jmathb.2020.100822

Font, V., & Contreras, Á. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics*, *69*(1), 33–52. https://doi.org/10.1007/s10649-008-9123-7

Font, V., Godino, J. D., & D'Amore, B. (2007). An Onto-Semiotic Approach to Representations in Mathematics Education. *For the Learning of Mathematics*, *27*(2), 2–14.

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, *82*(3), 439–453. https://doi.org/10.1007/s10649-012-9438-2

Harel, G. (2008). A DNR perspective on mathematics curriculum and instruction. Part II: With reference to teacher's knowledge base. *ZDM*, *40*(5), 893–907. https://doi.org/10.1007/s11858-008-0146-4

Harel, G. (2013). Intellectual Need. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 119–151). Springer. https://doi.org/10.1007/978-1-4614-6977-3_6

Lamb, L. L., Bishop, J. P., Philipp, R. A., Schappelle, B. P., Whitacre, I., & Lewis, M. (2012). Developing Symbol Sense for the Minus Sign. *Mathematics Teaching in the Middle School*, *18*(1), 5–9. https://doi.org/10.5951/mathteacmiddscho.18.1.0005

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176.

Nachlieli, T., & Elbaum-Cohen, A. (2021). Teaching practices aimed at promoting meta-level learning: The case of complex numbers. *The Journal of Mathematical Behavior*, *62*, 100872. https://doi.org/10.1016/j.jmathb.2021.100872

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, *44*(5), 624–640. https://doi.org/10.1080/0020739X.2013.795248

Park, J. (2015). Is the derivative a function? If so, how do we teach it? *Educational Studies in Mathematics*, *89*(2), 233–250. https://doi.org/10.1007/s10649-015-9601-7

Park, J., Martin, J., Oehrtman, M., & Rizzolo, D. (2023). Guided reinvention as teaching practices that aim to promote student learning to define. *Proceedings of the 45th Anniversary of the Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education*, *2*, 498–507.

Rubenstein, R. N., & Thompson, D. R. (2001). Discuss with Your Colleagues: Learning Mathematical Symbolism: Challenges and Instructional Strategies. *The Mathematics Teacher*, *94*(4), 265–271. https://doi.org/10.5951/MT.94.4.0265

Schüler-Meyer, A. (2020). Mathematical routines in transition: Facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and Its Applications: An International Journal of the IMA*, *39*(4), 237–247. https://doi.org/10.1093/teamat/hrz019

Sfard, A. (2007). When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning From a Commognitive Standpoint. *Journal of the Learning Sciences*, *16*(4), 565–613. https://doi.org/10.1080/10508400701525253

Sfard, A. (2008). *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Sfard, A. (2014). University mathematics as a discourse – why, how, and what for? *Research in Mathematics Education*, *16*(2), 199–203. https://doi.org/10.1080/14794802.2014.918339

Sfard, A. (2018). On the Need for Theory of Mathematics Learning and the Promise of 'Commognition.' In P. Ernest (Ed.), *The Philosophy of Mathematics Education Today* (pp. 219–228). Springer International Publishing. https://doi.org/10.1007/978-3-319-77760-3_13

Stewart, J. (2016). *Calculus, 8th Edition—9781285740621—Cengage*. Cengage Learning. https://www.cengage.com/c/calculus-8e-stewart/9781285740621/

Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: Prologue. *Educational Studies in Mathematics*, *91*(3), 299–306. https://doi.org/10.1007/s10649-015-9638-7

Thompson, D. R., & Rubenstein, R. N. (2000). Learning Mathematics Vocabulary: Potential Pitfalls and Instructional Strategies. *The Mathematics Teacher*, *93*(7), 568–574. https://doi.org/10.5951/MT.93.7.0568

Valenta, A., & Enge, O. (2022). Teaching practices promoting meta-level learning in work on exploration-requiring proving tasks. *The Journal of Mathematical Behavior*, *67*, 100997. https://doi.org/10.1016/j.jmathb.2022.100997

Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. *The Journal of Mathematical Behavior*, *43*, 98–110. https://doi.org/10.1016/j.jmathb.2016.07.001

Obligations and Norms: How Instructors Respond to Students' Correct But Non-Traditional Proofs

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Proof is a central pillar of mathematical practice, and so teaching proof to undergraduate students is a necessary responsibility for undergraduate mathematics instructors. Our study contributes to research on how instructors can achieve this goal by examining their responses to student-generated existence proofs. In particular, the simple logical structure of an existence proof allows for proof-writing norms to be the focus of attention. By analyzing instructors' responses to students' existence proofs in terms of norms, values, and professional obligations, our study attends to the ways in which norms are (or are not) supported by instructors in undergraduate mathematics classrooms.

Keywords: Proofs, Existence Proofs, Professional Obligations, Norms, Values

Among mathematicians, it is not uncommon to agree with Ziegler's (2013) observation that "proofs are the heart and mind of mathematics. They are the pillars upon which the structure of mathematics rests" (p. 130). Among students entering undergraduate mathematics programs, however, the centrality of proof may not be so self-evident; given the scarcity of proving activity featured at the secondary level, this limited perspective is perhaps understandable. It becomes the work of undergraduate mathematics instructors, then, to foster a fuller perspective on proof that more closely aligns with the actual work of mathematics—to illustrate to their students the nature and importance of proof by teaching them what proofs are, how to write them, and what role they serve.

This is not a simple task. Countless studies have explored the difficulties undergraduate students navigate as they come to understand proofs as a concept (e.g., Dawkins & Weber, 2017; Stylianides & Stylianides, 2009; Selden & Selden, 2013). Other researchers have spotlighted particular types of proofs, such as proof by contradiction (e.g., Chamberlain & Vidakovic, 2021), contraposition (e.g., Antonini & Mariotti, 2008) or mathematical induction (e.g., Norton et al., 2022), that are notoriously challenging for students.

By contrast, proving a statement of existence is perceived to be an approachable process for mathematical novices; such proofs are direct, typically constructive, and require only the provision of a single example (Buchbinder & Zaslavsky, 2019). Perhaps for this reason, there is comparatively little research on how undergraduate mathematics students develop an understanding of this type of proof.

But the simplicity of an existence proof is beneficial for researchers interested in how undergraduate students understand the form and function of proofs in mathematics. Because they are not encumbered by logical constructions that are difficult or unfamiliar to them, these students are free to produce existence proofs that adhere most closely to what they perceive a proof "should be like". In this way, existence proofs provide a clearer lens into the norms and values around proof-writing that students have begun to develop.

Simultaneously, the way that undergraduate mathematics instructors read and respond to students' proofs in the classroom—especially those that do not adhere exactly to conventional proof structure—can reveal the mathematical norms and values that inform their teaching. But undergraduate mathematics instructors also have responsibilities towards their institution or their students' learning that might affect how they respond to proofs. Our study examines instructors' reactions to non-traditional student-generated existence proofs to better understand what aspects of proving those instructors value and thus promote in their classrooms. This answers the research question: *What do mathematics instructors consider when evaluating student-written existence proofs and how do their considerations relate to normative conventions for proving?*

Background

To answer our research question, we must first consider what values and norms mathematicians might conventionally hold. Dawkins and Weber (2017) conducted a thorough review of literature in order to present such a list of values, and the associated norms, with respect to proving. The four values they identified were:

1. Mathematical knowledge is justified by a priori arguments.
2. Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
3. Mathematicians desire to increase their understanding of mathematics.
4. Mathematicians desire a set of consistent proof standards.

Each of these values begets a collection of relevant norms, that is, suggested practices for fulfilling the associated value when engaged in proving. For example, the norm "*Irrelevant statements are not presented in the proof"* is a convention used by mathematicians to support the value for increasing understanding; Dawkins and Weber argue that "adding irrelevant statements or assumptions will confuse the reader as she struggles to find out how this is relevant" (p. 131).

Rupnow and Randazzo (2023) extend the work of Dawkins and Weber (2017) by interviewing mathematicians about the values and norms they hold with respect to defining as opposed to proving. They propose an additional value: that mathematicians desire clarity in and for communication when writing definitions.

But when mathematics teachers must respond to student-written proofs or definitions, they are often at odds—they must introduce their students to conventional mathematical values and norms while still attending to the needs of their students, their classroom, and their institution. To understand how mathematics instructors might balance these conflicting needs, we turn to Erickson et al.'s (2021) concept of *professional obligations*. The authors explain: "If norms describe expectations for action held in common by members of a profession, professional obligations help us understand why actions at a particular point and time may deviate from those expectations" (p. 192). Mathematics teachers might experience obligations to the *individual student*, the *social classroom*, the *institution*, or to the *discipline of mathematics*. For example, a teacher might endorse a non-normative proof out of obligation to an individual student if they perceive that doing so would have a positive effect on that student's emotional or intellectual needs. Thus, a mathematics teachers' decisions in the classroom can be seen as aligning with a mathematical norm (and thus, promoting a particular mathematical value), or else as fulfilling a professional obligation to some other entity.

Methodology

Participants and Data Collection

Our data comes from two sources: an online survey and semi-structured interviews, both with mathematics instructors. The survey collected participants' ($n = 86$) reactions to five different student responses; these responses were proofs of the statement, "There exist four different prime numbers the sum of which is prime," and are provided in Table 1. The presented responses were inspired by student work submitted in a course intended to serve as an introduction to proofs.

Table 1. The five student responses assessed by participants in the study.

Participant	Proof
Alpha	True. In order to prove this exists we only have to show one valid example of the statement. *a, b, c, d, e* in "primes" *a + b + c + d = e* Example: 2 + 5 + 7 + 17 = 31 2, 5, 7, 17, and 31 are all primes. Therefore, there does exist a case where four distinct/ different primes sum to equal a prime number.
Beta	True. For example, consider the sum of prime numbers 2 + 3 + 7 + 11 = 23 which proves that there exists four different prime numbers the sum of which is prime. But there also exists four different prime numbers the sum of which is not prime. For example, 2 + 5 + 7 + 11 = 25.
Gamma	True. 2, 3, 5, and 7 are all different positive primes. Their sum, 2 + 3 + 5 + 7 = 17, is also a prime number. This example proves the initial statement that there exists a (at least one) solution. Note that 2 must be one of the primes in the sum: 2 is the only even prime number, so if none of the four primes is 2, then we have odd + odd + odd + odd = (2a + 1) + (2b +1) + (2c + 1) + (2d + 1) = 2(a + b + c + d + 2) = 2k where k is an integer, hence the result is even and cannot be prime (unless k = 2, except all odd primes are greater than 2, so the sum will be greater than 2 making this not possible).
Delta	True, 2 + 3 + 5 + 7 = 17, all primes.
Epsilon	This statement is true. Proving by contradiction, let us assume that this statement is false, then it becomes, "There don't exist four different prime (positive) numbers the sum of which is prime." We can disprove this with a single example, such as 2, 5, 7, and 17, which sum to 31, a prime number. As the inverse statement is false, we can see that the original statement, that there exists at least one group of prime numbers that sum to a prime, is true.

We note that, despite one minor (unintentionally included) error in Gamma's proof (the phrase "unless $k = 2$" should be "unless $k = 1$"), each proof is mathematically correct and logically sufficient to prove the existence of the requisite set of prime numbers. The proofs were chosen for the survey because they did not adhere to the previously observed norm for minimality found in existing research literature (or, in Delta's case, pushed the limits of what a lower bound on minimality might be).

Participants were prompted to "Imagine that you were teaching a math class, and you received the following five homework submissions as an argument or justification for whether the statement is true or false". Under these circumstances, participants were asked to rank the five student responses from "best to worst", under whatever criteria they deemed to be appropriate. More importantly, they were also asked to explain the rationality for their ranking.

The interviewees ($n = 5$) represent primarily a convenience sample, but one that targeted a variety of backgrounds both in learning and teaching mathematics. Interviewees completed the same survey as described above, but in the presence of a member of the research team. Doing so allowed the interviewees to explain in more detail their method for ranking the student responses; it also gave the research team member conducting the interview the opportunity to ask clarifying questions and prompt the interviewee to provide additional insight. Each of the interviews was audio-recorded and transcribed.

Data Analysis

Because the student responses were chosen for their relationship to the norm for minimality in mathematics, participants' explanation of their rankings were first coded by one member of the research team according to whether or not they indicated approval of the amount of content included in each student response. When participants endorsed a student's proof that was not minimal, their rationale was associated with a corresponding professional obligation. In other situations (e.g., when a non-minimal proof was rejected, or when Delta's proof was rejected despite its minimality) techniques from thematic analysis (Braun & Clarke, 2019) were used to identify commonalities between participants' responses. Through constant comparison, these commonalities developed into themes that sometimes aligned with existing norms or values that could be used to add context to the participants' reasoning. Next, a second member of the research team coded the participant explanations following the same procedure and the two researchers met to resolve any differences. An identical process was used to code and recode the interview transcripts.

Findings

In this report, we focus on three of the five responses that provide the most fertile analysis: Gamma, Delta, and Epsilon. We present participants' responses to each of the three existence proofs in separate subsections. Then, we provide a synthesized analysis of these responses.

Gamma

Gamma's response was ranked as the best by a majority of participants ($n = 63$) and by every interviewee. This overwhelming support of Gamma's proof, despite the fact that it is not normatively minimal, can be attributed to several different factors.

First, participants often identified Gamma as having produced the "best mathematical response" by "investigating and extending the problem." In particular, Gamma's observation that one of the prime numbers in the sum must be 2 was more than just an arbitrary mathematical fact appended to the end of the actual proof; participants considered this contribution a "corollary" that unpacks the "underlying structure of the statement rather than just providing an example." Participants stressed that Gamma's extra result was mathematically interesting and alluded to a more general explanation for how a set of primes meeting the requirements of the theorem could be found. This perspective was exemplified in Alan's interview when he said, "I don't particularly like the question. It doesn't do anything. It's nothing. There's no theory behind it. And Gamma, as I said, is the closest to coming up with a theory that makes the question interesting." Alan appears to accept Gamma's breach of the minimality norm because he feels an obligation to the discipline of mathematics, in the sense that it is a disservice to the subject to spend time proving inconsequential results. To Alan, mathematical proof is substantive and worthwhile if it is built on or contributes to existing theory; in his opinion, Gamma's corollary does exactly this.

Other participants ranked Gamma's proof as the best because they ordered the student responses according to the amount of mathematical understanding they revealed. These responses demonstrate an obligation to the student: unnecessarily longer proofs can be allowed when they provide an instructor with insight into their students' ways of thinking. Some participants reported that Gamma's extra work showed "the most conceptual understanding behind the answer"—that is, it revealed the extent of Gamma's mathematical subject matter competency. Other participants noted that Gamma probably "understands that they were done after the first sentence, but they enjoyed proving a little extra." In this sense, Gamma isn't displaying mathematical content knowledge but rather a "meta-theoretical understanding of the role of an example." Gamma's higher-level grasp of the role and methods of proof is evident, somewhat ironically, only because of Gamma's explicit violation of the norm for minimality in proving.

Finally, a number of participants recognized Gamma's proof as the best response because they quantified "best" according to how useful the student's proof would be as a pedagogical tool for promoting mathematical discourse in a classroom discussion. One participant explained that "Gamma […] immediately provided an example in the clearest way possible. The extra work they did actually put them above Delta because it gives me, as an instructor, ways to extend the question in possible follow-up discussion." This sentiment was echoed by Anthony in his interview: "So, in that sense, Delta and Gamma's responses are both good and I think it would be interesting to talk about which one would be preferred […] as a class." Anthony ranked both Delta and Gamma highly out of obligation to the social classroom. That is, he foresaw a chance for his students to collaboratively negotiate a sociomathematical standard for "how much" information should be included in a proof. The length of Gamma's response was acceptable exactly because it was not minimal, and thus could be juxtaposed against Delta's particularly extreme minimalism.

In fact, Gamma's response was never ranked as the worst by any participant and was only identified as the fourth best response a total of eight times. Most often, participants who ranked Gamma especially low took issue with his mathematical mistake (writing "unless $k = 2$" instead of "unless $k = 1$") rather than the length of their response. However, one participant linked the two: "The most concise answers are the best. [Answers like Gamma's], while to be encouraged since they are exploring the problem further, are more likely to lead to responses including incorrect statements."

Delta

Delta's response was ranked as the best by five participants. Their rationale for doing so was consistent: these participants valued Delta's "directness and simplicity," and variously described his response as "concise," "succinct and clear," and "short and to the point". This demonstrates a very clear preference for proofs that adhere as strictly as possible to the norm for minimality. Alan provided another perspective on Delta's work; although he also admitted that he liked it "for its brevity," he explained that brevity was only warranted because "this question is not significant in any way, so I think it's—the response given is on par for the level of the question."

On the other hand, Delta's response was ranked as the worst by half of the total participants ($n = 43$). These participants took a much more negative view of Delta's level of explanation, describing it as "incomplete" or "insufficient". Importantly, this appeared to be a mathematical shortcoming—one participant felt that Delta "provided no warrants at all for how this [example] justifies the claim." Another explained that, although "Delta's example is technically correct," he should "include more structure to make this a more formal argument." These responses indicate that participants were not necessarily concerned with Delta's understanding but instead with the understanding of those who might need to read and interpret his proof. This sentiment was outlined explicitly by Kenneth during his interview:

> For me, proof in mathematics is about communication of ideas. And for me, I—if I didn't have the question in front of me, I didn't know what we were trying to prove, I would have *no* idea what we were trying to do here, right?

This aligns with earlier results from Rupnow and Randazzo (2023) that mathematicians value communication but applied to proofs rather than definitions. However, it can also be interpreted as a type of obligation to the discipline of mathematics. Kenneth appeared to believe that proofs are not just a means of establishing deductive truths, but also a means of sharing those truths with fellow mathematicians. To do this, proofs need to be self-contained; this perspective is, in fact, one of the important mathematical values delineated by Dawkins and Weber (2017). From this perspective, Kenneth's reaction to Delta's proof is an example of competing mathematical norms: A proof must be minimal, but not so minimal that it cannot stand alone as an independent mathematical object.

Other participants who ranked Delta's response last were more concerned with Delta's personal understanding; in the most extreme case, one participant wondered if Delta had even copied their proof from a peer. These participants explained that Delta's work "gave very little insight into their thinking," and thus did not allow them to "assess what they understand." Kenneth's interview added a caveat to this assessment of Delta's understanding. Kenneth recognized that he "might accept this from a graduate student because I would assume that they have a certain baseline level of proof knowledge, but from an intro to proof course, my expectations are a little more, I guess." This illustrates the importance of context for identifying mathematical norms, especially in mathematics classrooms.

Epsilon

Epsilon's response was ranked as the best in eight participants' responses. Interestingly, despite the fact that Epsilon was ranked first in so few responses, there were still two clearly distinct schools of thought as to why it should be placed above its peers.

First, some participants thought that Epsilon's proof was simply the least incorrect. These responses typically took issue with Gamma's mathematical error and Delta's insufficient explanation, leaving Epsilon as the only "valid proof with no errors." Kenneth added some context to this perspective when he explained that

> Epsilon is clearly communicating everything that they're doing. So they—they're clear about what method of proof they're going to use. […] I don't love that they chose proof by contradiction, but they've done it in a nice way and it's clear to me what they're doing.

That is, Epsilon's proof may not be stylistically preferable but is still technically correct. Anthony also summarized this somewhat conflicted appraisal of Epsilon's proof by admitting that "if I was going to write a contradiction proof for this existence theorem, which is bizarre, it would probably look like Epsilon's. I just don't think that's a good idea."

On the other hand, some participants ranked Epsilon's response as the best specifically because of its logical complexity. One participant characterized Epsilon's proof as the "most sophisticated," while another described it as the "most formal." Importantly, this sophistication was seen as evidence of Epsilon's mathematical understanding: not only did he clearly understand how a proof by contradiction should be logically structured, but he was also able to use a proof by contradiction in a surprising and unexpected way. Alan provided a counterpoint to this interpretation of Epsilon's understanding when he observed that "it seems like Epsilon is, like, over formalizing. Maybe because they expect, you know, this is a math class and I'm supposed to be really formal." Then, the use of proof by contradiction could be seen as an explicit lack of understanding. In either case, however, the instructors are acting out of obligation to the student; Epsilon's clearly non-minimal proof is important to allow because it ultimately provides evidence that further conversation should take place with Epsilon.

Epsilon's response was the worst-rated proof in almost as many cases as Delta's ($n = 36$). Almost universal criticism was directed at Epsilon's choice of proof by contradiction, which participants found "labored," "unnecessarily complex," or simply "less clear" than other choices. This was a mathematical consideration; these participants were not concerned with why Epsilon felt that a proof by contradiction was warranted but were simply dissatisfied with the resultant mathematical object. Other participants, however, did identify Epsilon's choice of proof framework as insight into his (lack of) understanding. For example, one participant wondered whether Epsilon might be "uncomfortable with existence proofs". Another inferred that Epsilon was "proving things a bit ritualistically—they seem to be copying a known proof structure without considering whether it is the best way to prove the theorem in question". This was the basis of Gavin's poor appraisal of Epsilon's proof. During his interview, Gavin explained that Epsilon "complicated the solution unnecessarily. I don't think they understand clearly the process of solving—or sorry, proving this question."

Discussion

Our research question asked: *How do mathematics instructors respond to student-written existence proofs and how do their responses relate to normative conventions for proving?* Very commonly, non-minimal existence proofs that included technically extraneous information were accepted by participants. This is best illustrated by the overwhelming approval of Gamma's proof. Participants appreciated that Gamma's additional contribution was mathematically interesting, revealed a deeper understanding, and could serve as a foundation for classroom discussion. The latter two points are unique to the pedagogical context in which Gamma's work is embedded, but the former point is strictly a mathematical consideration. But minimality is also a mathematical consideration, and so in this way, reactions to Gamma's proof illustrate how two mathematical norms for proving can be in conflict even when they are both in service of the same value: that proofs should increase a mathematicians' understanding. A similar observation was made about reactions to Delta's proof, in which minimality was at odds with the need for the proof to make sense as an independent object.

Despite Gamma's positive reception, other non-minimal proofs (i.e., Epsilon's) were largely rejected by the same participants. This is in spite of the fact that, as pointed out by some interviewees, Epsilon's proof does in fact meet the standard for minimality (in terms of a proof by contradiction) and is revealing of Epsilon's understanding. Ostensibly, Epsilon's proof also could be used to stimulate a classroom discussion. We hypothesize that the difference between reactions to Gamma and Epsilon's proofs lies in the fact that Epsilon's proof added logical complexity without any accompanying mathematical insight.

Conclusion

Ultimately, participants in this study illustrated that mathematics instructors are often required to attend to non-mathematical obligations when considering student-written proofs. One limitation of this study was that much of the pedagogical context that might have led to these obligations was not provided in the survey—some participants commented that it was difficult for them to appraise the student responses without more information on the structure and purpose of the class, for example. Future research might include fewer student responses, but within a more elaborate fictional setting to allow instructors the opportunity to make more specific judgements.

References

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: what is specific to this way of proving?. *ZDM*, *40*(3), 401-412. https://doi.org/10.1007/s11858-008-0091-2

Braun, V., Clarke, V., Terry, G., & Hayfield N. (2019). Thematic analysis. In Liamputtong, P. (Ed.), *Handbook of research methods in health and social sciences* (pp. 843-860). Springer. https://doi.org/10.1007/978-981-10-5251-4_103

Buchbinder, O., & Zaslavsky, O. (2019). Strengths and inconsistencies in students' understanding of the roles of examples in proving. *The Journal of Mathematical Behavior*, *53*, 129-147. https://doi.org/10.1016/j.jmathb.2018.06.010

Chamberlain Jr, D., & Vidakovic, D. (2021). Cognitive trajectory of proof by contradiction for transition-to-proof students. *The Journal of Mathematical Behavior*, *62*, 100849. https://doi.org/10.1016/j.jmathb.2021.100849

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*, 123-142. https://doi.org/10.1007/s10649-016-9740-5

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, 1-27. https://doi.org/10.1007/s40753-022-00163-2

Rupnow, R., & Randazzo, B. (2023). Norms of mathematical definitions: imposing constraints, permitting choice, or both?. *Educational Studies in Mathematics*, 1-18.

Selden, A., & Selden, J. (2013). The genre of proof. In M. Fried & T. Dreyfus (Eds.), *Mathematicians and mathematics educators: Searching for common ground* (pp. 237–260). Springer.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, *40*(3), 314-352. https://doi.org/10.5951/jresematheduc.40.3.0314

Ziegler, G. M. (2013). *Do I count?: Stories from mathematics*. CRC Press. https://doi.org/10.1201/b15150

What Meaning Should We Attribute to the Colon in Set-builder Notation?

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In this study, we report one group of students' efforts to create a community meaning for set-builder notation collectively. Students' ability to develop and interpret set-builder notation is essential to transition-to-proof courses. Conventionally, a colon is used in set-builder notation to (1) separate the universe of discourse from the set's defining property and (2) indicate an ordering to these components, with the universe to the left and the property to the right of the colon. We describe one normative and non-normative interpretation of this notation and how the students' individual attribution of conventional meanings for the colon to different inscriptions within the notation helped (or inhibited) them from interpreting these expressions. We report how communicative discourse between the students affected their meanings and discussions.

Keywords: Set-builder notation, set-based reasoning, symbolization, transition-to-proof

The mathematical notion of a set is a crucial component of advanced mathematics. In the context of mathematical proof, students' understanding of sets and ability to posit appropriate relationships between sets can positively influence their interpretation of mathematical statements and proofs (Dawkins, 2017; Dawkins et al., 2023; Dawkins & Roh, accepted; Hub & Dawkins, 2018). Instructors often convey information about sets visually (e.g., Euler diagrams) or symbolically (e.g., set-builder notation). Still, many students struggle to reason viably about sets through these representational mediums. For example, Eckman et al. (2023) reported that some students create oval regions in Euler diagrams to (1) *gather* elements of the universe of discourse that fulfill a particular property or (2) *distinguish* between classes of elements when comparing two equal sets. In the symbolic sense, Eckman et al. (2023) reported that students can attribute various meanings to *arbitrary particulars* in set-builder notation (i.e., ΔABC).

This paper aims to investigate students' conceptions of an additional symbolic component of set-builder notation: the colon (:). In the conventional sense, mathematicians utilize the colon (sometimes written as a vertical bar |) in set-builder notation to (1) differentiate between the universe of discourse and the property by which the elements of the universe are partitioned into a set and its complement and (2) denote an ordinality to how students are supposed to create the set (i.e., first define a universe, then sort the elements of the universe). For example, a student considering the set $S = \{x \in \mathbb{Z} : x \text{ is divisible by } 4\}$ would first construe the universe as the set of all integers and then sort these integers into two sets: the integers divisible by 4 (set S) and the integers not divisible by 4 (set S^c, or the complement of S).

We report two instances where three students attempted to interpret set-builder notation to determine the relationship between two sets. Our data stem from the fourth day of a semester-long classroom teaching experiment we conducted to investigate the affordances of set-based reasoning for students' comprehension of transition-to-proof coursework. We provide the following research question to guide our discussion: *What do students' meanings for the expressions in set-builder notation reveal about their meanings for the colon?*

Theoretical Perspective

We adopt the framework proposed by Eckman (2023) to describe students' *symbolizing activity* or the process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. We use the term *symbol* to denote a personal artifact to which a student has attributed a meaning (Thompson et al., 2014) that she can re-present to herself through the artifact (c.f., Glasersfeld, 1995). We employ Eckman's (2023) framework, which involves three symbols: *personal*, *communicative*, and *conventional* expressions.

We use the following example to illustrate the difference between the three types of expressions. Suppose an instructor of a transition-to-proof course presents her students with the set $A = \{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$. We consider the set-builder notation $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ to constitute a *conventional* expression because the instructor presents a perceptible artifact to the students as an authorized representative of the mathematical community. In the moment of the presentation, each student attributes a meaning (Thompson et al., 2014) to the conventional notation $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ to organize or synthesize portions of their experience, forming a *personal expression*. As the students interact and negotiate a community-approved meaning, the expression $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ becomes a *communicative expression*. A vital component of a *communicative expression* is that individuals must interpret and reconcile the meanings others attribute to the expression (which may or may not reflect their thinking) with their personal meanings for the symbol.

Eckman (2023) described a *relational meaning* as one that students might attribute to their personal expressions. This paper focuses on relational meaning students may attribute to the colon in set-builder notation, which instructors conventionally use to express a relationship between the universe of discourse and the defining property for a set. There are two components to a relational meaning: *connector* and *comparator*. A *connector-oriented* meaning refers to students' conception of a relationship between two expressions and attributing this relationship to a symbol separating the two expressions. For example, the normative connector-oriented meaning an instructor may attempt to convey through the colon in the expression $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ would be that the set of integers, $\mathbb{Z}$, is the universe of discourse and "multiples of 3" is the property defining the elements in the set. A *comparator-oriented* meaning refers to (1) a comparison action the student attributes to the symbol separating the two related expressions and (2) an ordering in which this comparison must occur. For example, an instructor might portray the colon as denoting an ordered process by which the set is created: (1) define the universe as the set of integers, $\mathbb{Z}$, and (2) use the property "x is a multiple of 3" to separate the elements of the universe into set A and its complement. The instructor would expect her students to use the notation (generally) and the colon (specifically) as *communicative expressions* to convey to others their images of the relationships between the integers and the multiples of three.

Mathematicians often attribute multiple meanings to a mathematical expression (Gray & Tall, 1994). In this sense, we consider a conventional meaning for the relational inscription (:) in set-builder notation to include viable *connector-oriented* and *comparator-oriented* meanings. We further expect that students possessing these meanings can re-present them through their *personal* and *communicative* expressions. In the results section of this paper, we address how the students' various comparator-oriented meanings for the components of set-builder notation facilitated or hindered their construction of *communicative expressions*.

Methodology

The data we present in this paper come from an ongoing project to investigate how set-based reasoning might help students to access transition-to-proof coursework (Dawkins et al., 2023; Eckman et al., 2023; Roh et al., 2023; Ruiz et al., 2023; Tucci et al., 2023). Specifically, we report data from the fourth day of a semester-long constructivist teaching experiment (Steffe & Thompson, 2000) during the Fall 2022 semester at a large public university in the United States. During this session, the students worked in groups to determine relationships between sets defined using set-builder notation. The instructor had presented the conventional meaning of set-builder notation immediately before this activity, and we consider the students' group work to constitute their attempt to co-construct set-builder notation as a communicative expression. We focus on one group of three male students, Enrique, Simón, and Juan. The second author was the instructor for the course, and the first author served as the discussion facilitator.

We collected students' work through audio recordings, photographs of students' notes, and pictures of collective whiteboard work. Our analysis first consisted of identifying moments in the data when the students disagreed about the meaning of an expression and needed to negotiate a collective interpretation. We analyzed these key moments using the principles of grounded theory (Strauss & Corbin, 1998). For example, we initially modeled the coevolution of the students' meanings for the set-builder notation during the critical moments (i.e., open coding). After generating a set of initial codes, we attempted to coordinate our codes into an overarching idea, which we determined to be students' attribution of meaning to the colon inscription (i.e., axial coding). In the results section of this paper, we describe how students' relational meanings they attributed (or did not attribute) to the colon inscription allowed them to interpret set-builder notation appropriately or, in other instances, led to cognitive conflict.

Results

The results section comprises two subsections. First, we provide an example of Enrique, Simón, and Juan's productive reasoning about set-builder notation and the relationship between two sets. Second, we share an example where each group member interpreted an instance of set-builder notation differently. In the discussion section, we describe how the students might have leveraged the non-normative interpretations for components of the notation they exhibited to make the seemingly "correct" interpretations we describe in the first subsection.

Distinct Meanings for Set-builder Notation Producing a Conventional Interpretation

At one point during the class, the students compared sets $A = \{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ and $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. The mathematical relationship between these sets is $A = F$ (i.e., both sets contain, and only contain, the multiples of 3).

During this comparison exercise, Simón and Enrique quickly interpreted the elements of set F and posited a relationship between sets A and F:

Simón Ok, so they're saying $x^2 - 1$ leaves a remainder of either 1 or 2 (*unintelligible*). So either x is a multiple of 3 itself, or, no, I think that's the only option, x has to be like a multiple of 3 because ,like, yeah (…) Ok, so I guess it's the same thing [i.e., set A and set F are identical].

Eckman Juan is looking confused.

Juan I don't understand how you got there.

Simón Because it's like, x, so, like if x were a multiple of 3, then this [$x^2 - 1$] won't be [a multiple of 3], like this $x^2 - 1$.

Enrique Right, because x^2 would also be a multiple of 3, but if you subtract 1, then it's no longer a multiple of 3.

(omitted dialogue)

Simón Or, I guess, it's like you could factor $x^2 - 1$ as $(x - 1)$ and $(x + 1)$.

Enrique Right, since you have a plus 1 and minus 1. Like, the thing about the second number [i.e., the expression "$x^2 - 1$ not a multiple of 3"] is it [i.e., x] has to be a multiple of 3, so it's got to be a multiple of 3 in order for it to [work].

Juan Oh yeah, that makes sense. So they would be the same set.

In this excerpt, Simón's order of reasoning indicates that he considered the multiples of 3 (i.e., set A) to be a subset of all integers x with the property "$x^2 - 1$ is not a multiple of 3" (i.e., set F). In contrast, Enrique's order of reasoning indicates that he considered the integers fulfilling the property "$x^2 - 1$ is not a multiple of 3" (i.e., set F) to be a subset of the multiples of 3 (i.e., set A). Collectively, Simón's reasoning that $A \subseteq F$ and Enrique's reasoning that $F \subseteq A$ satisfy the conditions to show that $A = F$. However, there was no indication from this excerpt that (1) Enrique and Simón recognized the subtle (to them) difference in their thinking or (2) either student attributed both conceptions to their communicative expression $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. Juan's final comment indicates the possibility that he considered both qualifications (i.e., $A \subseteq F$ and $F \subseteq A$) when positing the elements of set F.

We purposefully made no direct reference to the colon (:) in this subsection. Instead, we chose to describe the possibility that Enrique, Juan, and Simón agreed on a collective meaning for the elements of the set F while maintaining distinct personal meanings for the set-builder notation denoting these elements. In the following subsection, we describe how these students' differences in *relational* meanings for their communicative expressions might be attributed to their meanings for the colon (:) inscription.

Distinct Meanings Producing a Mathematically Incorrect Interpretation

Immediately prior to comparing sets A and F, the students compared the sets $A = \{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ and $E = \{2x \in \mathbb{Z} : x \text{ is a prime number}\}$. The mathematical relationship between these two sets is that $E \subset A$ (i.e., both sets share only the number 6).

The students spent most of their discussion negotiating a meaning for the elements of set E. Similar to the comparison between sets A and F, each student consecutively expressed a different personal meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$:

Enrique To answer [the question] is anything in both sets, maybe there's nothing in E. Because you can't have a prime number that's a multiple of 2.

Simón Well, E, to me E was either the set of primes or the set of all the primes times two.

Juan I think [the set E is] all the prime numbers multiplied by two.

In contrast with their comparison between sets A and E, the students recognized the differences in their thinking this time. Enrique first claimed that set E is empty because (to him) no integers exist that are simultaneously even and prime (in actuality, the number 2 satisfies both conditions). Simón countered that the set E constituted one of two options (between which he could not decide): (1) the set of all primes or (2) the set of all primes multiplied by 2. Finally, Juan posited that the set E contains only the doubles of all primes (the normative interpretation). In the following subsections, we describe how each student's responses were influenced by their *comparator-oriented* meanings for the relationship between the expressions $2x \in \mathbb{Z}$ and "x is a prime number," which mathematicians conventionally attribute to the colon (:) inscription.

Enrique: The colon does not matter—the universe is determined by x. Enrique talked the least during the negotiation of meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$ and did not explicitly agree to Simón and Juan's final decision for the meaning of this notation. Still, Enrique's comments about his meaning for set E were relatively consistent, as evidenced by the following statements he made at various times during this discussion:

Enrique	Right. But, if you were to like plug in, I don't think this, it can't even exist.
Enrique	No, because then if three equals x, then you'd have six [for $2x$] and six is not a prime number.
Enrique	$2x$ is an element of the integers such that x is a prime number. Could you, does that even exist?
Enrique	To answer [the question] is anything in both sets, maybe there's nothing in E. Because you can't have a prime number that's a multiple of 2.

These excerpts indicate that to Enrique, membership in set E, as defined by the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, required an integer first to satisfy the property "x is a prime number" and then satisfy the property $2x \in \mathbb{Z}$ (where $2x$ is also prime). In other words, Enrique first selected the set of prime numbers, P, as the universe of discourse. Then, he attempted to classify primes whose double was also prime as the elements of set E. Conventionally, we might write Enrique's definition for set E as $\{x \in P : x \text{ is a multiple of } 2\}$. When he could find no integers that satisfied his personal meaning for set E, he claimed the set was empty.

Enrique also stated that "the colon [in the set-builder notation] is like a subset." His comment emerged in response to Simón reviewing an example of set-builder notation he had written down in his notes. Enrique's meaning for E and his subset comment about the colon implies that he attributed his personal *comparator-oriented* meaning for the set-builder notation for E to the inscription x. In effect, Enrique considered x in relation to the universe of discourse and the colon as a synonym for "subset," giving no indication that he considered the colon to be more than a dividing symbol between two expressions (e.g., $x \in P, x \text{ is a multiple of } 2$) whose subset-relationship could be utilized to define the elements of a set.

Simón: Does x or the colon relate to the universe? Simón often led group discussions and frequently presented conventional interpretations of sets and relationships. Simón's initial conception of set E was that it contained "the set of primes." After realizing that neither Enrique nor Juan agreed with his personal meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, Simón began to wonder whether the set E contained the set of primes or the set of primes doubled. At this point of the discussion, the group facilitator intervened and asked the students to consider whether individual integers were elements of set E.

Eckman	So I think in this case it might be nice to just pick some numbers and say like, "Oh, [Is] five in E? Is eight in E?" and see if you can figure it out that way.
Simón	Okay, so … like the number 5. So 5 is a prime number, and 10 is an element of the integers?
Enrique	Is that what it's saying that?
Simón	But so then is 5 or 10 the number that is in the set [E]?

Simón's words indicate that to him, membership in E, as defined by the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, required an integer to satisfy the properties "x is a prime number" and $2x \in \mathbb{Z}$. Unlike Enrique, who envisioned $E = \{x \in P : x \text{ is a multiple of } 2\}$, Simón considered two distinct sets: $E_1 = \{x \in P : 2x \text{ is an integer}\}$ and $E_2 = \{2x \text{ is an integer} : x \in P\}$. For set E_1, Simón considered the set of primes to be the universe of discourse and claimed E contained all primes whose doubles were integers. This was a different approach than Enrique, who insisted that the double of a prime must also be prime to merit inclusion in set E. For set E_2, Simón considered the universe to constitute the set of even integers, and the elements of set E to comprise those even numbers whose value, when divided by two, is prime.

Although Simón did not explicitly describe his meaning for the colon (:) in the expression for set E, we infer, based on his remarks, that he recognized the order of comparison mattered for the expressions $2x \in \mathbb{Z}$ and "x is a prime number." At the beginning of the discussion, Simón appeared to attribute this ordinality to the inscription x (similar to Enrique). His later cognitive confusion emerged from considering whether to attribute the ordinality to the positions of the expressions (and the colon dividing them) or the variable x. In this case, Simón's *comparator-oriented* meaning for the inscription (:) was in development from a simple connector to an indication of an ordered process and relationship.

Juan: The colon divides the universe (left) from the property (right). Juan actively participated in group discussions when he agreed with the claims of his fellow students and quietly interjected or listened if he did not agree with or understand others' comments. Throughout the discussion about sets A and E, Juan insisted (quietly at first, then more rigorously later) that the set E contained the set of all primes doubled (the normative interpretation of set E). After Simón attempted to discern whether 5 or 10 was an element of E, Juan took command of the discussion:

Juan	Is the variable x the number that's in the set [E]? I don't think it is.
(*omitted dialogue*)	
Juan	Yeah, so what I'm reading here in my notes for set-builder notation is that where the $2x$ is in the general form [of set-builder notation], there's an $f(x)$ that represents a format by which every element of the set can be represented.
Enrique	Oh, that's useful.
Juan	So, every element of the set [E] is 2 times a prime number.

In this excerpt, Juan pinpointed what he considered the central point of conflict in the discussion: whether the integers represented by x were given inclusion in set E or the integers represented by $2x$ were given inclusion in the set. After referring to his notes on the general form of set-builder notation that he had taken during direct instruction, Juan stated that the expression $2x \in \mathbb{Z}$ was the arbitrary "format" (i.e., universe) by which to define the set and concluded that "2 times a prime number" was the way to denote the elements in set E.

Although Juan used his written notes to justify his claim about membership in set E, he repeatedly commented throughout the discussion that the elements of set E were the set of primes multiplied by 2. His consistent comments imply his *comparator-oriented* meaning for the colon (:) included a distinct and consistent ordering. This ordering included a notion that the expression that comes to the left of the colon constitutes the universe, and the expression following the colon constitutes the defining property for determining membership in a set. Juan's description of set-builder notation in arbitrary form (e.g., the first expression is $f(x)$) could also indicate that he was beginning to construct a general form of set-builder notation, which has been called a *personal expression template* (Eckman, 2023) or a *symbolic form* (e.g., Jones, 2013).

Discussion

The purpose of this paper was to provide insight into the question: *What do students' meanings for the expressions in set-builder notation reveal about their meanings for the colon?* In this discussion, we address three distinct ideas: (1) the three types of comparator-oriented meanings revealed by our data, (2) how Simón and Enrique might have leveraged their meanings for set E to make their mathematically correct interpretation of set F, and (3) the relevance of this paper in the context of research literature and student instruction.

We have described three relational *comparator-oriented* meanings that Juan, Simón, and Enrique attributed to the expressions $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$ and $\{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. All three students appeared to attribute viable *connector-oriented* meanings to both expressions, perceiving that the expression on one side of the colon referred to the universe of discourse and that the other expression referred to the property by which elements of the set are identified.

However, the portion of the expression to which each student wished to attribute a *comparator-oriented* meaning differed. For instance, Enrique (and, at times, Simón) attributed a *comparator-oriented* meaning to the inscription x, which they considered to relate to the universe of discourse. Consequently, in these moments, the students merely attributed a *connector-oriented* meaning to the colon to divide two connected (to them) ideas. As the discussion progressed, Simón's *comparator-oriented* meaning developed so that he discerned two distinct ways to describe the elements of set E. In this moment, Simón began attributing a *comparator-oriented* meaning to the colon. Still, his meaning was tenuous and only achieved equal status with his prior meaning for x. Finally, Juan's intervention and description of his *comparator-oriented* meaning resulted in the group attributing exactly one order to the notation, indicating the relationship between a universe of discourse and a set defined within that universe.

Simón agreed, and Enrique did not disagree, with the Juan-proposed meaning for the communicative expression $E = \{2x \in \mathbb{Z} : x \text{ is a prime number}\}$. Still, it is possible that Simón and Enrique continued to leverage their personal meanings for x as an indicator of the universe to relate the two expressions for the set $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. For instance, these students might have (1) identified the expression containing x ($x \in \mathbb{Z}$) to define the universe of discourse and (2) selected elements in the domain of discourse that satisfied the other expression (i.e., $x^2 - 1$ is not a multiple of 3) to define the elements of F.

This study furthers previously reported research on students' understanding of sets, set-builder notation, and symbolization. For example, our report provides additional insight into how students might interpret set-builder notation previously reported by (Eckman et al., 2023). Additionally, our explanation of *comparator-oriented* meanings including an ordered component adds to the examples and constructs proposed by (Eckman, 2023). This study also supports (to some extent) that conventional meanings for mathematical topics and symbols are not merely transmitted to individuals. Instead, the individuals must construct their personal meanings and attribute them to a conventional symbol to achieve a normative meaning for a mathematical topic. The students' active participation in the group discussion seemed to serve as a catalyst to bridge their personal meaning and conventional meaning by engaging in community efforts to build a collective meaning for the notation as a communicative expression.

This study also informs efforts to improve the instruction of sets and set-based reasoning in the context of transition-to-proof courses. For example, our results provide further insight into how students come to interpret set-builder notation, which can further inform teaching-oriented research projects related to set-based reasoning (Dawkins & Roh, accepted; Hub & Dawkins, 2018; Roh et al., 2023). Specifically, we describe a potential idiosyncratic student interpretation of set-builder notation, students' attribution (or misattribution) of *comparator-oriented* meanings in the context of set-builder notations. We encourage transition-to-proof instructors to explicitly address the comparison order between expressions in set-builder notation and the inscription (i.e., the colon :) to which this meaning should be attributed.

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References

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 496–522. https://doi.org/10.1007/s40753-017-0055-4

Dawkins, P. C., Roh, K. H., & Eckman, D. (2023). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. *The Journal of Mathematical Behaviordawkins*, *70*(101043).

Dawkins, P., & Roh, K. H. (accepted). Undergraduate students' reinvention of the principle of proof by universal generalization. *Journal for Research in Mathematics Education*.

Eckman, D. (2023). *Students' Personal Algebraic Expressions as a Reflection of their Meanings: The Case of Infinite Series* [Unpublished doctoral dissertation]. Arizona State University.

Eckman, D. C., Roh, K. H., Dawkins, P. C., Ruiz, S., & Tucci, A. (2023). Two vignettes on students' symbolizing activity for set relationships. *26th Annual Conference on Research in Undergraduate Mathematics Education*.

Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140. https://doi.org/10.2307/749505

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *The Journal of Mathematical Behavior*, *50*, 90–102. https://doi.org/10.1016/j.jmathb.2018.02.001

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*, *32*(2), 122–141. https://doi.org/10.1016/j.jmathb.2012.12.004

Roh, K. H., Dawkins, P. C., Eckman, D. C., Ruiz, S., & Tucci, A. (2023). Instructional interventions and teacher moves to support student learning of logical principles in mathematical contexts. *25th Annual Conference on Research in Undergraduate Mathematics Education*.

Ruiz, S., Roh, K. H., Dawkins, P. C., Eckman, D. C., & Tucci, A. (2023). Differences in students' beliefs and knowledge regarding mathematical proof: Comparing novice and experienced provers. *25th Annual Conference on Research in Undergraduate Mathematics Education*.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Lawrence Erlbaum.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Sage Publications.

Thompson, P., Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (pp. 1–24). University of Wyoming. http://pat-thompson.net/PDFversions/2013MagsInAlg.pdf

Tucci, A., Dawkins, P. C., Roh, K. H., Eckman, D. C., & Ruiz, S. (2023). Student explanations and justifications regarding converse independence. *26th Annual Conference on Research in Undergraduate Mathematics Education*.

Accelerating Preservice Secondary Mathematics Teachers' Noticing of Student Needs

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Serving diverse student populations equitably is a focal concern for mathematics educators (Association of Mathematics Teacher Educators, 2017), particularly given recent teacher shortages in high needs schools. Teacher preparation programs are tasked with preparing new teachers to thrive in these settings. In this paper, we examine what preservice, secondary mathematics teachers found valuable engaging in structured mentoring and guided reflective opportunities that integrate theory into practice. Participants engaged in authentic experiences including learning assistantships along with traditional practicum experiences. Participants completed guided written reflections throughout the semester, in addition to meeting regularly with a faculty mentor. We utilized Wenger-Trayner and Wenger-Trayner's (2014) value framework to examine the data and share findings that suggest most participants developed an awareness of mathematical content knowledge, pedagogical content knowledge, and knowledge of students (Ball et al., 2008) at an earlier phase of their training than may be expected in traditional teacher preparation programs.

Keywords: preservice teacher development, reflection, mathematics teacher development

The recruitment of highly qualified mathematics teachers is a pervasive issue in K-12 education (Darling-Hammond et al., 2016; Ingersoll & Perda, 2010). Prior to the COVID-19 pandemic, nearly every state reported shortages of highly qualified mathematics teachers (United States Department of Education Office of Postsecondary Education, 2017), with attrition rates for math and science teachers nearly 70% greater than this average, particularly in high poverty, urban settings (Carver-Thomas & Darling-Hammond, 2019; Sutcher et al., 2019). A recent study (Institute of Educational Sciences, 2022) found 45% of public schools had one or more vacancies nationwide, with major disparities for schools serving students living in poverty. These figures, in conjunction with declining enrollment rates in teacher preparation programs (Partelow, 2019), highlight a critical need to recruit, train, and retain highly qualified mathematics teachers.

Serving students in urban, high needs settings requires educators keep students' background knowledge and cultures at the forefront of their planning and teaching. However, beginning teachers' concern often rests in their own inadequacies, situational responses, supervisor feedback, and classroom management (Fuller, 1969; Fuller & Brown, 1975). Further, mathematics teachers require high levels of pedagogical content knowledge (PCK) and mathematical content knowledge (MKT) (Ball et al., 2008; Schulman, 1986). In response to this need, our university developed a layered model of authentic teaching and learning experiences, where preservice teachers (PSTs) engaged in both learning assistantships in college math courses as well as in traditional K-12 classroom practicums. We paired these experiences with structured reflective opportunities, seeking to answer the question: *What do PSTs value about field experiences and learning assistantships when paired with targeted mentoring and reflection as part of their teacher development?*

Literature Review

Shulman (1986) developed the idea of pedagogical content knowledge (PCK) to differentiate among content knowledge, curricular knowledge, and effective teaching strategies and representations for students. In the field of mathematics education, PCK is defined in three parts: knowledge of content and the curriculum, knowledge of content and teaching, and knowledge of content and students (Ball et al., 2008; Hill et al. 2008). Master teachers with strong PCK can interweave students' prior knowledge with mathematical concepts and create a clear infrastructure for students to decipher and build their comprehension, which van Es et al. (2017) and Lebak (2022) describe as “ambitious pedagogy.”

Fuller (1969), and later Fuller and Brown (1975), identified three phases of concern for novice teachers: (a) self-survival (awareness of self), (b) teaching situation (awareness of task), and (c) pupil (awareness of student-impact). Over time, teachers gradually move away from solely focusing on their own actions and toward how students grapple with content. Specific to mathematics teaching, the goal is for PSTs to shift away from examining non-mathematical classroom matters, such as engagement or classroom management, and toward mathematical teaching and learning (Mewborn, 1999; Roller, 2016). These shifts often take time for novice teachers to make but can be supported through structures such as targeted mentoring and reflection (Feiman-Nemser, 2001; Mewborn, 1999). Powell (2014; 2016), for example, found that PSTs increased their take up of these stages of concern in undergraduate methods courses, particularly the task-oriented stage, however he found little change between or among stages across the span of a semester-long course. More research is needed about the contextual factors that influence PSTs’ stages of concern and impact their development over time.

Research on teacher preparation consistently points to the need for PSTs to have authentic teaching and learning experiences in classrooms to support retention in the profession (Darling-Hammond, 2005) and to help them develop effective teaching practices (McDonnough & Matkins, 2010), including culturally responsive pedagogies (Ladson-Billings, 2021). Traditionally, field experiences (practicums) are incorporated into teacher preparation programs as opportunities for PSTs to apply what they are learning in methods courses (Darling-Hammond et al., 2002; Ellerbrock et al., 2018; Garza et al., 2013). Learning assistantships offer an alternative model for engaging in teaching and learning experiences, whereby undergraduate students act as learning assistants (LAs), supporting undergraduate peers and faculty instructors in college courses versus K-12 classrooms (Ellerbrock et al., 2018). LA programs can reduce student-teacher ratios, support faculty in incorporating active learning practices, and offer increased supports for student-centered learning (Otero et al., 2006), and increase the content knowledge of LAs as they engage with near peers (Gomez Johnson et al., 2021; Closer et al., 2016; Talbot et al., 2015). Research also indicates that LAs later recruited to K-12 teaching positions exhibit more reform-based practices than their peers (Gray et al., 2016). Authentic teaching experiences, including both field experiences and learning assistantships, create avenues for PSTs to actively partake in community-based classrooms with experienced mentor teachers. When collaborating and reflecting with these mentors, PSTs can discover the realities of educating students in real time (Ellerbrock et al., 2018; Garza et al., 2013).

Opportunities to reflect *in* practice and *on* practice (Schon, 1987) can be facilitated through both formal and informal means. Effective reflective opportunities can help PSTs hone their skills in determining what is worth taking note of related to teaching and understanding students and making connections between pedagogical theory and practice (van Es et al., 2017; Kersting, 2008). One of the goals of reflective activities in PST programs is to develop their professional noticing to progress more quickly through Fuller and Brown's (1975) stages of concern in anticipation of working with students in diverse settings. Therefore, it is important for researchers and teacher educators to identify effective elements of such reflective opportunities to create robust opportunities for PST growth.

Theoretical Framework

Collaborative experiences support meaningful learning (Lave & Wenger, 1991), thus understanding how opportunities for PSTs to engage in authentic teaching experiences (field experiences and learning assistantships) supported through collegial structures is important to understand. In this paper, we investigate PSTs' reported value participating in such experiences and examine how engaging in structured reflections after the fact influenced their capacity to shift from focus on self to students during instruction (Fuller & Brown, 1975).

Situated learning theory (Lave & Wenger, 1991; Wenger, 1998) proposes learning is an intrinsic factor of collaborative participation between agents in an organization. Value can be viewed as connected and flexible such that researchers can examine participant experiences in their social and academic contexts (Wenger, 1998; Wenger et al., 2011). Wenger et al. (2011) and Wenger-Trayner & Wenger-Trayner (2014) identified five value cycles: immediate (in the moment), potential (for the future), applied (tested implementation), realized (actualized implementation), and transformative (broader dissemination to others) value. Naming specific aspects of learning experiences that PSTs find valuable, and interrogating how those are linked to their development along the stages of concern can help teacher preparation programs maximize opportunities for PSTs to evolve and develop their MCK and PCK. This study is part of a larger study at a mid-sized, urban, midwestern university investigating undergraduate PSTs participating in a STEM education scholarship program. In this paper, we analyze the guided reflective opportunities that augmented PSTs' early authentic teaching and learning experiences to support their mathematics teacher development.

Research Methodology

The NebraskaMath Noyce project, a National Science Foundation (NSF) Robert Noyce Teacher Scholarship project (Grant No. 1852908), seeks to recruit, train, and retain high-quality mathematics teachers. The authors of this study are members of the project leadership team. This study is part of a larger investigation into the value that PSTs experienced during program activities (e.g., professional development, community of practice, mentorship). Qualitative research methodology was utilized to allow us to uncover participant conceptions of value and developmental stages of concern (Charmaz, 2008) by exploring "how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences" (Merriam, 2009, p.5) through the collection of rich, descriptive data (Yin, 2018). We used qualitative methods to examine the research question: *What do PSTs value about field experiences and learning assistantships when paired with targeted mentoring and reflection as part of their teacher development?*

Context and Participants

In Fall 2020, the Noyce scholarship program began integrating learning assistantships in undergraduate mathematics courses as part of the experiential component of the program. By Fall 2021, the project leadership team began to proactively consider PST and faculty pairings so that each undergraduate student was placed in active learning mathematics courses where they would engage with students and observe evidence-based practices each day.

Participants included nine undergraduate students who served as Learning Assistants (LAs) in Fall 2021. Five participants had completed at least one formal field experience at the time of the study (however, these early experiences were limited by the COVID-19 pandemic). Two participants had completed one field experience, while the remaining four participants had not engaged in a field experience at the time of the study. Participants were placed as LAs in one to two mathematics courses where their roles primarily involved encouraging and supporting students through one-on-one or small group engagement. The participants were paired with eight mathematics faculty members who taught using active learning techniques in their undergraduate mathematics courses (e.g., college algebra, quantitative reasoning, precalculus). These faculty members also met with the LAs outside of class to mentor them in areas such as lesson planning, personal and academic life check ins, and reflection on events from the week.

Data Collection and Analysis

We collected 10 reflection journal entries from each participant (totaling 90 reflections). A member of the research team also conducted a focus group with participants at the end of the semester. We aggregated and de-identified all data from the journals and focus group transcript to ensure the anonymity of study participants and, where appropriate, pseudonyms are used in this report. Reflective journals were a program requirement not only as a data source, but also to support the literature on PST development (e.g., Collins, 2006; Shulman, 1987; van Es & Sherin, 2008). Reflections can frame PSTs' thinking around their teaching practice and can offer a unique opportunity for them to highlight both seemingly large and small occurrences during teaching moments across the semester. Participants were prompted to write their reflections around one of the eight following categories:

1. Learning Assistantship Reflections- based on…participation & engagement in campus math courses.
2. Professional Development (PD)- based on…Noyce Math or campus-offered PD workshops, book study, colloquium, etc.
3. Community of Learners- based on…interaction with other Noyce Math participants or study groups.
4. Outreach- based on…community or campus activity where you were able to work with youth or on a project.
5. Mentoring- based on…interactions with a faculty mentor, whether formally assigned or informal partnership (can also include practicum mentor).
6. Learning Mathematics- based on… experiences as a student of mathematics.
7. Leadership- based on…personal leadership experiences where you took initiative to organize your peers or create a new process/project/event.
8. Lesson Plans- based on…creation of a lesson for your learning assistantship experience, a practicum lesson reflection, or other tutoring planning.

We used a combination of directed content analysis (Hsieh & Shannon, 2005) and thematic analysis (Clarke et al., 2015), mapped onto our theoretical framing of the Wenger et al. (2011) value framework (Jakopovic & Gomez Johnson, 2021; Gomez Johnson et al., 2021), to derive meaning from participants' written reflections and the focus group interview. Via directed content analysis, we leveraged prior research related to key concepts (e.g., teacher preparation, PCK, MCK) as our initial *a priori* codes, using the five value types (immediate, potential, applied, realized, and transformative) to develop common working definitions and examples of each code (Hsieh & Shannon, 2005). We first deductively coded value to find areas of saturation (LeCompte & Schensul, 2013). We then reanalyzed the data using a thematic lens. Thematic analysis is an appropriate approach for answering research questions about people's lived experiences and perspectives on particular topics (Clarke et al., 2015). In the second cycle of coding, we examined data for themes that emerged, which we unpack in the following section.

Findings

Participants had access to both K-12 classrooms (practicum) and undergraduate math courses (learning assistantships) where they participated in authentic learning and teaching activities. While participants were intentionally placed in "active learning classrooms" with master teachers as LAs, their field experiences varied greatly in the level of mentorship and quality of instructional practices they observed and enacted. In our analysis, we found participants most often valued what they learned about their developing understanding of learners' needs. Participants reported immediate value (i.e., experienced in-the-moment awareness) in the following four areas as they participated in authentic experiences and guided reflection: the importance of building relationships, student motivation and confidence, the importance of students' mathematical background knowledge, and the role of differentiated instruction. PSTs discussed these topics, considering their impact in the classroom along with their level of satisfaction teaching and learning.

Building Relationships

Participants noticed the ways in which building relationships with students impacted the overall experience of working with students and helping them learn. For example, James reported that his "first goal was to learn every name," and while it was often a challenge, he continued "redoubl[ing] my efforts because during my third week there, I got the evil eye from a student when I asked for his name yet again." After learning names, he remarked on the impact that act had on students' reactions to his instruction. Along with learning names, participants also described how building relationships meant getting to know learners as "whole people," which involved creating a safe space for students to take risks and show that instructors can be caring adults who care about their success (Stipek, 2010). As Monica noted, "One thing that I am confident in is that my students will know... that I accept each of them just as they are... I am not just there to teach them Math." Monica, along with other participants, shared their increased awareness of students' perceptions of PSTs' investments in them as individuals and the positive impact this had on the classroom environment overall.

The Role of Student Confidence and Motivation

Additionally, participants identified in-the-moment situations where students lacked confidence or motivation, resulting in various student behaviors. Several participants recognized student engagement as a concern and made conjectures about why students might elect to not participate during instruction, as well as ponder their role in addressing classroom engagement. As Anthony wondered, "As a teacher, how can I approach a student who is having difficulties in a class to the point that they give up?" In our data other participants, like Nicole, indicated that specific strategies might be necessary to support and engage learners, whether it be offering one-on-one support, continuing to foster positive relationships, or noticing other aspects of a student's reality that might be influencing their willingness to participate in math class.

Participants indicated that at times this may be more about culture, context, and experiences (Ladson-Billings, 2021; Safir & Dugan, 2021), and less about a student's intrinsic interest in mathematics. Helen reported observing differences in the learning environments between upper-level mathematics classes and on-or below-level mathematics classes,

> The honors students seem eager to learn and the "general ed" classes... seem to lack motivation. It makes me wonder how much of that is due to school culture/teacher expectation and how much of it is the student themselves.

In her reflection, Helen pondered how the classroom environment and student backgrounds might influence both behavioral and educational experiences. Several participants noticed that larger, system level factors may lead to inequities in classrooms (Ladson-Billings, 2021).

Students' Mathematical Background Knowledge

Participants also valued opportunities that highlighted the role student background knowledge plays in their classroom interactions. For example, Leslie explained her struggle to identify the appropriate level of difficulty as a new teacher, "It is such a scary thing when you show something to them that's too advanced for them, they just shut down...but then if you give them something that's way too easy, they might get mad at you." She noticed that, without an accurate prior assessment of a student's mathematics level, significant consequences can occur without knowledge of where a student is situated on a learning trajectory. While PSTs learn about theory such as Vygotsky's zone of proximal development (1978), seeing the application when students have a broad range of prior knowledge raises a new level of consideration for PSTs. This can. be rare for new teachers, who typically are not yet at the stage of concern where they notice how students interact with content (Fuller, 1969; Mewborn, 1999).

Differentiation in Action

A final theme for several participants was the recognition that students often do not have the expected level of background knowledge, along with the fact that within any given class this understanding likely exists along a continuum. Some realized their role as a teacher was to respond with different strategies or approaches than they may have experienced in their own learning (Ball et al., 2008; Hill et al., 2008). Monica shared a moment where she wrongly assumed what an adult learner was struggling on a particular concept. "That was way more advanced than he was ready for, and I was like, oh okay, we have to take another step back and I looked at that." Monica noted that as a teacher she needed to reframe her thinking.

For many early career mathematics teachers, identifying moments where differentiation is needed can be challenging. Participants in the study identified the need to tailor content and identify specific, effective teaching strategies. Eddie shared that, "You have to have multiple different tools to be able to teach students. Some students just don't understand something that 15 people will, and you have to work a little harder...for that one student." James reflected on his observation of his mentor teachers' approaches to this, and talked about how their decisions impacted learners. James mused, "The students that got the easy one are worried they aren't properly prepared for the hard one...[and]...the group with the hard one felt rushed by everyone else." James' noticing of students' reactions to the differentiated problems allowed him to consider how he might approach differentiation in ways that were similar or different to this experience. This example helps to illustrate the depth at which James, among other participants, noticed and analyzed the impact of differentiation on learners.

Implications

Preparing the next generation of mathematics teachers requires both the art and science of teaching. For many PSTs, the realities of the myriad roles they play, along with the knowledge, skills, and dispositions they need to develop can be overwhelming. Early opportunities for PSTs to interact with students in a variety of environments and to reflect on these experiences can illuminate key areas of concern for teaching. In terms of immediate value, the most prevalent takeaways for our participants revolved around the students themselves. As they grew to understand how to teach content, they noticed key elements of understanding learners necessary for successful teaching. Pinpointing areas such as background knowledge, motivation and confidence, and engagement, can provide PSTs with a drawing board of essential components for their future planning. Furthermore, these features of understanding students can facilitate their development as effective teachers qualified to teach in diverse settings.

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References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. https://amte.net/standards

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Beswick, K., & Fraser, S. (2019). Developing mathematics teachers' 21st century competence for teaching in STEM contexts. *ZDM: The International Journal on Mathematics Education*, *51*(5). https://doi.org/10.1007/s11858-019-01084-2

Carver-Thomas, D., & Darling-Hammond, L. (2019). The trouble with teacher turnover: How teacher attrition affects students and schools. *Education Policy Analysis Archives, 27*(36), 1-32. https://doi.org/10.14507/epaa.27.3699

Charmaz, K. (2008). Constructionism and the grounded theory method. In J.A. Holstein & J.F. Gubrium (Eds.), *Handbook of constructionist research* (pp. 397–412). The Guilford Press.

Clarke, V., Braun, V., & Hayfield, N. (2015). Thematic analysis. In: Smith, J.A., Ed., *Qualitative Psychology: A Practical Guide to Research Methods*. *SAGE Publications*, *3*, 222-248.

Close, E. W., Conn, J., & Close, H.G. (2016). Becoming physics people: Development of integrated physics identity through the learning assistant experience. *Physical Review Physics Education Research*, *12*(1).

Collins, A. (2006). Cognitive apprenticeship. In R.K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 47-60). Cambridge University Press.

Darling-Hammond L., Chung, R., & Frelow, F. (2002). Variation in teacher preparation: How well do different pathways prepare teachers to teach? *Journal of Teacher Education*, *53*(4), 286-302.

Darling-Hammond, L., & Bransford, J. (Eds.). (2005). *Preparing teachers for a changing world: What teachers should learn and be able to do.* Jossey-Bass.

Darling-Hammond, L., Furger, R., Shields, P. M., & Sutcher, L. (2016). Addressing California's emerging teacher shortage: An analysis of sources and solutions. *Learning Policy Institute.*

Ellerbrock, C., Main, K., Falbe, K., & Pomykal Franz, D. (2018). An examination of middle school organizational structures in the United States and Australia. *Education Sciences, 8*(4). 168. https://doi.org/10.3390/educsci8040168
Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record*, *103*, 1013-1055. https://doi.org/10.1111/0161-4681.00141
Fuller, F. F. (1969). Concerns of teachers: A developmental conceptualization. *American Educational Research Journal, 6*(2), 207–226. https://doi.org/10.3102/00028312006002207
Fuller, F., & Brown, O. (1975). Becoming a teacher. In: K. Ryan (Ed.), *Teacher education, 74th Yearbook of the National Society for the Study of Education: Part 2*, 25–52.
Garza, R., Duchaine, E.L., & Reynosa, R. (2013). Preparing secondary STEM teachers for high-needs schools: Challenges of an urban residency program. *Teacher Education and Practice, 26*(4), 721-738.
Gomez Johnson, K., Jakopovic, P., Rech, J., & Zickerman, A. (2021). Learning assistants in college mathematics classes: Value for future teachers? *Transformative Dialogues, 14*(2), 75-92.
Gray, K.E., Webb, D. C., & Otero, V.K. (2016). Effects of the learning assistant model on teacher practice. *Physical Review Physics Education Research*, *12*, 020126.
Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4), 430-511.
Hsieh, H.F., & Shannon, S.E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research*, *15*(9), 1277–1288. Sage. https://doi.org/10.1177/1049732305276687
Ingersoll, R.M., & Perda, D. (2010). Is the supply of mathematics and science teachers sufficient? *American Educational Research Journal, 47*(3), 563-594.
Institute of Educational Sciences (2022, October). School Pulse Panel: Staffing Report 2022. https://ies.ed.gov/schoolsurvey/spp/
Jakopovic, P**.**, & Gomez Johnson, K. (2021). Beyond traditional teacher preparation: Value-add experiences for preservice secondary mathematics teachers. *Mathematics Teacher Education and Development, 23*(1), 5-31.
Kersting, N. (2008). “Using video clips as item prompts to measure teachers’ knowledge of teaching mathematics,” *Educational and Psychological Measurement, 68, 845-861*.
Ladson-Billings, G. (2021). *Culturally relevant pedagogy: Asking a different question*. Teachers College Press.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.
Lebak, K. (2022). Examining teachers’ understandings of their enactment of ambitious pedagogies in their classrooms. *Pedagogies: An International Journal.* DOI: 10.1080/1554480X.2022.2077343
LeCompte, M. D., & Schensul, J. J. (2013). Analyzing and interpreting ethnographic data: A mixed methods approach, (2nd ed.). *AltaMira Press.*
McDonnough, J.T., & Matkins, J.J. (2010). The role of field experience in elementary preservice teachers’ self-efficacy and ability to connect research to practice. *School Science and Mathematics*, *110*(1). https://doi.org/10.1111/j.1949-8594.2009.00003.x
Merriam, S.B. (2009). Qualitative research: A guide to design and implementation. Jossey-Bass.

Mewborn, D.S. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education*, *30*(3), 316–341. https://doi.org/10.2307/749838

Otero, V., Pollock, S., McCray, R., & Finkelstein, N. (2006). Who is responsible for preparing science teachers? *Science*, *28*(5786), 445-446.

Partelow, L. (2019). What to make of declining enrollment in teacher preparation programs. *Center for American Progress*. https://www.americanprogress.org/article/make-declining-enrollment-teacher-preparation-programs/

Powell, S.R. (2014). Examining preservice music teacher concerns in peer- and field-teaching settings. *Journal of Research in Music Education, 61*(4), 361–378. https://doi.org/10.1177/0022429413508408

Powell, S.R. (2016). The influence of video reflection on preservice music teachers' concerns in peer- and field-teaching settings. *Journal of Research in Music Education, 63*(4), 487–507. https://doi.org/10.1177/0022429415620619

Roller, S. (2016). What they notice in video: A student of prospective secondary mathematics teachers learning to teach. *Journal of Mathematics Teacher Education, 19*(5), 477-498.

Safir, S. & Dugan, J. (2021). *Street data: A next-generation model for equity, pedagogy, and School Transformation*. Corwin.

Schön, D.A. (1987). *Educating the reflective practitioner: Toward a new design for teaching and learning in the professions.* Jossey-Bass.

Shulman, L.S. (1986). Those who understand: Knowledge and growth in teaching. *Educational Researcher*, *15*(2), 4- 14.

Sutcher, L., Darling-Hammond, L., & Carver-Thomas, D. (2019). Understanding teacher shortages: An analysis of teacher supply and demand in the United States. *Education Policy Analysis Archives, 27*(35), 1-40. http://dx.doi.org/10.14507/epaa.27.3696

Talbot, R.M., Hartley, L.M., Marzetta, K., & Wee, B.S. (2015). Transforming undergraduate science education with learning assistants: Student satisfaction in large-enrollment courses. *Journal of College Science Teaching*, *44*(5), 24–30. http://www.jstor.org/stable/43631844

U.S. Department of Education Office of Postsecondary Education (2017). *Teacher shortage areas nationwide listings 1990–1991 through 2017–18.* https://www2.ed.gov/about/offices/list/ope/pol/ateachershortageareasreport2017-18.pdf

Van Es, E., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy, *Cognition and Instruction, 35*(3), 165-187. https://doi.org/10.1080/07370008.2017.1317125

Van Es, E., & Sherin, M.G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes.* Harvard University Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge University Press.

Wenger, E., Trayner, B., & de Laat, M. (2011). *Promoting and assessing value creation in communities and networks: A conceptual framework*. Open University of the Netherlands.

Wenger-Trayner, E., & Wenger-Trayner, B. (2014). Learning in a landscape of practice: A framework. In E. Wenger-Trayner, M. Fenton-O'Creevy, S. Hutchinson, C. Kubiak, & B.

Wenger-Trayner (Eds.), *Learning in landscapes of practice: Boundaries, identity, and knowledgeability in practice-based learning* (pp.13–29). Routledge.
Yin, R.K. (2018). *Case study research and applications: Design and methods* (6th ed.). Sage.

Student Agency and Identity in Reading Didactical Mathematics Texts

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Reading didactical texts, such as textbooks, requires particular sets of literacy practices. Researchers have proposed that these practices are related to aspects of identity and agency, and have described aspects of these constructs separately. In this study, we examine the interactions between students' agency and their macro- and micro-identities. We present data from two undergraduate students reading a section of a calculus textbook and explore the ways the interactions between the students' identity and agency shape their reading activity.

Keywords: Agency, Identity, Literacy, Textbooks

Didactical texts—that is, media such as textbooks and instructional videos—play an important role in college mathematics instruction. Many students are asked to read mathematics textbooks or watch instructional videos (e.g., Butler, 2019; Shepherd & van de Sande, 2014; Weinberg et al., 2022), and these texts constitute an important resource for student learning outside of in-person instruction.

Disciplinary texts have long been viewed as requiring particular sets of skills and literacy practices to read productively (e.g., Moje, 2007; Shanahan & Shanahan, 2008). Fang and Schleppegrell (2010) described their grammatical resources, Lee and Spratley (2010) described discipline-specific reading actions, and others (e.g., Morgan, 1998; Rezat & Rezat, 2017) have contrasted aspects of genres in mathematics texts with those of other disciplines. In addition, Shanahan et al., (2011) demonstrated that mathematicians, chemists, and historians enact distinct literacy practices when reading texts from their own disciplines.

Some studies of mathematics disciplinary literacy have focused on the identities of the readers (e.g., Wiesner et al., 2020) and the relationships between these identities, the text, and the discipline itself (e.g., Fulmer et al., 2022; Weinberg et al., 2022), while others have explicitly incorporated functions of agency into this exploration (e.g., Fang & Schleppegrell, 2010). Both identity and agency have been widely studied in mathematics education research. A close examination of theoretical descriptions of these concepts suggests that they are intertwined—for example, Boaler and Greeno (2000) suggested that "positional identity" is related to the enactions of these positions; Barton and Tan (2010) alluded to the "intersecting roles" of agency and identity; Cobb et al. (2009) suggested that identity is related to "the ways that students are able to exercise agency" (p. 44); and Hitlin and Elder (2007) stated that agency is grounded in ideas of "the self." Although some studies (e.g., Fulmer et al., 2022; Louie, 2020) have explicitly discussed the entanglement of agency and identity, few have explored their interaction in depth.

Our current research has two goals. First, we want to understand students' mathematical identities, agency, and potential interactions between these aspects. Second, we are curious about the extent to which students' agency and identity manifest themselves in the ways they read and attempt to learn from mathematics texts.

Theoretical Framework

Agency

Agency is generally described as an ability make independent choices and act on those choices (e.g., Anderssen & Norén, 2011), although Gresalfi et al. (2009) point out that agency exists in moments of action rather than being a permanent characteristic of a person. Some researchers (e.g., Barton & Tan, 2010; Emirbayer & Mische, 1998) regard these actions as existing in a structure-agency dialectic (Bourdieu, 1977), so that the structures promote and constrain particular types of action and the actions, in turn, influence the structure.

Many mathematics education researchers cite Pickering's (1995) notion of a "dance of agency" (e.g., Boaler & Greeno, 2000; Brown, 2018; Cobb et al., 2009; Grootenboer & Jorgensen, 2010; Sengupta-Irving, 2016; Wagner, 2007) involving human agency—individuals' acts of creating ideas and making choices—and the agency of the discipline, which refers to established mathematical practices. In this notion of a dance, disciplinary norms are the product of collective human activity and that activity is, in turn, constrained by the norms; thus, this dance mirrors the more general agency-structure dialectic.

Barton and Tan (2010) point out that disciplinary norms aren't immediately affected by individual acts of agency and, thus, Pickering's perspective doesn't allow researchers to account for more localized entailments of position and power. They called for a formulation of agency that attends to its "socially transformative" nature (p. 191) and integrates context, position, knowledge, and identity. Similarly, Fulmer et al. (2022) worked to bridge Pickering's formulation of agency with a more "emancipatory theory of constructing knowledge" that attends to the "interplay of autonomy, identity, power, and knowledge" (p. 632).

We draw on Barton and Tan's (2010) formulation of agency in which "Agency is at once the possibility of imagining and asserting a new self in a figured world at the same time as it is about using one's identity to imagine a new and different world" (p. 193). Agency is grounded in the individual's perception and enaction of control and choice. Thus, moments of agency can be identified by either affective or enactive dimensions. In the former, the participant expresses a perception of their ability to make a choice or feels in control of a situation; in the latter, the participant takes an action (or describes acting) in a way that clearly demonstrates a choice they have made. These feelings and actions exist within a set of locally negotiated and constructed interactional roles and structures that entail a status quo, so actions that run counter to the status quo are evidence of agentive moments.

Identity

Identity has been theorized in many ways within the discipline of mathematics education. Darragh (2016) distinguished between two broad perspectives in which identity has been framed. In the first perspective, identity is presented as a stable construct that you acquire over time. For example, Aguirre et al. (2013) describe mathematics identity in terms of "dispositions and deeply held beliefs" about participating in mathematical contexts. This perspective aligns with the structure-agency dialectic in which identities are described in relation to a socially established notion of "what it means to know and do mathematics" (Cobb et al., 2009, p. 44). Wood (2013) describes these long-term stable constructions as *macro-identities* and points out that their scale makes them difficult to use to examine relationships with moments of learning or to describe the multiple, dynamic identities a student might simultaneously inhabit.

The second perspective frames identity in terms of activity. For example, Boaler and Greeno (2000), citing Holland et al. (1998) described positional identity as "the way in which people comprehend and enact their positions in the worlds in which they live" (p. 173). This perspective focuses on interactions in which people "convey positions or identities for themselves and all others involved in the interaction" (Wood, 2013 p. 778) in a way that draws on the socially constructed context. Wood (2013) refers to the narratives that individuals construct to make sense of these interactions as micro-identities. This focus on positioning enables researchers to describe relationships between enactments of identity and moments of learning, and to consider students' sense of self as potentially multiple and divided.

In our research, we examine both students' macro- and micro-identities. The former enables us to consider the students' perceptions of themselves in relation to long-term narratives (e.g., "I feel like I'm good at calculus"), whereas the latter lets us resolve apparently conflicting identity-related narratives (e.g., "I'm not good at derivatives of trig functions") and to relate aspects of identity with the meanings they construct as they attempt to learn from a didactical mathematics text. Thus, statements in which students describe their self-perception provide evidence of identity, and the interactions—or described interactions—between students and other participants in their socially-constructed worlds also can be evidence of their identity.

Methods

Materials

Examining literacy practices for didactic mathematics texts required using text materials for which the participants would have the necessary background knowledge, yet could engage in the reading process for the purpose of learning a new concept. We selected Newton's method for finding roots as the target concept since it only requires knowledge of derivatives and tangent lines and is not always taught in first- or second-semester calculus. Our materials consisted of four pages from a standard calculus textbook (Stewart, 2016) that introduced Newton's method, provided examples of using Newton's method, and included exercises at the end of the section.

Participants

We sent invitations to participate in the study to two groups of students at our institution: (1) all students who had declared either a major or minor in mathematics and hadn't yet completed a course in numerical analysis; and (2) all students who had been enrolled in first-semester calculus the previous semester and earned at least a C-. All participants who replied and were available during the times the researchers were able to conduct the interviews were included in the study. This resulted in five math majors/minors and five non-math majors/minors. For the current study, the restricted length of the conference paper format led us to randomly select two student participants for analysis: one math major and one non-math major/minor. We expected that such a pairing would allow us to identify similarities and differences in their identities and the ways they reported interacting with pedagogical situations and didactical texts.

Data Collection

Each student participated in a 75-minute interview, conducted by the first author, who is a professor in the mathematics department. They were asked about their positionality with respect to calculus, college, learning, and textbook-use (e.g., "how would you describe your relationship with calculus?"). Then they described their knowledge of calculus and worked on a problem using linear approximation to estimate the value of a function at a point.

Next, the participants sat at a computer that displayed an electronic version of the textbook excerpt; participants were able to use the arrow keys to "turn" the pages. The participants were asked to read the textbook section for the purpose of learning about the concept of Newton's method. As they read, an eye-tracker recorded their gaze pattern. After reading the excerpt, the participants watched a recording of their eye movements, with a red dot on the screen indicating the area of the text they had been looking at each moment. As the recording played, the participants were asked to describe what they were looking at and what they were thinking about; both the participant and interviewer were able to pause and resume the playback.

In the final stage of the interview, the text was segmented into conceptual chunks consisting of (1) an example to motivate Newton's method; (2) the derivation of Newton's method; (3) a description of instances when Newton's method would fail; (4) an example of using Newton's method to approximate the roots of a polynomial; (5) a description of the idea of precision in Newton's method; (6) an example using Newton's method to approximate the root of a number; (7) an example using Newton's method to approximate the intersection of two functions; (8) a description of the impact of choosing a first approximation; and (9) a collection of exercises; each chunk was further sub-divided into a graph (if present), an equation in a blue box (if present), and the rest of the text. The interviewer first asked the participant to describe the main ideas of the text, and then to describe what was being shown or described within each of the indicated conceptual chunks, figures, and boxed equations.

Analytical Methods

We divided the text into areas of interest that matched the conceptual chunks, graphs, and equations described above. Then we used the eye-tracking data to record which of these parts of the text the participants were looking at and how long they looked at each part. We used these data to construct a holistic description of each participant's reading activity.

We used thematic analysis (Braun & Clarke, 2006) to describe the participants' identities and agentive moves. After segmenting each interview transcript into talking turns, we identified turns where the participants demonstrated agency by (1) indicating that they felt they were making a choice and expressed a feeling of control, (2) describing a previous situation in which they felt they were making a choice, or (3) took action that appeared to run counter to the status quo; this identification process took into account the positionality of the participant within the relevant social structures—specifically, the context of mathematics classrooms, independently learning mathematics for a class, and the interview itself.

We then identified turns where the student described or enacted aspects of their identity by (1) describing a stable narrative about their self (i.e., a macro-identity), (2) describing their feelings about their self in the moment, and (3) enacting or describing behavior that reflected particular forms of interactional relationships within the relevant contexts. After identifying turns that reflected aspects of agency or identity, we generated themes inductively by looking for similarities and patterns among the turns. Throughout the identification process, we worked collectively to identify, discuss, and resolve differences in our interpretations.

Results

Non-Math Major [NMM]

Activity. NMM spent approximately 63% of his reading time looking at the three examples, with nearly half of that time on the first example. He spent approximately 31% of his time reading the expository text; nearly 1/3 of this time was spent looking at the equation in the blue box and the surrounding text. He spent only 2% of his time looking at the graphs. He generally read the text in order, but there were 12 instances of him either glancing forwards/backwards or skipping to a different conceptual chunk.

Identity. NMM described being a person who enjoyed both school (e.g., "I've always kind of liked school") and calculus ("it is interesting, but hard, and parts of it are easy and cool"). However, he simultaneously contrasted himself with "people who are quicker with math," suggesting a multi-faceted macro-identity. NMM's interaction with the textbook suggested a micro-identity as somebody who could make sense of mathematical concepts (e.g., "I was kind of just trying to see like, all right, like what's the point of this? Like where is this going?"). He had a strong image of himself as a mathematics student, describing himself as "a numbers person" and "not a graph person," distinguishing himself from other "people who like to look at the graphs rather than the numbers." Entailments of this macro-identity included "lik[ing] equations" and "seeing people do something" that he could repeat. At several points in the interview, he described his activity in a classroom, saying that he would (e.g.) "want to be prepared for [problems on an exam]"; this implicit interaction between himself and an instructor suggested his position in the classroom as a student.

Agency. NMM described a lack of agency for reading mathematics texts, describing how he "steered away from" them and "found other textbooks easier to read." More locally, during the interview, he skipped some parts because he didn't "really understand the context." Although it is possible to view these decisions to avoid reading parts of the textbook, in the context of a mathematics classroom we interpret these as reflecting his feeling of a lack of agency.

Despite this feeling of a general lack of agency, NMM regularly offered critiques and evaluations of pedagogical aspects of the textbook (e.g., describing a figure as "a nice thing to include" and a paragraph as "important to add"), suggesting some feeling of agency. NMM regularly made choices about which parts of the textbook excerpt to read and when to read them (e.g., "I spent a second or two on these side graphs, but I wasn't really drawn to them"); if the status quo is to read all parts of the textbook in the order they are presented, then these actions reflect choice and control, and, thus, are agentic moves. Similarly, he made choices about what to write in his notes based on personal goals (e.g., "I kind of just wanted to give myself an example of how I would go about solving the problem like that"), reflecting feelings of agency.

NMM also demonstrated agency in relationship to the mathematical concepts described in the textbook excerpt. He felt capable of "figur[ing] out what [the equation] actually means" and looked for connections with other concepts he had learned (e.g., "it's kind of like with asymptotes…"). These agentic moves were underscored while he was working on the pre-reading problem, when he transformed symbols (e.g., "I converted [decimal numbers] to fractions because I feel like it's easier") and created his own examples (e.g., "I'm operating under the assumption that the equation f of x is equal to two x") to help him solve the problem; this "tinkering" demonstrates moments of agency.

Intersection of Agency and Identity. We noticed several aspects of NMM's identity and agency that built on each other to significantly influence his reading activity and learning. First, his identity as an "equations person" and as a person who can understand mathematical concepts prompted him to focus on the equations that were presented in the excerpt and enabled an agentive relationship to (agentively) "figure out" what things mean. For example, he described his sense-making activity: "I like equations and stuff like that. I was immediately drawn to the f of x equation up there. I kind of just figured, like, well, that's an equation. How can I figure out what it actually means?" His identity as an "equations person" was related to his identity as not a "graph person," which, coupled with his agency to choose which parts of the excerpt to read and which to skip. For example, he described his choice to skim part of the excerpt: "I'm sure for other people who are more like graph-based learners, it [text describing one of the graphs] would be helpful. just for me as a learner, like I'm not going to look at that graph."

NMM's identity as a student and as an "equations person" was, similarly, coupled with his choices about which parts of the excerpt to focus on, and guided his notetaking. For example, he described his rationale for taking detailed notes on the equations in one of the examples: "I kind of just wanted examples to look back on, like as if they were my notes for an actual test."

Math Major [MM]

Activity. MM spent approximately 62% of her reading time reading the expository text, with roughly half of this time reading the paragraph that described the derivation of Newton's method. She spent approximately 19% of her time reading the examples, with this time split roughly evenly between the three examples. She spent roughly 5% of her time looking at the equation in the blue box and only 2% looking at the figures. She generally read the text in order, but there were 7 instances of her either glancing forwards/backwards or skipping to a different conceptual chunk; when she was reading the paragraph describing instances where Newton's method failed, she looked back and forth between the text and the figure 4 times.

Identity. MM described herself as a person who enjoyed math (e.g., "I was taking math courses for fun") and, in particular, calculus, describing it as "my favorite form of like mathematics because I like solving the derivatives and the integrals and all that." She described herself as "an audio-visual learner," but didn't elaborate. MM described her interactions and position in a math classroom in terms of her status as a student. In particular, she described not being able to fully understand concepts until "the professor goes over it" the next day and wanting to know "what they'd be asking" to guide her reading and learning from the textbook.

Agency. MM "felt confident for the most part" about reading and using textbooks and reported developing her own strategy for reading the textbook by "skimming it for important aspects" and then "go[ing] back to find the answer" to problems. MM regularly made choices about which parts of the textbook excerpt to read and when to read them. She regularly described "skimming" various parts of the excerpt, looking "more at the equations" than the diagrams, and "looking back" at previous equations and explanations when she felt confused.

MM also demonstrated agency in relationship to the mathematical concepts. She felt capable of solving calculus problems in general (e.g., "I like solving the derivatives and the integrals and all that") and that she can do so "on [her] own." She worked to resolve uncertainty by revisiting confusing sections of the text and constructing her own interpretations. For example, when she felt confused about instances where Newton's method fails, she reported "just trying to make sense of that because the wording tripped me up" and went back to the first page "to try and double check." As part of this process, she looked for connections with other concepts she had learned (e.g., "it's kind of like Reiman sums, how it's like the more, the smaller and smaller you make the rectangles the more accurate you're going to get").

Intersection of Identity and Agency. MM's identity as a person who enjoyed calculus appeared to be closely connected to her sense of agency at solving calculus problems, noting that calculus is her "favorite form of mathematics because I like solving the derivatives and the integrals and all that." Similarly, her identity as a confident textbook reader appeared to be connected to her having chosen a textbook-reading strategy.

MM's identity as a student, coupled with her choices about which parts of the excerpt to to focus on appeared to be connected. For example, she described her choices as guided by the obligations she would face as a student: "I like looking at the exercises just to like, get a sense of like what the problems are like or like what they'd be asking and how similarly they are structured to one another."

There were several instances where MM moved on from a section of the text without understanding concepts. For example, she described her feeling when after reading the paragraph that described instances where Newton's method would fail: "I think I understood it enough, but at the same time, I think I was still confused that I was like—I'm just going to move on. Probably that would have been something if I were in class, like I would ask about the next day." In this moment, MM appeared to have been weighing her options for action based on her assessment of her understanding. Her identity as a student allowed her to position herself with respect to the textbook, the mathematical concepts, and the (imagined) professor, and this position facilitated her choice for a course of action.

Discussion

Both NMM and MM shared aspects of their macro- and micro-identities, as well as their feelings and enaction of agency. Both students enjoyed calculus, and they both felt and demonstrated agency for figuring out mathematical concepts. They made connections between the text and their prior learning and made decisions about which parts of the text to read and when. In terms of their reading activity, both students chose the order in which to read the text components and neither student spent much time looking at the graphs.

The two students also exhibited some differences in their identities, agency, and reading activity. While MM felt confident about reading textbooks, NMM expressed a lack of confidence. In their reading, NMM spent most of his time examining the examples, while MM focused on the expository text, examined the exercises, and did look back and forth between one of the figures and the corresponding text.

For each student, their identity, agency, and activity intersected in different ways. NMM's identity as a student prompted him to seek out formulas he could use to solve problems on a test and to consider the examples as helping him build a library of problem types he could use as templates for later solving problems; his identity as an "equations person" and his sense of agency with mathematical concepts appeared to enable him to try to make sense of the equations. At the same time, being not a "graph person" led him to avoid examining the graphs, which potentially contributed to his occasional feelings of confusion, which, in turn, led him to skip particular parts of the text.

MM's identity as a student connected with her sense of agency differently than NMM. At first, her identity as a student appeared to guide her read the text to help her solve problems on a test. However, MM also worked to understand the expository text; this could be connected to both her identity as an effective textbook-user and her sense of agency for mathematical concepts. MM's student identity also appeared to be connected to her perceived choices when she encountered confusing parts of the text: rather than exhibiting a lack of agency, she formulated a course of action that involved utilizing the expertise of her (imagined) professor, and choosing a course of action that she felt would help her resolve that confusion.

Taken together, these data show students making agentic moves based on their stated and enacted identities and their relationship to both mathematics (as a subject) and the text. This demonstrates the flow between identity and agency, and vice versa. What we find particularly interesting is that the students in this study are bringing their humanity to the experience of reading the textbook—that they *do* have identities relative to the experience, and that there are many moments of agency that they experience. Thus, it is relevant that their identity and agency affect their reading activity, but the central take-away is not that there is a causal relationship between these two concepts, but rather these two aspects work together to shape the students' activity.

References

Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics learning and teaching: Rethinking equity-based practices*. National Council of Teachers of Mathematics.

Andersson, A., & Norén, E. (2011). Agency in mathematics education. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of 7th Congress of the European Society for Research in Mathematics Education, CERME 7* (pp. 1389-1398). Rzeszów: University of Rzeszów.

Barton, A. C., & Tan, E. (2010). We be burnin'! Agency, identity, and science learning. *The Journal of the Learning Sciences*, *19*(2), 187-229.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple perspectives on mathematics teaching and learning*, *1*, 171-200.

Bourdieu, P. (1977). Cultural reproduction and social reproduction. In J. Karabel & A. H. Hasley (Eds.), *Power and ideology in education* (pp. 487-511). New York: Oxford University Press.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77–101.

Brown, S. (2018). E-IBL, proof scripts, and identities: An exploration of theoretical relationships. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 21st Annual Conference on the Research in Undergraduate Mathematics Education* (pp. 1-15). San Diego, CA: Mathematical Association of America, SIGMAA on RUME.

Butler, M. (2019). Preparing our students to read and understand mathematics. *Journal of Humanistic Mathematics, 9*(1), 158-177.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for research in mathematics education*, *40*(1), 40-68.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*, 19-33.

Emirbayer, M., & Mische, A. (1998). What is agency? *American Journal of Sociology*, *103*(4), 962-1023.

Fang, Z. & Schleppegrell, M. J. (2010). Disciplinary literacies across content areas: Supporting secondary reading through functional language analysis. *Journal of Adolescent & Adult Literacy, 53*(7), 587-597.

Fulmer, E. F., L. Dobbs, C., Weinberg, A., & Wiesner, E. (2022). Disciplinary literacy, agency, and didactical texts: Findings from a calculus textbook think-aloud study. *Reading Psychology*, *43*(8), 628-659.

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, *70*, 49-70.

Grootenboer, P., & Jorgensen, R. (2009). Towards a theory of identity and agency in coming to learn mathematics. *Eurasia Journal of Mathematics, Science and Technology Education*, *5*(3), 255-266.

Hitlin, S., & Elder Jr, G. H. (2007). Time, self, and the curiously abstract concept of agency. *Sociological Theory*, *25*(2), 170-191.

Holland, D., Skinner, D., Lachicotte, W., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge: Harvard University Press.

Lee, C. D. & Spratley, A. (2010). *Reading in the disciplines: The challenges of adolescent literacy*. New York, NY: Carnegie Corporation of New York.

Louie, N. (2020). Agency discourse and the reproduction of hierarchy in mathematics instruction. *Cognition and Instruction*, *38*(1), 1-26.

Moje, E. B. (2007). Developing socially just subject-matter instruction: A review of the literature on disciplinary literacy teaching. *Review of Research in Education, 31*, 1-44.

Morgan, C. (1998). *Writing mathematically: the discourse of investigation*. Bristol, PA: Farmer Press.

Pickering, A. (1995). *The mangle of practice: Time, agency and science.* University of Chicago Press.

Rezat, S., & Rezat, S. (2017). Subject-specific genres and genre awareness in integrated mathematics and language teaching. *Eurasia Journal of Mathematics, Science and Technology Education, 13*(7b), 4189-4210.

Sengupta-Irving, T. (2016). Doing things: Organizing for agency in mathematical learning. *The Journal of Mathematical Behavior*, *41*, 210–218.

Shanahan, T., & Shanahan, C. (2008). Teaching disciplinary literacy to adolescents: Rethinking content area literacy. *Harvard Education Review*, *78*, 40–59.

Shanahan, C., Shanahan, T., & Misischia, C. (2011). Analysis of expert readers in three disciplines: History, mathematics, and chemistry. *Journal of Literacy Research*, *43*(4), 393–429.

Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior, 35*, 74-86.

Stewart, J. (2016). *Calculus: Early transcendentals 8th edition*. Cengage Learning.

Wagner, D. (2007). Students' critical awareness of voice and agency in mathematics classroom discourse. *Mathematical Thinking and Learning*, *9*(1), 31-50.

Weinberg, A., Wiesner, E., & Fulmer, E. F. (2022). Didactical disciplinary literacy in mathematics: Making meaning from Textbooks. *International Journal of Research in Undergraduate Mathematics Education*, 1-33.

Wiesner, E., Weinberg, A., Fulmer, E. F., & Barr, J. (2020). The roles of textual features, background knowledge, and disciplinary expertise in reading a calculus textbook. *Journal for Research in Mathematics Education*, *51*(2), 204-233.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*(5), 775-808.

“Dear Math…” Letters: Insights into Mathematics Graduate Students’ Mathematics Identities

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Understanding mathematics identity development has yielded insights for undergraduate mathematics education. In this work, we build on existing conceptualizations of mathematics identity to explore this construct in the context of mathematics graduate students with academic career aspirations through analyses of “Dear Math…” letters. We identify and detail four dimensions of graduate students’ mathematics identity: Doer of mathematics, Sharer of mathematics, Feeler of mathematics, and Believer of mathematics. We conclude by discussing implications of these dimensions, as well as the unique opportunities of “Dear Math…” letters to provide valuable insight into graduate students’ mathematics identities.

Keywords: Graduate Students, Mathematics Identity, Methodology

Introduction

Mathematics identity development has emerged as a critical area of research, providing insight into students’ achievement (e.g., Gonzalez et al., 2020), persistence (e.g., Joseph et al., 2017), and career choices (e.g., Cribbs et al., 2021), and informing curriculum, pedagogy, and instruction that can better support students’ identity development (e.g., Clark et al., 2013, Fernández et al., 2022). Despite our increased understanding of mathematics identity in K-16 education, little is known about the mathematics identities of graduate students. Graduate education has historically served to socialize and develop graduate students’ professional identities as professors (Van Lankveld et al., 2017). However, within the mathematics discipline, this process may be more complex and nuanced. Scholars have explored graduate students’ professional identity formation as teachers (e.g., Beisiegel & Simmt, 2012) and as researchers (e.g., Hancock & Walsh, 2014). Still, we have yet to critically examine their sense of selves as mathematicians or attempt to understand how they make sense of and identify with the discipline in ways that do not conflate with their professional identities as mathematics professors. Mathematics graduate students have succeeded and persisted through years of rigorous mathematics studies, thus focusing on their identities allows us to both incorporate and extend beyond notions of competence, performance, and traditional markers of success to dive deeper into dimensions of mathematics identities that may not be as prevalent in K-16 students.

In this work, we also introduce the use of “Dear Math…” letters (e.g., Webb, 2020), or letters that students write to the mathematics discipline, and illustrate how this type of qualitative data provides new perspectives into graduate students’ mathematics identities. This can inform us about the formative and disruptive experiences in their mathematical journeys and we contend that a better understanding of graduate students' mathematics identities can position departments and institutions to better support their transition to academic positions. The research question that guided this study was: How did mathematics graduate students reflect on their mathematics identities through writing letters to the discipline?

Conceptual Framing

Scholars generally describe mathematics identity as the ways students relate to mathematics and position themselves within or around mathematics. This identity is dynamic and transformative, changing depending on the context and the accumulation of mathematical experiences (Gardee & Brodie, 2021). However, the exactitudes within different definitions vary. Cribbs et al. (2015) divide the concept into three sub-categories: ways in which students perceive their competence in mathematics, their interest in mathematics, and their recognition as mathematically capable by themselves and others (e.g., instructors, peers). Crossley et al. (2018) present a similar three-part definition, including students' math self-concept about their ability, their interest in mathematics, and the value of mathematics in terms of its utility in their lives. These two conceptualizations describe mainly individual perceptions and traits. In contrast, Voigt (2020) posits a conceptualization that explicitly draws on social and contextual aspects of mathematics education. He describes mathematics identity as encompassing students' perceptions of their ability to effectively perform and participate in mathematical environments, the ways they position themselves within or outside mathematical communities, and how they negotiate their identity within institutional, individual, and societal contexts. These are just three examples of the numerous avenues scholars conceptualize mathematics identity. However, the majority of contemporary research focuses on the undergraduate level, with very limited work on graduate students' mathematics identity. Given the many ways undergraduate and graduate school differ, in this study we leverage the wide scope of research on mathematics identity and the multiple definitions presented to examine graduate students' mathematics identity.

Method

Context and Participants

This study was conducted at a Minority-Serving Institution in California with a mathematics department that offers graduate programs in mathematics at the masters and doctoral levels. We used convenience sampling (Miles et al., 2020) to recruit graduate students in the mathematics department. Sixteen doctoral students completed a survey that elicited self-identified/self-selected demographic information and requested their participation in a 90-minute qualitative interview. Six students participated in the interview and are identified in this study by their pseudonyms and pronouns: Connor and Landon (he/him); Kayla, Kendall, and Morgan (she/her); and Martin (they/them). All participants reported actively pursuing mathematics faculty positions.

Data Collection and Analysis

We utilized Zoom to conduct the 90-minute interviews. In the first hour, we used a semi-structured interview protocol (Rubin & Rubin, 2011) focused on students' experiences in graduate school, including the supports and barriers they encountered, their perceptions of the mathematics professoriate, and their understanding of mathematics teaching. Within the last 30 minutes, participants wrote the "Dear Math…" letter in which they personified the mathematics discipline and wrote correspondence to it. They were prompted to reflect on their experiences with mathematics, their career choices, and how mathematics has impacted their sense-of-selves and then tell the discipline how they really feel. Although the first author was present in the Zoom call, participants wrote the letters without any interaction or interference. For this study, we utilized these "Dear Math…" letters as the primary data source.

We consider self-reflective, narrative activities as mechanisms of identity development. Prior scholars have emphasized that narrative activities (e.g., writing self-reflections, telling stories, etc.) are identity constructing activities and can be methodological tools for researchers to examine identities (e.g., Beijaard et al., 2004). We posit that writing letters to the discipline is one avenue for students to share stories that illuminate dimensions of their mathematics identities. The use of letters is a facet of correspondence and arts-based methodology (Leavy, 2020), and is different from asking participants to narrate their experiences in an interview. Most notably, writing letters provides participants with an avenue to privately share experiences and emotions they might not comfortably share in an interview (Harris, 2002; Stamper, 2020). Webb (2020) found that letter-writing methodology allowed students to critically self-examine their experiences with mathematics past and present.

To analyze the data, the research team first individually read through the letters and wrote memos about what we noticed in how graduate students described themselves, others, and mathematics. Then, as a group, we open-coded (Miles et al., 2020) the letters and found three initial themes: (1) *Doers* of mathematics, (2) *Feelers* of mathematics, and (3) *Sharers* of mathematics. The research team verified the relevance of the codes by re-coding the letters and in this round, another theme emerged: *Believer* of mathematics. In our final round of coding, we re-coded all the letters using the four themes listed above, which we conceptualize as dimensions of graduate students' mathematics identity.

Findings

We now highlight how our analysis of graduate students' letters to the mathematics discipline revealed four key dimensions of their mathematics identities: *Doers*, *Sharers*, *Feelers*, and *Believers* of mathematics.

Doer of Mathematics

The first dimension that emerged from our analysis was students positioning themselves as *Doers* of mathematics. In these instances, our participants described how they make use of mathematics or see its function in the world (i.e., its utility). For two of our participants, the utility of math was mostly a form of play, whereas one of our participants saw it as a means to gain valuable life skills. Morgan and Kendall referred to doing mathematics as fun and playful. For example, Morgan described how she "loved playing with numbers and coming up with my own problems that I could subsequently solve." Similarly, Kendall stated how "you are extremely fun to explore and play with." These two students viewed themselves as *Doers* of mathematics for the purpose of playing, exploring, and having fun.

Landon described a more practical application of what it means to do mathematics. For example, he wrote that "most importantly, I love how you have taught me how to think. Doing math has clearly changed the way my internal world model interprets and analyzes sensory data …equivalent to upgrading from an old computer to a new one." We see that Landon viewed himself as a *Doer* of mathematics for the purpose of developing cognitive skills like logical thinking and sensemaking. Further, he placed value on skills that he has acquired through doing mathematics. Interestingly, he explicitly specified that the act of doing mathematics was not as a requirement to fulfill in school or necessary to enter a mathematical career, but rather "these things all feel secondary to doing math as a way to learn how to think clearly about things and apply this clear thinking to every aspect of my life." Again, whereas Morgan and Kendall viewed themselves as *Doers* of mathematics for enjoyment, Landon viewed himself as a *Doer* of mathematics for the purpose of developing valuable and widely applicable skills for life.

Sharer of Mathematics

The second dimension that emerged pertained to how students positioned themselves as *Sharers* of mathematics. In their letters, the participants discussed mathematics both in terms of their role in sharing mathematics with others and their sense of mathematics as a community. This dimension frequently appeared as students assuming the responsibility to share the positive side of mathematics with others. An awareness of common negative perceptions associated with mathematics prompted many students to write about how they see themselves as advocates for the discipline. Kendall noted in her letter that she hopes she "can help more people see you [mathematics] as a friend." Many participants, like Kendall, identified teaching as a way they could take on this advocate role. Martin wrote, "I see [my students] grapple with your theorems and symbols, but I want to show them who you truly are – what you can really do." Similarly, Connor wrote, "I'm thankful for…the moments of teaching I remember when I finally showed someone something very cool about you." These participants saw teaching as a prime opportunity to share their love and admiration for the discipline with others.

Affiliation with a mathematics community and awareness of how mathematics is shared within that community also revealed how participants identified as *Sharers* of mathematics. For instance, Connor wrote how "you've led to the creation of some of my most treasured communities, you've led me to some of my best friends". Kendall wrote of a similar experience, writing "I appreciate how much I have learned from you and the people you've brought into my life. You really have the best friends." For these two participants, mathematics fostered community and friendships. For Landon, mathematics could be used to bring people together through a universal language, writing "I can communicate perfectly with someone that doesn't speak a word of english." For some, the notion of a mathematics community did not always come with positive experiences or connotations. Morgan wrote explicitly about how she felt excluded as a result of being a woman in a male-dominated field, writing "I learned, through thousands of little words and actions, that I wasn't welcome in the boys' club that is the world of math." She later wrote about persevering and finding her place within the community, but acknowledged how exclusionary it can be. Overall, we saw both positive and negative experiences in how graduate students described being *Sharers* of mathematics.

Feeler of Mathematics

We also found that participants described themselves as *Feelers* of mathematics. This dimension of their mathematics identities refers to affective factors relating to their experiences with mathematics, relationships with members of the mathematics community, and overall feelings toward the discipline. Participants described experiencing a range of positive and negative feelings towards mathematics over the course of their educational experiences. These feelings were not static and were often influenced by their graduate education experience. Some students revealed that early experiences with mathematics fostered curiosity, excitement, and interest, and this often came from being positioned as successful by their teachers and peers. However, positive early experiences were not always shared by the participants. For example, Martin wrote, "When we first met, we didn't get along – at least, I didn't understand you and wanted you to go. Multiplication was hard, and those timed worksheets made me anxious…Honestly, you scared me, or so I thought." Here, Martin expressed how their early feelings surrounding mathematics were marked by anxiety and fear, illustrating that graduate students can have widely different early emotions with mathematics that shape their identities.

Additionally, students detailed how graduate school marked a period of time when they contested with negative feelings about themselves and the discipline. This detrimentally impacted their mathematics identities and made them question their place within the discipline. A frequently cited area of dissent was when graduate students compared themselves to their peers and were made to feel like their mathematical abilities were not adequate for this space. For example, Kayla noted that entering graduate school involved confronting the idea that "there were other people that were just as good at you as I was," which made her feel "threatened" and "jealous." Kayla was faced with reconciling her mathematics identity, mainly related to her competence, in comparison to how she perceived her peers. Similarly, Morgan described feelings of exclusion due to gendered mathematics spaces, leading to feeling that "every success was handed to me for the sake of diversity, or so I was told, until I believed it." These feelings of exclusion constitute an affective component of Morgan's positioning within mathematics.

The most common sentiment across the letters was gratitude. The participants reflected on the experiences that mathematics afforded – including the connections and community that they developed. Kayla expressed gratitude to mathematics for "bringing [her] joy all through [her] school years" and for "your friendship." Similarly, Morgan wrote, "I wanted to take a moment to…express gratitude for the profound impact you made on my life." Despite the varying feelings expressed about themselves and mathematics, the participants described being drawn to mathematics and feeling motivated to continue their studies and pursue faculty positions.

Believer of Mathematics

The final dimension of mathematics identity which emerged was the *Believer* of mathematics. This theme echoes the work of scholars who investigate the notion of mathematical study and religion as conjuring parallel experiences, both in their embodiment and in their rejection (Jonker, 2012; Krajewski, 2021; Kurniawan & Hidayati, 2020). Kurniawan and Hidayati (2020) define religion in part as a way one "seeks the meaning of life, the value of truth, and the meaning of the world." Many of our participants described ways that mathematics provides universal meaning and truth. In the same way that understanding one's religious identity yields insight into their worldview, we view understanding the ways in which an individual "believes" in mathematics and its power as something which yields insight into a student's mathematics identity.

One way in which this theme appeared was through the participants' frequent assignment of powerful qualities to mathematics. In her letter, Kayla contrasted her initial negative feelings towards mathematics with her present feelings of "appreciating your beauty and mystery." Similarly, Morgan noted her passion for "studying the beauty of your theory", and Martin noted the "utility and richness of your subjects." Landon described mathematics as "foundational to understanding the world", and cited this as a reason he was drawn to the field. In some instances, participants expanded on these qualities in ways that went beyond ascribing a quality and characterized mathematics in a more holistic way. Connor made two such potent characterizations: that "the spirit of discovery resides in you", and that "we will unlock the secrets of the universe with your help." Both of these characterizations are not out of line with a religious profession of exalt; Connor is expressing his exaltation of mathematics through the ascription of spiritual characterizations to the subject.

The participants also described mathematics in ways that reflect omnipresent and omniscient properties often ascribed to religious deities. Connor explained how mathematics "is the hope I have for a brighter future…, because humanity has hope through you and the insights you hold." Regarding mathematics' far-reaching power, Martin further wrote that "you are everything, everywhere, all at once, and I think nothing else can explain the universe in a better way." Landon and Martin also positioned mathematics as a determinant of universal truth. For example, Martin noted that mathematics "offer[s] exactness in a time and place where everything is changing, and… keep[s] me grounded to this existence." In these ways, the participants' descriptions reflect a higher "belief" in the power and truth of mathematics.

Discussion and Implications

The participants' letters revealed four main themes that suggest important dimensions in mathematics graduate students' expressions of their mathematics identities. These themes both connect and diverge from literature focused on undergraduate students' mathematics identity.

The *Doer* dimension reflects mathematics identity conceptualizations that include utility, or the "degree to which a student thinks that math is or will be useful to their life" (Crossley et al., 2018, p. 13). Landon's letter, for example, reflects this, discussing how mathematics supported the development of critical thinking skills useful in numerous aspects of his life. The *Feeler* dimension relates to affective components within students' mathematics identity but expands on the usual limitation to students' level of mathematical interest. Cribbs et al. (2015) define mathematical interest as "a student's desire or curiosity to think about and learn mathematics" (p. 1052). The participants described feelings of curiosity, such as when Kayla wrote how "ever since I was of a young age, I always knew that I wanted to learn more about you." However, they also expressed numerous other emotions including excitement, inspiration, enjoyment, and appreciation around mathematics, as well as anxiety, fear, and annoyance. For example, Connor described how he is "annoyed with you so very often." Thus, while interest and curiosity appeared in some of the letters, participants described additional feelings that relate to their mathematics identity such as perceptions of ability and mathematical self-efficacy.

The *Feeler* dimension also demonstrates connections to perceptions of ability. Martin described not "getting along with" mathematics initially, implying that a lack of perceived ability hindered their relationship with mathematics. Kayla also describes changes in her perceived ability, reflecting back on grade school when "you made me stand out as a very smart and exceptional student," whereby in graduate school, "there were other people that were just as good at you as I was, and I became threatened and jealous." As a result of her lower self-efficacy, she questioned that mathematics "maybe… [was] never my friend to begin with." Thus, perceptions of ability still appear in graduate students' construction of their mathematics identity, whether as a consequence of earlier experiences or within graduate school itself.

The *Sharer* dimension exhibits similarities and differences with existing literature on undergraduate students' mathematics identity. Echoing existing literature, the participants frequently referenced their positioning within or outside of a mathematical community. Connor and Kendall described the friends they have gained through mathematics. However, Morgan cited gender-based exclusion within the "boys' club" of mathematics and Connor noted his dislike toward competition within mathematical spaces. These notions of community reflect conceptualizations of mathematics identity focused on belonging and recognition (Cribbs et al., 2015; Voigt, 2020). However, the *Sharer* dimension also included a unique component of graduate students' mathematics identity, namely their role in sharing mathematics as a teacher. Multiple participants described teaching as a way to change negative perceptions around mathematics and communicate their love for the discipline. For these participants, the co-construction of professional and mathematics identity was an important component of their graduate school experience. Beisiegel and Simmt (2012) note that "it is through the formative experiences of graduate school that mathematics graduate students' identities are shaped as mathematicians, researchers, teachers, or more generally as professors of math" (p. 35). The dimensions of *Doer* and *Sharer* are particularly relevant to Beisiegel and Simmt's assertion;

understanding how graduate students see themselves as doing and sharing mathematics in relation to their mathematics identity can illuminate their professional identity development.

The identity dimension that most diverges from existing literature on undergraduate students' mathematics identity is the *Believer* of mathematics. This appears to be a unique facet of mathematics graduate students' mathematics identity. The salience of this theme across all of the participants' letters suggests that a belief in the power of mathematics is a significant factor in their mathematics identity. The participants expressed this dimension in various ways, from admiration of mathematics' beauty and spirit to its bringing hope to humanity and grounding one to this existence. These statements do not align well with previously established mathematics identity components such as perceptions of ability, interest, utility, sense of belonging, participation, etc. Given the novelty of this theme, further investigation is needed to determine how fundamental or universal this dimension is among graduate students, whether undergraduate students who resonate with this aspect of mathematics identity are more likely to pursue graduate education, and if and how instructors could instill this within their undergraduate students.

The letters also reinforced how mathematics identity is a fluid construct, subject to change across time and experience. We see past mathematics identities present in Kayla's narrative, as she reflected on mathematics as bringing her her "greatest joys" and "lowest places" in a temporal way. She associated a shift with the beginning of graduate school, in which she began feeling "threatened and jealous" – something she identified as a "false narrative" that resulted in "ill feelings." In addition to considering previously held mathematics identities through descriptions of feelings and experiences, there was one instance of consideration of a future mathematics identity present in Kendall's letter in which she writes she "[has] no doubt we will be lifelong friends." These elements of both a past and a projected future mathematics identity are integral to how students position themselves with respect to mathematics as a discipline, and thus serve as informants of their mathematics identity. We see this as a possible area of application for Markus and Nurius's (1986) *possible selves* in which identities are given temporal dimensions, and past or possible considerations of identity are conceptualized as irrevocably intertwined with the current self. In the same way that possible selves have been leveraged to provide insight into professional development in other realms on education (e.g., Blaney et al., 2022; Park & Shallert, 2020; Quaisley et al., under review), so too could they provide insight in the developing mathematics identities of graduate students.

Conclusion

Overall, our work suggests that graduate students' mathematics identity development reflects and extends beyond traditional notions of identity. We recognize the salience of social identities (e.g., gender, race, etc.) in the shaping of mathematical experiences (e.g., Leyva et al., 2020; Voigt, 2020); future work is needed to more carefully examine the connections between these identities and the mathematics identity development of graduate students. We view examining letters as a novel and insightful approach to making sense of mathematics identities as they afforded students opportunities to articulate self-perceptions, stories, feelings, etc. that may have been difficult to convey through other data collection efforts. We posit that writing "Dear Math…" letters is one avenue that professional development efforts can take up to gain insight into the experiences that graduate students consider supportive or hindering in their progression through graduate school and their pursuit of academic positions. A continued examination of graduate students' mathematics identities can inform mathematics departments and higher education institutions on how to better support their transition to the mathematics professoriate.

References

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher education*, *20*(2), 107-128. https://doi.org/10.1016/j.tate.2003.07.001

Beisiegel, M., & Simmt, E. (2012). Formation of mathematics graduate students' mathematician-as-teacher identity. *For the Learning of Mathematics*, *32*(1), 34-39. https://www.jstor.org/stable/23391950

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing.* New York, NY: Basic Books.

Blaney, J. M., Wofford, A. M., Jeong, S., Kang, J., & Feldon, D. F. (2022). Autonomy and privilege in doctoral education: An analysis of STEM students' academic and professional trajectories. *The Journal of Higher Education, 93*(7). https://doi.org/10.1080/00221546.2022.2082761

Clark, L. M., Badertscher, E. M., & Napp, C. (2013). African American mathematics teachers as agents in their African American students' mathematics identity formation. *Teachers College Record*, *115*(2), 1-36. https://doi.org/10.1177/016146811311500201

Cribbs, J. D., Hazari, Z., Sonnert, G., & Sadler, P. M. (2015). Establishing an explanatory model for mathematics identity. *Child Development, 86*(4), 1048–1062. https://doi.org/10.1111/cdev.12363

Cribbs, J.D., Huang, X., & Piatek-Jimenez, K. (2021). Relations of mathematics mindset, mathematics anxiety, mathematics identity, and mathematics self-efficacy to STEM career choice: A structural equation modeling approach. *School Science and Mathematics*, *121*(5), 275-287. https://doi.org/10.1111/ssm.12470

Crossley, S., Ocumpaugh, J., Labrum, M., Bradfield, F., Dascalu, M., & Baker, R. (2018). Modeling math identity and math success through sentiment analysis and linguistic features. In K. E. Boyer & M. Yudelson (Eds.), *Proceedings of the 11th international conference on Educational Data Mining* (pp. 11-20). EDM.

Fernández, L. M., Nguyen, U., & Callahan, R. (2022). Learners' mathematics identity and achievement: Where does the teacher come in?. *International Journal of Mathematical Education in Science and Technology*, 1-26. https://doi.org/10.1080/0020739X.2022.2117657

Gardee, A., & Brodie, K. (2021). A framework for learners' mathematical identities: A critical realist perspective. *Didactica Mathematicae, 43*(1), 5-30. https://doi.org/10.14708/dm.v43i1.7114

Gonzalez, L., Chapman, S., & Battle, J. (2020). Mathematics identity and achievement among Black students. *School Science and Mathematics*, *120*(8), 456-466. https://doi.org/10.1111/ssm.12436

Hancock, S., & Walsh, E. (2016). Beyond knowledge and skills: Rethinking the development of professional identity during the STEM doctorate. *Studies in Higher Education*, *41*(1), 37-50. https://doi.org/10.1080/03075079.2014.915301

Harris, J. (2002). The correspondence method as a data-gathering technique in qualitative enquiry. *International Journal of Qualitative Methods*, *1*(4), 1-9. https://doi.org/10.1177/160940690200100401

Jonker, L. (2012). Mathematics and religion: Similar solitudes? *Antistasis*, *2*(1), 16-18. https://doi.org/10.3917/enje.018.0210

Joseph, N. M., Hailu, M., & Boston, D. (2017). Black women's and girls' persistence in the P–20 mathematics pipeline: Two decades of children, youth, and adult education research. *Review of Research in Education*, *41*(1), 203-227. https://doi.org/10.3102/0091732X16689045
Krajewski, S. (2020). Is mathematics connected to religion? In B. Sriraman (Ed.), *Handbook of the History and Philosophy of Mathematical Practice* (pp. 1–25). Springer International Publishing. https://doi.org/10.1007/978-3-030-19071-2_77-1
Kurniawan, W., & Hidayati, T. (2020). In search of the encounter between religion and mathematics. *International Journal of Social Science and Religion, 1*(3), 199–212. https://doi.org/10.53639/ijssr.v1i3.31
Leavy, P. (2020). *Method meets art: Arts-based research practice*. Guilford publications.
Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2020). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition and Instruction, 39*(1), 1–34. https://doi.org/10.1080/07370008.2020.1849218
Markus, H., & Nurius, P. (1986). Possible selves. *American Psychologist, 41*(9), 954–969.
Mendick, H. (2006). *Masculinities in mathematics*. New York, NY: Open University Press.
Miles, M. B., Huberman, A. M., & Saldaña, J. (2020). *Qualitative data analysis: A methods sourcebook.* SAGE.
Park, J. J. & Schallert, D. L. (2020). Reciprocity between doctoral students' emerging professional identity and their envisionment of a possible future self in real and imagined communities of practice. *Learning, Culture and Social Interaction, 26*(100434). https://doi.org/10.1016/j.lcsi.2020.100434.
Quaisley, K., Smith, W. M., Criswell, B., Funk, R., & Hutchinson, A. (2023). From becoming to being: How STEM teachers develop leadership identities [Manuscript submitted for publication].
Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data*. Sage.
Stamper, C. (2020). Letter writing: Life in letters—A method of qualitative inquiry. *Using arts-based research methods: Creative approaches for researching business, organisation and humanities*, 177-208. https://doi.org/10.1007/978-3-030-33069-9_7
Van Lankveld, T., Schoonenboom, J., Volman, M., Croiset, G., & Beishuizen, J. (2017). Developing a teacher identity in the university context: A systematic review of the literature. *Higher Education Research & Development*, *36*(2), 325-342. https://doi.org/10.1080/07294360.2016.1208154
Voigt, M. (2020). *Queer-spectrum student experiences and resources in undergraduate mathematics* (Order No. 27745009) [Doctoral dissertation, University of California, San Diego]. ProQuest Dissertations & Theses Global. http://www.proquest.com/docview/2435858844/abstract/608F986CC12E4095PQ/1
Webb, M. A. (2020). Dear math letter to self. *The Currere Exchange Journal*, *4*(2), 48-55. https://cej.lib.miamioh.edu/index.php/CEJ/article/view/62

An Exploration of Undergraduate Mathematics Education Measures and Their Validity Evidence

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This report discusses the validity evidence reported in undergraduate mathematics education measures. Through a comprehensive literature review of articles published from 2000-2019, we identified 166 measures that provided validity claims and evidence to varying degrees. Findings overall suggest that validity evidence can be more robust within measures of RUME constructs, although there are examples of rigorous validity arguments. Of the validity evidence, the major sources of validity were reliability, test content, and internal structure. We report on the number of sources of validity, by construct, and the types of evidence to support the validity claims.

Keywords: validity evidence, undergraduate, measures, reliability

Measures play a key role in research in undergraduate mathematics education. Measures provides a means to quantify elements of teaching and learning to investigate quantitative hypotheses. Types of measures might include cognitive measures (e.g., Precalculus Concept Assessment; Carlson et al., 2010), affective measures (e.g., Motivated Strategies for Learning Questionnaire; Pintrich & de Groot, 1990), or learning environment measure (e.g., Inquiry-Oriented Instructional Measure; Kuster et al., 2019) As researchers, we have observed the use of many different measures in our field, but it is less clear the degree to which they have sufficient validity evidence to assure that scores and interpretations of them are meaningful. Previous literature has shown a lack of sufficient validity and reliability evidence in mathematics education assessments, broadly (Bostic et al., 2019; Krupa et al., 2019). Thus, we aim to explore whether this claim holds true in undergraduate mathematics education research. This report addresses:

RQ1: What measures are currently in use by RUME researchers?

RQ2: What types of validity evidence support the validity of their usage?

Background

Measurement theory is the process of investigating varying forms of evidence to form the basis of a valid and reliable instrument (Crocker, 2012). For claims of an instrument to be meaningful the instrument needs to be assessed to ensure it is measuring what it was intended to. In doing so, researchers assess the reliability and the validity of the instrument. Reliability is the degree in which the results/scores of an instrument are replicable, consistent, and accurate (Brennan, 2006; Crocker & Algina, 2006). Some examples for evidence of reliability include test-retest, alternate test forms, item response theory, Cronbach alpha, and inter-rater reliability. Validation, on the other hand, has been described by Cronbach (1971) as the collection of evidence to support the claims drawn from the results of the instrument by the assessment creator or the user of the instrument. In education, validity is defined in the *Standards for Educational and Psychological Testing* as “the degree to which evidence and theory support the interpretations of test scores for proposed uses of tests” (American Educational Research Association [AERA] et al., 2014, p. 11). In the *Standards*, five forms of validity are noted: test content, response process, internal structure, relation to other variables, and consequence of testing (AERA et al., 2014). Primary sources of evidence for the five types of validity are provided in the *Standards* (AERA at al., 2014) and *Validation in Mathematics Education* (Krupa et al., 2019).

Five Sources of Validity

Test content validity is explained as "the relationship between the content of a test and the construct it is intended to measure" (AERA et al., 2014, p. 14). Evidence for test content stems from the judgments of experts, observation, and logical or empirical analyses (AERA et al., 2014, p. 14). In practice, such validity evidence includes, but not limited to, data from experts (i.e., experts in the field), field work, literature reviews, alignment of a learning trajectory, and participant generated content (i.e., the participant creating the data for the researcher) (Krupa et al., 2019). *Response process* validity is concerned with "the fit between the construct and the detailed nature of the performance or response actually engaged in by test takers" (AERA et al., p. 15). In other words, this form of validity ensures that the test takers engage in the appropriate thinking, processes, and analyses in which the construct was designed to be measured for proper claims to be made (Lavery et al., 2019). A non-exhaustive list of Validity evidence for response processes includes, cognitive interviews, focus groups, eye-tracking, predicted response patterns based on learning trajectories, and think alouds (AERA et al., 2014, Krupa et al., 2019; Lavery et al., 2019).

Validity from *internal structure* addresses "the degree to which the relationships among test items and test components conform to the construct" (AERA et al., 2014, p. 16). This validity type is focused on confirming that the construct adheres to the theories in which it was built upon (Lavery et al., 2019; Krupa et al, 2019). Further, validity evidence of internal structure confirms that the construct is unidimensional. A unidimensional construct assesses a single dimension or attribute (e.g., a student's level of multiplicative reasoning) (Bond & Fox, 2015). Validity evidence of internal structure includes, but is not limited to, factor analysis, Item Response Theory, Rasch modeling, and multidimensional scaling (AERA et al., 2014, Krupa et al., 2019; Lavery et al., 2019). *Relation to other variables* validity is defined as "the degree to which the relationships among test items and test components conform to the construct" (AERA et al., 2014, p. 16). Many instruments are developed with the expectations that the construct will relate to other constructs or variables in predictable ways or patterns (Lavery et al., 2019) thus, this source of validity evidence connects results to other constructs (Krupa et al., 2019). Such types of validity evidence include correlations analysis, regression analysis, structural equation modeling, hierarchical linear modeling, and alignment with expert opinion of user (e.g., teacher). Lastly, validity of *consequences of testing* is the process of "gathering evidence to evaluate the soundness of [the] proposed interpretations for their intended use" (AERA et al., 2014, p. 19). *Standards for Educational and Psychological Testing* discuss three main types of consequences of testing: unintended consequences (e.g., differences in test score based on race/ethnicity), claims made that are not based on test score interpretations (i.e., the need for new more information to address the results of the testing), and the interpretations and uses of the test scores intended by the instrument creators (AERA et al., 2014). Evidence for consequence of testing validity include explicit intended use/interpretations, motivational consequences, differential item functioning, and appropriate cut scores (AERA et al., 2014; Krupa et al., 2019).

Methods

This project began in 2020 as part of an NSF-funded collaboration (BLINDED) to catalog mathematics education measures that have been in use for the last twenty years. The authors of this paper have led the work on the undergraduate sub-team. To identify measures in use, we conducted a literature search spanning 2000-2019. We searched the mathematics education research journals identified by Williams and Leatham (2017) along with *International Journal of Research in Undergraduate Mathematics Education* as its newness means it was not contained in Williams and Leatham's list. We used the following Boolean string to identify articles: (tertiary OR post-secondary OR undergraduate OR college OR university OR "graduate student") AND (test* OR assess* OR instrument* OR measur* OR survey* OR inventor*). From this initial search, we identified 2,603 articles. At this point, we developed an inclusion and exclusion system with for criteria: (0) Must focus on undergraduate or graduate mathematics students and/or instructors (exclusive of primarily statistics or pre-service teachers as they fall under other subteams (1) Must be an empirical study (2) Used an instrument that is uniform across the population, that should be useable by other researchers with at most minimal modification, potentially in a different setting (3) That is associated with a named or described construct. After training with a set of 30 articles within the full set, the remaining articles within the 2,603 articles were coded according to these criteria by two members of the research independently, then coming to agreement about whether to include to the next stage. Here, 432 articles met these criteria. At this point, we switched to identifying measures. To be considered a measure, we required a composite score of some type. That is, an instrument that only reports data at an individual item level was not included. We identified 241 unique measures.

The final stage of this process was then to identify validity evidence. For each measure, we went into the article where it was found, then determined if that article contained its validity evidence. If not, we searched for another source of evidence (such as a reference validation study.) For any measure with validity evidence, we documented: focal construct (cognitive, affective, learning environment), usage information, and any validity evidence or claims provided categorized within the five types of validity evidence in the prior section. We aligned our classifications with the larger project teams' set of evidence types. In the next section, we present an overview of the types of instruments and most common types of validity evidence.

Results

Of the 241 unique measures identified to fit our criteria, only 166 measures were reported to have validity evidence for the constructs. Ultimately, 75 measures were eliminated due to having no apparent reported validity evidence within the manuscript. Our final list of 166 undergraduate mathematics articles contained three prominent construct types that were broken up into their focal construct: affective domain, cognitive domain, and learning environment (see Table 1). Our findings show that there is a skew in the type of constructs being implemented in undergraduate research. These results indicate that the majority of the instruments measure the affective (47%) and cognitive domain (45%) and only about a tenth (11%) are designed to measure an undergraduate student learning environment.

On average, two (M = 1.92, SD = 1.06) forms of validity evidence (including reliability) were reported for each measure. Only one measure provided all sources of validity and reliability evidence. Given that most of the articles only contained two sources of validity, on average, the most prevalent source of validity was computed. Table 1 illustrates the validity evidence broken down by construct type. The most reported form of validation was test content (n = 113) followed by reliability (n = 87). Very few undergraduate mathematics measures reported validity evidence for consequences of testing (n = 4) or response process (n =15).

Table 1. Source of Validity by Construct Type

	Construct Type [Total # of Articles]							
Source of Validity	AD [72]	AD / CD [2]	CD [72]	CD / LE [1]	LE [13]	LE / AD [4]	Other [2]	Total [166]
Reliability	46	1	25	0	9	4	2	87
Test Content	43	1	56	1	8	0	1	110
Response Process	3	0	9	0	3	0	0	15
Internal Structure	36	1	16	0	9	1	1	64
Relation to Other Variables	18	0	15	0	4	0	1	38
Consequence of Testing	0	0	4	0	0	0	0	4

Note: AD = Affective Domain, CD = Cognitive Domain, and LE = Learning Environments

Table 2 reports the types of validity evidence that were documented for the 166 measures. Our findings indicate that of our 87 measures that made claims for reliability, 72 (83 %) reported a Cronbach Alpha Coefficient as the source of evidence. Further, the 38 measures that provide evidence for relation to other variables predominantly used correlation analysis (n = 27) as their source of evidence (71%). As shown in Table 2, other than evidence type for test content, there is little variation to the evidence type documented undergraduate mathematics education research.

While in general, the validity evidence for measures was scarce, there were some notable exceptions in the set. Of the affective focused measures, *Mini-IPIP Scales* (Alcock et al., 2014) and *Attitudes to Technology in Mathematics Leaning Questionnaire* (Forgarty et al., 2001) had 7 and 6, respectively, forms of validity evidence (see Table 2 for validity evidence types). The cognitive focused measures with 6 or more types of validity evidence consisted of the *Group Theory Concept Assessment* (Melhuish, 2019), *Calculus Readiness Assessment* (Pyzdrowski et al., 2013), *Function Concept Inventory* (O'Shea et al., 2016), *College Readiness Assessment* (Bernbaum et al., 2011), *Calculus Concept Inventory* (Bison et al., 2016), and *Proof Comprehension Tests* (Mejía-Ramos et al., 2017). Lastly, of learning environment focused measures, *Survey of Calculator Usage Extensiveness and Subordinality* (Mao et al., 2017), *Concept Test* (Kaw & Yalcin, 2012), and *Inquiry-Oriented Instructional Measure* (Kuster et al., 2019). While we highlight these measures, we note that it is not the number of validity evidence types that matters so much as the integration of validity evidence into a validity argument. Nonetheless, when validity evidence is presented primarily as reliability, it is difficult to marshal this evidence into a robust argument that an instrument measures a particular construct as intended.

Table 2. Validity Evidence

Source	Evidence Type	
Reliability	Alternate Test Forms	1
	Inter-Rater Reliability (Kappa & Percent Agreement)	7
	Cronbach Alpha	72
	Item Response Theory and/or Rasch Modeling	2
	Item Remainder Correlations	2
	Kuder-Richardson Formula 20	4
	Sensitivity Analysis	1
	Split-Half Reliability	2
	Test-Retest	10
Test Content	Alignment with Framework/Theory/Learning	24
	Data from Experts	38
	Fairness of Content	2
	Field Work	4
	Literature Review	36
	Participant Generated Content	16
	Revision Process	15
	Standard Setting	1
Response Process	Cognitive Interviews	4
	Rater Agreement/Reliability	3
	Rater Training and Collaboration	2
	Student Written Work	5
	Think Aloud	2
Internal Structure	Factor Analysis*	58
	Item Difficulty	4
	Item Scale Correlations	8
	Item Response Theory (IRT)	6
	Rasch Modeling	5
Relation to Other Variables	Alignment with Expert Opinion	1
	Convergent or Divergent Association	4
	Correlation Analysis	27
	Discriminant Validity	1
	Hierarchical Linear Modeling (HLM)	1
	Statistical Testing (e.g., regression, t-test, chi-square)	5
	Triangulation	1
Consequence of Testing	Appropriate Cut Score	2
	Cost-Benefit Analysis	1
	Item Function (e.g., DIF)	1

*Includes Bifactor, Exploratory/Confirmatory Factor Analysis, Parallel Analysis, Principal Axis Factoring, and Principal Component Analysis

Discussion

Our analysis of measures used in the undergraduate mathematics education literature points to several findings. First, we found that Bostic et al.'s (2019) and Krupa et al.'s (2019) reflection on the state of measures in math education was largely true in the undergraduate context. While many instruments are in use many provided limited or no validity evidence at all. This is particularly problematic when scores from such measures are used in service of testing particular hypothesis or have impacts on students or instructors in practice. That is, the trustworthiness of an instrument is determined by the degree we can be assured it is valid.

If we consider our results more broadly, we can see that certain types of validity evidence are more frequently reported. For example, Cronbach's alpha is by far the most common type of validity evidence in our data set. This is not surprising as it is a relatively easy type of statistic to calculate with associated indices of what is acceptable for different measure purposes. We also documented other common psychometric and quantitative approaches to validity with the prevalence of factor analysis and correlation with variables. These common approaches still accounted for less than half of the measures reporting any validity evidence.

We note the measures were more frequently reported with quantitative evidence and less qualitative validity evidence. We suggest a robust validity argument (e.g., Kane, 1992) is made with mixed methods. For example, response process evidence was quite infrequent. This type of evidence stems from making sure that participant response to items actually reflects their thinking. This can be done through cognitive interviews, collecting written work, and other approaches. Without such evidence, we may be left wondering the degree to which an instruments' scores truly reflect participants.

We acknowledge some limitations of the current work. First, our exclusion and inclusion process mean that we may have overlooked instruments that have not been discussed in primary mathematics education journals. Additionally, it is possible that we overlooked a source of validity evidence that might be found in harder to access internal reports. There is also likely some variation in how our coders interpreted different types of evidence and therefore, we suggest caution when generalizing from frequencies that are similar.

Finally, we note directions for future research. In our own data set, we have made some initial notices that within our construct categories we can find some interesting patterns. For example, a full 28 of the affective instruments were focused on mathematics attitudes and belief scales. This means that there are many instruments in use targeting the same construct. We also note that within the cognitive construct measures, instruments that aligned themselves with concept inventory procedures (e.g., Group Theory Concept Assessment, Function Concept Instrument, Calculus Concept Inventory) often had more robust amounts of validity evidence likely due to commonalities in development approach. The next research direction will involve better understanding the nature of the instruments within each category and their relation to the types of validity evidence shared.

We conclude by suggesting that the field consider a variety of validity evidence types when creating measures. It may be a truism that the validation process is never complete. Each new sample and use require recalibration. The stronger the evidence is for our measures, the more we can trust the results of our quantitative studies.

References

Alcock, L., Attridge, N., Kenny, S., & Inglis, M. (2014). Achievement and behaviour in undergraduate mathematics: personality is a better predictor than gender. *Research in Mathematics Education, 16*(1), 1-17.

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education. (2014). *Standards for educational and psychological testing*. AERA.

Bernbaum Wilmot, D., Schoenfeld, A., Wilson, M., Champney, D., & Zahner, W. (2011). Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning*, *13*(4), 259-291.
Bisson, M. J., Gilmore, C., Inglis, M., & Jones, I. (2016). Measuring conceptual understanding using comparative judgement. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 141-164.
Bostic, J. D., Krupa, E. E., & Shih, J. C. (2019). Introduction: Aims and scope for assessments in mathematics education contexts: Theoretical frameworks and new directions. In J. D. Bostic, E. E. Krupa, & J. C. Shih (Eds.), *Assessment in mathematics education contexts: Theoretical frameworks and new directions* (pp. 1–11). Routledge.
Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145.
Kane, M. T. (1992). An argument-based approach to validity. *Psychological Bulleti*n, 112(3), 527–535. https://doi.org/10.1037/0033-2909.112.3.527
Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. G. (2019). The inquiry-oriented
Crocker, L., & Algina, J. (2006) *Introduction to classical and modern test theory*. Thomson Wadsworth.
Cronbach, L. J. (1971). Test validation. In R. L. Thorndike (Ed.). *Educational Measurement* (2^{nd} Ed.). Washington, D.C.: American Council on Education.
Fogarty, G., Cretchley, P., Harman, C., Ellerton, N., & Konki, N. (2001). Validation of a questionnaire to measure mathematics confidence, computer confidence, and attitudes towards the use of technology for learning mathematics. *Mathematics Education Research Journal*, *13*, 154-160.
Kaw, A., & Yalcin, A. (2012). Measuring student learning using initial and final concept test in an STEM course. *International Journal of Mathematical Education in Science and Technology*, *43*(4), 435-448.
Krupa, E. E., Bostic, J. D., & Shih, J. C. (2019). Validation in mathematics education: An introduction to quantitative measures of mathematical knowledge: Researching instruments and perspectives. In J. D. Bostic, E. E. Krupa, & J. C. Shih (Eds.), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives* (pp. 1–13). Routledge.
Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. G. (2019). The inquiry-oriented instructional measure. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 183-204.
Lavery, M, R., Jong, C., Krupa, E. E., & Bostic, J. D. (2019). Developing an instrument with validity in mind. In J. D. Bostic, E. E. Krpua, & J. C. Shih (Eds.), *Assessment in mathematics education contexts: Theoretical frameworks and new directions* (pp. 12-39). Routledge.
Mao, Y., White, T., Sadler, P. M., & Sonnert, G. (2017). The association of precollege use of calculators with student performance in college calculus. *Educational Studies in Mathematics*, *94*, 69-83.
Mejía-Ramos, J. P., Lew, K., de la Torre, J., & Weber, K. (2017). Developing and validating proof comprehension tests in undergraduate mathematics. *Research in Mathematics Education*, *19*(2), 130-146.
Melhuish, K. (2019). The Group Theory Concept Assessment: A tool for measuring conceptual understanding in introductory group theory. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 359-393.

O'Shea, A., Breen, S., & Jaworski, B. (2016). The development of a function concept inventory. *International Journal of Research in Undergraduate Mathematics Education, 2,* 279-296.
Pintrich, P. R., & de Groot, E. V. (1990). Motivational and self-regulated learning components of classroom academic performance. *Journal of Educational Psychology, 82*(1), 33–40. https://doi.org/10.1037/0022-0663.82.1.33
Pyzdrowski, L. J., Sun, Y., Curtis, R., Miller, D., Winn, G., & Hensel, R. A. (2013). Readiness and attitudes as indicators for success in college calculus. *International Journal of Science and Mathematics Education, 11*, 529-554.
Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 369-396.

What Makes a Proof a Proof?: It's Your *Job* to Decide

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This study investigates students' conceptions of what makes a particular written mathematical communication a proof. To that end, we implemented one-on-one task-based interviews involving mathematical communications. These communications were presented as trios, and we called each communication a job. We intentionally designed some of these jobs to sit in the vague space between proof and calculation. Our study investigated the reasons why participants believed certain jobs were proofs, allowing us to develop a model of students' conceptions of proofiness. This model revealed differences among students in terms of which attributes they thought contributed to or detracted from proofiness, as well as those they thought were and were not relevant to whether a job is a proof. These differences and how they relate to normative conceptions of proofiness point to aspects of proof that may require further attention in these students' introduction to proof courses.

Keywords: Proof, Introduction to proof students, Remote data collection.

What follows is the novice portion of a larger planned expert novice study (e.g., Inglis & Alcock, 2012). The overall project addresses the question "What makes a proof a proof?" We use the term *mathematical communication* to broadly capture any written work that establishes a mathematical fact or property. For example, a calculation that finds the roots of a polynomial might be considered a mathematical communication that establishes what the roots of the polynomial are. Similarly, a written proof is a mathematical communication that establishes the truth of some theorem.

This portion of the research addresses the research question: "What are undergraduate mathematics students' conceptions of what makes a mathematical communication a proof?" To facilitate our discussion, we borrow a construct name from Seife (2010), *proofiness*. In his work "Proofiness: the dark art of mathematical deception," Seife defines proofiness to be manipulations of mathematical norms that, "make falsehoods look like numerical fact" (Seife, 2010, p. 18). Instead, we use *proofiness* to refer to a person's personal conceptions of the attributes that make a written communication a (correct mathematical) proof; in other words, *proofiness* refers to the qualities of a text that distinguish proofs from non-proofs. Those are the attributes that keep a purported proof from being an instance of proofiness in Seife's sense of the word. So, from our perspective, our use of this construct name is more precise than Seife's usage and more consistent with the mathematical concept of proof (Weber, 2014).

To simplify the conveyance of ideas, we refer to mathematical communications as *jobs*. We used our *jobs*-based tasks, and data from clinical interviews with undergraduate mathematics students, to develop a *proofiness* framework. This framework can be implemented to classify the attributes of a mathematical communication which a person thinks are relevant to certify or discount it as a proof. In the future, this might allow us to implement more informed learning trajectories for guiding students toward normative conceptions of proof. As alluded to earlier, determining mathematicians' conceptions of *proofiness* is part of a planned followup to the current study. Thus, the utility of this framework will be demonstrated in a second order followup study. The students' conceptions of *proofiness* will inform the design of a learning trajectory (Simon, 2020; Simon & Tzur, 2012) that is intended to advance students' understanding of proof toward observed mathematicians' conceptions.

Literature Review

Proof is, out of necessity, not well defined (Inglis & Aberdein, 2016, Weber & Czocher, 2019). This means that, even among expert mathematicians, there is often no consensus on whether a particular mathematical communication is a proof (Weber & Czocher, 2019). Czocher and Weber (2020) addressed this fuzziness by describing proof as a cluster concept (Lakoff, 1987), which is when "a number of cognitive models combine to form a complex cluster that is psychologically more basic than the models taken individually" (p. 74). The cluster concept nature of proof means that novice provers' conceptions of proof are, inherently, a fuzzy picture of an already fuzzy picture. However, whether a mathematical communication is a proof is also context dependent (Stylianides, Stylianides, & Weber, 2017) and students, especially those in introduction to proof (ITP) courses (David & Zazkis, 2020), have only experienced proofs in the single context of their course. This means that asking an ITP student the question "is this a proof?" implicitly asks, "is this considered a proof within the context of your ITP course?" The existing research on conceptions of what a proof is and what counts as a proof is primarily focused on expert mathematicians (e.g., Inglis, & Aberdein, 2016, Weber & Czocher, 2019). Here, we shift toward studying students, since a good grasp of student conceptions of proof is needed to use this knowledge as part of well-informed learning trajectories for proof.

We argue here that, especially given both how much variety there is in approaches to ITP course work (David and Zazkis, 2020) and the importance of proof in mathematics, there is value in exploring ITP students' impressions of what makes a proof a proof. However, the nebulous nature of proof necessitates a shift away from traditional tasks and interview questions (Zazkis & Hazzan, 1998).

<u>Context</u>: For $f: R \to R$ defined by $f(x) = x^{"} + x - 6$,

Job (1): Justify that f has two roots.

<u>Claim</u>: f has roots -3 and 2.

<u>Justification</u>: Note that $f(x) = x^2 + x - 6 = (x + 3)(x - 2)$. If $(x + 3)(x - 2) = 0$, then $x = -3$ or $x = 2$. Thus, f has roots -3 and 2.

Job (2): Justify f is not one-to-one.

<u>Claim</u>: f is not one-to-one.

<u>Justification</u>: Suppose to the contrary that f is one-to-one. This means that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Note that $(-3, 0)$ and $(2, 0)$ are ordered pairs that satisfy f. In this case, we have $f(-3) = 0 = f(2)$, but $-3 \neq 2$. And this is a contradiction, since f is one-to-one. Thus, our supposition that f is one-to-one must be false. Therefore, f is not one-to-one.

Figure 1. An example task which presents two jobs for the subject to compare.

Method

Participants were undergraduate students, each of whom completed a zoom-based think-aloud one-on-one task-based clinical interview (Clement, 2000). Remote data collection was needed because of the COVID-19 pandemic. The ten participants were students in the same section of an ITP course offered at an Anonymous State University (ASU) in the Southwestern United States. Six of the ten were pursuing a degree in mathematics, and the rest were engineering or science majors.

Students were presented trios (or sometimes pairs) of *jobs* that spanned the range from proof to calculation (from our perspective). Throughout the interview, students were asked which of the *jobs* in front of them were proofs, and then asked to justify their assertions by referring to the *jobs* and their attributes. A pair of example *jobs* are shown above in Figure 1. As a reminder, most of the *jobs* in our study occurred in trios, but we include this pair due to space limitations.

Working in the spirit of grounded theory (Strauss and Corbin, 1997) allowed us to generate a framework of the components of *proofiness* that participants attended to. We refer to our organization and categorization of these components as the Personal Conceptions of *Proofiness* framework (PCP). Documenting which students attended to which segments of the PCP allowed us to create a personalized *proofiness* profile (PPP) for each participant.

Results

First, we present the components and definitions of our Personal Conceptions of *Proofiness* (PCP) framework. Then, we utilize the PCP to create personalized *proofiness* profiles (PPP) for each of our participants. We then narrow our focus to the parts of students' PPP that differ from student to student. These highlight differences in our participants' conceptions of what makes a *job* a proof. The differences in students' PPP are particularly relevant, since all participants were recruited from the same ITP section at the same institution, and thus had ostensibly comparable experiences with proof.

The PCP is divided into two *domains of proofiness*: *norms of communication* (how a mathematical communication is written and presented) and *purposes of communication* (why a mathematical communication is written, and how and whether it meets its mathematical aims). Figure 2 presents the *domains, subdomains, and dimensions of proofiness* that make up the PCP. Figure 2(a) displays the *norms of communication domain*. The *subdomains* in the left table of Figure 2(a) refer to mathematics-specific conceptions, while the *subdomains* on the right describe conceptions that pertain to argumentative writing in general. The primary difference between the left and right tables in Figure 2(a) is that the conceptions on the right are broadly applicable in written argumentation that is not mathematical (e.g., legal argumentation), while those on the left are more closely related to mathematical argumentation. Figure 2(b) details the *purposes of communication domain* of the PCP. Since the PCP was generated from student remarks in aggregate, the relative size of Figure 2(b) compared to Figure 2(a) is an important finding: it highlights that participants spoke relatively little about *purposes of communication*; most students focused their attention on *norms of communication.*

We used the PCP to classify each student's PPP. A subset of each student's PPP can be seen in Figure 3. The table is written so that + means the student believes that dimension *of proofiness* is necessary (though not sufficient) for a *job* to be a proof. We use ++ to indicate when the student was emphatic in their statement of a *dimension*'s importance that enthusiasm is not part of the analysis in this paper, we maintain the convention to be consistent with future work. We use - to indicate the student believes the *dimension* detracts from a *job's proofiness*. We use the symbol / to indicate that the student explicitly indicated that the *dimension* has no effect on whether a *job* is a proof. Finally, the symbol ~ means the student was inconsistent in their assessment of the *dimension*.

Domain	Sub-domain	Dimension	Description of Dimension
Norms of Communication	Sufficient Detail	Explicit Reference to Definitions	Stating "By definition of…" and especially followed by a verbatim statement.
		Assumptions Stated	Explicit statement of assumptions. As opposed to taking them as shared between author of claim and writer of Job. Even if assumptions are listed one line before, ensuring that assumptions are stated as a necessary first line.
		Fastidiousness	Thouroughness. Writing things out for the sake of being thorough, with attention to the detail in reproducing text they have read or written before, especially with definitions.
		Explicit Statement of Warrant / Backing	Stating why a certain inference was made, though no participant used that language. Explicit statement of warrant / backing (no participant used that language either). Participants talked about explaining why a step was taken (though not necesssarily why the outcome is true). If a function was factored, stating something like "By factoring, …". If the associate property was used, stating that it was. Ensuring that steps do not "come out of nowhere".
	Adherence to Norms of Mathematical Communication	Generality	Addressing more than a single case, preferably many or all cases.
		Arbitrariness	Acknowledgement of arbitrary choice.
		Variables	Letters as stand ins for either one or multiple numbers.
		Unidentified variables	Letters that are used but not intoruduced with words, described, or named.
		Specific Numbers	Using known, non-arbitary numbers with no variable label. Like "5".
		Symbols	Set notation. Elementhood / inclusion symbol. Blackboard R, N, and Z.
		Use of Examples	As stated by the participant, the presence of and use of an example. There were times when students called things examples that normatively were not.
	Mathematical Content and Context	Familiar Context	Contexts that participants have seen in their transition to proof course, including functions (but only special kinds like 1-1, and not polynomials), sets, divisibility, relations, number theory, etc.
		Unfamiliar Context	Contexts that participants have not seen in their transition to proof course that, importantly, remind them of their previous mathematics experiences. Especially calculus. No instances of mathematics contexts that the participants had never seen before. For instance, a Job whose mathematical context was in topology or graph coloring.
		Algebra	Presence of one or multiple lines of algebraic work like manipulation of equations, factoring, expansion of polynomials, etc.
		Higher Sophistication of Math Context	Lower sophistication: quadratic functions and calculus. Higher sophistication: number theory (divisibility) and classifying functions as 1-1. Mathematical concepts that were treated in their transition to proof courses felt more sophisticated to some participants.

Domain	Sub-domain	Dimension	Description of Dimension
Norms of Communication	Structure	Givens, Work, Goal	A structural concern. Starting with a statement of some assumptions and then taking steps (doing work) to somehow arrive at a goal.
		Proof Framework / Outline	The Job makes use of a recognizable proof framework or outline.
		Transition Words to Link Steps	Words like "hence", "thus", "therefore", "yields", "which means", "and then".
	Visual Cues	Paragraphs	As opposed to one block of text, the Job is written in paragraphs.
		Typesetting	Underlining of words like "Claim:","Justification:" and "Proof:", followed by indented text that contains the body of the Job.
		Centered Equations	As opposed to in-line equations.
		Pictures	The presence of pictures, especially graphs.
	Adherence to Norms of Written Argumentation	Conventionally Proofy Language	Words like "let", "assume", "suppose", "thus", and "therefore".
		Complete Sentences	The text of the Job is written in complete sentences.
		Long Length	More words. More lines. The Job takes up more physical space.
		Efficiency / Concision	The Job is lean. Does not take more steps than necessary. Brevity.
		The Word "If"	Presence of the word "if" in the Job.
		Voice	A composite dimension that is composed of various written English dimensions of *proofiness* . An understanding that proofs have or are written in
	Tone	Helps Reader	Effort is made by the Job to help or guide the reader, or otherwise advance the understanding of the reader.
		Convincing	The Job works to convince the reader of the truth of the claim or persuade the reader of something.

Figure 2(a): Norms of communication domain of PCP.

Domain	Sub-domain	Dimension	Description of Dimension
Purpose of Communication	Normative Mathematical Value of Proof	Inevitability	The claim is the logical conclusion that follows from the work done in the Job.
		Coherence	The conclusion of the Job matches the claim.
		Unpacking and Repacking	Definitions are used to make statements about some fact. Manipulation of those statements yields something of a sub-conclusion. This sub-conclusion is presented according to a definition, and this is the conclusion. Unpacking definition: writing a fact in a more useful form. Repacking definition: using that more useful form to make a conclusion, which is achieved by refrencing the language of a definition. Note that participants that explored this dimension of *proofiness* did not necessarily explicitly reference logical inference.
		Worthwile	The Job proves something that is nontrivial or proves something that requires a proof.
		Circular Logic	Assuming the conclusion. Or using rules that depend on other rules.
	Types of Claims	If / Then	The Job is an attempt ot justify a conditional statement.
		Answer-Getting	The Job is an attempt to solve a calculational problem. The Job amounts to a calculation.

Figure 2(b): Purposes of communication domain of PCP.

Participant	Major	Explicit Reference to Definitions	Inevitability	Coherence	Unpacking & Repacking	Worthwhile	Circular Logic	If / Then	Answer-Getting
Redmond	Mech E	++							
Megan	Math	++							-
Gem	Math			+					
Javon	Math	+	+		++	+	-		
Aidan	Math	++							
Eric	EE & Math			+			-	+	-
Claudette	Comm & Math	+							
Estrella	EE	/	++						
Han	Chemistry								-
Sabrina	*Professional*	~							

Figure 3: A subset of the participants PPP. Note the thick border, separating domains.

To illustrate these distinctions, consider the following two quotations from Han and Estrella, both of which are relevant to the Generality *dimension of proofiness* in the PCP framework:

Han: "Broader case feels *proofier* than talking about one specific case."

Estrella: "Not all proofs have to be arbitrary. You can still prove something for like a specific situation."

Han indicates he holds the conception that Generality is a dimension of *proofiness* that increases *proofiness*, while Estrella states that she does not believe that Generality affects *proofiness*. This indicates differences in their personal conceptions of proof. Since the mathematically normative conception of Generality is that a proof can be general or specific, depending on the claim, we see that not all students agree with that normative conception: two of them stated explicitly that to be more general was to be more *proofy*.

We further explore the PCP by comparing Claudette and Redmond's remarks associated with the Long Length dimension of *proofiness*, which relates to a *job* having more words, more lines, and taking up more space on the page. Claudette said she thinks proofs should have "lots of words", even referring positively to its "sheer amount of words" when claiming one *job* was *proofier* than another. Redmond, on the other hand, stated he did not like one of the *jobs* because, in his words, "It's too long." Therefore, Claudette has a + in the relevant space, while Redmond has a -.

The table in Figure 4 allows us to glean patterns and differences in the students' *personal proof profiles* (PPP). For example, we can notice that seven of the ten participants mentioned that they hold the belief that the inclusion of Conventionally *Proofy* Language contributes to *proofiness*. The three participants that did not verbalize this view on language did not bring up language at all, indicating de facto consensus. This stands in contrast to the group's conception of Algebra (the presence of one or multiple lines of algebraic work like manipulation of equations, factoring, expansion of polynomials, etc.)—some students believe that Algebra decreases a mathematical communication's (*job's*) proofiness, and others believe that Algebra is irrelevant to the *proofiness* of a mathematical communication. This indicates disagreement regarding whether the presence of algebra contributes to *proofiness*. More broadly, we found that, taken as a whole, there was consensus among our participants with respect to Assumptions Stated, Explicit Statement of Warrant / Backing, 'Givens, Work, Goal', Proof Framework / Outline, and Conventionally *Proofy* Language as positive contributors to *proofiness*. Moreover, we saw strong support for the detrimental effect to *proofiness* that Unidentified Variables, Unfamiliar Context, and Answer-Getting yielded (see Figure 2 for descriptions of these *dimensions of proofiness*). Further, students broadly disagreed about the *proofiness* of Long Length, Algebra, and Generality. Lastly, the Explicit Reference to Definitions *dimension of proofiness* is noteworthy: most participants agreed enthusiastically that this *dimension* contributed positively to *proofiness*, with only one stating it did not matter, and another being inconsistent. This is intriguing, if not worrisome, since many mathematicians believe that, across contexts, full statements of definitions should not be included in proofs (Lew and Mejia-Ramos, 2020). This discrepancy about whether to include definitions is related to lack of clarity (Ibid.) on what student-produced proofs should look like, according to mathematicians.

Dimension of Proofiness	Redmond	Megan	Gem	Javon	Aidan	Eric	Claudette	Estrella	Han	Sabrina
Explicit Reference to Definitions	++	++		+	++		+	/		~
Assumptions Stated	+		+		++		+			+
Fastidiousness			+		+		+			++
Explicit Statement of Warrant / Backing	+	+			+		+		+	+
Generality			+					/	+	/
Arbitrariness		+								
Variables	+		+				+			
Unidentified variables			-	-		-		-	-	-
Specific Numbers	-		-			/	-			
Symbols	+									
Use of Examples	~	-	-				-		~	
Familiar Context					+	+			+	
Unfamiliar Context			-	-	-	-			-	-
Algebra	-					-		/	-	/
Higher Sophistication of Math Context			+							/
Givens, Work, Goal	+				+		+	+	+	+
Proof Framework / Outline		+	+	+	+				+	+
Transition Words to Link Steps			+							
Paragraphs							+			
Typesetting							+			
Centered Equations							~			
Pictures				-			-			
Conventionally Proofy Language	+		+	+	+	+	++			+
Complete Sentences					+					+
Long Length	-	-	+			+	+		-	+
Efficiency / Concision			+	+			-	+	+	
The Word "If"		-								
Voice				/		+				
Helps Reader	+	+			+		++			~
Convincing		+					+			
Inevitability				+				++		
Coherence			+			+				
Unpacking & Repacking				++						
Worthwhile				+						
Circular Logic				-		-				
If / Then						+				
Answer-Getting		-				-			-	

Figure 4: PCP framework (applied to students). Note the thick border, separating domains.

One final observation merits consideration: Figure 4 illustrates sparse attention among the pool of participants to *purposes of communication.* In fact, only four participants had a + listed in any *dimension of proofiness* within this *domain*. Put another way, six out of the ten participants did not attend at all to the *purposes of communication domain.* This is understandable given that the participants are at the ITP stage in their mathematics education. However, we hope that future study can inform ways to support students like these to do more than "show their work" nicely, and in doing so they may consider more deeply exactly what the "work" of proof is.

Discussion and Conclusions

We used *job*-based tasks to study individual students' conceptions of *proofiness*. This allowed us to develop a framework for which attributes emerged in discussion of what makes something a proof. Compiling all these conceptions into one organized list of *dimensions of proofiness* is what we call the Personal Conceptions of *Proofiness* framework. Applying this to each of our participants creates a personalized *proofiness* profile, which allows us at a glance to notice patterns and trends in our data related to student beliefs about *proofiness*. For example, we were able to notice differences in our student participants regarding conceptions of generality in proof, and that there was consensus regarding students' belief that the use of a proof framework / outline was relevant to determining whether something is a proof–that is, it increased the *proofiness* of a *job*. Given that all participants were enrolled in the same introduction to proof (ITP) section, this information provides a kind of profile of conceptions of proof that emerged from that class as a whole.

In this connection, we could also envision using these same tools (our *jobs*-based tasks and the PCP) to compare the class profiles of two separate sections of the ITP course taught by different instructors with different instructional approaches. Documenting this could lead to richer understandings of what kinds of instructional approaches lead to conceptions of proof that are more/less aligned with normative mathematical perceptions of *proofiness*.

Finally, we plan on using and further developing both our *jobs*-based tasks and the PCP and getting a better sense of the range of mathematicians' PCP. This followup work should aid us in creating curricula aimed at, among other things, developing students' PCP and simultaneously tying this research to currently available research on mathematicians' conceptions of *proofiness*.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, *547*, 341-385.

Czocher, J. A., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education*, *51*(1), 50

David, E. J., & Zazkis, D. (2020). Characterizing introduction to proof courses: a survey of US R1 and R2 course syllabi. *International Journal of Mathematical Education in Science and Technology*, *51*(3), 388-404.

Inglis, M., & Aberdein, A. (2016). Diversity in proof appraisal. In *Mathematical Cultures: The London Meetings 2012-2014* (pp. 163-179). Springer International Publishing.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Lakoff, G. (1987). Women, fire, and dangerous things. Cambridge: Cambridge University Press.

Lew, K., & Mejía Ramos, J. P. (2020). Linguistic conventions of mathematical proof writing across pedagogical contexts. Educational Studies in Mathematics, 103(1), 43-62.

Seife, C. (2010). Proofiness: The dark arts of mathematical deception. *AMC*, *10*, 12.

Simon, M. (2020). Hypothetical learning trajectories in mathematics education. In *Encyclopedia of mathematics education* (pp. 354-357). Cham: Springer International Publishing.

Simon, M. A., & Tzur, R. (2012). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. In *Hypothetical learning trajectories* (pp. 91-104). Routledge.

Strauss, A., & Corbin, J. M. (1997). *Grounded theory in practice*. Sage.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. *Compendium for research in mathematics education*, 237-266.

Weber, K. (2014). Proof as a Cluster Concept. *North American Chapter of the International Group for the Psychology of Mathematics Education*.

Weber, K., & Czocher, J. (2019). On mathematicians' disagreements on what constitutes a proof. *Research in Mathematics Education*, *21*(3), 251-270.

Zazkis, R., & Hazzan, O. (1998). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, *17*(4), 429-439.

Inquiry-Based Calculus I: What is the impact on student performance?

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While active learning has been found to support student success, it has not yet emerged as the dominant instructional approach in undergraduate mathematics (Stains et al., 2018). This study focuses on student performance in undergraduate Calculus I where we compared the performance of students in inquiry-based (IBL) Calculus I with students in non-IBL Calculus I courses. Additionally, we examined the performance of first-year and upper-year students as well as students who were or were not pursuing a math-intensive major. Students in IBL Calculus performed on the final exam an average of nine percentage points higher than those enrolled in non-IBL Calculus. Differences in performance between first-year students and upper-year students and between students majoring in math-intensive fields and those in non-math-intensive narrowed in the IBL classes. This study aims to add to the ever-growing body of research showcasing the benefits of IBL in improving student course performance.

Keywords: inquiry-based learning, calculus, student performance

In 2016, the Conference Board of Mathematical Sciences (CBMS), a coalition of nineteen professional undergraduate mathematics organizations, issued a compelling statement urging "…institutions of higher education, mathematics departments, and the mathematics faculty, public policy-makers, and funding agencies, to invest time and resources to ensure that effective active learning is incorporated into post-secondary mathematics classrooms" (CBMS, 2016, p. 1). Research has consistently demonstrated the positive impact of active learning and student-centered approaches on improved student course performance (e.g., Abdi, 2014; Bruder & Prescott, 2013; Dunnigan & Harlow, 2021; Freeman et al., 2014), a deeper appreciation of challenging content (Cilli-Turner, 2017), and increased persistence and confidence in doing mathematics (Laursen et al., 2011).

Additionally, the attrition of students from STEM fields is a pressing concern, particularly when high-achieving students who are often successful in other subject areas are leaving STEM (Olson & Riordan, 2012). Troublingly, almost 90% of students cite poor instruction as a factor in leaving STEM (Kung & Speer, 2020). Moreover, Calculus has been noted as one of the largest gatekeepers and barriers for STEM majors, ultimately excluding students who would otherwise be successful (Bressoud et al., 2013; Rasmussen & Ellis, 2013). However, there are signs of improvement as active learning pedagogies have helped increase student performance (Freeman et al., 2014) and even significantly decreased achievement gaps for underrepresented students in STEM (Theobald et al., 2020).

Although mathematics departments have the support of organizations like the CBMS and MAA (Abell et al., 2018; CBMS, 2016), and there are many positive findings regarding the benefits of active learning in undergraduate STEM courses, active learning pedagogies are still not the dominant instructional method in undergraduate STEM (Stains et al., 2018). Because of active learning's overwhelmingly favorable results in supporting STEM students (e.g., Bruder & Prescot, 2013; Laursen et al., 2011), we studied the effects of inquiry-based learning (IBL), a specific type of active learning, on student course performance in an undergraduate Calculus I course. We contribute to the growing body of research on the effectiveness of IBL in improving student course performance. Additionally, this study explores the benefits of IBL on course performance for various groups of students who often struggle in STEM, such as upper-level students (Gregg-Jolly et al., 2016) and students majoring in fields that are not math-intensive such as the life sciences (Kokkelenberg & Sinha, 2010; Rasmussen & Ellis, 2013). In particular, we use final exam scores to measure differences in student performance between first-year students and upper-year students as well as between students in math-intensive majors and non-math-intensive majors. By shedding light on these aspects, we seek to contribute to the existing body of research on IBL implementation and, in turn, encourage other instructors to embrace active learning pedagogies. Specifically, this study seeks to answer the following research questions:

1. How does the final exam performance compare between students enrolled in inquiry-based (IBL) Calculus I and those in non-inquiry-based (non-IBL) Calculus I?
2. Do IBL and non-IBL final exam scores differ for first-year and upper-year students?
3. Do IBL and non-IBL final exam scores differ for students enrolled in math-intensive majors and students enrolled in non-math-intensive majors?

Inquiry-Based Learning

Inquiry approaches to undergraduate education have been gaining momentum (Laursen & Rasmussen, 2019) since the CBMS released their statement supporting active learning (CBMS, 2016). One such approach is IBL, where students work through a “carefully scaffolded sequence of mathematical tasks” (Ernst et al., 2017, p. 570) to construct their mathematical knowledge under the guidance of an instructor. In IBL, students actively communicate with their peers, posing questions and testing conjectures, fostering a collaborative and interactive learning environment. IBL differs from other active learning methods in three distinct ways. First, strategically sequencing tasks allows topics to build on each other throughout the semester instead of using a task unrelated to others. Second, IBL encourages students to actively reinvent mathematics they knew before or create mathematical ideas that are new to them. Third, IBL fosters critical reflection among students and instructors on their perceptions of mathematics and what it means to know, teach, and do math (Laursen & Rasmussen, 2019).

Table 1. The four pillars of inquiry.

	Mathematical Space	**Social Space**
Student Behavior	Engage deeply with meaningful tasks	Collaborate with classmates in processing ideas
Instructor Behavior	Inquire into student thinking and reasoning	Foster equity, respect, and responsibility

Adopted from Foley (n.d.).

The key tenets of IBL are built on the four pillars of inquiry (see Table 1), which consider the behaviors and actions of students and instructors mathematically and socially (Laursen & Rasmussen, 2019). In the mathematical space, students deeply engage with meaningful tasks and work collaboratively to process intricate mathematical concepts. Concurrently, the instructor tries to draw out and understand students’ thinking and reasoning, aiming to acknowledge each individual’s unique perspective. In the social space, students collaborate with their peers, exchanging mathematical ideas and collectively constructing their understanding of concepts, leading to deeper mathematics learning. The instructor’s social role is to cultivate an inclusive learning environment where every student's voice is respected, and students take responsibility for their learning.

Methods

Context

In this study, we analyzed three consecutive semesters of implementation of IBL and non-IBL Calculus I courses at a large, public, midwestern university. Instructors of the IBL and non-IBL sections varied each semester, and instructors had the freedom to choose which type of section they wanted to teach. Importantly, no explicit indication in the course offerings differentiated the IBL and non-IBL sections. Therefore, students enrolling in these sections were unaware of the pedagogical style beforehand. All sections shared the same learning objectives, met for the same amount of time per week, had approximately the same number of enrolled students, and used the same final exam, making them suitable for comparing the pedagogical differences between IBL and non-IBL Calculus and their impact on student performance.

The non-IBL sections had a single instructor in class, while the IBL sections incorporated graduate and undergraduate students as learning assistants (LAs) to support group work (see Webb et al. 2014), aiming for an instructor-to-student ratio of 15:1. In the IBL sections, students took an active role in their learning and spent most of their time working collaboratively in small groups, engaging with *Active Calculus* tasks (Boelkins et al., 2018). Instructors and LAs provided support by addressing difficulties, asking questions, checking comprehension, encouraging discussion, and fostering independent problem-solving skills rather than providing direct answers to student questions. On the other hand, the non-IBL sections primarily relied on lecture-based instruction from the instructor. While some non-IBL sections included elements of IBL Calculus, such as small group work, they did not emphasize students' responsibility for developing strategies, procedures, or conceptual understanding. Instead, instructors primarily directly presented these procedures, strategies, and key concepts to students.

Data Collection & Analysis

The participants in this study were students enrolled in either non-IBL (n = 381) or IBL Calculus I (n = 310). In both IBL and non-IBL Calculus sections, all students completed the same standards-based final exam each semester, designed to align with specific course objectives and assess conceptual and procedural understanding. While the exams varied slightly across semesters, they were created by the same calculus coordinator, who adhered to the same exam guidelines, making it suitable to compare final exam performance across semesters. We used Welch's t-test (α = .05), a robust test, to account for the unequal variances between the populations to compare exam scores for groups of students in both IBL and non-IBL sections. We also used Cohen's d (Cohen, 1988) effect sizes to help determine the practical significance of the differences observed by certain groups. For these analyses, we followed Cohen's recommended benchmarks for small (d = .20), medium (d = .50), and large (d = .80) effects. Finally, we ran these same tests on subgroups of students including first-year and upper-year students, in addition to students pursuing math-intensive (MI) majors, those that required a mathematics course beyond Calculus II, or non-math-intensive (NMI), majors where Calculus I or Calculus II was the terminal mathematics course required at our institution.

Results

Here, we present our findings organized by research question. First, we compare IBL and non-IBL participants' performance on a standards-based final exam. Then, we discuss how participants' scores differed based on whether they were first-year versus upper-year students or students majoring in a math-intensive field versus a non-math-intensive field.

IBL vs. Non-IBL Final Exam Performance

Overall, IBL students outperformed non-IBL students. As can be seen in Table 2, IBL participants' average final exam score was nearly a full letter grade higher than non-IBL participants' average final exam score. IBL participants' final exam scores ($M = 68.37$, $SD = 16.82$) were significantly higher, ($t(687.21) = 6.59$, $p < .001$), than the scores of non-IBL participants ($M = 59.24$, $SD = 19.65$). The effect size (Cohen's d = 0.50) indicated a moderate difference in the final exam performance of IBL vs. non-IBL participants. Thus, students in IBL scored significantly and noticeably better, on average, than their peers in non-IBL, averaging over 9 percentage points higher on the final exam.

Table 2. IBL vs. Non-IBL Participants' Exam Performance.

	n	*M*	*SD*	*t*	*p*	Cohen's d
IBL	310	68.37	16.82	6.59	<.001	0.50
Non-IBL	381	59.24	19.65			

First-Year vs. Upper-Year Final Exam Performance

The average final exam score of first-year students ($M = 67.64$, $SD = 17.45$) across both IBL and non-IBL sections was significantly higher ($t(679.373) = 5.80$, $p < .001$) than the average final exam score of students in their upper years of school ($M = 59.49$, $SD = 19.29$). The effect size (Cohen's d = 0.44) indicated a moderate difference in the final exam performance of first-year and upper-year participants. As seen in Table 3, first-year participants' final exam score average was nearly eight percentage points higher than upper-year participants' final exam average.

Table 3. First-Year and Upper-Year Participants' Exam Performance in IBL and Non-IBL

	First-Years	Upper-Years
IBL	71.58 (n=132)	65.42 (n=173)
Non-IBL	64.76 (n=181)	54.34 (n=199)
Total	67.64 (n=313)	59.49 (n=372)

We further examined the differences in exam scores of first-year and upper-year students between the IBL and non-IBL sections. We found that upper-year IBL students' average final exam score ($M = 65.42$, $SD = 16.99$) was significantly higher ($t(300.572) = 3.54$, $p < .001$) than upper-year non-IBL students' average final exam score. The effect size (Cohen's d = 0.60) showed a strong difference between the scores of upper-year students in IBL and non-IBL, indicating IBL may have better served upper-year students than did non-IBL. Additionally, first-year IBL students' average final exam score (*M* = 71.58, *SD* = 15.83) was significantly higher ($t(300.572) = 3.54$, $p < .001$) than first-year non-IBL students' average final exam score (*M* = 64.76, *SD* = 18.06). The effect size of (Cohen's d = .40) indicated a moderate difference in exam scores among first-year students, with those in IBL scoring about 7 percentage points higher on the final exam than the first years in non-IBL.

Furthermore, we compared final exam scores between first-year and upper-year students enrolled in the same type of section. The average final exam score for first-year non-IBL students (*M* = 64.76, *SD* = 18.05) was significantly higher ($t(377.983) = 5.38$, $p < .001$) than the average final exam score for upper-year non-IBL students (*M* = 54.34, *SD* = 19.72). The effect size (Cohen's d = 0.55) indicated a moderate difference in the final exam performance of first-year and upper-year students in non-IBL. In the IBL sections, the average final exam score for first-year students (*M* = 71.58, *SD* = 15.83) was significantly higher ($t(291.193) = 3.26$, $p < .001$) than the average final exam score for upper-year students (*M* = 65.42, *SD* = 16.99). Thus, in both IBL and non-IBL, first-year students were performing better than upper-year students. However, the effect size (Cohen's d = 0.37) was not as strong for IBL as it was for non-IBL (Cohen's d = 0.55), indicating a larger difference between first-year and upper-year students in non-IBL. This is evident in the percentage point gap of nearly 6 percent between first-year and upper-year students in IBL compared to over 10 percent between first-year and upper-year students in non-IBL. Therefore, it seems that IBL may have helped lessen the gap between first-year and upper-year students' final exam performance.

Math-Intensive vs. Non-Math-Intensive Final Exam Performance

Regardless of being in an IBL or non-IBL section, the average final exam score for MI majors (*M* = 63.34, *SD* = 19.55) was not significantly higher ($t(477.703) = 0.39$ $p = .697$) than the average final exam score for students in NMI majors (*M* = 62.74, *SD* = 18.50) (see Table 4). The effect size (Cohen's d = 0.03) indicated very little difference in the final exam performance based on whether students were required to take a course beyond Calculus II for their major. Thus, average exam scores did not differ based on the intensity of mathematics courses required for a major.

Table 4. Math-Intensive and Non-Math-Intensive Participants' Exam Performance in IBL and Non-IBL

	Math-Intensive	Non-Math-Intensive
IBL	67.18 (*n* = 104)	68.55 (*n* = 201)
Non-IBL	60.46 (*n* = 139)	57.77 (*n* = 235)
Total	63.34 (*n* = 243)	62.74 (*n* = 436)

We next examined the differences between MI and NMI majors' exam scores for students enrolled in IBL or non-IBL sections. Math-intensive IBL students' average final exam score was

(M = 67.18, SD = 18.54) significantly higher (t(299.453) = 2.72, p = .007) than the average final exam score for MI non-IBL students (M = 60.46, SD = 19.85). The effect size (Cohen's d = 0.35) indicated a small to moderate difference between MI majors in IBL and non-IBL, with a difference of about 7 percentage points on the final exam. Thus, the averages of MI majors in IBL were noticeably higher than MI majors in non-IBL. Similarly, NMI IBL students' average final exam score (M = 68.55, SD = 15.78) was significantly higher (t(433.295) = 6.428, p < .001) than the average final exam score for NMI non-IBL students (M = 57.77, SD = 19.22). The effect size (Cohen's d = 0.61) indicated a moderate and stronger effect than those students in MI fields. NMI majors in IBL scored an average of about 11 points higher on the final exam than NMI majors in non-IBL. These results indicate that IBL benefited both MI and NMI majors and it was not the case that MI majors outperformed NMI majors or masked poor performance from NMI majors.

Finally, we compared the exam scores of MI and NMI majors enrolled in courses with the same pedagogical approach. Within a type of instruction, typically MI majors and NMI majors performed about the same. The average final exam score for MI non-IBL students (M = 60.46, SD = 19.85) was not significantly higher (t(282.356) = 1.28, p = .201) than the average final exam score for NMI non-IBL students (M = 57.77, SD = 19.22). The effect size (Cohen's d = 0.14) indicated a small difference in the final exam performance of non-IBL MI and NMI participants. For IBL participants, the average final exam score for MI majors (M = 67.18, SD = 18.54) was not significantly lower (t(181.566) = – 0.64, p = .523) than the average final exam score NMI IBL students (M = 67.18, SD = 18.54). The effect size (Cohen's d = –0.08) indicated little difference in the final exam performance between the majors. The NMI majors scored over one percentage point higher than MI majors in IBL. In contrast, in non-IBL, NMI majors scored nearly three percentage points lower on average than MI students.

Discussion

This study provided insights into the effects of IBL on student performance in undergraduate Calculus I. We aimed to compare student performance on a common final exam between the following populations of interest: 1) students in IBL Calculus and those in non-IBL Calculus, 2) first-year and upper-year students, and 3) students in math-intensive majors and students in non-math intensive majors. We specifically choose to examine comparisons between year and major because both our own experience and research has shown that upper-years (Gregg-Jolly et al., 2016) and those not in math-intensive fields (Kokkelenberg & Sinha, 2010; Rasmussen & Ellis, 2013) struggle in calculus.

Overall, IBL participants outperformed non-IBL participants on the final exam, with a difference of nine percentage points (see Table 2). Although this study used a smaller sample than some prior studies (Freeman et al., 2014), the fact that the final exam was relatively consistent across semesters, made by the same course coordinator, using the same guidelines, and was administered to all Calculus I students in courses with similar student populations helped us draw strong conclusions about student performance. The significant difference and moderate effect size (Cohen's d = 0.50) reinforces the idea that IBL can improve student performance in Calculus I; students enrolled in IBL sections scored nearly a letter grade higher than students enrolled in non-IBL on the same final exam.

Unsurprisingly, overall, first-year students performed significantly better than upper-year students. However, the largest gap in final exam performance was between upper-year students in IBL and upper-year students in non-IBL. Thus, IBL significantly benefited both first-year and upper-year students' course performance. Moreover, we see these differences in performance between first-year and upper-year students narrow in IBL. The IBL structure provided a learning environment that focused on (1) deliberate practice, where students worked on scaffolded tasks and received immediate feedback from course instructors, and (2) a culture of inclusion that fostered respect among students and demonstrated confidence and interest in each student's individual success in calculus (Theobald et al., 2020). We claim this structure supported these upper-year students who often struggle in STEM courses as they feel the pressure of making academic and life decisions about their chosen career path (Gregg-Jolly et al., 2016) which helped lessen differences in performance between first-year and upper-year students.

In terms of the intensity of mathematics required for students' majors, we did not find a significant performance difference between whether or not a student's major required a mathematics course beyond Calculus II. However, both students who required a math course beyond Calculus II and students whose terminal math course was either Calculus I or Calculus II benefited significantly from the IBL pedagogical approach. Effect sizes indicated the largest difference to be between NMI majors in IBL and NMI majors in non-IBL with students in IBL scoring nearly 11 percentage points higher on the final exam, which is critical as NMI majors are those students who most frequently leave calculus (Rasmussen & Ellis, 2013). By incorporating IBL, students enrolling in these majors succeeded at a greater rate, and hopefully, in turn, persisted with their majors.

Conclusion and Limitations

Overall, our results indicate the potential for benefits of instruction informed by an IBL approach. Two groups of students in particular benefited: upper-year students and students in non-math-intensive majors, who are traditionally less successful in STEM (Gregg-Jolly et al., 2016; Kokkelenberg & Sinha, 2010; Rasmussen & Ellis, 2013). In some ways, we anticipated better performance of IBL students than non-IBL peers, as suggested by prior research (Freeman et al., 2014; Laursen et al., 2011). However, further data analysis revealed subtle differences in how those benefits played out. Notably, differences in performance between upper-year and first-year students was significantly smaller in the IBL classes than non-IBL. Similarly, there were no significant differences in performance for NMI majors and MI majors in IBL classes, debunking the notion that an IBL approach is only for mathematically strong students. Thus, not only was stronger course performance associated with the IBL approach, but an IBL approach may have also better supported students who traditionally leave STEM fields (Gregg-Jolly et al., 2016; Kokkelenberg & Sinha, 2010; Rasmussen & Ellis, 2013). By incorporating IBL, college math instructors can create a learning environment that supports overall student success in Calculus I and specific populations who traditionally struggle in STEM. Although we did not have adequate sample sizes for analyses in this study, future research could examine the effects of IBL on differences in performance of other traditionally underrepresented groups in STEM: women, black and Latinx students, first-generation students, and students of low socioeconomic status (Theobald et al., 2020).

References

Abdi, A. (2014) The effect of inquiry-based learning method of students' academic achievement in science course. *Universal Journal of Educational Research, 2*(1), 37–41. https://dx.doi.org/10.13189/ujer.2014.020104

Abell, M., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *MAA instructional practices guide.* The Mathematical Association of America.

Boelkins, M., Austin, D., & Schlicker, S. (2018). *Active Calculus 2.1*. Grand Valley State University Libraries.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685–698. https://dx.doi.org/10.1080/0020739X.2013.798874

Bruder, R., & Prescott, A. (2013). Research evidence on the benefits of IBL. *ZDM: The International Journal on Mathematics Education, 45*(6), 811–822. https://dx.doi.org/10.1007/s11858-013-0542-2

Cilli-Turner, E. (2017). Impacts of inquiry pedagogy on undergraduate students conceptions of the function of proof. *The Journal of Mathematical Behavior*, *48*, 14–21. https://dx.doi.org/10.1016/j.jmathb.2017.07.001

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed). Lawrence Erlbaum Associates.

Conference Board of Mathematical Sciences [CBMS]. (2016, July 15). *Active learning in post-secondary mathematics education.* https://www.cbmsweb.org/wp-content/uploads/2016/07/active_learning_statement.pdf

Dunnigan, G., & Halcrow, C. (2021). If you don't build it, they will leave: Reforming an applied calculus course by eliminating large lectures and incorporating active learning. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 31*(3–5), 413–433. https://dx.doi.org/10.1080/10511970.2020.1769234

Ehrenberg, R. G. (2010). Analyzing the factors that influence persistence rates in STEM field, majors: Introduction to the symposium. *Economics of Education Review*, *29*(6), 888–891. https://dx.doi.org/10.1016/j.econedurev.2010.06.012

Ernst, D. C., Hodge, A., & Yoshinobu, S. (2017). What is inquiry-based learning? *Notices of the American Mathematical Society*, *64*(6), 570–574. https://dx.doi.org/10.1090/noti1536

Foley, n.d.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Gregg-Jolly, L., Swartz, J., Iverson, E., Stern, J., Brown, N., & Lopatto, D. (2016). Situating second-year success: understanding second-year STEM experiences at a liberal arts college. *CBE—Life Sciences Education*, *15*(3), ar43. https://dx.doi.org/10.1187/cbe.16-01-0044

Kokkelenberg, E. C., & Sinha, E. (2010). Who succeeds in STEM studies? An analysis of Binghamton University undergraduate students. *Economics of Education Review*, *29*(6), 935-946.

Kung, D., & Speer, N. (2020). *What could they possibly be thinking!?! Understanding your college math students.* The Mathematical Association of America.

Laursen, S., Hassi, M. L., Kogan, M., Hunter, A. B., & Weston, T. (2011). Evaluation of the IBL mathematics project: Student and instructor outcomes of inquiry-based learning in college mathematics. [Report prepared for the Educational Advancement Foundation and the IBL Mathematics Centers]. Retrieved from https://www.colorado.edu/eer/sites/default/files/attached-files/iblmathreportall_050211.pdf

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–146. https://dx.doi.org/10.1007/s40753-019-00085-6

Olson, S., & Riordan, D. G. (2012). Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics. Report to the President. *Executive Office of the President*.

Rasmussen, C., & Ellis, J. (2013, July). Who is switching out of calculus and why? In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education, Vol 4.* (pp. 73–80). PME.

Stains, M., Harshman, J., Barker, M. K., Chasteen, S. V., Cole, R., DeChenne-Peters, S. E., Eagan, M. K., Esson, J. M., Knight, J. K., Laski, F. A., Levis-Fitzgerald, M., Lee, C. J., Lo, S. M., McDonnell, L. M., McKay, T. A., Michelotti, N., Musgrove, A., Palmer, M. S., Plank, K. M., & Rodela, T. M. (2018). Anatomy of STEM teaching in North American universities. *Science, 359*(6383), 1468–1470. https://dx.doi.org/10.1126/science.aap8892

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón. D. L. Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones II, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., ... & Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476–6483. https://dx.doi.org/10.1073/pnas.1916903117

Webb, D. C., Stade, E., & Grover, R. (2014). Rousing students' minds in postsecondary mathematics: The undergraduate learning assistant model. *Journal of Mathematics Education at Teachers College*, *5*(2), 39–47.

Groupwork as a Site for Servingness among Undergraduate Latin* Mathematics Students

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Undergraduate mathematics classrooms are racialized spaces for Latin students, even at Hispanic-Serving Institutions (HSIs) with educational missions of cultural affirmation. Instruction plays an important role in reinforcing and disrupting racial oppression in mathematics, which has significant implications for gateway courses (e.g., calculus) that impact STEM persistence. Groupwork is a widely-adopted practice in gateway mathematics courses with intentions to promote equitable access to content and participation; however, research has shown that groupwork can perpetuate inequitable experiences for historically marginalized groups in STEM, including Latin* students attending HSIs. The present study addresses these concerns of racial equity in undergraduate mathematics by exploring Latin* students' groupwork experiences in gateway courses at a HSI. Our findings capture how groupwork facilitated or removed access to a sense of racially-affirming community, which was central in Latin* students' visions of equitable support as mathematics learners at a HSI.*

Keywords: groupwork, Hispanic-Serving Institutions, Latin* students, race, servingness

Purpose of the Study

Latin*[1] students experience undergraduate mathematics as a racialized space due to instances of isolation and underestimation of ability (Leyva, 2016; Oppland-Cordell, 2014). Such realities in gateway courses (e.g., calculus, introduction to proofs) can limit racial equity in educational opportunities, including negative effects on Latin* students' mathematics identities as well as access to course content and STEM majors (Brown, 2018; Convertino et al., 2022). Instructional practices in gateway courses, such as groupwork, that are adopted to build equitable access to content and participation hold potential to disrupt racial oppression for Latin* learners (Laursen et al., 2014; Leyva et al., 2021; Fullilove & Treisman, 1990; Villa et al., 2023). However, research has shown that groupwork can perpetuate inequitable experiences for Latin* students and other minoritized groups (Johnson et al., 2020; MacArthur & Dobie, 2023; Oppland-Cordell, 2014). Researchers have called for work that centers minoritized student populations' groupwork experiences to better understand equitable practices (Ernest et al., 2019; Reinholz, 2018).

Even at U.S. colleges and universities with the federal designation of being Hispanic-Serving Institutions (HSIs), mathematics education remains a racialized environment for Latin* students (Leyva et al., 2022; McGee, 2016). This reality reflects a broader structural concern at HSIs – namely, their lack of institutional visions for serving Latin* students to promote racially-equitable outcomes and culturally-affirming experiences (Garcia, 2020). Research at HSIs that uncovers features of practices and policies for racial equity is a critical need (Vega et al., 2022). To guide such work, Garcia and colleagues (2019) conceptualized *servingness* as a framework that addresses various dimensions of culturally-enhancing opportunities for better serving Latin* students at HSIs, including outcomes, experiences, organizational structures, and external forces.

[1] *The term Latin* is inclusive of gender-nonconforming identities in the Latin American diaspora.*

Research on servingness in STEM has focused on student outcomes related to institutional structures, such as departmental policies (Burn et al., 2019) and support programs (Rodríguez Amaya et al., 2018). However, limited work has examined Latin* students' experiences of STEM instructional practices. Such work is important in gateway mathematics courses that impact STEM retention (Bhattacharya et al., 2020; Byrne et al., 2023; Convertino et al., 2022).

The present study addresses the two areas of needed research in undergraduate mathematics identified above specific to equity issues in groupwork and servingness through instruction. We explored Latin* students' groupwork experiences in gateway mathematics courses at a HSI to uncover features of peer collaboration that constrained and promoted identity-affirming learning opportunities. The study addresses two research questions: (i) How does groupwork in gateway courses reinforce and disrupt mathematics as a racialized experience for Latin* students? ; and (ii) In what ways does groupwork advance and limit opportunities for actualizing Latin* students' conceptions of servingness as mathematics learners at a HSI?

Methods

Our study is from a larger project exploring influences of faculty professional development about culturally-responsive pedagogy in promoting equity for Latin* mathematics students at Sonoma State University – a medium-sized, public university recently designated as a HSI. The university's undergraduate demographics are about 45% white, 35% Latin*, and 20% some other race. The larger project explores instructors' and Latin* students' perspectives on servingness in instruction across gateway courses for STEM majors, including calculus, statistics, introduction to proof, and developmental mathematics sequences. Our team has completed two years of recruitment and data collection since fall 2021. A total of 24 Latin* students were recruited in Year 1 to complete individual interviews via Zoom and journaling about instructional experiences. Fifteen students completed in-person group interviews at Sonoma State during Year 2. Only one student participated in both years. The present report focuses on group interviews.

We invited Latin* students in gateway courses to express interest as participants via email and class visits. We purposefully sampled from students who expressed interest to have multiple voices from different gateway courses as well as to ensure variation in ethnicity and gender. In fall 2022, a total of 31 Latin* students were invited to complete a 2-hour, semi-structured group interview that was audiotaped and transcribed. Fifteen students completed interviews, resulting in six conducted interviews (two for statistics, two for calculus, and two for other courses). The majority of our sample identifies as Mexican or Mexican-American, which is reflective of Sonoma State enrollment. Eleven of the 15 interviewed students are cisgender women. Students with nonbinary gender identities were invited to participate, but were unable to attend the interviews. To the best of our ability, we paired each participant with at least one student of the same gender to mitigate feeling tokenized and make space for variation in race-gender identities. One faculty and two Ph.D. student researchers from outside of Sonoma State conducted the interviews, each with 2-4 participants. To the extent possible, we matched interviewers and participants by race and gender as an effort to build comfort with discussing racism and other forms of oppression. A Latin* researcher was present for all interviews.

Interviews consisted of three parts: (i) students' views on servingness, (ii) responses to three prompts of instructional scenarios corresponding to themes of servingness from Year 1 data analysis (see Leyva et al, 2022 and McNeill et al., 2023), and (iii) responses to an excerpt from Sonoma State's HSI Task Force Report. One scenario featuring instructional practices addressed groupwork:

> My professor asks us to work in groups when solving a mathematics problem, either in class or as homework, followed by presenting our group solution to the class. I often feel concerned about collaborating with classmates in groups because I am unsure if I will have something meaningful to contribute and if my ideas will be welcomed or taken seriously. Groupwork can also be an isolating experience for me as a Latinx student in the mathematics classroom because students may choose to work with classmates of the same racial and gender identities.

For the prompts and report excerpt, we asked about the extent to which Latin* students related as mathematics learners, instructional aspects (dis)affirming of their identities, and ways to improve support. Our analysis focused on the first and third interview parts and the groupwork scenario.

A pair of researchers (one from Sonoma State and one from an outside university, at least one of whom was Latin*) coded each transcript. Sonoma State researchers coded de-identified versions to protect participant confidentiality. We independently and inductively coded to flag instructional and departmental features that students viewed as fostering or limiting servingness, including attention to Latin* students' intersectional identities and cultural backgrounds. A coder from each pair synthesized codes for each transcript to identify themes discussed as a team.

Our team approached the analysis with critical self-reflexivity. We have robust diversity among faculty and students across intersections of race (Asian/Filipinx, Black, Latin*, white) and gender (nonbinary, cisgender woman, cisgender man). Half of our team involved in the present analysis identifies as Latin*, and several members are first-generation college students. As individuals, we brought awareness of how our respective areas of privilege and oppression impact our study of servingness in mathematics at HSIs. We resisted deficit engagement with Latin* students' perspectives and constantly recognized how undergraduate mathematics is situated in broader systems of social power. We bracketed our lived experiences when interviewing and coding to avoid distorting students' perspectives, all while interrogating structures that limit servingness in groupwork and other mathematical contexts at Sonoma State. Our findings avoid essentializing portrayals of Latin* experiences by looking across three cases of groupwork for first-generation females from low-income, Mexican families enrolled in different gateway courses[2].

Findings

Our analysis uncovered how groupwork in gateway mathematics courses can be an opportunity to advance servingness, particularly in terms of Latin* students building a racially-affirming community. Participants shared different reasons for why they valued groupwork, such as accountability to complete assignments, decreased vulnerability of asking questions in class, and meeting new people. Appreciation for groupwork aligned with values of community that were central across participants' views of servingness at Sonoma State. Several students reported feeling served through Sonoma State's student support services (e.g., Educational Opportunity Program office), where they built communities that nurtured their mathematical success. Latin* students, however, also described a lack of race-conscious support in mathematics, including groupwork as an oppressive experience that limited access to content and participation as well as negatively impacted their mathematics identities. Despite an overall lack of servingness in mathematics, participants addressed possibilities through instruction that can disrupt groupwork as an oppressive context and promote a sense of community affirming of their Latin* identities.

[2] The findings do not specify the gateway mathematics course for each participant to protect confidentiality.

We now present findings from analyzing perspectives from three participants (Kayla, Indrid, and Oliva), which offer illustrative cases of how access to community in groupwork can impact servingness. Kayla is a third-year Mexican-American female[3] studying humanities. Indrid is in her first year and Oliva in her second year. Both identify as Mexican females in STEM majors.

The first two sections of the findings address our first research question by detailing racial (in)equity in Lain* students' groupwork experiences. First, we present how social forces, such as stereotypes and structural inequalities, shaped groupwork as a racialized experience. Next, we show how groupwork afforded or disrupted Latin* students' access to a racially-affirming peer community. The final section of the findings addresses how groupwork fell short or advanced each Latin* participant's perception of servingness, answering the second research question.

"If You're Latina… They Don't Take You Seriously If You're in a Group Project"

All three participants addressed how stereotypes of mathematical ability and structural inequities of access contributed to experiences of isolation, with groups often segregated by race and gender. Kayla felt white peers were assumed to be smarter and chosen more often as partners, "When students get to choose our own groups, sometimes for Latinx students, we would feel left out… It's hard to find a group because everybody pairs off with the smart people… because they're white." With strong representation of white students in Kayla's class, she often had white groupmates who seemed to undermine her ability and limit her contributions.

> A lot of Caucasian people, I try to put my inputs, try my best, but when I give the answers, they always look at me like it's wrong. Essentially, they do all the work and I just put my name on it. I still feel I'm not learning anything because then if I ask, 'Oh, how do you do this?' They're like, 'Well, it's simple. Just look at the notes.' I could look at the notes all I want. It's gibberish. Essentially, it's making me feel, again, like I don't know it.

Racialized assumptions of who is mathematically able made groupwork exclusionary for Kayla, restricting her access to learning opportunities and a positive sense of mathematical competence.

Oliva saw herself contributing to racialized segregation during groupwork because she was concerned that collaborating with classmates who did not share her Latinx or Mexican identity could limit her mathematical contributions, "I understand the self-isolation because I still self-isolate… If it were [a] choose-your-own kind of group, I would one hundred percent choose people that look like me, so I can feel related to and I can put forth what I feel like I need to put forth." Her intentional selection of Latinx or Mexican groupmates allowed her to connect with peers who understood her and to protect herself from racialized judgment. Indrid similarly shared how being stereotyped as a Mexican female shaped groupwork as a racialized-gendered space. When asked if being in a mathematics class impacted how students select collaborators, she said, "You unconsciously go towards people that look like you… Yeah, because… stereotypes. If you're Latina… 'Girls are not good at math. Latina, Hispanic girls are just raised to end up being housewives… [or] pregnant'... They don't take you seriously… in a group project." Groupwork was a racialized-gendered space where stereotypes disallowed productive peer collaboration. Indrid recalled an instance of racial stigma when a white female groupmate proposed working on textbook problems independently followed by assuming that she could afford to buy the book and not trusting her to borrow it. Being low-income also stigmatized Indrid during groupwork.

[3] We used language consistent with how students described their identities, including their interchangeable use of terms for their race (e.g., Latinx, Hispanic, Mexican) and gender (e.g., female, girl, woman) during the interview.

“Have a Community That Understands the Math and… Without Feeling Ashamed”

Participants viewed racialized dynamics in groupwork limiting opportunities to find and build community that supported their Latin* identities. We now show how exclusion in groupwork was reinforced or disrupted for Kayla and Oliva, impacting access to community.

As a white-passing Mexican-American female, Kayla grappled with tensions of concealing her Latin* pride (e.g., not speaking Spanish) to contribute more in groupwork, “[A white person] hears me speaking Spanish, they’re like, ‘Oh, you’re not white?’… I also grew up knowing what I can and cannot do [as a Latin* person], and sometimes the white-passing helps with the things I’m able to do… They treat you differently once they find out [you’re Hispanic].” In addition to such linguistic racism in groupwork, she saw white peers’ backgrounds with “parents having higher education” as reinforcing inequities of available support. As a first-generation college student, Kayla saw her family limited in offering support for her in mathematics because her parents did not attend college, “I can’t ask him [Kayla’s father] for help. He never finished high school… They [Kayla’s parents] don’t know the level of math that I’m trying to learn.” Racial inequities in terms of families’ educational backgrounds and access to peer support were left unchallenged in Kayla’s mathematics classroom where she lacked a community of support, “It would be nice to have a community that understands the math and I can go to them without feeling ashamed.” Even with groupwork, she was on her own to succeed mathematically.

In contrast to Kayla being denied community in groupwork, Oliva shared a recent classroom moment allowing her to overcome fears of racialized judgment that isolated her and to collaborate with a racially-diverse group who became close friends. Her instructors opened the course with a discussion about recognizing social diversity and prioritizing mutual respect.

> First day of instruction… they’re [instructors] like, ‘We’re going to have a conversation… We respect everybody.’ That changed the aspect of the class completely… We were able to sit down, have a conversation, and be like, ‘Okay, I am this, but I am also this. Intersectionality is a real thing.’… That’s probably why I feel really strongly about that class. I was able to talk to people in other races and genders…and still be able to collaborate.

Oliva perceived this practice of encouraging students to be identity-conscious and respect each other as establishing a “sense of community,” which facilitated positive groupwork experiences with classmates across social differences and thus disrupted racialized segregation in groupwork.

“There Was Just Much More Communication… That Opened Up A Lot of Doors For Me”

Latin* students’ groupwork experiences that lacked a racially-affirming sense of community constrained opportunities to experience servingness as mathematics learners. Kayla viewed having a community that understood her struggles and motivated her pursuits of academic success as central to being served as a first-generation Hispanic female at Sonoma State. She saw being in a multicultural sorority providing a racially-affirming community missing in mathematics. “The sorority that I am rushing, a lot of them [are] first-gens, low-income… which makes me feel more [of a sense of] belonging… I’ll have somebody that have gone through the same struggles as I have.” The racialized space of groupwork, where Kayla navigated white peers’ judgment for her questions and marginalization as a contributor, left her without a community where she felt seen and supported like in the sorority space. Kayla saw hiring Hispanic mathematics tutors fluent in Spanish as one way to increase Latin* students’ access to such a peer community, “A lot of the tutors… are students and not many of them are Hispanic, especially in math… I tend to use Spanish to get my points across and sometimes we can’t really do that with someone that doesn't speak Spanish." Mathematics groupwork free of racialized judgment and an openness to speaking in Spanish would promote Kayla's vision of servingness.

Indrid's oppressive experience in groupwork restricted access to a community where she felt seen for her financial struggles, which played a major role in her view of being served as a low-income Mexican college student, "Honestly, it's [servingness is] just more financial help… Luckily for me, I got FAFSA. That helped with my tuition… groceries, just basic needs. And then housing is a struggle sometimes." Racialized tensions with white groupmates, such as the female peer who denied loaning her a textbook, reinforced structural inequities that limited Indrid's access to learning opportunities and therefore servingness in mathematics. When asked the extent to which Sonoma State serves Latin* students in mathematics, she critiqued the lack of faculty diversity, which she saw impacting comfort in seeking support unavailable in groupwork.

> In the group, I was always the social one, so I had to email the teacher… We should have a more diverse faculty… If we had a Latino or Latina math teacher… that would be good too because Hispanic, Latino students will most likely be more comfortable talking to them and reach[ing] out for help than a white teacher that they are most likely intimidated by.

Indrid viewed having Latin* professors for mathematics increasing access to support for overcoming struggles in groupwork. Presence of Latin* mathematics faculty would enhance servingness for Indrid by expanding her community of racially-affirming support at Sonoma State, which can mitigate oppression due to stereotypes and structural inequities in groupwork.

Unlike Kayla's and Indrid's groupwork experiences that departed from their conceptions of servingness, classroom norms of mutual respect and social awareness that guided groupwork in Oliva's classroom aligned with her views of being served as a Mexican female at Sonoma State. She perceived open dialogue in her mathematics classroom as resonating with her Mexican family's value for communication, which she described as important to her success, "There was just much more communication [in the class], which I personally need, especially… coming from the family that's mostly just speaking to each other on how to get points across. I feel like, at least in that class, I thrived." With the classroom experience of "having that conversation" about mutual respect being likened to communication practices in her family, this instructional practice advanced servingness for her because it established a collaborative space attuned with her values in her family and culture as a Hispanic mathematics learner. Oliva perceived Sonoma State serving her as a Mexican female by providing transformative educational opportunities that go beyond those that her family can offer and will inspire future generations.

> I have the opportunity to continue in education… That's something I don't come from. My family, definitely not… It's what I've been told, 'This is what you can do to better yourself and everyone that comes before and after you.' So, just the fact that I'm given the opportunity to be able to do that… I'm truly grateful... When you say [Sonoma State] serves me… that's what it is… A home away from home, but with more opportunities… It changes you as a person… I'm also first-gen, low-income, have younger siblings and other people at home I gotta impress, pave the way, [and] get there so they can get there with a little more support. But it just makes you a stronger person at the end of the day.

Personal transformation through education in Oliva's perspective on servingness is evident in the long-term positive impact of open dialogue in her mathematics classroom. She shared, "That [the value of communication in class] opened up a lot of doors for me… That class has led me to where I'm at right now. I can talk to the professor, not having to be [about] math… That little seed right there has really been pushing me." Such open communication, which established classroom norms of respect, nurtured Oliva's personal growth as a Mexican female. She overcame fears of racialized judgment in peer collaboration and actively sought faculty support, both academic and personal, that motivated her persistence. Thus, the disruption of racial exclusion in groupwork contributed to transformative learning opportunities that Oliva sought in being served at Sonoma State and provided her with a community of support in mathematics.

Discussion

The scholarly significance of our findings is threefold. First, our findings add knowledge to address the lack of clarity about groupwork approaches for equity (Hwang et al., 2022; Reinholz, 2018). Our analysis contributes to research on racially-equitable collaborative learning in mathematics (e.g., Bhattacharya et al., 2020; MacArthur & Dobie, 2023) by elucidating how groupwork expanded and constrained identity-affirming support for Latin* students. Second, the study deepens understandings of equity-oriented instruction in gateway mathematics courses through its focus on a single racially-minoritized group (Latin* students) and a single type of classroom practice (groupwork). By centering three Latin* first-generation college women in the findings, we also shed light on intersectional complexities on how gender overlapped with race, language, and social class to impact equity in groupwork. Third, our study addresses limited inquiry on instructional experiences for servingness in STEM. The findings provide a novel view of how HSI structures that foster community can inform equitable groupwork in mathematics.

Our analysis raises implications for research. Future studies can examine perspectives from mathematics faculty at HSIs on designing groupwork opportunities that promote equity for Latin* students. Exploring how these views converge and diverge from students' conceptions of servingness can orient faculty learning in translating HSI missions of culturally-affirming support into instructional practices. Additional research on student experiences of groupwork and other classroom practices across various HSI contexts with different Latin* populations can inform more robust understandings of servingness in undergraduate mathematics education.

In terms of implications for practice, Latin* students' isolation in racially-segregated groups as well as limited access to faculty and peer support underscore how faculty play a key role in structuring groupwork to mitigate oppression (e.g., Oliva's instructors setting norms for socially-conscious collaboration). Faculty can make informed decisions about grouping arrangements by learning about students' backgrounds and collaboration histories using a short survey at the beginning of the semester. In addition, faculty can structure groupwork tasks with rotating roles and frequent check-ins to ensure Latin* students are collaborating well with peers, engaging deeply with the mathematical content, and receiving adequate support. Latin* participants' references to campus support structures at Sonoma State where they experienced servingness (e.g., Educational Opportunity Program office, multicultural sorority) raise implications for mathematics departments about building partnerships with such offices and units. Mathematics faculty and student support leaders can share their respective struggles and successes in providing Latin* students with a racially-affirming community. Such exchanges can guide faculty to structure instruction, including groupwork, that enhances servingness in mathematics.

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References

Bhattacharya, N., Ordaz, A. S., Mosqueda, E., & Cooper, C. R. (2020). Redesigning the gateway college algebra course with inclusive and asset-based pedagogy. In G. Garcia (Ed.), *Hispanic-Serving Institutions (HSIs) in practice: Defining "servingness" at HSIs* (pp. 61-77). Information Age.

Brown, S. (2018). E-IBL: An exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. In A. Weinberg & C. Rasmussen, J. Rabin, M, Wawro, & Brown, S. (Eds.). *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1–15). San Diego, CA.

Burn, H., Zamani-Gallaher, E., Mesa, V., & Wood, J. L. (2019). Transitioning STEM learners to calculus: Findings from a national survey of mathematics chairs in two-year colleges by Hispanic-Serving institutional designation, *MathAMATYC Educator, 10*(2), 5-13.

Byrne, M., Hobson, N., Kanaana, I., & Lahme, B. (2023). Stretching calculus: A yearlong calculus class that stretches minds. In M. Voigt, J. Hagman, J. Gehrtz, B. Ratliff, N. Alexander & R. Levy (Eds.), *Justice through the lens of calculus: Framing new possibilities for diversity, equity and inclusion* (pp. 184-193). Mathematical Association of America.

Convertino, C., Alvidrez, M., Pickett, J., & Camberos, D. (2022). Exploring worksheets and attendance-taking as gatekeeping features in a gateway course for Latinx STEM majors. *Journal of Latinos and Education*, 1-17.

Ernest, J. B., Reinholz, D. L., & Shah, N. (2019). Hidden competence: Women's mathematical participation in public and private classroom spaces. *Educational Studies in Mathematics*, *102*, 153-172.

Fullilove, R. E., & Treisman, P. U. (1990). Mathematics achievement among African American undergraduates at the University of California, Berkeley: An evaluation of the mathematics workshop program. *The Journal of Negro Education, 59*(3), 463–478.

Garcia, G. A. (2020). *Hispanic-Serving Institutions (HSIs) in practice: Defining" servingness" at HSIs*. Information Age.

Garcia, G. A., Nunez, A. M., & Sansone, V. A. (2019). Toward a multidimensional conceptual framework for understanding "servingness" in Hispanic-serving institutions: A synthesis of the research. *Review of Educational Research, 89*(5), 745-784.

Hwang, J., Castle, S. D., & Karunakaran, S. S. (2022). One is the loneliest number: Groupwork within linguistically diverse classrooms. *PRIMUS*, *32*(10), 1140-1152.

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504-516.

Leyva, L. A. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education, 9*(2), 81-121.

Leyva, L. A., Mitchell, N. D., McNeill, R. T., Byrne, M. H., Ford, B., Chávez, L. A., & Abreu-Ramos, E. D. (2022). Faculty and student perceptions of instructional servingness in gateway mathematics courses at a Hispanic-Serving Institution. In A. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 444-452). Nashville, TN.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition & Instruction, 39*(1), 1-34.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406-418.

MacArthur, K., & Dobie, T. E. (2023). Benefits and challenges of peer interaction in calculus group exams: Perspectives of Latina/Hispanic & mixed race women. *PRIMUS*.

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture?. *American Educational Research Journal*, *53*(6), 1626-1662.

McNeill, R. T., Leyva, L. A., Byrne, M. H., Mitchell, N. D., Lewis, R., & Abreu-Ramos, E. D., (2023). "Looking outside of my bubble": Whiteness-at-work in mathematics faculty sensemaking about serving Latin* students. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 270-278). Omaha, NE.

Oppland-Cordell, S. (2014). Urban Latina/o undergraduate students' negotiations of identities and participation in an emerging scholars calculus I workshop. *Journal of Urban Mathematics Education*, *7*(1), 19-54.

Reinholz, D. L. (2018). A primer on small group instruction in undergraduate mathematics. *PRIMUS*, *28*(10), 904-919.

Rodríguez Amaya, L., Betancourt, T., Collins, K. H., Hinojosa, O., & Corona, C. (2018). Undergraduate research experiences: Mentoring, awareness, and perceptions—A case study at a Hispanic-serving institution. *International Journal of STEM Education*, *5*(1), 1-13.

Vega, B. E., Liera, R., & Boveda, M. (2022). Hispanic-serving institutions as racialized organizations: Elevating intersectional consciousness to reframe the "H" in HSIs. *AERA Open*, *8*.

Villa, E. Q., Lesser, L. M., & Mien, E. (2023). Shifting pre-calculus from gatekeeper to gateway at a Hispanic-Serving Institution. In M. Voigt, J. E. Hagman, J. Gehrtz, B. Ratliff, N. Alexander & R. Levy (Eds.), *Justice through the lens of calculus: Framing new possibilities for diversity, equity, and inclusion* (pp. 364-372). MAA.

Opposing Dimensions in Mathematicians' Counter Narratives Written for Undergraduate Students

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In mathematics, counter narratives can be used to fight the dominant narrative of who is good at mathematics and who can succeed in mathematics. Eight mathematicians were recruited to co-author a larger NSF project (RAMP). In part, they were asked to create author stories for an undergraduate audience. In this article, we use narrative analysis to present five polarities identified in the author stories. We present various quotations from the mathematicians' author stories to highlight their experiences with home and school life, view of what mathematics is, experiences in growth in mathematics, with collaboration, and their feelings of community in mathematics. The telling of these experiences contributes towards rehumanizing mathematics and rewriting the narrative of who is good at and who can succeed in mathematics.

Keywords: Counter narratives, narrative analysis, mathematicians

Stories provide a particularly powerful way to make sense of our lives and provide structure to society. Stories may be personal narratives, the narratives we tell ourselves about our lived experiences, as well as dominant (master) narratives that are culturally shared. In the United States, there are prevalent deficit narratives in relation to students of color (e.g., Adiredja, 2019; Berry III et al., 2011) and women (e.g., Leyva, 2017) and their ability to do mathematics. As elaborated by Berry III et al.,

> Master narratives embody and dictate expectations about how things work and how stories are framed. Often, master narratives present contrasts between groups of people by advantaging dominant groups and disadvantaging members of marginal groups such as women and people of color (p. 11).

An important means of challenging dominant (or master) narratives are *counter stories* or *counter narratives* (e.g., Solórzano, D. G. & Yosso, 2002) which run counter to status quo and dominant narratives. Many scholars have presented such narratives as a way to disrupt prominent stories of who is good at math and who can succeed at math (e.g., Berry III et al., 2011; Harris et al. 2011; Langer-Osuna et al., 2016; Leyva, 2016; McGee, 2009). These narratives are often crafted by researchers or those marginalized and brought into spaces to perturb and make change.

Narratives themselves are constructed with a certain audience in mind. Kaasila (2007b) explains, "when we are telling a narrative (or narratives), we often take our audience into consideration and adapt what we say and how we say it accordingly" (p. 206). In this project, we examine a set of narratives that were crafted explicitly for students to read in undergraduate proof courses. Eight mathematicians whose stories challenge the dominant narratives of who can succeed in mathematics, provided two-page biographies for students.

In this proposal, we share an analysis of these narratives, identifying rhetoric and plots these mathematicians shared. For the scope of this paper, we present findings from the portion of our analysis focused on "key rhetoric" (Kaasila, 2007a) indicating opposing polarities internal to the author's stories. This analysis is led by the following research question:

What opposing dimensions are salient in mathematicians' career counter narratives as told for a student audience?

Narrative and Mathematical Identity

A narrative approach to identity emphasizes that identity is created by the stories we tell about our lives. Narratives are stories that include events that are "attach[ed] to a character" (Kaasila, 2007b, p. 206) and are organized into plots. Identity is then a "subjective, constructed, and evolving story of how one came to be the person one currently is" (McLean & Syed, p. 320). Furthermore, the author of a narrative imparts coherence to the story. Kaasila (2007a) identifies, "[k]ey rhetoric [a]s a coherence system through which different life events are connected and their relation is explained by dividing the narrated world into different dimensions of reality" (p. 377). Kaasila (2007b) further asserts that key rhetoric often points to expectations, whether met or subverted, adapting Tennen's (1979) notion that expectations determine how narratives are shared. That is, the author of a narrative may be framing their story as adhering to or subverting their own or societal expectations.

Relatedly, many scholars argue for distinctions between master (or dominant) narratives and alternative (or counter) narratives. McLean and Syed (2016) elaborated that, "Master narratives are culturally shared stories that tell us about a given culture and provide guidance for how to be a 'good' member of a culture; they are a part of the structure of society." These narratives may be unproblematic if one's life easily fits into the dominant narratives. However, one's personal narrative may involve adopting an alternative narrative that differs or resists the dominant narrative. We take on McLean and Syed's stance implying there is *negotiation* between self and society and *internalization* of dominant narratives. We would anticipate that subversion of expectations may be reflected in personal narratives when alternative narratives are provided of who is good at math and what it means to do mathematics.

Mathematics identity becomes salient when one tells stories of their mathematics experiences including "stories about one's relationship to mathematics, its learning and teaching. This means that a person's mathematical identity is also context bound and always under construction" (Kaasila, 2007b, p. 206). Martin (2007) suggests that mathematics identity is the "dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives" (p. 15). From a narrative perspective, this means considering how one's stories and experiences with mathematics are constructed within their institutions, communities, and in relationship to sociohistorical dominant narratives (Larnell, 2016). That is, mathematics identity is reflected in the stories told of how one arrived at their current dispositions and beliefs, and how expectations were subverted or maintained in this process.

Methods

Eight mathematicians joined the larger RAMP project team to develop curriculum materials for introduction to proof. The mathematicians were selected based on their reputation in the field and via snowball sampling (Parker et al., 2019). That is, mathematicians recommended colleagues and friends who are active in the field and dedicated to supporting a more diverse and inclusive mathematics community. Amongst the group of eight mathematicians, two identify as Black/African American men, one identifies a Black woman, one a Black and Chinese woman, one as a Latinx woman, one a Latino man, and one a woman and Native Pacific Islander.

The eight participating mathematicians were asked to exhibit mathematical results (without restriction on the origin of statements) through proof writing based on their own intended goodwill and mathematical expertise. The authors refined their work through team-led proof sculpting--a process to reduce the noise of mathematical exclusivity in favor of amplifying reception by vetting choices of transmission based on a form of accessible reasoning. This type of collective engagement recasts expertise. The resulting pieces run counter to the style prevalent in research mathematics: Concluding the veracity of a mathematical statement does not have to cost the reader (nor the author) their humanity as an act to pay homage to mathematical elitism. These costs and refutations were mentioned amongst other themes in author stories that were requested to accompany the proofs. A set of guiding (but not required) questions were provided to help shape the author stories which had an overarching aim of sharing how the mathematicians arrived at their current career. These included prompts about their life before becoming a mathematician, finding meaning and joy in math, struggles and overcoming struggles, and what it means to do mathematics. Each narrative was roughly two pages and designed to be part of the curriculum provided for students.

We took an “analysis of narratives” approach (Polkinghorne, 1995) analyzing the narratives for themes that hold across them. We focused on mathematical identity markers, mathematical socialization markers, key rhetoric reflecting polarities, and plot features. The plot features and key rhetoric were adapted from Kaasila (2007a) where plots were considered in terms of *outcomes*, *key events*, and *important people*, and key rhetoric to identify how incoherent or differing dimensions of life stories are connected. All instances of contrasting language such as “but”, “however”, or explicit language such as “contradiction” or “surprise” were identified in order to identify and analyze the “polarities” described in the mathematician stories. To focus on mathematical identity, we adapted Larnell’s (2016) coding scheme marking instances that reflected, “importance of mathematics”, “motivation”, “strategies”, “opportunities”, “constraints”, and “capacity to perform.” For socialization, we used Larnell’s categories: “institutional”, “sociohistorical”, “community-home.” We note that it is in the sociohistorical category we are most likely to observe dominant narratives endorsed or contradicted in the mathematician stories. Each story was read and analyzed independently by two members of the team. In the next section, we report on some of the polarities that were salient in relation to mathematical identity and socialization as indicated by key rhetoric in the narratives.

Results

For the scope of these results, we share a series of polarities identified and some quotes that illustrate the ways they were discussed in the narrative collection.

Polarity: Home and School Life

An early polarity observed through the narratives was a divide between home and school. This polarity difference was not always experienced the same, but the contrast between the two environments was salient. One mathematician shared that mathematics was an escape, elaborating, “In a whirlwind of instability, I could always rely on math to be a subject where I could solve problems and feel good about the world.”

However, more commonly, school was described as constraining with one mathematician explaining:

> As a child and teenager, I was diligent and shy at school, while being loud, goofy, and creative at home. Most of my weekends were spent gathering with our large extended family, celebrating our Indian heritage. I remember these times fondly. I felt, and still feel, at home with my cousins, in sharp contrast to how out of place I felt at school.

Similarly, another mathematician contrasted school as causing an “academic conflict” that became problematic as the rigidity of the “high school schedule” contracted with their earlier experiences both with the philosophy of Montessori and also the “freedom of viewing/connecting dots outside of the classroom.” That is, the school setting was presented as a constraint (in various ways) in their mathematics story.

Finally, we note another contrast that emerged in some of the stories: the math opportunities outside of school versus inside whether those be competitions, or camps, or in the case of one mathematician’s games,

> Whether it was finding the logic to solve a puzzle or simply counting dominoes to record the score of a game, I really enjoyed it all. In school, the story was a little different. I always performed well in math classes but found them repetitive and not very interesting.

It seemed frequently outside experiences of math were salient to supporting interest.

Polarity: Mathematics as Creativity and Computation/Speed

A related polarity can be found in mathematicians’ descriptions of what mathematics is. As seen in that last quote, school mathematics was often linked to its “repetitive” and closed nature. It was common for stories to contrast earlier understandings of what mathematics is, such as one mathematician noting, “I thought all the interesting questions had been answered and that being a professor meant knowing all the answers.” To follow up this statement with “Spoiler alert: none of that previous sentence is true.” There were often key events that led to this shift such as research opportunities.

Another relevant storyline is the way that the dominant narrative around mathematics, that what it means to be good at mathematics, is somehow tied to having absolute knowledge or speed. For example, one mathematician shared an experience learning a new topic that interested one of their students:

> I would imagine a few years back, I would be nervous about understanding a topic with which I was unfamiliar. I realized this was tied to my ego. I didn’t want to look like I didn’t know something about mathematics. This is a contradiction, it seems. If the mathematics I love is about ideas, then why is my ego involved? Why should I care about who did the problem fastest?

This quote describes two contrasts: the contrast between where the mathematician is now versus “a few years back” and the contrast between “speed” and “ego” versus “love” of mathematics.

Finally, we note one other way this polarity came out. One mathematician explained, “I try to convey that everyone (not only white men) can do math, that math is more than computation and arithmetic, and that math can be fun.” Here the contrast is explained in the context of making efforts to change a dominant narrative. It would be remiss not to note that nearly every author story did not conclude with them doing their mathematics work, but ways that they have integrated changing narratives for others into their lives.

Polarity: Mathematical Breakthroughs and Struggles.

Another polarity is part of what Larnell (2016) calls the strategies of mathematics. Related to ideas that math should be fast, with answers known immediately and quickly recited, is the inherent tension of breakthroughs and struggles. Across the stories, the mathematicians contrasted these elements with statements like: "To me, mathematics is messy and that is the beauty of it, because it allows me to be messy without judgment," "I became more comfortable with the fact that mathematics is very hard and that challenges are ok," and,

> The most glorious moments are always the breakthroughs of a new idea, or an epiphany. It's an indescribable moment that I hope we all get a chance to feel. One moment, the struggle is real, frustrating and long. I can feel like I am getting nowhere. There are times where it feels like I have wasted time. But when the moment of clarity arrives, it's worth it.

If we consider key rhetoric as bringing coherence to stories, we can see the ways that messiness, challenge, and struggle are all coherent parts of the beauty and joy of doing mathematics. While the polarities are in contrast, they are not in tension in the work of the mathematicians in their reflections on the present day.

Polarity: Isolation and Collaboration/People

Another dominant narrative frequently subverted in the stories was the idea that mathematics is not a human endeavor and that mathematical activity is meant to be done in isolation. This isolation was often highlighted as a part of their mathematical journey that contrasted with the joy they later felt. For example, mathematicians explained the challenges of feeling alone in school or the challenges related to research in their careers with one mathematician noting, "I found it very difficult to keep my research going since, at the time, I did not have collaborators."

Many of the mathematicians contrasted this isolation with feelings of joy in connecting with others through mathematics. For example, the mathematician above described a key event where they had given a talk and invited collaboration which led to:

> In the years that followed, we got together and had the best time cranking out some super neat results on posets and order dimension. It still brings tears of joy to my eyes when I think about those days in the basement of the math department at [blinded] with these phenomenal women because it was the first time I truly felt like an honest-to-goodness mathematician, working on some very cool math with some very cool people.

Others described the joy in collaboration or made similar comments about the integration of people and mathematics. One mathematician noted, "I came to realize that I find the most meaning in mathematics through human connection and interaction" or another reflected, "So what brought me back to this world of academic and research mathematics? People."

We note that the role of people went further than just people to do mathematics with, but also included the many role models, family members, and community in the lives of the authors. As one mathematician noted, "Who are the people who support us unconditionally? Surrounding yourself with these people can create a life more expansive and fulfilling than you can possibly imagine."

Polarity: Outlier and Belonging

Finally, we note one last polarity related to the human element: belonging or being an outlier. This polarity is intimately tied to representation and the people around you. The mathematics discipline is notoriously white and male. When taking classes, mathematicians mentioned ideas like, "this love for combinatorics did not remedy how alone and isolated I often felt not knowing other people like me in mathematics," "I am often the odd one out racially and/or culturally wherever I go," and,

> On the other hand, I often felt out of place in many of my classes at [blinded university] as a woman of color. I remember constant signaling from my peers and other professors that I didn't belong in math. I found refuge from these experiences through my hobbies and social life, and especially through my participation in a competitive collegiate dance.

One mathematician reflected that they still feel a degree of imposter syndrome despite their CV:

> You might never guess that is how I feel when you look at my CV and all of the math I have done and learned along my mathematical journey. Yet, that feeling lingers. Now that I am older, I understand that some of that feeling is often triggered because I did not see myself reflected in those who I considered mathematicians: my teachers and professors. The fact remains that being Latinx and a woman means I am often one of the few in a room who is not part of the dominant group in mathematics: male and white.

This feeling of "not seeing" oneself amongst others in mathematics was quite salient. Key events in the authors' stories would allude to seeing themselves represented or collaborating with others that are not part of the dominant group.

For example, one mathematician noted the impact of seeing a Black, male professor explaining a critical moment,

> A tall large Black man entered the room. He took his suit jacket off and said: "I am [name blinded]. This is analysis." He began to detail the course and what it was about. It was at this moment where I whispered to myself: "That is who I want to be."

Across the stories, mathematicians talked of building community and not just in the sense of collaboration above but building community amongst others like themselves. With one mathematician explaining,

> I worked to find a community of mathematicians where I do feel like I belong. The mathematicians I am closest to are not only my colleagues, but also my friends. We connect on shared values and views and work together to promote equity and justice in the mathematical community.

Further, as discussed above, this work was also done to open doors for others coming up, students, young scholars, and as one mathematician noted about the stereotype of who does math, "By dispelling this stereotype, we create a more inclusive culture for mathematics.

Discussion

An overarching goal of stories and narratives is to center the humanity of both the individuals and their communities. Humanizing mathematics has been a part of the endeavor to improve education of mathematics in service of justice and inclusion (e.g., Berry III, 2021; Tan et al., 2022; Yeh & Otis, 2019). If we want students to see themselves as doers of mathematics, it is necessary that we problematize popular narratives of mathematics in terms of both who is capable of doing mathematics and what mathematics is. Some of this work has been championed taking a humanistic approach to mathematics. In summarizing Hersch's contributions, Pais (2018) explained, "the purpose is not (only) to study mathematics in itself, but as an activity, developed by humans in a variety of different settings" (pp. 235-236). This approach stands in stark contrast to the materials often provided in proof-based classes. Davis and Hersh (1981) elaborate that proofs often obfuscate the humans involved with an "ideal" mathematician writing to "conceal any sign the author or the reader is a human being. It gives the impression that, from the stated definitions, the desired results follow infallibly by the purely mechanical procedure" (p. 36).

If we consider the narratives provided, we can identify many ways that mathematics, as engaged in by mathematicians, is a fully human experience. People and communities shaped the stories told. These narratives eschew not only dominant narratives that only certain types of people should and can succeed at mathematics, but also what mathematical success entails. The stories elaborated on how school mathematics geared towards speed and computation is not necessarily the mathematics of joy and discovery that supported and inspired the authors' journeys. The mathematicians were agents in the stories, and the proofs they produce are not authorless. As we consider ways to humanize mathematics for students, author stories can provide a powerful means to challenge the status quo. While it is unlikely that the norms of how proofs are written will shift any time soon, we can certainly take strides in pulling back the curtain and sharing not just the final product, but the process and the people involved.

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References

Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense making. *Journal for Research in Mathematics Education, 50*(4), 401-435.

Berry III, R. Q. (2021). Humanizing Mathematics to Broaden the Space of Participation. *Journal of Mathematics Education at Teachers College, 12*(2), 45-46.

Berry III, R. Q., Thunder, K., & McClain, O. L. (2011). Counter narratives: Examining the mathematics and racial identities of Black boys who are successful with school mathematics. *Journal of African American Males in Education (JAAME), 2*(1), 10-23.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Penguin Books.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics, 95*, 123-142.

Prieto-Langarica, A., Rivera Quiñones, V., Sordo Viera, L., Uscanga, R., & Vindas Meléndez, A. (2021). Testimonios: Stories of Latinx and Hispanic Mathematicians (Vol. 67). American Mathematical Soc..

Kaasila, R. (2007a). Mathematical biography and key rhetoric. *Educational Studies in Mathematics, 66,* 373-384.

Kaasila, R. (2007b). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM, 39*(3), 205-213.

Langer-Osuna, J. M., Moschkovich, J., Norén, E., Powell, A. B., & Vazquez, S. (2016). Student agency and counter-narratives in diverse multilingual mathematics classrooms: Challenging deficit perspectives. *Mathematics education and language diversity: The 21st ICMI study*, 163-173.

Larnell, G. V. (2016). More than just skill: Examining mathematics identities, racialized narratives, and remediation among black undergraduates. *Journal for Research in Mathematics Education, 47*(3), 233-269.

Leyva, L. A. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education, 9*(2).
Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 397-433.
Martin, D. B. (2007). Mathematics learning and participation in the African American context: The co-construction of identity in two intersecting realms of experience. In N. S. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 146–158). New York: Teachers College Press.
McGee, E. O. (2009). *Race, identity, and resilience: Black college students negotiating success in mathematics and engineering.* University of Illinois at Chicago.
McGee, E. O., & Martin, D. B. (2011). “You would not believe what I have to go through to prove my intellectual value!” Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, *48*(6), 1347-1389.
McLean, K. C., & Syed, M. (2016). Personal, master, and alternative narratives: An integrative framework for understanding identity development in context. *Human Development*, *58(*6), 318-349.
Pais, A. (2017). Book Review: The limits of pluralism. Bharath Sriraman (Ed.) (2017) Humanizing mathematics and its philosophy: Essays celebrating the 90th birthday of Reuben Hersh. *Educational Studies in Mathematics, 99*, 235–240 .
Parker, C., Scott, S., & Geddes, A. (2019). Snowball sampling. *SAGE research methods foundations.*
Polkinghorne, D. E. (1995). Narrative configuration in qualitative analysis. *International journal of qualitative studies in education, 8*(1), 5-23.
Solórzano, D. G., & Yosso, T. J. (2002). Critical race methodology: Counter-storytelling as an analytical framework for education research. *Qualitative inquiry, 8*(1), 23-44.
Tan, P., Padilla, A., & Lambert, R. (2022). A critical review of educator and disability research in mathematics education: A decade of dehumanizing waves and humanizing wakes. *Review of Educational Research, 92*(6), 871-910.
Yeh, C., & Otis, B. M. (2019). Mathematics for whom: Reframing and humanizing mathematics. *Occasional Paper Series, 2019*(41), 8.

A Comparison of Traditional and Active Learning Classroom Practices in Calculus Using Fixed- and Mixed-Effects Models

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Active learning in STEM classrooms has been shown to increase student outcomes in multiple ways. We present here a discussion of data analysis supporting these types of conclusions using data from a large-scale randomized study of the implementation of an active learning-based curriculum for Calculus I that collected 1019 observations of student outcomes over three semesters. We compare fixed-effects models including cluster levels of student learning outcomes to mixed-effects models with random cluster level effects. We discuss the differences between the models and the resulting effect sizes suggested by the different factors included in the models.

Keywords: calculus, regression model, fixed effects, mixed effects

Introduction

Evidence of the effectiveness of active learning (AL) classrooms in STEM includes increases in student achievement, attitudes, and persistence in college (Duran et al., 2022; Ellis et al., 2016; Freeman et al., 2014; Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023). Most of these studies have measured this effectiveness by conducting non-randomized experiments and focused on evidence gathered from observational studies or performed meta-analyses of studies with varying implementations in order to interpret the outcomes across a collection of studies with different sets of participants or contexts. In (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023), the authors present a large-scale study of the implementation of an active learning-based curriculum for Calculus I that collected 1019 observations of student outcomes over three semesters with a variety of instructors teaching at multiple times. On the one hand, this study presents compelling evidence that student learning in an active curriculum has the potential to be far more effective than in traditional lecture-based paradigms. The random allocation of students to the treatment and control conditions allows the comparison of student outcomes across populations while controlling for multiple demographic factors including incoming mathematics background, race and/or ethnicity, gender, and student enrollment status. At the same time, students are grouped into sections at the same time of day/day of week and instructors will be present in different sections, introducing random effects on student outcomes (Hedges, 2009).

In this work, we present a fixed effects model (FEM) of student learning outcomes dependent on multiple demographic factors and show that the differences between the treatment and control group outcomes are statistically significant. We then compare that FEM to the mixed-effects model (MEM) in (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023) and discuss the ways in which the random effects in that model compare to the FEM properties. We also discuss changes to the MEM and show that the MEM presented in (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023) and using the data set from that project here (Kramer, Fuller, Watson, Castillo, Duran

Oliva, et al., 2023) possesses the same structure as both the FEM as well as these other models. Finally, we provide a comparison of the effect sizes and confidence intervals for different models.

Methods

In (Castillo et al., 2022; Duran et al., 2022; Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023) the authors collected data from a trial of an active learning based curriculum where students were randomly allocated and presented a number of outcomes analyses. In this study, 1019 students were randomly allocated to either the treatment condition which utilized the Modeling Practices in Calculus (MPC) approach to student learning or the control setting based on existing instructional practices that were primarily lectured based. Student data was collected from institutional records that provided potential fixed demographic effects on student outcomes including mathematics background from standardized tests/high school GPA (*MBS*), gender (*Gender*), race and/or ethnicity (*RE*). Along with these data, the section of the course (*Section*) and the instructor of record (*TIDw*) were recorded as random effects that depended on student choices made during enrollment. In addition, student outcomes were recorded both as grade outcomes in the course as well as a measure of student learning, *LearningMeasure*, collected from embedded final exam questions implemented within the common final administered across all sections of the courses and assessed in a blinded form by multiple evaluators (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023).

Using these data, a fixed effects model with interacting terms can be constructed to determine the dependence of the measure of student learning from the study, *LearningMeasure*, on these factors including the cluster levels *SecPair* to represent the grouping of participants by their time of day and day of week classroom groups. Note that these sixteen pairs then split into 32 based on random assignment, and this is also a representation of the instructor level clustering. The linear model used here is then

$$\begin{aligned} LearningMeasure_i \sim & \beta_0 + \beta_{i1} Treatment_i + \beta_{i2} MBS_i + \beta_{i3} Gender_i + \beta_{i4} RE_i + \\ & \beta_{i5} SecPair + \\ & \beta_{jkl} Treatment_{ij} MBS_{ik} Gender_{il} + \epsilon_i \end{aligned}$$

where *LearningMeasure* is the vector of students' scores, *Treatment* is the predictor (a dummy variable, 0:Non-MPC sections; 1: MPC sections), β is the regression coefficient for the covariates measured, and ϵ is the vector of residuals, assumed to be distributed N(0,2I). Other variations involving either *Section* or *TIDw* for cluster levels as fixed effects were not considered, as they introduced collinearity with *Treatment*. Similarly, a variation of the mixed effects model presented in (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023) can be constructed using the fixed effects above but instead applying *SecPair* along with *Section* and *TIDw* as random effects. Models were implemented in R (R Core Team, n.d.) using the glm and lme4 (Kuznetsova et al., 2017) packages.

Results

Coefficients for the FEM with both interacting and non-interacting terms along with a model with Treatment as the only fixed effect were computed and found to have the values shown in Table 1. The model shown using the sixteen cluster levels of *SecPair* also has the interaction terms of the first model.

Table 1. Fixed Effects Model Coefficients with and without Interactions and Cluster Levels.

	Model With Interactions			**No Interactions**			**With SecPair**		
Predictors	*Est.*	*s.e*	*p*	*Est.*	*s.e*	*p*	*Est.*	*s.e*	*p*
(Intercept)	-8.20	9.06	0.365	-2.99	4.91	0.543	-0.54	9.41	0.954
Treatment	30.48	11.82	**0.010**	16.45	1.39	**<0.001**	28.91	11.63	**0.013**
MBS	0.89	0.13	**<0.001**	0.81	0.06	**<0.001**	0.84	0.13	**<0.001**
Gender	4.79	12.49	0.701	-3.64	1.39	**0.009**	0.25	12.24	0.984
RE2	-7.68	3.59	**0.032**	-7.57	3.58	**0.035**	-9.66	3.56	**0.007**
RE3	-1.04	2.64	0.694	-1.17	2.64	0.656	-2.30	2.60	0.376
RE4	2.21	3.38	0.512	2.43	3.37	0.471	-1.12	3.35	0.737
Treatment × MBS	-0.22	0.18	0.218				-0.19	0.17	0.287
Treatment × Gender	-23.97	16.74	0.152				-20.32	16.38	0.215
MBS × Gender	-0.13	0.18	0.479				-0.07	0.18	0.707
(Treatment × MBS) × Gender	0.37	0.25	0.136				0.30	0.24	0.213
SecPair2							-0.74	3.62	0.839
SecPair3							2.55	3.61	0.481
SecPair4							-12.79	3.90	**0.001**
SecPair5							-6.54	3.79	0.084
SecPair6							-0.93	3.65	0.799
SecPair7							4.97	4.23	0.240
SecPair8							-16.92	4.61	**<0.001**
SecPair9							-0.59	3.57	0.868
SecPair10							-3.00	3.43	0.382
SecPair11							2.19	3.54	0.536
SecPair12							-9.44	3.59	**0.008**
SecPair13							-5.19	3.37	0.123
SecPair14							2.24	3.78	0.553
SecPair15							-1.07	4.68	0.819
SecPair16							-4.27	4.97	0.390
Obs		671			671			671	
R^2		0.331			0.328			0.384	
AIC		5792.7			5787.4			5767.6	
logLik		-2884.3			-2885.7			-2856.8	

Using these FEMs we can also then estimate the effect sizes for the fixed effects, shown in Table 2.

Table 2. Effect Sizes for Fixed Effects in Interacting Model

Parameter	**Partial Cohens *f***	**95% CI low**	**95% CI high**
Treatment	0.45807	0.37679	0.539
MBS	0.54491	0.46202	0.627
Gender	0.10635	0.0287	0.184
RE	0.12148	0.00737	0.189
SecPair	0.29389	0.16633	0.338
Treatment:MBS	0.00917	0	0.079
Treatment:Gender	0.00445	0	0.063
MBS:Gender	0.03243	0	0.11
Treatment:MBS:Gender	0.049	0	0.126

Computing a mixed effects model incorporating interactions and random effects due to section (*Section, SecPair*) and teacher level (*TIDw*) as in (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023) yields the coefficients shown in Table 3.

Table 3. Mixed Effects Model Coefficients with and without Interactions.

	Model With Interactions			**No Interactions**		
Predictors	*Estimates*	*std. Error*	*p*	*Estimates*	*std. Error*	*p*
(Intercept)	-2.80	8.98	0.755	0.67	5.14	0.896
Treatment [TR]	28.07	11.71	**0.017**	15.76	2.78	**<0.001**
MBS	0.83	0.13	**<0.001**	0.78	0.06	**<0.001**
Gender [F]	0.09	12.02	0.994	-3.72	1.35	**0.006**
RE [2]	-9.42	3.49	**0.007**	-9.30	3.48	**0.008**
RE [3]	-2.18	2.55	0.395	-2.26	2.55	0.376
RE [4]	-0.51	3.28	0.877	-0.29	3.27	0.930
Treatment × MBS	-0.18	0.17	0.285			
Treatment × Gender	-18.83	16.09	0.242			
MBS × Gender	-0.05	0.18	0.769			
(Treatment × MBS) × Gender	0.28	0.24	0.247			
Random Effects						
σ^2	289.50			288.68		
τ_{00}	17.14 $_{Section}$			17.46 $_{Section}$		
	13.67 $_{TIDw}$			13.57 $_{TIDw}$		
	12.62 $_{SecPair}$			12.45 $_{SecPair}$		
ICC	0.13			0.13		
N	16 $_{SecPair}$			16 $_{SecPair}$		
	32 $_{Section}$			32 $_{Section}$		

	19 TIDw	19 TIDw
Observations	671	671
Marginal R^2 / Conditional R^2	0.304 / 0.395	0.303 / 0.394
AIC	5754.594	5746.831
log-Likelihood	-2862.297	-2862.415

Computing effect sizes within a MEM is more complicated and in general must be approximated (Hedges, 2009). The effect sizes for each section pair were computed and shown in Table 4, and the precision weighted effect size (DerSimonian & Laird, 1986) for the Treatment effect computed from those using the standard error for each.

Table 4. Effect Sizes for Each Section Level Pair of Treatment and Control groups

Section Pair	Hedges *g*	95% Confidence Interval lower	upper
1	1.097	0.483	1.748
2	-0.501	-1.094	0.076
3	1.003	0.400	1.639
4	2.221	1.412	3.127
5	-0.259	-0.903	0.375
6	0.569	-0.001	1.158
7	0.200	-0.591	1.004
8	1.662	0.655	2.800
9	0.673	0.104	1.262
10	0.993	0.453	1.560
11	0.932	0.362	1.530
12	0.899	0.309	1.518
13	1.226	0.699	1.782
14	1.021	0.375	1.706
15	0.740	-0.185	1.727
16	0.808	-0.142	1.828

In some cases, the variance of the outcome data due to the number of data points in some pairs increases the error and uncertainty in the outcomes for that cluster. This pattern is reflected in a forest plot of the effect sizes (Hedges' *g*) computed within the section pair clusters as shown in Figure 1.

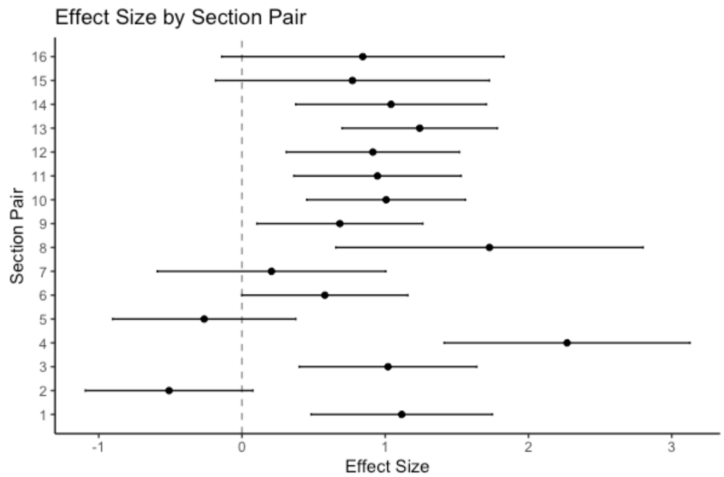

Figure 1. Forest Plot of Section Pair Effect Sizes

Of the 16 pairs, twelve have statistically significant increases in outcome for the Treatment section, and four do not. Two of the four with indeterminate outcomes are positive effects, and two are negative. The precision weighted effect size in this case is d_{pw}= 0.8008 (CI [0.4589, 1.1427], t=4.99 p=0.0002).

Discussion

The fixed effects model indicates that the treatment condition is statistically significant even in the presence of the other fixed effects. Specifically, the models show that the treatment is statistically significant and positive even in the model with interactions (β = 30.48, 95% CI [7.30, 53.65], t(660) = 2.58, p = 0.010) when controlling for mathematics background, race/ethnicity, and gender. The FEM interaction terms are found to be insignificant (the null hypothesis of zero-valued coefficients cannot be rejected). In addition, the model without interactions is found to have a better fit (AIC = 5787.4) than the model with interactions (AIC = 5792.7), but the model with interactions and the SecPair cluster levels is best (AIC = 5767.6) with a coefficient of β =28.91 which is almost equivalent to the non-interacting estimate. This model compares well to the mixed effects model that includes random effects from the *SecPair*, *Section* and *TIDw* clusters. In that model, fixed effects were found to explain 39.4% (with marginal R^2=0.303) of the outcome while the FEM was found to explain 38.4% (R^2 = 0.384) of the outcome with interactions and SecPair cluster levels. The MEM is perhaps preferred since it incorporates the impact of student choice of section and the presence of the instructor, but this analysis shows that these factors explain approximately ~10% (conditional R^2=0.1) of the outcome in this data. In determining the treatment effect on outcomes, the FEM gives a similar model for interpretation even though the MEM is a slightly better fit. One benefit of the MEM approach, however, is that it also characterizes the intraclass correlation (ICC = 0.13) for these underlying clusters and provides some insight into the degree to which those correlations might impact the outcomes. In (Kramer, Fuller, Watson, Castillo, Oliva, et al., 2023), the fixed effect Cohen's d = 0.774 (CI [0.618,0.930]), and cluster adjusted effect size from the random effects of the MEM d_T =0.771, CI [0.468,1.073] are consistent with the metanalytic approach here using only the section pair data.

The coefficient similarities between the linear FEM and the MEM with random effects in some sense reflect the fact that the data form distinctly linear structures when modeled on the strongest effect, *MBS*. As shown in Figure 2, a LOESS curve approximation of the data plotted within a scatterplot of *LearningMeasure* against *MBS*, we see that the majority of the curve is linear in nature with low levels of uncertainty, and that only in the outlier regions where few data points exist do the curves show high levels of non-linearity and uncertainty.

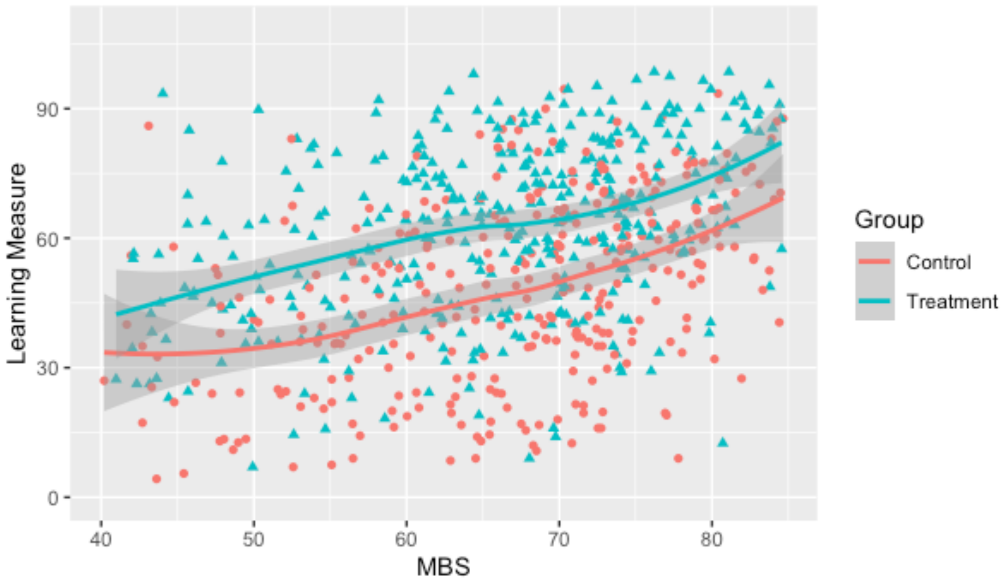

Figure 2. Scatterplot of LearningOutcome Against MBS (Math Background Score) for Treatment and Control Groups with LOESS Curve Approximations. Dark Grey Areas Indicate the 95% CI Surrounding the Approximating Curve

Conclusions

In the data analyzed in this article, the underlying cluster structure has a relationship to the outcome variable variance but this explanatory power is small when compared to the other fixed and random effects. The impact of the time of day/day of week of the courses was on the order of 50% of the effect found for *Treatment* and *MBS* in the FEM. The estimates for the coefficients of *Treatment* in all the models are similar and the computed effect sizes are consistent using either model. The MEM approach has the advantage of estimating the relative variance due to cluster levels in a way that represents their underlying random nature, while incorporating *SecPair* into the FEM approach yields similar results. Both analyses support the conclusion that the treatment condition led to medium to large increases in student learning outcomes.

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References

Castillo, A., Duran, P., Fuller, E., Watson, C., Potvin, G., & Kramer, L. H. (2022). Student attitudes in an innovative active learning approach in calculus. *International Journal of Mathematical Education in Science and Technology*, 1–29. https://doi.org/10.1080/0020739X.2021.2023771

DerSimonian, R., & Laird, N. (1986). Meta-analysis in clinical trials. *Controlled Clinical Trials*, *7*(3), 177–188.

Duran, P. A., Castillo, A. J., Watson, C., Fuller, E., Potvin, G., & Kramer, L. H. (2022). Student attitudes and achievement in active learning calculus. *International Journal of Mathematical Education in Science and Technology*, 1–22.

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 Times More Likely to Leave STEM Pipeline after Calculus Compared to Men: Lack of Mathematical Confidence a Potential Culprit. *PLOS ONE*, *11*(7), e0157447. https://doi.org/10.1371/journal.pone.0157447

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Hedges, L. V. (2009). Effect sizes in nested designs. *The Handbook of Research Synthesis and Meta-Analysis*, *2*, 399–416.

Kramer, L., Fuller, E., Watson, C., Castillo, A., Duran Oliva, P., & Potvin, G. (2023). *Establishing a New Standard of Care for Calculus Using Trials with Randomized Student Allocation* (10.5061/dryad.kkwh70s95; v3.0) [dataset]. Dryad; Dryad. https://doi.org/10.5061/dryad.kkwh70s95

Kramer, L., Fuller, E., Watson, C., Castillo, A., Oliva, P. D., & Potvin, G. (2023). Establishing a new standard of care for calculus using trials with randomized student allocation. *Science*, *381*(6661), 995–998. https://doi.org/10.1126/science.ade9803

Kuznetsova, A., Brockhoff, P. B., & Christensen, R. H. (2017). lmerTest package: Tests in linear mixed effects models. *Journal of Statistical Software*, *82*, 1–26. https://doi.org/doi:10.18637/jss.v082.i13

R Core Team. (n.d.). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing. https://www.R-project.org

Dynamical Systems and Mathematical Practices for Future Secondary Teachers

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The relevance of upper division mathematics courses for future secondary teachers is a longstanding thorny issue. Suggested improvements include capstone courses and revised upper division content courses to explicitly address future teachers' relevant secondary mathematics content knowledge, beliefs about teaching and learning, and experience with learning mathematics while engaging in authentic mathematical practices. In this report, we investigate prospective teachers' reflections on their opportunities in an upper division Inquiry-Oriented Dynamical Systems course to engage in the eight Common Core State Standards for Mathematical Practice. Analysis of students' self-reported engagement in the eight Practices revealed five practices that strongly resonated with them and the various ways that their experiences in an inquiry-oriented classroom supported meaningful and powerful engagement in these Mathematical Practices. We conclude with implications for practice.

Keywords: mathematical practices, inquiry, prospective teachers, dynamical systems

The relevance and usefulness of upper division mathematics courses for prospective secondary school mathematics teachers has long been of concern (Begle, 1972; Klein, 1932; 2016; Wasserman et al., 2019). A number of studies document that teachers find their advanced mathematics courses have little relevance to their teaching (e.g., Cofer, 2015; Wasserman, 2017, Zazkis & Leikin, 2010). While these challenges are longstanding and pervasive, professional organizations have outlined possibilities for improving the connection between university and secondary mathematics for prospective teachers (Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of the Mathematical Sciences [CBMS], 2001; 2012). Creating capstone courses is one approach. Another recommendation, and the one taken in our work, is to redesign upper division math content courses so intentionally strong connections to high school mathematics content and teaching are made.

In recent years progress has been made on the university-secondary mathematics connection. For example, a recent issue of *ZDM Mathematics Education* focuses on how the intersectional nature of mathematical and mathematics educational content might be addressed in a wide range of university courses in order to prepare better secondary mathematics teachers (Wasserman et al., 2023). We contribute to this uptick in progress by investigating the opportunities for prospective teachers in an Inquiry-Oriented Dynamical Systems and Modeling (IODSM) course to engage in the eight Common Core State Standards for Mathematical Practice (MP) (Common Core State Standards Initiative [CCSSI], 2010). The eight Standards are: 1) Make sense of problems and persevere in solving them, 2) Reason abstractly and quantitatively, 3) Construct viable arguments and critique the reasoning of others, 4) Model with mathematics, 5) Use appropriate tools strategically, 6) Attend to precision, 7) Look for and make use of structure, and 8) Look for and express regularity in repeated reasoning. In particular, we address the following research question: *How frequently do prospective teachers in an IODSM course report engaging in the eight Standards for MP and how do they describe their engagement in these Standards?*

In related prior work, Apkarian et al. (2023) investigated the impact of an IODSM course on prospective teachers' knowledge of rate of change, their shifting beliefs about learning and teaching, and their self-reported ways in which their emerging beliefs and knowledge would influence their future practice. The work reported here adds a new dimension to this prior work by examining IODSM student-reported connections to the eight Standards for MP.

Theoretical Background

The "inquiry" part of the IODSM course is heavily influenced by the following four pillars of inquiry described by Laursen & Rasmussen (2019): 1) Students engage deeply with coherent and meaningful mathematical tasks, 2) Students collaboratively process mathematical ideas, 3) Instructors inquire into student thinking, and 4) Instructors foster equity in their design and facilitation choices. In the IODSM course, students collaboratively reinvent mathematics by engaging in the kind of work that potentially reflects how mathematicians go about their work and which are embodied in several of the Standards for MP. Our use of reinvention is informed by the instructional design theory of Realistic Mathematics Education (RME), which views mathematical concepts, structures, and ideas as inventions that humans create to organize the phenomena of the physical, social, and mental world (Freudenthal, 1973).

Our work is also informed by the emergent perspective (Cobb & Yackel, 1996), which views learning as both an individual and social process. Of particular relevance for this report are the emergent perspective's constructs of social and sociomathematical norms. Social norms refer to regularities in discourse, such as students routinely explaining their own thinking, listening to and attempting to make sense of others' thinking, asking questions if something is unclear, and indicating their agreement or disagreement with reasons. We conjecture that social norms have considerable overlap with the first and third MPs (Make sense of problems and persevere in solving them and Construct viable arguments and critique the reasoning of others). Also related to these Standards is the sociomathematical norm that justifications be based on underlying concepts as opposed to appeals to procedures or external authorities such as the text or instructor. This particular norm may, for example, relate to the third and sixth Standards for MPs.

In this report we do not examine actual classroom interactions and hence the full power and full set of constructs of the emergent perspective cannot be leveraged. Instead, we use the constructs of social and sociomathematical norms to reflect on the extent to which students' report how often they engage in the various Standards and the nature of that engagement.

Methods

The participants were 30 students enrolled in an upper division IODSM course at a large, Hispanic-serving institution in the southwestern United States. This course fulfills an upper division math elective requirement and it was designed specifically for prospective secondary teachers by infusing content related to high school mathematics. We collected qualitative data from a survey taken at the beginning of the semester, a detailed homework assignment where students explored the Standards for MP (CCSSI, 2010) and their connection to their experiences in the IODSM course, and an hour-long interview with a subset of students. In this report, we only discuss findings from their written homework assignment.

On this homework assignment, students reflected on how their IODSM classroom experiences relate to the Standards for MP. Students were asked to read the eight practices and categorize each practice into one of three bins based on how often they experienced the MP in class, and to explain why they placed each practice into the Bin that they did. Bin 1 was the most opportunities, Bin 2 was some opportunities, and Bin 3 was the least opportunities. Students also provided an example from classwork to support their justification. The assignment provided insight into students' understanding and engagement with the Standards for MP. It allowed for an in-depth exploration of students' perceptions and experiences, contributing valuable qualitative data to the field. By linking their experiences to specific Standards, the students provided a rich and detailed view of their interaction with the mathematical concepts laid out in the Standards, helping us gain a nuanced understanding of their learning experiences. We note that the Standards for MP were never discussed in class. Therefore, the responses from students reflect their own interpretation of the MPs.

To analyze the data, we used a thematic analysis approach, as described by Braun and Clarke (2006), to identify, analyze, and interpret patterns within the data. Students' responses were separated by MP into a spreadsheet that included their bin classification, justification, and the example. Note, if students mentioned uncertainty in placing an MP between two bins, we coded them as the less often bin. This happened only two times and both were deciding between Bin 1 and 2; and thus, they were placed in Bin 2. After data were organized, two researchers read students' responses and took notes according to their interpretation of the students' explanation to identify interesting aspects and patterns. Then all researchers met to discuss meaningful ways to organize and code the data. We discussed interesting trends students demonstrated as a response to their classification of bins and possible explanations for them. Initial codes were created for each MP and highlighted specific aspects of the MP description. Common themes helped to identify what parts of the MP students considered to experience the most in the class and why they placed each practice in the corresponding bin. After collapsing overlapping themes, re-working and refining codes, all authors agreed on the coding of all of the MPs. Lastly, we found the frequency of common themes. We also calculated the standard deviation for each practice. This allowed us to see which practices students agreed on more about engaging in and those that they did not. In this report, we focus only on practices where there was more agreement (low standard deviation) or modest agreement (medium standard deviation).

Results

The average number of the bin placement for each mathematical practice ranged from 1.032 to 2.000 (i.e., for some practices nearly all students selected Bin 1 [the most often bin] while for some practices the average selection was Bin 2 [the second most often bin]). Standard deviations were between 0.1796 and 0.7878. Three natural groupings of the practices emerged by standard deviation (SD), with three practices having SD less than 0.5 (most agreement), two practices with SD between 0.5 and 0.75 (modest agreement) and three practices with SD between 0.75 and 1 (least agreement). As mentioned, we only report on the practices with the most or modest agreement. We hypothesize that practices that received a higher SD can be partly attributed to the ambiguity of wording of the practices and/or students' personal interpretations. Table 1 lists the mathematical practices by SD group, as well as the themes identified across student responses and the frequency of occurrence for each theme. Only themes that were identified in the responses of at least one third of the class are listed in Table 1.

Practices with Most Agreement (Low Standard Deviation)

MP 1. The first theme for MP 1 was *highlighting the emphasis on making sense of problems* which was when students mentioned the importance of taking a step back to read and understand the problem to make sense of things. The second theme was *the importance of working hard and* *not giving up* which was when students reflected on attempting problems and their perseverance towards the correct outcome. This student response exemplifies making sense of problems:

> ... for a lot of problems in the class, you just cannot look at the given information and just solve for one variable, and that's it. There has to be more meaning to it, what exactly is the solution we are looking for, how do we work for the problem in question, what can we do and then we can attempt a method and change up our attempt if needed.

The students addressed the importance of making sense of a problem by stating that when working on a problem their solution approach cannot be deciphered by just looking at "one variable" or applying a single technique and solving for it, instead they dive further into finding meaning and value in the text, and only then, continue working towards a solution. That is, students in the class described that solving problems requires more than purely manipulating variables, they said that the solution has to make logical sense and they need to question themselves continually to check if their solution approach is right.

Students' responses coded as *importance of working hard and not giving up* referred to the importance of being perseverant when solving a problem and if they committed a mistake that they could step back and try a different approach. For example,

> Some of the problems that I encounter in class can be particularly challenging and require a great deal of patience and perseverance to solve. It is important to remain focused and persistent in working through these problems, breaking them down into smaller parts and utilizing any available resources or strategies to find a solution.

Students also described that due to the nature of the course more challenging problems were constantly being encountered which forced them to persevere in working out solutions. Also, students recognize that mathematics is not about getting the correct answer at the first try, but that it is important to persevere and change methods as necessary.

Table 1. *Student themes from Mathematical Practices.*

SD	**Mathematical Practice**	**Themes Identified**	**Freq.**
Low	(MP1) Make sense of problems and persevere in solving them	*Highlights the emphasis on making sense of problems*	30
		Importance of working hard and not giving up	11
	(MP2) Reason abstractly and quantitatively	*Contextualization or decontextualization*	20
		Doing mathematics with meaning	16
	(MP4) Model with mathematics	*Mathematics applications*	16
		Mathematical techniques	10
Medium	(MP3) Construct viable arguments and critique the reasoning of others	*Critiquing and revision of ideas*	22
		Group work and collaborative learning	22
		Constructing Arguments and Justifying Answers	13
	(MP6) Attend to precision	*Precision in communication ideas*	28
		Group work and collaborative learning	13

MP 2. Themes identified for MP 2 included *contextualization or decontextualization* and *doing mathematics with meaning.* An example student response for the first theme is "A lot of times we are given problems with context, then we create mathematical models of the scenarios, solve the problem using math, then relate our solutions back to the context/scenario" which highlights the transition between abstract mathematics and contextual understanding of the problem and vice versa. Additionally, some students talked about giving meaning to the variables when working with equations and functions in class and such responses were coded under the second theme of *doing mathematics with meaning*. For example, a student reported,

> This standard most directly applies to our use of putting differential equations into words. We explored each piece of a differential equation individually, and then we worked on putting a differential equation into a sentence. We spoke of a differential equation with meaning, rather than speaking out the signs, numbers, and letters as they are. We have used these statements of meaning in most of our work in this class thus far.

In the excerpt above the student mentioned that it is important to be aware of the symbolic representations of models in regard to the context, emphasizing the need to provide meaning to them. The student expressed the necessity to read equations with intention and comprehension. In addition, students described their experience in the class with differential equations as translating them into meaningful sentences rather than mere symbol recitation, which exemplifies the practice's aim to ensure students can contextualize symbols and equations in real-world problems.

MP 4. The fourth practice revealed that students were seeing differential equations as a powerful tool to model real world phenomena and the two main themes identified were *mathematics applications* and *mathematics techniques.* The first theme was when students mentioned real-world scenarios in which mathematical concepts were applied. For example,

> …there have been many times in the class where our problem is a model of a real-world situation, with examples of the salty tank problem, the helicopter problem, and the list problem just to name a few. I also think that we have been given the opportunity to think about whether the model fits the situation as for example when we were asked if the model of the fish population matched the mathematical model…

Student responses highlighted the application of differential equations to model real-world systems, which aligns with the practice's focus on using mathematical knowledge to address real-life situations. The second theme in MP4 was the use of *mathematical techniques* in facilitating problem solving. Responses which discussed the use of Euler's Method (or as we called it, the Tip-to-Tail method), graphs, slope fields, phase lines, and other math techniques were coded in this theme.

Practices with Modest Agreement (Medium Standard Deviation)

MP 3. Three main themes were identified for the third practice. These included *critiquing and revision of ideas, group work and collaborative learning and constructing arguments and justifying answers.* For the first theme students acknowledged the importance of peer review and critiquing of ideas when developing their understanding of mathematics and underscored the value of the collaborative nature of the learning experience. For example,

> ... We are asked to come up with our own explanations for certain answers or approaches. We have to share them with the rest of the class and also take critiques if others don't agree until everyone has a satisfactory answer and explanation. Many examples where groups would have to come up with a graph tend to cause some disagreement thus leading to more discussion and ultimately to a well-backed understanding.

The student described the active engagement in constructing explanations, presenting them to peers, and refining them through critique until a shared understanding is achieved. This not only embodies MP 3's focus on constructing arguments and critiquing reasoning but also echoes the collaborative nature of mathematical exploration emphasized in MP 3. Other students' described the collaborative nature of the course. The next most common theme for MP 3 was *group work and collaborative learning* which was attributed to responses mentioning the value of discussing problems with classmates, sharing ideas, and collectively analyzing and working on problems. The following excerpt by students mentions the use of group work daily in class which leads to discussions and support from teammates:

> Since this class consists of almost entirely group work, constructing clear and viable arguments is crucial as we are always explaining our thought process to everyone else in our group, and sometimes the rest of the class. In order to do this successfully we must fully understand what we are doing and be able to explain why each step was made … This class is a team sport, and that quality of respect is crucial as we are all in a learning environment where mistakes are welcomed as long as we work through them together and help each other out along the way…

The student above describes daily group work and emphasizes the process of understanding and exploring mathematical problems and the "team sport" reference signifies the collaborative nature of learning in class. Finally, the third theme was attributed to students who emphasized reasoning and justifying their approach to problems. For example, "although critiquing the reasoning of others is very much one of the more common things that happens during class, the construction of arguments is done in a way more informal which is the reason it is in bin 2". Other student responses expressed this sentiment, or the feeling that there was not always enough time available in class to construct careful arguments.

MP 6. Two main themes emerged for the sixth practice including *precision in communicating ideas and group work and collaborative learning*. Note the second theme around group work also emerged for MP 3. The first theme was attributed to responses which emphasized the clarity and exactness in mathematical communication, including a need to refine mathematical language and avoiding vague terms. For example,

> ....I feel that I have had the most opportunity in this class to engage in this mathematical practice because communicating in a precise manner underpins all the work that we do in this class. Whenever I ask a question, present a result, or draw a graph, I strive to be accurate with my spoken words and written statements….

The students who brought up this theme addressed the importance of communicating mathematical ideas and concepts to others with precise language which relates to the second theme identified for *group work and collaborative learning*. For example,

> Communication is a huge part of this class. Group work and class discussion is what makes this class impactful. If we were to do things on our own all the time, there is a low chance that if we were to get something wrong, we'd understand why and how to find the right answer. Communication with others keeps each individual on track when it comes to using the correct definitions and meanings, symbols, math processes.

Students who mentioned group work and collaborative learning in relation to MP 6 mainly pointed out communication with others and being precise in their language in doing so to get their ideas across. Additionally, students mentioned the importance of working with others as time to get constructive criticism on delivering their ideas.

Conclusion

The AMTE (2017) Standards state that effective mathematics teacher preparation programs should provide opportunities for prospective teachers to learn mathematics that enable them to engage in mathematical practices, and that mathematics content should be taught using teaching methods that serve as models of effective teaching (AMTE, 2017). Consistent with this call, we investigated prospective teachers' reflections on their opportunities in an upper division inquiry-oriented mathematics course to engage in the eight Common Core State Standards for Mathematical Practice. We found that students' self-reported engagement centered five practices (MP 1, 2, 3, 4, 6) as strongly resonating with them. There were three practices (MP 5, 7, 8) that had higher standard deviations in terms of which bins students placed them in. This meant there was not as much agreement on how these practices were reflected, from the students' points of view, in class. We posit high SD may have been because these practices appeared in the latter half of the assignment (so perhaps not read as carefully) and/or that students may not have understood aspects of the educational terms in these practices. Recall that the Standards were never discussed in class.

Collaboratively processing ideas showed up in more than practice (MP3 and 6). This relates to social norms (Cobb & Yackel, 1996) in that central to students engagement in class was the time to collaborative process ideas. Students discussed how sometimes mathematical concepts did not make sense to them until another student explained something or provided more information. Relatedly was the concept of being precise with language. A sociomathematical norm in the class was speaking with meaning (e.g., avoid saying "it"). Being precise in language is not only critical as an MP but also important for future teachers to be precise in their language when engaging with their future students. Lastly, the idea of critiquing was often discussed. Importantly, some students took a negative connotation to critiquing in that they argued that they did not *critique* but they went back and forth discussing mathematics until concepts were agreed upon. Whereas some students fully embraced what it means to critique in their IODSM class. To them, it was important to critique because it meant that ideas were only getting better when the class critiqued reasoning to improve upon said reasoning.

Our next steps are to analyze the interviews already conducted which investigated student perceptions of the MPs in deeper detail, their beliefs about learning and teaching mathematics, and the connections they made between the upper division college mathematics content and secondary school mathematics. We also intend to conduct additional iterations of this course, at the same and different universities, expanding our focus to approximations of practice.

References

Apkarian, N., Habre, S., LaTona-Tequida, T., & Rasmussen, C. (2023). Prospective secondary teachers' emergent knowledge and beliefs: Inquiry-oriented differential equations contributing to teacher preparation. *ZDM Mathematics Education, 55,* 823-835. https://doi.org/10.1007/s11858-023-01469-4

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at https://amte.net/standards.

Begle, E. G. (1972). *Teacher knowledge and pupil achievement in algebra* (NLSMA technical report number 9). Stanford University, School Mathematics Study Group.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, *3*(2), 77–101.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3-4), 175–190.

Cofer, T. (2015). Mathematical explanatory strategies employed by prospective secondary teachers. *International Journal of Research in Undergraduate Mathematics Education, 1*(1), 63–90.

Common Core State Standards Initiative [CCSSI]. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers* (Vol. 11). American Mathematical Society.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II* (Vol. 17). American Mathematical Society.

Freudenthal, H. (1973). *Mathematics as an educational task.* Reidel.

Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (trans. Hedrick, E. R. & Noble, C. A.). Macmillan.

Klein, F. (2016). *Elementary mathematics from a higher standpoint: Volume I: Arithmetic, algebra, analysis*. Springer.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 129-146.

Wasserman, N. (2017). Making sense of abstract algebra: Exploring secondary teachers' understanding of inverse functions in relation to its group structure. *Mathematical Thinking and Learning, 19*(3), 181–201.

Wasserman, N., Buchbinder, O., & Buchholtz, N. (2023). Making university mathematics matter for secondary teacher preparation. *ZDM Mathematics Education*.

Wasserman, N., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*, *22*, 379– 406. https://doi.org/10.1007/s10857-019-09431-6

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263–281.

Networking Theories to Investigate Status and (In)equities in Small Group Proof Contexts

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Efforts have been made to study (in)equity in undergraduate mathematics education research. Across various fields, there is a foundation of work on how status impacts students' learning and participation in small groups as well as how differing patterns of interaction contribute to inequitable outcomes. This report contributes a networked theory for analyzing relationships between status and (in)equity in general. We argue that applying this methodology to proof-specific contexts has the potential to uncover how status hierarchies form in proof classrooms using group work components. We hope to promote conversations within the RUME community around taking actionable steps towards delegitimizing status hierarchies in proof classrooms.

Keywords: Status, Inequity, Small Group, Proof

Generally, increasing student-student interactions in classrooms changes the quality of interactions, creating more opportunities for societal narratives about who can do what kind of mathematics to influence who participates and how (Battey & McMichael, 2021). A dominant societal narrative is that 'doing mathematics' reflects 'doing masculinity' (Jaremus et al., 2020; Leyva et al., 2017; Mendick, 2006) and simultaneously privileges whiteness (Battey & Leyva, 2016; Martin, 2019) and Eurocentric perspectives (Rowlands & Carson, 2002). Scholars have argued that students who identify with the dominant culture of mathematics inherit this privileged status, particularly in advanced proof-oriented classrooms (Weber & Melhuish, 2022). The power that these narratives have can be corroborated by the fact that advanced courses are predominantly taught by (mostly white) men and the majority of students who take them are men (Blair et al. 2013). If a goal is to increase student-student interactions in these spaces (Saxe & Braddy 2015; the MAA Instructional Practices Guide, 2018), then it stands to reason that issues of inequities related to status and power must be considered.

Scholars have recently attended to status and power in proof-based contexts (e.g., Brown, 2018; Ellis & Alzaga Elizondo, 2023; Hicks et al. 2021). Adapting an authority framework, Hicks et al. (2021) explored how four students' mathematical authority was distributed while working on an abstract algebra task. They found discrepancies in authority relations and offered reasons why these discrepancies might have occurred, such as only some students authoring ideas in public spaces, students self-selecting to not participate, and assessing ideas to give or take away authority. Drawing on Shah and Lewis's (2019) work, Ellis and Alzaga Elizondo (2023) investigated how status attenuated/amplified (in)equities during small group work in an intro-to-proof course. Using relational and participatory equity as an analytic lens, they claimed that a 'less mathematically collaborative' episode exhibited a more balanced status relationship while a 'more collaborative' episode exhibited strained power relations. While intriguing, these claims could have been strengthened by a clearer theoretical connection between status, power, and inequity, which guided the empirical analysis of their data.

This report contributes a possible theoretical approach to empirically identify how status hierarchies form in small group interactions. That is, we claim that networking positioning theory (Harré & van Langenhove, 1999) with systemic functional linguistics (SFL; Halliday, 1978) may provide empirical evidence of discursive processes by which students attribute academic status to themselves and others. This approach may yield stronger claims regarding how inequities are attenuated/amplified in small groups (Shah & Lewis, 2019).

Background

In classrooms where students interact with each other regularly, issues related to status and positioning are more likely to occur (Cohen & Lotan, 2014; Esmonde, 2009b; Shah & Lewis, 2019). In general, 'status' refers to the idea that it is desirable to be in a higher position relative to another (Cohen et al., 1999; Ridgeway, 2018). For example, narratives around socioeconomic status allude to the idea that it is better to have more monetary capital than less, and people generally agree that is advantageous to be in a higher status position than a lower status position. In classroom contexts, academic and peer status greatly influence perceptions of where individuals fall along a status continuum (Cohen & Lotan, 2014). The former relates to perceptions of who is 'smart' or doing well in the class. The latter relates to perceptions of social standing (i.e., popularity, attractiveness). Then, 'diffuse status characteristics' refer to identity markers that are perceptible upon initial encounters with others, such as race (via skin color) or language use (via intonations or accents), gender expression, and certain forms of ability status. For our purposes, when we say "status" we mean academic status while acknowledging that peer and diffuse status characteristics influence attributions of academic ability or competency in proof classrooms.

Researchers studying student-student interactions during collaborative small group activities have made theoretical connections between status and positioning (Langer-Osuna, 2016; Esmonde, 2009a; Shah & Lewis, 2019). For example, Lange-Osuna (2016) argued that "students interactionally position themselves and one another with academic and social power that can affect collaborative mathematical work" (p. 108). To examine authority relations, they asserted that students who are positioned with more intellectual authority – that is, students positioned as valid sources of information directly related to the current task – accumulated more influence, and thus, academic status. In their coding scheme, influence was attributed to a student whenever their idea was "positioned as having become part of (or rejected from) the solution path" (p. 112). This aligns with our notion of influence as the combined result of being given opportunities to contribute that are subsequently evaluated positively by others.

An indirect link can be made between status and research on students' participation in groups since identity markers, such as gender, have been shown to influence participation (Ernest et al. 2019; Langer-Osuna, 2011; Reinholz et al., 2022). For instance, Ernest et al. (2019) found that men and women participated at relatively similar rates in private group talk, yet men dominated the public space in class discussions. Likewise, investigating gender roles in small groups using mixed methods, Langer-Osuna (2011) showed how Brianna's project-related conversation declined overtime and self-perception went from "good leadership" to "being bossy" while Kofi's project-related conversation increased, and he perceived himself as the "smart" student in the group.

At the undergraduate level, quantitative studies have pointed to gendered disparities in performance on proof-based tasks. Johnson et al. (2020) reported findings that indicated men fairing significantly better in inquiry-oriented proof classes compared to men in traditional classes, while there were no significant differences in performance for women. Reinholz et al.

(2022) built on this analysis by studying participation patterns, finding that performance disparities could be attributed to women's participation rates. We argue that there is a need to provide qualitative explanations for such results, and one possible direction is to explore how status hierarchies are de/legitimated (Adams-Wiggins et al., 2020) in proof contexts where small group work occurs, such as in inquiry-oriented classrooms.

Networking Theories to Evidence Status Relations

This report contributes a possible theory to analyze student-student interactions mediated by discourse – conceptualized as language-in-action used to communicate meaning, including written, verbal, gestural, and other forms of communication. We contend that networking positioning theory with SFL may provide useful analytic tools to document how status hierarchies form in small group interactions. With such evidence as a guide, we argue that situations in which status hierarchies are legitimated likely amplify inequities, while situations where status hierarchies are delegitimated likely attenuate inequities, with *in*equities defined as situations that prevent access to resources needed for learning (Shah & Lewis, 2019). It is worth noting that because human interactions and power relations fluctuate based on available positions, no situation will ever be 'status-free' or fully 'equitable'.

Positioning Theory

Positioning theory explains the processes underlying how participants in an interaction attribute rights and obligations to themselves and others (Harré & van Langenhove, 1999; Harré, 2012). Relationships between communication acts (i.e., the meaning embedded in speech and other forms of communication) and storylines (i.e., accepted sociocultural repertoires for how to interact in a situation) are what give rise to available positions or 'rights and duties' (Herbel-Eisenmann et al., 2015). In an educational setting, for example, a traditionally accepted teacher-student storyline is one where teachers are the authority and students are obligated to do what the teacher says. Such a storyline may be evidenced in the communication acts between participants; perhaps the teacher issues a command to students (e.g., "Please get out your notebooks and a pencil") and students accept the obligation by getting out their notebooks and a pencil. In positioning theory, when communication acts and storylines work together to make rights and duties available to participants, there is always a choice to accept, negotiate, or reject the positioning. For example, in the teacher-student scenario, perhaps a student does not get out a notebook or pencil. This could be interpreted in multiple ways; maybe they do not have the materials with them or perhaps they are rejecting the command intentionally.

Broad narratives about mathematics and who is perceived to belong to mathematics culture can operate as storylines associated with mathematics classrooms that students are likely aware of. For instance, mathematics culture is widely perceived as Eurocentric, white, and dominated by men (Battey & Leyva, 2016; Jaremus et al., 2020; Leyva et al., 2017; Martin, 2019; Mendick, 2006; Rowlands & Carson, 2002), which positions students who identify with this culture with higher status in mathematical spaces relative to others. A problematic exception is that students are well aware of the societal narrative that Asians are "good at math," which perpetuates the "model minority myth" (Poon et al., 2015; Shah, 2017). This is problematic since it positions students in a place of higher status (i.e., gifted academically) while also excluding them from communities of other minority students along with associated supports for those communities (Ng et al., 2007; Suzuki, 2002).

In sum, positioning theory provides conceptual grounding for why (and how) status hierarchies form in small group interactions. A limitation is the conceptual vagueness around interpreting the meaning embedded in participants' communication acts and the possible storylines at play (Herbel-Eisenmann et al., 2015). Therefore, we bring in analytic tools from systemic functional linguistics to evidence how positions are created and maintained through participants' discourse.

Systemic Functional Linguistics (SFL)

Broadly, SFL offers a learning theory centered on language use, with language operating as a complex, dynamic, and context-based system (Halliday, 1978). Three metafunctions of language comprise SFL: *interpersonal*, *ideational*, and *textual*. The textual metafunction "manages the flow of information to make extended discourse coherent and cohesive" and the ideational metafunction "constructs ideas and experiences" (Gebhard & Accurso, 2020, p. 1029). The interpersonal metafunction represents socially constructed positions and power structures. In our methodology for studying how status operates to organize interactions in classrooms, we are centrally concerned with the interpersonal metafunction which is mediated by "tenor" choices – resources including (but not limited to) the use of mood systems.

Within the mood system, statements made evoke a declarative mood, questions asked evoke an interrogative mood, and commands issued evoke an imperative mood (Gebhard & Accurso, 2020). Gebhard and Accurso (2020) assert that within the interpersonal metafunction and mood system, textual analysis can evidence how statements, questions, and commands influence social structures and power dynamics, particularly in classroom interactions. For example, such an analysis can capture who has the right to speak versus who remains silent, "who uses statements to construct authoritative 'facts'; who asks questions and engages in negotiating meaning; who gives commands and how commands are taken up or resisted" (p. 1032). For our purposes, the textual artifacts analyzed are transcripts of students interacting during group work.

We argue that the interpersonal metafunction of SFL, including tenor resources such as the mood system, provides concrete evidence of power relations emerging and shifting during group interactions through discourse patterns. We conjecture that documenting these processes can evidence how status hierarchies form in interactions because *who* issues commands, uses statements to convey authoritative 'facts' and maintains the right to speak will likely be perceived as having higher status relative to others in the interaction.

Interaction Process Related to Status Formation

Through the interaction process of creating opportunities to contribute and evaluating contributions, certain students are positioned as having more influence (e.g., Langer-Osuna, 2016). Those who acquire more influence during the interaction will have higher relative status compared to others, ultimately legitimating or increasing status hierarchies within the group. In what follows, we describe this process in the context of proof.

Opportunities to contribute. When assessing *opportunities to contribute* among members of a small group, we examine instances in which the group encounters a problematic situation and consider group members' attempts to offer a resolution, the timing of these offerings, how these offerings are solicited by other group members, and the time and attention members have when responding. For example, a group may need to interpret an assumption when working together to develop a proof. A group member may ask, "Does anyone know what [term] means?" followed by wait time. Such a generic solicitation often confers an opportunity to contribute upon the highest-status members of a group, who tend to experience the least psychological risk when offering a suggestion and will often be the first to respond. Alternatively, a member (or instructor) may ask a specific groupmate, "I remember you said something helpful about [term] before. Can you remind me what you said?" Depending on the relative status between the participants, this may elevate the academic status of the groupmate granted the opportunity to contribute, potentially countering the hierarchical effects of peer status and diffuse status characteristics.

Evaluations of contributions. Any contribution by a group member, unless interrupted by the end of a small-group activity, is implicitly or explicitly evaluated by the group. When analyzing *evaluations of contributions* from group members, we consider both explicit evaluations (immediate verbal and nonverbal responses to the contribution) by other members and implicit markers such as an attempt to reinforce or reconcile the contribution with other prior contributions, a follow-up question or suggestion building on the contribution, or a group moving on without appearing to give serious consideration to the idea suggested.

Students' evaluations may have status implications when they appear to agree or disagree with a peer's contribution. For example, when asked to provide a proof of a proposition about equivalence of two expressions involving set operations, a student in a group might suggest drawing a Venn diagram illustrating each sequence of set operations. Another group member, skeptical of the viability of this approach, might simply say, "[Instructor] said that a Venn diagram isn't a proof," implicitly dismissing the possibility that a diagram might support the group's thinking. Alternatively, they might say, "Can you show how you would use a diagram for this problem?", inviting the student to elaborate on their initial contribution. A third possibility is that a peer might ignore the Venn diagram suggestion entirely and say "We need to assume that x belongs to A and B but not C." We posit that each of these interactions has different implications for the status of the group member who suggested the diagram, both as evidence of their academic status within the group and as potential incremental effect on the student's status within the group and in the class.

Influence. When considering each group member's *influence* in a proof-oriented context, we examine instances in which a group must make a decision about a strategy, validity of a claim or contribution, or about the group's collective focus (such as a decision to move on to a different problem or task). In each such instance, we review group members' verbal and nonverbal communication to determine: (1) whether group members seem to defer to a specific member or subset of the group when making the decision; (2) whether a member's input into the decision is taken up by the group; and (3) whether a member suggests criteria or heuristics for decision-making that ultimately inform the group's actions. As an example of the latter, consider an episode in which a group must prove that a sequence converges to a limit, and a student starts by writing the inequality $|a_n - L| < \varepsilon$. At this point a peer might interject and say that this inequality is what the group must prove, and a proof is not allowed to begin with the statement to be proven. If this persuades the group to move away from this strategy, this would point to the peer's influence over the group's decision making. If on the other hand the student continues manipulating this inequality (perhaps as a strategy to discover a value of N corresponding to ε) and is able to enlist the group into helping, notwithstanding the peer's objection, this might point to the first student's influence. It is important to consider that the direction the group takes in this scenario is not purely a function of how well each group member argues in favor of their strategy; it depends on the group's implicit judgment about each member's propensity to identify and validate potential strategies for developing the proof.

Example of Theory in Action

In this section, we demonstrate the analytic potential of the proposed networked theory. Ellis and Alzaga Elizondo (2023) previously analyzed the role of status in two small-group episodes using the constructs *opportunities to contribute*, *evaluations of contributions*, and *influence over group decisions.* The participants were students in an introduction-to-proofs course taught remotely over Zoom with the second author attending class each day (see Alzaga Elizondo, 2022). Alison, a white woman, became a focal participant since interactions in one episode with Lee (East Asian man) evidenced differing participation patterns compared to an episode with Justin (white man). Based on daily observations of class interactions (including virtual breakout rooms), Lee was perceived to have higher academic status relative to Alison. Both Lee and Justin also benefited from belonging to demographic groups that granted them higher academic status in mathematical spaces.

Questions Support Leveling Academic Status Positions

In the following exchange, Lee and Alison used a balanced combination of statements and questions as they worked on a shared Google doc to prove that group isomorphisms preserve inverses.

Lee: From here can I just jump to like, therefore e_2- therefore phi(e_1) is the identity by definition or is that skipping some steps? (pause) [Question]
Alison: Hold on I'm thinking (pause) [Statement]
Alison: yeah I think that's good. [Statement]
Alison: (reads) "By definition of identity." So phi(e_1) must be the identity in-
Lee: Oh wait, but we're not saying for all H, we have to prove that that's all H. [Statement]
Alison: the identity- What do you mean all h? [Question]
Alison: Oh for the identity for all h? [Question]
Alison: But we've already proved that there is only one identity. [Statement]
Alison: Isn't that in the definition of identity? [Question]
Lee: Yeah, but like this is showing the identity for all these, some elements of H until we show- [Statement]
Lee: I guess we can use onto right? [Question]
Alison: Yeah. (nods affirmatively) Yeah, we probably have to use onto. [Statement]

Overall, the discursive move of asking questions to elicit evaluations of intellectual contributions functioned to level the perceived academic status between the pair. For instance, Lee, a relatively higher status student compared to Alison, started off the exchange with a question inviting Alison to evaluate his suggestion to "jump" to the desired conclusion. This attributed academic status to Alison as someone who had the right to validate mathematical ideas. Alison accepted this position by using a declarative statement that positively evaluated Lee's idea ("yeah I think that's good"). Further along, Lee began a talk turn with a statement declaring what he knew about a particular line in their proof, saying "this is showing the identity for all these." He then used a question to ask Alison how they should approach a refinement, saying "I guess we can use onto right?" Again, this attributed academic status to Alison by positioning her as a knowledgeable peer.

Declarative Statements Uphold Unbalanced Academic Status Positions

In the subsequent exchange, Alison and Justin engaged in a back-and-forth about how to move forward with their proof that elements of a Cayley table are unique (see Ellis & Alzaga Elizondo, 2023).

Alison: Yeah it's different actions but I think for the sake of our proof we need to somehow say we're limited to you know (pause) [Statement]

Alison: just four, well in this case we don't have- however many symmetries there are. (pause) [Statement]

Alison: I might be articulating that wrong. (pause) [Statement]

Justin: I think the original route we're going down is right, where we have this, I think this is definitely the right way. [Statement]

Justin: I'm just trying to make sure that we have the proper way saying that proper, like-

Alison: Yeah, I agree, we have to find the way to set it up before we can just say. (pause) [Statement]

Justin: Well, (cross talk with Abigail) I just want to make sure there's no holes, I guess- [Statement]

Abigail: (cross talking) Q is identical to W then [Statement]

Alison: (responding to Justin) I understand that. [Statement]

Using declarative statements as discursive moves seemed to assert each speaker's own knowledge and authority over the work. For instance, Justin made a declarative statement evaluating Alison's prior contribution ("I think the original route we're going down is right"), positioning himself with the right to validate the group's work. Alison responded with a declarative statement "Yeah, I agree, we have to find the way to set it up…" which functioned as a positive evaluation of Justin's contribution. Rather than offer or invite a suggestion for how to accomplish what they agreed they needed to do to move the proof forward, Justin used another declarative statement, "I just want to make sure there's no holes" maintaining his right to validate their mathematical work. Instead of using questions to elicit intellectual contributions, as in the exchange between Alison and Lee, this interaction pattern functioned to uphold Justin's higher relative status he entered the interaction with.

Discussion

This report contributes a potential theory to empirically identify how status hierarchies form in small group interactions. We have argued that exploring processes by which status hierarchies form is necessary in proof contexts because such qualitatively-driven empirical analyses can potentially explain prior inequitable quantitative results, particularly regarding gender. While this theoretical approach seems fruitful, it has limitations. Additional data about students' experiences in group work is needed to obtain a more complete picture of how de/legitimating status hierarchies attenuates/amplifies inequities in small group interactions (see Adams-Wiggins et al., 2020). Specifically, sociometric data about social relations in the classroom environment and interviews to allow students the opportunities to elaborate on their experiences would serve as appropriate data sources to confirm or disconfirm our interpretations from the status analysis. We hope this work sparks conversations around how status functions in small groups for the purpose of discussing and implementing possible instructional approaches that delegitimate status hierarchies in proof spaces.

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References

Adams-Wiggins, K. R., Myers, M. N., & Dancis, J. S. (2020). Negotiating status hierarchies in middle school inquiry science: Implications for marginal non-participation. *Instructional Science*, 1-25.

Alzaga Elizondo, T. (2022). Collective Proving Activity in a Synchronous Online Environment (Doctoral dissertation, Portland State University).

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, *9*(2), 49-80.

Battey, D., & McMichael, E. W. (2021). Inquiry-based practices: Opening possibilities for (in)equitable interactions in classrooms. In *International Handbook of Inquiry and Learning* (pp. 189-203). Taylor and Francis.

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2013). *Statistical abstract of undergraduate programs in the mathematical sciences in the united states.* American Mathematical Society. http://www.ams.org/profession/data/cbms-survey/cbms2010.

Brown, S. (2018). E-IBL: An exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. In A. Weinberg & C. Rasmussen, J. Rabin, M, Wawro, & Brown, S. (Eds.). *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1–15).

Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (3rd Ed.). Teachers College Press.

Cohen, E. G., Lotan, R. A., Scarloss, B. A., & Arellano, A. R. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory into Practice*, *38*(2), 80-86.

Ellis, B. M., & Alzaga Elizondo, T. (2023). Exploring how status influences relational and participatory equity in inquiry-oriented small group interactions. In Cook, S., Katz, B., & Moore-Russo, D. (Eds.). *Proceedings of the XXV Annual Conference on Research on Undergraduate Mathematics Education* (pp. 324–332).

Esmonde, I. (2009a). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *The Journal of the Learning Sciences*, *18*(2), 247-284.

Esmonde, I. (2009b). Ideas and identities: Supporting equity in cooperative mathematics learning. *Review of Educational Research*, *79*(2), 1008–1043.

Ernest, J. B., Reinholz, D. L., & Shah, N. (2019). Hidden competence: Women's mathematical participation in public and private classroom spaces. *Educational Studies in Mathematics*, *102*(2), 153-172.

Gebhard, M., & Accurso, K. (2020). Systemic Functional Linguistics. In C. A. Chapelle (Ed.). *The concise encyclopedia of applied linguistics*. (pp. 1029-1037). John Wiley & Sons.

Halliday, M. (1978). Sociolinguistic aspects of mathematics education. In M. Halliday (Ed.), *Language as social semiotic: The social interpretation of language and meaning* (pp. 194–204). Edward Arnold.

Harré, R. (2012). Positioning theory: Moral dimensions of social-cultural psychology. In J. Valsiner (Ed.), *The Oxford handbook of culture and psychology* (pp. 191–206). Oxford University Press.

Harré, R., & van Langenhove, L. (Eds.). (1999). *Positioning theory: Moral contexts of intentional action*. Blackwell Publishers.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in Mathematics Education: Revelations on an imported theory. *Educational Studies in Mathematics*, *89*(2), 185–204.

Hicks, M. D., Tucci, A. A., Koehne, C. R., Melhuish, K. M., & Bishop, J. L. (2021). Examining the distribution of authority in an inquiry-oriented abstract algebra environment. In S. S. Karunakaran & A. Higgins (Eds.), *2021 Research in Undergraduate Mathematics Education Reports* (pp. 109–116).
Jaremus, F., Gore, J., Prieto-Rodriguez, E., & Fray, L. (2020). Girls are still being 'counted out': Teacher expectations of high-level mathematics students. *Educational Studies in Mathematics*, *105*(2), 219–236.
Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune N. (2020). Inquiry and inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504-516.
Langer-Osuna, J. M. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education*, *11*, 207-225.
Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. *Mathematical Thinking and Learning*, *18*(2), 107-124.
Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 397–433.
Mathematical Association of America (MAA) (2018). MAA Instructional Practice Guide. Mathematical Association of America.
Martin, D.B. (2019) Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459-478.
Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, *67*, 100986.
Mendick, H. (2006). *Masculinities in mathematics*. Open University Press.
Ng, J. C., Lee, S. S., & Pak, Y. K. (2007). Contesting the Model Minority and Perpetual Foreigner Stereotypes: A Critical Review of Literature on Asian Americans in Education. *Review of Research in Education*, *31*(1), 95–130.
Poon, O., Squire, D., Kodama, C., Byrd, A., Chan, J., Manzano, L., ... & Bishundat, D. (2016). A critical review of the model minority myth in selected literature on Asian Americans and Pacific Islanders in higher education. *Review of Educational Research*, *86*(2), 469-502.
Reinholz, D., Johnson, E., Andrews-Larson, C., Stone-Johnstone, A., Smith, J., Mullins, B., ... & Shah, N. (2022). When active learning is inequitable: Women's participation predicts gender inequities in mathematical performance. *Journal for Research in Mathematics Education*, *53*(3), 204-226.
Ridgeway, C.L. (2018). Status construction theory. In P. J. Burke (Ed.), *Contemporary social psychological theories* (p. 301–323). Stanford University Press.
Rowlands, S., & Carson, R. (2002). Where would formal, academic mathematics stand in a curriculum informed by ethnomathematics? A critical review of ethnomathematics, *Educational Studies in Mathematics*, *50*, 79-102.
Saxe, K., & Braddy, L. (2015). A common vision for undergraduate mathematical sciences programs in 2025. Mathematical Association of America.

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record*, *119*(7), 1-42.
Shah, N., & Lewis, C. M. (2019). Amplifying and attenuating inequity in collaborative learning: Toward an analytical framework. *Cognition and Instruction*, *37*(4), 423-452.
Suzuki, B. H. (2002). Revisiting the model minority stereotype: Implications for student affairs practice and higher education. *New Directions for Student Services*, *97*, 21-32.
van Langenhove, L. & Harré, R. (1999). Positioning as the production and use of stereotypes. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 127–137). Blackwell Publishers.
Weber, K., & Melhuish, K. M. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education, 22*(2), 305-314.

Students' Productive Techniques for Approaching Well-Definedness and Everywhere-Definedness

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Functions are critical in mathematics but have received limited attention at the advanced level. Research in advanced contexts has primarily focused on students' reasoning about specific types of functions (e.g., isomorphisms) but not on the function concept itself. In this paper, we explore students' productive techniques involving the definitive properties of function: well-defined and everywhere-defined. We found that the techniques students productively employed extended far beyond canonical procedures (like the vertical line test) and largely drew upon function meanings involving coordination of the domain, codomain, and rule, which previous research has highlighted the importance of but stopped short of directly investigating. Two contributions of this work include our focus on productive, successful techniques (rather than challenges and difficulties), and explicit focus on everywhere-defined (which has not received direct attention).

Keywords: function, well-defined, everywhere-defined, advanced mathematics

Functions are an essential concept in advanced mathematics underscoring ideas like binary operation, continuity, and homeomorphisms. We treat functions as having two defining properties (Even and Tirosh, 1995): "(1) they should be defined on every element in the domain, and (2) for each element in the domain there should be only one element (image) in the range; this condition is also known as univalence" (p. 4). We refer to the first property as everywhere-defined and the second as well-defined. These properties can serve to support or constrain student reasoning about abstract types of functions in later mathematics (Melhuish et al., 2020). For example, many students determine that functions like $f(x) = 1/x$ are not continuous because of their asymptotes (e.g., Takači et al., 2006). Students may not be considering that a function must be everywhere-defined on its domain. Similarly, in abstract algebra, well-definedness plays a crucial role as students work with functions on structures such as quotient groups whose elements are equivalence classes. In fact, Rupnow (2021) has documented that instructors may spend an entire lesson on these properties (along with onto and one-to-one).

The larger body of literature about student understanding of function primarily illustrates students' incomplete or non-generalizable meanings and techniques (e.g., Dorko, 2017; Even & Bruckheimer, 1998; Martínez-Planell & Trigueros Gaisman, 2012). Even and Tirosh (1995) proposed that the concept of well-definedness is often viewed in a rote and procedural way which is found to be superficial upon probing. Indeed, the most common manifestation—procedural or otherwise—of well-definedness at the K-12 level is the vertical line test or checking listed set of elements. The vertical line test can be a useful technique when accompanied with meanings of well-defined (e.g., Clement, 2001; Melhuish et al., 2020; Thomas, 2003); however, researchers have also noted that students over rely on the vertical line test and such techniques remain salient at the advanced level (e.g., Zandieh et al., 2017).

Beyond the vertical line test, researchers have also documented that students may not attend to function properties at all (e.g., Melhuish et al., 2020; Vinner, 1983). Thompson (1994) argued that the "predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p. 5). Furthermore, introductory examples of function tend to draw on metaphors like the "function machine" (Tall & Bakar, 1992) which appeal to diagrams and tables where properties are readily identifiable by observation. Thus, even when students draw attention to the defining properties, it is not always in a way that is operable in terms of techniques for other contexts (e.g., Vinner 1983). We also note, despite the important role of everywhere-definedness (such as the continuity example above or in support of understanding homomorphism), few researchers have addressed this property explicitly, and we were not able to find any documented techniques. In this paper, we explore: What types of techniques do advanced mathematics students use to productively engage with well-defined and everywhere-defined properties when determining if a given relation is a function?

Theoretical Framing

We broadly take an approach aligned with Paoletti and colleagues' (2018) distinction between techniques and meaning. We share similar constructivist assumptions that students have developed their meanings for a particular concept based on an array of prior experiences. That is, the meanings students have for functions and the properties of well-defined and everywhere-defined reflect not an in-the-moment cognitive state, but rather more overarching understandings. In contrast, techniques refer to "student's words and observable activity as she addressed a single task" (p. 95). Students draw on their knowledge when engaging in problem-solving tasks, and techniques describe what they know-to (Mason & Spence, 1999) do in-the-moment. In our study, we were particularly interested in documenting the techniques used as students drew on their meanings for functions related to well-definedness and everywhere-definedness.

As noted in the introduction, it is essential that students move beyond a common meaning for functions as only a rule or formula. We suggest that function meaning requires coordination between the domain set, the codomain set, and rule that incorporates the two defining properties. If, when explaining why a proposed correspondence is or is not a function, students make explicit reference (in their language, gestures, or inscriptions) to (1) an element in the domain, (2) an element in the codomain, and (3) the rule, it would suggest that they are drawing on a meaning that has some coordination between these key components.

Methods

We conducted task-based clinical interviews (Clement, 2000; Goldin, 2000) with five students who (1) had already completed an advanced mathematics course (e.g., abstract algebra, cryptography, number theory) and (2) successfully approached basic "is this a function?" tasks. To identify such students, we reached out to several advanced mathematics courses at a large research university in the United States to ask for participants.

Data Collection and Task Design

We conducted two to three interviews with each student individually for an hour to an hour and a half each. Each interview was conducted and video-recorded over Zoom. The tasks were created with each category of non-examples from Uscanga and Cook (2022) being represented (sometimes in the modified form of an example). These categories are: *well-definedness – domain choice* (where an element in the domain can be represented in different equivalent ways and the symbolic rule maps these representations to distinct outputs), *well-definedness – codomain choice* (where the rule requires a choice of the outputs in the codomain, even though it does not invoke different equivalent representations), *everywhere-definedness – domain restriction* (where the symbolic rule maps an element in the domain to an output that is not contained in the proposed codomain), and *everywhere-definedness – domain expansion* (where the symbolic rule assigns to an element in the domain an output that is not contained in the proposed codomain but is contained in an easily accessible superset). The interviews were centered around four core tasks which asked students to identify if a proposed correspondence was an example or a non-example of a function. Additional tasks were designed for subsequent interviews to test and refine the hypotheses we established for each student, so they were adequately supported or refuted based on our observations.

Data Analysis

We engaged in both ongoing and retrospective analysis. Ongoing analysis occurred during and between sessions. This involved generating tasks and questions to test specific hypotheses about the students' reasoning. After data collection, we independently analyzed students' responses to five tasks. We identified excerpts from the transcripts that illustrated the techniques used by students to explore each task and coded these for definitional properties (well-defined or everywhere-defined), focus on domain, codomain, or rule, and the tool or approach utilized to solve the task. Codes were revised until agreement was reached and summaries of each student's approaches were created. This allowed us to compare techniques across students as well as across tasks. A second round of analysis involved grouping the data by type of property explored in order to identify the different techniques students used to productively engage with well-definedness and everywhere-definedness. Finally, we categorized these individual techniques into broader categories that described the general approaches students employed.

Results

Well-definedness

The well-defined property has been the subject of many empirical studies (e.g., Clement, 2001; Dorko, 2017; Even & Bruckheimer, 1998; Even & Tirosh, 1995). In this section, we highlight two overarching themes in students' activity: sameness and divergence. We begin by illustrating students drawing on prototypical examples in their open explorations, then consider the more sophisticated techniques used by students in more abstract settings.

Table 1. Students' techniques for engaging with well-definedness.

Technique	Focus	Method
Diagram or Table	Canonical Technique	Identify diverging outputs or repeats
Vertical Line Test	Canonical Technique	Identify multiple points along a same vertical line
Rule Ambiguity	Symbolic Rule	Identify if symbolic form assigns multiple outputs
Rule Equivalence	Symbolic Rule	Manipulate symbolic rule into known equivalent form
Attending to Function Notation Elements	Rule Focus, Domain Secondary	Determine if symbolic form of how an element is represented in input notation impacts output
Attending to Equivalence in the Domain	Domain Focus, Rule Secondary	Identify equivalence classes in the domain and explore their corresponding outputs
Attending to Equivalence in the Domain and Codomain	Domain, Rule, Codomain Focus	Identify equivalence classes in the domain and examine outputs via equivalence classes in the codomain

Canonical techniques. We suggest that the literature-based techniques (multiple outputs for one input identified via listing elements or vertical line test) might be deemed early canonical. Our advanced mathematics students drew on such examples when prompted to create functions and non-functions (see Figure 1). For example, Student B explained, "one of the guys from the domain goes to like, four different outputs, so that violates our condition." Student C stated, for instance, "if we drew like a vertical line, we know there's a lot of points where, you know, our one input value has two different output values." From the point-of-view of sameness and divergence, these techniques do not require identification of sameness, but rather rely on identifying divergence of outputs. They simplify the relationship between the domain, rule, and codomain to draw attention to inputs and outputs without substantial exploration of the surrounding sets.

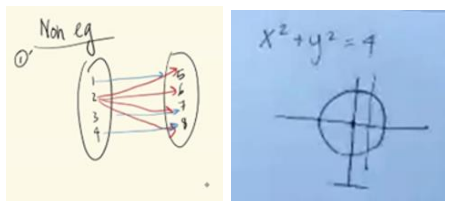

Figure 1. Left. Student B's function diagram of an example and a non-example of a function. Right. Student C's depiction of the vertical line test to establish a circle is not a function.

Attending to divergence. We begin by elaborating a technique that involves forefronting the symbolic rule which involves recognition of the potential for a rule to lead to divergence. Many of the students discussed this issue in relation to the correspondence $f: \mathbb{R} \to \mathbb{C}$ given by $f(x) = \sqrt{x}$. For example, Student A explained, "we have made a convention that if you just see a square root sign, you automatically decide that it's the positive square root sign." These students went on to provide a counterfactual that without this convention, it would not be a function, because, as noted by Student B, "our four could map to plus two or minus two. And then we don't know which one we're choosing, so that's an ambiguity." We note that the students are attending primarily to the symbolic rule but coordinating the role the rule plays diverging a single input to two outputs—drawing on the well-defined meaning. The students' language also reflects sophistication in their meanings with attention to the role of convention and ambiguity.

Attending to equivalent functions. When exploring the rule $f: \mathbb{Q} \to \mathbb{Q}$ taking $\frac{a}{b}$ to $\frac{a+b}{b}$, several students worked with this unfamiliar rule with a lens of sameness: finding a familiar well-defined function. After some algebraic manipulation, Student B explained, "my assignment can be abbreviated as x going to $x + 1$, which is, which is a function." We note that this approach is not explicating the coordination between sets and rule nor the definition of well-defined. However, we infer that the student using this technique likely had an awareness of that, as Student E stated, this function "is simply your input plus one" which is quite obviously well-defined. Thus, we suggest that manipulating the rule can be a productive technique if there is an obvious equivalent function.

Attending to function notation elements. Techniques that attend primarily to the rule are not sufficient in approaching a common area of concern for well-definedness: equivalence classes in the domain set. We identified two different ways that domain sameness was coordinated with rule divergence. The first attends primarily to the rule in terms of not just the outputs produced from an input, but also the notation of the input in function notation. For example, regarding the function $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$, Student B explained that "given an x, it could be in the form a divided by b … But the formula, the assignment does not need that information." This student is attending to elements of the domain (recognizing the set involved and potential representations), but then acknowledging that the function, as written, would not use that information. In contrast, for $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$, Student B explained, "the formula in this case, actually depends on the representation. Like, depends on how we have written the fraction." They go on to provide an example, $\frac{1}{2}$ and $\frac{2}{4}$ are the same, but do not give the same output (divergence). This student is drawing on meaning and coordination between the domain set ($\mathbb{Q}$) and the rule in order to determine if elements from the domain can have distinct representations and whether the rule, attending to function notation input, will act on those representations differently.

Attending to equivalence in the domain. Other techniques centered the domain to determine if an "equivalence" relation existed. This technique involved identifying equivalent elements (sameness) and then testing if the rule produced two outputs (divergence). Consider Student A's response to a modular arithmetic task ($f: \mathbb{Z}_3 \to \mathbb{Z}$ given by $[a]_3 \mapsto a$):

> Look, zero-three [writing $[0]_3$], I could write it a different way, right? … I could write this as six-three [writing $[6]_3$]. Those are the same element in $\mathbb{Z}_3$. … If I was gonna do six-three, your rule tells me that I have to map it to six. … the element zero-three maps to zero, six, three, nine … There's a lot of, a lot of outputs, and I'm only allowed to have one output.

This sort of exploration and attention to specific elements was found not just in the modular arithmetic tasks, but also the rational number tasks.

Everywhere-Definedness

We now transition to the everywhere-defined property which has received less attention in the literature. The results in this section are organized around three main themes that arose from students' activity: containment, existence, and set operation. We begin with a students' prototypical example from their open exploration and then explore more sophisticated techniques used by the different students throughout the interviews.

Table 2. Students' techniques for engaging with well-definedness.

Technique	Focus	Method
Diagram	Canonical Technique	Identify inputs that do not have corresponding outputs
Attending to Output Containment	Codomain and Rule Focus	Determine if outputs assigned by the symbolic rule are contained in the proposed codomain
Attending to Output Existence	Domain and Codomain Focus, Rule Secondary	Determine if symbolic rule assigns an output to every input
Manipulating Sets via Extending	Domain and Codomain Focus, Rule Secondary	Operate on domain or codomain by extending the set and focusing on a superset
Manipulating Sets via Subsetting	Domain and Codomain Focus, Rule Secondary	Operate on domain or codomain by subsetting the set and focusing on a subset
Manipulating Sets via Partitioning	Domain and Rule Focus, Codomain Secondary	Subdivide domain into sets where the symbolic rule causes known problem areas

Canonical technique. The students rarely evoked everywhere-defined while generating examples and non-examples. Only one student attended to it when asked to provide an example of a non-function. We deemed their approach canonical as it is the type of example students might be used to from K-12 settings. As seen below (Figure 2), the student drew a function diagram where one element in the domain had no corresponding output in the codomain. They explained:

> I was like "oh, well, could I have no outputs? … Would that be, a part, would that break it being a function?" Well, that wouldn't even count because a relation requires that each input has at least one output … So, I could make it not a relation by sticking this F in here, and it has no arrow. Well, then it's not a function because it's not a relation.

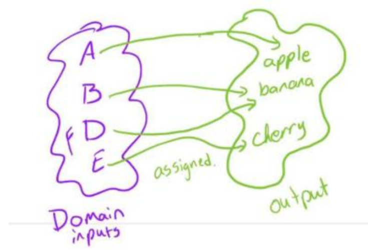

Figure 2. Student A's function diagram which fails the everywhere-defined property.

Attending to output containment. Several students attended to whether the outputs assigned by the rule were contained in the proposed codomain. They viewed the rule as assigning an output to each input and focused on whether the resulting outputs were contained in the codomain specified. For instance, when dealing with the non-example $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$, Student A explained, "this relation has its own meaning, and its meaning, says, something about, this $\mathbb{N}$ right here, having, to include the answer that the rule would go to." In more detail, they noted:

> What happens when I plug in negative two? Right? Well then the rule, says that, I have to go to negative two cubed … that has to be negative eight, and negative eight does not live in $\mathbb{N}$ … So I can't say if it's a function because it's a nonsensical thing … Negative two lives here, look it's right there, and it's telling me to go to negative eight, but negative eight, I can't, I can't go. It won't let me. This, this won't let me go there."

This reflects a two-stage process, attending to the rule and then coordinating with the codomain.

Attending to output existence. A slightly different technique focused on the correspondence not assigning outputs to certain inputs. For example, Student B used this technique to explain why $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ was not an example of a function:

> So the first thing that we need to have is, we have to have a relation between stuff in the range, the stuff in the domain and the stuff in the range. But, so if I tried to write down an ordered pair here, I will not have anything to write for minus one. So, that, we can't even get an ordered pair. So we can't even get a relation for minus one.

This student argued that there is no output assigned to -1. The attention is on the domain and codomain and the rule either relates the two sets or does not.

Manipulating sets via extending and subsetting. Two other techniques that arose involve operating on the domain or codomain by extending the set or subsetting it. For instance, Student E subsetted the codomain when working with the square root function and explained how they restricted their focus on a subset, "So if it was negative, it would result in a complex number, but when you're considering the function is mapping from reals to reals, then you have to restrict the domain. But it's still a function." They continued, "It's just your domain, the y values don't exist for this particular field that we're considering. But you defined the mapping as from reals to the complex numbers, which means that your domain is unrestricted here."

We also observed students extending the domain, as Student E did when discussing the function $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$, "since you restricted your domain, this is a function … if you would have said that you were just mapping from reals … x cannot equal to zero … you do not have a valid y value for that." The student extended the domain to explore the issues that arise with the correspondence and notes that the domain is not extended in this case so there are no issues. In this technique, the students' focus is on a superset of the specified domain or codomain, whereas in the first technique the student focuses on a subset of the proposed domain or codomain.

Manipulating sets via partitioning. The other related technique involved subdividing the domain as a result of a known problem area related to the symbolic form of the rule. Students then further made arguments about containment of the outputs. For example, Student C noted about the x^3 non-example:

> So our two sets here is the integers and the natural numbers. And so this, function that we wrote here, the x cubed, you know, there are some things that "okay, yeah we can put some numbers in for x and we can get some natural numbers, that's fine." But there's also some numbers, you know, our negative numbers, if we put negative numbers in here, we're not getting output values in the natural numbers.

Here, the student is subdividing the domain into the positive and the negative integers since negative integers could have issues with this rule. By exploring the places where possible issues might arise with the symbolic rule, students were able to determine whether the provided correspondences were or were not functions.

Discussion

In this paper, we described different types of techniques that advanced mathematics students used to productively engage with the properties of well-defined and everywhere-defined when determining if a given correspondence is a function. We found that the students in the study used techniques that extended far beyond the canonical techniques that rely primarily on observing irregularities (multiple outputs or lack of outputs). For well-defined, students used ideas of convention and ambiguity to attend to issues with symbolic rules and naming conventions, and sameness and divergences to consider equivalence classes and representations in the codomain and domain sets. For everywhere-defined, students used ideas of containment and existence to consider implications of certain rules and operated on sets to consider extensions and subsets that may complicate the relationship with the rule. This is a key contribution because while researchers had previously examined various facets of reasoning productively with functions in mathematics in general (e.g., Oehrtman et al., 2008; Tall & Vinner, 1981) and in advanced mathematics in particular (e.g., Melhuish et al., 2020; Zandieh, et al., 2017), investigation into well-definedness and everywhere-definedness has been rather limited.

These results can provide a foundation for additional research that could attend to ways students might develop these techniques. Future research can elaborate how techniques like the ones identified here might influence students' reasoning about more advanced concepts within the context of abstract mathematics where these properties have crucial roles in formal proofs.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-590). Lawrence Erlbaum Associates, Inc.

Clement, L. L. (2001). What do students really know about functions? *Mathematics Teacher, 94*(9), 745-748.

Dorko, A. (2017). Generalising univalence from single to multivariable settings: The case of Kyle. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 562-569).

Even, R. & Bruckheimer, M. (1998). Univalence: A critical or non-critical characteristic of functions? *For the Learning of Mathematics, 18*(3), 30-32.

Even, R. & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*, 1-20. https://doi.org/10.1007/BF01273897

Goldin, G. A. (2000). A scientific perspective on task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-546). Lawrence Erlbaum Associates, Inc.

Martínez-Planell, R. & Trigueros Gaisman, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics, 81*, 365-384. https://doi.org/10.1007/s10649-012-9408-8

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Forms of mathematical knowledge: Learning and teaching with understanding*, 135-161.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior, 60*, 1-16. https://doi.org/10.1016/j.jmathb.2020.100806

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Mathematical Association of America.

Paoletti, T., Stevens, I. E., Hobson, N. L. F., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics, 97*, 93-109. https://doi.org/10.1007/s10649-017-9787-y

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructor's descriptions for themselves and when teaching. *The Journal of Mathematical Behavior, 62*, 1-14. https://doi.org/10.1016/j.jmathb.2021.100867

Takači, D., Pešić, D., & Tatar, J. (2006). On the continuity of functions. International Journal of Mathematical Education in Science and Technology, *37*(7), 783-791.

Tall, D., & Bakar, M. (1992). Students' mental prototypes for functions and graphs. *International Journal of Mathematical Education in Science and Technology, 23*(1), 39-50. https://doi.org/10.1080/0020739920230105

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Thomas, M. (2003). The role of representation in teacher understanding of function. In N. A. Pateman, B. J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of*

PME and PMENA (Vol. 4, pp. 291-298). Center for Research and Development Group, University of Hawaii.
Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education, I* (Vol. 4, pp. 21-44). American Mathematical Society.
Uscanga, R., & Cook, J. P. (2022). Analyzing the structure of the non-examples in the instructional example space for function in abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, 1-27. https://doi.org/10.1007/s40753-022-00166-z
Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305. https://doi.org/10.1080/0020739830140305
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*, 21-38. https://doi.org/10.1007/s10649-016-9737-0

Two Students' Meanings for Partial and Directional Derivatives when Constructing Linear Approximations: The Cases of Alonzo and John

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In this paper, I describe two vignettes of two students' meanings for partial and directional derivatives. The data was collected in the Spring 2023 academic semester from two STEM students who had completed multivariable calculus at least two semesters prior to participating in the study. The interviews were conducted using task-based, exploratory teaching interviews with the goal of creating explanatory models of student thinking to highlight the nuance in their meanings for differential multivariable calculus ideas. The two students, Alonzo and John, each exhibited novel meanings for partial derivatives that were constructed over several semesters of mathematics and physics instruction. These results add to the field's understanding of students' thinking about a relatively underdeveloped area of the research literature, specifically multivariable calculus education.

Keywords: Students' meanings, partial derivatives, directional derivatives, multivariable calculus education

Introduction and Literature Review

Investigations of students thinking about multivariable and vector calculus ideas have been of increasing interest to undergraduate mathematics education researchers over the past several years (Jones, 2020, 2022; Mhkatshwa, 2021; Moreno-Arotzena et al., 2021). Despite a relative lack of mathematics education research literature investigating students' conceptions of multivariable calculus topics (Rasmussen & Wawro, 2017), Multivariable and Vector Calculus (MVC) is positioned at an important point in the sequencing of the undergraduate mathematics curriculum. Specifically, MVC is typically offered as the third semester of university-level Calculus and is sometimes required as a prerequisite course for upper-division mathematics courses such as Linear Algebra or Differential Equations. Further, the differential calculus ideas normally discussed in a MVC course are often necessary to make sense of certain concepts and systems of equations encountered in upper-division physics and engineering courses (Bajracharya et al., 2019; Dray & Manogue, 2006; Roundy et al., 2015). To better understand how undergraduate STEM majors think about partial and directional derivatives, I conducted a set of exploratory interviews in the Spring 2023 semester. In this paper, I share two vignettes from this study that highlight nuance across two students' meanings for partial and directional derivatives.

Theoretical Perspective

Discussions about students' meanings for mathematical ideas necessarily elicit recursive discussions about what we, as mathematics education researchers, "mean" by meaning.

The Meaning of "Meaning."

Thompson (2013), described the recursive nature of meaning, and the development of meaning, in the context of mathematics education in the United States. Thompson provided an in-depth analysis of what several philosophers meant by "meaning." As a genetic epistemologist, Piaget appeared to understand "constructing a meaning" as "constructing an understanding," and appeared highly related in Piaget's Genetic Epistemology (Glasersfeld, 1995; Montangero & Maurice-Naville, 1997; Piaget, 2001). In the context of Piaget's theory, *assimilation* is a biological notion in which an individual grafts new sensorimotor experiences, or mental activity, onto previously constructed mental structures, or *schemes*. For Piaget, assimilation was the source of all schemes, and *accommodation* was the driving mechanism behind the creation of new schemes and differentiations in existing schemes to create new schemes through the process of Reflective Abstraction (Glasersfeld, 1991; Piaget, 2001). In the context of the learning theory that emerges from adopting Radical Constructivism as a background theory or an epistemological stance (Simon, 1995), learning is conceptualized as the process of iterative cycles of assimilation and accommodation driving individuals, conceptualized by Radical Constructivists as individual cognizers, towards a temporary state of *cognitive equilibrium* (Cobb & Steffe, 1983; Glasersfeld, 1991, 1995). If having an understanding is therefore the result of assimilating to a scheme, then the corresponding understanding accompanied by an inference, or a set of inferences in the moment, would be someone's meaning that resulted from the assimilation to their scheme(s) (Thompson, 2013; Thompson et al., 2014).

Students' Classroom Experiences and the Negotiation of "Shared Meaning."

However, the construction of meaning is incomplete without consideration of the conveyance of one's personal meanings to another individual, which will always impact the nature of the interaction between two or more people (Blumer, 1986; Thompson, 2013). For instance, in classroom contexts, one aspect of a teacher's role is to act as a brokering agent between the local classroom context and the larger community of mathematicians and mathematics students (Zandieh et al., 2017). It makes sense that a teacher's meanings will necessarily impact the meanings that students construct, and the negotiation of meaning is often a subtle and difficult-to-grasp process in settings as dynamic as classrooms (Thompson & Thompson, 1994). As such, I decided to first analyze the meanings that students *had* constructed as a start to building second-order models of students thinking about partial and directional derivatives (Cobb & Steffe, 1983; Steffe & Thompson, 2000).

Further, while first-order models of student thinking can inform the initial set of tasks or instructional materials, engaging with students' genuine mathematical realities will result in more viable models of student thinking. Therefore, a researcher who adopts Radical Constructivism as a background theory takes seriously the notion that knowledge resides in the mind of the individual and rejects the notion that the researcher has "direct" access to the student's internal cognitive process, or *students' mathematics*. Thus, adopting Radical Constructivism implies an acceptance of the building models of student thinking is the best we can do as qualitative researchers to build theories about student thinking (Cobb & Steffe, 1983; Steffe & Thompson, 2000). Since I chose to adopt Constructivism as a background theory, I decided to follow the recommendations of Steffe and Thompson (2000) to engage in exploratory teaching with students to get in touch with their mathematical realities, or *the mathematics of students*.

A Quantitative Meaning for Partial Derivative as a Directional Rate of Change Function

In the context of MVC instruction, Mhkatshwa (2021) highlighted students' struggles with conceptualizing partial derivatives in non-kinematics contexts using covariational reasoning. Mhkatshwa's results corroborated other research results that students will conceptualize derivatives differently depending on the problem context (Jones, 2017; Zandieh, 2000; Zandieh & Knapp, 2006). For example, conceptualizing a partial derivative as a directional rate of change function would entail (i) a meaning for constant rate of change as a constant of proportionality which multiplicatively compares an amount of variation in the value of a single independent quantity and the corresponding amount of variation in the value of the dependent quantity, while mentally holding the value of the other independent quantity fixed, and (ii) a meaning for rate of change function as a record of the relationship between the value of the independent quantity and how quickly the value of the dependent quantity changes due to infinitesimal amounts of variation in the value of the independent quantity. Such a meaning would lead to the anticipation that the symbolization $f_x(x, y) = x^2y$ where $z = f(x, y)$ would imply that $f_x(2,1)$ represents the rate of change of the z value for any change in x away from $x = 2$ where the y value is fixed at $y = 1$. The rate of change would represent an equivalence class of ratios leading to the inference that the corresponding change in z is $f_x(2,1)$ times as large as the corresponding change in the x value away from $x = 2$, e.g., $\Delta z \approx f_x(2,1)\Delta x$.

Conceptions of partial and directional derivatives align well with recent instructional recommendations of research groups investigating student thinking about partial and directional derivatives by explicitly lecturing on 3D slopes (Martínez-Planell et al., 2017; Martínez-Planell & Trigueros Gaisman, 2021; McGee & Moore-Russo, 2015; Moore-Russo et al., 2011) and tactile manipulatives from the *Raising Calculus to the Surface* workshops (Wangberg, & Dray, 2022). The issue is that students' conceptions of rates of change as slope, e.g., as a graphical measure of the slantiness of a line or plane, cannot be assumed to automatically "transfer" to non-graphical contexts (Lobato & Siebert, 2002). As such, further research is required to investigate students' quantitative conceptions of partial and directional derivatives as rate of change functions whose dependent quantities have an equivalent measure to the 3D slopes of tangent planes as described by previously mentioned research groups.

Methods

The data presented here emerged from a study designed to highlight the nuance in STEM intending students' meanings for partial and directional derivatives across a set of symbolic, graphical, and dynamic geometry tasks (Bettersworth, 2023). The tasks were designed using a combination of literature on students' meanings for partial and directional derivatives (Jones, 2022; Martínez-Planell et al., 2017; Mhkatshwa, 2021) and my conceptual analysis for the two dynamic geometry tasks (Bettersworth, Year), to support students in engaging in the goal directed activity of using derivatives to create linear approximations for particular functions in graphical and symbolic settings.

Each student completed a 30-minute screening interview, which included questions about interpreting the limit definition for derivatives and partial derivatives in graphical and symbolic settings. Alonzo (he/him) and John (he/him) were invited to participate in up to five task-based, think-aloud style, exploratory teaching interviews with the goal of producing explanatory models of student thinking (Cobb & Steffe, 1983; Goldin, 1997; Steffe & Thompson, 2000). While the initial goal was to engage in *intuitive and responsive interaction* with each student, the interviewer would transition to making teacher moves if either student expressed discomfort or unfamiliarity with a particular subset of the ideas presented in my tasks or interview protocol. The teacher moves made by the teacher-interviewer were based on my a priori conceptual analysis for the previously mentioned geometry tasks (Bettersworth, 2021; Thompson, 2008).

I engaged in retrospective analysis of each interview recording by reviewing each video between interviews to determine any areas which required further clarification, or to inform the design of future tasks and teaching interventions. After the conclusion of the data collection, I engaged in ongoing analysis of the data through iterative rounds of open coding (Strauss & Corbin, 1998). Once an initial set of codes was established, the interview data was reanalyzed using quantitative reasoning (Thompson, 2022) as a lens for investigating the degree to which each student conceptualized variables (Carlson et al., 2002; Thompson & Carlson, 2017), function notation (Oehrtman et al., 2008; Thompson & Carlson, 2017), graphs (David Parr et al., 2018; Moore & Thompson, 2015), and rates of change (Thompson, 1994) as a record of their conception of the relevant quantities within each task.

Context of Student Data

Alonzo (he/him) and John (he/him) were selected from a group of five students who participated in a set of screening interviews in February 2023. Alonzo is a physics major at a large research university in the southwest United States. Alonzo completed the undergraduate calculus sequence at a local community college before transferring to the university. John is a double major in mathematics and physics at the same university. John completed the undergraduate mathematics sequence during high school and commented frequently about his positive STEM instructional experiences in high school. Both students were very vocal with their thinking and agreed to participate in the exploratory interviews after the completion of their individual screening interviews and after signing the approved study participation forms.

Results

Alonzo and John demonstrated varied and nuanced meanings for partial and directional derivatives that include interrelated conceptions of other MVC ideas (e.g., dot product, cross product, basis vectors, and projection) while constructing linear approximations. However, due to space limitation, I will share two vignettes related to Alonzo and John's meanings for partial and directional derivatives.

Vignette 1: Students' Meanings for Partial Derivatives

In the first vignette, I highlight Alonzo and John's meanings for partial derivatives which included: (i) curves on the surface in the x and y direction, (ii) tangent lines to each trace in the x and y direction, and (iii) slopes of the lines tangent to the curves on the surface in the x and y direction, or contained in a plane parallel to the xz or yz plane.

Meaning 1: Partial derivatives are the curves on the surface. Throughout the second and third day of Alonzo's interviews, it appeared that his meaning for partial derivative was evoked differently based on whether he anticipated mentally evaluating the partial derivative function for a particular set of values for both independent quantities. For instance, as demonstrated in the following transcript, when finding the partial derivatives of the function $f(x, y) = xy^2 - xy$ in task 2.1.2 (day 2, task 1, question 2), Alonzo interpreted $f_x(x, y)$ and $f_y(x, y)$ differently than he interpreted $f_x(2,1.5)$ and $f_y(2,1.5)$ in task 2.1.3. The task statements are pictured in Figure 1(b).

Alonzo: Okay, find the partial derivative of x of the function of f and the partial derivative of y with a function of f. What do these partial derivatives represent? So, find partial. I'll start with that, with partial of x. So, that would be f of x is equal to $y^2 - y$, f of y is equal to $2xy - x$.

Interviewer: Nice, and then how did you come up with those again?

Alonzo: So here [*points to* f_x], I held everything but the x variable is constant, so these y's are just constant in this case, and here [*points to* f_y] I held everything but the x variables everything with the y variable is constant. So, the x variables were constant.

Interviewer: Nice.

Alonzo: Okay, and then what do these represent? These represent I'd say curves on the 3D surface.

After I asked him to clarify his response, Alonzo sketched the diagram pictured in Figure 1(a). Alonzo explained that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ represented the curves on the surface in the x and y direction respectively. This meaning appeared highly associated with his attention to mentally fixing the value of one of the independent quantities by treating the variable as a constant when differentiating. Relatedly, when interpreting $f_x(2, 1.5)$ and $f_y(2, 1.5)$ in task 1.3.3, Alonzo explained that once he evaluated the partial derivative function, for example f_x, at $x = 2$ and $y = 1.5$, $\partial f/\partial x$ and $f_x(2, 1.5)$ represented the slope of the tangent line to the curve at the given point (as shown in Figure 1(b)) while mentally fixing the value of $y = 1.5$. This example of Alonzo's meaning for partial derivatives is interesting because it highlights how Alonzo evokes different aspects of his apparent scheme for partial derivative based on his current goal (either differentiating, evaluating the function at a point, or attending to graphical interpretations).

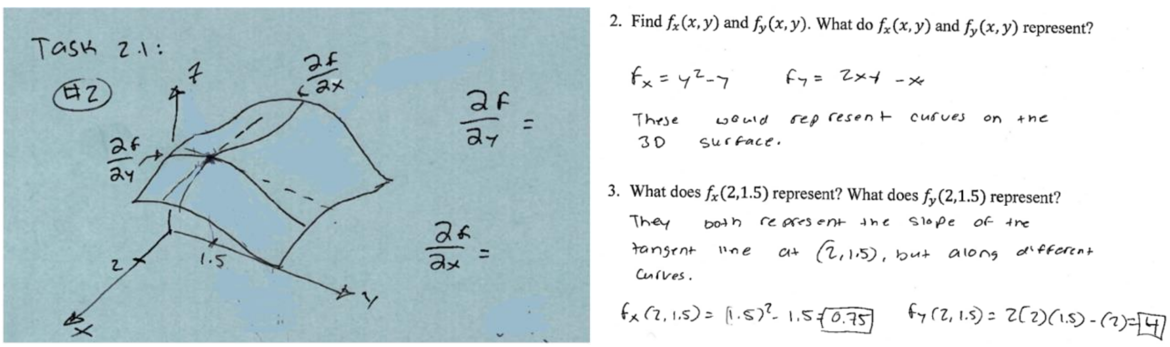

Figure 1: (a) Alonzo's sketch in task 2.1 accompanying his explanation for task 2.1.2 and 2.1.3. (b) Alonzo's response to task 2.1.2 and 2.1.3 demonstrating his dual meanings for partial derivative notation.

Meaning 2: Partial derivatives are the slopes of the tangent lines to each curve. John's meaning for partial derivative, on the other hand, consistently included a conception of partial derivatives as the slope of the tangent line contained within a plane parallel to either the xz or yz plane respectively. When responding to task 2.1.2, John's meaning for $f_x(x, y)$ and $f_y(x, y)$ included a graphical representation of the surface and slopes, as demonstrated by his gestures moving along an imagined surface graph. For John, mentally fixing the value of one of the independent quantities corresponded to isolating his attention on the graph of the function to a particular cross-section represented by intersecting the surface graph with a plane, as demonstrated in the following transcript.

John: Okay, so what do these represent? Well, if I had a sheet [*moves hands along an imagined bump which smooths out as he moves hands away from one another*] right? Then I can take a plane in the x direction [*chops hand down pointing to his left*] and if I fix the x value [*gestures with same hand to the right*] then I should be able to find the instantaneous slope [*moves pinched fingers up and to his right*] in the y direction. If I were to just look at that, that graph right there [*brings hands together flattened together at the fingertips*], then I would be able to figure out what the derivative would be with this, what the slope would be [*moves right hand along holding left hand fixed flat*].

While Alonzo's meaning for $f_x(x, y)$ and $f_x(2, 1.5)$ foregrounded different aspects of his more general meaning for partial derivatives, John's meaning for $f_x(x, y)$ and $f_x(2, 1.5)$ remained relatively consistent across all day 2 tasks.

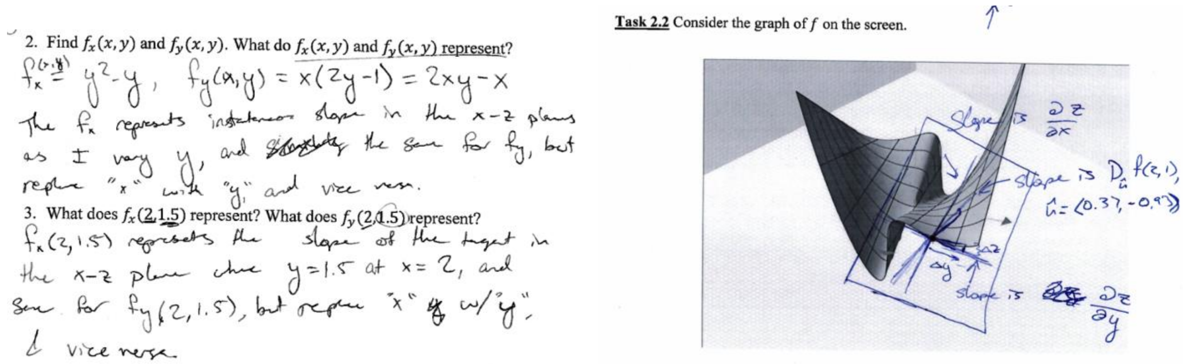

Figure 2: (a) John's response to tasks 2.1.2 and 2.1.3 demonstrating his consistent meaning for partial derivatives. (b) John's annotation of the surface graph in task 2.2 slope of the tangent line meaning.

Vignette 2: Student's Meanings for Directional Derivatives

Alonzo was not comfortable with the idea of directional derivatives, so we spent most of the remainder of his day 2 interview making teacher moves aligning with the meanings for directional derivative as outlined in my conceptual analysis. John, however, due to his recent Differential Geometry course, had revisited these ideas and exhibited numerous meanings for directional derivatives which included: (i) the rate of change of z in the $\hat{u}$ direction, (ii) the slope of the tangent line resulting from intersecting the graph of the surface with a new "shifted" uz plane, and (iii) as the amount of variation in z due to change in the $\hat{u}$ direction.

Meaning 1: The directional derivative is the slope of the tangent line to the curve in the new, shifted uz plane. As shown in Figure 2(b), John's meaning for directional derivative included a conception of slope in the $\hat{u}$ direction. While it may look like John conceived of partial and directional derivatives as the slopes of the tangent planes in task 2.2, this was not explicitly the case. In task 2.2, John was tasked with sketching the best local, linear approximation for the given function at a specific coordinate triple. He then discussed his meaning for $D_{\hat{u}} f(2, 1)$ using the relative size measurement of the horizontal and vertical segments he sketched in Figure 2(b). However, John's meaning for directional derivative is better demonstrated in the following excerpt from his work on task 2.1, where he describes his construction of the new "shifted" uz plane, which is also demonstrated in his response to task 2.1.5 in Figure 3(a).

John: Okay, this right here is a directional derivative. And so, if I were to want to find, turn my plane now [*rotates hands together*]. I'm going to turn my plane so that the u vector is pointing [*points thumb in a particular direction*] in the direction of my plane, in the xy plane, right? [***omitted***] And so, this u vector lies in here and so beforehand we were talking about this plane [*sketches xz plane*] we were talking about this plane right here [*sketches yz plane*] and now I want to talk about varying this plane right here [*sketches plane parallel to the z axis containing the u vector*].

Interviewer: Great.

John: And so, if I take that plane and I move it around however I need to [*moves vertically-held hands around in front of him*] and find the directional derivative that's the derivative in that direction [*points with pen*]. So, it represents the rate of change in the direction of u-hat. Yeah, I guess that's probably the easiest way to say that.

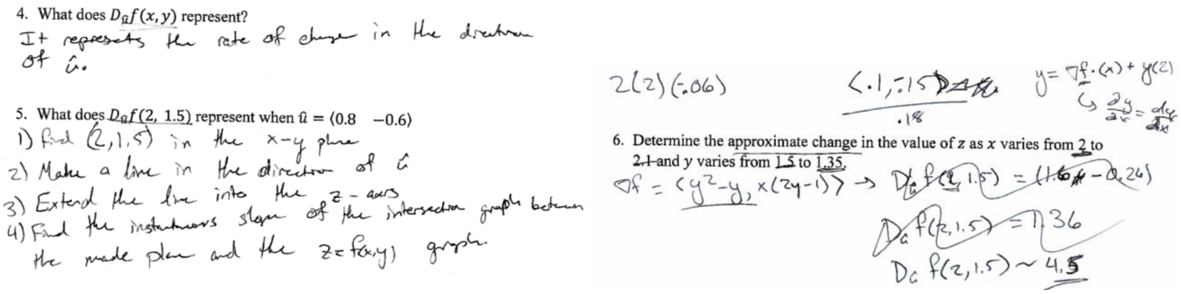

Figure 3: (a) John's response to task 2.1.4 and 2.1.5 demonstrating his slope in the uz plane meaning. (b) John's response to task 2.1.6 demonstrating his meaning for directional derivative as the amount of variation in z.

Meaning 2: The directional derivative is the amount of variation in z. While John appeared to have a robust meaning of linearization which was highly interrelated to his meaning for derivative, it appeared that, at times, John conceptualized rates of change as amounts of variation in the value of the dependent quantity, as shown in Figure 3(b). For instance, in task 2.1.5, John described the directional derivative as "the best approximation you can make if you're assuming that it's from here starting at this point [*underlines x=2 and y=1.5*] and goes to these points [*underlines x=2.1 and y=1.35*]." While it is not uncommon that students conceptualize rates of change as amounts of variation in the value of the dependent quantity of the function, it does highlight the importance of supporting students in differentiating between the concepts of rate of change and variation earlier in their mathematics instruction, as the complexity of attending to the values of more than two quantities as they change together, vectors and vector notation, and the calculus of MV functions is enough of a cognitive load for students without worrying about conflating rates of change with amounts of variation.

Discussion and Conclusion

In this paper, I shared two vignettes of two student's nuanced meanings for partial and directional derivatives. These vignettes highlight the importance of ensuring that students construct a strong foundational understanding of derivatives as rate of change functions in single variable calculus, to support the generalization of their meanings to MVC settings (Harel, 2021; Martínez-Planell & Trigueros Gaisman, 2021; Thompson, 2019). Second, these results highlight a need to support students in differentiating between amounts of variation and rates of change as distinct quantities (Thompson, 2022). This is an important, but difficult, conceptual distinction that students struggle to make in single variable calculus settings. Since students' meanings for derivatives of functions appear to influence the meanings they construct in MVC settings (Bettersworth, 2023), more research is required to support students in making this distinction.

Limitations

The mathematics of students presented here represent second-order models of student thinking emerging from the analysis of two students' responses within the context of exploratory teaching interviews. As such it is impossible to claim that these are representative of every possible meaning a student might possess or construct. Future studies could attempt to replicate or generalize the results presented here with a larger collection of students from a different or more varied cross-section of students majoring in STEM.

Bibliography

Bajracharya, R. R., Emigh, P. J., & Manogue, C. A. (2019). Students’ strategies for solving a multirepresentational partial derivative problem in thermodynamics. *Physical Review Physics Education Research*, *15*(2), Article 2. https://doi.org/10.1103/PhysRevPhysEducRes.15.020124

Bettersworth, Z. S. (2023). An Exploration of Students’ Meanings for Derivatives of Univariable and Multivariable Functions When Building Linear Approximations (Order No. 30566458). Available from ProQuest Dissertations & Theses Global. (2851697842). https://wku.idm.oclc.org/login?url=https://www.proquest.com/dissertations-theses/exploration-students-meanings-derivatives/docview/2851697842/se-2

Bettersworth, Z. (2021). Students' meanings for their symbolization about rate of change functions in dynamic geometric contexts. *2021 SIGMAA on RUME Reports*, 21-29.

Blumer, H. (1986). *Symbolic Interactionism: Perspective and Method* (1st ed.). University of California Press.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), Article 5. https://doi.org/10.2307/4149958

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, *14*(2), Article 2. https://doi.org/10.2307/748576

David Parr, E., Roh, K. H., & Sellers, M. (2018). How do undergraduate students make sense of points on graphs in calculus contexts? *40th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 524–531.

Dray, T., & Manogue, C. A. (2006). The geometry of the dot and cross products. *Journal of Online Mathematics and Its Applications*, *6*, 1–13.

Glasersfeld, E. von. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget’s approach. In *Epistemological Foundations of Mathematical Experience* (pp. 45–67). Springer.

Glasersfeld, E. von. (1995). *Radical Constructivism: A Way of Knowing and Learning*. Falmer Press.

Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education*, 40–177.

Harel, G. (2021). The learning and teaching of multivariable calculus: A DNR perspective. *ZDM Mathematics Education*, 1–13. https://doi-org.ezproxy1.lib.asu.edu/10.1007/s11858-021-01223-8

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior*, *45*, 95–110. https://doi.org/10.1016/j.jmathb.2016.11.002

Jones, S. R. (2020). Scalar and vector line integrals: A conceptual analysis and an initial investigation of student understanding. *The Journal of Mathematical Behavior*, *59*(2), Article 2.

Jones, S. R. (2022). Multivariation and students’ multivariational reasoning. *Journal of Mathematical Behavior*, *67*, 1–18. https://doi.org/10.1016/j.jmathb.2022.100991

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, *21*(1), 87–116.

Martínez-Planell, R., & Trigueros Gaisman, M. (2021). Multivariable calculus results in different countries. *ZDM Mathematics Education*. https://doi.org/10.1007/s11858-021-01233-6
Martínez-Planell, R., Trigueros Gaisman, M., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. *Journal of Mathematical Behavior*, *46*, 13–41.
McGee, D., & Moore-Russo, D. (2015). Impact of explicit presentation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. *International Journal of Science and Mathematics Education*, *13*(2), Article 2.
Mhkatshwa, T. P. (2021). A quantitative and covariational reasoning investigation of students' interpretations of partial derivatives in different contexts. *International Journal of Mathematical Education in Science and Technology*, 1–23.
Montangero, J., & Maurice-Naville, D. (1997). *Piaget of the Advance of Knoweldge*. Psychology Press Taylor and Francie Group.
Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, 782–789.
Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, *76*(1), Article 1. https://doi.org/10.1007/s10649-010-9277-y
Moreno-Arotzena, O., Pombar-Hospitaler, I., & Barragués, J. I. (2021). University student understanding of the gradient of a function of two variables: An approach from the perspective of the theory of semiotic representation registers. *Educational Studies in Mathematics*, *106*(1), 65–89. https://doi.org/10.1007/s10649-020-09994-9
Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection* (pp. 27–42). The Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.004
Piaget, J. (2001). *Studies in Reflecting Abstraction* (Campbell, Ed.; 1st ed.). Psychology Press.
Rasmussen, C., & Wawro, M. (2017). Post-calculus research in undergraduate mathematics education. In *Compendium for Research in Mathematics Education* (pp. 551–579). National Council of Teachers of Mathematics.
Roundy, D., Weber, E., Dray, T., Bajracharya, R. R., Dorko, A., Smith, E. M., & Manogue, C. A. (2015). Experts' understanding of partial derivatives using the partial derivative machine. *Physical Review Special Topics - Physics Education Research*, *11*(2), Article 2. https://doi.org/10.1103/PhysRevSTPER.11.020126
Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), Article 2.
Steffe, L., & Thompson, P. W. (2000). Teaching Experiment Methodology: Underlying Principles and Essential Elements. *Research Design in Mathematics and Science Education*, 267–307.
Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Sage Publications, Inc.
Thompson, P. W. (1994). The Development of The Concept of Speed and its Relationship to Concepts of Rate. In *The Development of Multiplicative Reasoning in the Learning of Mathematics* (In G. Harel&J. Confrey (Eds.), pp. 181–234). SUNY Press.

Thompson, P. W. (2013). In the Absence of Meaning…. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 57–93). Springer New York. https://doi.org/10.1007/978-1-4614-6977-3_4
Thompson, P. W. (2022). Quantitative reasoning as an educational lens. *Quantitative Reasoning in Mathematics and Science Education*.
Thompson, P. W. (2019). Making the Fundamental Theorem of Calculus fundamental to students' calculus. *Conference on Calculus in Upper Secondary and Beginning University Mathematics*.
Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In *Compendium for Research In Mathematics Education* (In J. Cai (Ed.), pp. 421–456). National Council of Teachers of Mathematics.
Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for Thinking with Magnitudes: A hypothesis about Foundational Reasoning Abilities in Algebra. In *Epistemic Algebra Students: Emergins Models of students' algebraic knowing* (In L.P. Steffe, K.C. Moore, L.L. Hatfield, S. Belbase (Eds.), pp. 1–24). University of Wyoming. http://pat-thompson.net/PDFversions/2013MagsInAlg.pdf
Thompson, P. W., & Thompson, A. G. (1994). Talking about Rates Conceptually, Part I: A Teacher's Struggle. *Journal for Research in Mathematics Education*, *25*(3), Article 3. https://doi.org/10.2307/749339
Wangberg, A., & Dray, T. (2022). Measuring the derivative using surfaces. *Teaching Mathematics and Its Applications*, *41*(2), 110–124.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103–127.
Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *Journal of Mathematical Behavior*, *25*, 1–17.
Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An Example of Inquiry in Linear Algebra: The Roles of Symbolizing and Brokering. *PRIMUS*, *27*(1), Article 1. https://doi.org/10.1080/10511970.2016.1199618

Online Mathematics Tutoring: Where Did We Come From; Where Did We Go?

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Online mathematics tutoring has become a fixture of the educational landscape in higher education. Using Lepper and Woolverton's (2002) INSPIRE model for effective in-person mathematics tutoring as a basis, we developed a codebook to analyze online tutoring, modifying existing codes to reflect how they manifest in the online environment and augmenting our codebook with behaviors that were not present in the in-person environment. We give a description of the current state of online mathematics tutoring and identify methods used by online tutors to actively engage students in the learning process. In addition to our results, we include a brief description of some challenges we faced as researchers when engaging with research on online tutoring and the solutions we reached.

Keywords: online tutoring, student engagement, affect, equity

Background

Access to high-quality online tutoring is an issue of equity. Student utilization of mathematics tutoring services has been correlated with an increase in final course grades (Byerley et al., 2018; Rickard & Mills, 2018; Xu et al., 2001) and with improvements in persistence, retention, and degree completion (Astin, 1993; Pascarella & Terenzini, 1991, 2005; Rheinheimer et al., 2010; Rheinheimer & Mann, 2000; Rouche & Snow, 1977). This is critical for academically fragile students as individual or small group tutoring allows instruction to be tailored to student needs (Anghileri, 2006) while helping students develop mathematical identities (Bjorkman & Nickerson, 2019). Peer tutoring centers can also provide a space for at-risk students to find support and build community, as first-generation college students have reported needing to work harder than their peers to attain the same level of achievement, and underrepresented minority students have reported that lacking a sense of belonging presented a significant challenge to persistence in college (Richardson & Skinner, 1992). Anecdotally, the authors have witnessed an increase in the need for high-quality mathematics tutoring among undergraduate students at their institutions since the widespread school closures in 2020, an effect that was predicted by Chetty et al. (2020) and Kuhfeld et al. (2020).

Despite its importance, online mathematics tutoring has not been extensively studied. Directors of online tutoring centers face myriad decisions related to drop-in vs. appointment-based tutoring, hours of operation, types of technology to use, tutor training and evaluation, and advertisement of services, among others. As funding continues to become a more pressing concern for many college tutoring programs (Johns & Mills, 2021), it is essential that tutoring center directors are able to make informed decisions about the types and quantity of services to offer. The body of literature on online tutoring provides insight into these topics (Johns & Mills, 2021; Turrentine & MacDonald, 2006), but there has been no comprehensive systematic implementation and evaluation of online tutoring practices for undergraduates. For that reason, we present a description of online tutoring in college mathematics describing ways that tutors engage students in online tutoring and outlining challenges particular to this type of research.

Research Questions

During the advent of the modern era of online tutoring via video conferencing in Spring 2020, the authors observed that online tutoring had become something like a "drive-thru" homework service without the benefits tutoring often offers when in-person, where tutors provide emotional and metacognitive support to students, provide community, and act as a repository of institutional knowledge (Blackwell et al., 2007; Boaler, 2013; Cohen & Sherman, 2014; Dweck, 2007; Dweck et al., 2014; Moser et al., 2011). Even during purely mathematical tutoring sessions, skilled tutors often place much of the problem solving in the hands of their students by asking thought-provoking questions and guiding students to exhibit more mathematical habits (Lepper & Woolverton, 2002).

Our main goal with this research was to find ways to restore these more "meta-mathematical" practices in our online tutors and to place more focus on the needs of their students. We began with simple observations of online tutoring sessions to gain a deeper understanding of what was already taking place in these sessions to determine how best to improve them. This led to the first research question we attend to below: *What does online mathematics tutoring look like?*

With a general sense of how online tutoring sessions typically operate, we returned to our original goal of improving the student experience in online tutoring. Our second research question focuses on just one aspect of the online tutoring experience: *How are tutors engaging students in the online tutoring process?*

Theoretical Framing

We viewed our data through the lens of Lepper and Woolverton's (2002) INSPIRE model for effective tutoring. This model identifies cognitive, metacognitive, and affective strategies and considerations employed by expert tutors. These strategies focus on academic content, but also emphasize the importance of study skills and student mindsets. The INSPIRE model, outlined in Table 1, categorizes the practices and dispositions of expert tutors into seven categories.

Table 1. INSPIRE Model category descriptions.

Category	Description
Intelligent	Knowledge of subject matter and pedagogical strategies
Nurturant	Developing a personal rapport with the tutee
Socratic	Asking questions to foster dialogue rather than telling
Progressive	Purposeful selection of problems, systematic feedback, predictable routines
Indirect	Providing appropriate types and amounts of feedback
Reflective	Asking the tutee to explain reasoning and generalize
Encouraging	Promoting confidence, challenge, curiosity, control of the learning

Although the INSPIRE model is a comprehensive summary of the practices of effective tutors, the model was developed from observations of tutors operating in the in-person tutoring environment. Using the INSPIRE model as an established baseline for "good tutoring," our work expands on these results by describing effective practices in the setting of online tutoring.

Methods

Data were collected from Spring 2021 through Spring 2023 at two universities, U1 and U2, and consisted of recorded online tutoring sessions. Both universities offered tutoring in two modalities: in-person during the day and online via Zoom in the evening. Each semester there were five tutors at each university providing online tutoring and participating in the research project. All tutors received tutor training, and those who were tutoring online received training specific to the online environment. Additionally, online tutors were provided with a tablet and stylus to facilitate writing capabilities in online sessions. All students, regardless of whether they were enrolled in in-person or online classes, could attend either in-person or online tutoring.

U1 Context

U1 is a research-focused institution in the mid-Atlantic region of the U.S. with a total enrollment of about 21,000 undergraduate students. It is approximately 50% residential, with mostly freshmen living on campus. U1 has two campuses, A and B, about 2 miles apart. The mathematics tutoring center is located on Campus A in the same building as the mathematics department and offers drop-in, in-person tutoring. Although most math courses are offered on Campus A, only about 9% attend their mathematics class in the same building as the tutoring center, and 25% of students enrolled in a math class attend their class on Campus B.

U2 Context

U2 is a research-focused institution in the southwest region of the U.S. with a total enrollment of about 23,000 undergraduates. It is 29% residential, with most freshmen residing on campus. U2's mathematics tutoring center offers drop-in, in-person tutoring. U2's tutoring center is located on the main floor of the building where about 75% of its math classes are offered and where all mathematics instructors' offices are located.

Codebook Development

Four researchers developed and participated in the development of our codebook and the analysis of our data. Data analysis began by applying codes that were adapted from Lepper and Wolverton's (2002) INSPIRE model presented above. As their study was focused on in-person tutoring sessions, the codes needed to be adapted for the online tutoring environment. The codebook underwent several revisions across 11 coding-specific meetings as we began adapting the codes to what was present in the data. Table 2 below describes our final broad category codes including their relation to the INSPIRE Model categories.

Table 2. Descriptions of coding categories used for study.

Code	Description
Technology	Technology and digital resources used during tutoring captured with subcodes to indicate tutor or student sharing screen, who writes, etc.
Multiple Students	Tutor work involving multiple students during a session
Modes of Interaction	Tutor's interaction modes: artifacts, gesture, inscriptions, and speech
Tutoring Content Knowledge	How the tutor approaches the student making meaning of the mathematics; how the tutor's understanding of the content drives the tutoring episode; how the tutor facilitates learning including how his explanations/questions drive the session, degree to which tutor tries to facilitate student understanding; related to the Intelligent and Progressive categories
Tutor Affective Interactions	How the tutor facilitates or drives the tone of the interaction; related to Nurturant and Encouraging categories
Tutor Communicative Interactions	How the tutor facilitates the student being an active participant engaging in communication; related to the Socratic, Indirect, and Reflective categories
Student Pursuit of Understanding	How the student enters the tutoring space; how student expectations drive the session; degree to which student seeks understanding

Units of Analysis

When data analysis began, we set the unit of analysis to be the interaction between tutor and student on a single problem. Although this fairly granular level of analysis allowed certain interactions to be captured, it missed others, especially affective interactions. There was also significant repetition in coding since interactions across problems in a setting were extremely similar. Hence, we zoomed out and tried using a single student-tutor interaction; using the problems they completed together in a single "sitting." Again, we found that some key interactions were overlooked such as how a tutor handles multiple students in the session and transitions between students. Thus, we settled on each individual recording that corresponded to the tutor's shift for the day, usually 1 to 2.5 hours, as our unit of analysis.

Results

The nature of the tutoring we observed among the tutors in our study has changed dramatically since data collection began in Spring 2021. At the inception of our study, online mathematics tutoring looked like a "drive-thru" homework service. *Student pursuit of understanding* was often minimal in that students arrived in the online tutoring space with questions like "I don't know how to do number 10 in my homework. How do you do it?" In response, *tutor communicative interactions* were lacking; as the tutors typically solved the problem for students, rarely involving the students in the process, and then students would leave the tutoring space. There were few *affective interactions,* and tutors and students were scrambling to figure out ways to share their thinking and writing (*modes of interaction*). This is not entirely surprising since tutors were not trained to operate in the online environment, and the technology to which tutors had access varied widely. After tutors were given tablets and training specific to tutoring mathematics online, the nature of tutoring sessions began to change. In this section, we will first discuss what online mathematics tutoring looks like when tutors have appropriate technology and training, and then we will identify specific strategies employed by tutors to engage students in the tutoring process in the online environment.

What Online Mathematics Tutoring Looks Like Now

Tutoring sessions begin differently now than they used to; tutors are more intentional about welcoming students into the tutoring space. It is very easy for students to feel like they are being ignored when they enter a video conferencing session, and a tutor is already deep in conversation with another student. Unless the tutor explicitly acknowledges that students have entered the tutoring space, the students may not know if anyone has noticed their presence. Moreover, tutors keeping their cameras on is particularly important, as students may not even know if a tutor is present when they enter the virtual tutoring space without this visual information.

Some in-person tutoring practices related to *tutor affective interactions* have translated well to the online environment. For example, when a new student enters the tutoring space while a tutor is already working with a student, tutors often ask what class the new student is in as well as more details on why they are coming to tutoring. It was common to hear, "What do you need help with today?" When the new student's needs overlap with the topic of the conversation in-progress, the tutor will invite the new student to work along with them so both students can be helped simultaneously. If the new student's needs differ significantly from what the tutor is working on with the first student, the tutor will then bounce between students, giving one student a task to work on while discussing progress that has been made by the other student. In this manner the *tutor communicative interactions* are attempting to help more than one student actively engage in learning mathematics during the tutoring session.

In some cases, we have observed a set of new practices develop that are better suited to the online setting than in-person – on the parts of both tutors and students. Students have expressed being able to learn by listening to conversations between tutors and students in which they were not actively taking part. In person, it is often difficult to listen in on other conversations in the tutoring space because of physical distance between students, background noise from other conversations, and other factors. Video conferencing software makes it especially easy for students to eavesdrop on other conversations. So, for some students, the *student pursuit of understanding* has shifted from attending online tutoring to get help with certain homework problems to attending to learn through listening to the conversations of others and piping in with questions about another's tasks.

Regarding *tutor communicative interactions* involving multiple students, there is one in-person tutoring practice that has not been noted in online tutoring. In person, tutors often encourage students to sit in pairs or small groups in the tutoring center to work on similar problems so they can collaborate while the tutor helps other students. Our data set does not contain any instances of students working together while the tutor works with another student. To do this online without disrupting others, the tutor would have to place students in a breakout room and then move between breakout rooms to check on different students. It is possible that our tutors either do not know how or are not comfortable doing this. It is also possible that, in the moment, they have decided that the costs in terms of time and efficiency outweigh the benefits.

Looking at the *modes of interaction*, gesture usage in the online environment is particularly interesting. Deictic (pointing), iconic, and metaphoric gestures (Alibali et al., 2014) are nearly absent from online tutoring. Tutors seem to realize that many gestures are hard to see and that others cannot identify the referents of their pointing gestures, rendering them ineffective. However, we observed a large number of what Alibali et al. (2014) call writing gestures: "writing or drawing actions that were integrated with speech in the way that hand gestures are typically integrated with speech but that were produced while holding a writing instrument (usually chalk or marker) and that involved writing to indicate or illustrate the content of the accompanying speech (e.g., underlining an equation on the board while saying 'this equation')" (p. 76). Tutors became adept at writing or drawing with different colors for the purpose of emphasizing or differentiating between different kinds of information, such as text written as part of a problem's solution and text written as an annotation of that solution for their student's edification. As with deictic gestures, tutors seemed to understand that students could not see where their stylus was when they were tracing over a figure, as when tracing over the graph of a parabola to emphasize its shape, or when repeatedly tracing over a tangent line to emphasize its slope. Tutors used color in these instances to call students' attention to the importance of certain features or to add new information to previously drawn figures.

How Tutors Engage Students in Online Tutoring

Tutors' engagement of students in the tutoring process has improved substantially since the inception of the modern age of online tutoring in Spring 2020, both in terms of *tutor affective interactions* and *tutor communicative interactions*. Tutors have improved in their abilities to create an inviting space for students to engage with them and with mathematics. Tutors have also become much more adept at using technology to facilitate interactions that allow students to become active participants in the conversation.

Tutors have adopted a subtle strategy whose power is often understated in the in-person tutoring environment: casual and welcoming conversation. The tutors in our sample began each session by intentionally greeting and welcoming students into the online tutoring space and engaging them in casual conversation by asking how they were doing or what their plans were for the weekend, commiserating about the challenges of being a student, or discussing events taking place around campus. More than just creating a comfortable space, this act seemed to prime students for social interaction and set a tone for the tutoring session in which both tutor and student were active participants.

Tutors now engage in much more questioning behavior than in the early days of online tutoring. After the tutors welcomed students into the session, the next tutor move was typically to ask students what they need help with and to describe the progress made so far, and in some cases reenact their thought process up to the point where they encountered difficulty. This allowed the tutors to identify what students already understood and to diagnose any misconceptions or gaps in understanding. Tutors continued to ask questions throughout their sessions, encouraging students to think instead of passively receiving information from tutors. When working on procedural tasks such as solving equations, tutors often asked students for procedural information with justification (i.e., next algebraic step and why that operation is the correct choice) or more conceptual questions related to task at hand.

Tutors' increased facility with technology has improved their ability to engage students. One thing we noticed after providing tutors with tablets was that, rather than replacing their original device with the tablet, they used the tablet in concert with the original device. The purpose of the original device shifted to become solely for face-to-face interaction with their students, while the tablet was used for writing and drawing. This experience mirrors the in-person tutoring environment by allowing tutors the ability to look at the students and their work simultaneously, as if they were working side-by-side with a sheet of paper or a whiteboard between them.

Tablets facilitated our tutors' ability to act as scribes during sessions in which their students were not able to write on their own devices. In the early days of synchronous online tutoring, in many cases, neither tutors nor students had a device that would allow them to work together in a visible collaborative space. Our tutors' access to tablets gives them the ability to ask students questions and keep a record of their work, allowing students to drive conversations about solving algebraic equations and computing derivatives and integrals while tutors simply keep track of their work for them.

Discussion

We have learned quite a bit about how to make the online mathematics tutoring experience better for students. Most importantly, we have learned that, with access to training and tools, tutoring practices improve. Since Spring 2020, the tutors in our sample have transitioned from simply serving as a quick homework solution service to providing highly interactive learning experiences for students. The practices our tutors now demonstrate are well aligned with established best practices for in-person tutoring – though more in-depth study is necessary to determine a set of best practices for online tutoring. Our work takes a necessary first step toward establishing these best practices in the research literature.

In person, nonverbal communication conveys a substantial amount of information. Through communication via video conferencing software, much of this nonverbal information is lost (particularly if a person's camera is turned off) or must be communicated in different ways. As we observed in our data, gestures, especially deictic gestures and beat gestures (which are often used for emphasis and to direct attention), do not seem to be used as frequently as they are during in-person communication. However, among the tutors in our sample, writing gestures seem to have supplanted many forms of gesture, and these writing gestures are often supplemented with strategic use of color to draw students' attention.

Challenges of Online Mathematics Tutoring Research

When working on such a multifaceted topic, we, as researchers, experienced more challenges than initially anticipated. Issues related to data collection, codebook development, selection of an appropriate unit of analysis, and other things arose as our project progressed. We conclude by discussing a selection of these problems and our solutions to them. This list is not exhaustive, but we present it for the benefit of others who may consider engaging in similar research.

As noted earlier, we revisited the process of selecting an appropriate unit of analysis several times. As we operationalized this decision, we quickly realized that tutoring sessions varied greatly in terms of length, with some lasting only 10-15 minutes and others lasting upwards of 2 hours, and tutor behaviors in the first 10 minutes of a session often look very different than they do after an hour. Eventually, we accepted that there was likely no choice that would allow us to uniformly capture all the information we wanted, and we settled on our current unit of analysis in the interest of advancing the project. Subsequent analyses may call for a different unit.

Development of an appropriate codebook was also challenging. Using the INSPIRE model as an initial guide, we set out to investigate the prevalence of a subset of these practices: the Nurturant, Socratic, and Encouraging aspects of the framework. As we coded, we realized that the difference in environment (online vs. in-person) introduced behaviors not accounted for by the original model (e.g., issues and behaviors related to technology), forcing us to expand the codebook. Furthermore, within each of the behaviors coded, we recognized that many were present to varying degrees: for some tutors, drawing several pictures per session was commonplace, while others might only draw one picture per session; though two tutors might both ask questions, one tutor may ask significantly more questions. This caused us to augment our codebook in places to include subcodes related to degree – strong, weak, or mixed.

Conclusions and Future Work

We have made great strides toward improving the online tutoring experience for students. The online tutoring environment has become much more like the in-person environment, and students are able to be much more interactive participants in the learning process instead of passive recipients of information. Despite our best efforts to make the online environment feel like the in-person environment, we must acknowledge that the online environment is not, and will not be, the in-person environment, and therefore some differences may persist. With access to training and appropriate technology, tutors can adapt the online environment to work just as effectively for students as the in-person environment.

Research into mathematics tutoring is a complicated process (see Johns et al., 2021), and this is doubly true for online tutoring. The online environment now seems to be a permanent fixture of the educational landscape; we must understand its similarities and differences with in-person environments to provide our students with the best learning experience possible.

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References

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Martinez, C. J., & Knuth, E. (2014). How teachers link ideas in mathematics instruction using speech and

gesture: A corpus analysis. *Cognition and Instruction*, *32*(1), 65-100. https://doi.org/10.1080/07370008.2013.858161
Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, *9*(1), 33-52.
Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*, 97-109. https://doi.org/10.1007/s10649-008-9163-z
Astin, A. (1993). *What Matters In College: Four critical years revisited*. Jossey-Bass.
Bjorkman, K., & Nickerson, S. (2019). This is us: An analysis of the second groups within a mathematics learning center. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro. (Eds.) *Proceedings of the 22nd Annual Conference on Research on Undergraduate Mathematics Education* (pp. 44-52). Oklahoma City, OK: RUME.
Blackwell, L. S., Trzesniewski, K. H., & Dweck, C. S. (2007). Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development*, *78*(1), 246-263.
Boaler, J. (2013). Ability and mathematics: the mindset revolution that is reshaping education. *FORUM*, *55*(1), 143-152.
Byerley, C., Campbell, T., & Rickard, B. (2018). Evaluation of impact of calculus center on student achievement. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown, (Eds.) *Proceedings of the 21st Annual Conference on Research on Undergraduate Mathematics Education* (pp. 816-825). San Diego, CA: RUME.
Chetty, R., Friedman, J., Hendren, N., Stepner, M., & Team, T. O. I. (2020). *How Did COVID-19 and Stabilization Policies Affect Spending and Employment? A New Real-Time Economic Tracker Based on Private Sector Data* [Working Paper].
Cohen, G. L., & Sherman, D. K. (2014). The psychology of change: Self-affirmation and social psychological intervention. *The Annual Review of Psychology*, *65*, 333-371. https://doi.org/10.1146/annurev-psych-010213-115137
Dweck, C. S. (2007). Is math a gift? Beliefs that put females at risk. In S. J. Ceci & W. Williams (Eds.), *Why aren't more women in science? Top researchers debate the evidence* (pp. 47-55). American Psychological Association. https://doi.org/https://doi.org/10.1037/11546-004
Dweck, C. S., Walton, G. M., & Cohen, G. L. (2014). *Academic Tenacity: Mindsets and Skills that Promote Long-Term Learning*. B. M. G. Foundation.
Johns, C., Byerley, C., Moore-Russo, D., Rickard, B., Oien, J., Burks, L., James, C., Mills, M., Heasom, W., Ferreira, M., & Mammo, B. (2023). Performance assessments for mathematics tutoring centres. *Teaching Mathematics and Its Applications*, *42*(1), 1–29. https://doi.org/10.1093/teamat/hrab032
Johns, C., & Mills, M. (2021). Online mathematics tutoring during the COVID-19 pandemic: Recommendations for best practices. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *31*(1), 99-117. https://doi.org/10.1080/10511970.2020.1818336
Kuhfeld, M., Soland, J., Tarasawa, B., Johnson, A., Ruzek, E., & Liu, J. (2020). *Projecting the Potential Impacts of COVID-19 School Closures on Academic Achievement*. Brown University.
Lepper, M. R., & Woolverton, M. (2002). The wisdom of practice: Lessons learned from the study of highly effective tutors. In J. Aronson (Ed.), *Improving Academic Achievement: Impact of Psychological Factors on Education* (pp. 135-158). Academic Press.

Moser, J. S., Schroder, H. S., Heeter, C., Moran, T. P., & Lee, Y.-H. (2011). Mind your errors: Evidence for a neural mechanism linking growth mind-set to adaptive posterror adjustments. *Psychological Science*, *22*(12), 1484-1489. https://doi.org/10.1177/0956797611419520
Pascarella, E., & Terenzini, P. (1991). *How College Affects Students: Findings and Insights From Twenty Years of Research*. Jossey-Bass.
Pascarella, E., & Terenzini, P. (2005). *How College Affects Students: A Third Decade of Research* (Vol. 2). Jossey-Bass.
Rheinheimer, D. C., Grace-Odeleye, B., Francois, G. E., & Kusorgbor, C. (2010). Tutoring: A support strategy for at-risk students. *The Learning Assistance Review*, *15*(1), 23-34.
Rheinheimer, D. C., & Mann, A. (2000). Gender matching, floor effects, and other tutoring outcomes. *Journal of Developmental Education*, *24*(2), 10-18.
Richardson, R. C., Jr., & Skinner, E. F. (1992). Helping first-generation minority students achieve degrees. *New Directions for Community Colleges*, *80*, 29-43.
Rickard, B., & Mills, M. (2018). The effect of attending tutoring on course grade in Calculus I. *International Journal of Mathematical Education in Science and Technology*, *49*(3), 341-354.
Rouche, J., & Snow, J. (1977). *Overcoming Learning Problems*. Jossey-Bass.
Turrentine, P., & MacDonald, L. (2006). Tutoring online: Increasing effectiveness with best practices. *NADE Digest*, *2*(2), 9-18.
Xu, Y., Hartman, S., Uribe, G., & Mencke, R. (2001). The effects of peer tutoring on undergraduate examination scores in mathematics. *Journal of College Reading and Learning*, *32*(1), 22-31. https://doi.org/10.1080/10790195.2001.10850123

Undergraduate Gender Differences in Knowledge of the GRE and Perception of the GRE as a Barrier to Applying to Graduate Mathematics Programs

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Graduate Record Examination (GRE) scores are commonly required in applications to graduate school in mathematics. We examine undergraduate mathematics majors' knowledge of the GRE and their perceptions of the GRE as a barrier to applying to these programs as part of a larger project studying student knowledge of the graduate school application process and how it contributes to lack of diversity in graduate mathematics programs. We found that there was an association by gender, and that women were less likely to report that they had heard of the GRE General and Subject Tests. Similarly, women were more likely to report that the GRE tests were a potential barrier to their decision to apply to graduate mathematics programs.

Keywords: Graduate Record Examination, Graduate School Applications, Gender, Social Cognitive Career Theory

The field of mathematics lacks diversity; this becomes more pronounced at higher levels of education. While 50.8% of the U.S. population identify as women and 31.9% as Hispanic/Latinx or African American (U.S. Census, 2020), in recent years only 39% of mathematics bachelor's degrees were earned by women (Golbeck et al., 2019) and 15.9% of mathematics and statistics (mathematics-only data unavailable) bachelor's degrees were earned by minoritized[1] students (National Center for Science and Engineering Statistics, 2019). At the highest levels of formal education, only 24.1% of new mathematics doctoral recipients were women and 7.4% were minoritized (Golbeck et al., 2020).

Many graduate programs in the U.S. require students to take the Graduate Record Examination (GRE) General Test, and some programs also require the discipline specific test as well (e.g. mathematics, physics). Graduate programs often use the GRE to gauge applicants' preparedness for graduate school. Despite recommendations against this practice from the ETS (Miller et al., 2019; Posselt, 2016), some programs advertised cut off scores for the Subject and General Test in order to apply (Miller et al., 2019; Petersen et al., 2018). Other programs weigh GRE performance heavily to speed up the review process (Petersen et al., 2018; Posselt, 2016). The frequent use of cut off scores and heavy weighting of the GRE led to investigation into whether these practices disadvantage certain groups. Studies found that minoritized students and women score lower on the GRE than their counterparts (Bleske-Rechek & Browne, 2014; Cochran et al., 2018; Miller et al., 2019; Petersen et al., 2018). In fact, women and minoritized students were less likely than their peers to score above a program's stated Subject Test threshold, which reduces their chances of being accepted to a given program (Miller et al., 2019; Posselt et al., 2019; Verostek et al., 2021; Young & Caballero, 2021). Several other studies found

[1] Minoritized is an alternative way of referring to people who are often labeled as "Underrepresented Minorities" in STEM. This alternative phrasing makes it clear that it is power imbalances and systematic oppression that cause these groups to be less represented in STEM (Wingrove-Haugland & McLeod, 2021).

similar results for the General Test (Petersen et al., 2018; Posselt et al., 2019). Building on these results, researchers have investigated whether the GRE predicts PhD completion, finding that GRE performance does not predict PhD completion or success in a PhD program (Miller et al., 2019; Petersen et al., 2018; Roberts et al., 2021; Wilson et al., 2018). So, while the usefulness of the GRE in predicting success is uncertain it still often serves as a barrier for women and minoritized students pursuing graduate education. In this paper we examine data related to what undergraduate mathematics majors know about the GRE as they prepare to apply to graduate school.

This study is part of the Undergraduate Knowledge of the Mathematics Graduate School Application Process (Knowledge-GAP) project which was created to explore undergraduate mathematics majors' knowledge about the graduate school application process and differences in perceived barriers to applying to graduate school across different demographic groups. This paper focuses on the results related to the GRE. Specifically, we examined differences in knowledge and perceptions of the GRE between graduate school applicants.

1. *What do undergraduate mathematics majors know about the GRE?*
2. *Do knowledge and perception of the GRE differ by gender identity of the students?*

Theoretical Background

Social Cognitive Career Theory (SCCT) incorporates Tinto's non-cognitive and contextual factors known to be important in retaining minoritized students and women and expands them for use in STEM career choice for these groups (Lent et al., 1994, 2000; Tinto, 1975, 1993, 2007). Tinto's sense of academic belonging is particularly important for student groups marginalized in STEM; SCCT refines this aspect and identifies several additional significant barriers affecting degree interest and completion for minoritized students and women, each of which are exacerbated by institutionalized environmental barriers at every level of education (Alexander & Hermann, 2016; Cutright et al., 2015; Estrada et al., 2016). The SCCT model incorporates gender as an individual characteristic and situates it within a person's context specific characteristics. Kanny et al., (2014) discussed SSCT studies focusing on individual characteristics (e.g. race, ethnicity), structural barriers (e.g., institutional and classroom climates), psychological factors (self-efficacy or sense of belonging), and family influences (including gender role socialization and self-concept), and perceptions of STEM careers. Within the SCCT framework, each of these contextual factors impacts career trajectory by acting as either a facilitator or a barrier and they may even be the key factors influencing lower participation of women and minoritized people in STEM careers. For example, racist and sexist systemic barriers may affect both the entrance and persistence of marginalized groups in STEM careers by negatively influencing their STEM self-efficacy and their STEM career outcome expectations.

Within this framework, we view the GRE as representative of a structural barrier for some groups of students wishing to enroll in graduate programs in mathematics. A recent study of mathematics graduate programs at three large research universities reported that only 18%, 15%, and 12% of applicants were women, respectively (Gevertz & Wares, 2020). Given the widespread use of the GRE as an application requirement, gender differences in knowledge of and perception of the GRE as a barrier to applying has the potential to impact the demographics of mathematics graduate education.

Methods

Instrument Development

The research team created a survey based, in part, on a survey used to determine undergraduate physics majors' interest in graduate school and how important they believed different aspects of the application process were (Chari & Potvin, 2019). Nineteen survey items were adapted from that instrument, though a notable difference in our survey was that we provided an opportunity for participants to express their lack of knowledge about different parts of the application process. The final survey had 57 items separated into four categories: (a) knowledge about different aspects of the application process, (b) barriers to applying, (c) interest in graduate school and what students look for in programs they apply to, and (d) demographic questions. Most questions were Likert scale or multiple choice, though four were open-ended and some of the multiple-choice items allowed participants to type in a text response. The full survey is available at this link: https://researchrepository.wvu.edu/faculty_publications/3291/

Data Collection

To ensure broad participation, the research team sent the survey to department chairs and undergraduate program directors at all undergraduate mathematics programs at colleges and universities in the U.S. with at least 1000 students total ($N = 985$). We asked programs to send the survey to all undergraduate mathematics majors. Initial emails were sent Fall 2022 through Spring 2023, via Qualtrics, and follow-up emails were sent to encourage a greater response rate. In addition to direct emails, the research team also posted the survey on social media, listservs, and in newsletters for several professional organizations in mathematics.

Data Analysis

We received 1090 responses from students at 181 colleges and universities, with 519 complete responses. Note that students could miss part of a question and still have their response marked as complete. Thus, the *N*s for different items are not always the same. Statistical tests were run in IBM SPSS.

To address our research questions, we analyzed responses to five survey items. Three were binary response items asking participants about the following aspects of both the GRE General Test and Mathematics Subject Test: if they had previously heard of, or taken, the tests; if they knew about the different sections on the test, testing modality options, testing frequency and locations, costs associated with taking exams and having scores sent to institutions, and availability of fee waiver codes. The final two were the Likert scale items: *To what extent are the following factors a potential barrier to your pursuit of graduate school?* and *How important are the following factors in choosing which schools you apply to?* Both items were adapted for our study from Chari and Potvin (2019). The first item had 17 sub-item topics, rated on a scale of 1 (not at all a barrier) to 5 (very significant barrier). The second item had 15 sub-item topics, rated on a scale of 1 (not at all important) to 5 (very important). For this paper we only analyze responses to the seven sub-item topics related to the GRE.

Results

Participant Demographics

Table 1 shows demographics of participants with complete responses. Participants were able to select more than one option for gender, so the total adds up to more than 100%.

Table 1. Self-identified Gender.

What is your gender?		
Gender	*N*	Percentage
Man	251	48.4%
Woman	226	43.5%
Genderqueer or Non-Binary	41	7.9%
Agender	12	2.3%
Transgender	21	4.0%
A gender not listed	2	0.4%
Prefer not to say	8	1.5%
Total	519	

Knowledge of the GRE

Overall participant knowledge of the GRE was incomplete at best. While a majority of the participants (379/518, or 73.2%) had heard of the test before, only half (270/518, or 52.1%) had heard of the GRE Mathematics Subject Test. More worryingly, when asking participants who had heard of the GRE what specifically they knew about the exam, there were large gaps in specific knowledge about the exam. Of the 346 participants who said they had heard of the GRE General Test before, only about half (50.7%) knew that the test had three sections: Verbal Reasoning, Quantitative Reasoning, and Analytical Writing. Only about half of those participants (168/346, or 48.6%) knew that the test is offered with regular frequency, and that it is possible to take it from home. For questions pertaining to the cost of the exam, only a third of participants (115/346, or 33.2%) knew the cost of the test ($220). Just over a fifth of participants (74/345, or 21.4%) knew that it costs $30 to send GRE scores to a graduate program after taking the test and just over a fifth (79/345, or 22.9%) knew that fee waivers were available for the GRE.

For the GRE Mathematics Subject Test, of the 267 participants who said they had heard of the test before, only about a third (96/267, or 36.0%) knew that the test is only available three times a year. About 40% (107/267) knew that at the time the survey was administered, the test was not available to take from home and you had to travel to a testing center to take it. Finally, only 30.5% (81/266) knew the cost of the test ($150).

It is necessary to mention that these results are for a subset of the larger sample. For example, 36% of participants who had heard of the Mathematics Subject Test knew how often the tests are available, but only 52% of participants overall had heard of the Mathematics Subject Test. Therefore, the percentage of participants overall who knew how often the Mathematics Subject GRE is offered was only 96/518 = 18.5%. In addition, only a small number of participants had taken either of the two GRE tests before taking the survey: 50/518, or 9.7%, for the General Test and 25/518, or 4.8%, for the GRE Mathematics Subject Test.

Based on the established literature on gender and GRE performance we tested if there was a difference in knowledge or perception of the GRE as a barrier by participant gender. One issue we encountered in our data analysis was that our participants were not limited to a gender binary like most previous studies of the GRE. To get results comparable to previous studies participants were separated into two groups based on their answer to the survey item asking for their gender. Participants who said they were women, regardless of whether they selected any additional gender identities were labeled as “Women” for our analysis. This includes women who are also cisgender, agender, transgender, and/or non-binary. Similarly, participants who did not select the women option were labeled as “non-women”. We use this categorization to have results comparable to studies that had a binary definition of gender, while also being inclusive of our participants’ other identities. We found that there was an association by gender, women were less likely to say they had heard of the GRE General Test before taking the survey $\chi 2(1, N = 518,) = 13.47, p = <.001$ ($V = .16$). Similarly, there was an association by gender, and women were less likely to say they had heard of the GRE Mathematics Subject Test before taking the survey $\chi 2(1, N = 518) = 14.95, p = < .001$ ($V = .17$). Both results had a small effect size. There were no associations by gender for the other survey items about knowledge of the GRE (All p’s > .05).

Perception of the GRE

We report here on the GRE-related sub-item topics for the two Likert scale items, five for the first item and two for the second item. A one-way analysis of variance (ANOVA) was not employed because for 3 of the 7 sub-item topics the Homogeneity of Variance assumption was violated. Thus, for ease of comparison and consistency, Mann-Whitney U tests were performed using the women/non-women variables for all sub-item topics. Table 2 contains Mann-Whitney U test results for the women/non-women groups for the 519 participants who responded to the selected sub-item topics from the first item. The output of a Mann-Whitney U test is a Z value on a normal distribution. The Z values in Table 2 indicate that the women group has greater means than the non-women group. These results show there is a statistically significant difference (all p’s < .05) between the women/non-women groups in the responses for all five sub-item topics. In all cases, women were more likely to view each sub-item topic as a potential barrier to their pursuit of graduate school than their peers. All these results had a small effect size (all r’s between 0.1 and 0.3).

Table 2. Mann-Whitney U test results for selected items for the question “To what extent are the following factors a potential barrier to your pursuit of graduate school?” using the women/non-women variable.

Item	Group	*N*	Mean	Mean Rank	*U*	*Z*	*p*	*r*
The need to do well on the GRE General Test	Women Non-Women	226 291	3.08 2.57	295.28 230.82	24684	-5.01	<.001	0.22
The need to do well on the GRE Mathematics Subject Test	Women Non-Women	226 291	3.37 2.75	301.06 226.33	23376.5	-5.80	<.001	0.26
Paying for the General GRE Test ($220)	Women Non-Women	225 290	3.10 2.57	289.42 233.62	25556	-4.32		0.19
Paying for the GRE Mathematics Subject Test ($150)	Women Non-Women	225 292	3.05 2.51	291.35 234.07	25571	-4.42	<.001	0.19
Sending GRE scores to programs ($30 per program)	Women Non-Women	224 292	2.84 2.43	283.58 239.26	27086.5	-3.44	<.001	0.15

							<.001	

For the second survey item, “How important are the following factors in choosing which schools you apply to?”, it should be noted that not all participants saw this item. Prior to this, survey participants were asked to state their interest in graduate school in mathematics. Only participants who responded with anything other than “Not interested in graduate school in mathematics” saw this item. Table 3 contains Mann-Whitney U test results for the women/non-women groups for the 438 participants who responded to the selected sub-item topics from the second survey item. The Z values in Table 3 indicate that the women group have greater means than the non-women group. Results show a statistically significant difference (all p’s $< .05$) between the women/non-women groups in the responses for all five sub-item topics. The women were more likely to view each sub-item topic as an important factor in choosing which school to apply to than their peers. All results had a small effect size (all r’s between 0.1 and 0.3). These Mann-Whitney U test results show that women are more concerned about all aspects of the GRE compared to their peers.

Table 3. This table provides Mann-Whitney U test results for selected items for the question “How important are the following factors in choosing which schools you apply to?” using the women/non-women variable.

Item	Group	N	Mean	Mean Rank	U	Z	p	r
No GRE General Test requirement or no minimum score requirement	Women Non-women	184 253	2.83 2.46	240.37 203.46	19344.5	-3.10	.002	0.15
No GRE Mathematics Subject Test requirement or no minimum score requirement	Women Non-Women	184 254	2.99 2.47	248.55 198.45	18022.5	-4.19	<.001	0.20

Discussion

Overall, we found that while students may have heard of the GRE, they rarely had detailed knowledge of the exam, including where and when it is offered, and its associated costs. Those last two points are especially problematic; if students miss the deadline or do not have the finances to afford the exam, they cannot apply to any program that requires GRE scores. The financial barrier is particularly a problem, since minoritized students often come from lower income families than their peers, and thus are more likely to have the cost of the GRE serve as a barrier to applying to graduate school (McEldowney et al., 2024). Since this survey was conducted, the Educational Testing Service recently changed many aspects of how they offer both the GRE General and Subject Tests, including offering them remotely (Educational Testing Service, 2023a, 2023b). This did not impact our results since these changes occurred after data collection was completed.

We contribute to the literature on gender differences in GRE test scores by finding gender differences in knowledge and perception of the GRE. Women were less likely to have heard of either GRE test, but for those who had heard of the exam their knowledge of the exams was not statistically different from other participants. More research is needed to determine the cause of this observed difference. As for perception of the GRE, women were more likely to state that the GRE, both the General Test and Mathematics Subject Test, were barriers to applying to graduate school. Women were also more likely to favor applying to programs with less rigorous GRE requirements. Given the established literature showing that womens' average scores are lower than their peers on the GRE, which disadvantages them in the application process, (Bleske-Rechek & Browne, 2014; Miller et al., 2019; Petersen et al., 2018) it is not surprising that women would view these exams negatively. Our results demonstrate that the GRE acts as an institutionalized environmental barrier, as proposed by SCCT, that affects degree interest and interest in applying to programs is perceived differently depending on the gender of the participant.

A recurring part of the conversation surrounding the GRE is whether it should be part of the graduate admissions process. During the height of the pandemic many programs dropped the GRE due to unavailability. Even now many programs have decided to continue not requiring the GRE (Google, n.d.). Many disciplines have dropped the subject GRE requirement altogether to the point where the ETS now only offers three subject tests: Mathematics, Physics and Psychology. There are very few studies of the Mathematics Subject GRE, though there is research on the reliability and impact of the Physics Subject Test (Miller et al., 2019, Young & Caballero, 2021). We challenge the research community to study the Mathematics Subject GRE to this level of rigor.

An important consideration we had for this paper was how to utilize the provided demographic information of our participants. Most existing research on the GRE assumes a gender binary while our results give a more honest and interesting reflection of gender among American college students. To tie our work back to the established literature we decided to use the women/non-women categories. While this categorization is imperfect, it was the most ethical solution we found to run statistically meaningful tests. That said, we call on future researchers to use gender beyond the binary in their quantitative research. New formulations and solutions will be needed to do this well, but we owe that to our participants.

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References

Alexander, Q. R., & Hermann, M. A. (2016). African-American women's experiences in graduate science, technology, engineering, and mathematics education at a predominantly White university: A qualitative investigation. *Journal of Diversity in Higher Education, 9(4),* 307-322.

Bleske-Rechek, A., & Browne, K. (2014). Trends in GRE scores and graduate enrollments by gender and ethnicity. *Intelligence, 46,* 25-34.

Chari, D., & Potvin, G. (2019). Understanding the importance of graduate admissions criteria according to prospective graduate students. *Physical Review Physics Education Research*, *15*(2), 023101.

Cochran, G. L., Hodapp, T., & Brown, E. E. (2018). Identifying barriers to ethnic/racial minority students' participation in graduate physics. In *Physics Education Research Conference*.

Cutright, T. Williams, L., Coats, L., & Puskas, J. E. (2015). Easing the tortuous road that Under-Represented Minorities travel to become engineering faculty. *In Proceedings of the 122nd ASEE Annual Conference & Exposition, American Society for Engineering Education, June.*

Educational Testing Service. (2023a) *Shorter GRE® Test Coming September 2023* https://www.ets.org/news/press-releases/shortened-gre-test-coming-september-2023.html

Educational Testing Service. (2023b) *Upcoming GRE Subject Test Changes*. https://www.ets.org/gre/score-users/subject-test-changes.html

Estrada, M., Burnett, M., Campbell, A. G., Campbell, P. B., Denetclaw, W. F., Gutiérrez, C. G., Hurtado, S., John, G. H., Matsui, J., McGee, R., Okpodu, C. M., Robinson, T. J., Summers, M. F.,Werner-Washburne, M., and Zavala, M. E. (2016). Improving underrepresented Minority student persistence in STEM. *CBE-Life Sciences Education, 15(3)*, es5.

Gevertz, J. L., & Wares, J. R. (2020). Fostering Diversity in Top-Rated Pure Mathematics Graduate Programs. *Notices of the American Mathematical Society*, *67*(2).

Golbeck, A. L., Barr, T. H., & Rose, C. A. (2019). Fall 2017 Departmental Profile Report. *Notices of the AMS*, 66(10) 1721-1730.

Golbeck, A. L., Barr, T. H., & Rose, C. A. (2020). Report on the 2017-2018 New Doctoral Recipients. *Notices of the AMS*, 67(8) 1200-1213.

Google. (n.d.). *USA/Canada Math Phd programs: GRE requirements and qualifying exams*. Google Sheets. https://docs.google.com/spreadsheets/d/1hmdO7af3-lLvtJQO-szayG6blTvAYBQ1JcYXFZ_6apE/edit#gid=0

Kanny, M. A., Sax, L. J., & Riggers-Piehl, T. A. (2014). Investigating forty years of STEM research: How explanations for the gender gap have evolved over time. *Journal of Women and Minorities in Science and Engineering, 20(2)*, 127-148

Lent, R. W., Brown, S. D., & Hackett, G. (1994). Toward a unifying social cognitive theory of career and academic interest, choice, and performance. *Journal of Vocational Behavior, 45(1)*, 79–122.

Lent, R. W., Brown, S. D., & Hackett, G. (2000). Contextual supports and barriers to career choice: A social cognitive analysis. *Journal of Counseling Psychology, 47(1)*, 36 - 49.

McEldowney, T., Townsend, E., Maldonado, D., Michaluk, L., & Deshler, J. (2024). The Price Does Not Appear the Same for Everyone: Racial Differences in Students' Perceptions of the

Mathematics Graduate School Application Process. *Proceedings of the 26th Annual Conference on Research in Undergraduate Mathematics Education*
Miller, C. W., Zwickl, B. M., Posselt, J. R., Silvestrini, R. T., & Hodapp, T. (2019). Typical physics Ph. D. admissions criteria limit access to underrepresented groups but fail to predict doctoral completion. *Science Advances, 5(1)*, eaat7550. https://doi.org/10.1126/sciadv.aat7550
National Center for Science and Engineering Statistics. (2019). *Women, Minorities, and Persons with Disabilities in Science and Engineering: 2019.* Special Report NSF 19-304. Alexandria, VA. Available at https://www.nsf.gov/statistics/wmpd
Petersen, S. L., Erenrich, E. S., Levine, D. L., Vigoreaux, J., & Gile, K. (2018). Multi-institutional study of GRE scores as predictors of STEM PhD degree completion: GRE gets a low mark. *PLoS One, 13(10),* e0206570.
Posselt, J. R. (2016). *Inside graduate admissions: Merit, diversity, and faculty gatekeeping*. Harvard University Press.
Posselt, J. R., Hernandez, T. E., Cochran, G. L., & Miller, C. W. (2019). Metrics first, diversity later? Making the short list and getting admitted to physics PhD programs. *Journal of Women and Minorities in Science and Engineering, 25(4).*
Roberts, S. F., Pyfrom, E., Hoffman, J. A., Pai, C., Reagan, E. K., & Light, A. E. (2021). Review of racially equitable admissions practices in STEM doctoral programs. *Education Sciences*, *11*(6), 270.
Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research, 45*, 89–125.
Tinto, V. (1993). Leaving college: Rethinking the causes and cures of student attrition (2nd ed.). Chicago, IL: University of Chicago Press.
Tinto, V. (2007). Research and practice of student retention: What next? *Journal of College Student Retention: Research, Theory & Practice, 8(1)*, 1–19.
U.S. Census Bureau QuickFacts: United States. (2020). Retrieved September 29, 2020, from https://www.census.gov/quickfacts/fact/table/US/PST045219
Verostek, M., Miller, C. W., & Zwickl, B. M. (2021, October). Time to PhD completion is no different between men and women despite score gap on physics GRE. *In Physics Education Research Conference 2021v.*
Wilson, M. A., DePass, A. L., & Bean, A. J. (2018). Institutional interventions that remove barriers to recruit and retain diverse biomedical PhD students. *CBE—Life Sciences Education*, *17*(2), ar27.
Wingrove-Haugland, E., & McLeod, J. (2021). Not "Minority" but "Minoritized". *Teaching Ethics*, *21*(1).
Young, N. T., & Caballero, M. D. (2021). Physics Graduate Record Exam does not help applicants "stand out". *Physical Review Physics Education Research, 17(1),* 010144.

Representations of the Derivative Valued by Post-Secondary Teachers

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Introductory calculus classes often serve as prerequisite classes for science and engineering majors. However, some researchers have questioned whether calculus courses, as currently taught, are filtering students appropriately (Black & Hernandez-Martinez, 2016; Williams, 2012; Williams & Choudry, 2016). Central to introductory calculus classes is the derivative (Kidron, 2019). Scholars have discussed the ways in which representations of the derivative differ between disciplines (Dray et al., 2019), but there has been no formal research investigating the representations of the derivative valued by teachers of calculus and teachers of subsequent non-mathematics classes. This study, through a new survey instrument, determined there are significant differences between the representations of the derivative valued by various post-secondary teachers in majors requiring calculus. By analyzing these differences, recommendations can be made to improve calculus as a preparatory course for non-mathematics majors.

Keywords: calculus, STEM education, derivatives, gateway courses

In a popular comedy sketch, a mad scientist reveals his latest depraved invention: junior high school (Studio C, 2014). As part of his invention, the mad scientist forces a student to take algebra and geometry. When asked by his assistant if a student will at least use what he learns later in life, the mad scientist replies, "Yes…if he ever teaches geometry or algebra" (Studio C, 2014, 1:33). In horror, his assistant asks, "You mean they exist only for themselves?" (Studio C, 2014, 1:42). Based on the laughter from the live studio audience in the video, these comments resonated with the audience's experiences.

Research has repeatedly found that students share the belief that mathematics is not valuable to them in their future personal or professional lives (Black & Hernandez-Martinez, 2016; Brown et al., 2008; Di Martino, 2019; Hernandez-Martinez & Vos, 2018; Rellensmann & Schukajlow, 2017; van der Wal et al., 2017; Vos, 2018). This feeling is encapsulated by a study that found a group of graduates from engineering programs "perceived their mathematics courses as islands, with no relation to the rest of their education" (van der Wal et al., 2017, p. S98). Mathematics classes existing only for themselves is concerning because mathematics classes, especially calculus, serve as filters for many science and engineering majors (Bressoud et al., 2013; 2015). As noted by Bressoud et al. (2015),

> At least one term of calculus is required for almost all STEM majors. For too many students, this requirement is either an insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics. (p. v)

There have been calls for calculus to serve as a pump for science and engineering majors, rather than a filter (Bressoud et al., 2015). However, as noted by Bressoud et al., there seems to be little evidence of this change taking place despite multiple efforts to reform calculus classes.

Purpose of the Study

The purpose of this study was to gather quantitative data to determine the representations of the derivative favored by teachers of calculus classes as well as the representations favored by teachers of classes in majors requiring calculus. The goal of this comparison is to make recommendations for improving the current content taught in calculus I. As noted by Dray et al. (2019), students have struggled with representations of the derivative used in upper-level physics classes that are not emphasized in calculus I classes. Since Dray et al.'s report was developed using anecdotal data from the teaching experiences of the researchers, this study aims to generate more general data that can be analyzed to determine which representations need to be emphasized more in general calculus I classes. Additionally, the focus of Dray et al. was on physics students. However, it is estimated that only about 4% of calculus I students are pursuing physical science degrees, which includes both physics and chemistry degrees (Bressoud et al., 2015). This study cast a wider net and surveyed teachers from faculty in multiple disciplines to better determine which representations should be emphasized to better serve the students taking mainstream calculus I.

Research Question

This study sought to answer the following research question: Is there a difference in the value post-secondary teachers of calculus and teachers of non-mathematics classes requiring calculus ascribe to different representations of the derivative?

This survey was distributed to post-secondary faculty at colleges in the state of Wyoming. Since Wyoming has only one public university and seven community colleges, surveying faculty at all institutions was a feasible project. Additionally, Wyoming law empowers the state community college commission to establish common course numbering across undergraduate courses throughout the state (Common College Transcripts, 2018). This made surveying teachers of common classes across the state of Wyoming easy and efficient, as the course numbering and most prerequisites are the same across the state.

Significance of this Study

Based on the work of Dray et al. (2019), I suspected that numerical and experimental representations of the derivative would be more highly valued by teachers of non-mathematics classes than by mathematics faculty. Additionally, it is likely these will be valued more than other representations by teachers of non-mathematics classes. As there have been multiple stated concerns with calculus instruction and curriculum as they currently stand (e.g., Bressoud, 2021; Bressoud et al., 2015, Dray et al., 2019; Tallman et al., 2021; Teague, 2017), this study can help determine how mainstream calculus I curricula can be improved to better prepare students for future non-mathematics classes. Specifically, this study can help determine how calculus I instruction can be made more authentic, to potentially better help prepare professionals to use calculus concepts in their fields.

Calculus: A Problematic Filter

There are many potential reasons for this failure to reform calculus classes. One concern postulated by researchers is that mathematics classes privilege those who know the "rules of the game" of math education rather than prepare students to use math outside of math class (Black & Hernandez-Martinez, 2016; Williams, 2012; Williams & Choudry, 2016). This suggests that proficiency in mathematics classes, as evidenced by good grades, is not of value for use in future classes or careers, but as a form of exchange that can be leveraged to enter competitive fields or prestigious post-secondary institutions (Williams, 2012). This form of exchange has been dubbed "mathematics capital" (Williams, 2012).

This notion of mathematics classes acting more as producers of mathematical capital rather than transferable skill is concerning considering results found by Tallman et al. (2021). In their study, they reviewed 254 calculus final exams to gain a better understanding of how calculus students were assessed. Tallman et al. found that the calculus final exams they reviewed did not generally assess students' understanding of calculus concepts, but their abilities to use mathematical procedures to answer particular types of calculus questions. Additionally, few questions in the final exams allowed students to engage meaningfully with real-world contexts.

The results found by Tallman et al. (2021) along with Williams' (2012) notion of mathematical capital suggest that calculus classes are not acting as a pump but as a filter that does not filter for attributes that help students succeed in science and engineering degrees. Some suggest this may be a problem as the number of engineering graduates has not seen real gains over the last three decades despite calls from political and advocacy groups for more students to obtain such degrees (Bressoud et al., 2017). Additionally, there are concerns that mathematics classes as they currently stand reinforce existing inequities rather than make it possible for all students, including those from marginalized populations, to succeed in science and engineering careers (Williams, 2012; Williams & Choudry, 2016). As students from marginalized populations are generally more likely to struggle in calculus classes (Bressoud et al., 2015), the filtering done by calculus classes, in their current form, presents serious concerns for educational equity.

These issues show that mainstream calculus is in need of improvement. For such improvement to happen, the content of calculus classes must be better examined. Some science education researchers, in conjunction with math education researchers, have pointed out that mathematicians and scientists often utilize different representations and understandings of important calculus concepts (Dray & Manogue, 2005; Dray et al., 2019). As noted by Dray and Manogue (2005), "The way mathematicians view and teach mathematics, and the way mathematics is used by physicists and other scientists, are completely different; we speak different languages, or at least different dialects" (p. 2). This suggests that calculus content should focus on future applications of calculus. However, more information is needed to determine which applications should be included in an introductory calculus class. This study aimed to begin the process of determining which applications should be included through a quantitative survey.

Theoretical Framework: Multiple Representations of the Derivative

Since mainstream calculus focuses mostly on using the derivative, this study will examine how the derivative is represented by calculus teachers, as well as by teachers of STEM classes. As noted previously, researchers have observed differences in the ways scientists and mathematicians use and discuss derivatives (Dray et al., 2019; Dray & Manogue, 2005). Although the formal definition of the derivative is given using limits, this definition is often not directly used when determining a derivative from real world data. Roundy et al. (2015) saw this when they asked groups of mathematicians, physicists, and engineers to find the derivative of a certain rate of change in a mechanical machine created for the purpose of the case study. The physicists and engineers found approximations of the derivative by looking at ratios of small changes in the input and output variables and were thus able to find "good enough" approximations of the derivative to the point they described their answers as "the derivative" rather than as an approximation of the derivative. However, the mathematicians were unable to find a derivative and were unsure of the approximation they were able to generate.

In later work, Dray et al. (2019) described this issue seen by Roundy et al. (2015) as an overreliance, by the mathematicians, on the limit definition of the derivative. Along with their firsthand experiences in working with derivatives in their research and in upper-level physics classes, Dray et al. (2019) used this difference between the results of the mathematicians and the physicists and engineers in Roundy et al. (2015) to develop the notion of a "thick derivative." A "thick derivative" is a derivative function or value developed using numerical data to approximate the formal limit process. Dray et al. (2019) postulated that this notion of a thick derivative is used by scientists and engineers in their work as an approximation of the limit definition of the derivative.

To operationalize these differences, Dray et al. (2019) created a framework describing the multiple representations of the derivative they encountered in their professional work. This framework was an extension of a framework developed by Zandieh (2000). The framework of Dray et al. (2019) lists five main representations of the derivative: (a) graphical, (b) verbal, (c) symbolic, (d) numerical, and (e) physical. These representations are all considered process-objects (Sfard, 1991) that begin as ratios that are then reified into limits and functions. This framework was used to help determine which of these five primary representations of the derivative are favored by post-secondary teachers in majors that require students to take calculus.

Methods

For this study, a quantitative, correlational study design was used (Field, 2018). Correlational studies are used to determine if there is a relationship between sets of independent and dependent variables. In this study, I specifically examined the relationship between the subject domain of a teacher and the value they ascribe to different representations of the derivative. As such, I considered the subject domain of collegiate teachers surveyed as the independent variable for this study. This independent variable is categorical. There are six dependent variables, each representing the value a teacher places on a certain representation of the derivative. These variables follow from the representations of the derivative described by Dray et al. (2019). Although Dray et al. describe only five representations of the derivative, the graphical, verbal, symbolic, numerical, and experimental representations, they do divide their symbolic representation into two parts. There is the symbolic representation of using a limit to determine a derivative and the symbolic representation of using derivative rules to find derivatives of known functions. Since derivative rules play a key role in calculus I classes (Tallman et al., 2021), I considered the limit representation of the derivative and derivative rules used to find derivative functions separately within this study. This resulted in a survey examining six representations, the graphical, verbal, symbolic, rules, numerical, and experimental representations. As the value respondents ascribe to each representation will be determined using Likert-scale questions, the value respondents assign to each variable can be considered scale variables (Field, 2018).

Data Analysis

Since the data collected consisted of one categorical independent variable and multiple dependent scale variables, data was analyzed using a multivariable analysis of variance (MANOVA) (Fields, 2018; Huck, 2012). This analysis is appropriate as it prevents possible type I errors that may result from repeated uses of an analysis of variance procedure for each separate dependent variable (Fields, 2018). Additionally, MANOVA can be used to reveal possible relationships between the dependent variables. To further analyze differences for the representations between individual subject areas, I used Tukey's post-hoc test.

To help measure the reliability of the items used to assess the value ascribed to each representation of the derivative, Cronbach's alpha was calculated for each set of items used to create the normalized representation scores (Fields, 2018; Huck, 2012). All analyses were carried out using SPSS.

Participants

For this study, I surveyed Wyoming public community college and university faculty who teach in a major that includes calculus as a requirement. Of those who were sent the survey, seventy responded and completed the survey.

Although surveying teachers within the state of Wyoming does not strictly represent a random sample, this sample is still representative of post-secondary teachers across the country. All colleges in the state of Wyoming are accredited by a national accrediting body (Higher Learning Commission, 2019), and thus educational requirements for these positions are the same as at other institutions in the United States. These colleges recruit faculty from all around the country, with some faculty even coming from international sources.

Participant anonymity was protected by collecting no personally identifying information. Furthermore, only I had access to the data on a password-protected cloud storage system. This research was also protected by having the study approved by the institutional review board of the University of Wyoming.

Data Sources

Respondents participated in this study by completing a survey I created. This survey was created by designing Likert-scale items based on the framework of Dray and colleagues (2019) previously described. Multiple items were created for each conception described in the framework. These questions were items to elicit respondents' values of each process-object layer for each conception. For each item, respondents were asked to share how important they thought various concepts were for students to understand at the end of an introductory calculus class. For example, respondents were asked how important it was for students to memorize basic derivative rules. For each statement, respondents could rank the statement, "Not at all important," "Slightly important," "Moderately important," "Very important," or "Extremely important."

I created this survey based on Dray and colleagues (2019) framework due to its suitability as a framework of the mathematical knowledge to teach derivatives, as discussed in the literature review. Since an overwhelming majority of calculus students have been found to declare non-mathematics majors (Bressoud, 2015), this framework is especially well-suited to assess the value non-mathematics teachers place on different conceptions of the derivative.

Results

As noted earlier, this survey had 70 responses. Participant breakdown by subject area is as follows: (a) Mathematics: 22 respondents; (b) Biology: 6 respondents; (c) Chemistry and physics: 8 respondents; (d) Engineering and computer science: 9 respondents; and (e) other: 25 respondents.

Reliability Results

Each subscale of this survey, the graphical, verbal, symbolic, rules, numerical, and experimental subscales, had high reliability scores. Additionally, to meet the assumptions of MANOVA, the graphical and verbal subscales were combined. This new, larger subscale also had high reliability scores. This resulted in five representations analyzed as part of the MANOVA: the graphical & verbal representation, the symbolic representation, the rules representation, the numerical representation, and the experimental representation. Each of these representations had a reliability score of 0.891 or better.

Test Results

Based on Pillai's trace, there was a significant association of subject area with the value ascribed to different representations of the derivative, $V = 0.758, F(20,256) = 2.690, p < 0.001$. For each representation, the mathematicians yielded the highest value for their mean score. In other words, mathematicians in this study valued each representation more than their counterparts. Separate univariate tests for each representation yielded significant associations within each representation as well. Tukey's Post-Hoc test was used to consider subject areas pairwise and determine which had significantly different values ascribed to the representations. Most notably, mathematics and biology teachers differed significantly on the graphical & verbal and rules representation ($p = 0.018$ and $p = 0.042$ respectively), with the mathematicians valuing each representation more than their biologist counterparts.

Discussion

Assuming the differences found in the values ascribed to different representations found in this survey are accurate, there are two main ways to address these differences to improve calculus I classes; curricular adjustments and creating calculus classes specifically for certain majors, often termed "siloed classes" (Ellis et al., 2021; Luque et al., 2022; Voigt et al., 2020).

Curricular Adjustments

A call to adjust the curriculum of calculus I is not new. There have been attempts to reduce the calculus curriculum in the past to focus less on rote skills and more on relational understanding (Douglas, 1986). More recent research, however, indicate that these reductions have not yet occurred (Jones & Watson, 2018; Tallman et al., 2021). Despite this lack of reduction, researchers involved in more recent studies still expressed a desire to reduce and focus the calculus curriculum. Jones and Watson (2018) suggested focusing on Zandieh's (2000) version of the derivative representation framework to help focus instruction, while Tallman et al. (2021) suggested that calculus exams should assess relational understanding of the derivative rather than solving specific types of calculus problems.

The results from this study lend additional credence to the suggestions of Jones and Watson (2018) and Tallman et al. (2021). Scholars often suggest lessening the emphasis on complex uses of rote procedures, especially with the advent of computer technology that can complete these procedures more accurately and efficiently than humans (Douglas, 1986; Gravemeijer et al., 2017; Tallman et al., 2021).

As mentioned before, non-mathematicians do not view such procedures as unimportant; they merely view them as less important. These data do not suggest a complete removal of rote procedures from calculus I curricula; instead, they suggest a partial removal of such material only. Furthermore, research literature suggests that building fluency with these procedures through relational understanding of these procedures, rather than through extensive practice of complex combinations of such procedures, can help ensure future professionals can effectively use technology to assist them with such procedures in the future (Gravemeijer et al., 2017; Hoyle et al., 2013; Tallman et al., 2021; van der Wal et al., 2017).

Siloed Classes

The significant difference in the rules category between mathematicians and biologists is both striking and practically important. As a reminder, the mathematicians reported valuing the rules category more than the biologists, with a difference of 1.5076. These differences between the biology and mathematics teachers is made practically important by the number of students from these majors in mainstream calculus. Based on a large national survey, about 30% of calculus I students are biology or life science majors (Bressoud et al., 2015). This group is second only to engineering majors, who make up about 31% of calculus I students. The group with the next highest percentage is business majors, with 7% of calculus I students. Thus, this disparity where biologists value the Rules category much less than mathematicians, while the engineers answer quite similarly to the mathematicians in that category, leads to a conundrum that is difficult to solve in a class serving both populations.

One possible solution to this conundrum is to create siloed calculus classes (Ellis et al., 2021; Klingbeil & Bourne, 2013; Luque et al., 2022; Voigt et al., 2020). These calculus classes are variations on the traditional calculus curriculum that focus on using calculus in a particular subject area. There are many types of siloed classes, but one that has received significant attention is a calculus class for future biologists and other life scientists. These commonly referenced siloed classes parallel the large numbers of such students in calculus I classes as well as the results of this study.

Research into siloed classes has shown such classes can lead to lower failure rates in calculus classes, especially those for biology and life science majors (Luque et al., 2022; Voigt et al., 2020). Siloed classes overall seem to have helped students who were less prepared for college math (Klingbeil & Bourne, 2013; Voigt et al., 2020). In one study, the drop, failure, and withdrawal rate (DFW) for such students was equal to that of their colleagues who entered college with better math skills, suggesting that siloed classes "levelled the playing field for less prepared students" (Voigt et al., 2020, p. 868). Siloed classes also tend to lead to lower failure rates as well as increased enrollment in such classes and majors (Klingbeil & Bourne, 2013; Luque et al., 2022). These findings suggest that siloed classes have the potential to help students proceed past calculus into major coursework.

This study adds additional evidence to the need for siloed classes especially for biology majors. The significant difference found between the value ascribed to the Graphical and Verbal representation and Rules representation between the mathematicians and biologists suggests mainstream calculus classes may not be properly serving biology majors. Further research, including targeted interviews with course coordinators, is needed to validate this notion.

Future Research

This study opens the door to several avenues of future research. In many ways, this study can be considered a pilot study for the derivative representations survey used herein. As mentioned before, the framework by Dray and colleagues (2019) used to design this survey has been used in the past to analyze differences in the ways mathematicians and non-mathematicians approach differentiation tasks in task-based settings. As piloted in this study, such differences can continue to be analyzed among larger groups of professionals using the derivative representations survey. However, before such research could be conducted, improvements would need to be made to the survey as it currently stands.

References

Black, L., & Hernandez-Martinez, P. (2016). Re-thinking science capital: The role of 'capital' and 'identity' in mediating students' engagement with mathematically demanding programmes at university. *Teaching Mathematics and its Applications: An International Journal of the IMA, 35*(3), 131-143. https://doi.org/10.1093/teamat/hrw016

Bressoud, D. M. (2015). The Calculus Students. In D. M. Bressoud, V. Mesa, & C. L. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus* (pp. 1-16). MAA Press.

Bressoud, D. M. (2017). Introduction. In D. Bressoud (Ed.), *The role of calculus in the transition from high school to college mathematics* (pp. 1-8). MAA Press.

Bressoud, D. M. (2021). The strange role of calculus in the United States. *ZDM, 53*(3), 521-533. https://doi.org/10.1007/s11858-020-01188-0

Bressoud, D. M., Camp, D., & Teague, D. (2017). Background to the MAA/NCTM statement on calculus. In D. Bressoud (Ed.), *The role of calculus in the transition from high school to college mathematics* (pp. 77-81). MAA Press.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insight from the Mathematical Association of America national study. *International Journal of Mathematics Education in Science and Technology, 44*(5), 685-698. https://doi.org/10.1080/0020739X.2013.798874

Bressoud, D., Mesa, V., Rasmussen C. (2015). Preface. In D. M. Bressoud, V. Mesa, & C. L. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus* (pp. v-viii). MAA Press.

Common College Transcripts, Wyo. Stat. § 21-18-202(a)(vi) (2018). https://wyoleg.gov/NXT/gateway.dll?f=templates&fn=default.htm

Douglas, R. G. (Ed.). (1986). *Toward a lean and lively calculus: Conference/workshop to develop alternative curriculum and teaching methods for calculus at the college level Tulane university, January 2-6, 1986*. The Mathematics Association of America. https://www.maa.org/sites/default/files/pdf/pubs/books/members/NTE6_optimized.pdf

Dray, T., Gire, E., Kustusch, M. B., Manogue, C. A., & Roundy, D. (2019). Interpreting derivatives. *Primus, 29*(8), 830-850. https://doi.org/10.1080/10511970.2018.1532934

Dray, T., & Manogue, C. A. (2005). Bridging the gap between mathematics and the physical sciences. In D. Smith and E. Swanson (Eds.), *Preparing future science and mathematics teachers* (pp. 39-41). Montana State University.

Ellis, B., Larsen, S., Voigt et al., M., & Vroom, K. (2021). Where calculus and engineering converge: An analysis of curricular change in calculus for engineers. *International Journal of Research in Undergraduate Mathematics Education, 7*(2), 379-399. https://doi.org/10.1007/s40753-020-00130-9

Epstein, J. (2013). The calculus concept Inventory–Measurement of the effect of teaching methodology in mathematics. *Notices of the American Mathematical Society, 60*(8), 1018-1026. https://doi.org/10.1090/noti1033

Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education, 15*(S1), 105-123. https://doi.org/10.1007/s10763-017-9814-6

Hernandez-Martinez, P., & Vos, P. (2018). "Why do I have to learn this?" A case study on students' experiences of the relevance of mathematical modelling activities. *ZDM, 50***,** 245–257. https://doi.org/10.1007/s11858-017-0904-2

Higher Learning Commission (2019) *Higher Learning Commission Policy Book.* Retrieved from http://www.hlcommission.org

Huck, S. W. (2012). *Reading statistics and research* (6th ed.). Pearson.

Jones, S. R., & Watson, K. L. (2018). Recommendations for a "Target understanding" of the derivative concept for first-semester calculus teaching and learning. *International Journal of*

Research in Undergraduate Mathematics Education, 4(2), 199-227. https://doi.org/10.1007/s40753-017-0057-2
Kidron, I. (2019). Calculus teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 1-8). Springer. https://doi.org/10.1007/978-3-319-77487-9_18-2
Klingbeil, N. W., & Bourne, A. (2013). A national model for engineering mathematics education: Longitudinal impact at Wright State University. In P*roceedings of the 120th ASEE conference & exposition* (pp. 23.76.1-23.76.12). American Society for Engineering Education. https://doi.org/10.18260/1-2--19090
Luque, A., Mullinix, J., Anderson, M., Williams, K. S., & Bowers, J. (2022). Aligning calculus with life sciences disciplines: The argument for integrating statistical reasoning. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 32*(2), 199-217. https://doi.org/10.1080/10511970.2021.1881847
Roundy, D., Weber, E., Dray, T., Bajracharya, R. R., Dorko, A., Smith, E. M., & Manogue, C. A. (2015). Experts' understanding of partial derivatives using the partial derivative machine. *Physical Review Special Topics. Physics Education Research, 11*(2), 1-21. https://doi.org/10.1103/PhysRevSTPER.11.020126
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36. https://doi.org/10.1007/BF00302715
Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511. https://doi.org/10.1007/s10857-008-9089-5
Studio C. (2014, April 7). *The Mad Scientist Creates Junior High School.* [Video]. YouTube. https://www.youtube.com/watch?v=9mR8-loJNpU
Tallman, M. A., Reed, Z., Oehrtman, M., & Carlson, M. P. (2021). What meanings are assessed in collegiate calculus in the United States? *ZDM, 53*(3), 577. https://doi.org/10.1007/s11858-020-01212-3
Teague, D. (2017). The song remains the same, but the singers have changed. In D. Bressoud (Ed.), *The role of calculus in the transition from high school to college mathematics* (pp. 41–45). MAA Press.
University of Wyoming. (2022). *2022-2023 University of Wyoming Catalog*. https://acalogcatalog.uwyo.edu/index.php?catoid=9
University of Wyoming Department of Mathematics and Statistics. (n.d.). *Wyoming Mathematical Association of Two Year colleges*. WYMATYC | Events | Mathematics and Statistics Department | University of Wyoming. Retrieved November 10, 2022, from http://www.uwyo.edu/mathstats/events/wymatyc.html
van der Wal, N. J., Bakker, A., & Drijvers, P. (2017). Which techno-mathematical literacies are essential for future engineers? *International Journal of Science and Mathematics Education, 15*(S1), 87-104. https://doi.org/10.1007/s10763-017-9810-x
Voigt et al., M., Apkarian, N., Rasmussen, C., & the Progress through Calculus Team. (2020). Undergraduate course variations in precalculus through calculus 2. *International Journal of Mathematical Education in Science and Technology, 51*(6), 858-875. https://doi.org/10.1080/0020739X.2019.1636148
Vos, P. (2018). "How real people really need mathematics in the real world"—Authenticity in mathematics education. *Education Sciences, 8*(4), 195-208. https://doi.org/10.3390/educsci8040195

Williams, J. (2012). Use and exchange value in mathematics education: Contemporary CHAT meets Bourdieu's sociology. *Educational Studies in Mathematics, 80*(1-2), 57-72. https://doi.org/10.1007/s10649-011-9362-x
Williams, J. & Choudry, S. (2016) Mathematics capital in the educational field: Bourdieu and beyond. *Research in Mathematics Education, 18*(1), 3–21. https://doi.org/10.1080/14794802.2016.1141113
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Shoenfeld, J. Kaput, C. Kessel (Eds) *Research in Collegiate Mathematics Education. IV* (103-127). Providence, RI: American Mathematical Society.

Finding Elements of Exploration in a Ritualized Calculus Discourse

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Undergraduate mathematics classrooms continue to experiment with active learning strategies as an alternative to the traditional lecture-based teaching model. This paper investigates one student's (Jacob) engagement in a poster session activity in a Calculus 1 course and explores how he reasoned about calculus concepts in this non-standard learning environment. Informed by Sfard's theory of commognition, an analysis of Jacob's discursive routines reveals a complex interplay between ritualized and explorative discourse. While Jacob's poster presentations appear highly ritualized on the surface, with routines recycled from class, indications of explorative activities emerged in his preparation with a partner, reflecting an active search for suitable routines. This research emphasizes the importance of considering the entire ritual-exploration continuum in mathematics education and raises questions about facilitating the transition from ritualized to explorative discourse.

Keywords: Commognition, Routines, Calculus, Exploration, Rituals

Introduction

In undergraduate mathematics education, there is a growing trend of incorporating forms of learning which differ from the standard lecture model. One notable study reported calculus instructors being encouraged to employ *active learning* strategies in their classroom (Rasmussen et al., 2014) as one of seven characteristics of successful calculus programs of doctoral granting institutions in the country. However, the term *active learning* is vague and has varying interpretations. Generally, it can refer to any form of student engagement in class which differs from the standard lecture model, ranging from the incorporation of worksheets to engaging in group discussions and problem solving (Freeman et al., 2014). Nevertheless, there have been recent calls to incorporate such strategies in tertiary mathematics learning (e.g., CBMS, 2016; Saxe & Braddy, 2015) and a call to investigate such "interventions in practice" (Rasmussen et al., 2014, p. 509). At Barrow University[1], a large research-intensive university in the southeastern United States, one intervention has been implemented in the Calculus 1 courses for several years and serves as the context for this research project. The weekly activity of students engaging in poster sessions (further described in the "Methods" section) was designed to have students become expert at solving one problem and then communicating their solution to their peers. Because communication is at the forefront of this activity, Sfard's (2008) theory of commognition, which considers thinking is communicating, is fitting to study how students reason about calculus concepts, especially in a non-standard learning environment. For the purposes of this paper, I focus on the discursive routines of one student, Jacob, and how he situated the poster session activity as explorative or ritualized. Preliminary findings indicate that the presentation of the poster was highly ritualized, yet there were moments of exploration evidenced in Jacob's discourse when speaking with his instructor. As well, there was also an indication that Jacob engaged in more explorative routines when designing the poster he presented, but the constraints of the assignment may have funneled Jacob into a ritualized routine.

[1] All names in this paper are pseudonyms.

Literature Review

Commognition has gained popularity recently with the framing's tenet that students' thinking is revealed in their communication. Sfard (2016) promoted that analyzing discursive activities stands in stark contrast to a more cognitivist approach which has "the tendency to view communication as a mere window to something else" (p. 41). With a commognitive lens, thinking *is* communicating and vice versa. Since its inception, a wide range of aspects of commognition have been trialed in various forms in the research of undergraduate mathematics education, particularly with intervening learning activities that disrupt the standard lecture model. Most recently, Kontorovich (2023) critically examined the panacea-like collaborative learning structure by attending to a triad of cognitive, social, and affective aspects of students' collaboration as they studied linear algebra. His research explored how two students, Jan and Sai, were able to position themselves as mathematizers (or not as mathematizers) and how they perceived each other as such. The careful commognitive analysis of the dynamics between the two students illuminated a great need for further research into how the triad of mechanisms plays a role in collaborative learning situations.

Another avenue of study in which the lecture model is disrupted through varying forms of classroom activity is the continuum of routines. Lavie et al. (2019) operationalized routines with a commognitive lens and made a distinction between rituals (process-oriented routines) and explorations (product-oriented routines). Whereas rituals typically are guided by the question, "how do I proceed," explorations are guided by the question, "what is it that I want to get" (Lavie et al., 2019, p. 166). Rituals should be in no way absent from a novice mathematics learner's discourse. They are the foundation of transitioning to an explorative discourse. Therefore, as a mathematics learner becomes a full-fledged member of the mathematics community, they undergo a gradual de-ritualization of their routines and move toward the exploration end of the spectrum. In fact, certain curriculum materials which are "explicitly designed to promote student-centered activities that shift classroom discourse away from lectures" (Barnett, 2022, p. 1579), such as Primary Source Projects (PSPs), were analyzed for their potential to de-ritualize mathematical discourse at the tertiary level. Barnett (2022) showed how the task situations in PSPs ranged on the whole of the ritual-exploration continuum and may be used to fuel the de-ritualization process (depending on how the PSPs are implemented).

Similarly, Nachlieli and Tabach (2019) and Sfard (2016) proposed that the nature of routines in which students engage is highly dependent on the ways in which a lesson or activity is implemented. Nachlieli and Tabach (2019) developed a framework to analyze mathematical learning tasks as either ritual-enabling opportunities to learn (OTL) or exploration-requiring OTL. They, too, stated that rituals are essential for students to enter the discourse because rituals help to develop students' object-level and meta-level discourse which is the foundation of explorative activities. However, Sfard (2016) analyzed the discursive routines of a teacher during a lesson and subsequently the routines of his students as they solved similar problems. She found that they mirrored his ritualized routines and exhibited some of the same features of his routines. Thus, it is not surprising that it should be essential to consider the routines afforded by and exhibited in such classroom activities.

Theoretical Framework: Commognitive Routines

Researchers of mathematical discourse are no stranger to Sfard's (2008) theory of *commognition*, a concatenation of *communication* and *cognition*, which posits that thinking and communication are one in the same. Sfard (2008) proposed that mathematical discourse is a special form of discourse that can be characterized by four properties: word use, visual mediators, narratives which can be composed of key words and visual mediators, and routines. Both word use and visual mediators may be rather trivially conceptualized as those special words and symbols we use to communicate mathematics and which are exclusive to mathematical discourse. For instance, in the context of evaluating a limit, key words and visual mediators may include the words *indeterminate form* and algebraic symbols like $\frac{\infty}{\infty}$ or graphs that show the end behavior of a function. Narratives are the stories we tell about mathematical objects using the key words and visual mediators (e.g., $\lim_{x\to\infty}\left(\frac{2^3}{3}\right)^x = \infty$ because $\frac{2^3}{3} > 1$). Finally, routines are the repetitive patterns found in mathematical discourse (e.g., evaluating a limit using algebra). For the purposes of this paper, routines will be characterized as explorative or ritualized.

Explorations may be considered by many in the professional mathematical community as the true form of "doing mathematics." That is, the goal of an exploration is to produce a narrative about mathematical objects which the mathematical community generally accepts as true. Thus, explorations may be considered as product-oriented routines (Lavie et al., 2019). On the opposite end of the routinization spectrum is the ritual. Those who perform rituals are completing discursive actions with the goal of "creating and sustaining a bond with other people" (Sfard, 2008, p. 241) and may not be concerned about telling stories about mathematical objects. A ritual is "appreciated for its performance and not for its product" (Lavie et al., 2019, p. 166) and therefore is considered a process-oriented routine.

Because pure rituals and explorations are truly rare in mathematics classrooms (Lavie et al., 2019), I will characterize elements of the mathematical discourse of a student presenting a poster to his peers and instructor as ritualized or explorative and seek to answer the question:

How might a student engage in discursive routines in the context of a Calculus 1 poster session?

Methods

Context of the course and poster session

Data from this study were collected from a Calculus and Analytic Geometry 1 course taught in the Spring 2023 semester at a large southeastern research-intensive university. The mathematics department employs graduate teaching assistants as instructors of record for the four-credit hour calculus course as well as a specialized teaching faculty member who serves as the course mentor and collaborates with TAs to implement a unified curriculum. Members of the teaching team are expected to adhere to the same schedule, covering specified content and implementing a hallmark classroom activity, poster presentations, weekly. The poster activity consists of students collaborating with a partner outside of class to solve one calculus problem and then presenting their solution as a poster during class. The problems on which students collaborate are comparable to what they may be given as a homework problem and are directly related to the content of a recent lecture. As stated in the activity description given to students, they are expected to become "expert" at solving the problem. Their expertise is demonstrated through the inclusion of four mandatory components of the poster: detailed steps to solve the problem, the "final answer" to the problem, clues from the problem that may lead to the solution, and mistakes made or mistakes students anticipate others might make while solving the problem. Once per week, the class engaged in two rounds of poster sessions. In the first round, one person from each pair stood at their respective poster while the partner visited other posters. The person standing explained their problem to anyone who visited and should have been able to answer any questions posed by their visitors. After 10–15 minutes, partners switched places, and the second round commenced. The poster session from which this analysis was taken occurred in the third week of the semester and consisted of problems related to calculating limits and finding the derivative at a point.

Data Collection

The episode analyzed for this paper is part of a larger research project in which 24 of 29 students consented to have their classroom interactions audio/video recorded and copies of their coursework collected as data. Field notes and recordings of the class lectures were captured. Additionally, six of the 24 consenting students participated in an interview designed to expand on their mathematical routines observed in the class. This paper focuses on one consenting student, Jacob, who also participated in an interview. Although Jacob initially enrolled in the course to be eligible for entrance in the computer science program, he revealed that after a few weeks of being in the course, he realized that he wanted to progress in a different direction entirely and major in Chinese Language and Culture. By personal admission, Jacob said that a heavy course load was one factor contributing to his change of heart. The other factor was a newfound interest in learning about Chinese culture. The episode of analysis is from Jacob's second poster session in which he was tasked with evaluating the limit using algebra: $\lim_{x \to \infty} \frac{2^{3x+1}}{3^{x+3}}$.

Data Analysis

When analyzing the poster episode, I drew upon Sfard's (2016) and Lavie et al.'s (2019) notion of the routine as a spectrum ranging from ritual to exploration. According to Sfard (2016), there are three main features of a routine that need to be considered when determining whether an episode may be explorative. First, the subjects of the narratives are typically abstract objects rather than more concrete symbols and signifiers. Second, the source of mathematical narratives in an explorative discourse "can be logically deduced from stories already endorsed" (Sfard, 2016, p. 45) as opposed to relying on memory or an authoritative source, such as the instructor or textbook. Lastly, the goals of mathematical activity must be considered. As described above, the goal of an exploration is to tell a story about mathematical objects while the goal of a ritual is to engage in the process itself. The data sources which inform the following analysis are the interview transcript, the poster presentation transcript, and the poster artifact.

Results

The following episode is taken from Jacob's poster presentation in which he presented to 13 persons including the instructor (Ms. Rebecca) and the researcher (Mark). Jacob's poster monologue was roughly the same for each presentation. He admitted in the interview that he did not rehearse a script, however, he said:

> as the poster session gets going, I do say like a similar spiel…. obviously, the prompts are laid out step by step; I go step by step. And, I say pretty much the same thing, although sometimes I change the wording a bit to make it a bit more interesting to myself.

Therefore, the components of his mathematical discourse are virtually identical in each presentation. Below is one representative example of Jacob's presentation which he would begin once a visitor appeared before him.

We have [the problem][2] find the limit of *x* to infinity of two to that over three to that [pointing in the general area of the upper left-hand corner of the poster]. So yes, and put very simply, no matter what we tried, we ended up with the indefinite[3] form. So, we went [with] *undefined*. We tried to divide by the conjugate because of those pesky exponents. And yeah, we still got infinity over infinity, so we went with *undefined*. To be honest, [we are] not sure if it's correct.

Figure 1 displays the poster that Jacob and his partner, Rachel, created to present their solution to the limit problem.

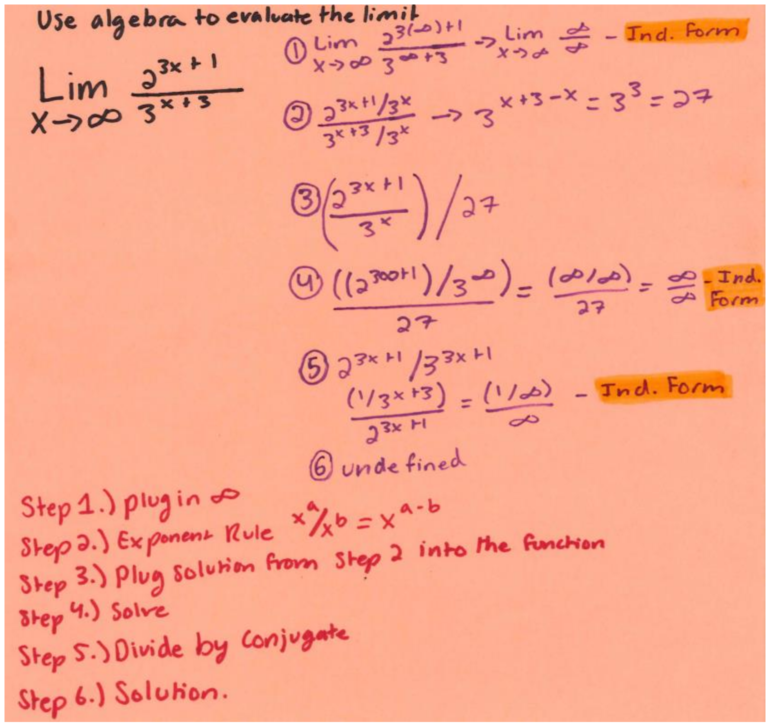

Figure 1. Jacob and Rachel's poster detailing their three attempts at evaluating the limit.

Objects of Mathematical Discourse

One distinctive component of mathematical discourse is characterized by how discursants communicate about mathematical objects. The objects of mathematical discourse were coded as either concrete or abstract. This distinction was dependent not only on the key word itself but also how the discursant communicated about the object. In other words, concrete objects were treated by the discursant as something tangible and that could be operated on by a person, whereas abstract objects existed primarily in the discourse and could take on discursive actions. For example, Jacob spoke about $\lim_{x \to \infty} \frac{2^{3x+1}}{3^{x+3}}$ as something "we have" in terms of a concrete problem which needed a solution. There is no perceptible indication that Jacob considered the

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[2] Words in brackets clarify Jacob's meaning based on his other instances of delivering virtually the same monologue.

[3] Jacob consistently used "indefinite" in place of "indeterminate." This seemed to have no bearing on what Jacob meant by the "indefinite" form. In fact, his instructor and his partner both used this term when speaking about the indeterminate form while reasoning through Jacob's poster. However, the instructor did not use "indefinite form" while lecturing.

function inside the limit notation as a quotient of exponential functions with certain properties. In fact, Jacob further concretized the limit in his recitation of the function by referring to the exponents as “that” and “that.” The reason for this word choice is not apparent; however, I conjecture that this was a form of colloquial discourse in which Jacob felt comfortable engaging with his classmates. This is supported by the fact that Jacob changed his discourse once I (who established myself as a researcher in the class) visited his poster. While explaining the poster to the me, Jacob recited the entirety of the exponents as “two to the three *x* plus one over three to the *x* plus three.” Jacob may have perceived me as a person with whom it would be inappropriate to engage in colloquial discourse. However, once I left, Jacob adopted this practice of reciting the exponents as he did for me in subsequent presentations to both students and the instructor. Regardless of the reason for the word choice, Jacob still treated the exponents as “things” rather than mathematical objects with special properties.

Perhaps the most concrete of objects Jacob mentioned were “those pesky exponents.” Jacob spoke of the exponents in terms of being physical obstacles imbued with the ability to cause trouble. More characteristic of an explorative discourse would be how Ms. Rebecca spoke about the exponents when she was trying to help Jacob reason through evaluating the limit. Ms. Rebecca posed the question, “are the exponents actually growing faster in the top or the bottom [of the fraction]” as if to model the types of questions her students should be asking when solving a problem. The way in which she spoke about the behavior of exponents and how that behavior affects the overall function is more indicative of discourse with abstract objects. Although Jacob’s discourse appeared highly ritualized in the poster sessions based on how he communicated about mathematical objects, the source of his narratives and the goals of his mathematical activity may shed light in a different direction.

Sources of Mathematical Narratives

The source of Jacob’s mathematical narratives both within the poster and the poster session itself was predominantly a recycled routine. This is more indicative of ritualized routines. Jacob’s poster showed that he and his partner tried to recycle three routines they learned in class: substituting a value for the variable, using exponent rules to simplify the expression, and multiplying by a quotient of conjugates. Although it may be a mere observation that Jacob and his partner attempted to recycle three routines, this was confirmed in Jacob’s admission to his instructor that he was looking for the appropriate routine to recycle but did not find one that matched the current situation. He said, “I don’t know if we’ve had a problem where we just straight up said *undefined*; so, that’s why I was kind of suspicious.” This explains why Jacob felt the need to end each of his presentations with a word of caution about the accuracy of his solution.

Moreover, Jacob verified this in his interview as an almost instinctive normal practice when completing the poster each week. When working on a poster problem, he would sometimes refer to his notes, “comparing it to previous examples of problems” which were similar or “problems that use similar methods.” This may not be an unfamiliar practice to professional mathematicians nor to the commognition literature (see Lavie et al.’s (2019) discussion on *precedents*), but Jacob’s seemingly automatic trial of the three routines based on key features of the problem (the presence of infinity, exponents, and a sum in a quotient function) suggests that he was acting without much purpose. Furthermore, it appears from “Step 5” (Figure 1) that Jacob and his partner were not entirely certain of what a conjugate is or how it may be used to help evaluate limits. This may be typical of a novice in the mathematical discourse who recognizes that some discursive action is necessary, but it is not apparent what the discursive action should be.

In contrast to Jacob's seemingly highly ritualized poster presentation, there was some indication that he engaged in a more explorative activity when preparing the poster with his partner. Jacob consistently said that he "tried" different routines showing that the source of these narratives was not entirely automatic recall; rather, he and his partner were actively searching for an appropriate routine that corresponded to a non-indeterminate form solution. Because the routine of ending with an indeterminate form was unacceptable according to Jacob's previously encountered routines, he knew that he needed to engage in a different routine. The actual "how" of that routine and "which" routine was unclear to him. This trialing of routines shows a mixture of Jacob asking the questions that Lavie et al. (2019) characterized as explorative and ritualized. Jacob appeared to be asking *how to proceed to get* a non-indeterminate form.

Goals of Mathematical Activity

The third aspect of a routine that can serve to distinguish between explorative and ritualized discourse is the goal of mathematical activity. According to Sfard (2016), the objective of performing a ritual is grounded in establishing or developing a relationship with another person and adhering to the goals set by others. The objective of engaging in an exploration is to know more about mathematical objects" (p. 44) and produce an endorsable narrative about those objects. The explorative goal is more intrinsically motivated while extrinsic motivation drives rituals. It is possible that Jacob and his partner "went [with] undefined," because providing a solution was one of the goals of the assignment. This is apparent because in four of Jacob's presentations, he ended with "we went with what we thought was best." In contrast, Jacob's discourse with his instructor showed that he genuinely wanted to know more about evaluating the limit in this case. Jacob began his poster presentation to his instructor by saying, "I'm pretty sure this is incorrect, so that's why I want to present it to you: to see if you can explain." There was no precedent that Jacob would be able to modify his poster to improve his grade, so the desire to know how to evaluate the limit in this case was more of an intrinsic motivation to simply "know more." Further evidence of this conjecture was found in Jacob's interview. When asked why he attempted to solve the poster problem, one of the reasons that Jacob gave was "part of it is showing to myself that, hey, I can actually do this stuff. You know, I'm actually not that bad at math, even though I feel like I am." The goal of mathematizing to establish and develop a relationship with his mathematical self is, by definition, ritualized discourse. Yet, it is apparent that the explorative spark is rooted in Jacob's discourse, and the ritualized discourse in which he engaged during the poster session was not wholly absent of mathematical exploration.

Discussion

This research offers insight into the complex nature of student engagement with mathematical concepts at the tertiary level and the role of discourse in shaping students' mathematical thinking. It underscores the importance of considering the entire ritual-exploration continuum in the mathematics classroom and highlights the interplay between the two extremes. Future research in this area could explore how instructors can design activities that intentionally promote the transition from ritualized to explorative discourse, such as PSPs (Barnett, 2022) or other exploration-requiring OTL (Nachlieli & Tabach, 2019). Additionally, investigating how students' discursive routines evolve over time as they progress in their mathematics learning would provide a deeper understanding of the developmental aspects of mathematical thinking.

References

Barnett, J. H. (2022). Primary source projects as textbook replacements: A commognitive analysis. *ZDM – Mathematics Education*, *54*(7), 1569–1582. https://doi.org/10.1007/s11858-022-01401-2

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the American Mathematical Society*, *62*(2), 144–146.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685–698. https://doi.org/10.1080/0020739X.2013.798874

The Conference Board of the Mathematical Sciences (CBMS). (2016). *Active learning in post-secondary mathematics education*. https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, *11*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Kontorovich, I. (2023). When learning stumbles upon identity and affect: A loaded student-student collaboration in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *54*(8), 1526–1540. doi.org/10.1080/0020739X.2023.2173102

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271. https://doi.org/10.1007/s10649-018-9848-x

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., Bressoud, D., & The Progress through Calculus Team. (2019). Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, *50*(1), 98–111. https://doi.org/10.5951/jresematheduc.50.1.0098

Rasmussen, C., Marrongelle, K., & Borba, M. C. (2014). Research on calculus: What do we know and where do we need to go? *ZDM – The International Journal on Mathematics Education*, *46*(4), 507–515. https://doi.org/10.1007/s11858-014-0615-x

Saxe, K., & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025*. The Mathematical Association of America. http://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2016). Ritual for ritual, exploration for exploration: Or, what learners are offered is what you get from them in return. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 41–63). Routledge. https://doi.org/10.4324/9781315643236

Is it Possible to Instruct College Students in the Role of Inventors?
A Case Study of a Mathematics Instructor in India

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This qualitative case study attempts to understand instructional goals that a mathematics instructor in India sets and envisions for engaging students. The instructor participated in a professional development program provided by the authors on the active role of students in learning mathematics. The authors' interpretations from three interviews were guided by two theoretical lenses of Realistic Mathematics Education and Rehumanizing Mathematics. In return, the interpretations offer a context to imagine how the lenses might interact.

Keywords: Realistic Mathematics Education, Rehumanizing Mathematics, Undergraduate mathematics,

Today, a substantial number of students worldwide experience a disconnection from mathematics and adopt a passive role as mere recipients of knowledge in classrooms. What if our mathematics instruction was a journey of students' inventions built on situations experienced as "realistic" based on what they know, their intuition, and even their misconceptions (Van den Heuvel-Panhuizen & Drijvers, 2020)? What if we approached students' non-traditional conceptions as avenues to expand mathematical knowledge, as a form of rehumanizing mathematics (Gutiérrez, 2018)? Such an approach has the potential to enhance students' senses of connection with abstract mathematical concepts, foster active engagement and participation, and enable them to view themselves as "authors of mathematics" (Povey & Burton, 1999, p. 235). This study aims to investigate how a mathematics instructor in India, who has been provided professional development on the active role of students in learning mathematics, might conceive of engaging students in classrooms. As a qualitative case, this study seeks to shine light on the meaning that this one instructor made about his instructional aims while engaging in learning about ways to highlight the human nature of mathematics. The interpretations were guided by two theoretical lenses of Realistic Mathematics Education and Rehumanizing mathematics.

Literature Review and Theoretical Framework

Realistic Mathematics Education (RME), conceptualizes the use of "realistic" situations to teach mathematical knowledge (Artigue & Blomhøj, 2013; Gravemeijer, 2020; Van den Heuvel-Panhuizen & Drijvers, 2020). An underlying goal of RME is an instructional approach beyond just exposing the students to mathematics as ready-made, which makes it different from traditional instruction. Realistic Mathematics Education is theorized through six principles: activity, interactivity, reality, intertwinement, level, and guidance (Van den Heuvel-Panhuizen & Drijvers, 2020). Being focused on instructional practices, this study will center on the guidance principle or guided reinvention process, but not in isolation from other principles because other principles interact with and influence guided reinvention. Guided Reinvention (GR) theorizes how teachers can provide opportunities for students by selecting "realistic" contexts to facilitate them to reinvent "higher" level formal concepts individually and collaboratively. (Gravemeijer, 2008; Van den Heuvel-Panhuizen & Drijvers, 2020). Because GR appears to be a set of prescribed practices to teach predetermined concepts, is there room for instructors to be led more by "alternative conceptions" or student's conceptions that differ from or conflict with standard understanding of mathematics concepts (Fujii, 2020, p. 625)?

Rehumanizing Mathematics (RM) is a framework that centers queer BIPOC students, as they are most harmed by standard practices and forms of mathematics (Gutiérrez, 2018). RM seeks to correct for an overemphasis on abstraction, objectivity, and binary logic. The framework advocates for humane practices, such as the incorporation of the cultural and linguistic resources of the participants of school mathematics. RM involves eight dimensions: (a) participation/positioning, (b) cultures/histories, (c) windows/mirrors, (d) living practice/futures, (e) creation, (f) broadening mathematics, (g) body/emotions, and (h) ownership/stewardship. To understand how instructors might be led more by "alternative conceptions" or student's conceptions that differ from or conflict with standard understanding, we choose the Creation dimension of RM. The Creation dimension (Cr) acknowledges students' conceptions and diverse forms of expressing mathematics, which relates to possible students' conceptions in group discussions in GR (Gutiérrez, 2018). In this way, Cr along with GR can be considered as a more comprehensive way of being able to expand beyond traditional knowledge forms or representations and that can then feel humanizing for students who might possess other ways of knowing and viewing the world and help affirm their relationships with mathematics as authors/creators of mathematics. The overarching question – What are the perspectives and intentions of a mathematics instructor in India about his roles/goals in classroom instruction in terms of engaging in abstract concepts? More specifically, [GR] How does the instructor conceive of and facilitate students to learn abstract concepts of mathematics? [Cr] In what ways does the instructor respond to his students when their work and conceptions do not align traditional mathematics or when they could be perceived as "wrong"? (e.g., How does the instructor address students' responses that are "misconceptions"?)

Methods

India's new National Education Policy 2020 strongly emphasizes the achievement of "learning outcomes," which is evident from its repeated mention of the term in the document (Government of India, n.d., p. 3). The primary focus in traditional classrooms of India is often limited to demonstrating the prescribed curricula and preparing students for semester final exams (Venkataraman et al., 2012). As such, many instructors feel pressured to complete the curriculum, even though it is too much content to cover in a meaningful manner. Student-student and student-instructor interactions in these classrooms are usually minimal (Ramanujam, 2012). Thus, students tend to find themselves disconnected from the course content.

In order to build rapport and reciprocity, we organized an online professional development (PD) program consisting of seven interactive sessions (each 90-120 minutes long) for 10 college/university mathematics instructors in India. Inspired by RME and RM literature, we designed activities such as creating reinvention-style lesson plans and grading intuition-driven hypothetical student responses that conflict with standard ideas in order to facilitate participants to share their meanings around GR and Cr in their classroom practices. Five participants who volunteered to be part of the research study gave us permission to interview and record them as they reflected on their experiences in each workshop. The first author interviewed each research participant twice in two 60-minute semi-structured sessions conducted and recorded through Zoom. The first interview centered on Guided Reinvention included questions such as "How was the lesson plan activity with your group partner to create a lesson that can facilitate students to discover math concepts?" to probe how they make sense of reinvention-based instruction and their related instructional goals. The second interview, focusing on Creation, included questions such as, "In the workshop, you discussed with your group partner about 'wrong responses.' What were some important points that arose in the discussion?" to probe for participants' viewpoints for alternative conceptions."

Thus, the PD program served as the entry point for questions we developed for individuals based upon their work in groups and perceptions of the PD. The interview questions avoided evaluation and instead emphasized ways participants made meanings in the program sessions. We sought explanatory responses and offered interviewees a chance to share their perspectives and practices as a teacher and learner. Questions that asked participants to reflect on their experiences as learners offered them a chance to talk about "poor" or ineffective instructional practices without referring to themselves. For example, "In your career as a student and then as a teacher, can you recall an event when a learner's own ways of solving mathematics problems were supported or discouraged because of something? What was that?"

Data Analysis

After analyzing responses from the first two interviews, we selected one instructor, whom we call Aditya [pseudonym], to focus upon for a case study to fully engage and delve into more complexities of his identity and context in making pedagogical decisions (Yin, 2017). As a case study design, this study is not generalizable "from samples to universes" (Yin, 2017, p.20), but its findings might be "transferable" (Finfgeld‑Connett, 2010, p. 246) to other instructors' contexts. The final interview utilized a "Member Checking and Intervention Interview" protocol that was aimed (1) to determine the accuracy of interpretations of this instructor's identified practices in the first two interviews through member checking questions that consisted of summaries of our interpretations and (2) to extend/challenge the participants' practices toward GR and Cr through intervention questions or joint wonders that consisted of the interviewee's own perspectives about teaching and utilizing questions asked within the literature (Gambill, 2022). We wanted to honor the relational piece in research and promote collaborative wondering to make research more humane, empathetic, less extractive, and non-evaluative.

The data were analyzed in multiple phases. We first identified interesting segments and interpreted them with memos. When the authors had different interpretations, we attempted to reach a consensus or an "interrater reliability" (Belotto, 2018, p. 2625). We then labeled the segments and interpretations with short phrases of 3-4 words, which became sub-codes. These codes served as the basis of the findings we present below.

Findings

We found four themes: (a) Avoiding faulty foundations through correct proofs, (b) Offering exploratory opportunities through minimal-to-incremental guidance, (c) Encouraging student engagement by building upon their prior knowledge, and (d) Increasing motivation by providing the history of mathematical concepts. Due to space constrains, we present below the first two themes in this section and mainly the second in the discussion section later.

Avoiding faulty foundations through correct proofs

In this section, we will explore Aditya's role, which appeared in the data analysis, to instruct his students to refrain from faulty foundations through formal or correct proofs. Interestingly, with this goal in mind, he seemingly incorporates informality and motivates students to share diverse "guess[es]" or "wrong" conceptions. In his grading of a student's solution that included informal terms presented in our PD program, Aditya explained:

> [W]e [as students] **assume some results** are obvious or trivial. We simply **write** it down "clearly," "obviously." I forbid them [students] to **write** "obviously" or "clearly." I tell them that if it is so obvious or so clear to you, why don't you **produce**

> a proof because it's not at all obvious to me? … if I could not do it, then either there's a problem in my understanding … or … what we are trying to solve is not true.

Like many mathematics instructors, Aditya seems to consider his instructional role to "forbid" or prevent students to always rely upon intuition/informality ("assume some results") so that they can understand the value of proofs. He might be recognizing that ideas that seem to be true informally might not be the case when they try to prove those ideas formally. And, in this case, he seems to be invested in supporting his students to learn how to prove something, as it will be an important skill in learning mathematics. Even if he stresses formal proofs, Aditya appears quite different from many instructors who consistently ignore intuition/informality in instruction to understand his goals toward both informality and formality. When he says, "I forbid them to write obviously, or clearly," he seems to value mathematical symbolic or formal proofs, especially when it comes to *writing* proofs. But contrasting with the traditional emphasis *only* on formal procedures, Aditya acknowledges, at another instance during an interview, informality's role to get a global bird's eye view or an "overall idea" of proofs. These views appear to underscore informalities for *thinking* processes in students' heads when they prove formally or even when they have acquired formal concepts. The way Aditya "forbids" his students to prevent errors is not necessarily instant rejection of their thoughts. At another moment, Aditya appears to value an open classroom space by motivating his students to propose "guess[es]" or conjectures and to ask questions without thinking of them as trivial and incorrect. This might be one way to accomplish the goal effectively because Aditya is able to correct more "wrong" points or "faulty" foundations. Thinking that an understanding based on a student's "guess" can differ from or conflict with an instructor's understanding, the first author asked him about contradictory understandings. He said:

> [If two understandings] are contradictory, one of the concepts must be wrong because **in maths the answer [to a question] is just one [unique]**. … if I realized that I was completely wrong, [or] I was on the wrong track, … [then it] would stay with me always, and I would never make that **mistake** [again]…. If it's not contradictory, then there are scopes [or chances] of learning. Even if it is contradictory …or one of us [with contradictory understandings] is completely wrong, [then] you can think of it as a pin [or reminder] or some kind of marking that would always stay with the student and would help him **not to get it wrong the next time**.

Saying "in maths the answer is just one [unique]" and "not to get it wrong the next time," Aditya appears to value correcting his students' "mistakes" or conceptions, which might otherwise be "contradictory" and ambiguous. Such an instruction can highlight a limited exploration of "wrong" things though because the instruction is only about identifying "mistakes" so that the students do not repeat them. The students' exploration through "guesses" or "questions" requires instructor's facilitation and guidance, which we understand in the next theme.

Offering exploratory opportunities through minimal-to-incremental guidance

Aditya expects his instructional role to offer students exploratory opportunities through his minimal-to-incremental guidance that encompasses exposing the least content possible in the beginning and providing students hints when believed necessary. When asked about students' exploration opportunities in the classroom, Aditya said:

> [W]hat I feel I would … follow in the class actually [offer the] **least bit of help**. So, as if there are several pages [of hints]. You flip the first page, you have [a hint on the first page] if you can solve [the given problem] ... If you can't, then move to the next

page and see some **more hints or more suggestions** or some more help. So, step by step it's up to the students. … I get a feeling … how much **help they need**. So, depending on that I try to provide help, but at the very beginning. I would give them the minimum. … let them **struggle and learn**.

He appears to target his instruction by offering the "least bit of help" at first so that the students can "struggle and learn" and can get his support of "more hints or more suggestions" when they are perceived unable to complete the given tasks. Through such minimal-to-incremental guidance, as opposed to instructors who give them maximum guidance, his students are likely to first be given the opportunity to learn with guidance as per the extent of the "help they need." Aditya's exploratory and discovery-oriented goals for his students were also reflected in his epistemological perspective in the quote below:

[Exploring] Proof first then [getting at] theorem statement should be a way [to teach] because that means students always would be **curious**. … How can we get this theorem? It is a **reverse process**; you don't get to eat your meal until, unless you cook it. … That's the way it should be natural.

When asked, Aditya could not recall any particular example of such a "reverse process" at that time but it appears worthy to mention. This "reverse process" of getting a theorem statement after a proof differs from a widespread perception among mathematics professors and in mathematics textbooks to follow Definition – Theorem – Proof format. If directly taught, students might not feel supported to explore in the traditional format. On the other hand, with Aditya igniting "curious[ity]," students might come up with brand new conjectures because they are free to explore and build knowledge. However, the final interview's member checking revealed the strategy does not always play out in the ways he hopes and revealed that his students just wonder individually in their heads instead of contributing to proofs by sharing aloud with others. This is understandable because it might be difficult for students to understand what is expected of them. By practicing minimal-to-incremental guidance, he values neither complete absence of guidance by letting them explore without facilitation nor excessive guidance.

The analysis of the first two interviews highlights his emphasis on asking questions to the students, which can appear as "hints." Thus, in the final interview, we asked how he might model questions to students. Some questions such as "what-if" questions were found to be open-ended and promote exploration while others such as "this-and that" questions could guide students toward a particular line of thinking. He said:

[I]magine that I have **changed one condition** from there or the other condition from there **[in a known result]**. [Then I ask them] do you still think the result is true? So, some questions of this form [I would ask]. Can we remove these conditions or can we add [some], can we **weak[en]** them? So, with [their present] knowledge they [students] can come up with … counterexample[s] … You said that both A & B would hold? Almost probably A would hold or B would hold. **Can you guess which of them [A and B] would hold?** … They are just **interpolating [guessing] from what they know**.

When he says, "Can you guess which of them [A and B] would hold?", he appears to prompt the students to consider specific and important ideas or concepts by "interpolating from what they know." Such a binary question is important so that students can understand the intended goals of their instructor. Moreover, Aditya's "what-if" questions like "chang[ing] one condition … in a known result" in particular can be seen as open-ended questions and beyond mere yes/no questions because the "students can come up with … counterexamples" to justify and there can be brainstorming on tweaking conditional statements, like “weaken[ing] them” in his classroom. Thus, the what-if questions in Aditya’s classrooms are likely to provide support for exploration.

Discussion

Aditya had been introduced to the concepts of Guided Reinvention (GR) and Creation (Cr) through a series of PD sessions and was offered the opportunity to reflect on these concepts and how they (might) play out in his teaching. With the objective of how he makes sense of GR and Cr in mind, we identified four main themes through the teaching practices that he employs or conceives: (a) Avoiding faulty foundations through correct proofs, (b) Offering exploratory opportunities through minimal-to-incremental guidance, (c) Encouraging student engagement by building upon their prior knowledge, and (d) Increasing motivation by providing the history of mathematical concepts.

Due to space restraints, we mainly focus on the theme of *minimal-to-incremental guidance* in this section. Even so, we believe it is important to highlight some of the interactions between the themes and with the conceptual tools of GR and Cr. We found that Aditya envisions his responsibility to provide opportunities for exploration through an instructional strategy of giving minimum assistance in the beginning and increasing the assistance gradually through “hints” and questions as per the perceived need of the students. Through the instructor’s hints, such an approach has potential to empower students to reinvent intended concepts that GR contends. Due to the lack of expository teaching, the “curious” students might also have opportunities to *explore* and develop brand new conjectures in such a minimal-to-incremental guided instruction; thus, the instruction can be seen consistent with Cr. Aditya’s technique appears unique, given this type of guidance does not appear in Learning Outcomes based Curriculum Framework (LOCF) and the standard faculty induction program called GURU-DAKSHTA published by University Grants Commision (UGC), a statutory body in India. Aditya’ questioning strategy in the classroom supports both GR and Cr. The binary question, observed in the findings section, is important so that the students can understand the intended and clear goals of their instructor, which provides support for GR because RME-instruction sequences appear to have a clear specific path to follow. Moreover, Aditya’s open-ended questions like what-if questions appear consistent with King and Rosenshine (1993) who argue for “guided cooperative questioning strategy” in which teachers ask generic thought-provoking questions that can provide support for exploration and, thus, Creation. Such an approach is more process-oriented (as opposed to product-oriented), which makes it different from the learning outcomes approach highlighted in India’s education policy and GR’s “intended mathematics” goals (Gravemeijer, 1999).

A minimal guidance might facilitate students' exploration, but there are chances that such explorations can lead to students’ proofs that are not “correct” or to their “contradictory understandings” that are revealed in the theme on correct proofs. We note from Aditya’s practice that if guidance is increased too quickly, it might restrict exploration or discovery. In other words, if his students solve mathematics problems using informal or intuitive language that does not align with standard proofs, he might feel prompted to make them aware of the inaccuracy and tell them or motivate them to use formal proofs.

More than simply showing how Cr and GR arise in Aditya's views and instruction, his case highlights how the two conceptual tools Cr and GR relate and support or expand the other theoretically. Both concepts acknowledge: (a) students as mathematicians, (b) students participating and connecting with each other, and (c) instruction as process-oriented. Let’s consider how they expand each other. First, Cr expands GR in following ways: (a) The GR process aims to facilitate students to reinvent instructor's or curriculum's predetermined concepts. Cr acknowledges that the foundations of the predetermined concepts can be contextual, for example, following one of several appropriate sets of axioms or a collection of self-evident statements. In this way, Cr might go beyond just considering predetermined concepts in teaching and learning mathematics by taking into account the "misconceptions" as alternative, yet viable, ways to think about or develop a concept. (b) GR appears to be a linear sequence of levels starting from situation → referential → general → formal in "Model of - Model for: emergent modeling" (Gravemeijer et al., 2000; Streefland, 1985; Van den Heuvel-Panhuizen, 2003). Creation can inform us that instructional practices are not necessarily sequential due to the probable presence of an interaction between correct and "wrong" conceptions. Second, GR expands Cr in the following ways: (a) Goal-oriented: Cr does not speak about achieving specific goals that can indicate success. Considering GR with Cr can be helpful because GR appears goal-oriented by focusing on facilitating students to reinvent a predetermined concept. (b) Concrete practice: GR can offer instructional steps, for example the linear sequence of levels, should an instructor be confused or uncertain about how to approach Cr in the classroom. Thus, Cr and GR show the potential to expand each other.

There are several strengths of this study. The study was not extractive because we attempted to build through (a) the PD program before we generated data with Aditya and (b) the joint wonders during interviews that included the first author's teaching experience to identify the resistance in practicing GR and/or Cr in Aditya's instruction during the interview. Second, the interview protocols were tailored to the participant's responses and according to the examples and comments they provided in preceding interviews. Even so, there are some limitations to this study. For example, although several interviews were conducted with each participant, there is a single data source: interview responses from Aditya. Multiple data sources, such as classroom observations and comments from students during focal groups or interviews, or survey responses would have allowed for triangulation and increased credibility and a more comprehensive understanding.

Conclusion and Implications

This case study investigated how a mathematics instructor in India conceived of his practice in relation to facilitating abstraction and addressing students' conceptions. This study raises several issues for future researchers. In this study, Aditya described what he does and what he thinks about his teaching through interviews and, in that sense, his case offers researchers a glimpse of the potential approaches and challenges of incorporating GR and Cr into mathematics instruction at the college level. However, the data fail to capture how his students feel about his instructional practices. Future research might consider observing the classroom in an ethnographic study in order to get some evidence from students on how GR and Cr are experienced or might be supported in the classroom. This approach is especially important in the case of RM because instructors alone cannot determine if their pedagogy is being felt in rehumanizing ways (Gutiérrez, 2018). Potential research questions for the study might include: (a) How does a mathematics instructor practicing both GR and Cr know how much informality is appropriate during a course primarily based on abstract concepts or in courses that feature "pure" mathematics? (b) What instructional practices, such as debates in history, might offer Nepantla moments for students so they can appreciate conflicting conceptions in classrooms (Scott & Tuana, 2017)? and (c) How might seeing informality and formality as part of a larger cycle affect a mathematization process in GR during the instruction?

References

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *Zdm*, *45*, 797-810.

Belotto, M. J. (2018). Data analysis methods for qualitative research: Managing the challenges of coding, interrater reliability, and thematic analysis. *The Qualitative Report*, *23*(11), 2622-2633.

Finfgeld‐Connett, D. (2010). Generalizability and transferability of meta‐synthesis research findings. Journal of advanced nursing, 66(2), 246-254.

Gambill, T. (2022, February 8). *How to lead with empathy through higher-quality questions*. LinkedIn. https://www.linkedin.com/pulse/how-lead-empathy-through-higher-quality-questions-tony-gambill/

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modelling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225–273). Mahwah, NJ: Lawrence Erlbaum.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of curriculum studies*, *32*(6), 777-796.

Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education*, 217-233.

Gravemeijer, K., & Stephan, M. (2002). Emergent models as an instructional design heuristic. *Symbolizing, modeling and tool use in mathematics education*, 145-169.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. Mathematical thinking and learning, 1(2), 155-177.

Gravemeijer, K. (2008). RME theory and mathematics teacher education.

Gravemeijer, K. (2020) Emergent Modeling: an RME Design Heuristic Elaborated in a Series of Examples. Educational Designer, 4(13)

Gutiérrez, R. (2018). The need to rehumanize mathematics. In Goffney, I. & Gutiérrez, R.. (Eds), Rehumanizing mathematics for Black, Indigenous, and Latinx students. National Council of
Teachers of Mathematics.

Gutiérrez, R. (2018). Rehumanizing mathematics: A vision for the future. A keynote address given in the Rehumanizing Mathematics conference. Santa Clara County Office of Education. San Jose, California. September 7, 2018.

Ministry of Education, Government of India. (2020). *All India Survey on Higher Education 2019-20*. Retrieved April 29, 2023, from https://www.education.gov.in/sites/upload_files/mhrd/files/statistics-new/aishe_eng.pdf

Povey, H., & Burton, L. (1999). Learners as authors in the mathematics classroom. [[From Burton, L. (ed.) (1999). *Learning mathematics: From hierarchies to networks, pp*. New York: Falmer Press.]]

Ramanujam, R. (2012). Mathematics education in India-An overview. *Mathematics education in India: Status and outlook*, 1-12.

Shulman Lee S. (1970). "Psychology and Mathematics Education." In National Society for the Study of Education, 69th Yearbook, edited by Edward G. Begle. Chicago: University of Chicago Press.

Scott, C., & Tuana, N. (2017). Nepantla: Writing (from) the in-between. *JSP: Journal of Speculative Philosophy*, *31*(1), 1-15.

Streefland, L. (1985). Search for the roots of ratio: Some thoughts on the long term learning process (towards... a theory) Part II: The outline of the long term learning process. *Educational Studies in Mathematics*, *16*(1), 75-94.

University Grants Commission. (2019, November). *GURU DAKSHTA*. Page 8 - guru dakshta english. Retrieved April 29, 2023, from https://www.ugc.gov.in/e-book/GURU%20DAKSHTA%20English.pdf

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. *Encyclopedia of mathematics education*, 713-717.

Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational studies in Mathematics*, *54*, 9-35.

Venkataraman, G. & Sholapurkar, V., & Sharma, B. K. (2012). Curriculum and pedagogy in mathematics: Focus on Tertiary Level. *Mathematics Education in India Status and Outlook*, 127-150.

Yin, R. K. (2017). *Case Study Research: Design and Methods*, 6th edition. Sage Publishing.

A Student's Self-Perceptions and Social Positions in an Undergraduate Precalculus Classroom

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Mathematics education research has linked students' development of positive mathematics identities to outcomes of academic success and persistence in STEM. I contribute to this research area by adopting a positional approach to identity development, focusing on the students' development of in-the-moment identities through the negotiation of acts and roles. To better understand the connection between the roles adopted by students in class and the students' mathematics identity development, I conducted classroom observations of an undergraduate precalculus classroom and interviewed four focal students on their beliefs about math and experiences in this course early and late in the semester. Student-instructor interactions from the observations were coded to produce positioning profiles for the focal students. Here, I present the positioning profile for one student, alongside a reflection of her transition from not identifying as a "math person" to identifying as one, as developed through narrative analysis of the interview data.

Keywords: positioning, identity, student-instructor interactions

Students' mathematics identity development has been a topic of mathematics education research, with studies linking the development of a positive math identity to students' academic success (e.g., Ma & Kishor, 1997; Lee & Anderson, 2009; Gonzalez et al., 2020) and persistence in STEM trajectories (e.g., Meece et al., 1990; Boaler & Greeno, 2000). This study intends to explore how student-instructor interactions in the classroom support or hinder the development of positive math identities. Identity has been conceived of in many different ways, and here I leverage perspectives with a focus on students' in-the-moment construction of their *positional identity* as they interact with others in math spaces, foregrounding the roles that students are allowed access to and choose to adopt in their math classes (Herbel-Eisenmann et al., 2015; Radovic et al., 2018). This positional take on identity has offered particular insights into the role of communication acts (Andersson & Wagner, 2019) and their associated power dynamics (Esmonde & Langer-Osuna, 2013) in affording (or denying) resources for students' participation in mathematical tasks.

In this contributed report, I utilize longitudinal classroom observation and interview data to describe the ways in which Olivia (pseudonym), a student in an undergraduate precalculus course, (a) adopted particular roles during interactions involving the instructor in the math class, and (b) experienced growth in her sense of self as a "math person" across the semester. The ultimate goal of this research is to better understand how an undergraduate student's positioning during interactions with their math instructor impacts the student's enduring sense of themself as a learner and doer of math. To that end, the next phase of analysis will involve linking repeated positions adopted by multiple students to changes in their self-perceived mathematics identity. A better understanding of the impacts of student positioning during instructor-led interactions on their more stable identification with math will support instructors' ability to consciously reframe their discussions (Louie, 2017) in ways to support their students' mathematics identity development.

Theoretical Framing and Relevant Literature

I define a participant's *positional identity* using Harré and van Langenhove's (1999) positioning theory. This perspective is focused on understanding the meanings behind different communication acts in an interaction through the storylines and positions available to participants. *Storylines* are sociocultural frames for interpreting the available rights and duties of participants in the interaction, and *positions* are the ways that elements of these storylines are taken up by the participants during the interaction to act in particular ways (Andersson & Wagner, 2019). One's identity is constructed through social interactions where (a) the individual acts to position themselves as having particular roles or qualities, and (b) others act to position the individual and accept (or reject) the individual's own acts of positioning (Langer-Osuna & Esmonde, 2017). With this understanding, I define an individual's *mathematics identity* as their personally held beliefs about their ability to participate appropriately in particular mathematics spaces. This framing of mathematics identity assumes a connection between students' acceptance (or rejection) of in-the-moment positions within the mathematics classroom, and their self-understandings about how they are supposed to participate in mathematics spaces.

Significant work has been done in K-12 mathematics education research to better understand students' identity development and the outcomes associated with it using a positional lens (Boaler & Greeno, 2000; Esmonde & Langer-Osuna, 2013; Bishop, 2012; Anderson, 2009; Andersson & Wagner, 2019; Gholson & Martin, 2019; Louie, 2017). In particular, the positional approach to identity has given researchers insights into students' enactment of different mathematics identities in the same local context (Bishop, 2012) and the positive influence of different positioning acts on students' engagement with certain mathematical practices (Esmonde & Langer-Osuna, 2013). However, this positional approach to mathematics identity has been uncommon in RUME (Radovic et al., 2018).

To characterize the varied positions that students could adopt in the classroom, I initially leveraged Andersson and Wagner's (2019) characterizations of high school math students, who highlighted three positions of *leader*, *follower*, and *rebel*. These archetypes focus on characterizing who is making requests and who is fulfilling those requests (or not), but they are somewhat limited. For instance, any interaction involving students and the instructor tends to position the instructor as the *leader* and the students as *followers*, given the nature of their formal relationship. As such, Andersson and Wagner's characterization of positions is more suited for student-student interactions and not student-instructor interactions. Additional approaches to characterizing positions from the literature range from lower inference documentation of the structure and function of an individual's communication actions (Bishop, 2012) to higher inference coding for the emotional content of actions (e.g., Heyd-Metzuyanim & Cooper, 2022). These various approaches to identifying students' positions in the math class collectively influenced the initial analytic coding process.

Methods

Data Collection and Context

Data for this preliminary report consist of (a) classroom observations of an undergraduate precalculus classroom, and (b) early- and late-semester interviews with one of four focal students and the instructor. The course was taught in Spring 2023 at a large research university in the Southwestern U.S. Class sessions were 75 minutes long, and students typically engaged in small group work at their tables while the instructor circulated the room, interspersed with whole class discussions (WCD). During WCD, the instructor frequently used an Initiation-Response-

Evaluation model for questioning (Mehan, 1979), where she initiated the interaction by asking a question to the class, a student responded to the instructor's question, and the instructor assessed and evaluated the student's response.

Three preliminary classroom visits occurred at the start of the semester, to identify and recruit focal students and to refine the video and audio recording setup. Following this, seven observations were conducted for study: three earlier in the semester and four later in the semester. After the preliminary classroom visits, students who had interacted with the instructor at least once were invited to participate in the study, and four students were chosen who had interacted with the instructor with varying frequency. Small group work was recorded visually using two tripod-mounted iPads and audibly using two recording devices placed on tables with the focal students; both visual and audio recording devices were adjusted throughout the class session to best capture student-instructor interactions in both small group and whole class discussions. While observing and recording class sessions, field notes were taken to provide context and to indicate times when there were interactions between the instructor and one of the focal students. The instructor was interviewed at the beginning and end of the semester to provide additional context regarding instructional practices and how she interacted with students.

The focal students were interviewed at the beginning and end of the semester, immediately following the early-term and late-term observations. All interviews were video- and audio-recorded using Zoom conferencing software; some interviews were conducted in-person while others were conducted over Zoom. Interviews were semi-structured, with questions about the students' (a) academic status and past experiences in math courses, (b) general interests in and beliefs about mathematics, (c) performance and participation in this particular course, and (d) perceptions of how they and significant others in their life and course (family, friends, instructor, classmates) viewed them as a "math person." At the end of each interview, participants were asked to reflect on particular moments from a recent class session, supported by clips from the class recording in which the participant was interacting with the instructor.

Analysis of Observation Data

Analysis of the observation data began with the identification of notable moments from class recordings. A *notable moment* is one where (a) an interaction occurred involving the instructor and a focal student, and (b) the interaction was *not* focused on course logistics (e.g., asking about exam dates). Although this report focuses on the notable moments involving one particular student, a total of 17 notable moments were identified between all four focal students, ranging from one to three minutes in length. These clips were transcribed using Descript software and manual cleaning, with notes added to indicate associated physical movements and gestures. Analysis of these notable moments was done in accordance with thematic analysis (Braun & Clarke, 2012) to develop roles for these focal students and the instructor. Following familiarization with the data, initial codes were assigned to each talk-turn, with multiple codes sometimes associated with the same turn. Codes were developed in a predominantly deductive fashion, where prior positioning studies were leveraged to construct (a) *structural codes* (Andersson & Wagner, 2019; Bishop, 2012) that capture the discursive function of the talk-turn in the interaction, and (b) *subjective codes* (Heyd-Metzuyanim & Cooper, 2022) that incorporate inferences about turn-taker's emotional hue. Though informed by the literature, these codes were refined through an iterative reflection with the data to more accurately capture the roles that were common in the class. Once refined, the position codes were grouped by student, and *positioning profiles* were created for each student and the instructor to reflect any patterns noted in the individual's positions throughout the class. In the next section, the developed codes are presented alongside a checklist to indicate if Olivia or her instructor adopted any of those roles during notable moments involving Olivia.

Analysis of Interview Data

Interview recordings were automatically transcribed by Zoom and manually checked. Analysis of interview data was conducted in accordance with narrative analysis (Polkinghorne, 1995) to configure the elements and events of Olivia's interview into a developmental account of her feelings as a "math person" throughout the semester. The use of "math person" here is due, in part, to prior findings that the term is associated with mathematics identity for some undergraduate students (Guglielmo et al., 2023). Initially, events relevant to Olivia's perceptions of herself as a learner and doer of math were identified and summarized. These events were then organized to construct an initial recounting of Olivia's math identity development from beginning to end of the semester. This narrative was refined through a recursive movement between the data and the emerging thematic plot, to ensure it succinctly brought meaning to the interview data while maintaining accuracy. In the next section, a condensed version of the narrative along with relevant excerpts from the interview transcript are presented.

Results

Olivia's Positioning Profile

Olivia was involved in six notable moments over four class periods (two early-class, two late-class). Five of these moments occurred while students were working in small groups, the sixth occurred during whole class discussion. Table 1 describes all nine position codes that were developed during analysis. In this table, both the position code and its description are provided, alongside a checklist that acknowledges if the instructor or Olivia exhibited that role during *any* of Olivia's six notable moments. The first six codes are structural codes and the last three are subjective codes.

Table 1. Codes developed for Olivia and the instructor's roles during Olivia's six notable moments, with a checklist for which individuals exhibited these roles in at least one notable moment.

Code: Description	**Instructor**	**Olivia**
Requester (general): Making a request expecting a response from members of a work group or the entire class.	x	
Requester (specific): Making a request expecting a response from a specific individual.	x	x
Requester (support): Making a request looking for approval or support for their prior responses or actions.	x	
Responder: Responding to a previous request by another individual, with or without justification for response.	x	x
Subject: Being the explicit or implicit subject of another individual's request or response.	x	x
Revoicer: Rephrasing or restating the response of another person out loud.	x	x

Approver: Responding to another individual's request or response, with explicit or implicit approval of some part of their contribution.	x	x
Critiquer: Responding to another individual's request or response, with explicit or implicit critique of some part of their contribution.	x	
Apologizer: Expressing regret for a prior response or action.	x	x

The following excerpt illustrates how these codes were applied, using a small group interaction between Olivia, the instructor, and StudentA (a non-focal student in Olivia's small group), discussing the domain of the sine function:

		Code [*Subject*]
Instructor	So, do we have any restriction on our angle measure?	Requester [*General*]
Olivia	Well, I was thinking, wouldn't, wouldn't it be like zero to two pi?	Responder [Instructor]
Instructor	Can we go (*upward vocal inflection*) above two pi?	Critiquer [Olivia] Requester [*General*]
StudentA	Yeah, yeah, if you do multiple rotations.	Responder [Instructor]
Instructor	Okay. Can we go the (*upward vocal inflection*) other direction?	Approver [StudentA] Critiquer [Olivia] Requester [*General*]
Olivia	Yeah.	Responder [Inst.]
Instructor	So (*expectant pause*)	Requester [*General*]
StudentA	Kind of (*pause*) infinity?	Responder [Instructor] Requester [Instructor]
Olivia	Okay.	Approver [StudentA]
Instructor	Yeah, we can go in both directions. We'll still get out something, right, from our function. Yeah, we haven't been working too much in terms of, like, more than one full rotation. But, um, we definitely can.	Approver [StudentA] Revoicer [*General*]

Initially, Olivia responds to the instructor's request for the domain of the sine function (talked about as an angle measure) as zero to two pi. The instructor, instead of approving Olivia's contribution, makes a request for further information from the whole group, implicitly critiquing Olivia's response. StudentA responds to the instructor's first request with support, and then Olivia responds to the instructor's second request without support. After the instructor's final request, StudentA provides a hesitant response that also functions as a request for support, to which both Olivia and the instructor approve. This was one of three notable moments where Olivia made a mathematical error in her response, and the instructor adopted a Critiquer role by questioning the student's response. In the two other instances, one similarly involved an implicit critique of the student's response mediated by another student, while the other was an explicit critique of the student's algebra.

Olivia's Narrative Reflection on Mathematics Identity

Olivia's mathematics identity narrative was constructed relative to three phases of Olivia's participation in her precalculus course: the pre-precalculus narrative, the initial reflections on precalculus, and the final reflections on precalculus. Before enrolling in her precalculus course, Olivia had some positive associations with mathematics, based on prior experiences in math courses and her own interests in the sciences. In high school, Olivia found particular math subjects interesting and "easy" (trigonometry, statistics), while also developing a preference for math classrooms with an active instructor who provides step-by-step explanations and examples. In her transition to university, mathematics played a peripheral role in her choice to major in biological sciences, with Olivia desiring a biology-focused major that was not "fully set on just math." In relation to her biology identity, Olivia views math as a useful tool for interpreting graphs, analyzing data, and making predictions, with its application to many areas and its integral role in the advancement of our society. In sum, Olivia has an appreciation for mathematics, an interest in learning (at least some areas) of mathematics, and a view of mathematics as significant (in some regards) to her future academic and career motivations.

Despite these positive associations with mathematics, Olivia explicitly did *not* describe herself as a math person during the initial interview, in which she said:

> **I want to be a math person**. I associate a lot of like math people, **they're really smart**. Like they're really smart. They think really well, like the way their minds work, they're very calculated […] whereas, like, with English, it's okay, there's more leniency in it […] I feel good knowing that, like, my family would probably see me as [a math person], but **I just don't see myself as it probably due to some, like, I see, I'm like, I'm hard on myself. I'm a pretty, like, when it comes to that, I have a pretty high standard**.

Olivia often contrasted her sense of being an "English person" with being a "math person" during this first interview, as noted in the excerpt above. She viewed herself as an English person, in part because it "just comes more naturally," while not viewing herself as a math person due to her own "high standard." During this first interview, Olivia thought of "math person" as a title reserved for "really smart" individuals, of which she did not view herself.

While reflecting on her classmates' perceptions of her as a math person during the first interview, Olivia noted the mixed perceptions they likely have. She mentioned some students in the class that likely *don't* view her as a math person due to her getting multiple answers wrong, but also one particular student who likely *does* view her as a math person, because Olivia opens up more around her and explains her process. This focus on opening up and reaching out to others was brought up multiple times by the student, mostly expressing a desire to contribute her ideas more during discussions. Specifically in previous math courses, Olivia acknowledged that she often would not speak out during class due to her "low self-esteem when it comes to math related subjects." However, with her precalculus course, Olivia stated that "[the instructor] makes it to where you wanna talk to the tables," and that "this is the perfect moment for me, in my mind, to like, try and branch out more." Though she still has to "fight the feeling" of being afraid to be wrong, Olivia acknowledges that she has started sharing her ideas more openly with others in her precalculus class, due in part to the interactive nature encouraged in the class.

During the second interview, and after asking Olivia "How has this class affected your level of interest in math as a whole?" Olivia immediately responded, "Okay, this is crazy, but I've actually, I went from thinking that I wasn't a math person to actually thinking I am a math person now." Throughout the interview, the reasons behind this change in the student's self-perception as a math person were explored. One particular reason, highlighted in the excerpt below, had to do with the students' shift in thinking about being wrong as necessary in the process of learning mathematics:

> I'm, I'm feeling a lot more comfortable with being wrong. Just like being able to discuss in groups, if I'm wrong, I kind of, I, I used to just care about it so much. **It was all I would think about. I, I was afraid of like speaking.** But like now it's, I don't know. I, I'm feeling a lot more comfortable in doing that, because **I've seen everyone get something wrong before and I'm like, I'm not the only one** […] It's not like I care that they got that wrong, so why would others care about me getting stuff wrong, you know? It's all part of the process. **It's all part of learning. That's why we're learning it**.

Olivia more explicitly notes her instructor's influence on this shift, through her "sweet and genuine" approach to teaching and her ability to "understand where [Olivia's] coming from." In contrast, Olivia states that with her current English instructor, "I'm afraid that if I say this, I'm gonna be wrong, and she's gonna scream at me." Unlike other classes, Olivia consistently looks forward to going to her precalculus lessons, sees the importance in them, and feels "happy" and "motivated" when she learns something new. Overall, Olivia now predominantly sees a math person as one who enjoys doing math and participating in math contexts. As such, she now views herself as a math person.

Discussion and Next Steps

During early- and late-semester class observations, Olivia adopted relatively consistent positions during interactions involving the instructor. In five of the six notable moments, Olivia made a mathematical error in her response to the instructor's request, and another individual would implicitly or explicitly critique her response. In these interactions across the semester, Olivia took on the role of a quiet yet persistent member of her precalculus class, whose contributions were open to critique by others. This prevalence of critiquing contributions was mirrored in Olivia's reflections during both interviews, where she often mentioned a fear she had of speaking out and being wrong in previous math classes. By the end of her precalculus course, Olivia saw the process of trying and getting things wrong as a necessary part of learning mathematics. This shift coincided with her coming to claim the title "math person," as Olivia found enjoyment and motivation in her precalculus class. In accordance with positioning theory (Harré & van Lagenhove, 1999) and mathematics identity, I conjecture a link between Olivia's frequent in-class role as the subject of others' critique and her development of understanding wrong answers as necessary for learning.

Additional positioning profiles and math identity narratives for the other focal members of this class will contribute to a broader understanding of ways in which students' mathematics identities may shift in relation to the positions they adopt during interactions with their instructor. These will inform future research intended to better understand the ways in which undergraduate students' positioning in their math class affects their mathematics identity development, including exploring the storylines from which they draw their various roles and classroom norms. Ultimately, this will lead to a more comprehensive understanding of the factors that influence the students' positioning and mathematics identity development.

References

Anderson, K. T. (2009). Applying positioning theory to the analysis of classroom interactions: Mediating micro-identities, macro-kinds, and ideologies of knowing. *Linguistics and Education, 20*(4), 291-310.

Andersson, A., & Wagner, D. (2019). Identities available in intertwined discourses: Mathematics student interaction. *ZDM, 51*(3), 529-540.

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education, 43*(1), 34-74.

Boaler, J., & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. *Multiple Perspectives on Mathematics Teaching and Learning, 1*, 171-200.

Braun, V., & Clarke, V. (2012). *Thematic analysis*. American Psychological Association.

Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education, 44*(1), 288-315.

Gholson, M. L., & Martin, D. B. (2019). Blackgirl face: Racialized and gendered performativity in mathematical contexts. *ZDM, 51*(3), 391-404.

Gonzalez, L., Chapman, S., & Battle, J. (2020). Mathematics identity and achievement among Black students. *School Science and Mathematics, 120*(8), 456-466.

Guglielmo, J., Phelps, J.R., Yu, F., Heyer, N., & Apkarian, N. (2023). What is a "Math Person?": Students' Interpretation of Identity Terminology. In Cook, S., Katz, B. & Moore-Russo D. (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 661-668). Omaha, NE.

Harré, R., & van Langenhove, L. (1999). *Positioning theory* (pp. 32-52). Blackwell: Oxford.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics, 89*, 185-204.

Heyd-Metzuyanim, E., & Cooper, J. (2022). When the Problem Seems Answerable yet the Solution is Unavailable: Affective Reactions Around an Impasse in Mathematical Discourse. *International Journal of Research in Undergraduate Mathematics Education*, 1-27.

Holland, D., Lachicotte, W., Jr., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, MA: Harvard University Press.

Langer-Osuna, J. M., & Esmonde, I. (2017). Identity in research on mathematics education. *Compendium for research in mathematics education*, 637-648.

Lee, J. S., & Anderson, K. T. (2009). Negotiating linguistic and cultural identities: Theorizing and constructing opportunities and risks in education. *Review of Research in Education, 33*(1), 181-211.

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education, 48*(5), 488-519.

Ma, X., & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for research in mathematics education*, 26-47.

Meece, J. L., Wigfield, A., & Eccles, J. S. (1990). Predictors of math anxiety and its influence on young adolescents' course enrollment intentions and performance in mathematics. *Journal of educational psychology, 82*(1), 60.

Mehan, H. (1979). *Learning lessons: social organization in the classroom*. Cambridge, MA: Harvard University Press.
Polkinghorne, D. E. (1995). Narrative configuration in qualitative analysis. *International journal of qualitative studies in education, 8*(1), 5-23.
Radovic, D., Black, L., Williams, J., & Salas, C. E. (2018). Towards conceptual coherence in the research on mathematics learner identity: A systematic review of the literature. *Educational Studies in Mathematics, 99*, 21-42.

Examining Relationships Among Aspects of Secondary Teachers' Potential Competence and Instructional Characteristics

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Within secondary mathematics teacher preparation, recent scholarship points to the efficacy of strong links between university mathematics course content and secondary mathematics teaching practice. Studies aligned with this view typically focus on overarching course design, rather than instructional characteristics. This investigation builds on a prior such study that found improvements in teachers' potential competence for teaching across four different content areas. In the present study, we analyze 137 teachers' expectation for success in and valuing of core instructional practices and their gains in content knowledge for teaching, as well as a purposive sample of videos from 6 courses enrolling a subset of these teachers. Based on our analysis, we suggest that different aspects of university instruction may have differential influence on attitudinal and cognitive aspects of teachers' potential competence. Moreover, instructional practices that most benefit attitudinal aspects may be in tension with those that most benefit cognitive aspects.

Keywords: teaching practice, secondary mathematics teacher education, competence

For decades, the narrative of secondary mathematics education was one of disconnect. Klein (1924/1932, as cited in Kilpatrick, 2019) lamented a "discontinuity" between undergraduate mathematics experiences and secondary teaching. Empirical findings documented that many secondary teachers found their undergraduate mathematics courses irrelevant to their teaching (Goulding et al., 2003; Zazkis & Leikin, 2010). There is recent promise in an approach to designing mathematics courses for prospective secondary mathematics teachers that connects undergraduate course mathematics *content* to secondary mathematics *teaching practice* (see Lai et al., in press and Wasserman et al., 2023 for reviews). These results conclude that curricular materials that intentionally promote such connections beneficially serve prospective secondary mathematics teachers (PSMTs). However, these studies lack an investigation of the undergraduate pedagogy through which the intended curricula are enacted. Yet different enactments of the same curriculum may have differential impacts (Remillard, 2018).

Our purpose is to hypothesize teaching practices that may impact PSMTs' development of competence for teaching. We operationalize competence in terms of expectation for success and valuing of evidence-based teaching practices, and the development of content knowledge for teaching. In doing so, we account for relationships among these traits. We address: (RQ1) *To what degree do attitudinal and cognitive aspects of competence associate?* (RQ2) *What aspects of evidence-based teaching practice may most influence PSMTs' development of attitudinal and cognitive aspects of competence for teaching?* We analyzed teaching practices with the Mathematics Classroom Observation Protocol for Practices (MCOPP; Gleason et al., 2017) applied to a purposive sample of 6 instructors' videos. Hence, we contribute an analysis of relationships between PSMTs' development of competence and instructional practices they experienced.

Terminology. In this study, *secondary* refers to Grades 6-12, though we note that 90% of participants intended to teach Grades 9-12. *PSMT* refers to prospective secondary mathematics teachers, *instructor* refers to an undergraduate instructor, and *student* refers to a secondary student. *Content* refers to mathematics and statistics.

Conceptual Perspective

We take competence for teaching to include teachers' capacity to harness knowledge and attitudinal traits (Blömeke et al., 2015). With respect to knowledge, we focus here on *content knowledge for teaching* (*CKT)*, defined as disciplinary knowledge entailed in the recurrent work of teaching mathematics or statistics (Ball et al., 2008; Baumert et al., 2010; Thompson & Thompson, 1996). With respect to attitudinal traits, we focus on expectancy and value for enacting evidence-based teaching practices. A person's *expectancy* is their expectation of success at carrying out a task in a particular situation (Wigfield & Eccles, 2000). *Value* is the importance of carrying out a task well, and can encompass utility, enjoyment, and personal fulfillment (see Eccles & Wigfield, 2020, for a review). We use evidence-based teaching practices to refer to teaching practices that promote discussion and elicit student thinking about content in ways rooted in disciplinary norms (e.g., Gleason et al., 2017; Grossman et al., 2009). Such teaching practices are associated with instructional quality and student learning outcomes at the secondary levels (e.g., Baumert et al., 2010).

Competence is influenced by socialization (Eccles & Wigfield, 2020), and PSMTs' socialization includes their experiences in the mathematics courses they take (e.g., Buchholtz & Kaiser, 2013; Werler & Tahirsylaj, 2022). In this study, all participating PSMTs enrolled in courses using written curricula developed by the Mathematics Of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE(S2)) project, which promoted explicit links between university course content and secondary mathematics teaching practice. We posit that even with this similarity across PSMTs' experiences, there may still be differences due to instructors' enactment of the written curriculum (Remillard, 2018).

One theme in studies of undergraduate mathematics instruction in the past decade has been the effectiveness of *evidence-based teaching practices*, particularly those consistent with inquiry-based mathematics education (Laursen & Rasmussen, 2019; Freeman et al., 2014). We operationalize the term "evidence-based teaching practices" as teaching for conceptual understanding, including opportunities to examine mathematical structure, through teacher interactions and students' engagement. One characterization of evidence-based teaching practices (among others) is represented by the Mathematics Classroom Observation Protocol for Practices (MCOPP), which has been validated to capture the presence of evidence-based practices in undergraduate mathematics instruction (Gleason et al., 2017). The principles behind MCOPP items are consistent with guidance at the elementary, secondary, and undergraduate levels for mathematics teaching (e.g., Mathematical Association of America, 2018; National Council of Teachers of Mathematics, 2014). Gleason et al. (2017) reported that MCOPP items fall into two dimensions: Teacher Facilitation and Student Engagement.

Background

Evaluation of effectiveness in instruction is typically conceptualized in a process-product paradigm: "the search for relations between classroom processes (instruction) and products (what students learn" (Gage & Needels, 1989, p. 254). For instance, one of the main results cited in support of evidence-based practices in undergraduate mathematics education is Freeman et al.'s (2014) meta-analysis. They concluded from their analysis of 225 studies that "active learning" pedagogies (process) increase undergraduates' scores on concept inventories and course examinations (product) in STEM fields. Indeed, the term "evidence-based practices" is rooted in this paradigm: they are teaching practices for which process-product research indicates positive impacts on student outcomes.

Studies assessing the impact of evidence-based teaching practices as a process tend to focus on one product at a time. For instance, Laursen et al. (2014) examined the impact of inquiry-based methods on undergraduates' assessments of their own learning gains. Johnson et el. (2020) investigated the effect of active learning methods on undergraduates' knowledge of group theory. Freeman et al. (2014) focused on outcomes in course examinations and concept inventories. These results do not account for the possibility that particular teaching practices may have differential impacts on different aspects of student outcomes, such as attitudinal in comparison to cognitive outcomes.

Although process-product studies of inquiry-based mathematics education have included mathematics courses for PSMTs in their data, their analyses typically consider PSMTs' experiences in aggregate with undergraduate mathematics students more generally (e.g., Laursen, et al., 2014). However, PSMTs have needs distinct from other undergraduate mathematics students, because content knowledge for teaching is different from content as needed in other mathematically-intensive careers (e.g., Krauss et al., 2008; Hill et al., 2007). Moreover, although prior studies in undergraduate mathematics education have examined undergraduates' attitudes toward mathematics, they have not addressed attitudes specific to future teaching practice.

This study addresses the gap in examining the potential for differential impacts as well as the need for studies specific to PSMTs. We analyze a combination of teaching practices experienced by PSMTs in a mathematics course, content knowledge for teaching, as well as expectancy and value for carrying out evidence-based teaching practices.

Data & Method

The context for this study is the MODULE(S2) project, a research and development effort that produced written curricular materials in algebra, geometry, mathematical modeling, and statistics. Materials for each content area are intended to span one term of a university mathematics course for PSMTs. All materials featured opportunities for PSMTs to apply course content to teaching situations such as responding to student ideas. In a previous study, the MODULE(S2) project examined pre- and post-term data for PSMTs' expectancy and value for enacting evidence-based teaching practices, and their CKT. This research found mean increases in PSMTs' outcomes across each of these constructs (Lai et al., 2023). Further, this prior study indicated that PSMTs noticed their instructors' enactment of evidence-based practices. One significant limitation of this prior study is that it examined instruction indirectly, using PSMTs' perception of the instruction they experienced.

There are two parts to the present study. For RQ1, we examined the association among PSMTs' gains in expectancy, value, and content knowledge for teaching. For RQ2, we compared ratings for videos of 6 instructors' use of evidence-based practices to mean gains of participating PSMTs enrolled in their courses. Our rationale was that if we were going to examine the impact of instructional practices on PSMTs' development of three different traits (expectancy, value, and content knowledge for teaching), we should also examine the relationship among these constructs in the population. If significantly associated, instructional processes influencing one trait may influence others in a similar way. If association is more limited, we may be looking for differences in the influence of instructional practices on traits.

Participants for RQ1 were 137 PSMTs enrolled in university courses using MODULE(S2) materials, with 50 in algebra, 25 in geometry, 16 in mathematical modeling, and 46 in statistics. For RQ2, we purposively sampled 6 instructors' courses in geometry and statistics, the areas with the most video data available for analysis. Following the process-product model, we posited that PSMTs' mean gains in CKT represented variation in instruction. Our purposive sample consisted of videos from instructors with high and low relative mean gains among participating PSMTs, resulting in a selection of 19 videos from 4 geometry instructors' courses (12 videos; mean CKT gains ranged from 10% to 19.50% of the maximum possible CKT score) and 2 statistics instructors' courses (7 videos; CKT gains 10.6% and 26.88%). Video lengths ranged from 40 minutes to 75 minutes and documented a single class period instruction.

As for instrumentation, we measured pre-/post-term expectancy and value for carrying out evidence-based practices using a Likert-item survey, whose phrasings were drawn from Wigfield and Eccles (2000). In the prior study, internal reliability (Cronbach's α) was $\alpha = 0.91$ for the expectancy assessment, and $\alpha = 0.79$ for the value assessment. A common guideline for Cronbach's α is to consider values over 0.7 as acceptable and values over 0.9 as excellent (Nunnally, 1978). We measured (CKT) in each area with instruments featuring applications of the specified content to teaching (see Lai et al., 2023 for more details on instrument validity for capturing PSMTs' CKT). We analyzed instructional videos using MCOPP, which consists of 20 items describing aspects of two categories: undergraduate student engagement (SE) (e.g., assessing strategies, communicating ideas to peers) and teacher facilitation (TF) (e.g., allows for wait time; encourages independent thinking) (Gleason et al., 2017). Each item is rated on a scale of 1 to 3, representing less to more intense enactment.

For RQ1, the first three authors analyzed relationships between PSMTs' expectancy, value, and CKT. To do so, we used Pearson's correlation coefficient r to measure effect size of correlations. We used p-values to determine if there is evidence of a non-zero correlation in the theoretical population, but focus on practical significance.

For RQ2, four researchers (including the fifth and sixth authors) coded videos with MCOPP items, reconciling in pairs any difficult evaluations. We intended to analyze relationships between these dimensions, items, and mean PSMT gains per course in expectancy, value, and CKT. We decided to focus solely on TF as the videos often followed the instructor, affording limited data on student engagement. Of note, after 2017, items addressing equitable teaching practices were added to MCOPP from outside the original factor analysis. Based on parallelism in phrasing, we posited how these items fell into the teacher facilitation dimension; we call this dimension *TF+*.

Here we emphasize the phrasing of RQ2: "What aspects of evidence-based teaching practice <u>may</u> most influence PSMTs' development of attitudinal and cognitive aspects of competence for teaching?" Although we use Pearson's correlation coefficient r in this analysis, we do so in an exploratory way. Our findings are potential hypotheses. We posited that $r \sim 0$ ($\sim 0.1 < r < \sim 0.1$) can be interpreted as the hypothesis that the MCOPP item or dimension has little relationship to the PSMT gain in the trait (expectancy, value, or CKT). Otherwise, positive r indicates a hypothesis that the teaching practices described in the MCOPP item or dimension benefits PSMTs' development in that trait. And negative r indicates a hypothesis that those teaching practices may have less benefit for PSMTs' development.

We also emphasize the study context: we already know that PSMTs on average gained expectancy, value, and CKT. Thus a negative correlation indicates that the dimension may have had less impact on the PSMTs' development than others for the specified trait, rather than indicating that the dimension is detrimental to the PSMTs' development.

Results

RQ1: PSMTs' Gains in CKT Versus Expectancy or Value Show Mostly Statistically Insignificant Correlations with Negligible Effects

The first three authors used Pearson's correlation coefficient *r* to analyze relationships between PSMTs' gains in CKT versus their gains in expectation to implement and valuation of core instructional practices. Each PSMT's scores on these metrics were first converted to a percent change from pre to post out of total possible points on the relevant assessment or survey. Figure 1 shows Pearson's correlation coefficient *r* and the p-values for this group of PSMTs, both partitioned by content area and as a whole group. Note that both results for the combined group were statistically significant with $p < 0.05$, though their *r* values were small. The only content-specific significant result was for statistics and value, with a higher *r* value of +0.41. These results imply that, for the most part, PSMTs' learning of mathematics content for teaching was separate from their expectation and valuation of core instructional practices.

	CKT vs. E *r* value	CKT vs. E p-value	CKT vs. V *r* value	CKT vs. V p-value
Algebra	+0.107	0.458	+0.022	0.882
Geometry	+0.137	0.514	+0.153	0.466
Modeling	+0.342	0.195	+0.336	0.203
Statistics	+0.188	0.212	+0.41	0.005
All Content Areas Combined	+0.168	0.0497	+0.199	0.019

Figure 1. Pearson's correlation coefficient r and the p-value for correlations between PSMT's gains in content knowledge for teaching (CKT) and gains in expectancy (E) or value (V). Statistically significant results are highlighted in green.

RQ2: We Hypothesize that the Process as a Whole (of Evidence-Based Practices) is the Sum of Parts in Tension

Overall, Pearson's correlation coefficients with the TF dimension with expectancy, value, and CKT ranged from 0.022 to +0.49. Our estimate for standard error over the 19 observations (for 19 videos) is 0.45 assuming independent observations. This assumption is necessary for the analysis even though these observations are not independent, with multiple videos originating from the same instructor. Nonetheless, we infer that these coefficients are relatively small. The exceptions are the correlations with gains in expectancy, which were +0.45 (TF) and +0.49 (TF+).

When examining correlation for individual MCOPP items with expectancy, value, and CKT, we find wider variation. With expectancy, *r* ranged from -0.12 to +0.73. With value, *r* ranged from 0.42 to 0.57 actual. With CKT, *r* ranged from -0.62 to +0.51. For some items with larger relative *r* in terms of absolute value (operationalized as $|r| > \sim 0.4$), the sign for the coefficient flipped between expectancy and CKT, or between value and CKT, or the coefficient for CKT was relatively small (for examples, see Figure 2).

	7) The teacher promoted modeling with math.	11) The teacher's talk encouraged student thinking	13) There was a climate of respect for what others had to say	14) In general, the teacher provided wait-time
E	-0.08	+0.18	+0.58	+0.73
V	-0.42	+0.42	+0.57	+0.56
CKT	+0.51	-0.62	-0.09	+0.01

Figure 2. Examples of largest correlation coefficients (in absolute value) obtained among MCOPP items and PSMTs' change in expectancy (E), value (V), and content knowledge for teaching (CKT)

Discussion

We examined pre/post-gains in PSMTs' expectancy, value, and CKT as well as instruction experienced by the PSMTs, for which our analysis found insignificant correlations between PSMTs' gains in these aspects. We also hypothesized that teacher facilitation would have the greatest additional impact on PSMTs' gains in expectancy, in the context that we know students on average experienced gains across each trait across all courses.

Our study was limited in several ways. The band of CKT gains that determined the purposive sampling was relatively narrow, limiting differences to detect. The videos coded, though purposively selected, may not represent the widest variation in instruction; the courses with the highest and lowest mean CKT gains in a content area sometimes had no video data available. Additionally, we completed coding in two of the four original content areas. For this reason, we view our results as hypotheses for further consideration.

One hypothesis we make is that in the presence of gains in competence, teaching practices may make more difference in expectancy for carrying out evidence-based teaching practices than other traits. One reason for this being that experiencing evidence-based teaching can foster confidence by providing a model for what such teaching looks and feels like.

We also hypothesize that different teaching practices have differential benefits to distinct intended outcomes for competence (gains in expectancy, value, and CKT). Here we distinguish teaching practice, conceived as the whole of a teacher's instruction, from teaching practices, conceived as routines and aspects within instruction (Lampert, 2010). Further, in any given moment, an instructor can only focus on so many individual practices; emphasizing one practice may necessitate doing so at the expense of another. How do we conceive of "effective" teaching practices when faced with these kinds of results? Perhaps effective teaching practice is composed of teaching practices in tension.

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References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, 47(1), 133–180.

Blömeke, S., Gustafsson, J. E., & Shavelson, R. J. (2015). Approaches to competence measurement in higher education. *Zeitschrift für Psychologie*, 223(1), 3-13.

Buchholtz, N., & Kaiser, G. (2013). Improving mathematics teacher education in Germany: Empirical results from a longitudinal evaluation of innovative programs. *International Journal of Science and Mathematics Education*, 11, 949-977.

Eccles, J. S., & Wigfield, A. (2020). From expectancy-value theory to situated expectancy-value theory: A developmental, social cognitive, and sociocultural perspective on motivation. *Contemporary Educational Psychology*, 61, 101859.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 111(23), 8410-8415.

Gage, N. L., & Needels, M. C. (1989). Process-product research on teaching: A review of criticisms. *The Elementary School Journal*, 89(3), 253-300.

Gleason, J., Livers, S., & Zelkowski, J. (2017). Mathematics classroom observation protocol for practices (MCOP2): A validation study. *Investigations in Mathematics Learning*, 9(3), 111-129.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, 6, 361-393.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice*, *15*(2), 273-289.

Hill, H. C., Dean, C., & Goffney, I. M. (2007). Assessing elemental and structural validity: Data from teachers, non-teachers, and mathematicians. *Measurement*, 5(2-3), 81-92.

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, 51(4), 504-516.

Kilpatrick, J. (2019). A double discontinuity and a triple approach: Felix Klein's perspective on mathematics teacher education. *The legacy of Felix Klein, ICME-13 Monographs*, 215-225.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM*, 40, 873-892.

Lai, Y., Strayer, J. F., Ross, A., Adamoah, K., Anhalt, C. O., Bonnesen, C., Casey, S., Kohler, B., & Lischka, A. E. (2023). Enhancing prospective secondary teachers' potential competence for enacting core teaching practices—through experiences in university mathematics and statistics courses. *ZDM–Mathematics Education*, 1-15.

Lai, Y., Wasserman, N., Strayer, J., Casey, S., Weber, K., Fukawa-Connelly, T., & Lischka, A. (in press). Making advanced mathematics work in secondary teacher education. In B. M. Benken (Ed.), *AMTE Professional Book Series* (Vol. 5): *Reflection on past, present and future: Paving the way for the future of mathematics teacher education* (Chapter 6). Association of Mathematics Teacher Educators.

Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean? *Journal of Teacher Education,* 61(1-2), 21-34.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, 45(4), 406-418.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 5, 129-146.

Mathematical Association of America (2018). *Instructional practices guide*. Mathematical Association of America.

National Council of Teachers of Mathematics (2014). *Principles to actions*. National Council of Teachers of Mathematics.

Nunnally, J. C. (1978). An overview of psychological measurement. *Clinical Diagnosis of Mental Disorders: A Handbook*, 97-146.

Remillard, J. (2018). Mapping the relationship between written and enacted curriculum: Examining teachers' decision making. In Invited lectures from the 13th international congress on mathematical education (pp. 483-500). *Springer International Publishing*.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27(1), 2-24.

Wasserman, N. H., Buchbinder, O., & Buchholtz, N. (2023). Making university mathematics matter for secondary teacher preparation. *ZDM–Mathematics Education*, 1-18.

Werler, T. C., & Tahirsylaj, A. (2022). Differences in teacher education programmes and their outcomes across Didaktik and curriculum traditions. *European Journal of Teacher Education*, *45*(2), 154-172.

Wigfield, A., & Eccles, J. S. (2000). Expectancy-value theory of motivation. *Contemporary Educational Psychology*, 25, 68–81.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, 12(4), 263-281.

Connecting Unique Factorization Domains with the Teaching of High School Algebra in Approximations of Practice

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We designed approximations of practice (AoPs; Grossman et al., 2009) that pose hypothetical classroom scenarios of high school students learning about factoring polynomials. We investigated the ways in which prospective and in-service teachers leverage their understanding of unique factorization domains to inform their teaching practices of attending, interpreting, and deciding how to respond to students' thinking in the AoPs with the goal of making explicit connections between abstract algebra and the teaching of secondary mathematics.

Keywords: approximations of practice, noticing, unique factorization domain, abstract algebra

Connecting secondary mathematics and abstract algebra can support in-service and pre-service teachers (IPSTs) in their teaching. For IPSTs to recognize advanced mathematics courses, like Abstract Algebra, as beneficial for their teaching, they should be explicitly given opportunities in their mathematics courses to reflect on how the advanced content is connected to secondary mathematics and how those connections can inform their teaching. Some researchers and teacher educators (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Serbin & Bae, 2023) have provided such opportunities to IPSTs by using *Approximations of Practice* (AoPs) in advanced mathematics courses, which are activities situated in hypothetical classroom scenarios that require IPSTs to simulate certain teaching practices (Grossman et al., 2009). AoPs often include scripting tasks (Zazkis et al., 2013) and noticing tasks (Jacobs et al., 2010) that engage IPSTs in attending, interpreting, and deciding how to respond to students' thinking in samples of written student work or scripted class discussions.

Before teachers' understanding of connections between secondary and abstract algebra can become useful in their teaching or AoPs, those connections should first serve to fundamentally change the teachers' understandings of the content they teach (Wasserman, 2018). This principle guided our design of an instructional task sequence that guided IPSTs in reinventing (Gravemeijer, 1999) unique factorization domains (UFDs), connecting its properties to the factorization of integers and polynomials (commonly used in secondary algebra), and applying their understandings of those connections in AoPs. A *UFD* is an integral domain R in which every nonzero element $r \in R$ which is not a unit has the following two properties: (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1 p_2 \dots p_n$ and (ii) the decomposition in (i) is unique up to *associates*: namely, if $r = q_1 q_2 \dots q_m$ is another factorization of r into irreducibles, then $m = n$ and there is some renumbering of the factors so that p_i is associate to q_1 for $i = 1, 2, \dots, n$ (Dummit & Foote, 2003). After reinventing UFDs, the IPSTs were asked to complete a set of AoPs (Grossman et al., 2009) that were designed to lead them to use their understandings of UFDs and factorization while they noticed (attended to, interpreted, and decided how to respond to) student thinking or scripted class discussions in hypothetical classroom scenarios (Jacobs et al., 2010; Zazkis et al., 2013). We address this research question: *What aspects of IPSTs' understanding about UFDs do they leverage as they interpret and decide how to respond to students' mathematical thinking in AoPs?*

Literature Review

It is essential for teachers to have knowledge of the content they teach, as well as knowledge of advanced mathematics that is related to, but is still outside the scope of, secondary mathematics. Wasserman (2018) defines the mathematical knowledge outside of the scope that teachers teach as *nonlocal mathematics*. Nonlocal mathematics can influence the teachers' perceptions and interpretations of their students' thinking. Thus, it is important that preservice teachers are prepared beyond the local mathematics they plan to teach. However, it often occurs that IPSTs do not see any connections between the advanced mathematics courses they take with the secondary mathematics they eventually teach (Zazkis & Leikin, 2010).

Some researchers have tried to resolve this disconnect by studying how to support IPSTs in making connections between the content they learn and the content they teach on the secondary level (Alvarez et al., 2020; Wasserman, 2018; Wasserman et al., 2017; Larsen, 2013). However, there has not been much research done on how these connections actually influence the way graduate students teach mathematics (Wasserman et al., 2017). Abstract Algebra is one of the primary advanced mathematical courses required for IPSTs, where they can learn about algebraic structures (Pramasdyahsari, 2021; Ticknor, 2012). To make explicit connections between the abstract algebra and secondary mathematics, Álvarez et al. (2020) designed pedagogical tasks for use in abstract algebra courses that generated student-teacher interaction and helped students construct their own knowledge. Similarly, Zbiek & Heid (2018) proposed the Mathematical Understanding for Secondary Teaching framework where IPSTs connect abstract algebra and school algebra through the mathematical activities of mathematical noticing, mathematical reasoning and, mathematical creating, which leverage the practices of noticing structure and symbolic forms, proving and conjecturing, as well as representing and defining. Herbst et al. (2014) explored how AoPs using online learning can engage prospective secondary teachers as part of their practice-based teacher development. These studies exemplify efforts being made toward supporting IPSTs in connecting advanced mathematics to teaching.

Theoretical Background

Building Up From Practice – Stepping Down to Practice

To help teachers make connections between advanced and secondary mathematics, Wasserman et al. (2017) proposed the *building up-stepping down* model for a real analysis course. The model consists of providing IPSTs with pedagogical situation tasks that can help them recognize the utility of advanced mathematics, in this case, real analysis, in their teaching. Thus, they start from a practical scenario in the context of teaching mathematics and build up the advanced mathematical content from that practice. Once the students learn advanced mathematics, they step down to teaching practice, where the ideas of real analysis are explicitly connected to secondary mathematics, and IPSTs can use their knowledge about real analysis to support their teaching practices such as attending and responding to hypothetical students' ideas.

Approximations of Practice

Teaching requires multifaceted skills, like interpreting and responding to students' mathematical thinking, explaining mathematical definitions, posing questions, and responding to questions that can advance the students' reasoning (Álvarez et al. 2020). Therefore, it is crucial that IPSTs have opportunities to practice these skills without the complexity of teaching in a real-life scenario. Grossman et al. (2009) defined *AoPs* as "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (p. 2058). AoPs, like scripting and noticing tasks, serve as practice for IPSTs to implement the pedagogical moves they learn (Crespo, 2018). For instance, Zaskis & Marmur (2018) argue that scripting tasks provide IPSTs the opportunity to "explore erroneous or incomplete approaches of a student, revisit and possibly enhance personal understanding of the mathematics involved and enrich the repertoire of potential responses to be used in future 'real' teaching" (p. 294). Scripting tasks also prompt prospective teachers to reflect and explain why they would continue the discussion in that manner (Campbell & Baldinger, 2021). By having these opportunities in advanced mathematics courses, IPSTs can develop teaching practices in connection to the advanced mathematical content they learn in the courses (Álvarez et al., 2020, Zaskis & Marmur, 2018).

Noticing Students' Mathematical Thinking

In our task design, we leveraged the three components of professional teacher noticing of mathematical thinking: attending students' mathematical thinking, interpreting students' understanding and deciding how to respond (Jacobs et al., 2010), as we consider them key pedagogical moves that teachers can implement to guide a class discussion and answer their students' questions. Zambak et al. (2023) recognize that "noticing provides pedagogical readiness for PSTs to correctly identify students' mathematical conceptions and misconceptions and guides their instructional strategies" (p. 4). In noticing tasks, prospective teachers are posed with a hypothetical classroom scenario where the mathematical activity and thinking of a student are described. The goal of such tasks is for IPSTs to attend, interpret, and decide how to respond to the student's ideas by reasoning about the strategies the student used to solve the given task. We investigate the aspects of IPSTs' understandings of UFDs that IPSTs leverage as they interpret and respond to students' mathematical thinking in AoPs.

Therefore, our study design is informed by Wasserman et al.'s (2017) *building up-stepping down* model for designing curricula that connects advanced mathematics to teaching, along with the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999), informed our creation of local instructional theory, in which we guided graduate student IPSTs' reinvention of UFDs. We connected the first tasks in the sequence to the teaching of factorization in high school algebra by using algebra tile manipulatives. We built up the advanced mathematical content by guiding the IPSTs to reinvent UFDs (see Bae et al., 2024). We then guided the IPSTs to step down to practice by using their understandings of UFDs in their responses to AoPs situated in hypothetical classroom situations. The design of the AoPs was informed by Grossman et al.'s (2009) AoPs and Jacobs et al.'s (2010) noticing framework.

Methods

We designed a sequence of tasks to guide IPSTs to reinvent the reducibles and irreducibles (Unit 1), reinvent the definition of UFD (Unit 2 & 3), and apply their knowledge to AoPs (Unit 4). In Unit 4, we created AoPs where IPSTs engaged in noticing students' mathematical reasoning about concepts related to factorizations of integers and polynomials. The tasks were designed to elicit IPSTs' use of their knowledge of UFD from the previous units as they attended to, interpreted, and decided how to respond to students' thinking in written work and class discussion scripts. In this paper, we focus on their work on the AoPs tasks in Unit 4.

We administered the AoPs to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in the Southern US. Six mathematics graduate student IPSTs participated in this study: Josie and Raul (Group A), Javier and Roberto (Group B), and Kim and Taylor (Group C) (pseudonyms). They all had taken an abstract algebra course but had not yet learned about UFDs. They had varied teaching experiences at different school and college levels ranging from student teaching to 20 years. Each group participated in five 90-minute sessions of teaching experiments with the research team. Groups A and B participated in person, and Group C participated in Zoom. We collected and transcribed the session recordings.

We implemented the AoPs in Figures 1, 2, and 3. The IPSTs responded to questions about their interpretation of what the students' approaches and understandings, and how they would respond to such student ideas if they were the students' teacher and how their responses were informed by their understanding of UFDs, providing us with evidence about how they used their knowledge about UFD to inform their pedagogical actions. We used inductive coding (Miles et al., 2013) to analyze how IPSTs use their knowledge about UFD as they worked on the AoPs. We coded the properties of UFDs the IPSTs referenced as they in noticed and responded to students' mathematical thinking, i.e., (Jacobs et al., 2010).

In Mr. Garcia's class, students were asked to factor a quadratic polynomial $-2x^2 - 4x + 6$ for solving a quadratic equation $-2x^2 - 4x + 6 = 0$. Mr. Garcia had four of his students, Aden, Brian, Cassy, and David come to the board and share how they factored the polynomial. The following is the solutions of the four students that involve different final forms of factorization of the polynomial $-2x^2 - 4x + 6$.

Aden	Brian	Cassy	David
$-2x^2 - 4x + 6$ $= -2(x^2 + 2x - 3)$ $= -2(x + 3)(x - 1)$	$-2x^2 - 4x + 6$ $= -(2x^2 + 4x - 6)$ $= -(x - 1)(2x + 6)$	$-2x^2 - 4x + 6$ $= x^2 + 2x - 3$ $= (x + 3)(x - 1)$	$-2x^2 - 4x + 6$ $= (-x + 1)(2x + 6)$ $= (-x + 1)2(x + 3)$

Jaime, one of the students in Mr. Garcia's class, said "So who's got it right? They all got it different, so someone must be wrong." Monica followed, "That doesn't matter, you have to check if they all give you the same roots".

Figure 1. Approximation of Practice Task 1 Scenario

1. In Ms. Gonzalez's last class, students went over problems that required factoring polynomials. Ms. Gonzalez wants her students to think about the factorization of polynomials. She said: "Today, we are going to explore how polynomials and integers are similar. What similarities do you think polynomials and integers have?"
2. A student Amy asked the following question to Ms. Gonzalez: That problem told us to factor a polynomial, and I was trying to remember where I saw factoring before. I remember that numbers have prime factorizations. Do polynomials have prime factorizations too? Or is that a different kind of factorization?

Figure 2. Approximation of Practice Task Scenarios from Task Set 2

The teacher asks students to find the factors of the polynomial $x^2 + 5x + 6$.
When asking the students to share their answers, he receives the following answer:
Luis: The factors are 3 and 2.
Teacher: Can you explain your answer?
Luis: The factors of 6 are 3 and 2 because 3·2 is equal to 6 and 3 plus 2 gives me 5 which matches the polynomial.
Alex: I think he's right, he just needs to add x, so we have $(x + 3)$ and $(x + 2)$.
Mia: I have $(x + 2)$ and $(x + 3)$. Does the order for this matter?

Figure 3. Approximation of Practice Scenario from Task Set 3

Results

Episode 1: Reasoning About the Uniqueness Axiom of UFD to Inform their Interpretation of and Response to Hypothetical Students' Thinking

In the AoP task set 1, four hypothetical students, Aiden, Brian, Cassy, and David obtained different factorizations $-2x^2 - 4x + 6$ (see Figure 1). In this hypothetical scenario, the students' work was on the classroom board, and the class was discussing the factorizations. Two students, Jaime and Monica, shared their ideas (see Figure 1). The participants were asked to attend to and interpret the student understanding evident in Jaime's and Monica's comments. Josie and Raul agreed that Jaime's comment was valid, arguing that his understanding of factoring is strong. For instance, Josie claimed: He knows that… *when you're factoring something to the irreducibles, it should be unique no matter what… The order doesn't matter, but it still should be unique factorization. So, they should still end up with the same solution if the task is to factor it out.* Josie recognized that Jaime thought only one of the distinct solutions from his classmates must be correct because he might know that the problem must have one unique solution. Raul interpreted Jaime's understanding of factorization as "*on point*" for this same reason. Both IPSTs connected Jaime's comment to the uniqueness axiom of the unique factorization domain.

The participants were prompted to decide how they would respond to the class following the comments by Jaime and Monica. Josie responded:

> Just check each student's solutions of like, you have student [work] on the board, and you might ask, who got it right? It's a two-part question, the factoring and solving the equation. So, to *check if you got the factoring right, distribute it, see if you got the original polynomial*. And then the second part of solving the equation, actually solve it. Set it equal to 0. Find the roots for each one. And that will give you the checkmark or the X for each one, whether they got both parts of the question right or wrong.

Josie and Raul decided that they would ask students to go back and check their classmates' answers for them to find whether the factorizations were correct.

Finally, the participants were asked how their understanding of UFD helped them interpret and decide how to respond to the students' thinking. Josie and Raul said:

> *Josie*: *It helps you address Jaime's comment, of the unique factorization domain tells us that the elements can be written as a unique product of irreducibles up to the order and associates of the elements*. So, it helps us address his comment right away of, *he understands that the factorizations should be unique*, so every student should have gotten the same factorization if the question is to factor it all the way down to irreducibles.
>
> *Raul*: [...]. So, I kind of push already for like, *you need to factorize it as much as possible, which is like this unique factorization idea, prime factorization idea*. Um, so, I guess my push is like, with this, with *this unique factorization domain idea in mind is that, we want it to be as small as possible, on what the product is that.* […]
>
> *Josie*: *Because we know… how the integral domain works, we can answer their question like well, but how? But why?* …I think that's the question. Which is why I thought that the question would be more like appropriately addressed to Jamie's comment, and maybe student Aiden for that matter too, that they understand that the factorization, you're going to get it down to like, *you want to break it down as much as possible, like the primes*. And that *factorization must be unique, which is the property of the UFD we're looking at***.**

Josie recognized that the way she could address the students' comments was by knowing how some mathematical structures such as rings and integral domains work. Specifically, she mentioned that the UFD axiom about uniqueness helped her respond to the students' mathematical thinking. Raul also referenced the UFD axiom regarding the existence of prime factorizations, as evident in his response of pushing students to "factorize it as much as possible" and "be as small as possible." Thus, Josie's and Raul's understandings of the axioms of UFDs informed their interpretation of and response to student thinking in this AoP.

Episode 2: Reasoning about the Reducibility and Irreducibility of Polynomials to Inform their Interpretation of and Response to Hypothetical Students' Thinking

In Task Set 2, the IPSTs were given the classroom scenario in Figure 2, where a teacher prompts her students to identify the similarities between integers and polynomials to help them develop an intuitive understanding of factoring polynomials. The IPSTs were asked to anticipate what similarities students might find between integers and polynomials. Taylor and Kim said that students would probably recognize that integers and polynomials have numbers, and they could do operations with them such as addition, subtraction, and multiplication. Both agreed that students would probably not think about factorization as a similarity between them. Kim and Taylor were then asked about how they would respond to these anticipated student ideas and how they could connect those ideas to the factorization of polynomials. Kim replied, "I think the big thing is… branching the ideas together. Right, okay, so…this is surface-level. Let's get a little bit deeper, to get the point of the factorization at least. Okay, yes, so you can factor polynomials." Next, they were asked about the characteristics of unique factorization domains that would help inform their response to students in this question. They answered as follows:

Kim: I think for that one, just the fact that you *had the unique factorization domain of like integers but also the ring of polynomials. Knowing that they are, that that is the same, inherently those structures, allows one I think to be more confident in saying, okay, they are similar, because they are the same structure*, right?

Taylor: *I still like the "breaking them down", that they can be broken down into irreducible things. […] Just knowing that you can break these things down*, I think that would be really beneficial, at least for that level. So that I guess the first point is the most important one for me, not so much that they're unique, but that you can do it to begin with.

Taylor related the reducibility of elements in UFDs as the main property that was connected to factoring polynomials, arguing that students could understand factoring as the action of "breaking things down" Therefore, Taylor's knowledge of the reducibility in UFDs helped her have an intuitive understanding of factoring in a high school setting and informed her way of sharing these ideas with the student.

The participants were next asked about how they would decide to respond Amy's question if they were her teacher (see task 2 in Figure 2). They replied as follows:

Taylor: Yeah, they do have prime factorizations, *and I don't think it's a different kind of factorization*. Because like, 2 and 3, you know, that's 6, but the factorization of 2 and 3 can be the same as a polynomial as long as you have the correct x value. So it's, I don't know if I would call it different kind of factorization. *I would say they're the same, but they're performed differently and for different reasons*.

Kim: …I just immediately am like…what do you mean by prime, right? Like prime, just give the definition of prime... You can't reduce it, right? So, *talk about reducibility, make that connection, and then you can get the polynomial to a reducibility* as well, kind of depending on like that factorization and with the root finding.

Taylor agrees with Amy's reasoning about the similarities between the prime factorizations of integers and polynomials, and Kim relates such property of polynomials to the reducibility in the elements of UFDs given that both integers and polynomials with integer coefficients are UFDs. We asked the participants what aspects of UFD informed their responses to students. Kim said:

Just knowing that the UFD is both a definition for the integers and the polynomials ring is, is important for this response, like with the last one. Because, okay, if you know they are the same, you can confidently say yes, of course, without having that doubt. Because if you have that… second of doubt of like, oh, are they the same or are they not? And you're discouraging the student from like, but they were right, and they had the right idea. *So setting a foundation of yes, I know for a fact, because it's the same structure, okay. And it instills confidence in the student as well.*

Kim's understanding of the structure of the integers and polynomials as UFDs gave her the confidence to assert that the factorizations the student Amy asked about were indeed similar because they have the same UFD structure. Having this understanding about the structure of factorization in $\mathbb{Z}$ and $\mathbb{Z}[x]$, informed Kim's response to the hypothetical student in the AoP.

Episode 3: Reasoning About the Commutativity of Polynomial Factors to Inform their Interpretation of and Response to Hypothetical Students' Thinking

The last set of tasks consisted of the hypothetical classroom scenario in Figure 3. When asked to interpret Mia's understanding, Javier replied*: "Mia's just trying to I guess clarify if commutativity is going to work here also."* We asked the participants how they would address and respond to the students' comments if they were their teacher. Javier and Roberto responded:

Javier: For Mia, *I'd probably just tell her to multiply those out,* $(x+2)(x+3)$*, and then multiply* $(x+3)(x+2)$*, to see if it really does matter if we multiply them in any order.*

Roberto: *…Mia to expand both terms and see if they give you the same answer*. I think the way the task is given is that I think she knows how to factor everything out. She knows how to get those two answers, but *she's just not sure if commutativity matters or not*. It can be solved just by *telling her to distribute*. *It doesn't matter in this case, it's the same*.

The IPSTs were asked about how their knowledge of unique factorization domains was informing their responses to students. Javier responded, "Seeing the commutativity in the tiles," and Roberto agreed. The IPSTs explained that their knowledge of the commutativity property of UFDs helped them explain with confidence why the order of the factors does not matter when factoring a quadratic. They both referred to the "uniqueness up to the order" of the factors, which results from the multiplicative commutativity that can be applied to the factors in a factorization.

Discussion and Conclusion

We exemplified how IPSTs' understandings of the connections between the abstract algebraic concept of UFD and the concepts of integer and polynomial factorization from secondary mathematics seemed to support them in their pedagogical practices. Specifically, we identified how the participating IPSTs seemed to leverage their reasoning about the existence and uniqueness axioms of UFDs to inform their responses to AoPs tasks that prompted them to interpret and decide how to respond to hypothetical students' thinking or classroom scenarios. Our work contributed to the literature on how IPSTs' knowledge of abstract algebraic concepts can inform their pedagogical practices. Our findings also contribute to the literature on the utility of AoPs in supporting IPSTs in connecting advanced and secondary mathematics. Álvarez et al. (2020) suggested that these AoPs can be "used both as a vehicle for bridging undergraduates' advanced mathematical knowledge to secondary school mathematics and for strengthening undergraduates' understanding of the advanced mathematics from an encounter with school mathematics" (p. 16). Additionally, AoPs can give IPSTs experience in using their knowledge of advanced mathematics in their teaching practices, which can contribute to improvements in their perceptions of advanced mathematics as being useful for their teaching of school mathematics (Serbin & Bae, 2023). Future research can address how IPSTs use their knowledge of other advanced mathematical concepts, as well as their pedagogical or mathematical practices (e.g., Wasserman, 2022), as they perform AoPs. Lastly, we propose that mathematics instructors who prepare IPSTs should implement AoPs in their assignments to support IPSTs in developing coherent understandings of the connections between abstract algebra and the teaching and learning of secondary mathematics.

References

Álvarez, J. A., Arnold, E. G., Burroughs, E. A., Fulton, E. W., & Kercher, A. (2020). The design of tasks that address applications to teaching secondary mathematics for use in undergraduate mathematics courses. *The Journal of Mathematical Behavior*, *60*, 100814.

Bae, Y., Espinosa, S., & Serbin, K. S. (2024). Reinventing the definition of unique factorization domains. *Proceedings of the 26th annual Conference on Research in Undergraduate Mathematics Education.*

Burroughs, E. A., Arnold, E. G., Álvarez, J. A., Kercher, A., Tremaine, R., Fulton, E., & Turner, K. (2023). Encountering ideas about teaching and learning mathematics in undergraduate mathematics courses. *ZDM–Mathematics Education*, 1–11.

Campbell, M. P., & Elliott, R. (2015). Designing Approximations of Practice and Conceptualising Responsive and Practice-Focused Secondary Mathematics Teacher Education. *Mathematics Teacher Education and Development*, *17*(2), 146–164.

Campbell, M. P., & Baldinger, E. E. (2022). Using scripting tasks to reveal mathematics teacher candidates' resources for responding to student errors. *Journal of Mathematics Teacher Education*, *25*(5), 507–531.

Crespo, S. (2018). Generating, appraising, and revising representations of mathematics teaching with prospective teachers. *Scripting approaches in mathematics education: Mathematical dialogues in research and practice*, 249–264.

Dummitt, D.S. and Foote, R.M. (2004). *Abstract algebra*: 3rd Edition. John Wiley & Sons, Inc.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055–2100.

Herbst, P., Chieu, V. M., & Rougee, A. (2014). Approximating the practice of mathematics teaching: What learning can web-based, multimedia storyboarding software enable?. *Contemporary Issues in Technology and Teacher Education*, *14*(4), 356–383.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Pramasdyahsari, A. S. (2021, July). Developing the scripting task for mathematical connection between the university and school mathematics content. In *Journal of Physics: Conference Series* (Vol. 1957, No. 1, p. 012003). IOP Publishing.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education* (pp. 267-306).

Serbin, K. S., & Bae, Y. (2023). How a graduate mathematics for teachers course "helped me see things from a different perspective." *Proceedings of the 25th annual Conference on Research in Undergraduate Mathematics Education.*

Ticknor, C. S. (2012). Situated learning in an abstract algebra classroom. *Educational Studies in Mathematics*, *81*, 307–323.

Wasserman, N. H. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science, Mathematics and Technology Education*, *16*, 28–47.
Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS, 27(6),* 559–578.
Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior, 49*, 116–128.
Wasserman, N. H., Fukawa-Connelly, T., Weber, K., Ramos, J. P. M., & Abbott, S. (2022). *Understanding analysis and its connections to secondary mathematics teaching*. Springer Nature.
Zambak, V. S., Kamei, A., & Sorochka, K. (2023). Noticing elementary students' mathematical reasoning and diverse needs: investigating the impact of different noticing tasks. *International Journal of Mathematical Education in Science and Technology*, 1–25.
Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281.
Zazkis, R., & Sinclair, N. (2013). Imagining mathematics teaching via scripting tasks. *International Journal for Mathematics Teaching and Learning*.
Zazkis, R., & Marmur, O. (2018). Scripting tasks as a springboard for extending prospective teachers' example spaces: A case of generating functions. *Canadian Journal of Science, Mathematics and Technology Education*, *18*, 291–312.
Zbiek, R. M., & Heid, M. K. (2018). Making connections from the secondary classroom to the abstract algebra course: A mathematical activity approach. In N. H. Wasserman (Ed.) *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*, 189–209.

In Transition from Undergraduate to Postgraduate Mathematics: A Case of Re-learning to Learn

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The transition from undergraduate to postgraduate studies has been underexplored in mathematics education research. Several studies showed that this transition is challenging as it requires countless changes in familiar practices. In this paper, we present the case of Jordan—an honors student who was developing mathematical background as a stepping stone toward his research project. With the commognitive framework, we analyzed the data from three interviews to reveal Jordan's mathematical learning network of routines. The findings detail the network's components and illuminate the role of agency in Jordan's network development. Agency appears as necessary to navigate learning routines common in undergraduate studies and enables Jordan to be less dependent on externally provided resources shaping his learning.

Keywords: mathematics learning, postgraduate studies, commognition, learning routines

Introduction and Background

The preparation of new researchers is vital for the flourishment of any discipline. Mathematics departments initiate students to research through a range of programs such as honors, master's and doctorates. Each of these postgraduate contexts is distinct, and their structures differ substantially from what undergraduate students are used to. Thus, a shift from undergraduate to postgraduate mathematics constitutes a critical transition, that is, one that involves "a noticeable change of point of view […and] a necessity for entering into a different type of discourse […] or more broadly […] changing 'lenses'" (Yerushalmi, 2005, p. 37). Yet, research into students' experiences of the undergraduate–postgraduate transition in mathematics is scarce.

Higher education research identified important obstacles in students' transition to postgraduate studies. For instance, O'Donnell and colleagues challenged the assumption that undergraduates are experts in disciplinary practices and elaborated on multiple changes postgraduate students must navigate simultaneously (O'Donnell et al., 2009; Tobbell & O'Donnell, 2013a; Tobbell & O'Donnell, 2013b). Adjusting to a higher level of independence has been reported to be particularly challenging for students (Tobbell et al., 2010). Lovitts (2005, 2008) detailed a range of factors underpinning the complexity of the shift from being expected to "[learn] what others know and how they know it" to "conducting original research and creating knowledge" (p. 140). Among other factors, the microenvironment composed of the department, advisors, and peers appeared to have a significant impact on the shift to independent research. Overall, the growth of independence constitutes a central theme in the literature on the transition to postgraduate studies. While the existing findings are relevant to many postgraduate programs, they may be difficult to appreciate without accounting for the discipline within which the transition occurs. Therefore, the transition from undergraduate to postgraduate is a topic of interest for mathematics education research.

A few mathematics education studies discussed the experience of postgraduate students (e.g., Herzig, 2002, 2004; Morton & Thornley, 2001) and even fewer focused on the undergraduate–postgraduate transition. Duffin and Simpson (2006) listed a few differences between undergraduate and postgraduate studies. Within the former, students' learning mainly occurs through courses, where lecturers determine syllabi, teach the material, set problems for students to solve, and suggest textbooks and references. The research component features in some undergraduate programs, but it is typically small and comes with comprehensive support (Dorff, Henrich & Pudwell, 2019). In turn, postgraduate students are expected to learn in an independent and self-directed manner. While research supervision is a typical component of many postgraduate programs, this role differs from that of a course lecturer. Accordingly, students' adaptation to the circumstances of postgraduate mathematics learning appears unavoidable.

An example of such adaptations comes from Geraniou (2010). She observed different stages in students' doctoral studies and explored factors contributing to students' success. Geraniou describes the first stage, which she calls *adjustment*, as "reading and gaining the background knowledge so as to be in a position to start their research" (p. 286). As this study was undertaken in the UK, coursework was not required, and students experienced learning without the structured environment of courses. Geraniou reports that in the absence of clear learning aims and imposed assessments, the students drew on their personal interest in the material and turned to their supervisors for motivation and guidance.

Some studies addressed the distinctions between how undergraduates and postgraduate students carry out specific learning-oriented practices. For instance, Shepherd and van de Sande (2014) explored students' and mathematicians' learning from mathematical texts. The researchers found that undergraduates limit their attention to the provided reference, relying only on the material presented inside the text for support. Postgraduate students and mathematicians both used the provided reference and external resources when seeking clarifications.

The existing literature indicates that the undergraduate–postgraduate transition requires new ways to learn mathematics. This study was initiated to enrich the existing literature by exploring how these ways of learning come about. Specifically, we aim *to characterize the processes of students' adjustment to the learning of mathematics at the postgraduate level.* We pursue this aim in the case of an honors student who was developing mathematical background outside of courses as a preparation for his research project.

Theoretical Framework

Research has shown the usefulness of the commognitive framework (Sfard, 2008) in exploring mathematics learning and teaching in the university context (e.g., Nardi et al., 2014; Karavi et al., 2022; Kontorovich & Ovadiya, 2022). In this study, we rely on the framework to cast light on students' learning of mathematics at the postgraduate level.

Commognition construes different mathematical areas as *discourses*, defined as "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants, thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Consistently, *learning* is associated with one "becoming a participant in certain distinct activities" (Sfard, 2008, p. 23). This may be evident in one's capability to operate with broadly accepted discourse-specific narratives, such as definitions, theorems, and proofs.

Postgraduate studies are expected to prepare students to contribute to a targeted mathematical discourse. This preparation imposes expectations on students, but students are "contributors to their life circumstances, not just products of them" (Bandera, 2006, p.1). In the context of postgraduate studies, students' contributions are captured through *agentive learning.* Brennan (2012) defines a learner's agency as an "ability to define and pursue learning" (p. 24). Knowles (1975) maintains that agentive learners take initiative, are sensitive to their learning needs, set relevant goals, determine helpful resources, and monitor their learning outcomes. Similarly, Lavie et al. (2019) specify that growth in agentivity, the commognitive version of agency, can be identified via the increasing number of self-determined decisions. Lavie et al. highlight the importance of one initiating activities without invitations from others and in the capability to set "the relevant [tasks] for herself, in response to her own needs" (p. 170). These activities can be captured in terms of one's *routines*.

Commognition defines routines as patterned courses of action. Initially, the construct referred to capture one's participation in a specific mathematical discourse (Sfard, 2008). Lavie et al. (2019) expanded this approach, arguing that "whatever we do involves routines. From the simplest, most mundane of our activities to the most abstract and sophisticated of them, we cope with the task by repeating something that we did or have seen done before" (p. 154). Sfard (2023) demonstrated that one's routines of teaching can be construed as "a tightly interconnected system" (p. 5) or a *network*, where one routine feeds into another or where several routines constitute part of a greater routine. Sfard argues that networks have a fractal-like structure due to the recursive nature of routines. In her words, "routines are built from parts that, in themselves, constitute routines" (Sfard, 2023, p. 5).

Following Sfard's (2023) steps, we propose that *mathematics learning*, that is an activity targeted as becoming an insider to a mathematical discourse, can be considered through the lens of routines. Indeed, students' progression through school and university mathematics can be seen as a journey across different discourses and joining these discourses requires the performance of various routines of learning. Thus, we propose that learning involves a repertoire of routines students employ to come to grips with new mathematics time and again. In undergraduate studies, this repertoire may include engaging with the course resources, working on the assigned exercises, and consulting with the lecturer during office hours. Identifying what routines students employ to learn mathematics at the postgraduate level is at the heart of our study.

Sfard (2008) distinguishes between the "how" and the "when" of a routine. The former refers to the *procedure* or the course of action, whereas the latter captures applicability conditions within which one *initiates* a routine and brings it to *closure*. Lavie et al. (2019) proposed that an analysis of one's routines should account for the *task*—an interpretation of a goal a routine performer sets for themselves in a specific *task situation*, that is, in circumstances in which they consider themselves bound to act.

Lavie et al. (2019) maintain that in task situations, people replicate actions they either performed or observed others performing in circumstances they deem sufficiently similar to the current one. Common features of situations, called *precedents,* enable one to cope with unfamiliar task situations. As no two task situations are identical, drawing on precedents requires one to decide which aspects of past performance should be replicated as is and which elements need to be amended to new circumstances. Accordingly, the implementation of familiar routines still requires some degree of agency. It is especially needed when one realizes that existing routines are insufficient to navigate the current task situation. Routines must then be substantially revised or abandoned in favor of new habits. Given the systemic differences between undergraduate and postgraduate contexts, it seems reasonable to expect students to develop new learning routines deliberately and agentively. We attend to the agentive component when analyzing postgraduate students' routines when learning new mathematics.

Method

This study is part of the first author's doctoral research on mathematics students' transition from undergraduate studies to postgraduate research. In this paper, we report on the case of Jordan (pseudonym), a mathematics major enrolled in a one-year honors program. The study unfolded in a small department of mathematics at a research-intensive university in New Zealand. Jordan's program consisted of six semester-long graduate courses and a research project that he carried out in parallel under the guidance of two supervisors. Jordan's project revolved around group theory and topology. Group theory was part of Jordan's undergraduate and graduate coursework, but topology was not. Nonetheless, Jordan had to develop the mathematical background needed for his research project in the first few months of his program.

Our data came from three interviews, around 50 minutes each. The interviews took place when Jordan was in the third, fifth and sixth months of his program. The first interview focused on Jordan's work with the supervisors and his general impression of research so far. Through his responses, Jordan elaborated on his self-guided learning of topology, and we used the opportunity to delve into this process. In the following interview, Jordan elaborated on his typical practices of learning new content, now including group theory, and demonstrated some of them in the third interview.

We engaged with the interview transcripts with the commognitive apparatus, searching for routines Jordan employed to join new mathematical discourses. Our data mainly consisted of Jordan's reflections, and thus, we operated with *routines-narratives*, i.e. self-descriptions of patterned actions. Different routines-narratives varied in the level of detail, but we attempted to delineate Jordan's procedures and tasks where possible. In the last stage of the analysis, we structured the identified routines into a network, with particular attention to how Jordan's postgraduate learning routines compare to typical actions students employ in undergraduate mathematics courses.

Findings

To report our results on Jordan's adjustments to the learning of mathematics at the postgraduate level, we report on one network of learning routines central to Jordan's joining mathematical discourses. Building on the fractal nature of networks, we then zoom in on one component of this network, which is a network in its own right.

Agentive Navigation of Network of Learning Routines

As mentioned, Jordan had to develop the background needed for his research in topology and group theory. Thus, the overarching task of the learning network we identified was to become an insider of these discourses or, in Jordan's words, to "get the background knowledge." Here is an example of how Jordan described this endeavor in the case of topology:

> I spent the first couple of weeks learning some topology and I read [some] lecture notes from a graduate course. It's a sort of introduction to topological groups, and it had like exercises. So, yesterday, I met with my adviser, and we just went over the exercises that I did and checked if they were roughly correct, and tried to approach the ones that I didn't get correct or just didn't know how to do. This was all building up to this theorem called Van Dantzig theorem. The goal of the first couple weeks was to understand that because it's apparently a foundation theorem in this area of topological groups.

The synthesis of similar self-reflections and observations of Jordan's work showed that Jordan's learning activity included finding references online; engaging with them with a special attention to definitions, examples, and exercises; discussing selected exercises and their solutions with his supervisor; and writing up selected solutions in LaTeX. The activities had a patterned nature and fed into each other, giving rise to a network of learning routines.

Some elements of the network are not foreign to routines undergraduates employ to join mathematical discourses in their coursework. Indeed, it is not rare for students to engage with provided references, solve exercises, and attend office hours to clarify issues with the course lecturer. At some point in the first interview, Jordan reflected on this similarity explicitly:

> I remember when I was taking algebra and calculus theory, and you're doing line integrals or decomposition of matrices, I could just do exercises, and there were loads of exercises, like hundreds of exercises. I could just sort of go, do them quite quickly, and I would get a good understanding.

This excerpt suggests that engaging with exercises provided in lecture notes and textbooks was a routinized and valuable way for Jordan to learn in at least some of his undergraduate courses. Hence, we propose that features of undergraduate studies and the learning routines they shape constitute precedents informing Jordan's routines as he joins the topological discourse.

Notwithstanding, it is hard to ignore the differences between typical ways of learning in undergraduate courses and Jordan's learning network. During the first months of his program, Jordan studied topology and group theory in an independent, self-directed manner, without a teacher providing carefully selected readings, exercises, and assessments. This new situation did not enable the replication of routinized habits and necessitated Jordan to be agentive and make multiple decisions. Let us elaborate on some of them regarding his engagement with multiple references.

Engaging with multiple references The supervisors directed Jordan to the main concepts and theorems in the area, which focused on Jordan's search for references. Consider an example of Jordan reflecting on his engagement with one reference:

> *Jordan*: Right now I'm reading through this particular document [lecture notes]. I'll have like, 20 tabs opened on my computer, and it's a couple of different textbooks and this paper and I'm also looking up Wikipedia articles, looking up math stack exchange articles, or posts to explain certain things.
>
> *Rox-Anne*: And you use all those resources to understand better the notes that you have, right?
>
> *Jordan*: Yeah. It explains sparsely the notes [my supervisors] want me to go through. I have a couple of textbook PDFs that I have downloaded, and also a friend of mine in a course. He's doing a master's, and he has some notes on topology, so I'm also using his notes…just referencing all this stuff. Sometimes, I want a particular definition, so I have to go to another reference.

In an undergraduate course, students are typically offered single references to learn from. These resources cover the relevant material and provide explanations that lecturers deem satisfactory and appropriate to support students' learning. However, in the case of Jordan, he has to be the one to search for relevant references. The process occurs online, requiring particular search procedures (e.g., choosing keywords). Jordan needs to estimate the usefulness of each reference to decide whether to engage with it more deeply or search for an alternative. No single reference appears to totally fulfil Jordan's needs, which yields to search-engagement cycles. Furthermore, Jordan also has to be the one to decide when the reference has fulfilled its purpose. For instance, once "a particular definition" has been found, he can return to his main notes. In this way, we argue that Jordan's performance of search-for-references and engagement-with-references routines are more agentive than those typically expected from students in undergraduate courses. Agency pertains to all components of these routines, including their initiation, procedures, and closure.

Detachment from Externally Proposed Exercises in Favor of Self-Posed Ones

In the first interview, Jordan shared that working on exercises he finds in textbooks and lecture notes plays a major role in his learning. He also recognized the limitations of learning through exercises. In his words:

> I guess I am not really sure what I'm going to do when the exercises run out because that's kind of the main way I've always learned. So if there are no exercises, I don't really know, I'll have to ask my advisors to, like, create exercises for me or something, or maybe I'll just try to come up with it myself.

Working on externally assigned exercises constitutes a central component of Jordan's learning network. By engaging with them, Jordan puts himself in task situations he encountered at the undergraduate level: an authoritative source, such as a lecturer or a textbook, ensures that the exercise is well-formulated, relevant, and potentially contributes to students' learning. The relevant material that is sufficient to solve the exercise is typically presented in the preceding text, whereas some learning resources may even provide final answers, hints, or complete solutions. A self-generated solution is often interpreted as a marker of understanding and progress, whereas difficulties can be clarified with a course lecturer or a supervisor in Jordan's case. In this way, exercises contribute to Jordan's becoming a participant in his target mathematical discourse.

However, exercises also confine Jordan to a particular type of resource. Indeed, Jordan was cognizant that exercises are not characteristic of the literature found in postgraduate mathematics, or at least not in the way Jordan is used to. This may potentially impede his engagement with that literature since he is "not really sure [what] to do when the exercises run out." As in undergraduate courses, he considers the supervisors as potential setters of exercises in these cases. Jordan acknowledges the possibility of coming up with his own exercises, but this possibility was mentioned in passing.

Later interviews revealed a number of routines Jordan developed to reduce his dependency on externally set exercises. He labelled one of these routines as *filling in the gaps* and gave the following example:

> *Jordan*: For the Van Dantzig theorem, they say that the translation map is a continuous map and they just make a little remark about it, but the reason it works is because it comes from the fact that the product map is continuous as well.
>
> *Rox-Anne*: But it's not that obvious, right?
>
> *Jordan*: I mean, it kind of is, I don't know, I feel like I'll write it out for myself eventually. Just write out a little proof in full like we show that the translation is just a product of [continuous maps].

Filling in the gaps emerged as a network of routines where Jordan sets new task situations for himself. Within this network, Jordan engaged with a reference with attention to "gaps", such as statements that were not justified in full or inferences that were unclear to him. Then, he turned the generation of a justification or proof into a task for him to pursue. This routine was discussed more deeply as Jordan's research project became more oriented on group theory. To develop the background for his research, Jordan engaged with various scientific papers, and the new routine of filling in the gaps gained importance.

Jordan specified that he initially filled in gaps that the references' authors often indicated with phrases such as "we omitted the proof because it's not difficult." He interpreted these indicators as a "signal that ok, I should probably fill that in, for the details, to put in my research [thesis]". At some point, Jordan started to create gaps on his own by pausing his reading and attempting to prove or justify mathematical statements before engaging with how this was done in the reference. In his words, I "create exercises by looking at the statement, then [trying] to prove it, and then reading what their proof is." Gaps of this sort can be opened in any mathematical paper, turning it into a prolific source of learning-inducing task situations. Whether noticed or created, the routine of filling in the gaps enabled Jordan to distance his learning network from the dependence on externally set exercises.

Summary and Discussion

Mathematics departments invest considerable efforts and resources to involve their majoring students in mathematics research through postgraduate programs. However, only a few studies in mathematics education explored the transition from undergraduate to postgraduate studies. With all the research attention that transitions received in mathematics education (Gueudet, 2016), the undergraduate-postgraduate transition appears to be overlooked.

In this paper, we focused on Jordan—an honors student who was developing mathematical background on his own as a stepping stone toward his research project. Higher education research emphasizes the shift in independence in students' transition from undergraduate studies to postgraduate research (Lovitts, 2005, 2008). Jordan's case illustrates that undergraduate studies do not always prepare students to embark on research right away, and students may need to develop the necessary background related to their research area. Geraniou (2010) referred to such situations as the *adjustment* stage and also identified this stage among doctoral studies. Our study shows that an adjustment stage may also be necessary in transitioning to other types of postgraduate research.

Using the commognitive framework, we identified a network of learning routines Jordan employed and developed in his adjustment stage. We saw similarities between some network components and ways of learning students often employ in undergraduate courses. Instances of these similar network components are the engagement with resources and the discussions with the advisors who provided clarifications about the material. These correspondences are somewhat expected considering that Jordan majored only a few months ago and was offered learning aims similar to those found in undergraduate courses. Additionally, we note that the main reference used by Jordan when joining the topological discourse was lecture notes, which may have facilitated the replication of routines common in undergraduate courses. Nonetheless, a distinctive feature of Jordan's network pertained to agency. In particular, agency was needed to navigate the network routines since each required him to make decisions regarding the initiation of a routine, the course of action, and a decision that the targeted outcome was achieved. Initially, the network depended on Jordan having access to exercises from textbooks and lecture notes, which confined his learning opportunities to specific references. Then, he demonstrated agency when introducing the routine of filling in the gaps that enabled him to become a poser of his own exercises. This new routine had a major impact on the whole network, broadening its applicability and enabling Jordan to learn from research papers. Accordingly, we propose that in mathematics, the shift to independence mentioned by Lovitts (2005, 2008) may be characterized by a growing distance from externally provided resources supporting learning.

The network development we observed fully exemplifies what has been called agentive learning (Brenan, 2012; Knowles, 1975). Our findings show that in the case of Jordan, the shift to this kind of learning, which was required at the postgraduate level, was a gradual and time-consuming process and a part of his adjustment stage. This makes us wonder about the role of agentive learning in the transition from undergraduate to postgraduate mathematics, specifically considering students' expected contribution to research.

References

Bandura, A. (2006). Toward a psychology of human agency. *Perspectives on Psychological Science*, *1*(2), 164–180.

Brennan, K. (2012). Best of both worlds: Issues of structure and agency in computational creation, in and out of school (Ph.D. Thesis). Cambridge: Massachusetts Institute of Technology.

Dorff, M., Henrich, A., & Pudwell, L. (2019). *A mathematician's practical guide to mentoring undergraduate research.* American Mathematical Society.

Geraniou, E. (2010). The transitional stages in the PhD degree in mathematics in terms of students' motivation. *Educational Studies in Mathematics*, *73*(3), 281-296.

Gueudet, G., Bosch, M., DiSessa, A. A., Kwon, O. N., & Verschaffel, L. (2016). *Transitions in mathematics education*. Springer Nature.

Herzig, A. H. (2002). Where have all the students gone? Participation of doctoral students in authentic mathematical activity as a necessary condition for persistence toward the Ph.D. *Educational Studies in Mathematics, 50*(2), 177-212.

Herzig, A. H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research, 74*(2), 171-214.

Knowles, M. S. (1975). *Self-directed learning: a guide for learners and teachers*. New York: Association Press.

Karavi, T., Mali, A., & Avraamidou, L. (2022). Commognition as an approach to studying proof teaching in university mathematics lectures. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(7), 2132.

Kontorovich, I., & Ovadiya, T. (2022). Secondary-tertiary transition and undergraduate tutoring: Novice Tutors make sense of their teaching of first-year students. In Karunakara, S. S., & Higginis, A. (Eds) *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education.* (pp. 314-322). Boston, MA.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*(2), 153-176.

Lovitts, B. E. (2005). Being a good course-taker is not enough: A theoretical perspective on the transition to independent research. *Studies in Higher Education, 30*(2), 137-154.

Lovitts, B. E. (2008). The transition to independent research: who makes it, who doesn't, and why. *The Journal of Higher Education, 79*(3), 296-325.

Morton, M., & Thornley, G. (2001) Experiences of doctoral students in mathematics in New Zealand, *Assessment & Evaluation in Higher Education, 26*(2), 113-126,

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16*(2), 182-198.

O'Donnell, V. L., Tobbell, J., Lawthom, R., & Zammit, M. (2009). Transition to postgraduate study: Practice, participation and the widening participation agenda. *Active Learning in Higher Education, 10*(1), 26-40.

Tobbell, J., O'Donnell, V., & Zammit, M. (2010). Exploring transition to postgraduate study: shifting identities in interaction with communities, practice and participation. *British Educational Research Journal*, *36*(2), 261-278.

Tobbell, J. and O'Donnell, V. (2013a). Entering postgraduate study: A qualitative study of a neglected transition. *International Journal for Cross-Disciplinary Subjects in Education, 4* (1).

Tobbell, J. and O'Donnell, V. (2013b). Transition to postgraduate study: postgraduate ecological systems and identity, *Cambridge Journal of Education*, *43*(1), 123-138

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematicising.* Cambridge University Press.

Sfard, A. (2023). The devil's finest trick: routines that make teachers matter against their better judgement. *Journal of Curriculum Studies*, *55*(1), 21-36.

Shepherd, M. D., & Van De Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior*, *35*, 74-86.

Yerushalmy, M. (2005). Challenging Known Transitions: Learning and Teaching Algebra with Technology. *For the Learning of Mathematics*, *25*(3), 37-42.

Reinventing the Definition of Unique Factorization Domains

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In this study, we conducted a teaching experiment to test hypothetical learning trajectories of reinventing the definition of the unique factorization domain. Six graduate mathematics students who are preservice and in-service teachers were given experientially real tasks of examining factorizations of integers and quadratics using algebra tiles. The experiment showed that their intuitive and informal reasonings about the reducibility and unique factorization were leveraged to formalize two defining axioms of the unique factorization domain by the emergent model of algebra tiles and by the pedagogical moves of the teacher-researcher who led the sessions.

Keywords: Unique factorization domains, Mathematics for teachers, Defining, Conjecturing

The importance of making connections between advanced mathematics courses required in teaching preparation programs and the secondary mathematics that students may teach has been highlighted in recent research (Álvarez et al., 2020; Wasserman, 2018; Wasserman et al., 2017). Abstract Algebra is a course where students learn about algebraic structures, including the properties of number systems, sets, and operations, including those commonly found in secondary mathematics contexts. Abstract Algebra serves as a course where in-service and prospective teachers (IPSTs) can learn the structural and fundamental reasons why secondary mathematics works, e.g., why the factors of a polynomial $(a + b)(c + d)$ are equal to $(c + d)(a + b)$. One concept typically covered in a graduate algebra course is unique factorization domain (UFD), which is defined as an integral domain R such that for every nonzero non-unit element $r \in R$, (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1 p_2 \ldots p_n$ and (ii) the decomposition in (i) is unique up to *associates*: namely, if $r = q_1 q_2 \ldots q_m$ is another factorization of r into irreducibles, then $m = n$, and there is some renumbering of the factors such that p_i is associate to q_i for $i = 1, 2, \ldots, n$ (Dummit & Foote, 2004). The structure of UFD is commonly used in secondary mathematics contexts related to the fundamental theorem of arithmetic and the fundamental theorem of algebra.

There exists a gap in the literature on students' understanding and their reinvention of UFDs. In our study, grounded in the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1973; Gravemeijer, 1999), we designed a local instructional theory (Gravemeijer, 2004) with the goal of students reinventing UFDs. This concept of UFD is particularly useful for IPSTs, as it can support them in understanding that polynomials and integers have similarities in their structure and properties, which can be useful knowledge in their teaching of high school and college algebra. In this paper, we describe two hypothetical learning trajectories that comprise this local instructional theory where graduate students, who are IPSTs, engage in the mathematical activities of conjecturing and defining UFD by intuitively reasoning about the side lengths of rectangles represented in arrays and quadratics represented with algebra tiles. The students construct the axiom about the factorization of an element into a product of irreducibles. This helps students recognize the structural similarities of factorization in Z and $Z[x]$, which leads them to understand that a factorization of an element reduced to a product of irreducibles is unique up to associates and the reordering of the factors. We aim to answer the following questions: *1) Which aspects of the task sequence and which teacher-researchers' pedagogical moves support students' reinvention of unique factorization domains? 2) As students engage in the task sequence, which ways of reasoning are productive to leverage to guide them toward reinventing unique factorization domains?*

Literature Review

One major strand of research in students' reasoning about abstract algebra is based on students' guided reinvention (Freudenthal, 1991) of algebraic structures. Larsen (2009; 2013) developed a local instructional theory by which undergraduate students were guided to reinvent the definitions of group and group isomorphism by reasoning about the properties of the group of symmetries of an equilateral triangle. Larsen and Lockwood (2013) extended this work by developing another local instructional theory by which undergraduate students were guided to reinvent quotient groups. Cook (2012) also developed a local instructional theory for students' reinvention of rings, integral domains, and fields. These studies illustrated how students' intuitive reasoning could be leveraged for reinventing and defining formal mathematical concepts in abstract algebra. Researchers have not yet developed local instructional theories for unique factorization domains. Our study, therefore, contributes to the literature through its design of a hypothetical learning trajectory of how IPSTs reinvent the defining axioms of a UFD.

There has been minimal research done on students' understanding of UFDs. UFDs are special kinds of integral domains that have additional axioms about the existence and uniqueness of the factorization of each nonzero, non-unit element as a product of irreducibles. When exploring students' understandings of the factorization of elements in algebraic structures, researchers have focused on investigating student thinking in UFDs including the integers, Z, and the polynomials with integer coefficients, $Z[x]$. They have demonstrated the productivity of students considering the juxtaposition of elements in these sets, as doing so led them to recognize the overarching structure shared by the sets of integers and polynomials (e.g., Lee, 2018; Lee & Heid, 2018). We leverage this same juxtaposition of integers and polynomials with integer coefficients in this study to help guide students to make conjectures about the properties that integers and polynomials with integer coefficients share, which can be generalized to other integral domains that also satisfy the axioms of UFDs. One of the defining axioms of UFDs is the uniqueness of a nonzero non-unit element's factorization. Conceptualizing the factorization of integers and polynomials as unique has been shown to be a non-trivial endeavor for students (Zazkis & Campbell, 1996) This uniqueness property of irreducible factorizations is essential for secondary teachers to understand, as it is commonly used in problems that relate to the fundamental theorem of arithmetic or the fundamental theorem of algebra. Our task sequence, therefore, focuses on supporting IPSTs in reinventing the existence and uniqueness properties of the factorizations of integers and polynomials that are commonly used in the secondary mathematics content that they may teach.

Theoretical Background

Design Research and RME

We ground our teaching experiment by the design research methodology (Gravemeijer, 1999) that consists of designing and implementing a set of tasks that will be tested and iteratively modified, so that the task sequence aligns with the research team's learning goals. We designed a task sequence consisting of four units. We refer to the collection of four units as a local instructional theory (LIT), and each individual unit as a hypothetical learning trajectory (HLT). Our LIT had the overall goal of guiding students to reinvent the concept of UFD, which is why our task sequence was designed guided by the heuristics of RME (Freudenthal, 1991) which involve implementing: a) experientially real tasks where students can solve context problems that develop their intuitive understanding and guide them to formalize their understanding, b) an emergent model that will serve as a *model-of* students' informal understanding that can become a *model-for* students' formal and generalized understanding of an object, and c) guided reinvention, where students construct their own knowledge through their intuitive understanding with the guide and scaffolding of the tasks and/or an instructor. Each of the HLTs of the LIT, describes the intended learning goals, the learning activities, and the hypothesized student's understanding and learning (Simon, 1995). Once researchers design a proposed task sequence, they implement it in a small-scale teaching experiment where the tasks are administered to pairs of students, who with the help of a teacher-researcher, complete the tasks. The researchers then analyze students' thinking, testing the effectiveness of the tasks depending on how the students engage in such tasks and accomplish the intended goals. Through this process, the HLTs are edited and modified accordingly to help the students meet the hypothesized learning outcomes. Once this occurs, the LIT can be implemented in a larger-scale teaching experiment. Through this process of designing, implementing, and revising HLTs, researchers can discover students' ways of thinking that anticipate formal mathematics and identify ways to evoke and leverage those ways of thinking to support students' development of formal concepts.

Defining and Conjecturing through Guided Reinvention

In order to leverage students' ways of thinking, students need to be engaged in mathematical activities that can help them come up with ideas that describe their informal activity. The tasks in an LIT contribute to this process, but the teacher-researcher in charge of guiding the LIT has the role of evoking and leveraging those mathematical practices. In our case, the participants engage in the process of reinventing an abstract algebraic concept. Dawkins (2015) defined a successful reinvention as "an instance of psychological explication whenever students already possess some intuitive or less formal understanding from which they construct formalized meanings" (p. 67). To achieve this reinvention, the students engage in mathematical activities of defining and conjecturing where students can sense mathematics as a human activity (Zandieh & Rasmussen, 2010). To guide students in the process of conjecturing, researchers can generate examples and counterexamples that can help students "enrich their concept images and enable them to judge the probable truth of conjectures" (Selden, 2012, p. 403). For definitions consisting of axioms, students axiomatize while transitioning from a model-of to model-for (Larsen, 2013), where students start to formalize their intuitive reasoning into a more structured and refined conjecture. Vroom and Alzaga Elizondo (2023) described college instructors' ways to guide students in the process of refining and editing a definition by suggesting an edit with implicit/explicit mathematical reasoning and by presenting students with problematic situations. We explore students' conjecturing and defining activities, as well as the pedagogical moves that scaffold those activities, in this study.

Methods

In this study, we designed RME-based HLTs with the goal of guiding IPSTs to reinvent the definition of UFD and to connect their knowledge to the teaching of mathematics. The HLTs include task sequences that are designed to guide students to reinvent the definition of reducibles and irreducibles (Unit 1), reinvent the definition of UFD (Unit 2 & 3), and apply their knowledge to pedagogical scenarios in hypothetical classroom situations (Unit 4). In this paper, we focus on the IPSTs' work in Units 2 and 3. The IPSTs were given experientially real tasks to guide their reinvention of the two defining axioms of UFD following the two HLTs that we initially hypothesized in the task design stage of this research. They were asked to investigate composite integers and factorable quadratics using algebra tiles and generate conjectures about the reducibility of elements in Z and $Z[x]$ in Unit 2. They were asked to examine different factorizations of the same elements that are essentially the same and conjectured about the uniqueness of factorization in Z and $Z[x]$ in Unit 3.

We tested the HLTs by administering tasks to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in a Hispanic-serving institution in the Southern US. Six mathematics graduate student IPSTs participated in this study: Josie and Raul (Group A), Javier and Roberto (Group B), and Kim and Taylor (Group C). They took abstract algebra courses at the undergraduate and/or graduate levels but had not learned UFD in their coursework. Each group participated in five 90-minute sessions of teaching experiments with the research team including a teacher-researcher (TR), secondary instructor, and research assistant. The TR's role was to guide the session, provide students with prompts to work on, and ask follow-up questions to better understand and clarify the IPSTs' thinking. The research assistant operated the video camera and took field notes. The secondary instructor observed the sessions, took field notes, and asked questions to the IPSTs when needed. Groups A and B participated in person, and Group C participated on Zoom due to their availability. In-person groups were given printouts with physical algebra tiles, and the virtual group was provided with an online whiteboard with tasks and virtual algebra tile manipulatives.

We collected and transcribed the video recordings of fifteen 90-minute teaching experiment sessions. The groups' written work on handouts and virtual whiteboards were collected. We used inductive coding (Miles et al., 2013) to analyze how IPSTs engaged in conjecturing and defining as they were guided to reinvent the definition of UFD. We focused our analysis on their ways of reasoning that were leveraged by using the algebra tiles and by interacting with the TR.

Results

Reducibility of Elements in Z, $Z[x]$, and UFD

In the first task set of Unit 2 (see Figure 1), the IPSTs observed side lengths of rectangular arrays of algebra tiles that represent 12 and its factors. They were asked to keep reducing the side lengths of rectangular arrays and to conjecture about the reducibility of integers. For instance, groups arranged tiles to represent $12 = 3\times4$ or $12 = 2\times6$ then reduced the composite factors of 12 into other rectangular arrays such as $4 = 2\times2$ and $6 = 3\times2$. The IPSTs transitioned their use of the emergent model (i.e., algebra tiles) from being a model of factoring integers to a model for abstracting the reducibility property of composite integers. IPSTs initially conjectured *reducible integers can be reduced to a product of irreducible integers*. Their reasoning was leveraged when prompted to think of counterexamples to their conjectures. When needed, the TR suggested what to consider such as 0 and units in Z (1 and -1). This also involves referencing the definition of irreducibles from the previous unit where units and zero are not included. The IPSTs found that *the zero and units cannot be represented as a product of irreducibles because they are not irreducibles by the definition*. This investigation allowed them to revise their conjectures by limiting the scope of integers in the statement: *all non-zero, non-unit integers can be reduced to a product of irreducible integers*. Examining precise definitions of irreducibles and products (of elements) was productive to leverage their reasoning in revising their conjectures. For instance, they observed 2×1 arrangement of algebra tiles and asked if this could be a product of irreducibles. It allowed them to consolidate their definitions of irreducibles (e.g., irreducibles do not include units and zero) and products (e.g., a product can be a single element). In sum, IPSTs' initial conjectures about the reducibility of integers were elicited and leveraged by the prompts in the task sequence with support from the TR when needed. Counterexamples and formal definitions were useful resources for their progression in the practices of conjecturing and revising their conjectures.

Task Set 1	1. Look at the side lengths of the rectangles of 12. Are those side lengths (factors) of 12 reducible or irreducible? How do you know? 2. If the side length is a reducible, arrange the tiles representing that side length into a rectangular array. Are those side lengths of the new rectangle reducible or irreducible? 3. What do you notice about the reducibility of integers? Make a conjecture. 4. Can we always write an integer as a product of irreducibles? Try to find a counterexample where an integer cannot be factored into a product of irreducibles.
Task Set 2	1. Look at the side lengths of the rectangles that represent these quadratics. Note that we consider polynomials with integer coefficients only: (a) $3x^2 + 9x$, (b) $x^2 + 3x + 3$, (c) $2x^2 + 4x$, (d) $x^2 + 4x + 3$, (e) $x^2 + 5x + 7$. Are those factors of the quadratic reducible or irreducible? How do you know? 2. If the side length is a reducible polynomial, arrange that side length polynomial into a rectangular array. Are those new side lengths reducible or irreducible? 3. Create a conjecture about the reducibility of quadratics. 4. Can we always write a quadratic as a product of irreducibles? Try to find a counterexample where a reducible cannot be factored into a product of irreducibles. 5. What about a higher degree polynomial? Can we write any polynomial as a product of irreducibles? Try to find a counterexample where a reducible cannot be factored into a product of irreducibles.
Task Set 3	1. The Common Cores State Standards of Mathematics suggest that students should "Understand that polynomials form a system analogous to the integers." Thus, polynomials have similar structure to the integers. In what ways are polynomials and integers similar? 2. Integers and rings of polynomials with integer coefficients are special kinds of integral domains with special properties about their elements' reducibility. Let's make a conjecture about the reducibility of integers and polynomials. What similarities do you see between these examples of the side lengths of the rectangular array that represents an integer and the side lengths of the rectangular arrangements of algebra tiles that represent quadratics?

Figure 1. Unit 2 task sequence

Task Set 2 used the same structure of the task sequence to guide IPSTs to conjecture about the reducibility of quadratics over $Z[x]$ (see Figure 1). This time, they used algebra tiles to represent quadratics and their factorizations if they are reducibles. For instance, IPSTs reduced $3x^2 + 9x$ into $3x\times(x + 3)$, then further reduced $3x$ into $3\times x$ using algebra tiles (see Figure 2). The progression in their reasoning seemed similar to those from the previous unit: generating initial conjectures, looking for counterexamples (e.g., units and zero), and refining their conjectures by eliminating units and zero from the initial conjectures. When examining the array of $x\times1$ that cannot be reduced further, the IPSTs needed the precise definition of units in a ring to determine if x is a unit in $Z[x]$ or not. The TR guided the IPSTs to reference the definition and pushed them for precision in their conclusion that x is not a unit because its multiplicative inverse $1/x$ does not exist in $Z[x]$.

TR: ... why is x not a unit?

Kim: x is not a unit because, um, its multiplicative inverse, it does not have one that is within that ring. Because the power of x would need to be outside of the natural numbers, so it would not be the unit.

In this task set, the IPSTs' reasoning was leveraged along the task sequences with the support of the TR and the emergent model of the algebra tiles. In particular, those supports were productive in identifying non-unit and non-zero elements in $Z[x]$ that were less explicit for them than those in Z. So, the IPSTs concluded the conjectures about the reducibility of integers in Z and polynomials in $Z[x]$ with essentially the same structure: *A non-zero, non-unit element in R can be written as a product of irreducible elements in R.*

In Task Set 3, the IPSTs were asked to identify common algebraic properties of Z and $Z[x]$ and write a conjecture of the reducibility in UFDs that works for both Z and $Z[x]$. The TR supported the IPSTs to recall and synthesize algebraic properties of Z and $Z[x]$ and formalize their final conjecture. In sum, the task design in Unit 2 used algebra tiles, and the TR's pedagogical moves leveraged the IPSTs' reasoning of generating, revising, and abstracting conjectures of the reducibility in the respective algebraic structures.

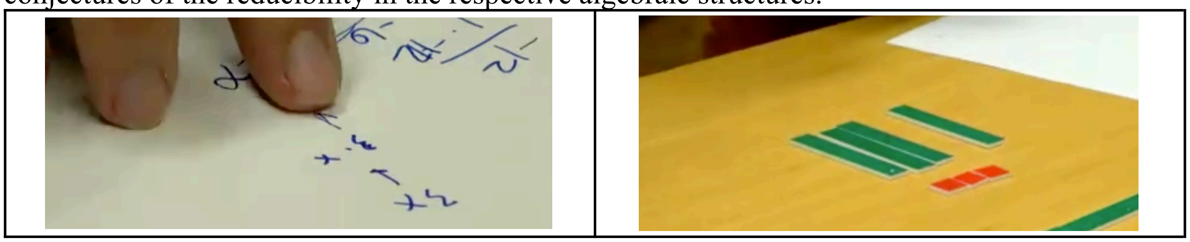

Figure 2. Group A used algebra tiles to represent the factorization of $3x = 3\times x$

Unique Factorization of Elements in Z, $Z[x]$, and UFD

Unit 3 tasks began with investigating two factorizations of an integer or a quadratic that differ by the order of factors in the products or by -1. Then, the IPSTs were guided to reinvent the concept of the uniqueness of factorizations, i.e., determine when two factorizations are essentially the same. For example, when asked to convince why two factorizations $(x + 3)(x + 2)$ and $(-x - 3)(-x - 2)$ are essentially the same, the IPSTs showed algebraic procedure to convert one into the other by using $(-1)^2 = 1$. They initially conjectured *two factorizations of a quadratic in* $Z[x]$ *are essentially same when a factor in one factorization is the negative of the respective factor in the other factorization.* This reasoning was leveraged in the following set of tasks that guided them to reinvent the definition of associates. Later, it allowed the IPSTs to use the formal language to capture their ideas about the relationship between factors in two essentially the same factorizations. When defining associates, the TR helped the IPSTs to generalize 1 and -1 to units for a general integral domain when they wrote the formal definition. The TR suggested to generalize their initial definitions, asked a leading question, and provided scaffolding. For instance, Javier (Group B) first wrote a conjecture that non-zero and non-unit elements a and b in an integral domain are associates if and only if their greatest common factors are 1 and -1. The TR leveraged his reasoning about associates with common factors of 1 and -1 but also pointed out a limitation of the conjecture by providing counterexamples.

TR: […] the only divisor I guess, the 1 and -1, then we could have something like $x + 3$ and $x + 2$ are associates? […] So you're right in that they do have this shared factor of 1 and -1, and it could be that it's just $a = 1\times b$ or it could be that $a = 1\times(-b)$. So it's looking like there's this shared kind of structure here, where a is equal to something times b. So now we have to figure out, what's that something?

Group B wrote a formal definition of associates using the idea of $a = ub$ where u is a unit in R that generalizes their initial conjecture to work for arbitrary integral domains (see Figure 3a). Following the problems in Task Set 2, the IPSTs revised their conjectures of the uniqueness of factorizations by using the definition of associates: *Two factorizations of an element in an integral domain are essentially the same if one factorization can be converted to the other by rearranging the order of factors or by replacing factors with their associates*. This involves using symbolic notations representing two possible factorizations $a_1 \cdot a_2 \cdots a_n$ and $b_1 \cdot b_2 \cdots b_m$ to express their reasoning of the uniqueness up to the associations and the order of factors. The TR helped the IPSTs to come up with precise subscripts in their writing. For instance, the TR suggested different subscripts n and m for the last factors in the given factorizations, a_n and b_m and different subscripts i and k for arbitrary factors from each factorization that are associates to each other, a_i and b_k. The TR pointed out an unintended assumption of correspondence between a_i and b_i when using the same subscripts for associates that are not necessarily ordered the same way (see Figure 3b).

TR: I like how you symbolized that, the $r = a_1 a_2 \cdots a_i$ and then the $u_1 b_1 u_2 b_2 \cdots u_i b_i$. So you're kind of pairing the a_1 with the b_1 and the u_1. So you're numbering them, that's kind of implying there's this correspondence between the a_i and the b_i.

This pedagogical move of the TR elicited the IPSTs' attention to the precision in mathematical language and leveraged their reasoning of what it means two arbitrary factorizations are essentially the same in the context as well as how to express precisely. The IPSTs' reasoning about the reducibility and the uniqueness of factorizations in Z and $Z[x]$ were leveraged in Task Set 3 where they were asked to synthesize two conjectures they made about the existence and uniqueness of factorizations of elements as a product of irreducibles to create the two axioms in the definition of UFD. The TR guided them to formalize their statements for the axioms in this process and helped them with making sense of the definition they created by using examples and non-examples (e.g., $Z[\sqrt{5}i]$) of UFD.

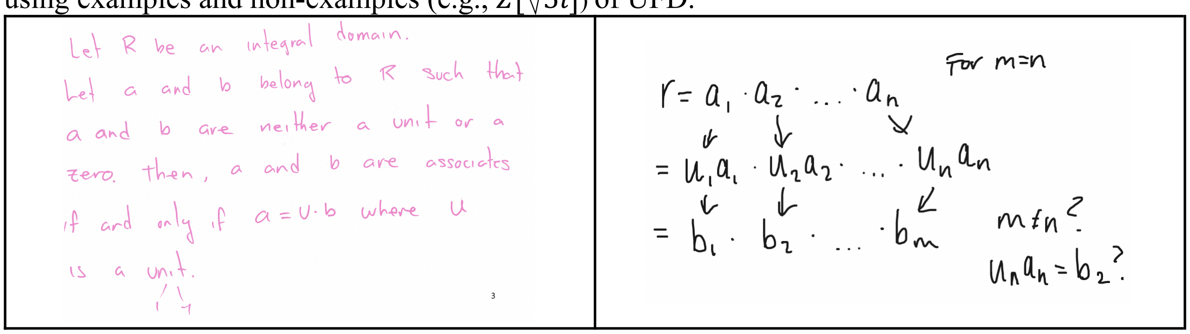

Figure 3. (a) Roberto's final definition of associates (left), (b) Kim's use of symbolic notations for two arbitrary factorizations that are essentially the same (right).

Conclusion

The findings of this study showed evidence of HLTs (Simon, 1995) in reinventing the definition of UFD by leveraging students' reasoning of the reducibility and the unique factorization of elements in Z and $Z[x]$. The sequence of experientially real tasks provided guiding prompts for the IPSTs' mathematical activities of conjecturing and defining in the progression of reinventing the two defining axioms of UFD. The emergent model of algebra tiles served as a *model-for* structuring and formalizing their intuitive reasoning about factorizations of integers and quadratics. The TR's pedagogical moves provided sociomathematical scaffolding that facilitates the IPSTs' horizontal and vertical mathematization (Gravemeijer & Doorman, 1999) of their informal reasoning and mathematical language. We plan to continue the design research on refining the hypotheses of student reasoning and task design of the HLTs in the following iterations of teaching experiments and classroom implementation.

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References

Álvarez, J. A., Arnold, E. G., Burroughs, E. A., Fulton, E. W., & Kercher, A. (2020). The design of tasks that address applications to teaching secondary mathematics for use in undergraduate mathematics courses. *The Journal of Mathematical Behavior*, *60*, 100814.

Cook, J. P. (2012). *A guided reinvention of ring, integral domain, and field*. [Doctoral dissertation, The University of Oklahoma]. ProQuest Dissertations Publishing.

Dawkins, P. C. (2015). Explication as a lens for the formalization of mathematical theory through guided reinvention. *The Journal of Mathematical Behavior*, *37*, 63-82.

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (3rd ed.). John Wiley & Sons, Inc.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Netherlands.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, 6(2), 105–128.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111–129.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior*, *28*(2-3), 119-137.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712-725.

Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *The Journal of Mathematical Behavior*, *32*(4), 726-742.

Lee, Y. (2018). *University Students' School Mathematics Understanding and Its Growth in Their Learning of Collegiate Mathematics: Through the Lens of a Transformative Transition Framework*. [Doctoral dissertation, The Pennsylvania State University]. ProQuest Dissertations Publishing.

Lee, Y., & Heid, M. K. (2018). Developing a structural perspective and its role in connecting school algebra and abstract algebra: A factorization example. In *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 291–318). Cham: Springer.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education* (pp. 391–420). Springer.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education* (pp. 267-306).

Vroom, K., & Alzaga Elizondo T., (2023). What Are Instructors Doing to Initiate Refinements of Students' Definitions? *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 1012–1017.

Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, *49*, 116–128.

Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *Primus*, *27*(6), 559-578.

Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57–75.

Zazkis, R., & Campbell, S. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, *27*(5), 540-563.

Exploring Graduate Teaching Assistants' Teaching Practices in a Purposefully Designed Active Learning Environment

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This study explores the instructional practices of four graduate teaching assistants (GTAs) – Ursula, Ava, Phoenix, and Cole – in an active learning Calculus I course. Each GTA's unique pedagogical background is discussed in relation to their teaching methods and their students' classroom actions. Ursula, a seasoned high school math teacher, brings a wealth of experience in active teaching, while Ava, having encountered active learning as a student, reflects on her positive experiences. Phoenix, a novice GTA, emphasizes group work and individual student exploration, while Cole, with limited teaching and active learning experience, promotes student independent work. The contrasting approaches of these GTAs within the same purposefully designed active learning environment provides valuable insight into the connection between previous educational experiences and teaching practice, especially with respect to student engagement in a Calculus classroom.

Keywords: Active Learning, Calculus, Teaching Practices, Graduate Teaching Assistants

Calculus teaching and learning is an important and much-discussed topic in post-secondary mathematics education. Many educators and administrators are concerned about high DFW (the percentage of students who receive a D, an F, or withdraw) rates in introductory mathematics courses. Up to 50% of students nationally fail their first mathematics course, and 50% of students who struggle in mathematics courses in their first year leave STEM majors (APLU, n.d.). One way that institutions have attempted to combat these high DWF rates is through the introduction of active learning into first-year mathematics courses (APLU, n.d.). The Conference Board of the Mathematical Sciences (CBMS) defines active learning as "classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote higher-order thinking" (CBMS, 2016, para. 1). Building on this, we define active learning as a productive learning environment that provides meaningful mathematical learning opportunities for all students. Such active learning environments include three key components: the inclusion of cognitively demanding mathematical tasks, the delivery of high-quality mathematics instruction, and the promotion of both small and whole-group discussions.

While much of the focus of active learning is on students – especially engagement and higher-order thinking – active learning also requires instructors to take on a prominent role. This role differs significantly from a traditional role in a lecture-based classroom. Instructors must choose rich tasks which promote higher-order thinking, facilitate classroom discussions, and manage a classroom in which student-centered activities occur. This shift often presents a challenge, as most university instructors have not experienced this type of classroom in either prior professional settings or their own education.

At large universities, Graduate Teaching Assistants (GTAs) are a significant part of the implementation of active learning. Consequently, there has been a push to provide GTAs with professional development opportunities to help them make the transition to facilitators of active learning in their classrooms (Deshler et al., 2015). Significantly, "effective training of graduate teaching assistants" is one of the seven characteristics of a successful calculus program, as put forth by Bressoud and Rasmussen (2015, p. 145). GTAs have an important role in implementing student-centered pedagogy and active learning. Therefore, it is important to understand how the GTAs themselves enact and support active learning in the classroom. It is particularly important to understand how their backgrounds may influence their actions. This leads to the question we explore in this study: what teaching actions do GTAs with different pedagogical backgrounds take in an active learning Calculus I course and how do they discuss these actions?

Literature Review

Because of the current focus on student-centered activities in entry-level mathematics classes, much has been written on active learning strategies and their effect on student learning (Freeman et al., 2014; Kim et al., 2013). As GTAs play an important role in enacting active learning, there has been a focus on GTA professional development (PD). GTA PD programs provide an important opportunity for GTAs to engage with and understand research-based instructional strategies, including exposure to active and student-centered pedagogy (Archie et al., 2022, Deshler et al., 2015). Many GTAs come into programs with little to no teaching experience, and both pedagogical content knowledge and knowledge of student thinking are important for GTAs' transition from learner to teacher (Kung & Speer, 2009). GTA PD provides them with opportunities to develop this knowledge and connected skills.

Regardless of whether they have participated in a PD program, GTAs may not have experienced active-learning approaches as learners in their own coursework, leading to a limited comfort with using these approaches in their teaching (Deshler et al., 2015). Patrick et al. (2021) studied GTAs across a College of Science and College of Engineering to determine their perceptions and practice of active learning strategies as a student and teacher. They found that GTAs' ranking of teaching strategies changed depending on whether they were thinking about their learning or their teaching. "TAs were currently teaching differently from the way they preferred to learn and how they thought undergraduates learned best" (Patrick, et al., 2021, p. 11). GTAs reported a desire for more active learning strategies in their own classes and felt that they should use more active learning strategies in their own teaching (Patrick, et al., 2021). However, GTAs' feelings about active learning were not sufficiently reflected in their classroom practices. While the link between experience and teaching practices seems to be straightforward, empirical evidence on mathematics GTAs' implementation of active learning based on their perspectives about and background with active learning is sparse.

Even experienced instructors can encounter challenges when implementing active learning strategies, requiring deep understanding of effective teaching practices and productive pedagogical strategies. The Mathematics Association of America National Study on College Calculus calls teaching which incorporates active learning ambitious teaching, stating that "they are ambitious in the sense that they require substantial institutional supports and advanced knowledge, skills, and beliefs on the part of instructors" (Larsen et al., 2015, p. 104). These complexities in implementing active learning also affect GTAs, who play pivotal roles in undergraduate mathematics courses. GTAs, often lacking training and prior classroom experience, face these challenges despite the growing emphasis on active learning in undergraduate mathematics education, as their own exposure to active learning is limited

(Manzanares et al., 2023). Therefore, we need studies that focus on GTAs conceptions and teaching practices of active learning.

Methods

Active Learning Context

This study takes place within a wider research effort to transform introductory courses in STEM at a large public university in the northeast USA. This study represents one component of the overall project and examines the actions and ideas of GTAs as they integrate active learning into their Calculus I teaching. The course follows a lecture-recitation model of instruction in which students attend a large lecture (~150 students) taught by a faculty member and a smaller recitation (~20 students) taught by a GTA. Implementation of active learning was purposefully focused on recitation sections and included a new classroom structure (introduction, small group work, class discussion), researcher-designed activities, and near-peer tutors called Learning Assistants (LAs). GTAs were given the freedom to implement these active learning methods in whatever way they saw fit.

While all recitations involved active learning, there were six special researcher-designed activities spread over the course of the 15-week semester. These activities were designed to foster the core practices of quantitative reasoning (Thompson, 1994) and representational fluency (Fonger, 2019). Students worked on these activities in small groups of three to four students, supported by the GTA and LA. The GTA was responsible for leading the class, meaning they facilitated the introduction and class discussion and were responsible for administrative tasks such as proctoring quizzes and collecting homework. The LAs' responsibility was to facilitate small group discussion, typically by asking students questions and attending to student ideas.

Data Collection and Analysis

Data for this study includes video recordings of recitation sessions and interviews with GTAs. Recitations were recorded six times throughout the semester on the days the researcher-designed activities were implemented. Structured interviews with GTAs were conducted at the beginning and the end of the course and were focused on GTAs' experiences with and ideas about active learning. This analysis focuses on a subset of these recordings for four GTAs. These four GTAs were selected for this analysis because they highlight distinct ways in which classroom actions and conceptions of teaching manifest in this active learning context.

To capture the classroom actions of these GTAs across the semester, two video recordings (one from the first half of the semester and one from the last three weeks) were analyzed for each GTA using the Classroom Observation Protocol for Undergraduate STEM (COPUS) (Lund et al., 2015). The COPUS framework describes the types of activities both instructors (Figure 1) and students (Figure 2) are engaged in throughout a lesson. Codes from this framework were applied to every two-minute interval of the recorded video. For each GTA, the codes were aggregated across the two videos, and the percentage of each coding category out of total number of codes was calculated. For robust descriptions of the COPUS action codes used in this study see Lund et al. (2015). Each lesson was coded by at least two members of the research team and disagreements were resolved by consensus of the team.

Interviews were analyzed in order to contextualize each GTAs' conceptions of their practice in the reformed classroom. These interviews were analyzed through open coding and thematic analysis to capture GTAs' definitions of and experiences with active learning (Miles et al., 2018; Strauss & Corbin, 1998).

Results

In this section, we will introduce the background of four GTAs, Ursula, Ava, Phoenix, and Cole, compare their instructional actions (Figure 1), and examine the classroom actions of their students (Figure 2). We aim to understand the teaching approaches adopted by the GTAs with diverse pedagogical backgrounds and explore how they discuss these actions.

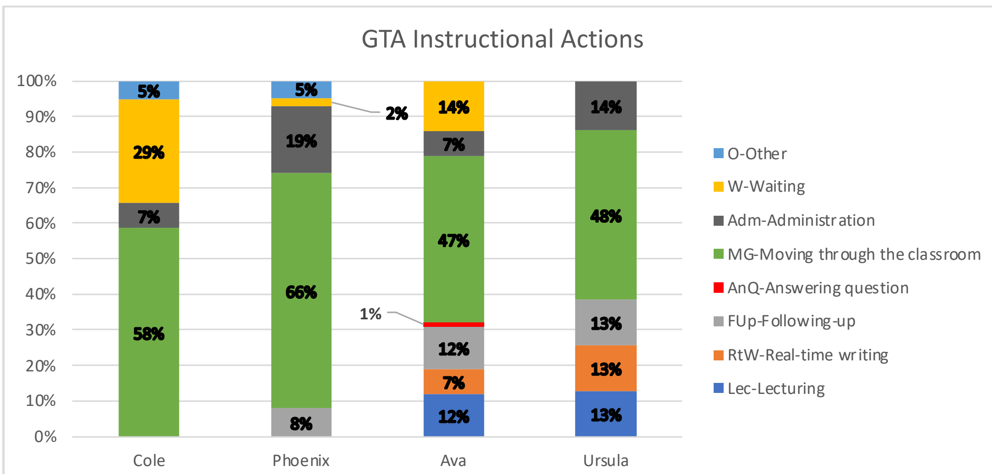

Figure 1: Instructional Actions of each GTA

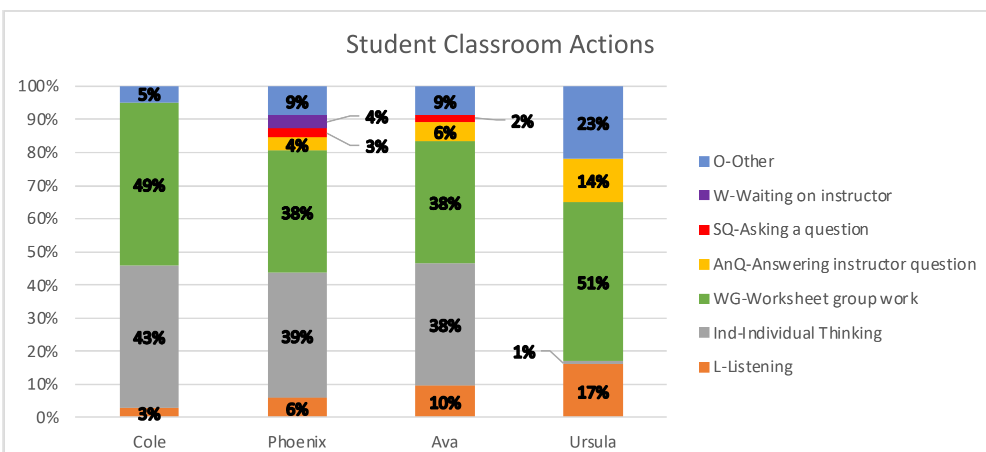

Figure 2: Actions of Students in each GTA's classroom

Ursula

Ursula was a graduate student in the Mathematics Education Ph.D. program and in her fifth year as a GTA at the university. Prior to her GTA experience, Ursula had substantial experience teaching mathematics to high school students. She described her high school teaching practice as "active" and said that she often had students work in groups or work independently. In Figure 1, we see that approximately half of classroom action codes are *moving*, meaning that she devotes significant attention to facilitating student work on activities in the classroom. She described that, when she moves through the classroom, “I’m not going to solve the activity for them, but if they got stuck I will ask them the question to lead them to the next step”.

Outside of facilitating group work, Ursula’s actions are still notably student centered; 13% of her actions were coded as *following-up,* meaning that she responded to student ideas in front of the class. Even though 12% and 13% of her actions are respectively focused on *lecture* and *real time writing*, these codes primarily occur during the lesson introduction. Ursula explained that, in this introduction, she “reviews concepts” and “examples” to make students more “comfortable” participating in group work by fostering “confidence” in their mathematical knowledge. While lecturing and writing are not typically considered as active, Ursula used them selectively and strategically in her teaching to support the more active portions of the lesson and to create a “balance between the instructors and the student”. While a significant number of codes were *administration,* even these actions were frequently student-centered, consisting of such things as organizing student groups and assigning students problems to write on the whiteboard.

This emphasis on group engagement and instructor collaboration was also reflected through the actions of students in Ursula’s classroom (Figure 2). Her students’ actions were primarily worksheet *group work,* indicating that students typically worked together on class activities. Also, the students in Ursula’s classroom were *answering instructor questions* (14%) and *listening* (17%) meaning that they shared their ideas in front of their peers and listened to one another and to Ursula. The *other* (23%) category in this case represents students writing work on the classroom whiteboards, again emphasizing the opportunities for students to share their mathematical ideas in this classroom.

Overall, Ursula’s conceptions and practices typify how a GTA with previous active-learning teaching experience was able to effectively implement such methods at the undergraduate level and provide reasons why such methods are appropriate for supporting student learning.

Ava

Ava was a mathematics Ph.D. student who at the time of the study was in her third year as a GTA at the university. Unlike Ursula, she had no previous independent teaching experience, but Ava did encounter active learning several times as an undergraduate student. She described that she “struggled” to interpret mathematical content independently as a learner in a flipped classroom, but she had more positive experience in a class where “the teacher would present the material, and then we'd work on it in groups”.

In her own practice as a GTA, Ava’s actions as a teacher mirror her positive experiences as a student. We see that her actions as a teacher are similar to Ursula’s (Figure 1), but 14% of Ava’s actions are *waiting,* meaning that she was merely standing on the side during class time. This presence of wait time could be because Ava frames active learning as “when students bounce ideas off of each other” and hinges upon “the student thinking more deeply about the topic”. Since student thinking is central to active learning in her view, Ava allowed students time to work on their own without her intervention.

Additionally, we see that Ursula and Ava’s students primarily engage in the same types of activities (Figure 2), except that Ava’s students engage in *independent thinking* (38%) while Ursula’s students rarely work independently. Again, this is probably rooted in their different conceptualizations of active learning; Ava has a broad view of active student engagement while Ursula takes group work as essential in such an environment.

From Ava's actions and discussion of these actions, we see that she used her own experiences as a student with active learning to guide her teaching as a GTA in an active environment. Because of her own positive experiences with group work and guided work during class, she implements such practices in her classroom, supporting students to work in groups or independently on classroom activities.

Phoenix

Phoenix was a graduate student in the mathematics education PhD program; in her second semester as a GTA. She began the graduate program directly following her undergraduate degree at the same institution, meaning that she had no independent teaching experience prior to her time as a GTA. She did, however, have experience as a peer-tutor and had taken several university-level courses focusing on teaching and learning.

Phoenix's teaching actions are notably distinct from the relatively similar pattern seen in the actions of Ursula and Ava. Unlike these two, Phoenix did not lecture or write on the board; she was mostly involved in *moving* (66%) and *administration* (19%). That is, Phoenix's classroom actions primarily focus on working with students and organizing student groups, rather than instructing students from the board. This is reminiscent of her own experiences with active learning as a student as she and her peers spent time "working in groups and kind of investigating the material ourselves". She further explained that she views active learning as rooted in "the students and the students wanting to learn the material" with her role being to "help each individual student" through this process. Phoenix recognized her responsibility as an instructor to guide student thinking, but she provided this support at an individual and group level, while Ursula and Ava provided support at the classroom level through short periods of direct instruction at the board.

While Phoenix's actions are distinct from Ursula and Ava's, we see that her students' actions are relatively similar to Ava's students (Figure 2). We do see that Phoenix's students are *listening* less than Ava's students (6% vs 10%), which reflects Phoenix's emphasis on group work time. Similar to Ava, we see that Phoenix's students had the opportunity to share their thinking as they are engaged in *asking questions* and *answering instructor questions.*

Through these actions and commentary, we see that Phoenix's status as a novice GTA with limited active learning experience led to a strong emphasis on group work in her classroom. While Ava and Ursula, who have more experience with active learning, chose to provide their students with some direct instruction, Phoenix chose to provide students more time to work on their own or in groups. While this alternate focus in teaching actions was not coupled with major shifts in student actions, these alternate approaches of GTAs in the same environment are notable.

Cole

Cole was a first-year Ph.D. student in statistics and, like Phoenix, was in his second semester as a GTA at the university. Cole had some online experience as a one-on-one tutor before his time as a GTA, but, unlike the other GTAs discussed in this study, he had no experience with active learning as a teacher nor as a student. He stated that in his time as a student he would "pretty much study alone, individually" and "usually just listen to the teachers and copy, paste" notes written on the board.

Accompanying this difference in experience, we see that Cole's classroom actions were distinct from those of Ursula, Ava, and Phoenix. While Cole's proportion of *moving* is similar to the other GTAs, he has a much higher proportion of *waiting* (29%) than the other GTAs. It is also important to note that Cole's *other* code marked times when he left the classroom entirely, contrary to other GTAs for whom the *other* code was content related, representing such activities as discussing study skills and reviewing student work. Again, these actions reflect Cole's view of active learning; he describes active learning as "communicating with others... exchanging your thoughts" making his role in the classroom "more like a tutor". Cole's actions are consistent with this idea of active learning as he lets students work with a high degree of independence and intervenes in this process sparingly. This strong emphasis on student independence could also be a reactionary effort to move away from his "copy, paste" experience as a learner.

Similarly, we see differences in the actions of Cole's students in comparison to the students of the other three GTAs. Cole's students are almost always engaged in *worksheet group work* or *individual thinking* (92%). Again, this echoes Cole's conceptualization of active learning as students working without much direct instruction.

Overall, Cole's actions show one way in which active learning can be enacted by a GTA with little prior experience with such methods as a student or as a teacher. In Cole's classroom, active learning manifests as student independence, with Cole offering students support at the individual or group level, rather than providing direct instruction.

Discussion

From the four GTAs examined above, we see that there is a connection between previous active learning experiences and instructional practices. Ursula, a GTA with experience teaching in an active learning setting, gives attention to students' work within groups, but supports this work through some direct instruction and following up on group work in front of the class. Ava and Phoenix, who had some experience with active learning as students, still emphasize group work, but allow students to work individually. Cole, a GTA without previous active learning experience, attends solely to group work, occasionally leaving students to their own devices.

While all GTAs in this study were provided with the same active learning teaching guidelines and supports (i.e., LAs, researcher-designed activities, suggested classroom structure), we observe that GTA actions and the actions of their students are distinct within each classroom. Previous literature acknowledges the need for GTAs pedagogical training, but we emphasize that GTAs educational needs as teachers vary depending on their previous experiences as both teachers and learners. That is, "effective training" (Bressoud & Rasmussen, 2015, p. 145) of GTAs must account for the lived educational experiences and resulting pedagogical ideas of GTAs. Without such background taken into account, GTAs may have difficulty implementing active learning practices in the classroom.

Since GTAs often serve important roles in facilitating student-centered instruction within Calculus courses, it is important to acknowledge factors which enable or inhibit them accomplishing this type of instruction. Here, we identify background experiences with active learning as one such factor and encourage both researchers and educators to pursue practical means of integrating GTAs' previous educational experiences into GTA pedagogical training.

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References

Archie T., Hayward C.N., Yoshinobu S., Laursen S.L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *PLoS ONE,* 17(4): e0267097. https://doi.org/10.1371/journal.pone.0267097

Association of Public and Land-Grant Universities (APLU). (n.d.) *Student engagement in mathematics through an institutional network for active learning (SEMINAL).* Association of Public and Land-Grant Universities: Our Work. https://www.aplu.org/our-work/2-fostering-research-innovation/seminal/

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the American Mathematical Society*, 62(2), 144-146.

Conference Board of the Mathematical Sciences (CBMS). 2016 *Active learning in post-secondary mathematics education.* https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

Deshler, J. M., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. *Notices of the AMS*, 62(6), 638-643.

Fonger, N. L. (2019). Meaningfulness in representational fluency: An analytic lens for students' creations, interpretations, and connections. *The Journal of Mathematical Behavior*, *54*, 100678. https://doi.org/10.1016/j.jmathb.2018.10.003

Freeman, S., Eddy, S., McDonough, M., Smith, M., Okoroafor, N., Jordt, H., and Wenderoth, M. (2014) Active learning increases student performance in science, engineering, and mathematics. *PNAS,* 111(23), 8410-8415.

Kim, K., Sharma, P., Land, S. M., & Furlong, K. P. (2013). Effects of active learning on enhancing student critical thinking in an undergraduate general science course. *Innovative Higher Education*, *38*(3), 223-235. https://doi.org/10.1007/s10755-012-9236-x

Kung, D., Speer, N. (2009) Mathematics teaching assistants learning to teach: Recasting early teaching experiences as rich learning opportunities. *Studies in Graduate and Professional Student Development,* 12, 133-152.

Larsen, E., Glover, E., Melhuish, K. (2015) Beyond good teaching: The benefits and challenges of implementing ambitious teaching. In D. Bressoud, V. Mesa & C. Rasmussen (eds.) *Insights and recommendations from the MAA national study of college calculus* (pp. 93-105). MAA Press.

Lund, T..J., Pilarz, M., Velasco, J.B., Chakraverty, D., Rosploch, K., Undersander, M., & Stains, M. (2015). The best of both worlds: Building on the COPUS and RTOP observation protocols to easily and reliably measure various levels of reformed instructional practice. *CBE Life Sciences Education,* 14, 1-12.

Manzanares, R., Webb, J., Harrell-Williams, L. M., Olson, G. A., Houston, S., Lanius, M., ... & Merchant, L. (2023). A Case Study of a Multi-Institution Replication of a Comprehensive GTA Teacher Training Program: Comprehensive GTA Teacher Training Program. *PRIMUS*, (just-accepted), 1-26.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2018). Qualitative data analysis: A methods sourcebook. Sage Publications

Patrick, L., Howell, L.A., Wischusen, E.W. (2021) Roles matter: Graduate student perceptions of active learning in the STEM courses they take and those they teach. *Science Progress*, 104(4), 1–16.

Strauss, A., & Corbin, J. (1998). Basics of qualitative research: Techniques and procedures for developing grounded theory. Sage Publications, Inc.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics,* 26(2–3), 229–274.

Instructor and Coordinator Perspectives within First-Year Mathematics Courses

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The perspectives, habits, and mindsets of experienced instructors and coordinators serving within course coordination systems are valuable for effective coordination. In this paper, we present perspectives and recommendations from coordinators and instructors being coordinated with the goal of helping those who are new to coordination. We utilize Martinez et al.'s (2022) work on two coordinator orientations, Humanistic-Growth and Knowledge-Managerial, to frame the common perspectives of instructors and coordinators of first-year mathematics courses. Most participants gave reasonings for their decisions surrounding coordination with a healthy balance of Humanistic and Managerial considerations in mind.

Keywords: Course coordination, first-year mathematics, coordinator orientations

Research on course coordination has defined the ways in which perspectives, habits, and mindsets of coordinators can improve student and instructor experience and outcomes. In this paper, we synthesize the perspectives of mathematics course coordinators and instructors across one region, for the purpose of supporting (a) instructors and coordinators with little prior experience with course coordination, (b) math departments seeking to build/improve course coordination, and (c) for instructors and coordinators seeking to improve coordination.

The paper contributes to the research of course coordination by providing replication and expansion of Rasmussen & Ellis's (2015) work exploring coordination at universities with successful calculus programs and Martinez et al.'s (2022) work on coordinator orientations. There have been calls in the field for more replication studies (e.g., Melhuish, 2018; Melhusih & Thanheiser, 2018) and published replication studies (e.g., Borji et al., 2022; Melhuish, 2018) indicate that the field values replication studies. Our work replicates Rasmussen and Ellis' (2015) and Martinez et al.'s (2022) work by studying common perspectives of coordinators and instructors who work under coordinators within a regional but expanded context of first-year mathematics courses. These instructors' voices are critical for understanding what makes coordination successful.

Literature Review

As part of an extensive research project involving five PhD-granting universities known for having successful calculus coordination programs, Rasmussen and Ellis (2015) characterized aspects of effective course coordination in Calculus I. Their work highlights the role of one serving as a coordinator as someone who serves in a multi-year or semi-permanent position, is knowledgeable and respected in teaching, and who is oriented to serve both as a resource and facilitator. Rasmussen and Ellis describe the importance of coordination in which the course is viewed as community property with concrete actions taken to this end. Regular meetings are held to focus on the needs and priorities of the instructors, compare progress, share difficulties and ideas for problems, and develop common grading practices. Additionally, drawing on work in behavioral economics (Thaler & Sunstein, 2008), Rasmussen and Ellis identify the role of effective coordinators in serving as a choice architect - "someone who is responsible for organizing the context in which people make decisions" (p. 112), not someone who makes decisions for others. They identified the role of coordinators in (a) making life easier by setting default options (providing resources), (b) providing feedback to users, (c) making mappings easy to understand, and (d) providing information about what others are doing. "If many people do something or think something, their actions, and their thoughts convey information about what might be best for you to do or think" (Thaler & Sunstein, 2008; p. 54)

Martinez et al. (2022) extended this work to studying coordinator *orientations* or views that "give a sense of direction for what one does, implicitly or explicitly" (p. 330). In their analysis of interviews with course coordinators and instructors, Martinez identified two coordinator orientations: a Humanistic-Growth Orientation and a Knowledge-Managerial Orientation (referred to here as *Humanistic* and *Managerial* orientations for simplicity). Each orientation was characterized by five themes. The Humanistic orientation involves the following five themes: (a) attending to student experience, primarily in receiving the same opportunity to succeed, (b) being concerned about others and expressing concern for others, (c) seeking to build community as a form of professional support, (d) being attentive to instructor differences in that "one size does not fit all", and (e) providing instructor support through professional development opportunities and as a resource for general concerns. The Managerial orientation involves (a) drawing on knowledge of the course, department, and institution, (b) bringing knowledge and experience of a variety of pedagogical practices, (c) providing materials related to content and curriculum, (d) communicating clearly, end (e) exhibiting strong communication skills. Martinez et al. (2022) conjecture that "a combination of both coordinator orientations and their corresponding professional development approaches (community-based and material-based) is more likely to lead to substantive and sustained improvement" (p. 344).

Both Martinez et al.'s (2022) and Rasmussen and Ellis' (2015) results draw from an extensive data corpus (over 92 interviews and 95 hours of audio recordings) of programs with established and effective support for course coordination. This study serves as a smaller-scale replication study (Melhuish, 2018) in studying the ways in which regional perspectives regarding course coordination are consistent with broader theoretical perspectives. In particular, this paper explores a broader category of mathematics course coordination in which there may be less structure or departmental support for course coordination, yet a growing local community of support for course coordination within and across mathematics departments.

Research Questions

In this context of both structured and semi-structured course coordination, we ask the following research questions:

1. What activities do coordinators devote their time and attention to in coordinating a course?
2. What perspectives, habits, and mindsets do coordinators and coordinated instructors find most conducive to effective course coordination?
3. In what ways do perspectives shared regarding coordination embody theoretical perspectives of Humanistic and Managerial coordinator orientations?

Methods

The authors of this paper recruited 14 instructors teaching and/or coordinating courses from five medium to large-enrollment universities in Oklahoma. Participants were recruited through an initial survey of participants of their primary role as coordinator or instructor. Of these instructors, a first subgroup of five instructors also served as coordinators of courses with structured coordination. These instructors were involved in coordinating various aspects of their course throughout the semester. A second subgroup of four instructors served as coordinators of courses involving semi-structured coordination, where coordinators were part of a team that mostly made decisions before a semester as opposed to during it. A third group of participants involved instructors who have taught one or more coordinated courses with structured coordination.

The authors lead focus-group-style interviews in 6 separate sessions of 1-5 participants per session. Interviews were conducted on Zoom and recorded, running about 1.25 hours each. The authors recorded field notes of responses to each question. Recorded detailed interview field notes, along with transcriptions of targeted segments of the interviews were transcribed. The first author open-coded responses based on the intended context of the perspective and advice offered (Strauss & Corbin, 1998). After this, a second cycle of axial coding was used to segment comments by similar perspectives (Saldaña, 2013). A third round of coding involved identifying instructor perspectives, habits, and mindsets aligned with themes of Martinez et al.'s (2022) coordinator orientations. Finally, a cross-comparison across the second and third round of coding were analyzed.

Results

The following sections provide a summary of what coordinators and instructors shared regarding (a) the purpose and affordances of coordination, (b) the structure and roles of coordination, (c) the goals of coordination meetings, and (d) recommendations for coordinators and instructors in both structured and semi-structured coordination courses. Following this, we provide the cross-comparison of these four categories of perspectives with Martinez et al.'s (2022) coordinator orientations.

Purpose and Affordances of Coordination

One of the primary purposes cited by coordinators for developing a coordinated structure for a course was to increase consistency of course content and delivery. One coordinator cited that before coordination, "students had wildly different experiences." Across contexts of coordination, instructors described how coordination enables a range of instructors to "bring the rigor [of the course] more closely together." Coordinators also described coordination as a factor in improving the quality of course materials and design.

Instructors described the importance of coordination in providing resources for improving teaching and learning, as well as handling student issues. Coordination appeared to be a factor in increasing instructors' competence in teaching by providing support for instructional development and a community for discussing and handling issues of concern. One instructor stated, "If the course hadn't been coordinated, I would not have survived my first semester." Another instructor stated that the coordinated structure "helped me focus more on teaching [and] what kind of questions I was asking; was I allowing enough time to absorb the material?"

All instructors described having an increased sense of relatedness to other instructors because of the coordination. They said coordination provided a venue to talk and ask questions of other instructors which was very beneficial. One instructor said they felt very appreciated by their coordinator, but not by the department's administration. This instructor had worked hard to flip the class, but in the following semester, was placed in a corequisite section where he was not allowed to flip the class. In the face of this loss of autonomy, he said "I'm a creative person, but I cannot bring this to the students - what I want to bring to them will not work out." Regarding the increased structure and loss of control, another instructor said she felt like she still had enough autonomy, and another instructor said the added support of coordination afforded more autonomy to work on what she wanted.

Structure and Roles of Coordination

We acknowledge that coordination presents itself in a variety of ways, from the selection of a textbook and topics only, to more heavily coordinated courses where the vast majority of aspects are decided on by instructors. From the pre-interview survey, the coordinator participants indicated coordinating textbooks, topics, pacing, syllabi, exams, online homework, grading schemes, rubrics, written homework, and quizzes. They also cited responsibilities of instructor training, instructor observation, and feedback, fielding student complaints, instructor mentorship, instructor evaluation, setting up LMS, communicating with an accessibility resource office, administering makeup assessments, and scheduling exam rooms. In the interviews, several coordinators took particular concern for instructor training and mentorship for graduate students and new teachers. When concerns regarding teachers' instruction and behavior arose, one coordinator described these as opportunities for professional development and learning, assuring responsive instructors that they were not in trouble and that "this is all normal."

Coordinators who held regular coordination meetings spoke in detail about how they conducted these meetings since it was a primary form of communication between coordinators and instructors. Thus, the following section describes the goals coordinators held regarding meetings and how they approached them.

Goals and Approaches for Coordination Meetings

One of the primary ways of developing and carrying out coordination was through instructor meetings. While some coordinators preferred online communication and one-on-one meetings with instructors, the majority of coordinators focused substantial time and care on coordination meetings. Coordinators used coordination meetings weekly, biweekly, or monthly to (a) discuss course policies, (b) guide instructors, (c) provide training on the use of course materials or technology, (d) provide opportunities for professional support, and (e) distribute feedback. Some coordinators described the importance of using meetings to convey a vision or overarching story for the course that they wanted instructors to see and follow. Coordinators described using the start of meetings to check in, to "Look everyone in the eye and make sure they are doing what they are supposed to be doing. 'You all are in the same place? Good! Let's go on.'"

In their discussion, coordinators agreed that they wanted instructors to perceive meetings as necessary and helpful in meeting practical needs. To this end, they discussed the importance of planning and sending out an agenda prior to the meeting so that instructors can offer suggestions on what to add. However, meetings also furthered the humanistic goals of building relationships and providing mutual learning opportunities. One coordinator emphasized that she was "not 'the one' with the answers." Although the "young ones" were learning and observing what other instructors were doing, discussions often focused on sharing ideas, experiences, and suggestions in an environment where everyone could ask questions about the course and learn and buy into why the course was set up in the way it was. One coordinator emphasized being sensitive to people's time; if meetings are becoming off track or unproductive, they sought to address the cause individually or change the structure of the meetings.

Specific Recommendations

Advice for Coordinators. In addition to advice for meetings, coordinators and instructors alike shared advice directly related to organization, interpersonal skills, getting feedback and support, and beginning a new coordinator position. Instructors gave emphatic advice that coordinators work on organizational skills by building detailed lists of what tasks were important at what time of the semester. One instructor described how important it was to receive resources in a timely way so that they could have sufficient time to prepare for class.

Coordinators advised focusing on interpersonal skills, especially in valuing the people you are working with, and discerning when to give ground to instructors and when to hold your line. Some coordinators described the importance as a coordinator of staying humble, asking questions, and continuing to learn from other instructors and coordinators. One coordinator advised framing the coordinator structure of their course as beneficial to new instructors. She said, "I am doing this to make things easier for you, and make things fair for students."

Coordinators also emphasized the value of collecting feedback from instructors. One advised that "especially when you have inexperienced instructors who may chafe under authority, get administrators on board with this." One instructor found instructor evaluations helpful for seeking feedback on what they did well and what didn't, which spurred them to also seek feedback outside the formal system of evaluations.

Two coordinators described the complexity and burden of the responsibilities in their first year as coordinators, advising new coordinators to be careful about what they take on. One coordinator advised: "Come in and get your bearings first. Try it out the way it exists, if it's not broken." Another coordinator echoed this advice: "Learn what the prior person did, and just do it; wait to start making changes until after the first and second semester."

Advice for those working in semi-structured coordination. While the above perspectives and principles may apply to a broad range of contexts, several course coordinators offered advice, particularly for those working in contexts where structured course coordination doesn't yet exist. In seeking the benefits of coordination (aligning the content and rigor of courses, improving course resources, and increasing collaborative support), they shared some lower-stakes ways to add coordination into a course. One instructor suggested in addition to a common textbook and syllabus, providing suggested homework assignments, and problems that could be used as a guide to new instructors, but would not restrict established instructors. Another suggested collaboratively working on building a common final exam. One coordinator was chair of a committee of those who regularly teach a course and hold monthly meetings; she suggested that if instructors find meetings effective and helpful, they can lead to small but beneficial iterative changes in the course. Another coordinator suggested, "Don't be afraid to add one thing per semester. And then keep it up."

Advice for Instructors. Since coordination offers affordances and constraints that differ from teaching without coordination, coordinators offered the following advice for instructors teaching within a coordinated structure: "Don't mistake the coordinator structure of the course for needed preparation," and "Don't underestimate the amount of time needed to properly prepare."

When offering advice to those teaching a coordinated course for the first time, they emphasized asking questions and talking to other instructors. One instructor said, "It doesn't matter how many years you have been teaching, you will encounter things that will miserably fail." This instructor emphasized how, as new instructors, we forget how to explain the basics and are not used to commanding attention and creating rapport with students: "Anything that you don't know how to do, it is helpful to have a team to talk to."

Advice to both instructors and coordinators. One instructor noted that both instructors and coordinators share the common goal of working well with students clearly communicating expectations. Thus, he advised both instructors and coordinators to see themselves as a team and work as a team.

Cross-Coding of the Results with Coordinator Orientations

Of a total of 308 comments assigned specific codes, 292 spoke positively of various aspects of coordination, and 16 spoke negatively of the coordinated structure. Of these 16, seven comments negatively described the specific structure and roles of the curriculum in relation to student experience (e.g., regarding the instructor's loss of autonomy in being assigned by the department a corequisite section in which he was not allowed to flip his classroom). The remaining 297 comments were distributed as presented in Table 1, which are further discussed in the next section. Table 2 provides a legend of the coordinator orientation codes used in Table 1.

Table 1: Cross-coding of coordinator orientations with axial coding of positive perspectives (number of instances coded with a positive perspective of coordination)

		Coordinator Orientations											
		H1	H2	H3	H4	H5	M1	M2	M3	M4	M5	NA	Total
Axial Coding	Purpose & affordances	9	2	3	4	6	2	4	6	1	2	9	48
	Structure & roles	10	7	5	7	14	5	2	24	4	7	3	88
	Coordination meetings	4	5	8	5	13	0	2	3	1	4	4	49
	Recommendations	3	3	6	2	3	5	2	1	2	11	6	44
	NA (no cross code)	13	4	4	9	6	1	3	8	6	9		63
	Total	39	21	26	27	42	13	13	42	14	33	22	292

Table 2: Codes of Coordinator Orientations (Martinez et al., 2022)

Humanistic-Growth	Knowledge-Managerial
H1: Attends to student experience	M1: Draws on knowledge of the course, department, & institution
H2: Concerned about others	M2: Demonstrates experience & knowledge of teaching the course
H3: Takes action to build community	M3: Provides material related to content & curriculum
H4: Attentive to instructor differences	M4: Communicates clearly
H5: Provides instructor support	M5: Exhibits strong administrative skills

Discussion

The above cross-coding of common perspectives regarding coordination and Martinez et al.'s (2022) coordinator orientations highlight the following in regard to this particular sample of instructors and coordinators. In describing the purpose, affordances, structure, and roles of coordination, these instructors particularly valued orientations of attending to student experience (H1), providing instructor support (H5), and providing course materials (M3). When offering explicit advice for coordinators and instructors, participants were more likely to emphasize attitudes and behaviors reflecting the importance of coordinators having strong administrative skills.

The most frequent theme found in the cross-coding was the Structures & Roles theme coupled with M3: Provides material related to content & curriculum. This is unsurprising in that a primary requirement of coordination in the context studied is that materials of some kind are provided. What may be more interesting within the Structure & Roles theme, however, is the noticeable presence of Humanistic perspectives. The coordinators spent a significant portion of time discussing how their decisions affected the students' experiences and whether or not they supported their instructors. The key role of coordination in how it affects the student experience was about keeping the course fair for all students, regardless of which instructor they had. The coordinator's concern for supporting their instructors was evident in how they discussed teacher training, observations of classes, and management of assessments and the LMS.

Course coordination meetings have the potential to emphasize Managerial aspects of coordination; however, Table 1 shows that the participants viewed the purpose of meetings more in terms of Humanistic perspectives. The purpose of the meetings for the participants was more about support and building a community than it was about relaying materials and rules for the course. Both coordinators and instructors were eager to offer recommendations for future coordinators and instructors. Frequently, their advice involved administrative skills that a coordinator needs to have in order to be effective and they made practical suggestions like making explicit to-do lists for each semester.

In their experiences with coordination, instructors described, in varying degrees, gaining an increased sense of competence, autonomy, and relatedness for teaching. However, in one case, coordination constrained an instructor's sense of autonomy, and in turn their creative engagement and motivation for teaching. Thus, it can be especially important for coordinators to give attention to specific decisions that may impact instructors' sense of autonomy. While decisions of course assignments and scheduling decisions may never be perfect, fostering a team mindset can mitigate negative attitudes regarding loss of autonomy. This instructor's experience also highlights the importance of coordinators holding team meetings and building relationships in which instructors can express and feel supported in their concerns. In line with Martinez et al.'s (2022) conjecture that a combination of coordinator orientations is more likely to lead to improvements, the participants in this study described effective coordination as involving decisions led by both coordinator orientations.

Conclusion

Research serves to provide a context for instructors and coordinators with little prior experience with coordination for understanding how to understand the affordances and constraints of coordination and to quickly and effectively adapt efforts within this system. Therefore, one of the purposes of this work is to create a practical guide for those coordinating for the first time, as well as for those teaching a coordinated course for the first time. We hope that categorizing the participants' perspectives by coordinator orientations can help others with new coordination systems and navigating initial decisions.

Coordination can support first-time instructors to progressively focus their attention on where they can grow the most in their effectiveness as instructors, by focusing first on (a) their own roles and responsibilities as an instructor, then as questions/issues arise, (b) their role in relationship to coordination, and ultimately (c) their role in as part of a team of instructors delivering consistent and quality instruction. Coordination also supports coordinators to allocate energy most effectively at various stages in their service as coordinators. Especially in light of the cyclical nature of course-coordinator-responsibilities (pre-semester preparation, course implementation, post-semester reflection and evaluation, revision, repeat), these perspectives can help coordinators evaluate where their skills, expertise, and experience can help people the most.

Acknowledgments

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References

Martinez, A. E., Gehrtz, J., Rasmussen, C., LaTona-Tequida, T., & Vroom, K. (2022). Course coordinator orientations toward their work and opportunities for professional development. *Innovative Higher Education*, *47*(2), 327-346.

Melhusih, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*, pp. 9-38.

Rasmussen, C., & Ellis, J. (2015). Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus* (pp. 111–120). Washington, DC: MAA Press.

Saldaña, J. (2013). The coding manual for qualitative researchers. Sage, 84-147.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Thaler, R. H. & Sunstein, C. R. (2008). *Nudge: Improving decisions about health, wealth, and happiness.* New Haven, CT: Yale University Press.

Exploratory Factor Analysis for a Measure of Fundamental Instructor Behaviors for Inclusion in Undergraduate Math Classes

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A recent movement has developed to emphasize the importance of inclusive teaching practices in undergraduate STEM education. Mathematics courses have one of the largest impacts on students' persistence and success in STEM majors, making efforts to improve diversity and inclusion in these classes all the more important. In this paper, we discuss our initial steps to develop a measure of students' perceptions of instructor behaviors that promote inclusivity in their classrooms. This paper briefly discusses the initial development of items for our measure and focuses on our exploratory analysis of the latent factor structure and our process for defining these latent factors in the context of math instructors' actions.

Keywords: Mathematics, instructor behaviors, factor analysis, instrument development, inclusion

Introduction

Efforts to diversify STEM fields, both in academia and industry, will benefit from efforts to cultivate inclusive introductory STEM courses. Mathematics courses in particular are notorious for contributing to disparities as they are often required in order for STEM undergraduates to graduate (Ellis et al., 2016; Leyva et al., 2021, Battey et al., 2022). Multiple efforts to promote inclusive teaching practices in higher STEM education have tackled aspects such as curricula, classroom interpersonal relationships, and pedagogy. (Dewsbury & Brame, 2019; Hogan & Sathy, 2022; Inclusive STEM Teaching Project, 2023; Johnson, 2019; Kachani et al., 2020; Tanner, 2013). While these works highlight the importance of inclusion in STEM classrooms, we find little work done to identify and functionally measure aspects of inclusivity in post-secondary STEM classrooms. This report details our team's effort to develop an instrument that would capture students' perceptions of how their instructor's behaviors support or diminish inclusivity in undergraduate mathematics classrooms. For our work, we conceptualize inclusiveness as students' feeling that they belong in and can achieve success in the math class. Our current findings are part of a long-term instrument development project (a more detailed description is available in Brown et. al. (2023)). The aim of this project is to create an instrument that can be used primarily for research purposes in investigating relationships between students' feelings of inclusion and other student-level variables (sense of belonging, mathematical confidence, etc.) or for instructors' personal use in improving their teaching practice. Our instrument should not be used for institutional assessment of instructors as such an application is likely to introduce racial and gender bias, among other concerns. In this report, we present an abbreviated description of the exploratory process for developing the 28-item Fundamental Instructor Behaviors for Inclusion (FIBI) instrument followed by the details of the exploratory factor structure, our process of naming the latent factors, and concludes with a discussion of future work.

Methods and Data Collection

Instrument

Development of the FIBI instrument began with the creation of 79 items that started with the common stem "My mathematics instructor…". Each item required a response on a 5-point Likert scale (1-Strongly Disagree, 2-Disagree, 3-Neither agree nor disagree, 4-Agree, 5-Strongly Agree). We created these items based on students' responses during focus groups discussions of the following questions:

- What has your instructor done or said that conveyed a sense you can succeed in math?
- What have they done or said that conveyed a sense you can't succeed in math?
- What have they done or said that conveyed a sense you belong in math?
- What have they done or said that conveyed a sense you don't belong in math?

(Focus groups are more fully reported in Brown et. al. (2023).) In addition to these items, we included nine items from two published instruments for measuring teacher behaviors: the Teacher Communication Behavior Questionnaire (TCBQ; She & Fisher, 2002) and the Questionnaire on Teacher Interaction (QTI; Wubbels et al., 1991). We included those items which covered behaviors not specifically mentioned by our focus groups, but that we thought could also impact students' feelings of inclusion in their classroom. These items were modified to fit the sentence structure of our previous 79 items before being added to the instrument, bringing the total number of items to 88.

Data Collection

In the Fall 2021, we collected data using the 88 items as well as demographic questions at a large, predominantly white, research-intensive university in the Northeastern United States. We intentionally recruited students from introductory chemistry and biology courses but asked them to provide responses based on their current mathematics instructor. Our goal for this strategy was to reduce the students' potential fears/concerns that their mathematics instructor would have access to their responses and to encourage students to provide honest responses to the items. Participants in the study received extra credit (equivalent to 0.5% of final grade) in their biology or chemistry course for completing the study. A total of 1,276 students participated in the study.

Due to the predominantly white makeup of this university and our desire to center the voices of students underrepresented in STEM classrooms, namely women and persons excluded because of their ethnicity or race (PEERs; Asai, 2020), we implemented a secondary strategy to specifically recruit PEER participants from other universities via emails to university chapters of student organizations such as the National Society of Black Engineers (NSBE) and the Society for the Advancement of Chicanos/Hispanics and Native Americans in Science (SACNAS) as well as colleagues at other institutions. Participants in this wave of recruitment were incentivized with a raffle for Amazon gift cards (20 $50 prizes, five $100 prizes, and two $250 prizes). This recruitment led to an additional 149 responses, bringing the total number of participants to 1,425.

Following the data collection, we cleaned the data by removing participants who (a) did not consent to participate (n = 161), (b) did not complete all FIBI items (n = 72), (c) were not enrolled in a mathematics course in Fall 2021 (n = 260), and (d) any responses identified as duplicates via student emails (n = 62). Thus, 870 participants remained in our sample. While the majority of our sample identified as female (n = 551), we still had a lack of participation from students who identify as PEERs (21%; n = 185). Approximately 71% of participants (n = 614) identified with at least one of these two groups.

Methods

To conduct our exploratory factor analysis (EFA), we used two sets of software: SPSS (version 28) and R (version 4.2.0). Analysis of the data was completed using multiple factor extraction methods (principal components analysis and principal axes) as well as using both Pearson and polychoric correlations (Holgado–Tello et al., 2010) to investigate the impacts of these choices on our analyses and to understand the FIBI instrument's structure in more detail. Across these different design choices, we consistently used an oblique rotation method (promax) to allow for correlations between factors which we suspected from our prior work with student focus groups (Brown et. al., 2023). In addition to considering multiple methodological choices, we were also interested in any potential differences in the factor structure for the subsets of women, PEERs, and the combined subset of women and PEERs and thus, we repeated these analyses for each subset as well as the whole sample. Due to the limited sample size, analysis of the PEER subgroup failed to converge. However, the other three analyses produced generally similar results (similar factor structure, number of factors, primary loading of items, and factor correlations). Thus, only the results for the whole sample are reported here.

Our EFA relied on several heuristics in tandem to determine the appropriate number of latent factors to retain including (a) the Kaiser-Gutman criterion (eigenvalues greater than one), (b) parallel analysis (Horn, 1965), and (c) analysis of Scree plots (R. A. Johnson & Wichern, 2007). After using these heuristics to limit the number of factors to consider, we then eliminated any items which did not have a factor loading with a magnitude greater than 0.4 to any of the remaining factors (Stevens, 1992). Following these removals via benchmarks, we moved into more detailed item-to-item analysis where we investigated the Pearson and polychoric correlations between items with primary loadings on the same factor. In doing so, we identified clusters of items containing correlations with magnitude of 0.6 or higher and then selected representative items from each cluster to remain in the instrument while removing the others. (See Brown et. al., 2023) for our guiding principles for examining item clusters.) Our process of exploratory analysis and item reduction was iterative to ensure the stability of the measure. Thus, after each wave of item removal by item benchmarks and item-to-item analysis, EFA was rerun on the subset of items to ensure similar performance (factor structure—item assignment to factors—and cumulative variance explained) after the removal of those items.

Results

Exploratory Factor Analysis and Item Reduction

For our initial EFA containing all 88 items, the Kaiser-Gutman criterion suggested a 12-factor solution which explained approximately 63% of the total variance in responses while parallel analysis suggested a slightly less complex, 10-factor model which explained 56% of the total variance. Unfortunately, with such a large number of factors, our initial benchmark for removal (magnitude of the factor loading greater than 0.4) only flagged 9 items for removal. Due to our desire for a more parsimonious model, we considered the Scree plot of our factors, which suggested a model containing four to six factors, and thus we conservatively chose to include six factors. With this smaller subset of factors, our benchmark for factor loadings identified a total of 23 items for removal while still explaining approximately 55% of the total variance. After the removal of these items, we began analysis of item-to-item correlations and using our research team’s guiding principles we identified another 23 items for removal, leaving 42 items left after our first wave of EFA and item reduction.

After completing the first wave of analysis, EFA was re-run to confirm that items remained in similar clusters and the number of factors was as we expected after removing those items. Scree plot analysis now suggested a seven-factor model. However, the seventh factor contained only a single item while the other six factors remained thematically intact from our previous analysis. Thus, the item was removed, and we retained the six-factor structure which still explained approximately 55% of the total variance. Again, employing the factor loading benchmark of 0.4, we identified 7 additional items for removal which no longer loaded significantly onto any of the factors. Item-to-item analysis was used again to identify six final items for elimination, resulting in a 28-item instrument.

Once more, to ensure our factor structure was stable, we conducted EFA on the subset of 28 items. Again, we arrived at the same six-factor model which explained 55% of the variation in responses. All items remained in their initial factors from the previous wave, except for the item, “My mathematics instructor talks too fast.” Additionally, the item “My mathematics instructor gives exams which are hard to finish in the allotted time,” now had a factor loading of 0.339, which did not meet the original benchmark of 0.4. However, our research team believed that the item provided student perceptions that were not equally addressed by any other items in the instrument and thus chose to retain the item. The details of the factor structure for the final 28-item instrument are discussed in the following section.

Resulting Exploratory Factor Structure

Following our iterative EFA process followed by item reduction, we arrived at a six-factor solution containing 28 items and explaining approximately 55% of the total variation in responses. A summary of the variation explained by each factor can be seen in Table 1. Factor labels (Factor 1, 2, …) were assigned in order based on the proportion of variance explained by the factor in the 28-item EFA. After reducing the instrument to 28 items, we see that Factor 1 explains the most variation in responses (12.7%) with Factor 2 explaining a similar amount of variation (12.6%). We note that Factor 1 and Factor 2 contain strictly negatively worded or positively worded items respectively, but both emphasize the irritation/willingness that instructors may exhibit when it comes to engaging with students and their questions which may have resulted in both factors explaining essentially identical amounts of variation.

Table 1: Summary of 6-Factor Solution on 28 Item Instrument

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6
SS Loadings	3.55	3.52	2.74	1.95	1.92	1.59
Proportion of Variance Explained	0.127	0.126	0.098	0.070	0.069	0.057
Cumulative Variance Explained	0.127	0.253	0.351	0.421	0.490	0.547

Factor are labeled (Factor 1, 2, …) in order based on the proportion of variance explained in the 28 item EFA.

Unlike the other factors, Factor 3, which explained 9.8% of the variation in responses, was the only factor to contain both positively and negatively worded items. Item assignments as well as their factor loadings are provided in Table 2. For Factor 3, we can see that both negatively worded items associated with this factor, Q20 and Q21, have negative factor loadings while all other items have positive loadings, which is consistent with our expectation that negatively

Table 2: Primary Factor Loadings 6-Factor Solution

Question	Factor Loading	Question	Factor Loading
Factor 1: My math instructor is disrespectful to us sometimes.		Factor 4: My math instructor models the math-person archetype.	
Q28: Blames us when we don't understand something.	0.816	Q22: Gives overly complex or technical explanations.	0.680
Q5: Puts us down.	0.694	Q1: Teaches too fast.	0.601
Q6: Criticizes our answers to their questions.	0.761	Q12: Is too advanced to be teaching this course.	0.535
Q25: Smirks or laughs at our questions.	0.781	Q18: Gives exams which are hard to finish in the allotted time	0.339
Q26: Says the material is simple, easy, or obvious.	0.458	Factor 5: My math instructor doesn't prioritize teaching us.	
Q27: Seems annoyed, frustrated, or exasperated by our questions.	0.766	Q14: Doesn't manage class time well.	0.611
Factor 2: My math instructor is open and adaptable to us.		Q11: Makes mistakes which confuse students.	0.469
Q15: Makes sure all questions get answered before moving on.	0.831	Q13: Pays more attention to some students than others.	0.495
Q2: Explains things in different ways when we ask.	0.675	Q17: Takes too long to grade and/or return homework, quizzes and/or exams	0.534
Q3: Pauses to let us absorb material or formulate questions.	0.522	Factor 6: My math instructor does more than just lecture at us.	
Q9: Teaches to all students, not just those who've already seen this material.	0.562	Q19: Checks in on us when we work in groups.	0.758
Q10: Is helpful during office hours.	0.586	Q8: Encourages us to work together.	0.745
Q16: Makes us feel comfortable asking questions.	0.592	Q23: Provides supplementary materials like handouts or worksheets.	0.508
Factor 3: My math instructor isn't apathetic toward us.			
Q4: Jokes with the class.	0.858		
Q7: Talks enthusiastically about math.	0.699		
Q20: Has little or no enthusiasm for teaching.	-0.728		
Q21: Has little or no interest in students' success.	-0.410		
Q24: Engages in small talk with the class.	0.661		

Item labels (Q1, Q2,…) are the item order in the survey. The first item listed for each factor has the largest factor loading for the factor; subsequent items are listed in order by item label.

worded items would have a negative relationship with an overall positive factor. When naming the factors, we devised a positively framed name for this factor, incorporating the antithesis of Q20 and Q21 as themes within the factor. For example, we considered “an interest in student success” as a theme defining Factor 3 rather than the lack of interest.

Based on the item-factor associations presented in Table 2, our team took a collaborative approach to develop names to define our latent factors. To begin this process, each team member created their own name/phrase to describe each factor. Then, as a team, we discussed the proposed names/phrases, identifying consistencies between them and any potential weaknesses. Our aim was to create a unified name that could unite all items under a common theme rather than naming based on a few of the higher loading items. Thus, after creating a group name for a factor, each item was compared to the factor name to ensure that the name represented the item well. As we continued this process for multiple factors, our team decided to implement a common structure for each factor name to emphasize that these factors represent students’ perceptions of their instructor. Thus, each factor name is written using the same structure as the FIBI items: “My mathematics instructor …” This commonality is meant to provide FIBI users with six collective statements, from the students’ perspective, about their instructor’s behaviors that may be increasing/decreasing the students’ feeling of inclusion in their classroom. In this respect, we phrased the titles to not be a definitive judgement of the instructor, but rather a current understanding that the students have developed with respect to their own class. For example, for Factor 5, a few of our initial phrases defining the factor were: “poor time management” and “inattentive to teaching.” As we worked to incorporate both the issues of timing in Q14 and Q17 and attention to details in Q11 and Q13, our discussion turned to the idea of having priorities above providing quality instruction for students. While the factor could have been given a definitive title such as, “My mathematics instructor doesn’t know how to manage their time well,” we wanted to avoid the declaration that an instructor “doesn’t know” or “lacks” some ability or effort. Instead, we chose the title, “My math instructor doesn’t prioritize teaching us,” because it addresses the students’ current perception of their instructor’s lack of attention and time management skills through the lens of “having priorities other than teaching” which an instructor can then address without feeling as if they have been labeled as “inattentive”.

Table 3: Factor Correlations for 6-Factor Solution

	Factor 1 (–)	Factor 2 (+)	Factor 3 (+)	Factor 4 (–)	Factor 5 (–)	Factor 6 (+)
Factor 1 (–)	1	-0.61	-0.44	0.51	0.62	-0.33
Factor 2 (+)		1	0.71	-0.59	-0.58	0.65
Factor 3 (+)			1	-0.35	-0.31	0.6
Factor 4 (–)				1	0.51	-0.46
Factor 5 (–)					1	-0.43
Factor 6 (+)						1

(+)/(–) after each factor denotes whether the factor emphasizes positive/ negative impacts on inclusivity.

After providing names for each of our factors, we have three positively framed and three negatively framed factors. Based on previous work, we expected the factors of inclusion to be dependent on each other. Thus, we also investigated the correlations between our factors which are presented in Table 3. Of our six factors, three are focused on positive behaviors to support inclusion and the other three address negative behaviors that may inhibit feelings of inclusion in a mathematics classroom. So, we expected that factors addressing negative behaviors will have an inverse relationship with positively focused factors. Moreover, two factors that are both positively (negatively) focused were expected to have a positive association between the two factors. This relationship did indeed present itself in our data. Additionally, we note that the magnitude of the associations between factors ranged from 0.31 to 0.71, suggesting that all of our factors are moderately dependent on one another and thus the different aspects of students' perceptions of inclusive instructor behaviors are linked to each other.

Discussion

This paper addresses the beginning stages of development for the FIBI instrument, a measure of students' perceptions of inclusive instructor behaviors in undergraduate mathematics classrooms. Development of this instrument began with an emphasis on the inclusion of women and PEER students in defining inclusive practices through targeted focus groups which became the basis for items included in the FIBI instrument. After developing these items, we proceeded through an iterative process of exploratory factor analysis and item reduction to arrive at our current 28-item instrument which contains six factors, of which three describe instructor behaviors that promote inclusion and three describe behaviors that inhibit feelings of inclusion.

One limitation of our work is the lack of diversity present in our sample. Although we were able to obtain a majority of our responses from women, our sample did not contain enough students who identified as PEERs to assess the factor structure for this subgroup alone. Additionally, other underrepresented identities such as Indigenous, Native Hawai'ian, Pacific Islander, non-binary gender identities, are not well represented in our sample and we did not investigate sexual identity as an identifier for underrepresented students in mathematics class. It may be possible that students from these groups would identify alternative factors which we did not see in this study. However, for our future confirmatory work, we plan to make a more targeted effort to include more PEER participants and to understand any possible differences in the factor structure. Another limitation of our study comes from our focus on deriving items from the responses of students in our focus groups. Because many students have only experienced traditional, lecture-based classrooms, it is doubtful that they have experienced any teaching strategies that may promote feelings of inclusion. Hence, we have aimed our instrument at defining only "fundamental" behaviors that may be present in a wide variety of teaching styles.

While this paper has presented a basic factor structure for the FIBI instrument, future analysis is needed to confirm this factor structure. Thus, our future steps for instrument development include a secondary data collection to use for confirmatory factor analyses (CFA) to validate our factor structure. Additionally, to continue centering women and PEERs in defining inclusive practices, we plan to run a multiple groups CFA model to analytically compare the factor structures and factor loadings based on gender and race/ethnicity. Finally, the FIBI instrument must be compared to other measures of similar constructs to assess construct validity and we plan to investigate the relationship between FIBI factor scores and other constructs such as math confidence, belonging, as well as course grades. These next steps will be crucial for establishing the validity of the FIBI so it can then be used by instructors aiming to assess and improve feelings of inclusion in their math classrooms.

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References

Asai, D. J. (2020). Race matters. Cell, 181(4), 754-757

Battey, D., Amman, K., Leyva, L. A., Hyland, N., & McMichael, E. W. (2022). Racialized and gendered labor in students' responses to precalculus and calculus instruction. *Journal for Research in Mathematics Education*, 53(2), 94–113. https://doi.org/10.5951/jresematheduc-2020-0170

Brown, N., Hatfield, N., Lewis, K., Sine, A., Palmer, L., Workman, D., & Esan, O. (2023). Development of an instrument for fundamental instructor behaviors for inclusion in undergraduate math classes. Manuscript in preparation.

Dewsbury, B., & Brame, C. J. (2019). Evidence based teaching guide: Inclusive teaching. *CBE Life Science Education*. https://lse.ascb.org/evidence-based-teaching-guides/inclusive-teaching/

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PLOS ONE*, 11(7), e0157447. https://doi.org/10.1371/journal.pone.0157447

Hogan, K. A., & Sathy, V. (2022). *Inclusive teaching: Strategies for promoting equity in the college classroom* (First edition). West Virginia University Press.

Holgado–Tello, F. P., Chacón–Moscoso, S., Barbero–García, I., & Vila–Abad, E. (2010). Polychoric versus Pearson correlations in exploratory and confirmatory factor analysis of ordinal variables. *Quality & Quantity*, 44(1), 153–166. https://doi.org/10.1007/s11135-008-9190-y

Horn, J. L. (1965). A rationale and test for the number of factors in factor analysis. *Psychometrika*, 30(2), 179–185. https://doi.org/10.1007/BF02289447

Inclusive STEM teaching project. (2023). Inclusive STEM Teaching Project. https://www.inclusivestemteaching.org/

Johnson, K. M. S. (2019). Implementing inclusive practices in an active learning STEM classroom. *Advances in Physiology Education*, 43(2), 207–210. https://doi.org/10.1152/advan.00045.2019

Johnson, R. A., & Wichern, D. W. (2007). *Applied multivariate statistical analysis* (6th ed). Pearson Prentice Hall.

Kachani, S., Ross, C., & Irvin, A. (2020). 5 principles as pathways to inclusive teaching. *Inside Higher Ed*. https://www.insidehighered.com/advice/2020/02/19/practical-steps-toward-more-inclusive-teaching-opinion

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition and Instruction*, 39(1), 1–34. https://doi.org/10.1080/07370008.2020.1849218

She, H.-C., & Fisher, D. (2002). Teacher communication behavior and its association with students' cognitive and attitudinal outcomes in science in Taiwan. *Journal of Research in Science Teaching*, 39(1), 63–78. https://doi.org/10.1002/tea.10009

Stevens, J. (1992). *Applied multivariate statistics for the social sciences* (2. ed). Erlbaum.

Tanner, K. D. (2013). Structure Matters: Twenty-One Teaching Strategies to Promote Student Engagement and Cultivate Classroom Equity. *CBE—Life Sciences Education*, 12(3), 322–331. https://doi.org/10.1187/cbe.13-06-0115

Wubbels, T., Brekelmans, M., & Hooymayers, H. (1991). Interpersonal teacher behavior in the classroom. In *Educational environments: Evaluation, Antecedents, and Consequences*. (pp. 141–160). Pergamon Press.

Teaching Proofs in a Second Linear Algebra Course: A Mathematician's Resources, Orientations, Goals, and Continual Decision Makings

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In this research study, we employed Schoenfeld's (2010) theory of Resources, Orientations, and Goals (ROGs) to examine an instructor's (and co-author's) textbook in a second course in linear algebra. The course was proof-based and the instructor anticipated that many students would have difficulty. Having the knowledge of students' difficulties from a first course in linear algebra they created tailored resources with many hints and pedagogical attributes for their students. The instructor's goal was to encourage students to construct their own proofs to understand and learn the materials, hence promoting exploration, conjecturing, and proving mental mathematical actions.

Keywords: linear algebra, proof, second course, belief, pedagogy, knowledge of content and students, mathematical knowledge for teaching

Literature Review

Linear algebra is a key topic for many mathematics majors and other fields. The Linear Algebra Curriculum Study Group (LACSG) recommended that "at least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 45). The LACSG 2.0 recommends that mathematics departments offer a variety of second courses (e.g., numerical linear algebra) and include wider topics (Stewart et al., 2022). However, research on topics in second courses of linear algebra, which contain more abstract content, is rare. In addition, in a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors found that more research on how students make sense of linear algebra proofs are needed. Hence, in this paper, we focus our attention on linear algebra proofs. For example, Stewart and Thomas (2019) aimed to uncover linear algebra students' perceptions of proofs in a first course. The results revealed that many students expressed their need for understanding. Studies also showed that the number of new definitions which linear algebra students must learn to begin writing proofs is overwhelming and makes learning proofs difficult (Hannah, 2017; Britton & Henderson, 2009). Malek and Movshovitz-Hadar (2011) employed one-on-one workshops to examine the effect of using their Transparent Pseudo Proofs (TPPs) in teaching first-year linear algebra proofs. They found for non-algorithmic proofs, students with familiarities in the TPPs wrote more in-depth and satisfactory answers than students who learned proofs traditionally, however, both groups of students performed equally for algorithmic proofs. Uhlig (2002) developed a novel approach compared to the traditional Definition, Lemma, Proof, Theorem, Proof, Corollary (DLPTPC) to teach linear algebra proofs. He posed the following questions: "What happens if? Why does it happen? How do different cases occur? What is true here?" (p. 338). He believed "Such a WWHWT sequence of presentation quickly leads students to understand, construct, reason through, enjoy, and actually demand 'salient point' type proofs" (p. 338).

Further expanding on the ideas of proof education, Melhuish et al. (2022) synthesized 104 papers from a range of countries and methods to describe what is known about proof-based learning and what is still missing. In their exploration, they uncovered that two schools of thought existed for teaching proofs, lecture-based proofs and student-centered learning. They concluded that while student-based learning is promising, a large gap remained in "understanding of the theoretical mechanisms which specific instructor moves may encourage or scaffold student activity is only beginning to emerge" (Melhuish et al., 2022, p. 17). Attempting to understand the theoretical mechanisms of an instructor is not something that is entirely new despite not being well explored. Using Schoenfeld's (2010) Resources, Orientations, and Goals (ROGs) framework, Hannah, Stewart, and Thomas (2011) attempted to describe the instructors' overarching goals to present linear algebra topics to their students. The paper found that analyzing the instructor's goals proved useful in uncovering his decision making and how the instructor approached teaching the "big picture." These results validated the instructor's decision making as a useful tool for improving the teaching ability of the instructor. Furthermore, the feedback from the students in this case of reflective teaching was positive, reinforcing the idea that this analysis provided enrichment for the instructor and students.

Theoretical Framework

The theoretical aspects of this study are based on Schoenfeld's (2010) Resources, Orientations and Goals (ROGs). He claims that "if you know enough about a teacher's knowledge, goals and beliefs, you can explain every decision that he or she makes, in the midst of teaching" (2012, p. 343). By resources Schoenfeld focuses mainly on knowledge, which they define "as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (2010, p. 25). Goals are defined as what the individual wants to achieve. The term orientations refer to "dispositions, beliefs, values, tastes, and preferences" (2010, p. 29). Although, the theory was originally applied on school teaching, (e.g., Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sriraman, 2010), it also has applicability to teaching at university (e.g., Hannah, Stewart & Thomas, 2011; Paterson, Thomas & Taylor, 2011; Stewart, Troup, & Plaxco, 2018). In addition, as we are interested in understanding the teaching and learning of proof in a formal second linear algebra course, we approach this problem with the Mathematical Knowledge for Teaching framework (MKT) by Ball, Thames, and Phelps (2008). This framework identifies several distinct domains of knowledge that instructors must use when they teach mathematical content.

In particular, for this paper, we restrict ourselves to one domain: *knowledge of content and students (KCS).* According to Ball et al., (2008, p. 401):

> *[K]nowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. … Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

In this study, our goal is to network these theories by associating Ball's instructor anticipation KCS and Schoenfeld's instructor beliefs about student resources. The research question guiding this research is: What knowledge of content and students do experienced instructors leverage for teaching proofs in linear algebra?

Methods

This narrative study (Creswell, 2013) is part of a larger study working with a mathematician (the instructor and first author), whose research is in geometry and algebra, and his linear algebra students. In the rest of the paper, we shall refer to the mathematician as "the instructor." The research team consisted of the instructor, a mathematics education researcher in linear algebra education, and a graduate student who is working on linear algebra education. The course was self-contained, proof-based, and constructed the theory of formal linear algebra from basic set-theoretic assumptions. Incoming students all had experience with formal proofs in other courses. The author approached this course with the core pedagogical perspective that *students should have the opportunity to construct all proofs themselves*.

Students were supplied with a self-contained textbook written by the instructor that had all definitions, propositions, and reflections, carefully sequenced, that students would progress through during the entire term. Recognizing the difficulty of creating proofs on their own, the instructor supplied *scaffolding* (via hints, questions, and reflections) to support the students in the completion of the proofs without reducing the cognitive demand of the proofs themselves.

Class met twice a week for 75 minutes. Class time was split into three components: lecture (~15min), group time (~45min), and presentations (~15min). Throughout lecture, the instructor would recall concepts from the past class and introduce new ideas for that day's investigations. During group time, groups would work, while standing up at and writing on white boards, to construct (and convince one another) of proofs to that day's propositions. During this time, the author would walk among the groups, attempting to facilitate discussions and provide hints for groups that were stuck. Towards the end of this time, the author would recruit groups to share their thinking for their proofs. During presentations, groups would present their proofs (complete or incomplete) to the class for discussion.

Each week, students were then required to LaTeX up and submit all proofs in their "course journal," a single cumulative document. Students were not allowed to look up proofs from other resources, but they were encouraged to work with classmates outside of class.

Data collection and analysis

We collected data from the textbook created by the instructor, videos of the instructor during lecture, videos of the students presenting, videos of student board work, and the final course journal submissions for each student. In this paper we only analyzed the instructor's textbook. The team wanted to look at a small sample and identify structures of linear algebra proofs. The members of the team collectively decided to analyze Investigation 12 (Linear Combinations and Span), both because it related to previous work (Madden et al 2023) and the team determined it was the first instance of propositions which contained sophisticated linear algebraic reasoning. The team then determined it was beneficial to also analyze Investigation 9 (Subspaces), as Investigation 12 frequently referenced it. The research team individually analyzed the data, identified emergent themes, and then the team met to share, discuss, and settle upon identified themes. The team was especially interested in the hints. The team continually consulted with the instructor to gain insights into his knowledge and beliefs that led him to make the decisions that were observed in the investigations. In this paper, we will only analyze the instructor's beliefs.

Results and Discussion

We analyzed two investigations on subspaces and span that the instructor provided his students (see Figures 2, 3, and 4). These investigations, as well as relevant references to earlier investigations, are provided below (see also Figure 1). These investigations were created by the instructor prior to his students using them. As such, these investigations contain a significant amount of information about the instructor's knowledge and beliefs about the content, his students, and teaching the subject, as well as his goals.

Instructor's Knowledge and Beliefs About Learning Proof in a Student-Centric Course

The instructor has taught a proof-based second course in linear algebra many times, and based upon those experiences, wrote his own self-contained textbook that he based his course upon and that he distributed to his students. As a consequence of his experiences teaching such a course, together with his beliefs concerning how people learn, the instructor came to hold certain beliefs regarding the structure of such a second course, and which caused him to make certain teaching decisions regarding the nature and structure of his textbook. Here are some of the instructor's beliefs and corresponding teaching decisions.

Belief (B1). Students should learn formal linear algebra in a student-centric course, which would include that the students should have the opportunity to construct all proofs themselves. Belief B1 resulted in the teaching decision (TD1): he created a self-contained textbook that contained no proofs, but rather a collection of carefully sequenced propositions and reflections.

Belief (B2). Teachers play an important role in student learning by (1) providing students with cognitively demanding problems and (2) helping students access knowledge and heuristics to solve the problem. In this context, the problems provided were proofs and Belief B2 resulted in teaching decision (TD2): for each proof, anticipating where students would struggle, he carefully wrote hints, which typically included a suggestion for the type of proof to pursue, the definitions to retrieve, and some intermediate results to prove. This can be observed in Figure 4 in the hints to the proofs of Propositions 4.48 and 4.50.

Belief (B3). Central to the story of a proof-based second course in linear algebra is (1) a solid understanding of linear algebra for the object $\mathbf{R}^n$, and (2) continual enactment of structural abstraction (Tall, 2013). Belief B3 resulted in the teaching decision (TD3): prior to a (formal) object being constructed, the antecedent object in $\mathbf{R}^n$ was revisited and discussed, as was explicitly the action of structural abstraction. The terminology of structural abstraction was introduced early in the textbook (see Figure 1) and then continually used throughout the term.

> A fundamental mathematical action is **structural abstraction**, in which we take a *specific* object within a *specific* context, observe its meaningful structures, and then construct an abstract object endowed with these structures, no more and no less. We then may mentally act upon this new abstract object, bringing it to life as a new mathematical entity, of which our original object is now a mere example.

Figure 1. Excerpt from Investigation 3 of the instructor's textbook.

Teaching decision 3 (TD3) can be seen having been implemented in the introduction to subspaces (see Figure 2).

> **Investigation 9**. **Subspaces.** Once again, think back to your first course in linear algebra. We were constantly looking at spans and solving matrix equations. Geometrically, we were looking it, lines, planes, and more generally, k-planes, through the origin. We now perform a structural abstraction on these objects, which we now formalize with subspaces.
>
> **Definition 4.24.** If V is a vector space over the field F, then a **vector subspace** of V is a subset $W \subset V$ satisfying:
>
> **(S1)** W is nonempty,
> **(S2)** W closed under addition: if $\mathbf{w}_1, \mathbf{w}_2 \in W$, then $\mathbf{w}_1 + \mathbf{w}_2 \in W$,
> **(S3)** W is closed under scaling: if $a \in F$ and $\mathbf{w} \in W$, then $a\mathbf{w} \in W$.
>
> I will often use the symbol W for a subspace.

Figure 2. Excerpt from Investigation 9 of the instructor's textbook.

Instructor's Knowledge and Beliefs About the Learning of Mathematical Mental Actions

Belief (B4). There are many distinct types of mathematical mental actions, including proving, conjecturing, generalizing, abstracting, modeling, and interpreting (Harel, 2008). It is a goal of the instructor that in a formal course such as this, students internalize these actions.

Beliefs (B5-B7) of the instructor is that students (1) come into such a course not being explicitly aware of such mental actions, (2) students are more likely to internalize such actions if these actions are explicitly named and discussed, and (3) students are more likely to internalize such actions if they have opportunities to engage in them.

Reflection 4.30. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Carefully explain your thinking.

Proposition 4.31. *If F is a field, $n \in \mathbb{Z}_+$, and $i \in \{1,2,\dots,n\}$, then the set* $W_i := \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in F^n \;\middle|\; v_i = 0 \right\}$ *is a vector subspace of F^n.*

Generalization is another important mathematical action we perform, distinct from, but related to structural abstraction. See, for example, the work of Harel and Tall, The General, the Abstract, and the Generic in Advanced Mathematics. Roughly speaking, **expansive generalization** occurs when you *expand the applicability* of an object or process with which you are already familiar.

Reflection 4.32. A common example of expansive generalization is given by the movement from linear combinations of vectors in $\mathbb{R}^2$ to linear combination of vectors in $\mathbb{R}^n$. Carefully explain the process of finding linear combinations in $\mathbb{R}^2$. Then, explain how this familiar process is expanded to $\mathbb{R}^3$, and eventually, $\mathbb{R}^n$. Explain how this fits the criteria for expansive generalization.

Reflection 4.33. Compare and contrast expansive generalization with structural abstraction. You may find it useful to use as examples, $\mathbb{R}^2 \to \mathbb{R}^n$ for expansive generalization and $\mathbb{R}^n$ to an abstract vector space V for structural abstraction.

Explore-Conjecture-Prove 4.34. Using expansive generalization upon the subspaces $W_i \subset F^n$ in Proposition 4.31, find, clearly conjecture, and then verify a more general collection of subspaces of F^n.

Figure 3. Excerpt from Investigation 9 of the instructor's textbook.

The course is built around proving, however, the instructor identified and highlighted other mental actions where he could. In this instance (Figure 3), the instructor made the teaching decision (TD4) to have students reflect upon and practice generalizing and conjecturing. This sequence comes at the end of the initial investigation into subspaces. Here, in Proposition 4.31, students are initially asked to prove that the subset W_i of F^n, where the ith entry is zero, is a subspace. The instructor provided a hyperlink to a mathematics education resource to reflect upon expansive generalization by Harel and Tall (1989) and then asked students to use expansive generalization to conjecture and then prove new propositions (see Figure 3). More precisely, students are asked to reflect upon the method in which they proved Proposition 4.31, and then in Explore-Conjecture-Prove 4.34 to determine other subsets that are subspaces. Here students were free to consider many generalizations, the two most obvious being: (1) if the ith entry is held a different (nonzero) constant (which will not be a subspace) or (2) if more than one entry is held to be zero (which is a subspace).

Investigation 12. **Linear Combinations and Span.** One core idea of linear algebra is *mixture*. Vector space axioms were were precisely the structures necessary to allow for mixtures, which we describe formally as linear combinations.

Throughout this investigation, V will denote a vector space over the field F.

Definition 4.45. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$, then a **linear combination** of S is an expression of the form

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n,$$

where $x_1, x_2, \dots, x_n \in F$.

Note that we only take *finite sums* in linear combinations.

Definition 4.46. The **trivial** linear combination is the one with all 0 scalars. Otherwise it is **nontrivial**.

Definition 4.47. The **span** of S, denoted $\text{span}(S)$, is the set of all linear combinations of vectors in S. (Note that S can be infinite, but the combinations are finite.) In symbols,

$$\text{span}(S) := \{x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \mid \mathbf{v}_1, \dots, \mathbf{v}_n \in S \text{ and } x_1, x_2, \dots, x_n \in F\}.$$

Sometimes if $\mathbf{v} \in V$, then I will write $\text{span}(\{\mathbf{v}\})$ as $F\mathbf{v}$ which reminds us that the span of a single vector is just the line created by all possible scalings of that vector.

Proposition 4.48. *If $S \subset V$ is nonempty, then* $\text{span}(S)$ *is a vector subspace.*
(Hint: Try a direct proof. Show that $\text{span}(S)$ satisfies (S1)-(S3). To show (S2), you will need to show that, if $\mathbf{w}_1, \mathbf{w}_2 \in \text{span}(S)$, then $\mathbf{w}_1 + \mathbf{w}_2 \in \text{span}(S)$. To do this, you need to interpret and unravel definitions. Similar for (S3). You may find Definition 4.47, Definition 4.45, Definition 4.24, and Definition 4.1 useful.)

A useful alternative characterization of span is that it is the *smallest* subspace containing S, which is made precise by the following proposition.

Lemma 4.49. *If $S \subset V$ is nonempty, $W \subset V$ is a subspace, and $S \subset W$, then* $\text{span}(S) \subset W$.
(Hint: Try a direct proof. You need to show that any linear combination of vectors in S is contained within W. You may find Definition 4.47, Definition 4.45 and Definition 4.24 useful.)

Proposition 4.50. *If $S \subset V$ is nonempty, then* $\text{span}(S)$ *is equal to the intersection of all subspaces of V which contain S.*
(Hint: Try a direct proof. Let $\mathscr{A}$ denote the collection of all subspaces of V containing S. In symbols, $\mathscr{A} = \{W \subset V \mid W \text{ is a subspace and } S \subset W\}$. Then you want to show $\text{span}(S) = \cap_{W \in \mathscr{A}} W$. You will need to show both inclusions. To show that $\cap_{W \in \mathscr{S}} W \subset \text{span}(S)$ is immediate (why???). To show the other inclusion, observe S is in each W for which $W \in \mathscr{A}$, and deduce $\text{span}(S) \subset W$ (how???). You may find Proposition 4.48, Lemma 4.49, and Definition 2.5 useful.)

Figure 4. Excerpt from Investigation 12 of the instructor's textbook.

Instructor's Knowledge and Beliefs About the Learning of Span

The instructor identified two characterizations of the concept of the span of a set of vectors S, span(S), as we now outline.

The first conception of span is a "bottom-up" construction. One first conceives of a linear combination. Based on the instructor's APOS (Dubinsky & McDonald, 2001) theoretical knowledge, the action of computing an explicit linear combination of vectors in $\mathbf{R}^n$ for small n, is a task that is introduced early in a first course in linear algebra. Through repeating this action, a student may internalize this action to create an internalized procedure. Eventually the student may encapsulate this procedure as a single object on its own. Only once a student has internalized the notion of a single linear combination as an object can the students conceive of many linear combinations of S at once. The student is then asked to conceive of not one, nor two nor three, but all linear combinations at once. They are asked to hold within their mind the infinitude of all such objects. That is span, built from the bottom up.

The second conception of span is a “top-down” construction. Here, one first conceives of a subspace. Unlike the above pathway, a subspace begins life as a formal object, defined in terms of properties. Furthermore, instantiations of this formal object are themselves typically infinite and there is no finite construction in the way there is for a linear combination. Once a student has internalized the formal object of subspace, the student can then begin to conceive of many subspaces at once. The student is then asked to conceive of not one, nor two, nor three, but all subspaces containing S at once. They are asked to hold within their mind the infinitude of all such objects and conceive of their intersection. That is span, built from the top down.

Both approaches, linear combination and subspace, are mathematically advantageous in appropriate contexts. For example, the former is useful for computation, while the latter is more tractable for formal proofs. The mathematician uses both. The student typically only confronts the first conception of span in their first course.

Concluding Remarks

This study examined a linear algebra instructor’s textbook through the lens of ROGs, and, in particular, the instructor’s beliefs and their knowledge of content and students. The instructor’s textbook is a product, an accumulation of a sequence of numerous teaching decisions, motivated by their beliefs and in line with their goals to help students succeed in understanding linear algebra, to promote mental mathematical actions, namely, exploration, conjecturing, and proving.

When tasked with problem solving, the instructor had knowledge from his experiences as a mathematician of the importance of being able to sort through and retrieve relevant knowledge (in particular, definitions and propositions) in the context of proving. Furthermore, from his experiences teaching, as well as familiarity with mathematics education literature, the instructor anticipated that students would significantly struggle (to the detriment of their learning) with the spontaneous retrieval of relevant definitions and propositions. The instructor therefore made the teaching decision to include these references in the hints. Anticipating the complexity of certain full proofs, the instructor made the decision to recommend specific intermediate results in the hints. Based on their beliefs as a mathematician and their knowledge of mathematics education literature and teaching experiences, the instructor made the decision to make pedagogical content knowledge available to their students. We question, what impact engagement with mathematics education literature might have on student learning (metacognition)?

The instructor got the impression that the hints, in harmony with their other teaching actions, had some long-term impact upon (1) students’ understandings of linear algebraic concepts and (2) how students approach proofs. Certain common definitions and propositions, such as the definition of subspace (Definition 4.24 in Figure 2) were invoked often throughout the term, and the students seemed to internalize them and speak of them fluently. After the course, students told the instructor that they learned that a well-written proof should include proper invocation of definitions and propositions.

Currently we are in the process of analyzing students’ data, for example, the course journals, and exit surveys. The team intends to look at the structure of students’ proofs, and how they constructed their proofs from the provided hints/resources. Without a doubt, proof is an important component of an abstract second course in linear algebra and more research on instructors and students is needed.

References

Aguirre, J., & Speer, N. M. (2000). Examining the relationship between beliefs and goals in teacher practice. *Journal of Mathematical Behavior*, *18*(3), 327–356.

Ball, L. D., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*, 59(5), 389–407. https://doi.org/10.1177/0022487108324554

Briton, S. & Henderson, J. (2009). Linear algebra revisited: an attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 963-974.

Carlson, D., Johnson, C. R., Lay, D. C., & Porter, A. D. (1993). The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. *The College Mathematics Journal*, *24*(1), 41–46. https://doi.org/10.2307/2686430

Creswell, J. W. (2013). *Qualitative Inquiry and research design: choosing among five approaches*, (3rd ed). SAGE.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Krichgraber, J. Hillel, M. Niss & A. Schoenfeld (Eds.), The teaching and Learning of Mathematics at University Level: An ICMI Study (pp. 273-280). Dordrecht: Kluwer Academic Publishers.

Hannah, J. (2017). Why Does Linear Algebra Have to Be So Abstract? In S. Stewart (Ed.). *And the Rest is Just Algebra* (pp. 205-217). Springer, Cham.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2011). Analysing lecturer practice: the role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, *42*(7), 975-984.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In B. Gold & R. Simons (Eds.). *Proof and other dilemmas: Mathematics and Philosophy*, Washington, DC: Mathematical Association of America, pp. 265-290.

Harel, G., & Tall, D. O. (1989). The general, the abstract and the generic in advanced mathematical thinking. *For the Learning of Mathematics*, *11*(1), 38-42.

Madden, A., Stewart, S., & Meyer, J. (2023). A Linear Algebra Instructor's Ways of Thinking of Moving Between the Three Worlds of Mathematical Thinking Within Concepts of Linear Combination, Span, and Subspaces. In Cook, S., Katz, B. & Moore-Russo D. (Eds.). (2023). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 761-769). Omaha, NE.

Malek, A. & Movshovitz-Hadar, N. (2011) The effect of using Transparent Pseudo-Proofs in Linear Algebra, Research in Mathematics Education, 13:1, 33-58, DOI: 10.1080/14794802.2011.550719

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, 67, 100986.

Paterson, J., Thomas, M. O. J., & Taylor, S. (2011). Decisions, decisions, decisions: What determines the path taken in lectures? *International Journal of Mathematical Education in Science and Technology*, 42(7), 985-996.

Schoenfeld, A. H. (2010). *How we think. A theory of goal-oriented decision making and its educational applications*. Routledge: New York.

Schoenfeld, A. H. (2012). How we think: a theory of human decision-making, with a focus on teaching, *Proceedings of ICME 12*. Korea, Seoul.

Stewart, S., Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: Themes from recent research and evolving research priorities. *ZDM Mathematics Education*, *51*(7), 1017-1030.
Stewart, S., Axler, S., Beezer, R., Boman, E., Catral, M., Harel, G., McDonald, J., Strong, D., & Wawro, M. (2022). The linear algebra curriculum study group (LACSG 2.0) recommendations. *The Notices of American Mathematical Society*, 69(5), 813-819.
Stewart, S., & Thomas, M. O. J. (2019). Student perspectives on proof in linear algebra. *ZDM Mathematics Education*, Springer, *51*(7), 1069-1082.
Stewart, S., Troup, J., & Plaxco, D. (2018). Teaching Linear Algebra: Modeling one instructor's decisions to move between the worlds of mathematical thinking. In Weinberg, A., Rasmussen, C., Rabin, J., Wawro, M., & Brown, S. (Eds.) *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1014-1022). San Diego, CA.
Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*, Cambridge University Press.
Thomas, M. O. J., & Yoon, C. (2011). Resolving conflict between competing goals in mathematics teaching decisions. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 241-248), Ankara, Turkey.
Törner, G., Rolke, K., Rösken, B., & Sririman. B. (2010). Understanding a teacher’s actions in the classroom by applying Schoenfeld’s theory teaching-in-context: Reflecting on goals and beliefs. In. B. Sriraman, & L. English (Eds.), *Theories of mathematics education, Advances in Mathematics Education* (pp. 401-420). Berlin: Springer-Verlag.
Uhlig, F. (2002). The role of proof in comprehending and teaching elementary linear algebra, *Educational Studies in Mathematics*, 50(3), 335-346.

“Really, you’re a math major?!”: Students’ Descriptions of Racial and Gendered Microaggressions and Sense of Belonging in Mathematics

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Microaggressions (MAs) are intentional or unintentional messages that communicate hostile, derogatory, or negative messages towards a recipient (Sue et al., 2007). MAs that students receive in a math class can impact a students’ learning experience and can often lead to feelings of exclusion (Cawley et al., 2023). This paper expands on previous work, discussing two types of MAs—racial and gendered—while also discussing students’ overall sense of belonging in a math classroom. This study analyzes the reflections of 133 undergraduate math students who were asked to reflect on an article about mathematical MAs (Su, 2015). Findings show that a majority of students have felt like they do not belong in the math classroom, and that racial and gendered MAs contribute to this. This research supports the need to develop initiatives at departmental and institutional levels to encourage more inclusive spaces in math classrooms.

Keywords: Microaggressions in the Mathematics Classroom, Sense of Belonging

There is much research related to microaggressions, and their impact within society. *Microaggressions* (MAs) have been characterized as the intentional or unintentional forms of insulting, disrespectful communications that occur during everyday exchanges (Yang & Carrol, 2018). These indignities can communicate “hostile, derogatory, or negative” messages that target a person and/or their marginalized group (Sue et al., 2007). The concept of *MAs* was introduced by Pierce and colleagues in the 1970’s to describe “subtle, stunning, often automatic, and non-verbal exchanges which are ‘put downs’ of Blacks by offenders” (Pierce et al., 1977, p.65). While these offenses themselves can be innocuous, they have a cumulative effect on the victim. Pierce felt that these MAs were essential to understanding how African Americans experience racism (Pierce, n.d.). While MAs were founded to describe experiences that Black people faced within society, these have expanded to other racial and ethnic groups such as Latine[1], Asian Americans, and indigenous people. In recent years other types of MAs that people have experienced have been identified by researchers to include women, persons with disabilities, ethnic and religious minority groups, and LGBTQ people (Nadal, 2011).

MAs have been studied within education to understand their effect on a students’ sense of belonging. Sense of belonging pertains to a person’s belief that they are an accepted member of an academic community, whose presence and contributions are valued (Good et al., 2012). The experience of mattering or feeling cared about, accepted, respected, valued by, and important to the classroom and campus community define ways that students feel a sense of belonging (Strayhorn, 2018). When thinking of students within a university context, sense of belonging can also include the perceived social support a student has while pursuing their degree. Steele (1997) implied that it is important for students to feel a sense of belonging to a domain, in other words, an area of study like math or STEM. Yet, Steele also highlighted how societal barriers, such as stereotypes around race or gender, actively reduce feelings of being accepted or valued. MAs have been well reported as widespread in academic spaces and detrimental to student outcomes

[1]*Latine* is a gender-neutral replacement of the term Latino. It has been used as a more linguistically natural alternative to Latinx or Latin@ for Spanish-speakers (Celis Carbajal, 2020).

(Sue et al., 2009). This paper focuses on three types of MAs that STEM students face—racial, gendered, and mathematical—which we discuss in more detail below.

Sue and colleagues (2007) defined *racial microaggressions* as "brief and commonplace daily verbal, behavioral, and environmental indignities, whether intentional or unintentional, that communicate hostile, derogatory, or negative racial slights and insults to the target person or group" (p. 271). Solórzano and colleagues (2000) revealed that MAs were prevalent in classrooms and inhibit students' sense of belonging, especially for students coming from marginalized groups. Nadal and colleagues (2014) reported that racial MAs negatively impact student self-esteem and self-worth which in turn can impact mental health and student achievement. Students' reports of racial MAs within STEM departments are distinctive, which is problematic as many STEM departments already face issues in representation from marginalized communities (Burke, 2007; McGee, 2016). Students of color, especially Black students, report racial MAs from STEM instructors, advisors, and peers (Lee et al., 2020). Specifically focusing on STEM contexts, Marshall et al. (2021) noted that MAs especially impact students who have been historically excluded on the basis of race/ethnicity.

Gendered microaggressions are defined as "nuanced and brief everyday exchanges that communicate sexist denigration and slights toward women, which can be conveyed verbally and/or nonverbally through expression, gaze, and other gestures (Yang & Carroll, 2018). Rainey et al. (2018) found that women more often lacked a sense of belonging in STEM and left the major at higher rates than men, citing reasons such as feeling socially different or feeling like they did not fit in. Sekaquaptewa (2019) shared that while receiving a gendered MA can greatly impact a person, witnessing other women receive MAs can also have a negative impact on a student. Intersections of gender and race create a nuanced space where women of color experience an even more elevated exposure to MAs during their college experience (Lewis et al., 2013; McGee & Bentley, 2017).

As an extension from discussions of racial MAs, Su (2015) used the term *mathematical microaggression*, which refers to the subtle ways that mathematical authorities (such as instructors, classmates, or textbook authors) communicate that one does not belong in math. Su offered examples such as "It is obvious/clear/trivial that…" and "The rest is just algebra," and elaborated that such comments can convey negative messages towards students (e.g., their knowledge is lacking, their questions are unwelcome, their potential in math is limited). Cawley and colleagues (2023) documented three different types of mathematical MAs students experienced, which included microslights, microinsults, and environmental MAs. They found that most students did experience a mathematical MA, and that these were often received from both their instructors and peers.

Our study focuses on the types of racial or gendered MAs that students share they have experienced in their math learning experiences. Specifically, this paper focuses on the following research questions: 1) How do math students describe experiencing racial or gendered microaggressions? 2) How have math students been made to feel like they do not belong in a math class?

Methods

This study took place at a large public university on the West Coast of the U.S., designated as a Hispanic-Serving Institution. Data for this paper were collected between Fall 2019 and Spring 2022. The sample included 133 participants enrolled in calculus 1 (46 students) or abstract algebra (87 students), taught by the same instructor (Author 2). Course modalities included both in-person and virtual instruction. These classes incorporated inquiry-based learning and active learning and provided a significant amount of time for student collaboration and discussion during instruction. Based on institutional data of the 133 students[2], 46% were female, 54% were male, and one student was non-binary. Table 1 shows the racial/ethnic demographics of the students in the study. Students in the study were mostly STEM majors.

Table 1. Student Demographics

	Female (N = 61)		**Male (N = 72)**		**Non-Binary (N = 1)**		**Total (N=134)**	
	Total	%	Total	%	Total	%		
Latine	33	55%	28	38.9%	1	100%	62	46.6%
Black	1	1.6%	5	6.9%			6	4.5%
Asian	10	16.6%	15	20.8%			25	18.7%
Native	0	0%	1	1.4%			1	0.7%
White	11	18.3%	10	13.9%			21	15.8%
Mixed Race	1	1.6%	3	4.2%			4	3.0%
Unknown	4	6.6%	10	13.9%			9	6.7%
Total	**60**		**72**		**1**		**133**	

Throughout the course, students were asked to submit reflection assignments. In one of these assignments, students were asked to read Su's (2015) article on mathematical MAs, to reflect on the paper, and to respond to the following prompt: Have you ever been made to feel like you don't belong in a math class? Students could reflect on any moments in their math experience. The 133 reflections form the data set for this study.

The data were analyzed using constant comparative methods (Corbin & Strauss, 2008). In the first round of coding, all de-identified reflections were coded to identify if the student had explicitly written if they had ever been made to feel like they do not belong in the math classroom. We categorized it as 1) yes, they have been made to feel they do not belong, 2) no, they have never been made to feel like they do not belong, and 3) does not say. After reviewing the data, we completed a second round of deductive coding to identify MAs that students experienced in STEM, specifically mathematical, racial, and gendered MAs. We marked a MA as present if a student wrote about an incident that related to any of the three types of MAs. The authors coded the journals in sets of 20, meeting to discuss coding to ensure agreement.

Findings

This section is separated into two parts. First, we will share about how students wrote about racial and gendered MAs. 21 students discussed racial MAs while 13 students wrote about experiencing a gendered MA. Seven students discussed how both racial and gendered MAs impacted them. Then, we discuss the ways that students have shared whether they have/have not been made to feel like they do not belong in a math classroom.

Racial Microaggressions

21 students discussed racial MAs in their writing. The racial identities of the students included Latine, Asian American, Black, and unknown race/ethnicity. Seven students explicitly

[2] Because racial- and gender-identity data were not collected during the course each semester, the authors used institutional data provided for each student to identify race/ethnicity and sex. This may not properly reflect the students' gender-identity or racial/ethnic identity.

mentioned how they related to Su's experience as an Asian American in Texas explaining scenarios that were similar to those that Su shared about himself. Each student shared some sort of experience that affected their sense of belonging. Most of these racial MAs represented experiences in the math classroom, while others reflected general experiences that they had.

A majority of the responses were written by Latine and Asian American identifying students. Three of these students expressed comments that reflected the Model Minority Stereotype (Chou & Feagin, 2008), indicating that they have experienced the notion that because they are Asian American, they should be good at math. Some students' writing reflected experiences they faced with MAs both inside and outside of the classroom. For example, Alan, an Asian American math major described many ways his intelligence was assumed based on his race.

> Math aside, microaggressions suck. One thing I've heard until I was a senior in high school was, "oh you're Asian, you should [just] be good at math." And lately (as of 2020) as a mathematician among engineers, my new boss has said "you're a math major, you should be able to do [algebra] in your head." Albeit that these examples aren't exactly like Su's examples, hearing these statements made me feel anything but a positive emotion.

Alan acknowledged how these comments negatively affected him, appearing to follow him in many aspects of his schooling and career. Another student, Lucia, a Latine female math major shared an experience where being a math major was questioned.

> I remember speaking to a man about where I go to school and what major I was studying. I don't know if it was because I was in my Domino's Pizza uniform, my ethnicity, being a woman, or a combination of all three, but the man's response was "really?! You go to [university name]? You're a Math major?!! Isn't that major super hard and for smart people?" I remember feeling really annoyed and almost anger at the response.

Lucia faced incredulous remarks from strangers regarding her major, and described how her complex intersectionality made it appear that she could not possibly be a math major.

Gendered Microaggressions

Thirteen female students wrote about experiencing a gendered MAs. Two main themes occurred in this group: 1) female students noticed how male-dominated the STEM space is and 2) how male peers do not take them seriously and often do not believe that they are capable of doing math. Almost half of the women coded with a gendered MA described the memory of being one of only a few in their math classes. Monica, a Latine female math major explained her feeling of being the only female student,

> As a female in a male-dominated field, there have been times that I have felt I do not belong in the classroom. One of the first times was my first quarter…in my Calc 3 class. My Calc 3 had 30 students and only 3 were female. I did not feel unwanted but I will say that it felt uncomfortable at times.

This sensation of being one of the only women in the math classroom amplified other MAs. Six students described ways in which their male peers made them feel as though they were not capable of contributing to the math space, often not taking them seriously. Flor, a Latine female engineering major expressed ways that she coped with this type of treatment.

> [My pre-calculus] class was predominantly male students, with just a handful of us female. From what I remember, after lecture, it seemed as if all the guys just "got it". Myself and the other five girls usually sat together because it took us longer to get the material. We felt more comfortable sitting next to each other because we all knew we wouldn't judge each other for it. It was as if we created a safe place for ourselves within our pre-calculus class; we could ask each other questions rather than asking our teacher

or the guys, who'd just make us feel stupid. That was the class that made me hate math again. I struggled the entire semester, but with the help and support of the other girls, I'd managed to pass with a C.

Other women shared similar experiences as Flor, describing how in group work their contributions would often be overlooked, or their male peers would get exasperated when they would ask clarifying questions.

Racial, Gendered, and Mathematical Microaggressions

Seven students listed having experienced all three of types of MAs, all of which were female students of color. These women identified different moments within their educational experiences that relate to these MAs. Leticia and Claudia, both Latine female math majors, wrote about how the choice of name that the instructor uses for an example can demonstrate a lack of diversity and inclusion. Leticia wrote,

> Choosing male white/European names was a weird thing in my opinion because when names are used in class the teacher tends to just say the first name that comes to mind, and I assume it is the same when making homework/test problems. Many of my non-white and/or female teachers just use the basic names (i.e., Ann, John, Max, Julie, etc.) because that isn't the important part of the problem.

Leticia implies that the names selected for word problems or examples usually relate to European names. What is more, she indicated that even her marginalized instructors, who she may assume would diversify this component, also use these types of names. Claudia shared that "as a person of color and a female in STEM, I have constantly felt like I didn't belong in my field or in college." She explained that utilizing more diverse examples in class would help to alleviate this feeling, and that it "is important to the students of color who are constantly told that they do not belong in higher education because the statistics said so or because society keeps showing the negative stereotypes of the community."

Shenise and Nina, both women of color, expressed the challenges of being a math major. Shenise shared her concerns of being both black and a woman in math.

> I know as a black woman in the Math field, white men or men in general don't think you're capable of understanding the material the way that they do. There are countless times where I had ideas or different approaches to a problem and was ignored or overlooked and would eventually have to do it the way that I had suggested. They treat women as if we did not take the same classes to get where we are now.

Shenise's experiences reflected how men in math radiate the sense that she, and others like her, do not belong in math. Being ignored is a MA against the person contributing ideas and approaches, especially in a math class; this is amplified when Shenise explained that it was because she is a black woman. Nina shared,

> It is especially hard to feel like I belong as a Latinx woman. Walking into a class and seeing that you are one of maybe 2 or 3 girls in the class is both interesting and scary. Interesting because now I realize that I am seeing the "leaky pipeline" happen in real-time, and scary because I remember when my previous math classes would have at least 8-9 girls in them. Women and minorities are still underrepresented in STEM fields and the numbers have not been exponentially growing even as the years have changed and people have gotten far more progressive.

Nina explained how she felt seeing few women, which is known as an environmental MA (Marshall et al., 2021), relating to MAs that occur in the environment in which a person exists, not directly received by another individual.

Sense of Belonging

72 students indicated that they have been made to feel like they do not belong in a math classroom while 19 students indicated that they have never been made to feel like they do not belong. We will refer to the first group as Low Sense of Belonging (LSB) and the second group as High Sense of Belonging (HSB). 43 students did not indicate an explicit answer. We discuss each of these groups in this order.

86% of students that indicated LSB also discussed a mathematical MA. Only ten students who stated they had LSB did not include any examples of receiving or witnessing a mathematical MA. Ezekial, a male student (whose race/ethnicity was unknown), shared an experience where a math teacher actively tried to push him out of the STEM field. He shared,

> In the summer before my senior year of high school, after my retake of algebra 2, the teacher called me aside and asked me what my plans were for the future. He told me that he would only give me a passing grade if I didn't plan to major in anything involving math. While I remember my grade was a 66, and of course I should consider it an act of benevolence that he moved me up to a 70, I know I didn't deserve it. I felt like a fraud.

At the time of writing this, Ezekial was in his final semester as an economics major, a field that required a lot of mathematical skill and reasoning. His reflection indicates that his teacher provided a kind act, to provide him with a passing grade. However, he reflects that this act made him feel a LSB because he did not feel like he belonged in math.

Twelve students who indicated LSB also described instances of racial MAs while five students described gendered MAs. Carson, a Black male student majoring in physics and minoring in math explained how being one of a small minority of Black students creates LSB.

> Being an underrepresented minority is difficult enough, as there are so few that we feel we do not belong. The lack of inclusive language in these subjects creates a subconscious feeling that we do not belong. It is interesting with microaggressions, those who inflict them often don't know that their statement or attitude was harmful. I have been told by others that I speak so well after doing presentations, and while they meant it as a compliment, it creates the feeling that they weren't expecting me to speak well.

Carson highlights that inclusive language is valuable to feeling a sense of belonging. He continues to say, "Mathematics, and Physics, tends to be full of language that is discouraging." While he persisted with the degree, he acknowledged that he did not feel as though he belonged.

Interestingly, of the 19 students who indicated HSB, 15 documented having received mathematical MAs while four did not express any examples of having received or witnessed a mathematical MA. This implies that while students experience mathematical MAs, they can still have HSB in the classroom. This number is very small, only representing 11% of students. Aron, a Latine male, described why he feels HSB,

> I may have felt discouraged but not as if I didn't belong. I have always had a passion for math, and no one can ever take that away from me. At times it has proven to be challenging. However, that is part of the beauty. I don't think there is any other place I do belong in.

Aron finds a sense of belonging within the subject itself – he finds that a strong part of his identity is tied to the beauty of math itself.

While three people with HSB were coded as discussing racial MAs, they still felt like they belonged in the math class. Dane, an Asian American male student described,

> I have not ever felt that I didn't belong in a math class. Being an Asian American there are a lot of stereotypes revolving around my ability to do math solely because of my race.

> Growing up I was somewhat decent in math whether or not it was my genetics giving me an advantage or the cultural upbringing that came with being Asian.

Dane expressed that an awareness of the Model Minority Stereotype, yet persisted in his math endeavors maintaining a sense of belonging in the classroom. None of the students who described a gendered MA indicated HSB.

43 students did not state explicitly whether they had been made to feel like they don't belong. However, a majority of these students described mathematical MAs that they directly experienced; only five of these students did not write about a mathematical MA. Leticia, who did not explicitly say if she had LSB wrote,

> The calculus three teacher I had should be required to read this five times. He is an aero-space engineer teacher that likes to make math majors feel awful about themselves if they aren't doing well or have questions. One other math student asked if he wrote an *r* or an *n* and he gave the most sarcastic answer. I was never the target because when he asked who was a math major he didn't see my hand…There is nothing worse than feeling belittled when you are learning something difficult.

Leticia described here a moment that felt like a macroaggression, where she and her peers experienced very direct interactions from a faculty member, creating an experience where many may not have felt a sense of belonging in the classroom. She points out that the Su (2015) paper would be a useful tool for some faculty to read to understand the possible experiences a student may have when trying to learn such a difficult subject. Paired with her acknowledgement earlier of the use of white, European names in math word problems, Leticia made clear how she may not have experienced a sense of belonging in math.

Discussion

Our study showed that students experience traumatizing MAs in their math learning experiences. While the frequency was low regarding racial (16%) and gendered (10%) MAs discussed in the journals, the fact that these experiences still exist implies a dire need to reflect and consider our classroom spaces and the math field in general. When considering all students who enroll in math courses, this affects many students. These are also the accounts of students who "made it"; what would we see if we collected data from students who left STEM or never made it Calculus 1? What is more, over half of the students in this study indicated that they have been made to feel like they do not belong in math. Many more wrote accounts that would indicate the same but did not state explicitly whether or not they have been made to feel that way. If we consider those who did not explicitly say it yet implied it in their writing, it *could* be possible that at some point in time about 85% of students have been made to feel like they do not belong in a math classroom. While this is an important finding, future work will explore the frequency and longevity with which students feel LSB in their math classrooms.

Understanding the racial, gendered, and mathematical MAs that students face in classrooms may help us to understand ways to make students feel a strong sense of belonging in their math classes. Leyva and colleagues (2021) have documented instructional mechanisms that may impact a students' experience in college math, which include limiting within-group peer support as well as activating exclusionary ideas of who belongs in STEM (p. 27). As we continue to develop best practices for supporting students, we need to keep in mind the ways to develop a stronger sense of belonging for math students. We note here that the intersectionality of students' identities as both women and students of color is an important aspect to consider, as negative experiences at the intersection of these identities can amplify the lack of sense of belonging (Crenshaw, 1991), specifically in math.

References

Burke, R. J. (2007). Women and minorities in STEM: A primer. In R. J. Burke, M.C. Mattis, & E. Elgar, *Women and minorities in science, technology, engineering and mathematics: Upping the numbers* (pp. 3-27), MPG Books LTD.

Cawley, A., Aniceto, R., Ricarte, S., & Wilson, R. (2023). "Well, it's obvious": Students' experiences with mathematical microaggressions. *Proceedings of the 25th Annual Conference for the Research in Undergraduate Mathematics Education*. Omaha, NE.

Celis Carbajal, P. (2020, September 29). From Hispanic to Latine: Hispanic heritage month and the terms that bind us. https://www.nypl.org/blog/2020/09/29/hispanic-heritage-month-terms-bind-us

Chou, R. S., & Feagin, J. R. (2008). The myth of the model minority: Asian Americans facing racism. Boulder, CO: Paradigm Publishers.

Corbin, J., & Strauss, A. (2008). Basics of qualitative research: Techniques and procedures for developing grounded theory (3rd ed.). Thousand Oaks, CA: Sage Publications.

Crenshaw, K. W. (1991). Mapping the margins: Intersectionality, identity politics, and violence against women of color. In The public nature of private violence (pp. 93-118). Routledge.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, 102(4), 700-717.

Lee, M. J., Collins, J. D., Harwood, S. A., Mendenhall, R., & Huntt, M. B. (2020). "If you aren't White, Asian or Indian, you aren't an engineer": Racial microaggressions in STEM education. *International Journal of STEM Education*, 7(1), 1-16.

Lewis, J. A., Mendenhall, R., Harwood, S. A., & Browne Huntt, M. (2013). Coping with gendered racial microaggressions among Black women college students. *Journal of African American Studies*, 17, 51-73.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition and instruction*, *39*(1), 1-34.

Nadal, K. (2011). The Racial and Ethnic Microaggressions Scale (REMS): Construction, Reliability, and Validity. *Journal of Counseling Psychology*, *58*, 470–480. https://doi.org/10.1037/a0025193

Nadal, K. L., Wong, Y., Griffin, K. E., Davidoff, K., & Sriken, J. (2014). The adverse impact of racial microaggressions on college students' self-esteem. *Journal of College Student Development*, *55*(5), 461–474. https://doi.org/10.1353/csd.2014.0051

Marshall, A., Pack, A. D., Owusu, S. A., Hultman, R., Drake, D., Rutaganira, F. U., ... & Hinton Jr, A. O. (2021). Responding and navigating racialized microaggressions in STEM. *Pathogens and disease*, 79(5). https://doi.org/10.1093/femspd/ftab027

McGee, E. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture? *American Educational Research Journal*, 53(6), 1626–1662.

McGee, E., & Bentley, L. (2017). The troubled success of Black women in STEM. *Cognition and Instruction*, 35(4), 265–289.

Pierce, C. M., Carew, J. V., Pierce-Gonzalez, D., & Wills, D. (1977). An Experiment in Racism: TV Commercials. *Education and Urban Society*, *10*(1), 61–87. https://doi.org/10.1177/001312457701000105

Pierce, Chester. (n.d.). *Offensive Mechanisms*.

Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International journal of STEM education*, *5*(1), 1-14.

Sekaquaptewa, D. (2019). Gender-based microaggressions in STEM settings. *NCID Currents*, *1*(1).

Steele, C. M. (1997). A threat in the air: How stereotypes shape intellectual identity and performance. *American Psychologist*, 52, 613–629.

Strayhorn, T. L. (2018). College students' sense of belonging: A key to educational success for all students. Routledge.

Su, F. E. (2015). Mathematical Microaggressions. MAA FOCUS, 36-37.

Sue, D. W., Capodilupo, C. M., Torino, G. C., Bucceri, J. M., Holder, A. M. B., Nadal, K. L., & Esquilin, M. (2007). Racial microaggressions in everyday life: Implications for clinical practice. *American Psychologist*, 62(4), 271–286. https://doi.org/10.1037/0003-066X.62.4.271

Sue, D. W., Lin, A. I., Torino, G. C., Capodilupo, C. M., & Rivera, D. P. (2009). Racial microaggressions in the classroom. *Cultural Diversity and Ethnic Minority Psychology*, 15, 183–190. doi:10.1037/a0014191

Solórzano, D., Ceja, M., & Yosso, T. (2000). Critical race theory, racial microaggressions, and campus racial climate: The experiences of African American college students. *Journal of Negro Education*, 69(1/2) 60-73.

Yang, Y., & Carroll, D. W. (2018). Gendered Microaggressions in Science, Technology, Engineering, and Mathematics. *Leadership and Research in Education*, 4, 28-45.

Exploring the Adoption of Research-Based Instructional Strategies in Undergraduate Mathematics with the Teacher Centered Systematic Reform Model

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Studies show that Research-Based Instructional Strategies (RBIS) help students learn, however their adoption has been slow. The Teacher Centered Systematic Reform Model (TCRM) is a general model for organizing enablers and barriers to adoption of new teaching methods that includes departmental, personal and teacher thinking factors. We used the TCRM model as a framework to assess the amount of formal lecture reported by 634 mathematics instructors in their undergraduate courses. Regression analyses found that instructors who participated in Project NExT (a professional development workshop) during their early careers were less likely to use lecture than non-participants. Other significant predictors of lecture less included evaluation expectations emphasizing active teaching methods, involvement in equity and diversity efforts, and prior experience with RBIS. Factors with a positive correlational association with lecture included evaluation efforts by departments where lecture was expected. Results confirmed some prior models in different disciplines.

Keywords: Undergraduate Education, Adoption of New Teaching Practices, Research Based Teaching Strategies, Mathematics Education, Inquiry-Based Learning

Reaching back over twenty years, many studies show that Research-Based Instructional Strategies (RBIS) help students learn in college, but that adoption and integration into undergraduate classrooms is stubbornly limited (American Association for the Advancement of Science, 2013; Laursen et al., 2019; Stains et al., 2018). While the use of RBIS can help provide student understanding of STEM (and other disciplines), their effectiveness is limited if their use is constrained. The Teacher Centered Systematic Reform Model (TCSR) provides a general structural model to assess incentives and barriers to adoption of RBIS (Gess-Newsome, 2003). The broad categories of enablers and barriers to adoption in this model include contextual, personal, and teacher thinking; these factors have been researched in studies that assess the relative contribution of factors in the implementation of active learning in college classrooms (Yik et al., 2022).

The departmental context is an important focus of many adoption studies. Departmental norms and expectations, both supporting and hindering active learning, have been studied as one factor for instructors adopting active teaching methods (Hora & Anderson, 2012). Some of the factors constraining adoption were related to teaching load and a lack of time to prepare lessons, perhaps related to prevailing practices in university departments (Henderson & Darcy, 2007). Pressures to achieve tenure and to publish, found in more research-intensive universities, act to constrain the amount of effort instructors can devote to course design (Lund & Stains, 2015). Prevailing norms and expectations linked to academic evaluation can also encourage or limit adoption of RBIS; if teaching is not valued or if alternative teaching methods are discouraged, instructors may be less willing to take risks with new teaching methods. More practically, classroom context, mainly large class sizes and classroom layout (e.g., places designed for group work), have also worked against adoption (Yik et al., 2022).

The personal characteristic of university instructors also plays into implementation decisions. These can include prior experience with active teaching, participation in professional development and beliefs and values related to teaching. Instructors who experienced or practiced inquiry-based teaching as graduate students were more likely to teach the same way later (Fukawa-Connelly et al., 2016), as are instructors who experienced RBIS as students (Yik et al., 2022). Participation in larger professional development initiatives such as Chem Connections, National Academies Summer Institutes on Undergraduate Education, or POGIL have been linked to greater use of active teaching methods (Dertling et al., 2016). Our current study used data from Project NExT (PN), a teaching initiative in mathematics. Participation in more short-term professional development efforts, usually conducted on college campus, have also been linked to greater adoption and implementation of alternative pedagogies.

While many studies have been conducted on factors influencing adoption, there are few that focus on teaching undergraduate mathematics. Yik et al. (2022) conducted the largest quantitative study to date on adoption, testing a wide range of factors from the TCSR model. These researchers compared the amount of lecture used by instructors in mathematics to chemistry and physics. While physics instructors spent significantly less time lecturing than mathematics instructors, instructors in mathematics lectured at similar rates to those who taught chemistry. However, no separate or interactive models were made to learn how other factors worked within the mathematics discipline. Johnson et al. (2019) examined many of the same factors found in the TCSR model with survey data from 219 algebra instructors, although they did not incorporate a regression model in their analysis. The researchers made comparisons between high, medium and low lecturing groups on a range of variables from the TCSR model. Significant group comparisons were found in teacher beliefs about student learning (e.g., "I think students learn better when they struggle with the ideas prior to me explaining the material to them."). Johnson's team found only small differences for departmental support between lecture groups, with instructors who were given more latitude in course design lecturing at lower rates than instructors whose teaching was more constrained by their departments.

Rationale for Study & Research Questions

Widespread adoption and implementation of RBIS provide the key to their effective use. Understanding what helps and hinders this adoption may spur adoption by addressing policies and practices that may encourage or stymie the use of more active instruction. Our research questions included:

1) What factors related to the TCRM model are associated with the amount of time instructors engage in formal lecture in their mathematics courses?
2) Is participation in the professional development workshop Project NExT associated with the amount of time instructors engage in formal lecture in their mathematics courses?

Method

Participants

Six-hundred-thirty-four (634) mathematics instructors answered the Alumni Survey, 492 former participants in Project NExT and 142 from a comparison group. On the survey, participants were asked to choose one course they had taught recently and report on their teaching practices. Calculus was the most frequently chosen course (31%), followed by Special Topics (e.g., higher division mathematics), Algebra (12%), and Other courses (28%). Demographically, participants were white (79%), with Asian (4.6%), Hispanic (3%) and Black instructors (1.1%) making up only a small percentage of respondents. Gender representation was nearly even with 50% of those answering the survey identifying as male, 44% as female and another 6% non-binary or preferring not to answer. Instructors reported teaching at the college level from three to 44 years with an average teaching career of 18 years. Of those instructors responding, 28% of respondents had been department chairs at one time during their time teaching, but only 3% had been deans. Having an advanced degree was a requirement for entering Project NExT; 96% indicated their terminal degree was a Ph.D.

Instrument

The Project NExT Alumni Survey was administered to a list of 1532 Project NExT Alumni obtained from the project with a response rate of (71%). The sampling frame for the Alumni Comparison Group Survey contained 882 names with response rate of 28%. While 902 mathematics instructors answered the survey, the current analyses used only 643 responses of those providing complete information about their teaching and described in-person and non-virtual classes. The survey was administered during the winter and spring of 2022.

The survey contained 47 items (many items with long lists of choices) asking about a range of topics related to careers as math instructors. These included: 1) the benefits of Project NExT or an alternate professional development project, 2) academic career activities, 3) participation in professional development, 4) participation in professional societies, 5) involvement in receiving grant money and research, and 6) expectations for evaluation of their work from their department, and a parallel section asking about the activities that brought respondents personal career fulfillment. The teaching component of the survey was based on the TAMI-S survey developed by our research team (Hayward et. al., 2017) and asked instructors to choose a course they had taught recently and estimate the amount of time they spent in a range of teaching activities (e.g., Lecture or Group Work). We also asked basic demographic questions as well as questions about time spent teaching at the college level, and the characteristics of their institution.

Analysis

Ordinary Least Squares (OLS) Linear Regression was used to estimate the individual contribution of variables on instructor responses to the item: *Please choose a course you have taught that represents your best teaching. What approximates the amount of time you spend in Formal lecture? (1, Did not use this activity, 1/3 or less, 1/3 to 2/3, 2/3 to all of class time).* The dependent variable was similar to that used in Yik et al. (2022) which asked instructors to report percentages of time spent in lecture.

The final regression model used 564 listwise responses from 26 variables. These variables were included as representing components of the TCSR model. For the sake of space, we reported only statistically significant predictors, and listed other variables tested but which were not statistically significant. We also compared both raw and adjusted mean differences for Project NExT using a simple ANCOVA procedure, adjusting the means for each group with the propensity covariate. We used the covariate in this manner only after checking the assumptions of the ANCOVA procedure.

The survey took place in the context of research on professional development at project NExT. As well as selection of PN Alumni, we selected a comparison group of instructors. These respondents were chosen through the Math Genealogy Project, a website that tracks the history of mathematics Ph.D's and their advisors. Instructors were chosen for the sampling frame by matching PN alumni with their graduate school colleagues who shared the same advisor and attended the same university program at the nearly the same time.

For comparisons between PN Alumni and the comparison group mentioned below, we used a propensity matching analysis to control for differences between groups (Benedetto et al., 2018). The propensity model provided information on how groups differed along many of the same variables used in the main analysis.

Results

Fifteen variables entered the regression model; the resulting R-squared value was $R^2 = 0.34$. Five-hundred and sixty-four (564) cases were used in the regression model. The greatest inverse predictors of time spent lecturing were academic evaluation expectations to use techniques other than lecture (Beta= -0.22) and participation in Project NExT (Beta = -0.19). Evaluation expectations to lecture (Beta= 0.15) and to use a variety of teaching methods were associated with greater use of formal lecture. Lecture was used less frequently in courses for education majors or non-majors (Beta = -0.13), and instructors who used active or inquiry-based methods during their early career tended to lecture less (Beta = -0.09). Other inverse predictors of lecture included involvement in equity and diversity in department or institution (Beta = -0.12) and collaborating with other instructors to promote changes in math teaching practices (Beta = -0.12). The number of campus professional development workshops instructors participated in predicted greater use of lecture (Beta = 0.08). Table 1 presents the Linear Regression Model for class time spent in formal lecture.

Other variables were tested from the TCSR model but did not enter our regression model. Non-significant variables included: Years teaching at college/university level, Department head or chair (past or present), Gender, Academic Department Expectation: Receiving High Evaluations of Teaching from Students (as evaluation criterion), Tenure Track Position, Teaching Load, Member of Minoritized Population, Highest Degree Offered at Institution, Member of (Specific) Professional Societies, and Participation in (Specific) Campus Professional Development efforts.

To better assess the association of participation in Project NExT with formal lecture we created a propensity matching model. This model used logistic regression to predict group membership in PN or the comparison group. The resulting probabilities of group membership derived from the logistic model were then used as a covariate in an Analysis of Covariance model (ANCOVA) that tests the differences in the mean estimates of time spent lecturing between groups and adjusts each mean to reflect the logistic probabilities. The propensity logistic model found statistically significant differences between groups favoring the PN group for the following variables: Served on National Committee, and Participation in Campus Professional Development. The variables favoring the comparison group in the propensity model included Highest Degree Offered, Receiving an Endowed Professorship or Other Honorary Post, and Years Teaching at University Level.

The ANCOVA comparison returned a statistically significant result for the main effect for program status (PN or Comparison) of $F = 16.02$, df 1,497, $p < .0001$**. This result tested the difference in means for Formal Lecture between groups with PN = 2.01 and Comparison = 2.94. These means were adjusted by the covariate to PN = 2.09 and Comparison = 2.62. As effect sizes, the difference between raw means was ES = -0.88; for adjusted means the effect size was

ES = -0.50. The result indicated that having participated in Project NExT was associated with less use of formal lecture in the classroom.

Table 1 Linear Regression Model Predicting Class time Spent in Formal Lecture

	B	SE	Beta	t	p
(Constant)	2.72	0.46		5.89	<.001**
Participated in Project NExT	-0.48	0.10	-0.19	-4.86	<.001**
Collaborated with colleagues to promote changes in math teaching practices	-0.27	0.09	-0.12	-3.06	.002**
Agree/Disagree: I am involved in efforts at my institution to promote equity and inclusion in teaching practice	-0.20	0.08	-0.09	-2.42	.016*
Personal Expectation: Teaching in more active and engaging ways	-0.22	0.07	-0.12	-2.98	.003**
Personal Expectation: Promoting equity and diversity in your department and institution	-0.15	0.06	-0.12	-2.72	.007**
Academic Department Expectation Giving academic talks at conferences	0.13	0.05	0.10	2.60	.010*
Academic Department Expectation: Expectation to use techniques other than lecture	-0.21	0.06	-0.22	-3.47	<.001**
Academic Department Expectation: Expectation to use a variety of teaching methods	0.17	0.07	0.16	2.55	.011*
Academic Department Expectation: Expectation to lecture	0.16	0.04	0.15	3.98	<.001**
Sum of professional development involvement at institution	0.05	0.02	0.08	2.13	.034*
Agree/Disagree: “In my department, I am mostly free to teach however I want.”	-0.15	0.07	-0.08	-2.16	.031*
Agree/Disagree: “I taught using active or inquiry-based methods during my early career”	-0.20	0.08	-0.09	-2.35	.019*
Agree Disagree: “I taught large introductory-level courses during my early career”	0.17	0.08	0.08	2.18	.030*
Content: Education and Non-Major	-0.46	0.13	-0.13	-3.65	<.001**
Content: Geometry	-0.41	0.18	-0.08	-2.33	.020*

R^2 (Adjusted) = .34, p<.05* p < .01** , N for model = 564

Discussion

The analysis of the Project NExT Alumni Survey data found significant effects predicting the amount of time instructors reported using formal lecture for a range of variables related to departmental expectations, personal expectations, campus professional development, course content and equity and diversity. The results support some of the previous findings related to the TCSR model (Gess Newsome, 2003).

The large attenuated effect seen for participation in the workshop professional development Project NExT reflected those seen in Yik et al. (2022) and various other research about teacher focused professional development (Dertling et al., 2016). Although this comparison cannot be considered causal in any way, the effect for participation in the PN program suggests that those who have participated in NExT tend to lecture less than those from the matched comparison group. This counters a previous assessment of the program that found little or no benefits for the PN program (Fukawa-Connelly et al., 2016). Participation in campus wide professional development was not predictive of lecture time by specific program although those participating in more types of professional development were slightly more likely to lecture. This countered findings from Benabentos et al. (2021) who found higher uptake of research-based methods for those using campus professional development services. Our result may have been related to the type of professional development we asked about including training in non-teaching activities such as grant writing.

Perceived departmental expectations for active teaching were also predictive of less time spent lecturing. Instructors who reported that their department evaluated them with the "expectation to use techniques other than lecture" lectured less. Conversely, departments with expectations for lecture reported lecturing more. Johnson et al. (2019) found only small effects for departmental expectations for instructors in designing their own courses, with those having more latitude less likely to lecture. This was similar to the small effect for our agree/disagree survey item: "In my department, I am mostly free to teach however I want". The findings that departmental evaluation criteria are predictive of time spent lecturing was seen in work by Seymour et al. (2011), generally with instructors feeling blocked from implementing active learning due to a lack of incentives for doing so and pressure to publish for tenure. Items asking about activities that instructors find personally fulfilling were also predictive of time spent lecturing including valuing "Teaching in more active and engaging ways". Other practical teaching expectations such as teaching load, tenure status and the importance of student teaching evaluations did not enter into our regression model, a result mostly consonant with the Yik et al. study (2022).

Collaboration with others was also found to be predictive of lecture time. Those who collaborated with colleagues to promote changes in math teaching practices were less likely to lecture. This reflects other studies on professional development and teaching practices (Bressoud & Rasmussen, 2015) where instructors shared experiences with innovative pedagogical practices. This perhaps extends to other areas of collaboration such as promoting student equity and diversity; those instructors who were involved in these efforts with colleagues were also less likely to lecture. Early career experiences seemed to impact current teaching for those who took our survey. Those agreeing with the statement : "I taught using active or inquiry-based methods during my early career" tended to lecture less, and those who agreed with "I taught large introductory-level courses during my early career" lectured more. The influence of early career experiences on teaching style is found in studies by Yik et al.(2022) and Lund and Stains (2015).

Like many models ours is under-identified. Because of constraints on data collection, we were not able to gather information about teaching thinking related to student growth mindset, an important part of the TCSR model (Gess Newsome, 2003) and a significant predictor of less lecture time in Yik et al. (2022). Similarly, variables of class size and class layout were not included in our model but were found to predict greater use of lecture in previous studies (Lund & Stains, 2015; Yik et al.,2022). Inclusion of these factors in future research would provide a more complete picture of enablers and barriers to implementation of active learning.

References

American Association for the Advancement of Science (AAAS) (2013). *Describing and measuring undergraduate STEM teaching practices. A report from a national meeting on the measurement of undergraduate science*. AAAS.

Benabentos, R., Hazari, Z., Stanford, J. S., Potvin, G., Marsteller, P., Thompson, K. V., ... & Kramer, L. (2021). Measuring the implementation of student-centered teaching strategies in lower-and upper-division STEM courses. *Journal of Geoscience Education*, *69*(4), 342-356.

Benedetto, U., Head, S. J., Angelini, G. D., & Blackstone, E. H. (2018). Statistical primer: propensity score matching and its alternatives. *European Journal of Cardio-Thoracic Surgery*, *53*(6), 1112-1117.

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the AMS*, *62*(2), 144-146.

Derting, T. L., Ebert-May, D., Henkel, T. P., Maher, J. M., Arnold, B., & Passmore, H. A. (2016). Assessing faculty professional development in STEM higher education: Sustainability of outcomes. *Science Advances*, *2*(3), e1501422.

Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can math education research improve the teaching of abstract algebra. *Notices of the AMS*, *63*(3), 276-281.

Gess-Newsome, J., Southerland, S. A., Johnston, A., & Woodbury, S. (2003). Educational reform, personal practical theories, and dissatisfaction: The anatomy of change in college science teaching. *American Educational Research Journal*, *40*(3), 731-767.

Hayward, C. N., Laursen, S. L., & Weston, T. J. (2017). TAMI-OP: Toolkit for Assessing Mathematics Instruction –Observation Protocol. Boulder, CO: University of Colorado Boulder, Ethnography & Evaluation Research. https://www.colorado.edu/eer/content/hayward-tami-op-description-updated-jul21

Henderson, C., & Dancy, M. H. (2007). Barriers to the use of research-based instructional strategies: The influence of both individual and situational characteristics. *Physical Review Special Topics-Physics Education Research*, *3*(2), 020102.

Hora, M. T., & Anderson, C. (2012). Perceived norms for interactive teaching and their relationship to instructional decision-making: A mixed methods study. *Higher Education*, *64*, 573-592.

Johnson, E., Keller, R., Peterson, V., & Fukawa-Connelly, T. (2019). Individual and situational factors related to undergraduate mathematics instruction. *International Journal of STEM Education*, *6*, 1-24.

Laursen, S., Andrews, T., Stains, M., Finelli, C. J., Borrego, M., McConnell, D., Johnson, E., Foote, K., Ruedi, B., & Malcom, S. (2019). *Levers for change: An assessment of progress on changing STEM instruction*. Washington, DC: American Association for the Advancement of Science.

Lund, T. J., & Stains, M. (2015). The importance of context: an exploration of factors influencing the adoption of student-centered teaching among chemistry, biology, and physics faculty. *International Journal of STEM education*, *2*, 1-21.

Seymour, E., DeWelde, K., & Fry, C. (2011). Determining progress in improving undergraduate STEM education: The reformers' tale. A White Paper commissioned for the Forum, "Characterizing the Impact and Diffusion of Engineering Education Innovations". https:// www. nae. edu/ File. aspx?id= 36664

Stains, M., Harshman, J., Barker, M. K., Chasteen, S. V., Cole, R., DeChenne-Peters, S. E., ... & Young, A. M. (2018). Anatomy of STEM teaching in North American universities. *Science*, *359*(6383), 1468-1470.

Yik, B. J., Raker, J. R., Apkarian, N., Stains, M., Henderson, C., Dancy, M. H., & Johnson, E. (2022). Evaluating the impact of malleable factors on percent time lecturing in gateway chemistry, mathematics, and physics courses. International Journal of STEM Education, 9(1), 15.

U-Substitution through Quantitative Reasoning: A Conceptual Analysis

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Research has shown how crucial quantities-based meanings are for calculus concepts. While past work has developed important quantitative approaches to integration, the major topic of u-substitution typically has not been fully detailed in these paradigms. This theoretical paper extends this past work to clearly define and elaborate u-substitution through a quantitative perspective. We use conceptual analysis based on quantities, starting with a concrete example of a solar panel producing energy. We abstract from this example to define u-substitution as a transformation from one quantitative relationship, via nested multivariation, to another quantitative relationship. We also detail a three-part structure within this transformation.

Keywords: calculus, integrals, u-substitution, quantitative reasoning, conceptual analysis

A growing body of work has established that quantitatively-based meanings for calculus concepts such as derivatives and integrals are essential for robust student understanding and productive usage outside of math classes (Byerley, 2019; Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). For integrals, this means moving away from the purely "area under a curve" meaning in favor of a "sum of small bits" meaning (Ely, 2017; Jones, 2015; Sealey, 2006). Much work has been done in introducing integrals through this meaning and in helping students reason and model with integrals (Bajracharya et al., 2023; Blomhøj & Kjeldsen, 2007; Chhetri & Oehrtman, 2015; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). Yet, integration chapters typically conclude with the major "u-substitution" method, which is the first in a long line of substitution techniques, and which is also utilized in the sciences and engineering to convert between quantitative expressions (see Koretsky, 2012). Unfortunately, the current literature does not adequately depict how to incorporate u-substitution into a quantitative paradigm. It would be detrimental to work toward quantitative meanings for integrals only to have this major method exist outside them. Treating u-substitution quantitatively would not only allow students to effectively *use* it, but to understand the mechanisms for *why* it works. To build on the important prior work on quantitative reasoning in calculus, in this paper we induct u-substitution into it as well. As the first key step, this paper focuses on the *theoretical* side of a quantitatively-based approach to u-substitution. We use a quantities-based conceptual analysis to define the quantitative relationships within u-substitution and its three-part quantitative structure. In a separate paper, we build on this theoretical work to engage in empirical examinations of learning u-substitution through a quantitative approach.

Brief Review of Closely Related Literature

There is a proposed quantitative structure for definite integrals called *adding up pieces* (AUP) (Jones, 2013; Jones & Ely, 2023). AUP is comprised of three parts: *partition*, *target quantity*, and *sum* (see also Dray & Manogue, 2023; Sealey, 2014; Von Korff & Rebello, 2012). *Partition* is taking a quantitative object (e.g., a length, a space, a time duration) and segmenting it into tiny pieces. If the pieces are essentially infinitesimal in size, they are called "differentials" and are denoted with a "*d*", as in dx or dt (Ely, 2020). In other words, a dx or dt can be thought of as an incredibly tiny Δx or Δt. While differentials can be rigorously formalized through limits or hyperreals (see Jones & Ely, 2023), the more loosely-defined "essentially infinitesimal" is quite common across the sciences and is a crucial idea for quantitative reasoning, even if no specific threshold is defined for passing from "Δ" to "*d*" (Amos & Heckler, 2015; Hu & Rebello, 2013; Pina & Loverude, 2019; Thompson & Dreyfus, 2016; Von Korff & Rebello, 2014).

Target quantity refers to determining the sought-after quantity within each partition piece (Jones & Ely, 2023). For example, if one is determining the distance travelled under variable velocity, and time has been partitioned into infinitesimal *dt* pieces, one conceptualizes that each *dt* piece corresponds to a tiny bit of distance travelled, *dD*. In AUP, any quantitative relationship can be used to determine the target quantity. The distance example uses a simple product, $D = v \cdot t$, but other quantitative relationships can be used when appropriate (Simmons & Oehrtman, 2017). Oehrtman and Simmons (2023) called the quantitative relationship that holds for constant values, such as $D = v \cdot t$, the *basic model*. If some quantities vary, such as a varying velocity, the essentially infinitesimal nature of the partition pieces allows this quantity to be considered essentially constant over each piece. Thus, the basic model can be applied to the infinitesimal partition piece, as in $dD = v \cdot dt$, which Oehrtman and Simmons (2023) called the *local model*.

With a conceptualization of tiny bits of the target quantity in each essentially-infinitesimal partition piece, *sum* refers to the literal summation of these target quantity bits to obtain the total amount of the target quantity. We follow Leibniz's convention (Katz, 2009) in using the integral symbol, $\int$, as a literal "sum" symbol. As an example, $\int_a^b v(t)dt$ is the summation of tiny distances (produced by $v \cdot dt$) across the many *dt* pieces, yielding the total distance.

Theoretical Lens: Quantitative Relationships and Multivariation

Thompson (Smith & Thompson, 2007; Thompson, 1990) defined a "quantitative relationship" as a system of three quantities where any two can determine the third. Thompson's three-quantity relationships can be represented as a triangle with a quantity on each vertex (Figure 1a). Jones (2022) elaborated on these quantitative relationships to create a set of distinct relationship types called *multivariation* (MV) relationships (though MV can also include relationships with more than three quantities). One MV relationship type is *independent MV*, where multiple inputs influence a single output, but where the inputs are independent from each other (Figure 1b). Another type is *dependent MV*, where all the quantities are co-dependent and a change in a single quantity implies changes in all others (Figure 1c). A third type is *nested MV*, where the quantities are structured in a chain of influence, congruent to the structure in function composition (Figure 1d). Our conceptual analysis of u-substitution in this paper relies heavily on Thompson's idea of a quantitative relationship and on the dependent MV and nested MV types.

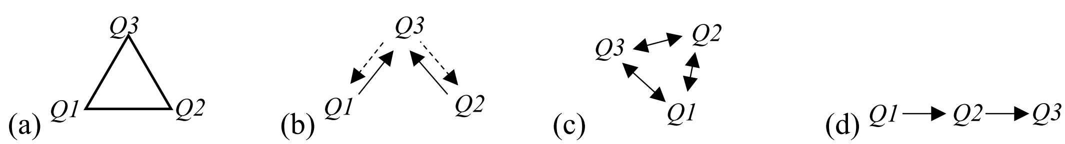

Figure 1. (a) Thompson's quant. relationship; with Jones' (b) independent MV, (c) dependent MV, (d) nested MV

Conceptual Analysis of U-Substitution: Starting with an Example Context

We begin our conceptual analysis with an example, from which we can abstract the general relationships. For our example, consider Figure 2a of a solar panel, with the sun rising at 6 am and reaching its zenith at 12 pm. The main quantities in this context are time (in hours, hr), the energy produced by the solar panel over a period of time (in kilojoules, kJ), and the power, or the rate at which energy is generated over time (in kilojoules per hour, kJ/hr). While a "watt" is a standard power unit (joules per second), we use kJ/hr to match the unit of time (hr), and to make the rate structure more obvious. Time, power, and energy are related by $E = P \cdot t$ (Figure 2b).

Notice that early in the morning there is less power because the sun makes a shallow angle with the panel. The power increases as the sun approaches its zenith, meaning that power is a function of time. In fact, we call time the main *input quantity* because it is an input both for the power, $P(t)$, and for the energy, $E(P, t)$. For this context, if we let 6:00 am be $t = 0$, we can use $P(t) = 1500 \sin\left(\frac{\pi}{12} t\right)$ kJ/hr as a reasonable model. The sine function suits the sun's increasing angle over time: it is zero at $t = 0$, increases from $0 \le t \le 6$, and is maximized at $t = 6$.

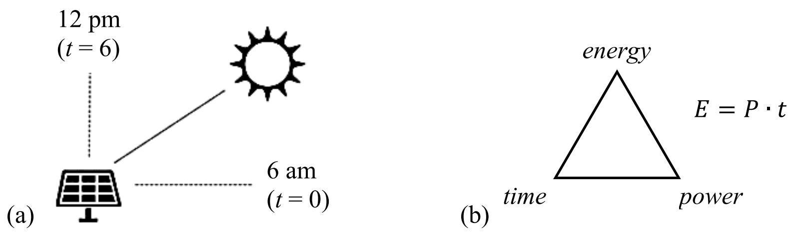

Figure 2. (a) A solar panel with varying power as the sun rises, and (b) the quantitative relationship in this context

The following initial question leads to a regular definite integral: How much energy is produced by the solar panel in this six-hour window? If power were constant over the six hours, we could use the basic model to calculate it: $E = P \cdot t$. But because power varies, we use the quantitative AUP structure to set up an integral. If we partition the six hour interval into essentially infinitesimal time intervals, *dt*, then we can assume power to be essentially constant over each interval. We can then use the local model $dE = P \cdot dt$ to capture the tiny bit of energy generated over a *dt* interval. Power, $P(t)$, is determined by a *t* value associated with the *dt* interval, so we can write our local model as: $dE = 1500 \sin\left(\frac{\pi}{12} t\right) \cdot dt$. We can then add up these bits of energy between $t = 0$ and $t = 6$ to find the total energy: $E = \int_{t=0}^{t=6} 1500 \sin\left(\frac{\pi}{12} t\right) dt$.

With this definite integral in place, we can now ask a different question that leads to u-substitution: What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock? This is the fundamental quantitative question we claim u-substitution is based on: How do you convert from one main *input quantity* to a different main *input quantity*? This conversion affects each of the three parts of the AUP structure. For the first part (partition), we need to determine what each infinitesimal *dt* piece corresponds to in terms of infinitesimal angles, *dθ*. For this conversion, notice that six hours corresponds to $\pi/2$ radians, three hours to $\pi/6$ radians, one hour to $\pi/12$ radians, and so on. In fact, for any time interval, the corresponding angle will always be $\pi/12$ as big, even at very small scales. Thus, $d\theta = \frac{\pi}{12} dt$, or equivalently $dt = \frac{12}{\pi} d\theta$, where the conversion factor $\frac{12}{\pi}$ has units of $\frac{\text{hr}}{\text{rad}}$. Switching from time pieces to angle pieces gives: $E = \int_{t=0}^{t=6} 1500 \sin\left(\frac{\pi}{12} t\right) \cdot \frac{12}{\pi} d\theta$. Of course, for this simpler context, the relationship between time and angle is linear at all scales. Later in the paper we examine a context where the relationship is *not* linear at all scales.

In tracking the units up to this point, we have $\left[1500 \sin\left(\frac{\pi}{12} t\right)\right] \frac{\text{kJ}}{\text{hr}} \cdot \left[\frac{12}{\pi}\right] \frac{\text{hr}}{\text{rad}} \cdot [d\theta]$ rad, which still yields a measure of kJ. This is valid quantitatively, though there is now a mismatch between defining part of the integral (the integrand) in terms of time and another part (the differential) in terms of angles. We also want to convert from power defined by time to power defined by angles. To do so, we note the nested MV relationship from *time → angle → power*, given by the function composition $\theta(t) = \frac{\pi}{12}t$ and $P(\theta(t)) = 1500\sin(\theta(t))$. However, if we no longer care about time, and only want to track the power in angles, this can become $P(\theta) = 1500\sin(\theta)$, leading to the integral expression: $E = \int_{t=0}^{t=6} 1500\sin(\theta)\frac{12}{\pi}\,d\theta$. Note that $1500\sin(\theta)$ still has units of kJ/hr, with the kJ/hr rate now determined by the angle. The factor 12/π still has units of hr/rad, so it is the entire "$1500\sin(\theta)\cdot 12/\pi$" that has units of kJ/rad.

While energy bits are now determined purely through the angle, $dE = 1500\sin(\theta)\cdot\frac{12}{\pi}d\theta$, the sum itself is still defined in terms of time. We want the sum to also be described in terms of the angle. Because $0 \le t \le 6$ hours corresponds to $0 \le \theta \le \frac{\pi}{2}$ radians, a sum running across *dt* pieces between $0 \le t \le 6$ hours covers the same ground as a sum running across $d\theta$ pieces between $0 \le \theta \le \frac{\pi}{2}$ radians. Thus, $E = \int_{\theta=0}^{\theta=\pi/2} 1500\sin(\theta)\frac{12}{\pi}\,d\theta$ and we have successfully described the total energy completely in terms of the *angle* as opposed to *time*.

Abstracting from the Example to a Quantitative Definition of U-Substitution

Before proceeding, we make some terminology clear. We have already used the term *input quantity* for the quantity that the other two depend on and that gets partitioned into infinitesimal pieces (*time*, in our example). We have also used *target quantity* to represent the quantity whose bits are added up by the integral and whose total amount is given by the value of the integral (*energy*, in our example). We introduce a new term, *integrand quantity*, for the quantity that functionally depends on the input quantity and that also combines with the input quantity to produce the target quantity. In our example, the integrand quantity was *power*.

With this terminology in place, recall that a definite integral answers the following question: If a target quantity is some combination of an input quantity and an integrand quantity, how can we determine its total amount if the integrand quantity varies over the input quantity? Relatedly, we claim that u-substitution now answers this question: How can we determine the total amount of the target quantity if we want to switch from tracking it in terms of the original input quantity to a new, different input quantity? This conversion is based on an inherent nested MV structure going from *original input → new input → integrand quantity*. In our solar panel example, the nested MV was $time \rightarrow angle \rightarrow power$, described by the function composition $P(\theta(t))$.

To provide a clear quantitative definition, we wish to use generic symbols for these quantities that are not specific to any context. We use "*a*" for the original input, "*b*" for the new input, "*Q*" for the integrand quantity, and "*T*" for the target quantity. In our solar panel example, "*a*" was time, "*b*" was angles, "*Q*" was power, and "*T*" was energy. Recall that for a basic integral, *a*, *Q*, and *T* are in a dependent MV quantitative relationship described by a Thompson triangle (Figure 3a). The nested MV relationship resides along the triangle's edge between the input quantity and the integrand quantity: *a* → *b* → *Q* (Figure 3b). Thus, *b* is always an intermediary between *a* and *Q*. We can now define u-substitution quantitatively as the act of shifting the vertex "*a*" of the

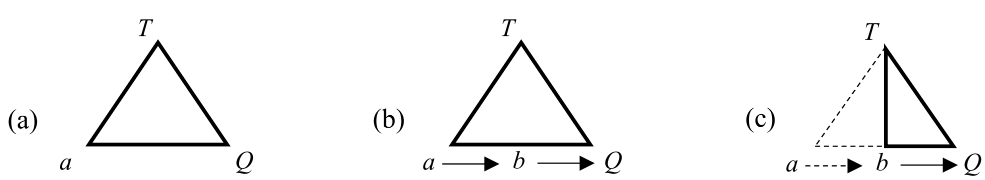

Figure 3. U-substitution as the transformation of the original dependent MV relationship (a); through the nested MV relationship between a, b, and Q (b); to a new dependent MV relationship (c).

original three-quantity triangle to the new input along this edge, “*b*,” thereby creating a new three-quantity triangle between *b, Q,* and *T* (Figure 3c).

The Three-part Structure within this Quantitative Definition

Having defined u-substitution as a transformation from one quantitative relationship to another closely-related quantitative relationship, as in Figure 3, we now examine the detailed structure contained within this transformation. Recall that the quantitative AUP structure for definite integrals contains three parts: *partition*, *target quantity*, and *sum*. U-substitution similarly contains a three-part structure, with the three parts corresponding to those in AUP.

The first part corresponds to AUP’s *partition* and involves converting the partition pieces from the original input quantity to *equivalent* partition pieces for the new input quantity. In the solar panel example, we saw that one cannot simply substitute *dθ* for *dt*, because it would change the size of the partition pieces. That is, a 0.001 second interval is not equivalent to a 0.001 radian interval, and a product between power and a 0.001 second interval does not yield the same energy as the product between power and a 0.001 radian interval. Rather, one must determine how big an infinitesimal piece of the new input quantity is in relation to an infinitesimal piece of the original input quantity. We call this part of the u-substitution structure: *differential* (Table 1).

The second part corresponds to AUP’s *target quantity*. It involves taking the integrand quantity that functionally depends on the original input and converting it to a functional relation based on the *new* input. In the solar panel example, power was dependent on time, and we converted it to a functional dependency on angle. This conversion permits the infinitesimal bits of the target quantity in each partition piece to be completely determined by the new input quantity, since both differential and integrand are now in terms of the new input quantity. We call this second part of the u-substitution structure: *integrand* (Table 1).

The third part is to re-describe the *summation*. The original sum is in terms of infinitesimal pieces of the original input quantity, and we need to instead describe it in terms of the new input quantity. In the solar panel example, we originally summed across infinitesimal time pieces between $0 \leq t \leq 6$, and we needed to switch to an equivalent sum across infinitesimal angle segments between $0 \leq \theta \leq \frac{\pi}{2}$. We call this third part of the structure: *bounds* (Table 1).

These three parts are exactly what enable the transformation from the original quantitative relationship to the new quantitative relationship by shifting the triangle’s vertex from *a* to *b*. If only some of these parts are enacted, the original input still exists as a vertex in the triangle, making it a four-quantity relationship in the triangle. While this is not incorrect, it creates issues for computation. In the solar panel example, when the differential in time was converted to a differential in angle, the quantitative structure still worked, but it was odd to have parts of the integral defined in one input quantity with other parts defined in another input quantity. Once the

Table 1. The three-part u-substitution structure, with original input a, new input b, and integrand quantity Q

Initial AUP parts	Corresponding parts in the u-substitution structure	
1. Partition: da	1. Differential: Convert partition from *da* pieces to equivalently-sized pieces in *db*.	$da \rightarrow [conversion]db$
2. Target quantity: $Q \cdot da$	2. Integrand: Convert integrand quantity, *Q*, from depending on *a* to depending on *b*. Target quantity now determined entirely by new input.	$Q(a) \rightarrow Q(b)$ $Q(a)da \rightarrow Q(b)[conv]db$
3. Sum: $\int_{a1}^{a2} Q(a)da$	3. Bounds: Convert sum from running across original input pieces to new input pieces.	$a_1 \leq a \leq a_2 \rightarrow$ $b_1 \leq b \leq b_2$

three parts of the u-substitution structure are enacted, the original input quantity's vertex is eliminated and the triangle transforms to the new vertex. Table 1 summarizes this three-part structure of u-substitution. Of course, we are quick to point out that in reasoning about u-substitution, one does *not* need to follow these three parts in a specific, prescribed order. Rather, it is possible to work through them in any order (see Oehrtman & Simmons, 2023).

Applying this Definition and Structure to Another (Non-Linear) Example

Having exposited the quantitative theory, we feel it beneficial to zoom back out to another contextual example to apply our quantitative definition and the three-part structure of the quantitative relationship transformation. We also use a context in which the relationship between the original input and the new input is *not* linear. Suppose a sphere is growing over time and we want to know the accumulated volume between $t = 5$ and $t = 10$ mins (Figure 4a, below). Note that at a given time the volume will grow at a rate proportional to its surface area, meaning there is a three-quantity relationship between *time*, *surface area*, and *volume* (Figure 4b). Here, time is the *input quantity*, volume is the *target quantity*, and surface area is the *integrand quantity*. Just to have something to work with, suppose the sphere's radius (in cm) is given by the function of time: $r(t) = t^2 + 5$ cm. Thus, surface area at any moment is $S(t) = 4\pi(t^2 + 5)^2$. Also, the instantaneous rate of radius growth is $dr/dt = 2t$ cm/min, so that for an essentially infinitesimal interval of time, dt, the radius grows by $dr = 2t \cdot dt$ cm. To more quickly get to u-substitution, we simply state that for an infinitesimal dt, volume will grow by the product of the surface area and the corresponding change in radius, leading to the integral: $V = \int_{t=5}^{t=10} 4\pi(t^2 + 5)^2\, 2t\, dt$.

We now ask the question specifically relevant to u-substitution: While volume grows over time, could we track its accumulation with respect to growth in the *radius* rather than *time*? This question is exactly what we claimed the domain of u-substitution to be. That is, we want to determine the total amount of the target quantity (volume), but we want to switch from one main *input quantity* (time) to another main *input quantity* (radius). Further, there is a nested MV relationship between *time* → *radius* → *surface area*, or $S(r(t))$ (Figure 4c). U-substitution will transform the original quantitative relationship into the new desired relationship (Figure 4d).

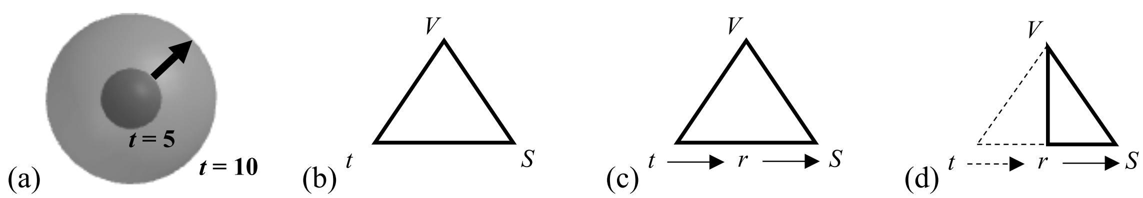

Figure 4. (a) growing sphere context; (b) time, surface area, volume quantitative relationship, (c) nested MV relationship between time, radius, and surface area, and (d) transformation to the new quantitative relationship

How do we enact this transformation? By using the three-part structure we proposed for u-substitution based on AUP. To show these can be done in any order, we purposefully use a different order here than for the solar panel example. First, the summation currently runs over *dt* pieces from $5 \leq t \leq 10$ min, but we could cover the exact same ground if we instead ran a summation over *dr* pieces from $30 \leq r \leq 105$ cm. This is the "*bounds*" part, making the integral so far: $V = \int_{r=30}^{r=105} 4\pi(t^2 + 5)^2\, 2t\, dt$. But the sum now is in terms of the radius, while all else is still described in terms of time. Second, we could re-describe the *integrand quantity* in terms of the radius. From the nested MV relationship, we have $4\pi(t^2 + 5)^2 = 4\pi\big(r(t)\big)^2 = S\big(r(t)\big)$, but since we want to eliminate time, we can rewrite this dependence simply as $S(r) = 4\pi r^2$. This is the "*integrand*" part, and makes the integral: $V = \int_{r=30}^{r=105} 4\pi r^2\ 2t\ dt$. Lastly, we need to convert the partition pieces in *dt* to partition pieces in *dr*. As explained before, a *dr* piece is not equivalent to a *dt* piece, so we need to know the conversion factor. While *t* and *r* do not have a linear relationship, we know that for any time *t*, dr will be exactly 2*t* times as big as *dt*: $dr = 2t \cdot dt$. This is the *differential* part and we now have a fully transformed integral from the original input quantity to the new input quantity: $V = \int_{r=30}^{r=105} 4\pi r^2\ dr$ (see Figure 4b-d).

Comparison to "Pure Math" U-Substitution

In a traditional "pure math" sense, u-substitution is depicted as undoing the chain rule: $\int_{x_1}^{x_2} f'\big(g(x)\big)g'(x)dx$. The procedure is to label the "inside" function, $g(x)$, as an arbitrary symbol "u", and to label $du = g'(x)dx$ in order to get $\int_{u_1}^{u_2} f'(u)\ du = f(u_1) - f(u_2)$. However, these symbols tend to be just notational conveniences and have little (if any) meaning, especially *du*. Here we show how our quantitative definition for u-substitution corresponds to this pure-math procedure and how it can provide conceptual meaning. If we think of *x*, f', and f as representing three quantities in a relationship, *u* is an intermediary quantity between *x* and f'. This is why *u* is always in a nested MV relationship with *x* and f', as in $x \rightarrow u \rightarrow f'$. In the quantitative paradigm, instead of *du* being just a notational convenience, $du = g'(x)dx$ actually represents the conversion factor from *dx* partition pieces to *du* partition pieces. Lastly, since we are converting from one input quantity to another, the switching of the bounds indicates a re-describing of the sum as it now ranges across infinitesimal pieces of the *new* input quantity.

Discussion

The contribution of this theoretical paper is to induct u-substitution into a quantitative paradigm, thereby extending the crucial prior work on quantitative reasoning in calculus (Jones & Ely, 2023; Oehrtman & Simmons, 2023; Thompson, 1994). In short, u-substitution deals with taking a quantitative relationship (Smith & Thompson, 2007; Thompson, 1990) in a definite integral and re-describing it through a *new* input quantity, where a nested MV exists between *original input → new input → integrand quantity* (Jones, 2022). We defined u-substitution as transforming this quantitative relationship by sliding the vertex from the original input to the new input. To fully accomplish this transformation, we described a three-part structure for u-substitution (*differential, integrand,* and *bounds*) that correspond to the three-part quantitative structure of AUP for definite integrals (Jones & Ely, 2023). These parts do not need to be executed in one specific order (Oehrtman & Simmons, 2023). In fact, a benefit to quantitative reasoning is actually reasoning about the context to flexibly carry out operations (Ely, 2017).

Our contribution allows for a quantitative reasoning paradigm in calculus based on informal infinitesimals (Ely, 2020) and AUP (Jones & Ely, 2023) to extend all the way through the introductory calculus curriculum, including u-substitution. This extension permits coherence across the entirety of the calculus course, founded on a reasoning type that has been shown to be extremely important for understanding and productive usage (Jones, 2015; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Pina & Loverude, 2019; Von Korff & Rebello, 2012). We are also building on this theoretical work through teaching experiments, teaching students u-substitution based on these quantitative meanings (which we report on in a separate paper). We believe that understanding u-substitution quantitatively in this way will enable students to actually understand the mechanisms behind it and to become more flexible users of it, as they can see what it means and the three-part structure for executing the quantitative transformation.

References

Amos, N. R., & Heckler, A. F. (2015). Student understanding of differentials in introductory physics. In A. Churukian, D. Jones, & L. Ding (Eds.), *Proceedings of the 2015 Physics Education Research Conference* (pp. 35-38). College Park, MD: American Association of Physics Teachers. https://doi.org/0.1119/perc.2015.pr.004

Bajracharya, R. R., Sealey, V. L., & Thompson, J. R. (2023). Student understanding of the sign of negative definite integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00202-y

Blomhøj, M., & Kjeldsen, T. H. (2007). Learning the integral concept through mathematical modelling. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the 5th congress of the European Society for Research in Mathematics Education* (pp. 2070-2079). ERME.

Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *The Journal of Mathematical Behavior, 55*, article #100694. https://doi.org/10.1016/j.jmathb.2019.03.001

Chhetri, K., & Oehrtman, M. (2015). The equation has particles! How calculus students construct definite integral models. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th annual Conference on Research in Undergraduate Mathematics Education* (pp. 418-424). SIGMAA on RUME.

Dray, T., & Manogue, C. A. (2023). Vector line integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00206-8

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152-167. https://doi.org/10.1016/j.jmathb.2017.10.002

Ely, R. (2020). Teaching calculus with infinitesimals and differentials. *ZDM--The International Journal on Mathematics Education, 53*(3), 591-604. https://doi.org/10.1007/s11858-020-01194-2

Hu, D., & Rebello, N. S. (2013). Understanding student use of differentials in physics integration problems. *Physical Review Special Topics: Physics Education Research, 9*(2), article #020108. https://doi.org/10.1103/PhysRevSTPER.9.020108

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R. (2022). Multivariation and students' multivariational reasoning. *Journal of Mathematical Behavior, 67*, article #100991. https://doi.org/10.1016/j.jmathb.2022.100991

Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00203-x

Katz, V. J. (2009). *A history of mathematics* (3rd ed.). Pearson Education.

Koretsky, M. D. (2012). *Engineering and Chemical Thermodynamics* (2nd ed.). Wiley.

Nguyen, D., & Rebello, N. S. (2011). Students' difficulties with integration in electricity. *Physical Review Special Topics: Physics Education Research, 7*(1), article #010113. https://doi.org/10.1103/PhysRevSTPER.7.010113

Oehrtman, M., & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00209-5

Pina, A., & Loverude, M. E. (2019). Presentation of integrals in introductory physics textbooks. In Y. Cao, S. Wolf, & M. B. Bennett (Eds.), *2019 PERC Proceedings* (pp. 446-451). AAPT.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 46-53). PMENA.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*(1), 230-245. https://doi.org/10.1016/j.jmathb.2013.12.002

Simmons, C., & Oehrtman, M. (2017). Beyond the product structure for definite integrals. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th annual Conference on Research in Undergraduate Mathematics Education* (pp. 912-919). SIGMAA on RUME.

Smith, J. P., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). Erlbaum.

Stevens, B. N., & Jones, S. R. (2023). Learning integrals based on adding up pieces across a unit on integration. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00204-w

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. San Diego State University.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274. https://doi.org/10.1007/BF01273664

Thompson, P. W., & Dreyfus, T. (2016). A coherent approach to the Fundamental Theorem of Calculus using differentials. In R. Göller, R. Biehler, & R. Hochsmuth (Eds.), *Proceedings of the Conference on Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 355-359). KHDM.

Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics: Physics Education Research, 8*(1), article #010125. https://doi.org/10.1103/PhysRevSTPER.8.010125

Von Korff, J., & Rebello, N. S. (2014). Distinguishing between "change" and "amount" infinitesimals in first-semester calculus-based physics. *American Journal of Physics, 82*(7), 695-705. https://doi.org/10.1119/1.4875175

A Framework for Time and Covariational Reasoning

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Within the line of work on students' quantitative reasoning, researchers have alluded to the significance of time in constructing covariational relationships. I draw on this body of literature and return to Piaget's perspective on time to provide a framework for the role of time in students' (co)variational relationships. The framework also clarifies the nature of the multiplicative objects underlying students' (co)variational relationships. In support of illustrating the framework and capturing its emergence from building second-order models of students' mathematics, I also describe a task and how its design reflects the framework.

Keywords: Covariational Reasoning, Time, Piaget, Quantitative Reasoning

Time has long been a topic of contemplation for researchers and philosophers, and ontological and epistemological considerations of time are certainly not restricted to the academy. Kant (1781/2003) considered time to be so ubiquitous as to be given a priori. Differing from Kant, Piaget viewed time as a concept an individual constructs. Accordingly, Piaget dedicated several studies to developing conceptual models of that construction (e.g., Piaget, 1954; Piaget, 1970). Based on his findings, Piaget proposed that the mental operations involved in constructing time are inseparable from space, motion, and objects (Piaget, 1970; von Glasersfeld, 1984). Building on researchers who have alluded to covariational reasoning being connected to time, I return to Piaget's (1970) conceptual models for time to further develop the role of time in students' (co)variational reasoning. In doing so, I elaborate on the constructs of *experiential time* and *conceptual time* (Castillo-Garsow, 2012; Thompson & Carlson, 2017) to provide a framework for characterizing students' (co)variational reasoning in relation to concepts of time. Reflecting its empirical roots, I illustrate the framework by describing a task designed to provide insights into the role of time with respect to students' (co)variational reasoning.

Covariational Reasoning and Time

The connection between motion, variation, and the concept of time has been indicated within work on students' covariational reasoning (e.g., Ellis et al., 2020; Johnson, 2015b; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015; Thompson & Carlson, 2017). Covariational reasoning—defined as the cognitive activities involved in reasoning about how quantities vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998)—is an emergent area of research within the landscape of quantitative reasoning. Researchers exploring covariational reasoning have illustrated its importance for the learning of concepts spanning middle, secondary, and undergraduate mathematics (Byerley & Thompson, 2017; Carlson & Oehrtman, 2004; Ellis, 2011; Ellis et al., 2015; Johnson, 2015a, 2015b; Moore, 2014; Paoletti et al., 2023; Thompson et al., 2017), with other researchers identifying its broader importance in STEM (Gantt et al., 2023; Rodriguez et al., 2019; Sokolowski, 2020; Yoon et al., 2021).

With respect to relationships between time and covariation or function, researchers have primarily focused on time as a parameter (Keene, 2007; Kertil et al., 2019; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015; Trigueros, 2004). These researchers have focused on the extent to which time is held implicitly or explicitly in mind by students as they construct and reason about relationships between quantities. For instance, Patterson and McGraw (2018) explored student meanings in the context of dynamic situations and their graphing quantitative relationships that did not include elapsed time as a graphed quantity. Relatedly, Paoletti and Moore (2017) explored how graphing experiences with quantitative relationships not explicitly involving elapsed time can create an intellectual need for time as a parameter. Taking a different approach, Stalvey and Vidakovic (2015) focused explicitly on students constructing relationships between elapsed time and two other quantities, and then their subsequent construction of a relationship between those two quantities.

Some of the aforementioned studies drew on notions of conceptual and experiential time, which Castillo-Garsow (2012) and Thompson (2011, 2012) introduced to characterize students' (co)variation. Having roots in Piaget's (1970) framing of time and Newtonian mathematics (Thompson, 2012), conceptual and experiential time are akin but not identical to explicit and implicit parametric distinctions. Whereas parametric distinctions focus on time as a distinct quantity, conceptual and experiential time are organic to quantities' (co)variation. Rather than framing time as implicit or explicit attribute in and of itself, time is framed as an emergent, intrinsic property of (co)variation that differs based on the (co)variation conception. Thompson (2012) described experiential time as "felt time that [passes]" in an experience, while conceptual time is part of the "flowing" of quantities and "Not time on a clock, but an imagined, smoothly changing, quantified time—a measured duration that grows in extent" (p. 147). The distinction between experiential and conceptual time is situated in how a phenomenon's attributes are conceived, reflecting Piaget's (1970) distinction between intuitive time and operational time.

Linking Piaget's Cognitive Account of Time and Covariation

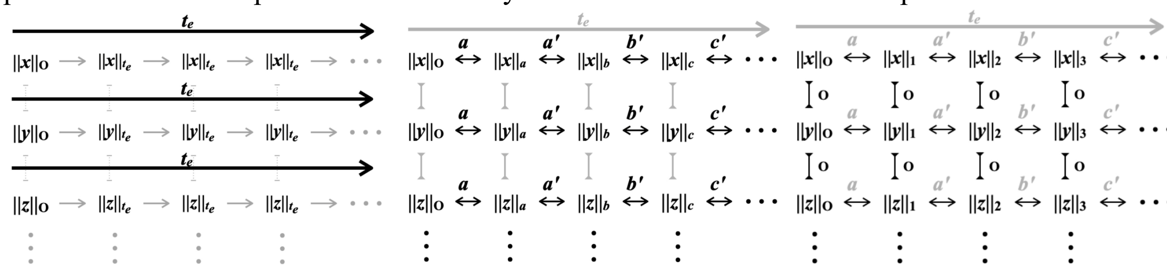
"We are far too readily tempted to speak of intuitive ideas of time, as if time, or for that matter space, could be perceived and conceived apart from the entities or the events that fill it" (Piaget, 1970, p. 1). Piaget considered time to be an emergent property of the co-ordination of simultaneous positions and the co-ordination of successive, spatial states. He referred to these co-ordinations as simultaneity and succession (with displacement), respectively, with their development occurring in the context of motions with different velocities. Piaget's view of time's link to conceptions of space and motion reflects his stance that concepts arise from the coordination and abstraction of mental actions. To Piaget, our temporal experience and memory of a situation are constructions subject to mental actions. We transition from intuitive to operative conceptions of time as we develop ways for organizing our experience that foreground operative forms of thought over experiential or figurative forms of thought (Piaget, 1970).

$$\begin{array}{ccccccccc}
\mathbf{O_1} & \xrightarrow{a} & \mathbf{A_1} & \xrightarrow{a'} & \mathbf{B_1} & \xrightarrow{b'} & \mathbf{C_1} & \xrightarrow{c'} & \cdots \\
\downarrow o & & \downarrow o & & \downarrow o & & \downarrow o & & \\
\mathbf{O_2} & \xrightarrow{a} & \mathbf{A_2} & \xrightarrow{a'} & \mathbf{B_2} & \xrightarrow{b'} & \mathbf{C_2} & \xrightarrow{c'} & \cdots \\
\downarrow o & & \downarrow o & & \downarrow o & & \downarrow o & & \\
\mathbf{O_3} & \xrightarrow{a} & \mathbf{A_3} & \xrightarrow{a'} & \mathbf{B_3} & \xrightarrow{b'} & \mathbf{C_3} & \xrightarrow{c'} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Figure 1. Piaget's co-seriation model of events, simultaneity, and succession. (Piaget, 1970, p. 264)

Piaget (1970) formalized the construction of simultaneity and succession of events as a grouping (i.e., co-seriation) shown Figure 1. $O_{\#}$ represents the initial state of event # (e.g., an attribute of an object/phenomenon like position in visual field, weight, or color). $A_{\#}$, $B_{\#}$, $C_{\#}$, and so on represent successive states of event #. a, a', b', c', and so on represent durations such that $b = a + a'$, $c = b + b'$, and so on. Piaget used $\downarrow o$ to link states of events occurring simultaneously (e.g., an object's weight and height), which can be thought of as a null vector due to the events' simultaneity. Piaget's (1970) model captures the multiplicative basis of co-seriation, in which events are united to form a *multiplicative object*—the cognitive uniting of attributes so that an object is simultaneously all of them (Inhelder & Piaget, 1964). As I illustrate below, constructing such an object is fundamental to the covariation of quantities (Saldanha & Thompson, 1998).

Drawing on Piaget's model of time and the simultaneity and succession of events, I present three conceptual models of time as it relates to an individual's conception of a phenomenon that entails quantities' magnitudes varying (e.g., $||x||$, $||y||$, $||z||$,…). The first model (Figure 2a) conveys a conception tied to experiential time. The second and third models (Figure 2b-c) each convey a conception tied to conceptual time. The second foregrounds the quantities as conceived with respect to elapsed time, while the third involves disembedding the quantities from the phenomenon and elapsed time so that they exist in an invariant relationship with each other.



(a) (b) (c)

Figure 2. Conceiving a phenomenon and quantities (a) with respect to experiential time, (b) with respect to conceptual, elapsed time, and (c) so they are disembedded with respect to time and understood in terms of their invariant relationship.

Adopting expression notation and restricting the focus to two quantities, we can represent Figure 2a, Figure 2b, and Figure 2c with $||x||_{t_e} \vee ||y||_{t_e}$, $(||x||_t \vee ||y||_t)$, and $(||x||_\Delta \wedge ||y||_\Delta)$, respectively. I use $||x||_{t_e} \vee ||y||_{t_e}$ with $\vee$ (OR) and no parentheses to indicate that when a phenomenon and its constituent quantities are conceived with respect to experiential time, the quantities are both understood as present and varying in experience. They are observed to co-occur, but they are not cognitively linked beyond that. A conception of their relationship involves sequentially recalling and possibly, but not necessarily, comparing the intuitive, in-the-moment experience of each quantity's variation. This is captured by the weak link between $||x||$ and $||y||$ in Figure 2a and foregrounding experiential time, t_e, with each quantity's variation.

I use $(||x||_t \vee ||y||_t)$ and $(||x||_\Delta \wedge ||y||_\Delta)$ to indicate a phenomenon and its constituent quantities conceived with respect to conceptual time, whether elapsed (t) or their relationship disembedded and understood with respect to variation (Δ) from another state. With respect to $(||x||_t \vee ||y||_t)$, I use parentheses to indicate that the quantities are understood as occurring simultaneously, but I use $\vee$ to indicate that elapsed time is the driver of the relationship such that each quantity exists in a multiplicative object with elapsed time but not with each other. The two quantities are related through their sharing a relationship with elapsed time. This is captured by the link between $||x||$ and $||y||$ in Figure 2b, which is stronger than that in Figure 2a but mitigated by the connection to elapsed time. With respect to $(||x||_\Delta \wedge ||y||_\Delta)$, I use $\wedge$ (AND) and parentheses to indicate that the quantities are understood as occurring simultaneously and persistently. One quantity's magnitude is held in mind with the "immediate, explicit, and persistent realization that, at every [magnitude], the other quantity also has a [magnitude]" (Saldanha & Thompson, 1998, p. 298). The quantities' magnitudes are the driver of the relationship, and thus properties of the relationship are understood as defining the multiplicative object formed by joining the two quantities' magnitudes and thus are sustained irrespective of elapsed time or figurative aspects of experience. This is captured by the link between $||x||$ and $||y||$ in Figure 2c, which indicates their simultaneous and persistent co-existence so that their covariation is defined precisely by their simultaneous variations. Figure 2b and Figure 2c each indicates a bi-directional relationship between states to reflect the operational nature of conceptual time (Piaget, 1970). Figure 2c indicates measured durations fade to the background so the relationship is not tied to any particular experience or measured duration.

Illustrating the Framework - Time and Task Design

The task illustrated here emerged during a teaching experiment with undergraduate mathematics education students as part of a larger project focused on capturing middle grades and undergraduate students' reasoning within dynamic situations (see Liang and Moore (2021), Lee et al. (2019), Tasova and Moore (2020), and Moore et al. (2019)). With respect to the task below, the project team drew on two sources of inspiration beyond the second-order models of student thinking that emerged during the teaching experiment (Steffe & Thompson, 2000; Thompson, 2008). As one source, we drew on the tasks demonstrated by Saldanha and Thompson (1998) and Carlson et al. (2002) that involve covarying quantities other than time. Tasks that prompt students to construct graphs with respect to time make it difficult for a researcher to tease out whether the student is reasoning with respect to conceptual or experiential time (Thompson & Carlson, 2017). The task below includes two distances (i.e., magnitude bars that provide figurative material to enact quantitative and covariational operations) with no reference to elapsed time. Piaget's (1970) aforementioned work on time provided the second source of inspiration for the task. Piaget described, "It is only by the co-ordination of at least two motions with different velocities that purely temporal relationships can be distinguished from spatial relationships or from intuitive ideas about motion" (p. 26). The task foregrounds relations of simultaneity and succession via prompting the participants to coordinate two objects in motion, with the two objects varying at different rates with respect to elapsed time.

The Task: Which One? – Going Around Gainesville (GAG)

"Which One? – GAG" is from a series of tasks titled "Which One?" A "Which One?" task is designed to be implemented after a participant constructs a covariational relationship within phenomenon or a graphical representation (Liang & Moore, 2021). A "Which One?" task provides several representations of covariational relationships, including magnitude bar sets that vary simultaneously or a collection of static or dynamic graphs. With the representations provided, the researcher asks the participant which of the representations, from none to all, accurately capture the relationship they identified previously (whence the name, "Which One?").

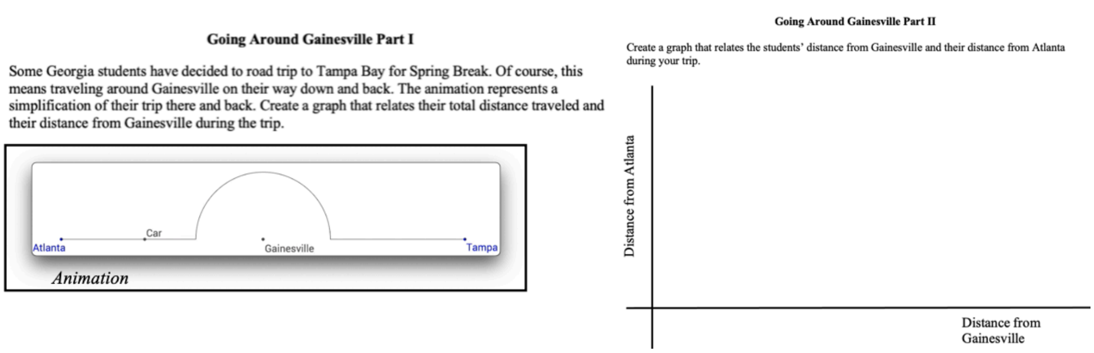


Figure 3. The Going Around Gainesville (GAG) task, video at: https://youtu.be/v2yc55Z9WV8.

The part preceding "Which One? – GAG" involves a video depicting a car starting in Atlanta and traveling back and forth from Tampa (Figure 3, see Moore et al. (2022) and Moore et al. (2019) for empirical data). After viewing the animation, the participant is sequentially asked two graphing tasks (Figure 3). After a participant engages in each part and has constructed what the research team perceives to be a stable understanding of the covariational relationship, the researcher implements the three-part task "Which One? – GAG". Each part consists of three pairs of magnitude bars presented in a *dynamic geometry environment* (DGE). As support for the reader, https://tinyurl.com/4v9ma7pc hosts videos illustrating each part and pair of the task. For Part I of the task (see Figure 4a for a snapshot), the participant is presented with three tabs, each containing a pair of magnitude bars. For each pair, one magnitude bar represents the *distance from Atlanta* (*dfA*) and one magnitude bar represents the *distance from Gainesville* (*dfG*). For each pair, the student can push "Drive" to start or stop the bars changing together, and the student can push "Reset" to return the pair to a zero-magnitude *dfA* and corresponding initial *dfG*. The participant is tasked with determining which, if any, of the pairs covary as to accurately capture the determined relationship between the *dfA* and the *dfG*. Table 1 describes the design of each magnitude pair. Pair B and C capture the normative relationship between the two distances.

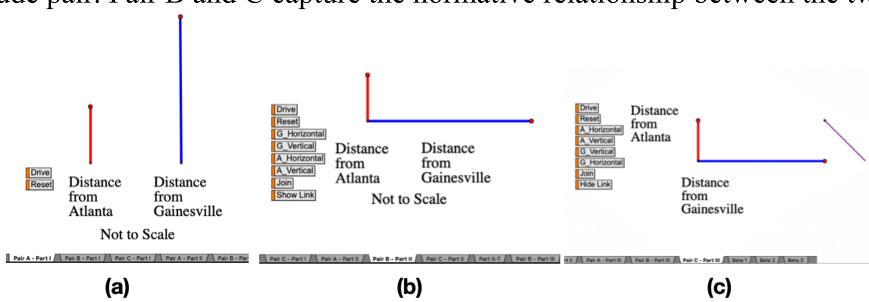
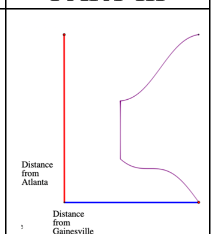
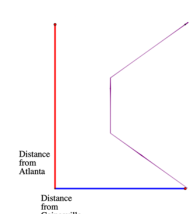
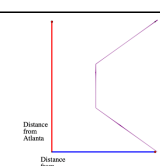


Figure 4. Example still shots for (a) Pair A – Part I, (b) Pair B – Part II, and (c) Pair C – Part III.

Table 1. The design of "Which One? – GAG".

RELATIONSHIP DESIGN	PART III
Pair A: With respect to *dfA*: *dfG* decreases at an increasing rate, decreases at a decreasing rate, remains constant, increases at a decreasing rate, and then increases at an increasing rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate. (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	
Pair B: With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate. (ii) *dfG* decreases at a decreasing rate, decreases at an increasing rate, remains constant, increases at an increasing rate, and then increases at a decreasing rate.	
Pair C: With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a constant rate. (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	

Part II of the task (see Figure 4b for a snapshot) presents the same three pairs of magnitude bars, but they can reorient the magnitude bars, join then, and show a "link" between them. This link represents the process of joining two orthogonal magnitudes to form a Cartesian point. A participant is told that each pair matches its respective pair from Part I (e.g., Pair A in Part I, II, and III covary equivalently), and that Part II of the dynamic sketch is designed to help them further explore the extent the two magnitude bars capture the determined relationship between the two distances. For Part III of the task (see Figure 4c for a snapshot), the participant is again presented with the same three pairs of magnitude bars. In this case, each pair is oriented orthogonally, a Cartesian point is displayed, and a trace of the point is recorded as the magnitude bars covary. Like Part II, the participants are told that each pair matches its respective pair from Part I, and that Part III is to aid further exploring the extent the two magnitude bars capture the appropriate relationship between the two distances. During Part II and Part III, a participant is also prompted to reflect on and describe any changes in their assessment of the paired magnitudes. They can return to the previous parts if desired. They are also asked to reflect on difficulties from previous parts and how subsequent parts assist their assessment. Said frankly, Part I is intended to be difficult, both conceptually and in functional design, with the hopes of both eliciting their thinking and affording spontaneous requests for other representations.

Connecting the Task to the Framework

First focusing on Figure 2a (i.e., $||x||_{t_e} \vee ||y||_{t_e}$), and reflecting quantities' variations occurring in experiential time, a student reasoning in such a way attends to the variation of each magnitude separately, and they primarily do so through the experience of watching the DGE animated continuously using "Drive". With respect to Pair A, the student might conclude that *dfG* varies appropriately due to its smooth decrease, constancy, and then increase, while concluding that *dfA* varies incorrectly. For the latter, they anticipate that *dfA* increase at a smooth rate, which reflects the manner in which it increases during the experience of watching the road trip animation. With respect to Pair B, and consistent with their response to Pair A, the student might conclude that *dfG* and *dfA* vary inappropriately due to anticipating both increases or decreases at smooth rates, again reflecting how they experience the variations with the road trip animation. With respect to Pair C, the student is likely to conclude that both *dfG* and *dfA* vary appropriately due to the smooth variation of each. Across all of the pairs, the student primarily focuses on each magnitude separately and draws on intuitive or experiential notions of rate to draw conclusions.

For Figure 2b (i.e., $(||x||_t \vee ||y||_t)$), due to the basis in conceptual time, a student reasoning in such a way attends to the variation of each magnitude separately, but they coordinate the variation of each using successive durations of elapsed time. This might be accomplished by stepping through states of the DGE and tracking the variation of each quantity with anticipated properties in mind. With respect to Pair A, as the student tracks through successive, equal duration states of the DGE, the student might conclude that although *dfG* varies by constant amounts, *dfA* does not vary by constant amounts and thus the magnitude bars do not capture the appropriate relationship. With respect to Pair B, the student might comment on the difficulty assessing the pair using the DGE and thus seek to step through the DGE state by state. Reflecting that the quantities are cognitively linked through their shared relationship with elapsed time in this form of covariation, the student might attempt to "Drive" the bars for equal durations of time and then compare the variations of the magnitudes to each other. With respect to Pair C, the student is likely to conclude that the pair covaries appropriately due to the smooth variation of each, and they might further test this by using successive, equal durations of "Drive". Across all of the pairs, the student coordinates each magnitude with equal durations in order to draw comparisons across the magnitudes. Because of this, Pair B can lead to a perturbation that stems from the student anticipating equal variations in each quantity for equal variations in duration due to the piecewise linear relationship between *dfG* and *dfA*.

For Figure 2c (i.e., $(||x||_\Delta \wedge ||y||_\Delta)$), due to the basis in a disembedded invariant relationship, a student reasoning in such a way foregrounds coordinating a quantity's variation with respect to the other quantity's variation. Whether Pair A, B, or C, the student is likely to attempt to vary one quantity's magnitude in a systematic way while tracking the variations in the other quantity's magnitude. For instance, the student might use "Drive" to step *dfA* through successive, equal increases, and then assess the appropriateness of the pair by investigating whether the *dfG* magnitude follows the pattern of constant decrease, constant, and constant increase. A student engaging in such covariational reasoning might experience a perturbation stemming from the functionality of the DGE (e.g., it is difficult to use "Drive" to step through equal amounts of *dfA* increase), but they would not be significantly perturbed by how a single bar varies as the animation plays. They persistently foreground how the bars simultaneously covary, which can lead to expressing annoyance at Part I and motivating a need for Parts II-III and a graph.

Closing

The three forms of (co)variational reasoning differentiate (co)variation based on the role of time and, hence, the extent a multiplicative object is formed between the two quantities. The three forms invite questions regarding their developmental and hierarchical nature. The three forms emerged from work conducted primarily with undergraduate students, and I do not have second-order models of their developmental trajectory and relationships. I hypothesize the continued work by colleagues such as Ellis, Johnson, Lee, Paoletti, and Tasova will provide such insights. With respect to hierarchy, there is a relative increase in sophistication and generativity from Figure 2a to Figure 2c that is reflected in Piaget's exposition of time, as well as Carlson, Castillo-Garsow, Saldanha, and Thompson's descriptions of (co)variation. This relativeness is captured by Patterson and McGraw (2018), who described,

> We hypothesize that it is advantageous to be able to envision the covariation between two dynamically changing quantities and, to some degree, decouple this image of covariation from a unidirectional, experiential image of the passage of time. This process is essential for developing an understanding of an invariant relationship between two quantities and explaining how changes in one variable constrain changes in another variable. (p. 320)

The authors hedge in their hypothesis, as the process of decoupling quantities' covariation from experiential time is intrinsic to the form of covariation captured in Figure 2c and, more broadly, that suggested by Carlson, Castillo-Garsow, Saldanha, and Thompson. Constructing a multiplicative object between quantities' magnitudes necessarily involves decoupling images of variation from experiential or specific passages of elapsed time. It is then that two quantities' variations are taken as objects of thought and united so that an invariant relationship is constructed to constrain the two quantities' simultaneous variations. Although the forms have a hierarchical nature, the implications of such remain an open question. This is particularly true as it relates to how the forms of (co)variation play a role in students constructing concepts in which covariational reasoning provides a foundation, such as rate of change and accumulation.

References

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior*, *48*, 168-193.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378. https://doi.org/10.2307/4149958

Carlson, M. P., & Oehrtman, M. (2004). Key aspects of knowing and learning the concept of function. In A. Selden & J. Selden (Eds.), *MAA Notes Online, Research Sampler Series*. The Mathematical Association of America. http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/9-key-aspects-of-knowing-and-learning-the-concept-of-function

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context* (pp. 55-73). University of Wyoming.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early Algebraization* (pp. 215-238). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_13

Ellis, A. B., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: supporting students' algebraic reasoning. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-020-09951-6

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, *39*, 135-155. https://doi.org/10.1016/j.jmathb.2015.06.004

Gantt, A. L., Paoletti, T., & Corven, J. (2023). Exploring the Prevalence of Covariational Reasoning Across Mathematics and Science Using TIMSS 2011 Assessment Items. *International Journal of Science and Mathematics Education*. https://doi.org/10.1007/s10763-023-10353-2

Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child: Classification and seriation*. Routledge & Kegan Paul.

Johnson, H. L. (2015a). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, *17*(1), 64-90.

Johnson, H. L. (2015b). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, 1-22. https://doi.org/10.1007/s10649-014-9590-y

Kant, I. (1781/2003). *Critique of pure reason* (M. Weigelt, Trans.). Penguin Classics.

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*(3), 230-246. https://doi.org/https://doi.org/10.1016/j.jmathb.2007.09.003

Kertil, M., Erbas, A. K., & Cetinkaya, B. (2019). Developing prospective teachers' covariational reasoning through a model development sequence. *Mathematical Thinking and Learning*, *21*(3), 207-233. https://doi.org/10.1080/10986065.2019.1576001

Lee, H. Y., Moore, K. C., & Tasova, H. I. (2019). Reasoning within quantitative frames of reference: The case of Lydia. *The Journal of Mathematical Behavior*, *53*, 81-95.

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: A student's meanings for amounts of change in covarying quantities. *Mathematical Thinking & Learning*, *23*(4), 291-317.
Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102-138.
Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education* (pp. 35-69). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_3
Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*. https://doi.org/10.1016/j.jmathb.2019.01.008
Paoletti, T., Gantt, A. L., & Corven, J. (2023). A Local Instruction Theory for Emergent Graphical Shape Thinking: A Middle School Case Study. *Journal for Research in Mathematics Education*, *54*(3), 202-224.
Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior*, *48*, 137-151. https://doi.org/10.1016/j.jmathb.2017.08.003
Patterson, C. L., & McGraw, R. (2018). When time is an implicit variable: An investigation of students' ways of understanding graphing tasks. *Mathematical Thinking and Learning*, *20*(4), 295-323. https://doi.org/10.1080/10986065.2018.1509421
Piaget, J. (1954). *The construction of reality in the child* [doi:10.1037/11168-000]. Basic Books. https://doi.org/10.1037/11168-000
Piaget, J. (1970). *The child's conception of time* (A. Pomerans, Trans.). Basic Books.
Rodriguez, J.-M. G., Bain, K., Towns, M. H., Elmgren, M., & Ho, F. M. (2019). Covariational reasoning and mathematical narratives: investigating students' understanding of graphs in chemical kinetics [10.1039/C8RP00156A]. *Chemistry Education Research and Practice*, *20*(1), 107-119. https://doi.org/10.1039/C8RP00156A
Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
Sokolowski, A. (2020). Developing Covariational Reasoning Among Students Using Contexts of Formulas. *Physics educator.*, *2*(4). https://doi.org/10.1142/S266133952050016X
Stalvey, H. E., & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *The Journal of Mathematical Behavior*, *40*, 192-210.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Erlbaum.
Tasova, H. I., & Moore, K. C. (2020). Framework for representing a multiplicative object in the context of graphing. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North*

American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 210-219). Cinvestav/PME-NA.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57).

Thompson, P. W. (2012). Advances in research on quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *WISDOMe monographs (Vol. 2) Quantitative reasoning: Current state of understanding* (pp. 143-148). University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421-456). National Council of Teachers of Mathematics.

Thompson, P. W., Hatfield, N., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior*, *48*, 95-111. https://doi.org/10.1016/j.jmathb.2017.08.001

Trigueros, M. (2004). Understanding the meaning and representation of straight line solutions of systems of differential equations. Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,

von Glasersfeld, E. (1984). Thoughts about space, time, and the concept of identity. In A. Pedretti (Ed.), *Of of: A book conference* (pp. 21-36). Princelet Editions.

Yoon, H., Byerley, C. O. N., Joshua, S., Moore, K., Park, M. S., Musgrave, S., Valaas, L., & Drimalla, J. (2021). United States and South Korean citizens' interpretation and assessment of COVID-19 quantitative data. *The Journal of Mathematical Behavior*, *62*, 100865. https://doi.org/https://doi.org/10.1016/j.jmathb.2021.100865

Genre Theories and Their Potential for Studying Proof

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In this theoretical report, we outline several major traditions of genre theory, argue that proof qualifies as a genre as defined by these traditions, and propose how genre analyses could be used to further our field's understanding of proof.

Keywords: genre, genre theory, proof, move analysis, critical genre analysis

Mathematics education researchers have engaged intensively with proof and the learning and teaching thereof, but rarely have they drawn on genre theories in this pursuit. In this theoretical report, we outline several major traditions of genre theory, argue that in line with these traditions proof is a genre, and propose how genre analyses could be used to further our field's understanding of proof.

A Brief Overview of Genre Theories

Since the 1980s, "genre" has received significant attention in the Anglophone world of applied linguistics from scholars of three research areas: English for Specific Purposes (ESP), Rhetorical Genre Studies (RGS) (sometimes referred to as "New Rhetoric" studies), and Systemic Functional Linguistics (SFL) (sometimes referred to as the "Sydney School") (Hyon, 1996, 2018b; Swales, 2012). All three traditions are united in viewing genre as a social practice (Tardy, 2012): ESP scholars define genre as communicative events (within discourse communities) with shared communicative purpose(s) (Swales, 1990), RGS scholars define genre in terms of the "action it is used to accomplish" (Miller, 1984, p. 151), and SFL scholars define genre as "a staged, goal oriented social process" (Martin et al., 1987, p. 59). As Tardy (2011) summarized:

> [The three traditions] agree on several general characteristics of genre as a category of discourse:
> - Genres are primarily a rhetorical category
> - Genres are socially situated
> - Genres are intertextual, not isolated
> - Genres are carried out in multiple—and often mixed—modes of communication
> - Genres reflect and enforce existing structures of power. (p. 55)

That said, the three traditions continue to differ, for example, in terms of: (a) the genres studied, (b) the attention paid to linguistic features, (c) the attention paid to institutional contexts, and (d) methods employed (Hyon, 1996, 2018b).

In addition to the three aforementioned traditions for studying genre (i.e., ESP, RGS, SFL), Swales (2012)—father of ESP genre theory—noted that there also exist "[t]he Brazilian approach to genre (Vian, 2012) and the Academic Literacies movement, sometimes known as the 'New London School'" (p. 113). Although it is incorrect to think of the Brazilian tradition as a monolith (Vian, 2012), it is correct to note that genre research in Brazil has been heavily influenced by Bakhtin's (1986b) work on genre (Gomes-Santos, 2003) and, albeit it to a lesser extent, socio-discursive interactionism (Bronckart, 1997). The Academic Literacies movement finds the ESP tradition too textual and pushes for a greater focus on academic practices (Swales, 2012). For discussion of further traditions of genre theory, see Vian (2012).

The Genre of Proof

In this section, we argue that proof satisfies Tardy's (2011) five characteristics of genre. We thereby wish to justify applying genre theories (particularly ESP, RGS, and SFL) and genre analyses to the genre of proof.

Proof Is Primarily a Rhetorical Category

By being "primarily a rhetorical category," Tardy (2011) meant that "what makes a text a genre is not its linguistic form but the rhetorical action that it carries out in response to the dynamics of a social context" (p. 55). As Hanna (2000) listed, proofs can be used to, for example, verify, explain, systematize, discover, communicate, construct, explore, and incorporate.[1] And, Hanna (1989) argued, some of these functions (i.e., rhetorical actions) may be more appropriate in certain social contexts—"proofs that explain should be favored in mathematics education over those that merely prove" (p. 45). That dynamics of different social contexts can lead to different rhetorical actions was also demonstrated by Schifter (2009): In one elementary school classroom, students used a proof as an aid to understanding; in the other, students used a proof to convince. In short, we argue that proofs serve a set of communicative purposes that depend on the discourse community in question, that is, as primarily a rhetorical category.

Proof Are Socially Situated

To assert that proofs are "socially situated" according to Tardy (2011), we should be able to observe that conventionalized forms of proof are tied to their socio-rhetorical context (e.g., a class of 4th graders and their teacher, Poisson geometers). Stylianides's (2007) frequently cited definition of proof acknowledges this centrality of the socio-rhetorical context by qualifying that accepted forms of statements, modes of argumentation, and modes of argument representation used in proofs are only "accepted" in that they are endorsed *by the classroom community*—or, as we would amend using ESP terminology, a given discourse community.

To underline that proofs are socially situated and that their form is tied to their socio-rhetorical context, consider the following three examples: (a) a two-column proof in a high school geometry classroom, (b) a proof by Gauß in *Disquisitiones Arithmeticae*, and (c) a proof told as a first-person singular narrative (e.g., "First, I noticed that if you take …") in an elementary school classroom. All three proofs vary significantly in form but, assuming their veracity, are valid forms of proof within their socio-rhetorical contexts.

Proof Are Intertextual, Not Isolated

"[T]he communicative work that genres do is almost never carried out by isolated, single texts. Rather genres work in coordination to accomplish complex tasks and social goals" (Tardy, 2011, p. 58). Below, we outline two ways in which proofs are intertextual.

First, proofs often explicitly reference previously proven statements (e.g., "By Proposition 4.2, …"). Thus, deliberate connections are made in proofs to other texts. All such intertextual references in proofs must be carefully organized so as to avoid circular reasoning. More generally, all proofs implicitly reference axioms and/or previously proven statements due to how mathematical theories are logically organized as inverted pyramids.[2]

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[1] We acknowledge that although "verifying" is not the only communicative purpose of the genre of proof, it is a central one, and it is hard to make sense of proof without thinking about proving that a statement is true or false.

[2] In this analogy, the point of the inverted pyramid is axioms, upon which a progressively broader set of theorems is built.

Second, proofs are intertextual in that they work in coordination with other genres, such as definitions, theorems, and remarks. (Mariotti, 2006, argued that proofs must be understood as a triad of proof, statement, and background theory.) Often, a definition is followed by the statement of a theorem (or proposition or lemma) about the just-defined object or property, which is then proven. After the proof, an author may make a remark, which offers, for instance, observations about the just-finished proof. (For a genre analysis of remarks that illustrates different communicative purposes of remarks, see Küchle, 2023.) Sequences of definitions, theorems, proofs, remarks, and other mathematical genres (e.g., examples, exposition) can combine to form larger narratives (e.g., a sequence culminating in Lagrange's Theorem)—narratives that university instructors can backwards engineer.

Proofs Are Carried Out in Multiple—and Often Mixed—Modes of Communication

As O'Halloran (2015) noted, mathematics is multimodal: Mathematics uses linguistic, symbolic, and visual forms of representation. We assert that proof is also multimodal: Proofs consist of words, symbols, and images (e.g., graphs, diagrams). Yet, we acknowledge that although the value of images during proof-production has been suggested (e.g., Alcock & Weber, 2008; Weber & Alcock, 2004), the question remains to what extent a valid proof may be visual. Larvor (2019) addressed this question by showing that although not all types of diagrammatic inferences are typically allowed in proofs, there is a class of such inferences that are often allowed. In addition to satisfying certain diagram characteristics (Larvor, 2019), we believe that a proof's socio-rhetorical context—context that includes level and subdiscipline of mathematics—contributes to the permissibility of images in proof.

Proofs Reflect and Enforce Existing Structures of Power

As Dawkins and Weber (2017) noted, there are values and norms at play in proof-writing. Further, where there are values and norms at play, power dynamics emerge. Thus, Tardy (2011) deduced, "[genres] must be viewed as not just a reflection but also a reinforcement of the power structures that exist in the community within which they are used" (p. 60). Indeed:

> Genres are forms of "symbolic power" (Bourdieu, 1991, p. 163) and could be forms of "symbolic violence" (Bourdieu, 1991, p. 139) if they create time/spaces that work against their users' best interests and their users perceive them as naturalized or "just the way we do things around here." (Schryer, 2002, p. 76)

Thinking about the educational setting, Tardy (2011) observed: "School genres […] inherently situate students in low-power positions, subject to the evaluation and preferences of teachers, who serve as gatekeepers" (p. 60).

Tardy's (2011) observation about school genres also holds for proofs in educational settings. Even proofs written by research mathematicians for publication are subject to a power difference between the authors in low-power positions and the editors and reviewers as gatekeepers. In educational and academic settings, the person(s) in high-power positions uses their set of values and norms to evaluate the proof of the person(s) in low-power positions (see Tanswell & Rittberg, 2020). For instance, a proof may be considered "too verbose" or "make too many omissions." That these types of critiques are subject to bias and can (re)enforce existing structures of power—in our case, white supremacist capitalist patriarchy (hooks, 2015)—can be seen by considering the example of Su's (2020) student Akemi, whose proofs stopped receiving full marks once the teaching assistant found out Akemi was a woman.

To see how proofs may reflect existing structures of power, consider the language in which they are written:

> [T]he mode in which mathematic is usually written and spoken is one of advocacy, of claims and assertions, one which generally ignores its audience. It is a language which I feel is more easily adopted by men than women, if we can believe what we are told about women using fewer declarative sentences in conversation. And it is a language, which particularly, when spoken, is frequently abused with impatience, frustration, and defensiveness. (Keith, 1988, p. 8)

Thus, proofs—and the language they are written in—may reflect stereotypical traits of masculinity (Hottinger, 2016; Weber & Melhuish, 2022).

Summary

We contend that proof satisfies Tardy's (2011) list of five genre characteristics on which the three central Anglophone traditions of genre theory can agree. Thus, we feel justified in applying these theories' genre analysis tools to the genre of proof.

Analysis Possibilities

In this section, we explore two types of genre analysis from the three traditions and see how they could be applied to the genre of proof.

Move Analysis

In the ESP tradition, a genre is often studied by conducting a move analysis, which focuses on moves, "those textual segments that make up a genre's organizational structure and help the genre achieve its purposes" (Hyon, 2018a, p. 27). They are functional units (i.e., they are not about what a text is saying but about what the text is doing) and can be thought of as mini communicative purposes (Hyon, 2018a). For example, a common move in research article introductions is "establishing a niche," which can be realized by, for instance, indicating a gap in research (Swales, 1990). As Bhatia (1993) noted, moves need not occur linearly; they can also be interactive (e.g., in legislative texts, legislating and specifying can be thought of as a two-part interactive move).

A common goal of performing a move analysis is to determine a move structure that can be shared with students to aid their learning (Hyon, 2018a). To this end, mathematics education researchers could use, for example, Hyon's (2018a) stages of a move analysis—based on Biber et al.'s (2007) and Upton and Cohen's (2009) approaches. A move such an analysis might yield is *indicating deduction*, which authors realize in many ways (e.g., "[X]. Thus/hence/therefore, [Y]", "Since/Because [X], (it follows that) [Y]", "If [X], then [Y]", "From [X], we get that [Y]", or "[X] implies that [Y]"). This move can serve several common communicative purposes of proof: by *indicating deduction*, the author is able to more easily *convince* the reader since the text has been *systematized* and aids the reader's *verifying*. Other example moves might include setting a goal or initiating a proof technique.

A move analysis may also shed light on cross-cultural differences, be they differences between, what Holliday (1999) termed, "large cultures" (e.g., proofs in Germany versus those in the U.S.A) or "small cultures" (e.g., proofs by undergraduates in an introduction to proof [ITP] course versus those by research mathematicians) (Hyon, 2018a).

To illustrate what a cross-cultural difference might look like, consider our following informal observation. While skim-reading proofs in research papers and in first-year lecture notes, we noticed that mathematicians-as-teachers frequently *(re)express the statement to be proved*—a move seen but once in the research papers. Consider the following examples from Linear Algebra, Abstract Algebra, and an ITP course:

Proposition 5.3. Let $T : U \to V$ be a linear map. Then:
(*i*) im(T) is a subspace of V;
(*ii*) ker(T) is a subspace of U.
Proof. For (*i*), we must show that im(T) is closed under addition and scalar multiplication. […]

Theorem VI.6. *Let G be a group and a, b* $\in G$. *Then* $(ab)^{-1} = b^{-1}a^{-1}$.
[…]
PROOF. We're being asked to prove that $b^{-1}a^{-1}$ is the inverse of ab. So we want to show that $(b^{-1}a^{-1})(ab) = 1 = (ab)(b^{-1}a^{-1})$. […]

Example 8.11. State whether the following sets are bounded or unbounded and prove your answer. […]
(b) $B = (-\infty, \sqrt{2})$. […]
Solution: […]
(b) $B = (-\infty, \sqrt{2})$ is unbounded. We will prove this by showing that for all $M \in \mathbb{R}$ we can find $x_0 \in B$ such that $|x_0| > M$. […]

(Re)expressing the statement to be proved seems pedagogical in nature as it breaks down what needs to be shown. Although this move may seem innocuous, we would like to point out its potential for "deceiving" the student into thinking that nothing has happened. Yet, (re)expressing, or "unpacking," what needs to be shown is often a key step in first-year proofs, which frequently amount to rewriting/seeing something in a new way. Therefore, this move can significantly simplify the proof. Although the move might be thought of as the instructor modeling "good" proof behavior, it also removes the opportunity for students to do this work themselves. The study of e-Proofs suggests that in the interplay between work done for the student reader by the author and work done by the student reader to make sense of the proof, sometimes learning is supported better when the student reader's work is not co-opted by the text (Alcock et al., 2015).

In short, given differences in intended audience and discourse community, proofs written by mathematicians-as-researchers and mathematicians-as-teachers will serve slightly different communicative purposes—recall Hanna (1989) asserting that "proofs that explain should be favored in mathematics education over those that merely prove" (p. 45). Differences in communicative purposes entail differences in moves (i.e., mini communicative purposes), and we posit that the field of mathematics education may benefit from studying mathematicians-as-teachers' proof moves and the extent to which these indeed facilitate student learning.

Critical Genre Analysis

As argued above, the genre of proof reflects and enforces existing structures of power. To study how genres do so, some genre researchers have performed critical genre analyses (e.g., Hyatt, 2005; Levina & Orlikowski, 2009). Rather than being a singular type of analysis, critical genre analyses differ in their methodologies but are connected by their focus on power structures, relations, and imbalances. For example:

- Hyatt (2005) performed a corpus analysis of tutor feedback (and noted the potential for a critical discourse analysis [Fairclough, 2003] to complement the analysis);

- Levina and Orlikowski (2009) performed a qualitative study, as part of which they identified genres in the management consulting field and then "examin[ed] how control over genre enactment was exercised […], as well as who had competence in and participated in the various genre enactments, and how." (p. 678); and
- Schryer (2002) combined a rhetorical analysis with a critical discourse analysis (Hodge & Kress, 1993) to create a chronotopic (Bakhtin, 1981) analysis of "bad news" letters by an insurance company.[3]

Below, we discuss bias and author–reader–proof-text relations[4], before offering our thoughts on what a critical genre analysis of proof could look like.

First, consider that different readers receive a text differently, that is, a text is not reader-independent. A reader's past experiences with a genre shape their expectations for contemporary realizations of the genre. More broadly, discourse communities have and use their own genres, which are understood against a horizon of expectations that has developed over time (Swales, 2016). In addition to a reader's past experiences with a genre, Akemi's example from before suggests that the reader–author relation (shaped by the reader's context and biases) also affects the reader's reading of text. Given how heavily U.S. mathematics departments lean non-Hispanic White (77%) and male (70%) (Blair et al., 2018)[5], women and people of color are most likely to be targeted by readers' implicit and explicit biases.[6]

"Properties" of proof—like *convincing*, *transparent*, or *perspicuous* (Czocher & Weber, 2020)—are not so much properties as they are author–reader–text relations. They can be used to weave a set of contradictions that will always ensure that a proof is not good enough for a biased reader. The reader might claim the proof was unconvincing because it was too short or that the proof—perhaps amended and increased in length—is not perspicuous and untransparent. Minoritized students receiving negative (and possibly contradictory) feedback is concerning as it can lead to a loss of confidence and contribute to an unsupportive culture—reasons identified as leading to undergraduate and graduate students leaving STEM, particularly minoritized students (Herzig, 2004; Thiry et al., 2019).

Second, note that an author's awareness of the (hypothetical) reader(s) shapes their writing. SFL scholars would observe that proofs do not merely serve an ideational function but also an *interpersonal* function (and a textual function). ESP scholars would note that a communicative purpose is shaped by *discourse community* and *intended audience*. RGS scholars would point to genre being a *social* action. Finally, Bakhtinian scholars would point to the importance of an utterance's *addressivity*. For example: An author of a journal article will be trying to convince the reader that the proof is novel, important, and gives due credit to prior work, whereas a student seeks not only to convince the instructor of the veracity of their proof, but also of their knowledge. In short, regardless of how much proof genre conventions suppress agency, a proof is not author-independent, and the author's shaping of the text is affected by their awareness of the (hypothetical) reader(s). (Whether the shaping achieves the author's goals is a different matter.)

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[3] Although Schryer (2002) did not label her work a critical genre analysis, she termed it a "critical approach to genre theory" (p. 76) and a way to "address the issue of genre and power" (p. 76).

[4] The triangle of author–reader–text relations (within contexts) is sometimes referred to as the rhetorical triangle.

[5] Percentages refer to full-time faculty in mathematics departments of four-year colleges and universities in fall 2015.

[6] Proof readers' bias is not just concerning with regard to university students, but also with regard to researchers. Single-blind journals are prevalent in mathematics alongside a culture of publishing preprints on arXiv. As a research mathematician friend of ours remarked, "I've never reviewed a paper I didn't know the author of."

Combining our observations about readers' biases and authors' awareness of their (hypothetical) reader(s), we note that an author's awareness may also include an awareness of their (hypothetical) reader(s)'s biases. For example: A friend of ours vividly remembers changing a proof to include more "powerful" mathematical ideas ("sledgehammers") to make her proof more convincing and beyond (anticipated) reproach.

Given the breadth of what constitutes a critical genre analysis, there are many possible avenues to pursue with the goal of studying power structures, relations, and imbalances as they pertain to the genre of proof. Drawing on our above observations and the view that criteria such as "convincing," "perspicuous," and "transparent" are prone to biased application, we believe it might be fruitful to ask: "What criteria do readers of proof apply when evaluating proofs, and how do they apply these in biased ways?" By studying the feedback that proof readers give—and to whom they give it—awareness can be fostered for the subjective nature of proof evaluation criteria. Further, revealing how biases interact with the application of proof evaluation criteria can provide a starting point for raising awareness of this problem within the discourse community of (research) mathematicians.

Finally, as aforementioned, proofs may reflect existing structures of power and stereotypical traits of, among others, masculinity (Hottinger, 2016; Weber & Melhuish, 2022). This observation raises the question of how proofs can be reenvisioned and rehumanized (Gutiérrez, 2018). For some examples thereof, consider Harron (2016), Sinclair (2005), Leron (1985), and Tymoczko (1993). A critical genre analysis could be used to identify the (invisible) constraints of the genre of proof (which hinder reenvisioning and rehumanizing proof) and how they reflect and enforce existing structures of power (e.g., Bowers & Küchle, 2020).

Conclusion

In this theoretical report, we hope to have given the reader a brief overview of contemporary genre theories, argued that proof is a genre for the purposes of (at least) three major genre theories, and outlined the potential of move analyses and critical genre analyses for advancing the field of mathematics education. But there are many more types of genre analysis mathematics educators could pursue, such as:

- a rhetorical appeals analysis (i.e., appeals to logos, pathos, and ethos);
- a lexico-grammatical analysis (e.g., of frame markers, attitude markers, hedges, and boosters) (SFL, in particular, offers a plethora of lexico-grammatical analysis tools);
- a historical or diachronic study of genre to understand the evolution of a genre;
- an ethnography to learn about the relevant discourse community;
- a genre systems analysis (or genre network analysis) to learn about the relationship among genres (and the relationship between genres and community);
- an intertextual analysis to, for example, understand references to other texts;
- a multimodal genre analysis. (Tardy, 2011)

Further, Bakhtin (1981, 1986a) has served as inspiration to genre theorists—particularly of the Brazilian tradition—and offered a whole set of possibly useful concepts, such as, carnivalesque, chronotope, dialogism, heteroglossia, and polyphony. In short, genre theories offer a whole host of analytic tools that, we argue, may be applied to the genre of proof to advance our field.

References

Alcock, L., Hodds, M., Roy, S., & Inglis, M. (2015). Investigating and improving undergraduate proof comprehension. *Notices of the American Mathematical Society*, *62*(7), 742–752. https://doi.org/10.1090/noti1263

Alcock, L., & Weber, K. (2008). Referential and syntactic approaches to proving: Case studies from a transition-to-proof course. In *CBMS: Issues in mathematics education, Vol. 16* (pp. 101–123).

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M. M. Bakhtin* (M. Holquist, Ed.; C. Emerson & M. Holquist, Trans.). University of Texas Press.

Bakhtin, M. M. (1986a). *Speech genres and other late essays* (C. Emerson & M. Holquist, Eds.; V. W. McGee, Trans.). University of Texas Press.

Bakhtin, M. M. (1986b). The problem of speech genres. In C. Emerson & M. Holquist (Eds.), *Speech genres and other late essays* (pp. 60–102). University of Texas Press.

Bhatia, V. K. (1993). *Analysing genre: Language use in professional settings*. Longman.

Biber, D., Connor, U., & Upton, T. A. (2007). *Discourse on the move: Using corpus analysis to describe discourse structure*. John Benjamins Publishing Company.

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2015 CBMS survey*. American Mathematical Society.

Bourdieu. P. (1991). *Language and symbolic power* (J. B. Thompson, Ed., G. Raymond & M. Adamson, Trans.). Harvard University Press.

Bowers, D. M., & Küchle, V. A. B. (2020). Mathematical proof and genre theory. *Mathematical Intelligencer*, *42*(2), 48–55. https://doi.org/10.1007/s00283-019-09926-3

Bronckart, J.-P. (1997). *Activité langagière, textes et discours: Pour un interactionisme socio-discursif*. Delachaux et Niestlé.

Czocher, J. A., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education*, *51*(1), 50–74. https://doi.org/10.5951/jresematheduc.2019.0007

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123–142. https://doi.org/10.1007/s10649-016-9740-5

Fairclough, N. (2003). *Analysing discourse: Textual analysis for social research*. Routledge.

Gomes-Santos, S. N. (2003). A lingüística textual na reflexão sobre o conceito de gênero. *Cadernos de Estudos Lingüísticos*, *44*, 315–324. https://doi.org/10.20396/cel.v44i0.8637085

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx students* (pp. 1–10). National Council of Teachers of Mathematics.

Hanna, G. (1989). Proofs that prove and proofs that explain. *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 45–51.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*(1/2), 5–23. https://doi.org/10.1023/A:1012737223465

Harron, P. A. (2016). *The equidistribution of lattice shapes of rings of integers of cubic, quartic, and quintic number fields: An artist's rendering* (Publication No. 10010774) [Doctoral dissertation, Princeton University]. ProQuest Dissertations & Theses Global.

Herzig, A. H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research*, *74*(2), 171–214. https://doi.org/10.3102/00346543074002171

Hodge, R., & Kress, G. (1993). *Language as ideology* (2nd ed.). Routledge.
Holliday, A. (1999). Small cultures. *Applied Linguistics*, *20*(2), 237–264. https://doi.org/10.1093/applin/20.2.237
hooks, b. (2015). *Feminism is for everybody: Passionate politics*. Routledge.
Hottinger, S. N. (2016). *Inventing the mathematician: Gender, race, and our cultural understanding of mathematics*. State University of New York Press.
Hyatt, D. F. (2005). "Yes, a very good point!": A critical genre analysis of a corpus of feedback commentaries on Master of Education assignments. *Teaching in Higher Education*, *10*(3), 339–353. https://doi.org/10.1080/1356251050012222
Hyon, S. (1996). Genre in three traditions: Implications for ESL. *TESOL Quarterly*, *30*(4), 693–722. https://doi.org/10.2307/3587930
Hyon, S. (2018a). Analyzing genre moves. In *Introducing genre and English for specific purposes* (pp. 27–50). Routledge.
Hyon, S. (2018b). *Introducing genre and English for specific purposes*. Routledge.
Keith, S. (1988, June). Women and communication in mathematics: One woman's viewpoint. Paper presented at the *10th Annual Meeting of the National Women's Studies Association*, Minneapolis, MN.
Küchle, V. A. B. (2023). A genre analysis of mathematical remarks. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1228–1233). The Special Interest Group of the Mathematical Association of America (SIGMAA) for Research in Undergraduate Mathematics Education.
Larvor, B. (2019). From Euclidean geometry to knots and nets. *Synthese*, *196*(7), 2715–2736. https://doi.org/10.1007/s11229-017-1558-x
Leron, U. (1985). Heuristic presentations: The role of structuring. *For the Learning of Mathematics*, *5*(3), 7–13.
Levina, N., & Orlikowski, W. J. (2009). Understanding shifting power relations within and across organizations: A critical genre analysis. *Academy of Management Journal*, *52*(4), 672–703. https://doi.org/10.5465/amj.2009.43669902
Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Sense Publishers.
Martin, J. R., Christie, F., & Rothery, J. (1987). Social processes in education: A reply to Sawyer and Watson (and others). In I. Reid (Ed.), *The place of genre in learning: Current debates* (pp. 58–82). Deakin University Press.
Miller, C. R. (1984). Genre as social action. *Quarterly Journal of Speech*, *70*(2), 151–167.
O'Halloran, K. L. (2015). The language of learning mathematics: A multimodal perspective. *The Journal of Mathematical Behavior*, *40*(A), 63–74. https://doi.org/10.1016/j.jmathb.2014.09.002
Schifter, D. (2009). Representation-based proof in the elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 71–86). Routledge.
Schryer, C. F. (2002). Genre and power: A chronotopic analysis. In R. Coe, L. Lingard, & T. Teslenko (Eds.), *The rhetoric and ideology of genre: Strategies for stability and change* (pp. 73–102). Hampton Press.

Sinclair, N. (2005). Chorus, colour, and contrariness in school mathematics. *Technology, Humanities, Education, and Narrative*, *1*(1).
Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.
Su, F. (2020). *Mathematics for human flourishing*. Yale University Press.
Swales, J. M. (1990). *Genre analysis: English in academic and research settings*. Cambridge University Press.
Swales, J. M. (2012). A text and its commentaries: Toward a reception history of "Genre in three traditions" (Hyon, 1996). *Ibérica*, *24*, 103–116.
Swales, J. M. (2016). Reflections on the concept of discourse community. *ASp*, *69*, 1–12. https://doi.org/10.4000/asp.4774
Tanswell, F. S., & Rittberg, C. J. (2020). Epistemic injustice in mathematics education. *ZDM*, *52*(6), 1199–1210. https://doi.org/10.1007/s11858-020-01174-6
Tardy, C. M. (2011). Genre analysis. In K. Hyland & B. Paltridge (Eds.), *The Continuum companion to discourse analysis* (pp. 54–68). Continuum.
Tardy, C. M. (2012). A rhetorical genre theory perspective on L2 writing development. In R. M. Manchón (Ed.), *L2 writing development: Multiple perspectives* (pp. 165–190). De Gruyter Mouton.
Thiry, H., Weston, T. J., Harper, R. P., Holland, D. G., Koch, A. K., Drake, B. M., Hunter, A.-B., & Seymour, E. (2019). *Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education* (E. Seymour & A.-B. Hunter, Eds.). Springer.
Tymoczko, T. (1993). Value judgements in mathematics: Can we treat mathematics as an art? In A. M. White (Ed.), *Essays in humanistic mathematics* (pp. 67–77). The Mathematical Association of America.
Upton, T. A., & Cohen, M. A. (2009). An approach to corpus-based discourse analysis: The move analysis as example. *Discourse Studies*, *11*(5), 585–605. https://doi.org/10.1177/1461445609341006
Vian, O. (2012). Beyond the three traditions in genre studies: A Brazilian perspective. In N. Artemeva & A. Freedman (Eds.), *Genre studies around the globe: Beyond the three traditions* (pp. 95–114). Inkshed Publications.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2–3), 209–234. https://doi.org/10.1023/B:EDUC.0000040410.57253.a1
Weber, K., & Melhuish, K. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education*, *22*(2), 305–314. https://doi.org/10.1007/s42330-022-00221-6

Student Happiness in Mathematics: Antecedents of Positive Emotions

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Students often express negative emotions in mathematics classes. Positive emotions feel good, but they are important beyond just feeling good. Positive emotions can benefit learning, attitudes, and perceptions of mathematics. In this paper, we discuss why positive emotions are beneficial along with a literature review of common themes which are associated with positive emotions among students. For example, increases in students' perceptions of personal control and value in the classroom have been associated with increased positive emotions. Many practices already encouraged by mathematics education researchers (e.g. teaching conceptually, helping students find mathematics important) have the added benefit of potentially increasing positive emotions in students. Positive emotions have been shown to be associated with engagement and motivation in mathematics. By increasing positive emotions in the classroom, instructors can encourage the gradual growth of positive attitudes and beliefs about mathematics to students which are currently lacking among many students.

Keywords: positive emotions, affect, happiness

Happiness and wellbeing in education have become increasingly emphasized in psychological and educational research. The Programme for International Student Assessment (PISA) has been including measures for student affect and wellbeing in their assessments of students around the world since 2015 (Govorova et al., 2020). In a recent survey of 21,678 US high school students, 75% of students reported that they typically experience negative feelings at school, with the most frequently mentioned feelings including tired, stressed and bored (Moeller et al., 2020).

It is well documented that students have similar negative feelings towards mathematics. For example, students report finding mathematics boring, unimportant to their lives, and a generally negative experience (Brown et al., 2008; Hernandez-Martinez & Vos, 2018; Zazkis, 2015). Emotions in the classroom have been found to be connected to engagement and motivation with mathematics and attitudes towards mathematics both in the short and long term (Brown et al., 2008; Middleton et al., 2017; Middleton et al., 2023). Negative emotions have been associated with reduced motivation and negative attitudes (Brown et al., 2008; Middleton et al., 2023). While, positive emotions have been connected to increased motivation (Middleton et al., 2023). Given the prevalence and potential consequences of negative emotions about mathematics, more work is needed to consider how students' emotions about and during mathematics can be improved. We suggest that cultivating positive emotions in mathematics would offer new perspectives for this pervasive problem.

Robust evidence demonstrates the positive association between positive emotions and achievement (Camacho-Morles et al., 2021; Coffey, 2020; Villavicencio & Bernardo, 2016). For example, Coffey (2020) found that positive affect at at age 1.5 years directly predicted higher educational attainment at age 29. In a meta analysis of 68 studies, Camacho-Morles et al. (2021) found a positive relationship between enjoyment and academic performance. Thus, positive emotions are associated with lasting benefits for academic achievement.

The benefits of positive emotions in mathematics likely accrue over time. According to the Broaden and Build Theory, positive emotions are associated with two categories of benefits (Fredrickson, 2001). First, positive emotions *broaden* awareness. People experiencing positive emotions are more aware of the "big picture" and able to take in a wider range of information. For example, smiles have been associated with increased breadth of attention (Johnson et al., 2010). Second, when in a broadened state of awareness from positive emotions, people are able to *build* more resources for themselves. This building of rescources has been associated with greater cognitive flexibility in an experimental task among participants in a positive emotion condition (Wang et al., 2017). Further, genuine smiles (Duchenne smiles) have been shown to predict increased flexibility in attention on an orienting task (Johnson et al., 2010).

Given these potential benefits of positive emotions in mathematics, an important area of research also involves understanding the factors that may predict positive emotions. We now review existing literature exploring the correlates of positive emotions in educational settings, which may inform strategies for educators or researchers interested in promoting positive emotions in mathematics classrooms. We begin by discussing student characteristics on an individual level that are associated with positive emotions. Then, we zoom out and discuss predictors within the classroom environment which are associated with positive emotions.

Student Cognitive Appraisals of Control and Value

Control Value Theory (CVT) posits that students' personal appraisals of a) *control* and b) *value* of the material are proximal antecedents of emotions related to achievement (Pekrun, 2006; Pekrun, 2021). Students' control appraisals describe how much control or influence students perceive they have over events, and value appraisals describe how much importance or worth students give to a particular object of focus (e.g. an exam, mathematics in general; Pekrun, 2006; Pekrun, 2021). Control and value appraisals can operate independently; thus, students can experience a combination of high and low appraisals of control and value. For example, a student may feel that they are able to answer every question on their homework assignment easily (high control), but the same student may feel that the homework is boring and unimportant (low value). According to CVT, students experience more positive emotions when they have high levels of both control and value (Pekrun, 2006; Pekrun, 2021).

Control Appraisals

Higher perceptions of control are generally associated with more positive emotions from students in math classes (Bieg et al., 2017; Bieleke et al., 2023; Buff, 2014; Buff et al., 2017). However, students on average report low feelings of control in the classroom (Bieg et al., 2017). These reported low feelings of control may be related to the many aspects of the classroom that are decided by teachers or administrators (e.g. assignments, exams; Bieg et al., 2017; Boehme et al., 2017). Research has shown that changes in perceived control are positively associated with changes in enjoyment of learning (Buff et al., 2017). Thus, perhaps interventions to increase student perceived control may have benefits by increasing positive emotions. For example, Bieg and colleagues (2017) demonstrated that working in groups and working individually increases perceived control when compared to learning from direct instruction.

However, it is important to note that too much control can lead to negative feelings for students (Buff et al., 2011; Pekrun, 2006, 2021). For example, if a student has very high control over what is happening in the classroom, they may feel bored and unengaged in mathematics (Pekrun, 2006). Buff and colleagues (2011) demonstrated that the relationship between control and positive emotions is nonlinear with the benefits of control tapering off at the highest levels.

Value Appraisals

Similar to control, it has been demonstrated that changes in perceived value are positively predictive of changes in enjoyment of learning (Buff, 2014; Buff et al., 2017). Math students generally report low values for math and feel that math is unimportant to their lives and boring (Brown et al., 2008; Hernandez-Martinez & Vos, 2018). Low value for math is associated with lower levels of motivation as well which contributes to negative emotions (Brown et al., 2008). Thus, interventions to increase student value could be beneficial to increase positive emotions for students.

However, just as with feelings of control, increasing value appraisals too much may be damaging to some students which has been demonstrated empirically (Boehem et al., 2017; Lauermann et al., 2017). High levels of value among families was associated with greater math anxiety in one study (Boehme et al., 2017). This view is also supported theoretically by CVT. For example, a student who believes succeeding in math is necessary for success in their future career has a high value of math, and anything that gets in the way of the success that they feel is necessary could lead to anxiety.

Control and Value Appraisals Occur Together

As discussed above, appraisals of control and value should not be increased for every student. Potential intervention studies are made more nuanced by the fact that control and value appraisals often co-occur. Thus, control and value appraisals become a type of optimization problem. For each student, there is a different optimal level of control and value which leads to the most success and positive emotions in the class. For example, Boehme et al. (2017) found that high family values of math were associated with test anxiety, yet high family values were associated with lower test anxiety through increased feelings of control among students. This finding suggests that increasing only value may worsen math anxiety; however, when an increase in value occurs with an increase in control, students may experience less math anxiety. Similarly, students with high math anxiety may benefit from an increase in control in the classroom while gifted students may benefit from classroom practices designed to decrease feelings of control as this may reduce feelings of boredom and increase positive emotions (Pekrun et al., 2010).

Classroom Environment Predictors of Positive Emotions

Research has also shown many factors within the classroom environment that can contribute to students experiencing positive emotions.

Learner-Oriented Instruction

General education research has demonstrated the benefits of learner-oriented teaching practices (e.g. group work, project based learning; Bruder & Prescott, 2013; Cevikbas & Kaiser, 2020; Laursen et al., 2014). However, direct instruction in a lecture format is still one of the most common teaching practices (Bieg et al., 2017; Jiang et al., 2021; Stigler & Hiebert, 2009). Direct instruction has also been shown to be associated with negative emotions in students (Bieg et al., 2017; Jiang et al., 2021). However, learner-oriented instruction, like working in pairs or small groups) has been shown to give students more feelings of control (e.g. control over discussions, control over pace of the material; Bieg et al., 2017).

Bieg et al. (2017) used experience-sampling methodology to measure students' emotions during math class. Students received notifications sent to an iPod Touch with surveys to complete during class. The survey asked them to report the current type of instruction (e.g. lecture, small group work), feelings of control in the moment, and current emotions (enjoyment, pride, anger, anxiety, and boredom). Results revealed that working in groups or working individually was associated with greater student enjoyment and perceived student control.

Collaborative work through learner-oriented instruction also offers the benefits of encouraging positive social interaction among students. Working in groups and having positive interactions with others is associated with positive emotions in math classes (Satyam, 2020). Also, giving students the opportunity to work in groups allows students to experience mathematics in a way similar to mathematicians, who also collaborate frequently (Burton, 1998). This approach is consistent with current approaches to science education, which has been dedicating more attention to cultivating experiences for students to make them feel like scientists (Jaber & Hammer, 2016). Burton (1998) reports that mathematicians report feeling increased joy and creativity through their collaborative work. Thus, group work could have a twofold effect of increasing positive emotions and allowing students the opportunity to work in ways similar to mathematicians and experts.

Learning Materials and Curriculum

Just as instruction type can influence student emotions, classroom activities and curriculum can influence student engagement which in turn can influence student emotions (Middleton et al., 2017). For example, giving students opportunities to work on challenging problems has been associated with more positive emotions and fewer negative emotions (Dettmers et al., 2011; Greensfeld & Deutsch, 2016; Greensfeld & Deutsch, 2022; Liljedahl, 2005; Satyam, 2020; Schukajlow & Rakoczy, 2016). For example, Dettmers et al. (2011) found that when students perceived homework assignments as challenging with well-selected problems, students put more effort into their homework and experienced fewer negative emotions.

Satyam (2020) describes “satisfying moments” as moments that create significant positive emotions for an individual. Eleven undergraduate students in a transition-to-proof class were interviewed to determine moments of satisfaction. The most common satisfying moments included overcoming a challenge. Thus, overcoming difficulties may create positive moments for students.

Alternative forms of curriculum and assessment can also decrease the common negative emotion of test related anxiety (Krispenz et al., 2019; Walen & Williams, 2002). Walden and Williams (2002) found that their participants had anxiety specific to timed assessments. These undergraduate students enjoyed mathematics, but the time limit of exams created anxiety. Further, using timed assessments likely deters students from using a problem solving approach as procedural methods will likely lead to faster solutions (Walden & Williams, 2002). So, even if teachers are teaching conceptual lessons and using well developed materials with challenging problems, timed assessments may undermine the work that went into those lessons and create more negative emotions for students as well.

Conclusion

Many practices already encouraged by mathematics education researchers (e.g. teaching conceptually, helping students find mathematics important) have the added benefit of potentially increasing positive emotions in students. By increasing positive emotions in the classroom, instructors can encourage the gradual growth of positive attitudes and beliefs about mathematics to students which are currently lacking among many students (Brown et al., 2008; Zazkis, 2015).

Although we have focused on positive emotions for this paper, it is important to not neglect the prevalence of negative emotions in the classroom as well (Deshler et al., 2019; Di Leo et al., 2019; Di Leo et al., 2020; D’Mello et al., 2014; Zazkis, 2015). Research has demonstrated that negative emotions can also be beneficial for students' learning (D'Mello et al., 2014). For example, although being confused is an uncomfortable feeling, and thus a negative emotional state, confusion has been shown to be beneficial to learning if the confusion is resolved (D'Mello et al., 2014). In D'Mello et al. (2014), the participants who experienced confusion were more successful in remembering information presented to them than participants who learned without experiencing confusion. Anxiety has also been found to have a positive relationship with student performance when students have high prior knowledge of mathematics (Schukajlow et al., 2021).

Just as negative emotions can be beneficial, positive emotions can also be detrimental. Villavicencio et al. (2016) found that pride among engineering students in the Phillipenes had a negative association with final course grades. Studies have also shown that positive emotions such as excitement and enjoyment can be barriers to perseverance in mathematical reasoning (Barnes, 2021; Ben-Eliyahu & Linnenbrink-Garcia, 2015).

Thus, there are benefits and detriments to both positive and negative emotions. However, on average, positive emotions have been shown to have more beneficial effects in mathematics learning than negative emotions (Bieg et al., 2017; Greensfeld & Deutsch, 2022; Greensfeld & Deutsch, 2016; Jiang et al., 2021; Liljedahl, 2005; Middleton et al., 2023; Satyam, 2020; Schukajlow & Rakocy, 2016). For example, Middleton et al (2023) found that positive emotions were associated with more motivational outcomes than negative emotions. Thus, in this study, perhaps an intervention to increase positive emotions would provide more benefit than an intervention to decrease negative emotions. With an increased focus on increasing positive emotions in the classroom, mathematics may become a more enjoyable place for students filled with creativity and curiosity.

References

Barnes, A. Enjoyment in learning mathematics: its role as a potential barrier to children's perseverance in mathematical reasoning. *Educ Stud Math* 106, 45–63 (2021). https://doi.org/10.1007/s10649-020-09992-x

Ben-Eliyahu, A., & Linnenbrink-Garcia, L. (2015). Integrating the regulation of affect, behavior, and cognition into self-regulated learning paradigms among secondary and post-secondary students. *Metacognition and learning*, 10, 15-42.

Brown, M., Brown, P., & Bibby, T. (2008). "I would rather die": Reasons given by 16-year-olds for not continuing their study of mathematics. *Research in mathematics education*, 10(1), 3-18.

Bieg, M., Goetz, T., Sticca, F., Brunner, E., Becker, E., Morger, V., & Hubbard, K. (2017). Teaching methods and their impact on students' emotions in mathematics: An experience-sampling approach. *ZDM*, 49, 411-422.

Bieleke, M., Goetz, T., Yanagida, T. et al. Measuring emotions in mathematics: the Achievement Emotions Questionnaire—Mathematics (AEQ-M). *ZDM Mathematics Education* 55, 269–284 (2023). https://doi.org/10.1007/s11858-022-01425-8

Boehme, Katharina Luisa, Thomas Goetz, and Franzis Preckel. "Is it good to value math? Investigating mothers' impact on their children's test anxiety based on control-value theory." *Contemporary Educational Psychology* 51 (2017): 11-21.

Bruder, R., Prescott, A. Research evidence on the benefits of IBL. *ZDM Mathematics Education* 45, 811–822 (2013). https://doi.org/10.1007/s11858-013-0542-2

Buff, A., Reusser, K., Rakoczy, K., & Pauli, C. (2011). Activating positive affective experiences in the classroom:"Nice to have" or something more?. *Learning and Instruction*, 21(3), 452-466.

Buff, A. (2014). Enjoyment of learning and its personal antecedents: Testing the change–change assumption of the control-value theory of achievement emotions. *Learning and Individual Differences*, 31, 21-29.
Buff, A., Reusser, K., & Dinkelmann, I. (2017). Parental support and enjoyment of learning in mathematics: Does change in parental support predict change in enjoyment of learning?. *ZDM*, 49, 423-434.
Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics?. *Educational studies in mathematics*, 37(2), 121-143.
Camacho-Morles, J., Slemp, G. R., Pekrun, R., Loderer, K., Hou, H., & Oades, L. G. (2021). Activity achievement emotions and academic performance: A meta-analysis. *Educational Psychology Review*, 33(3), 1051-1095.
Cevikbas, M., Kaiser, G. Flipped classroom as a reform-oriented approach to teaching mathematics. *ZDM Mathematics Education* 52, 1291–1305 (2020). https://doi.org/10.1007/s11858-020-01191-5
Coffey, J. K. (2020). Cascades of infant happiness: Infant positive affect predicts childhood IQ and adult educational attainment. *Emotion*, 20(7), 1255.
Deshler, J., Fuller, E., & Darrah, M. (2019). Affective states of university developmental mathematics students and their impact on self-efficacy, belonging, career identity, success and persistence. *International Journal of Research in Undergraduate Mathematics Education*, 5, 337-358.
Dettmers, S., Trautwein, U., Lüdtke, O., Goetz, T., Frenzel, A. C., & Pekrun, R. (2011). Students' emotions during homework in mathematics: Testing a theoretical model of antecedents and achievement outcomes. *Contemporary Educational Psychology*, 36(1), 25-35.
Di Leo, I., Muis, K. R., Singh, C. A., & Psaradellis, C. (2019). Curiosity… Confusion? Frustration! The role and sequencing of emotions during mathematics problem solving. *Contemporary educational psychology*, 58, 121-137.
Di Leo, I., & Muis, K. R. (2020). Confused, now what? A Cognitive-Emotional Strategy Training (CEST) intervention for elementary students during mathematics problem solving. *Contemporary Educational Psychology*, 62, 101879.
D'Mello, S., Lehman, B., Pekrun, R., & Graesser, A. (2014). Confusion can be beneficial for learning. *Learning and Instruction*, 29, 153-170.
Fredrickson, B. L. (2001). The role of positive emotions in positive psychology: The broaden-and-build theory of positive emotions. *American psychologist*, 56(3), 218.
Govorova, E., Benítez, I., & Muñiz, J. (2020). How schools affect student well-being: A cross-cultural approach in 35 OECD countries. *Frontiers in psychology*, 11, 431.
Greensfeld, H., & Deutsch, Z. (2016). The Centrality of Positive Emotions in the Field of Mathematics. *Athens Journal of Education*, 3(4), 345-364.
Greensfeld, H., & Deutsch, Z. (2022). Mathematical challenges and the positive emotions they engender. *Mathematics Education Research Journal*, 34(1), 15-36.
Hernandez-Martinez, P., Vos, P. "Why do I have to learn this?" A case study on students' experiences of the relevance of mathematical modelling activities. *ZDM Mathematics Education* 50, 245–257 (2018). https://doi.org/10.1007/s11858-017-0904-2
Jaber, L. Z., & Hammer, D. (2016). Learning to feel like a scientist. *Science Education*, 100(2), 189-220.

Jiang, Z., Mok, I. A. C., & Yin, H. (2021). The relationships between teacher emotions and classroom instruction: Evidence from senior secondary mathematics teachers in China. *International Journal of Educational Research*, 108, 101792.
Johnson, K. J., Waugh, C. E., & Fredrickson, B. L. (2010). Smile to see the forest: Facially expressed positive emotions broaden cognition. *Cognition and emotion*, 24(2), 299-321.
Krispenz, A., Gort, C., Schültke, L., & Dickhäuser, O. (2019). How to reduce test anxiety and academic procrastination through inquiry of cognitive appraisals: A pilot study investigating the role of academic self-efficacy. *Frontiers in psychology*, 10, 1917.
Lauermann, F., Eccles, J. S., & Pekrun, R. (2017). Why do children worry about their academic achievement? An expectancy-value perspective on elementary students' worries about their mathematics and reading performance. *ZDM*, 49, 339-354.
Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, 45(4), 406-418.
Liljedahl, P. G. (2005). Mathematical discovery and affect: The effect of AHA! experiences on undergraduate mathematics students. *International Journal of Mathematical Education in Science and Technology,* 36(2-3), 219-234.
Middleton, J.A., Wiezel, A., Jansen, A. et al. Tracing mathematics engagement in the first year of high school: relationships between prior experience, observed support, and task-level emotion and motivation. *ZDM Mathematics Education* 55, 427–445 (2023). https://doi.org/10.1007/s11858-022-01432-9
Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In J. Cai (Ed.), Compendium for Research in Mathematics Education (pp. 87-119). Reston, VA: National Council of Teachers of Mathematics.
Moeller, J., Brackett, M. A., Ivcevic, Z., & White, A. E. (2020). High school students' feelings: Discoveries from a large national survey and an experience sampling study. *Learning and Instruction*, 66, 101301.
Pekrun, R. (2006). The control-value theory of achievement emotions: Assumptions, corollaries, and implications for educational research and practice. *Educational psychology review*, 18, 315-341.
Pekrun, R. (2021). Self-appraisals and emotions: A generalized control-value approach. In T. Dicke, F. Guay, H. W. Marsh, R. G. Craven, & D. M. McInerney (Eds.), Self-A multidisciplinary concept (pp. 1-30). Information Age Publishing
Satyam, V. R. (2020). Satisfying moments during the transition-to-proof: Characteristics of moments of significant positive emotion. *The Journal of Mathematical Behavior*, 59, 100784.
Schukajlow, S., & Rakoczy, K. (2016). The power of emotions: Can enjoyment and boredom explain the impact of individual preconditions and teaching methods on interest and performance in mathematics?. *Learning and Instruction*, 44, 117-127.
Schukajlow, S., Blomberg, J., Rellensmann, J., & Leopold, C. (2021). Do emotions and prior performance facilitate the use of the learner-generated drawing strategy? Effects of enjoyment, anxiety, and intramathematical performance on the use of the drawing strategy and modelling performance. *Contemporary Educational Psychology*, 65, 101967.
Stigler, J. W., & Hiebert, J. (2009). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. Simon and Schuster.

Villavicencio, F.T., Bernardo, A.B.I. Beyond Math Anxiety: Positive Emotions Predict Mathematics Achievement, Self-Regulation, and Self-Efficacy. *Asia-Pacific Edu Res* 25, 415–422 (2016). https://doi.org/10.1007/s40299-015-0251-4

Walen, S. B., & Williams, S. R. (2002). A matter of time: Emotional responses to timed mathematics tests. *Educational Studies in Mathematics*, 49, 361-378.

Wang, Y., Chen, J., & Yue, Z. (2017). Positive emotion facilitates cognitive flexibility: An fMRI study. *Frontiers in psychology*, 8, 1832.

Zazkis, D. (2015). Monsters, lovers and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics*, 35(1), 33-38.

A Framework for Characterizing and Identifying Playful Mathematical Experiences

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Play has been recognized as a component of mathematical practice; accordingly, this manuscript explores the potential role of play in undergraduate mathematics education. The manuscript introduces a novel framework for characterizing play, wherein a person is said to be playing in a scenario if they are (1) making non-routine, voluntary, and free choices/actions, (2) experiencing an appropriate level of challenge/uncertainty, and (3) experiencing amusement, satisfaction, or excitement. This framework is applied to provide a definition of playful mathematical experiences and consider how playful mathematical experiences relate to other concepts, like mathematical play and authentic mathematics.

Keywords: mathematical play, characteristics of play, rehumanizing mathematics

In the traditional mathematics classroom, students learn mathematics through some combination of teacher demonstration and repetitive practice of key computational skills While there is no consensus as to what mathematics *is*, articulations of the nature of mathematics tend to refer to concepts like problem solving, logical reasoning, modeling, or creative exercise (Halmos, 1980; Mura, 1993; Sfard, 1998; Strauss, 2011) – characterizations which stand in stark contrast to the operation of traditional mathematics classrooms (Boaler, 2002; Mann, 2006). Many approaches under the umbrella of Inquiry-Based Mathematics Education (IBME) have sought to incorporate more components of authentic (cf. Brown et al., 1989) mathematical practice – i.e., ways mathematicians do mathematics – into undergraduate mathematics classrooms as part of larger efforts to improve undergraduate mathematics instruction and learning (Laursen & Rasmussen, 2019).

One component of authentic mathematical practice – that is, something which mathematicians actually *do* – not included in Laursen and Rasmussen's (2019) synthesis of IBME approaches is the notion of *play*. Play is a near-universal human experience, and also one with much ambiguity (Sutton-Smith, 1997); much of the remainder of this manuscript will be dedicated to characterizing it more fully. When mathematicians like Su (2020) and Lockhart (2008) discuss play as part of mathematical practice, they refer to the pleasure and whimsy of freely exploring mathematical structures. Su (2022) argues that "contemplating patterns, playing with ideas, exploring what's true, and enjoying the surprises that arise along the way" (p. 53) is key to mathematical research, while Lockhart (2008) states that in mathematics, "we get to play and imagine whatever we want and make patterns and ask questions about them" (p. 4). Beyond mathematicians' descriptions of their own practice, Horne et al. (2023) recently examined the problem-solving practices of professional mathematicians and found evidence that playfulness is at least sometimes invoked within their mathematical practice.

Now, play being a part of authentic mathematical practice does not necessitate the inclusion of play at all levels of learning mathematics. Reviewing journal articles and attending faculty meetings are practices of mathematicians, but I suspect most would agree that first year undergraduate students need not participate in these practices. So, while I *assert* that play is legitimate for inclusion in mathematics classes due to its role in authentic mathematical practice (Holton et al., 2001; Horne et al., 2023; Lockhart, 2008; Su, 2020), I go further and *conjecture* that play may be a useful element to introduce to undergraduate mathematics classrooms for reasons beyond its validity.

My evidence for this conjecture is the current status of participation in STEM. Many more students enter college with interests in STEM than complete degrees in STEM fields (Bressoud, 2020; Seymour & Hunter, 2019). One reason for this is that undergraduate mathematics classrooms can be dehumanizing (cf. Gutiérrez, 2018) spaces where students are not able to interact with mathematics as their whole authentic selves; this is especially so for students from minoritized groups (Leyva et al., 2021, 2022). Recently, Caldwell et al. (2023) have argued that play can contribute to rehumanizing mathematics classrooms. According to their argument, play can challenge traditional mathematics classroom practices and create space for students to assert their own understandings of mathematics (Caldwell et al., 2023). While Caldwell et al. (2023) placed their work in the context of elementary schools, the fact that people of any age can and do engage in play means the same ideas may apply to the undergraduate mathematics classroom.

So far in this manuscript, I have provided a brief overview of *why* we might want to integrate play into undergraduate mathematics classrooms. Before we can proceed to fully considering *how* play might be integrated into undergraduate mathematics classrooms, further clarity is needed on *what* play actually *is* and how I suggest we should operationalize it in this context. The remainder of this manuscript serves to articulate a framework for play oriented towards identifying whether someone is playing or not in a given scenario, using that framework to define *playful mathematical experiences*, and considering how this definition relates to and differs from existing work on play in mathematics.

Characterizing and Contextualizing Play in Mathematics

Characterizing exactly what play *is* is a challenge. The conceptual ambiguity of play makes it difficult to define in a way which captures all applications of the word while being operationalizable. As a result, diverse definitions and frameworks exist which vary greatly depending on both the backgrounds of the authors developing the frameworks, as well as the audience they are speaking to (Sutton-Smith, 1997; Salen and Zimmerman, 2004; Bergen, 2015). Zoologists, philosophers, psychologists, and game designers all study play, each emphasizing different aspects of it which are particularly relevant to their domains and motivations. Some characterizations of play conflict somewhat with the idea of play in the classroom – for example, Huizinga (1949) articulates that play must be separate from “ordinary life” and involves no material gain. As a further complication, within mathematics, the concept of mathematical play exists and has been operationalized differently by different researchers (Holton et al., 2001; Horne et al., 2023; Williams-Pierce, 2019), but typically as a very specific type of play.

As a transdisciplinary scholar, I felt it was important to leverage the collective knowledge about play from diverse perspectives. Moreover, as I examined many frameworks of play, I found that they often were oriented to be describing the environment that a person was playing in as much as the actions of the player – and in some cases, seemed to imply that environments which meet certain conditions demand play. Under certain understandings of play, this is sensible. However, this orientation is not useful if one’s goal – as mine is – is to clearly evaluate the extent to which the design and implementation of an activity intended to be playful results in playful experiences. In other words, in the environment of the undergraduate mathematics classroom (or indeed, any classroom), where both playful and non-playful behaviors are possible, how do we differentiate between those? So, to resolve this, I have pragmatically drawn on articulations of what play is from diverse authors and perspectives (Bergen, 2015; Callois, 2001/1961; Csikszentmihalyi & Bennett, 1971; Henricks, 2008; Holton et al., 2001; Huizinga, 1949; Salen & Zimmerman, 2004; Sutton-Smith, 1997; Sylva et al., 1976; Williams-Pierce & Thevenow-Harrison, 2021) to develop a framework of characteristics of play.

The Framework

Within any given scenario – whether designed or naturalistic – a person is likely engaging in play if, at some point, their interaction with the scenario meets each of the following criteria:

1. The person is making non-routine, voluntary, and free choices and/or actions.
2. The person is experiencing an appropriate level of challenge and/or uncertainty.
3. The person is experiencing amusement, satisfaction, or excitement.

In the following subsections, I describe in more detail how these characteristics draw on existing literature on play from multiple disciplines and authors. I also suggest (non-exhaustively) a few actions we may observe that would indicate that each characteristic is met in a scenario.

Making Non-Routine, Voluntary, and Free Choices and Actions. Across diverse authors and contexts, play is strongly associated with the idea of freedom (Henricks, 2008; Holton et al., 2001; Huizinga, 1949; Salen & Zimmerman, 2004; Williams-Pierce & Thevenow-Harrison, 2021). Although play is always bound by some sort of structure – be it an implicit societal structure or the more explicit rules of mathematics and games – play is defined by the player's ability to interact with that structure in ways that they choose. Moreover, for these choices to be truly free, they must be made voluntarily – they must not be imposed upon the potential player by an outside source (Huizinga, 1949; Williams-Pierce & Thevenow-Harrison, 2021). Of course, the degree to which a choice or action is taken freely and voluntarily may be challenging to determine as an observer. Observing kids at a playground at a public park, perhaps we could assume that their choices are all free and voluntary. In a classroom or even a playground attached to a school, this is a harder assumption to make – the rules of schools and classrooms make them spaces where students' freedom and agency are inherently reduced.

To allow for identification of playful behaviors under these environments of reduced freedom, I draw particularly on discussion by Sylva et al. (1976) about the relationship between routines and play. Sylva et al. (1976) articulate this by stating that play offers an "invitation to the possibilities inherent in things and events. It's the freedom to notice seemingly irrelevant detail" (p. 396). When operating under a routine, we do not give ourselves space to notice alternate possibilities (Sylva et al., 1976). In the undergraduate mathematics classroom, this is reflected in the difference between exercises and problems (cf. Schoenfeld, 1985): exercises, like completing a worksheet of u-substitution integration tasks after instruction on the technique, involve implementing routines and therefore have less space for freedom. Problems, like asking students who have not yet learned about integration to estimate the area under a curve, are non-routine and create more space for students to freely choose their strategy.

This is not to say that play never involves routine behaviors. Consider a pitcher's role in a baseball game – while the pitcher has practiced their routine for throwing fastballs and curveballs, how they implement these pitches in response to the state of the game and the actions of the offense is, at least occasionally, non-routine. In the undergraduate mathematics classroom, consider the prior example of asking students to estimate the area under a curve. As part of this task, students will likely compute the areas of figures they have well-established routines for – but the choices of how to deploy those routines is subject to students' choices. So, the notion of non-routine is useful to identify actions which are done freely and voluntarily in environments with reduced freedom, like the undergraduate mathematics classroom.

At this stage, I suggest three observable actions which would indicate to me that the person is making non-routine, voluntary, and free choices and/or actions. The first is when a person takes an action with an indication that the action is exploratory and that the outcome will be unknown. Routine actions inherently have at least expected outcomes, so an action taken with an explicit reference to uncertainty of outcome is indicative of a non-routine action. Second is when a person takes actions to test or push the bounds of a scenario. Testing and potentially adjusting the limitations of a scenario is reflective of asserting freedom over the scenario by specifically challenging it. An example of this is in the case of Calvin and the educational mathematical game *Boone's Meadow* (Wisittanawat & Gresalfi, 2021). While playing the game, Calvin tested the bounds of the scenario by seeing if he could hijack a vehicle and flying in different ways to see how the outcome of the game changed. Finally, the creation of individual goals would also indicate free action. These goals could be expressed explicitly as a goal or could be any desire to accomplish something that is not specifically implicated by the scenario.

Appropriate Level of Challenge or Uncertainty. In order to engage in playful behavior in a scenario, a person should find some challenge within that scenario (Csikszentmihalyi & Bennett, 1971; Salen & Zimmerman, 2004) or some uncertainty of how to act within that scenario (Holton et al., 2001; Salen & Zimmerman, 2004). Whether the play is provoked by challenge or by uncertainty roughly corresponds to Callois' concepts of ludus and paidia (2001/1961). When the play is provoked by challenge, the player is trying to accomplish some sort of specific goal within (and to some extent, perhaps against) the scenario – which is inclusive of the rule-based play of ludus. Conversely, if the play is provoked by uncertainty, the player may simply be trying to explore or understand the scenario and what the bounds of the scenario are – which would be more in alignment with the improvisational play of paidia. Of course, these are not mutually exclusive. How challenging a scenario is can fluctuate depending on a person's uncertainty, and the process of exploring uncertainty may lead to finding or creating challenges.

For this, I assert three observable actions which may indicate that a person is experiencing an appropriate level of challenge or uncertainty. The first is self-explanatory: a direct statement or express that they are experiencing challenge or uncertainty, like saying "this is so hard" or "I'm not sure if this will work." The second regards the potential of frustration. Sylva et al. (1976) indicate that play comes with a "moratorium on frustration" (p. 245), but Holton et al. (2001) suggest that frustration is actually a common feature of play. Indeed, in my own experiences, it is typical that working on appropriately challenging tasks will involve some frustration, even as it gives way to excitement when a 'lightbulb moment' is achieved. So, I draw on Holton et al. (2001) to suggest that frustration may be experienced while engaging playfully in a scenario – but, if it is experienced, it is not an obstacle to the person's engagement. What this would manifest as, then, is that a person showing signs of frustration persists in the task. Finally, I suggest that if the person makes multiple attempts at interacting with the scenario – by changing an action midcourse, somehow resetting the scenario, or otherwise repeating their experience with the scenario – this indicates an appropriate level of challenge and/or uncertainty. In the world of games, this may be reflected in restarting a level or loading an earlier save file before a certain action was taken; in the world of mathematical problems, this could involve crumpling up a sheet of paper or erasing everything on a whiteboard up to a certain point.

Experiencing Amusement, Satisfaction, or Excitement. The affective component of play – that is, the idea that play is fun – is broadly acknowledged (Csikszentmihalyi & Bennett, 1971; Henricks, 2008; Sutton-Smith, 1997). In my view, it is minimized and under-addressed in importance in most frameworks of play. Indeed, many of the definitions and frameworks did not mention fun; some who did mention fun did so only to state that fun alone is insufficient to understand play (Csikszentmihalyi & Bennett, 1971). I would argue, however, that if a person is engaging in free action and experiencing a challenge or uncertainty, but is not "having fun," then they are not engaging in play. Therefore, if the goal of observing an experience is to determine if that experience was playful, then whether they are having fun ought to be considered. Of course, fun is too broad a concept, so I suggest that having fun can be operationalized as experiencing amusement, satisfaction, or excitement.

These do not represent the total of all feelings that can be experienced during play – I have already mentioned frustration, and other emotions like annoyance or even sadness can occur while playing. My exclusion of these experiences in this framework is not to say that they are never part of play. However, if the experience of frustration, annoyance, or sadness *never* gives way to amusement, satisfaction, or excitement, that it would be incorrect to characterize that activity as play. In other words, play need not *always* be fun, but *at some point*, it must.

Even just with the experiences of amusement, satisfaction, and excitement, there are limits to how I might observe these. People do not always outwardly express these emotions even as they are experiencing them. Moreover, the way emotions are expressed and recognized varies from culture to culture and person to person (Elfenbein & Ambady, 2003), although there is some evidence that expressions of joy may be broadly recognized (Sauter et al., 2010). Observing for this may be particularly challenging, and is the part of this framework most in need of further development. For now, I draw on things I am likely to do when I am enjoying my experience of playing a game and/or solving a mathematical problem: smiling, laughing, and engaging in celebratory moves (e.g. a fist pump, a 'woop', or self-applause).

Playful Mathematical Experiences, Mathematical Play, and Mathematical Problem Solving

Given my framework for play, I define a playful mathematical experience as an experience where a person or group of persons is (a) interacting with mathematical rules, ideas, and objects, and (b) where their interaction otherwise meets each of the three characteristics in the prior subsection. Playful mathematical experiences thus overlap with several different kinds of experiences discussed in this manuscript (Figure 1). Here, I wish to interrogate the relationship

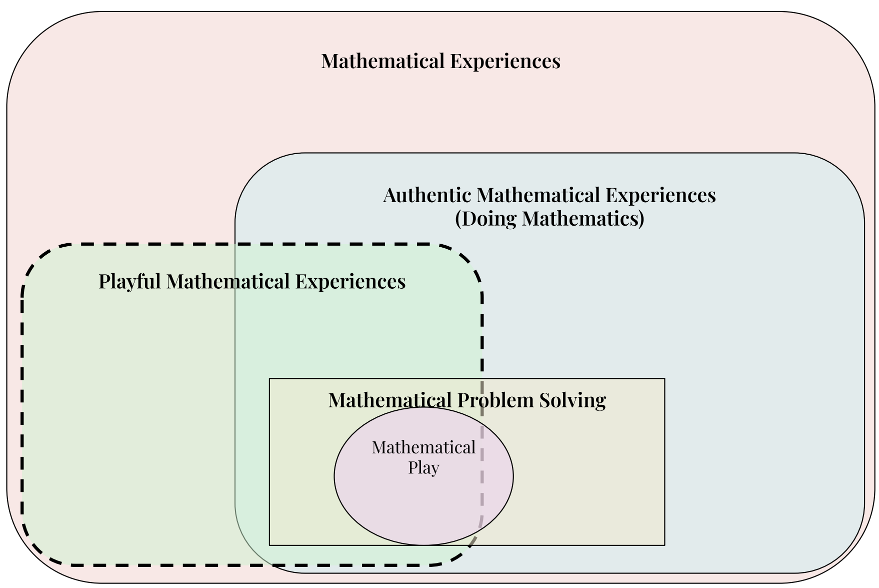

Figure 1. An Euler diagram showing the relationship between playful mathematical experiences (dashed lines) and other concepts discussed in this manuscript.

between playful mathematical experiences and mathematical play, and how these relate to authentic mathematical experiences.

Mathematical play is conceptualized differently by different scholars (Holton et al., 2001; Horne et al., 2023; Williams-Pierce, 2019). I start by considering those of Williams-Pierce (2019) and Holton et al. (2001), which I see as having the most differences from my definition of playful mathematical experiences. Williams-Pierce (2019) defines mathematical play as "voluntary engagement in cycles of mathematical hypotheses with occurrences of failure," (p. 591), where cycles of mathematical hypotheses entails "shift[ing] regularly between generalizing activity and generalizations…in which players learn to adjust their behavior and conceptualization" (p. 592). Holton et al. (2001) conceptualize it more generally:

> By mathematical play we mean that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion. Mathematical play involves pushing the limits of the situation and following thoughts and ideas wherever they may lead. (p. 403)

Both of these invoke ideas of freedom and exploration within mathematics, specifically with the solving of mathematical problems or development of mathematical concepts. I drew on both characterizations of mathematical play articulate in developing my own understanding of playful mathematical experiences.

Horne et al. (2023), meanwhile, operationalize mathematical play (reframed as playful math) in a way which is very well-aligned with my framework. In examining mathematicians' problem solving for evidence of play, they looked for evidence of three characteristics: "(a) agency in exploration or goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment" (Horne et al., 2023, p. 99). The first two of these characteristics align with my first characteristic of play, and their third characteristic lines up well with my third characteristic.

However, each of Williams-Pierce (2019), Holton et al. (2001), and Horne et al. (2023) specifically locate mathematical play within or very tightly adjacent to mathematical problem solving. While mathematical problem solving is a fertile ground for playful mathematical experiences, my definition for playful mathematical experiences is more broad. By way of example, consider a group of people who are telling a series of mathematical jokes to one another. Their experience may be playful in that (a) creating the jokes to tell is likely a non-routine process involving free exploration of different ideas, (b) the group members may view it as a "challenge" to try to get a bigger laugh with each joke, and (c) the shared laughter is an indication of their enjoyment of the situation. Since making jokes about mathematics involves interacting, in some manner, with mathematics, this would qualify as a playful mathematical experience. But, unless the jokes also represent a sequence of well-reasoned mathematical argumentation, it would likely not be construed as mathematical play.

I am motivated by the authenticity of play to mathematics. Based on the substantial overlap between these different characterizations, it seems to me that nearly any playful mathematical experience which involves doing mathematics (i.e., that is an authentic mathematical experience) would necessarily fit these definitions of mathematical play. Indeed, as I proceed in developing this framework, I am looking to mathematical problem solving experiences to better understand what play might look like in mathematics. However, I think it is important to include other ways of playing with mathematics, like my previous joke-making example, within the definition of playful mathematical experiences for two reasons.

First, students may hold beliefs about mathematics which preclude the possibility of playing in mathematics classrooms (cf. the case of Keegan in Wisittanawat & Gresalfi, 2021) or have anxiety about mathematics. Such students may not be 'ready' to engage playfully with mathematics while doing mathematics until they become more comfortable with doing mathematics itself. If playfulness is a goal, it may be useful to use activities which provoke playfulness but which are not necessarily authentic to mathematics as a bridge. In the classroom, this could involve an activity sequence which first prompts students to play more generally, then adding mathematics to their play, before finally inviting them to engage in mathematical play through mathematical problem solving.

Second, in a humanized mathematics classroom, students must be able to "draw upon all of their cultural and linguistic resources" and to "see aspects of themselves" (Gutiérrez, 2018, p. 1) in their mathematical work. To me, this means that students are not just invited to participate in existing practices of mathematicians, but are also invited to bring practices from other communities to bear on mathematics. Since one of my motivations for integrating play into undergraduate mathematics classrooms is to contribute to rehumanizing them, then it seems appropriate to include space for playful mathematical experiences beyond doing mathematics.

Discussion

The framework of play and definition of playful mathematical experiences in this manuscript started being developed when I, as a mathematics educator and gamer, first noticed that the traits of many of my favorite games were much the same as those of my favorite mathematical experiences. Since then, I have developed my understanding of play and playful mathematical experiences through the careful transdisciplinary examination of diverse sources of literature. While this has been a substantive endeavor, this is still an early draft of this framework, and further development is needed before it is used in the evaluation of designed learning activities. Even in between my first articulation of this framework and the writing of this manuscript, recently-published works (i.e., Caldwell et al., 2023; Horne et al., 2023; Su, 2020) led me to further develop my ideas.

The greatest weakness of my framework for play and my associated definition of playful mathematical experiences is that they are informed primarily by research literature and secondarily by my own experiences, and they are not yet informed by data. To address this, I am presently in the process of conducting a study which will use this framework to examine students' interactions with digital puzzle games and mathematical problems in a clinical interview environment. From this process, I expect this framework will evolve and deepen greatly. In particular, this process should allow me to better understand the relationship between playful experiences in general, playful mathematical experiences, and mathematical play, by examining how student interactions are similar and different in more- and less-explicitly mathematical situations. Even as a work in progress, I present this framework with the intent to prompt mathematicians, mathematics educators, and researchers of mathematics education to consider for themselves: what is, can be, and should be the role of play in undergraduate mathematics classrooms?

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References

Bergen, D. (2015). Psychological approaches to the study of play. *American Journal of Play*, *7*(3), 101–128.

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. *For the Learning of Mathematics*, *22*(1), 42–47.

Bressoud, D. (2020). Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education (Book review). *Notices of the American Mathematical Society*, *67*(09), 1. https://doi.org/10.1090/noti2145

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32–42. https://doi.org/10.3102/0013189X018001032

Caldwell, B., Gresalfi, M., & Vescio, J. (2023). What counts? Play as a mechanism for disrupting participation patterns in school mathematics. *Proceedings of the 17th International Conference of the Learning Sciences - ICLS 2023*.

Callois, R. (2001). *Man, play, and games*. https://doi.org/10.1080/21594937.2020.1778277

Csikszentmihalyi, M., & Bennett, S. (1971). An exploratory model of play. *American Anthropologist*, *73*(1), 45–58.

Elfenbein, H. A., & Ambady, N. (2003). Universals and cultural differences in recognizing emotions. *Current Directions in Psychological Science*, *12*(5), 159–164. https://doi.org/10.1111/1467-8721.01252

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Annual Perspectives in Mathematics 2018: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). National Council of Teachers of Mathematics.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly*, *87*(7), 519–524.

Henricks, T. S. (2008). The nature of play: An overview. *American Journal of Play*, 157–180.

Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology*, *32*(3), 401–415. https://doi.org/10.1080/00207390118654

Horne, D., Bloodworth, A., & Ellis, A. B. (2023). Modeling mathematicians' playful enagement in task-based clinical interviews. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 99–107.

Huizinga, J. (1949). *Homo ludens: A study of the play-element in culture*. https://doi.org/10.4324/9781315824161

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–149.

Leyva, L. A., McNeill, R. T., Balmer, B. R., Marshall, B. L., King, V. E., & Alley, Z. D. (2022). Black queer students' counter-stories of invisibility in undergraduate STEM as a white, cisheteropatriarchal space. *American Educational Research Journal*, *59*(5), 863–904. https://doi.org/10.3102/00028312221096455

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *Journal of Higher Education*, *92*(5), 784–814. https://doi.org/10.1080/00221546.2021.1879586

Lockhart, P. (2008). A mathematician's lament. *MAA Online*, *March 2008*, 25.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236–260. https://doi.org/10.4219/jeg-2006-264
Mura, R. (1993). Images of mathematics held by university teachers of mathematical sciences. *Educational Studies in Mathematics*, *25*(4), 375–385.
Salen, K., & Zimmerman, E. (2004). *Rules of play: game design fundamentals*. The MIT Press.
Sauter, D. A., Eisner, F., Ekman, P., & Scott, S. K. (2010). Cross-cultural recognition of basic emotions through nonverbal emotional vocalizations. *Proceedings of the National Academy of Sciences of the United States of America*, *107*(6), 2408–2412. https://doi.org/10.1073/pnas.0908239106
Schoenfeld, A. H. (1985). A framework for the analysis of mathematical behavior. In *Mathematical problem solving* (pp. 11–45). https://doi.org/10.1016/b978-0-12-628870-4.50007-4
Seymour, E., & Hunter, A. (2019). *Talking about Leaving Revisited*. https://doi.org/10.1007/978-3-030-25304-2
Sfard, A. (1998). The many faces of mathematics: Do mathematicians and researchers in mathematics education speak about the same thing? In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 491–511). https://doi.org/10.2307/749348
Strauss, D. (2011). Defining mathematics. *Acta Academica*, *43*(4), 1–28.
Su, F. (2020). play. In *Mathematics for human flourishing* (pp. 48–66). Yale University Press.
Sutton-Smith, B. (1997). *The ambiguity of play*. Harvard University Press.
Sylva, K., Bruner, J. S., & Genova, P. (1976). The role of play in the problem-solving of children 3-5 years old. In J. S. Bruner, A. Jolly, & K. Sylva (Eds.), *Play: Its role in development and evolution* (pp. 244–257). Penguin.
Williams-Pierce, C. (2019). Designing for mathematical play: failure and feedback. *Information and Learning Sciences*, *120*(9–10), 589–610. https://doi.org/10.1108/ILS-03-2019-0027
Williams-Pierce, C., & Thevenow-Harrison, J. T. (2021). Zones of mathematical play. *Journal of the Learning Sciences*, *30*(3), 509–527. https://doi.org/10.1080/10508406.2021.1913167
Wisittanawat, P., & Gresalfi, M. S. (2021). The “tricky business” of genre blending: Tensions between frames of school mathematics and video game play. *Journal of the Learning Sciences*, *30*(2), 240–278. https://doi.org/10.1080/10508406.2020.1817747

Conceptualizing Students' Ways of Understanding Bijections in Combinatorics

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Pairing and one-to-one correspondences are natural activities, which can be formalized using bijections. Despite bijections being central to mathematics, few studies have explicitly attended to how students construct, understand, and use them. Because bijections show up naturally in combinatorial arguments, I situate this work in combinatorics. In this theoretical paper, I propose three ways of understanding bijections in combinatorics—relabeling, representational, and relational bijections—and discuss possible affordances and constraints of each. I also provide combinatorial examples using binomial coefficients to exemplify each way of understanding.

Keywords: Combinatorics, bijection, ways of understanding

In elementary combinatorics there are two main proof techniques: double counting and constructing a bijection or correspondence (Benjamin & Quinn, 2003; Mazur, 2010). While there has been work done with counting arguments (Lockwood, 2013; Lockwood et al., 2015; Lockwood & Purdy, 2019; Lockwood & Reed, 2020) and double counting proofs (Engelke Infante & CadwalladerOlsker, 2011; Engelke & CadwalladerOlsker, 2010; Erickson & Lockwood, 2021; Lockwood et al., 2021), there has been little work done with bijections in combinatorics. In this theoretical paper, I define a bijective way of thinking and introduce three ways of understanding (Harel, 2008a) bijections in combinatorics: *relabeling bijections*, *representational bijections*, and *relational bijections*. For each of these, I provide one or more examples to illustrate the way of understanding and discuss the potential affordances and constraints.

Literature Review

In early childhood, one-to-one correspondences play an important role in children learning number concepts such as the cardinality principle and exact number. Mix (2002) found that children 12-38 months can experience and use one-to-one correspondences in a wide range of contexts and that certain types of correspondences may serve as bridging activities for students to develop numerical equivalence. For example, pairing objects that have some conceptual relationship, such as frogs and lily pads, serve as cues of equinumerosity for young children (Muldoon et al., 2005). Different types of correspondences may foster different skills. For example, item-to-item correspondences help with set pairing, whereas word-to-item correspondences help children with cardinality (Muldoon et al., 2009). However, just because children can understand cardinality does not mean they understand exact number, i.e., cardinality is an invariant for equinumerous sets (Muldoon et al., 2009). In fact, the relationship between exact number, cardinality, and one-to-one correspondences is complex (e.g., Izard et al., 2014; Sarnecka & Carey, 2008; Sarnecka & Gelman, 2004; Sarnecka & Wright, 2013).

As students progress in mathematics, one-to-one correspondences are formalized as the study of bijections, which are important for discussing cardinality of infinite sets, defining equivalent mathematical objects via isomorphism, and in making combinatorial arguments. In regards to infinite sets, research has found that students experience paradoxes related to arithmetic with infinite sets (Mamolo, 2014), which may arise because students rely on their knowledge of finite sets to reason about infinite sets (Fischbein et al., 1981; Tirosh, 1991). Moreover, students can often hold multiple conflicting criteria for comparing infinite sets, including bijection, single infinity, and subset inclusion (Tsamir, 1999). In fact, the presentation of two infinite sets can affect whether students reason using bijections or some other criterion (Tirosh & Tsamir, 1996; Tsamir & Tirosh, 1992). Even when students use bijections as a comparison method, they often rely on informal arguments rather can constructing explicit bijections (Hamza & O'Shea, 2011), which may be because students are limited in the bijections they can construct based on the functions they know (Tsamir & Dreyfus, 2005).

In combinatorial contexts, the research show mixed results related to students' bijective reasoning. Muter and Uptegrove (2011) found that a group of high school students could recognize and connect different combinatorial problems, such as pizza toppings and block towers, by constructing bijections, whereas Mamona-Downs and Downs (2004) found that students did not construct bijective solution unless supported in this approach, and even then "students did not make a clear distinction between the bijection itself and the result it was justifying (i.e., the number of elements of the two sets corresponded are equal)" (p. 244).

One-to-one correspondences are a natural object of study that are central to young children's development of number and play an important role in comparing infinite sets. Furthermore, bijections are central to discussing equivalent mathematical objects because an isomorphism (of graphs, groups, rings, topological spaces, etc.) is a structure-preserving bijection. I hypothesize that studying bijections in combinatorics, where the sets are typically discrete and easily visualized or listed, may provide students a space to understand and reason with bijections, injections, and surjections that could be drawn on and leveraged in other mathematical domains.

Theoretical Perspective

In this theoretical paper, I draw on Harel's dual constructs *ways of understanding* and *ways of thinking* (Harel, 2008a, 2008b). According to Harel, "a person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's *way of understanding* associated with that mental act" (Harel, 2008b, p. 490, emphasis original). For example, a solution is a product of the problem-solving mental act, and a particular proof of an assertion is a product of the proving mental act. Thus, a particular solution or proof is a way of understanding with respect to the corresponding mental act. Instead of focusing on the product of a single mental act, one can also look across ways of understanding to identify a way of thinking: "Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a *way of thinking* associated with that mental act" (Harel, 2008b, p.490, emphasis original). Sticking with the previous example, problem-solving strategies and proof schemes (Harel & Sowder, 1998) are examples of ways of thinking associated with the mental acts of problem solving and proving, respectively. These constructs are central to Harel's Duality-Necessity-Repeated Reasoning (DNR) instructional framework, and the Duality Principle notes that ways of understanding and ways of thinking are interdependent. As students engage in a mental act, they may use a way of understanding multiple times, which can contribute to the development of a way of thinking. Similarly, they may draw on a way of thinking to understand a particular idea or problem.

Researchers have used Harel's ways of thinking and ways of understanding to make sense of students' counting activity in enumerative combinatorics. Lockwood has repeatedly concluded that attending to sets of outcomes is an important way of thinking (Lockwood, 2014; Lockwood et al., 2017; Lockwood & Gibson, 2016). She calls this way of thinking a set-oriented perspective, which she has defined as "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). In her dissertation, Halani (2013) identified three broad categories of ways of thinking about combinatorial solution sets. Lockwood and Reed (2020) defined an equivalence way of thinking, which allows a student to frame "division as a way of accounting for duplicate outcomes, allowing a counter to identify such duplicates as belonging to the same equivalence class" (p. 4). This paper extends this work into bijective combinatorics.

Three Ways of Understanding Bijections in Combinatorics

Here, I will define and discuss three ways of understanding bijections in a combinatorial context: relabeling, representational, and relational. For each, I will present one or more combinatorial examples and discuss the potential affordances and constraints of thinking about a bijection from the perspective of each way of understanding. These three ways of understanding are associated with a bijective way of thinking, which is marked by a student conceptualizing two sets (possibly the same) and imagining how the elements of one set can be transformed or mapped to elements of the other set via a bijection.

Relabeling Bijections

Suppose two students are asked to solve a coin flipping problem such as the following: "If a coin is flipped five times in a row, what are the possible outcomes?". Student A solves the problem by denoting the possibilities of a single flip, using "H" for heads and "T" for tails. Student B, in contrast, decides to denote heads as "1" and tails as "0". Each student lists the 32 possible outcomes, and the question arises as to whether their answers are the same. The two solutions can be related to each other using a bijection that changes "H" to "1" and "T" to "0" and vice versa, and thus the students realize they have the same answer differing only by notation. This is an example of a *relabeling bijection*, which is a bijection that connects two sets by changing the surface features of those sets, typically by changing notation.

Relabeling bijections can serve as a starting point for students to recognize situations as similar, which can be leveraged to connect different representations. This way of understanding bijection may serve as a starting point into an intuitive understanding of injectivity, surjectivity, and invertibility, since the function is relatively simple. Additionally, relabeling bijections may be a way to begin to discuss permutations as a bijection from a set to itself. However, if a student only understands bijections as relabeling, it may difficult for them to construct a bijection between two sets whose elements differ in more substantive ways, such as subsets and lattice paths (discussed below).

Representational Bijections

In combinatorics, students often encounter or are introduced to a variety of different problems whose solutions are the same. As an example, let A be the set of k-element subsets of an n-element set, let B be the set of binary strings of length n with exactly k ones, and let C be the set of north-east lattice paths[1] from (0, 0) to (k, $n-k$) (see Figure 1). The cardinality of each set is $\binom{n}{k}$. This can be shown by starting with the standard combinatorial definition of $\binom{n}{k}$ as counting the number of k-element subsets of an n-element set. Thus, by definition, $|A| = \binom{n}{k}$. To also see that sets B and C have the same cardinality, I will construct two bijections.

The first bijection is between sets A and B. A k-element subset of A can be mapped to a binary string of B by labelling the n positions of the binary string 1 through n and placing a 1 in

[1] A north-east lattice path is a path on an integer lattice involving the steps (0, 1) and (1, 0).

the i^{th} position of the binary string if the i^{th} element is in the subset and a 0 otherwise. So, if n = 6 and k = 3, the subset $\{2, 4, 5\}$ will map to the binary string 010110 and the subset $\{1,2,3\}$ will map to 111000. Similarly, a given binary string can be uniquely mapped back to a subset by inverting the process, i.e., 110110 maps to $\{1, 2, 4, 5\}$ and 000100 maps to $\{4\}$. Since bijections preserve cardinality, $|B| = \binom{n}{k}$.

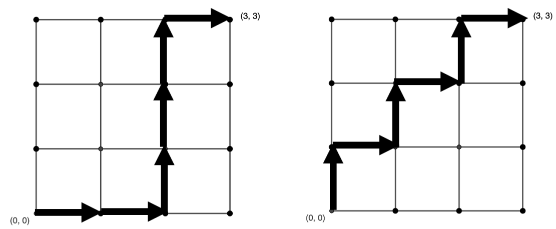

Figure 1. Two examples of a north-east lattice path from (0,0) to (3,3).

Finally, I will construct a bijection between sets *B* and *C*. Any binary string of *B* can be relabeled by changing each 1 to an "E" and each 0 to an "N," which can be turned into a lattice path by reading left to right and taking the step (0, 1) when there is an "E" and (1, 0) when there is an "N." For concreteness, I will continue to let n = 6 and k = 3. Consider the binary strings 110001 and 010101. These map to the binary words EENNNE and NENENE, which are exactly the two lattice paths shown in Figure 1. This mapping can be reversed by following the steps in reverse.

These two bijections are examples of what I call *representational bijections*. A representational bijection is a bijection between two sets represented in mathematically different ways that may afford different operations and ways of reasoning. In our example above, the elements of set *A* are subsets, which can be operated on naturally using intersections, unions, complements, and symmetric differences. In contrast, the elements of set *C* are lattice paths, which are geometric objects that can be reflected, rotated, and assigned an area (such as the area between the lattice path and the x-axis). It is not clear what the area of a subset would be, nor how to take the symmetric difference of a lattice path; however, it may be possible to interpret these through a representational bijection.

This way of understanding bijections enables students to connect different representations, which may allow them to better understand what a combinatorial expression such as $\binom{n}{k}$ counts. Additionally, having a range of contexts in which to interpret combinatorial expressions afford students choosing a representation with which they are comfortable working, which could support them in proving combinatorial identities (Lockwood et al., 2021). In the examples above, I have demonstrated bijections, and thus I know that every set can be counted by $\binom{n}{k}$. However, that does not answer the more fundamental question, which is, if I encounter a new context or representation, how do I recognize it as $\binom{n}{k}$? By examining the examples above, one can look for similarities and differences. In each case, there are n slots to be filled and two choices to be made:

1. **Set *A*:** To construct a subset of *A*, one decides whether to include the first element, the second element, and so forth. In order to get a subset with k elements, one must choose which k elements are included, at which point the $n - k$ excluded elements are determined.

2. **Set *B*:** To construct a binary string, one decides where to put a 0 or a 1. To ensure the binary string is an element of *B*, one has to choose *k* places to put a 1, which forces where a 0 can be put.
3. **Set *C*:** At each step of the lattice path, a choice to go east or north is made. The given constrains of the lattice (starting at (0, 0) and ending at (*k*, n – k)) imposes the restriction that *k* norths must be chosen, which will determine where the east steps must be.

In each of these cases, one decides between two choices (include or not include, 1 or 0, and east or north), and once *k* choices of one type have been made, the remaining *n* – *k* choices must be of the other type. This affords a student being able to look for similar reasoning in other problems and contexts, which they connect to this counting process (Lockwood, 2013), and by extension, to $\binom{n}{k}$.

When a combinatorial expression, such as $\binom{n}{k}$, is examined across the multiple contexts and representations, one can gain insight into the underlying counting process. However, this does not immediately afford an understanding of how a set counted by $\binom{n}{k}$ can be structured: Is there some recursion that determines the set? Does the set exhibit some symmetry? Can the set be restructured and counted according to some property? Being able to understand how a set can be structured, partitioned, refined, and counted is important for proving combinatorial identities, a student with a representational way of understanding bijections may overlook these finer processes.

Relational bijections

One goal of combinatorics is to understand the structure of the sets counted by a combinatorial sequence. As mentioned above, $\binom{n}{k}$ counts subsets, binary strings, and lattice paths; however, what properties do these objects have and how are they related to the binomial coefficients that count them? To be more concrete, I will discuss two binomial identities and highlight how each identity captures some information about the structure of sets counted by $\binom{n}{k}$.

The first identity is $\binom{n}{k} = \binom{n}{n-k}$. If I take the left-hand and right-hand side to be the cardinality of two sets, then this identity suggests a symmetry among the sets. To illustrate this, let us interpret these coefficients as counting the *k*-elements subsets of an *n*-element set and as counting the (*n* – *k*)-element subsets of the same *n*-element set. Taking the set complement (with respect to the *n*-element set) defines a bijection between the two sets, because the complement of *k*-element subset is an (*n* – *k*)-element subset and vice versa. What is this bijection really saying about objects counted by $\binom{n}{k}$? It indicates that if any *k* elements are chosen, then the other *n* – *k* element are determined. Thus, I could either count which *k* elements are included in a subset or count which *n* – *k* elements are excluded from the subset, which indicates a type of symmetry of $\binom{n}{k}$.

As a second example, consider the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Once again, I interpret these in the contexts of subsets in order to describe a bijection. The left-hand side counts the number of *k*-element subsets of an *n*-element set while the right-hand side counts the number of subsets with either *k* or *k* – 1 elements chosen from an *n* – 1 element set. Given a *k*-element subset of *n*-elements, it either contains the n^{th} element, or it does not. If it does not contain the n^{th} element, then it is already a *k*-element subset of an (*n* – 1)-element set and can be mapped to itself. Otherwise, it contains the n^{th} element and can be mapped to a set counted by $\binom{n-1}{k-1}$ by removing the n^{th} element. This process is reversible and thus defines a bijection. In this case, this bijection illustrates a recursive structure of the sets counted by $\binom{n}{k}$, since each element of this set can be built from an element of the set counted by $\binom{n-1}{k}$ or by $\binom{n-1}{k-1}$ as described in the bijection.

Each of these examples illustrate a *relational bijection*. Understanding a relational bijection entails thinking not only about the sets being counted but also about a property of the sets being counted, which can be related to some other property of the sets. This provides additional information about how sets counted by combinatorial expressions, such as sets counted by $\binom{n}{k}$, are structured. In the first example, the bijection provided information about the symmetry between the subsets of an *n*-element set, i.e., one can either focus on which elements the set contains or which elements the set does not contain. In the second example, the bijection describes a systematic process for building new subsets out of old subsets; thereby, it defines a two-variable recurrence for binomial coefficients.

Conceiving of bijections in this way pushes students to go beyond pairing two sets and arguing they are equinumerous. It pushes students to examine what properties the bijection may be preserving or what structure the bijection is hinting at, e.g., a partitioned structure, a refinement, or a recursive structure. All of these properties are related to size and cardinality, since that is what a bijection is guaranteed to preserve. However, this way of understanding may provide a chance for students to build intuition about preserving structure and connecting (counting) processes, which could be leveraged in future discussions about isomorphisms of rings, fields, and groups, homeomorphisms of topological spaces, and isometries in geometry.

One constraint with this way of understanding relates to the difficulty of constructing these types of bijections. A priori, it is not clear what property of the sets one should be exploring and trying to connect, and thus it takes time to explore and understand the sets.

Discussion

These three ways of understanding bijection are not mutually exclusive; it is possible for a student to simultaneously understand a bijection in two or more of these ways. To illustrate, I return to the identity $\binom{n}{k} = \binom{n}{n-k}$. Instead of interpreting the binomial coefficients as counting subsets, I could interpret them as counting binary strings. Then, $\binom{n}{k}$ counts the number of binary strings of length *n* with *k* ones, and $\binom{n}{n-k}$ counts the number of binary strings of length *n* with $n - k$ ones. One can form a bijection between these two sets by interchanging zeros and ones. Certainly, this bijection can still be understood as relational for the reasons noted above; however, a student may not attend to a counting process and instead see this bijection as relabeling. Alternatively, a student could interpret $\binom{n}{k}$ as counting binary strings and $\binom{n}{n-k}$ as counting *k*-element subsets. In this case, a bijection between the two sets may be interpreted as (1) relational if the student understands how the bijection is relating the structuring of the two sets; (2) representational if the student sees subsets and binary strings as affording different mathematical ideas and approaches; and (3) relabeling if the student has internalized the relationship between binary strings and subsets and views the bijection as changing notation from binary strings to a more familiar context of subsets (or vice versa). Thus, it is important to attend to how the student is reasoning about the bijection in order to understand what way of understanding the student may have.

I see these ways of understanding as hierarchical in the sense that there is a natural progression from relabeling to representational to relational. Relabeling bijections are conceptually easier for students to construct and reason about since they usually involve notational changes, which can be written down explicitly and properties like injectivity and surjectivity are easier to check. Representational bijections can be more challenging because different sets of objects have different properties, and it is not immediately obvious what the relationship between two sets of objects is, if any. Thus, an instructor may leverage relabeling bijections to begin a discussion of representational bijection. For example, a bijection between lattice paths and binary strings can be thought of as a relabeling bijection where each east step is relabeled as 1 and each north step is relabeled as 0. However, a discussion can be focused around how the lattice paths and binary strings are mathematically different by focusing on, for example, the geometric properties of lattice paths and whether there are corresponding properties in the binary strings. Discussing properties and operations of a set can progress to relational bijections. For example, a student may wonder what happens to a lattice path that is reflected across the line $y = x$. Using the representational bijection, a student could convert each lattice path and its reflection to binary strings and recognize that the two binary strings have their zeros and ones swapped, i.e., the ones in the first binary string are precisely the zeros in the second binary string and vice versa. With some further exploration or guidance, this could lead to a justification of the identity $\binom{n}{k} = \binom{n}{n-k}$ using binary strings and using lattice paths.

In this paper, I have provided a theoretical account of three ways of understanding bijections in combinatorics: relabeling, representational, and relational. I provided combinatorial examples to exemplify each way of understanding, and I discussed some potential affordances and constraints of reasoning with each type of bijection. My hope is that these ways of understanding can serve as a foundation for future empirical research and conceptual analyses, both to provide evidence and to suggest refinements and additions to these three categories. Moreover, conceiving of combinatorial bijections in terms of relabeling, representational, and relational provides one avenue for designing tasks to target a bijective way of thinking in combinatorics. By elaborating ways of understanding bijections in combinatorics, I hope to inspire thought and research into bijections and how this research can be connected to other areas of mathematics education research, such as research in combinatorics education, functions, equivalence and sameness, and topics such as isomorphism, homeomorphism, and isometry.

References

Benjamin, A. T., & Quinn, J. J. (2003). *Proofs that really count: The art of combinatorial proof*. Mathematical Association of America. https://doi.org/https://doi.org/10.5948/9781614442080

Engelke Infante, N., & CadwalladerOlsker, T. (2011). Student difficulties in the production of combinatorial proofs. *Delta Communications, Volcanic Delta Conference Proceedings*.

Engelke, N., & CadwalladerOlsker, T. (2010). Counting two ways: The art of combinatorial proof. *Proceedings of the 13th Annual Research in Undergraduate Mathematics Education Conference*.

Erickson, S. A., & Lockwood, E. (2021). Investigating undergraduate students' proof schemes and perspectives about combinatorial proof. *Journal of Mathematical Behavior*, *62*. https://doi.org/10.1016/j.jmathb.2021.100868

Fischbein, E., Tirosh, D., & Melamed, U. (1981). Is it possible to measure the intuitive acceptance of a mathematical statement? *Educational Studies in Mathematics*, *12*, 491–512.

Halani, A. (2013). *Students' ways of thinking about combinatorics solution sets* [Unpublished doctoral dissertation]. Arizona State University.

Hamza, S., & O'Shea, A. (2011). Students' misconceptions concerning infinity. In T. Dooley, D. Corcoran, & M. Ryan (Eds.), *Proceeding of the Fourth National Conference on Research in Mathematics Education* (pp. 192–202). Dublin: St. Patrick's College.
Harel, G. (2008a). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold & R. Simons (Eds.), *Proof and other dilemmas: Mathematics and philosophy* (pp. 265–290). Mathematical Association of America.
Harel, G. (2008b). DNR perspective on mathematics curriculum and instruction. Part I: Focus on proving. *ZDM - International Journal on Mathematics Education*, *40*(3), 487–500. https://doi.org/10.1007/s11858-008-0104-1
Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education*, *7*, 234–283.
Izard, V., Streri, A., & Spelke, E. S. (2014). Toward exact number: Young children use one-to-one correspondence to measure set identity but not numerical equality. *Cognitive Psychology*, *72*, 27–53. https://doi.org/10.1016/j.cogpsych.2014.01.004
Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior*, *32*(2), 251–265. https://doi.org/10.1016/j.jmathb.2013.02.008
Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics*, *34*, 31–37.
Lockwood, E., A. Swinyard, C., & S. Caughman, J. (2015). Patterns, sets of outcomes, and combinatorial justification: Two students' reinvention of counting formulas. *International Journal of Research in Undergraduate Mathematics Education*, *1*(1), 27–62. https://doi.org/10.1007/s40753-015-0001-2
Lockwood, E., & Gibson, B. R. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics*, *91*(2), 247–270. https://doi.org/10.1007/s10649-015-9664-5
Lockwood, E., & Purdy, B. (2019). Two undergraduate students' reinvention of the multiplication principle. *Journal for Research in Mathematics Education*, *50*(3), 225–267.
Lockwood, E., & Reed, Z. (2020). Defining and demonstrating an equivalence way of thinking in enumerative combinatorics. *Journal of Mathematical Behavior*, *58*, 100780. https://doi.org/10.1016/j.jmathb.2020.100780
Lockwood, E., Reed, Z., & Caughman, J. S. (2017). An analysis of statements of the multiplication principle in combinatorics, discrete, and finite mathematics textbooks. *International Journal of Research in Undergraduate Mathematics Education*, *3*, 381–416.
Lockwood, E., Reed, Z., & Erickson, S. (2021). Undergraduate students' combinatorial proof of binomial identities. *Journal for Research in Mathematics Education*, *52*(5), 539–580. https://doi.org/10.5951/jresematheduc-2021-0112
Mamolo, A. (2014). Cardinality and cardinal number of an infinite set: A nuanced relationship. In P. Liljedahl, S. Oesterle, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 4* (pp. 169–176). PME.
Mamona-Downs, J., & Downs, M. (2004). Realization of techniques in problem solving: The constructions of bijections for enumeration tasks. *Educational Studies in Mathematics*, *56*, 235–253.
Mazur, D. R. (2010). *Combinatorics: A guided tour*. Mathematical Association of America.
Muldoon, K., Lewis, C., & Freeman, N. (2009). Why set-comparison is vital in early number learning. *Trends in Cognitive Sciences*, *13*(5), 203–208. https://doi.org/10.1016/j.tics.2009.01.010

Muldoon, K., Lewis, C., & Towse, J. (2005). Because it's there! Why some children count, rather than infer numerical relationships. *Cognitive Development*, *20*(3), 472–491. https://doi.org/10.1016/j.cogdev.2005.05.008

Muter, E. M., & Uptegrove, E. B. (2011). Representations and connections. In C. A. Maher, A. B. Powell, & E. B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying, and building isomorphisms* (pp. 105–120). Springer.

Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition*, *108*(3), 662–674. https://doi.org/10.1016/j.cognition.2008.05.007

Sarnecka, B. W., & Gelman, S. A. (2004). Six does not just mean a lot: Preschoolers see number words as specific. *Cognition*, *92*(3), 329–352. https://doi.org/10.1016/j.cognition.2003.10.001

Sarnecka, B. W., & Wright, C. E. (2013). The Idea of an Exact Number: Children's Understanding of Cardinality and Equinumerosity. *Cognitive Science*, *37*(8), 1493–1506. https://doi.org/10.1111/cogs.12043

Tirosh, D. (1991). The role of students' intuitions of infinity in teaching the Cantorian theory. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 199–214). Kluwer Academic Publishers.

Tirosh, D., & Tsamir, P. (1996). The role of representations in students' intuitive thinking about infinity. *International Journal of Mathematical Education in Science and Technology*, *27*(1), 33–40. https://doi.org/10.1080/0020739960270105

Tsamir, P. (1999). Prospective teachers' acceptance of the one-to-one correspondence criterion for comparing infinite sets. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference (Vol. 4)* (pp. 305–312).

Tsamir, P., & Dreyfus, T. (2005). How fragile is consolidated knowledge? Ben's comparison of infinite sets. *Journal of Mathematical Behavior*, *24*(1), 15–38. https://doi.org/10.1016/j.jmathb.2004.12.001

Tsamir, P., & Tirosh, D. (1992). Students' awareness of inconsistent ideas about actual infinity. In W. Goeslin & K. Graham (Eds.), *Proceedings of the 16th Annual Meeting for the Psychology of Mathematics Education, Vol III* (pp. 90–97).

Ways of Reasoning About Inverse Functions

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Prior research on student understanding of inverse function has primarily focused on deficits in these understandings without explicitly addressing the mental actions and operations that researchers or instructors would like students to engage in. In this report, I present three ways of reasoning about inverse functions: a covariational approach, a mapping approach, and a set theoretic approach. I argue that all three entail productive ways of thinking about inverse functions, and that the coordination of the three can support students in reasoning about inverse functions in a variety of contexts.

Keywords. Conceptual Analysis, Inverse Functions, Precalculus

To intentionally design curriculum and plan instruction that targets the development of strong mathematical meanings on a particular topic, mathematics educators must first decide what specific mathematical meanings are most beneficial for students to construct (Steffe, 2007; Thompson, 2013). This includes a need to understand the mental operations and ways of thinking that are entailed in these desired ways of reasoning (Thompson, 2008; Thompson, 2013). Previous literature has primarily focused on student difficulty with inverse functions instead of providing a coherent description of how someone might reason productively about them. There is ample evidence that students often do not have coherent meanings for inverse function across contexts, but there are few detailed descriptions of what coherent meanings might entail. In this paper, I propose three ways of reasoning about inverse function and argue that all three are compatible with each other and useful for students, and that they should be targeted in teaching.

Literature Review

Although many researchers have studied student learning and understanding of function (e.g., Breidenbach et al., 1992; Carlson, 1998; Monk, 1992; Doorman et al., 2012) there has been less of a focus on inverse function in particular. Early work by Even (1990; 1992) examined the use of the metaphor of "undoing" to understand inverse function and several research groups have used characterizations of students' conception of function developed by Breidenbach et al. (1992) to extend to the context on inverse functions. Breidenbach and colleagues defined an action versus process conceptualization of function, where an action level of function involves the manipulation of objects (either physical or mental) and might allow a student to plug numbers into an algebraic expression to calculate a result but would limit the student to consider the result of such a calculation separately. A student with a process conception of function thinks about functions as processes that transform objects into other objects via some repeatable transformation but does not need to calculate any particular values to be able to conceptualize this process (Breidenbach et al., 1992). Vidakovic (1996) claimed that students with a process or object concept of function can develop concepts for inverse function. She argued that coordinating inverse function as the reverse of a function process, as the coordination of two function processes to get identity, and as an action of switching x and y and solving for y, is necessary for a deep conceptual understanding of inverse function but did not detail how a student might coordinate these different ways of thinking about inverse. Engelke and colleagues (2005) used the action versus process distinction and argued students who have developed a process view of function understand that this process can be reversed to obtain an inverse which would allow them to make sense of inverse function notation such as $f^{-1}(f(x)) = x$.

Oehrtman and colleagues (2008) discussed the implications of an action versus process conception of functions, and noted that an action view of functions makes it impossible to reason about functions dynamically because each input and output must be considered separately. Oehrtman et al. (2008) identified three distinct conceptions of the inverse of a function: as an algebra problem, as a geometry problem, and as the reversal of a process. They argued that students with an action view of function can understand inverse functions in the context of algebra (as a procedure to find the inverse of a given function) and the context of geometry (as a procedure to reflect a given graph over the line $y = x$), but do not understand why these procedures work or their connection to the ideas of composing or reversing functions. In contrast, students with a process conception of function can understand the inverse of a function to be the reversal of a process, which maps a set of outputs of the original function back to the associated inputs (Oehrtman et al., 2008). This way of reasoning requires students to be able to consider a function as a general process without having to coordinate individual input and output values as would be necessary for students with an action concept of function.

Paoletti and colleagues (2018) conducted task-based interviews that included contextualized and decontextualized inverse function tasks in analytic and graphical representations. Many participants tried to use the "switch and solve" procedure in contextualized contexts, which proved problematic because switching the variables led to a non-normative relationship between the two quantities. For example, when finding the inverse of a function that converts degrees Fahrenheit (F) to degrees Celsius (C), interchanging the variables F and C leads to a function that does not maintain the standard relationship between Fahrenheit and Celsius. Many of these participants concluded that their inverse was not meant to represent the same relationship as the original function or were unsure of how to interpret the inverse function that they found. Only one student demonstrated what the researchers inferred to be well connected meanings between analytic and graphical tasks in decontextualized and contextualized settings. This participant used "switching" techniques for all of the tasks, but she identified that she would need to change which quantity her variables referred to when switching in the tasks with quantitative context.

While most research on students' understanding of inverse functions has focused on the deficits and disconnected nature of this knowledge rather than on explicating potential ways that students might reason productively, Paoletti (2020) presented an example of a student successfully reorganizing her previously disconnected meanings for inverse function. The participant originally used switching techniques to solve tasks that did not support her in making sense of contextualized inverse function tasks. Throughout the course of the teaching experiment, the participant came to understand that a function and its inverse represent the same relationship in a contextualized situation, and that the switching and solving technique is used to maintain the convention that the quantity seen as the independent variable is represented on the x axis. Based on the student's actions, along with an analysis of research on covariational reasoning, Paoletti (2020) provided a detailed conceptual analysis of inverse functions that emphasized the invariance of the relationship between the two quantities related by a function and its inverse. This conceptual analysis will be discussed in more detail below.

More recently, Cook et al. (2023) conducted a conceptual analysis on the construct of inverse more generally (rather than specific to inverse functions), and proposed three ways of reasoning about inverse that can support students' understandings in a variety of algebraic contexts. Their first characterization is "inverse as undoing", where "inverse is viewed as a relationship between operations" and where "the purpose of the operation (or sequence of operations) in question is to undo the effect of the original operation(s)" (Cook et al., 2023, p. 757). They also argued that inverse can be thought of as "inverse as a manipulated element" in which students conceptualize an inverse as an element rather than an operation, and where this inverse element is "associated with a procedure by which a given element is manipulated into its inverse element")" (Cook et al., 2023, p. 757). Lastly, they proposed "inverse as a coordination of the binary operation, identity, and set" (Cook et al., 2023, p. 757) where students view inverse as a relationship between a pair of elements such that their interaction yields the identity, with whatever binary operation is relevant in the given context. This conceptual analysis provides potential ways for students to see commonalities in the broader construct of inverse across mathematical contexts but does not provide explicit ways that students might reason about particular inverse functions.

Three Ways of Reasoning About Inverse Function

To develop the ways of understanding inverse functions presented here, I conducted a conceptual analysis. Conceptual analysis is a method grounded in a radical constructivist epistemology that involves a fine-grained examination of the concepts surrounding a particular topic (Thompson, 2008). Importantly, the analysis is not situated in the abstract but instead the researcher considers how an individual might think about the concepts. Conducting a conceptual analysis can be a way to develop answers to the question "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). Thompson (2008) identified conceptual analysis as a fruitful technique for describing productive understandings for students to develop, and for examining the coherence of different ways of understanding a collection of ideas. To this end, I conducted a conceptual analysis on inverse functions to describe potential productive ways for students to reason about inverse functions, and to examine the coherence (or lack thereof) of these ways of reasoning. Based on my own mathematical understanding and my review of the literature on inverse functions, I have identified three distinct approaches to thinking about inverse function: a covariational approach, a mapping approach, and a set theoretic approach.

A Covariational Approach to Inverse Function

Covariational reasoning, as defined by Carlson et al. (2002), involves "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). This way of reasoning has been shown to be powerful for helping students reason about dynamic situations (Moore & Carlson, 2012), exponential growth (Ellis et al., 2012), calculus (Thompson et al., 2013) and trigonometry (Moore, 2014). Thompson and Carlson (2017) proposed a covariational meaning for the concept of function:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 444).

This construct was extended to define the idea of a "covariational relation" (Paoletti, 2020), in which a student has conceptualized two quantities varying together where one quantity varies first and the other quantity changes along with the first. This can then become a covariational function once the student constructs the property that each value of the first quantity corresponds to exactly one value of the second (Paoletti, 2020). A student who reasons about functions from a covariational perspective can imagine the first quantity changing and knows that the second quantity will also change correspondingly, without having to think about or calculate actual values of either quantity. For example, a student might construct a covariational relation between height and volume when considering water being poured into a pool, where they consider the height of the water first and then the associated volume at that moment. If the student realizes that for any height value, there is a unique corresponding volume, they have then constructed a covariational function of volume with respect to the height of the water.

If a student then were to consider the volume changing first instead of the height, they would have constructed an inverse covariational relation (Paoletti, 2020). If they realized that for each value of the volume there is exactly one associated value of the height of the water, they would have constructed an inverse covariational function (Paoletti, 2020). Once a student has constructed both a covariational function and an inverse covariational function to describe the same two quantities varying, they can see that both the original function and its inverse represent the same relationship between the quantities, and the only aspect that changes is which quantity is considered to be varying first, with variation in the second dependent on the variation of the first. This does not imply an actual causal relationship between the two covarying quantities, but instead refers to the fact that the amount of variation in one quantity that is considered is dependent on the amount of variation considered in the other. From this perspective, there is no need to consider the inverse relationship as a separate entity from the original relationship, it is simply the same relationship viewed from a different perspective.

In a quantitative context in which one quantity is not uniquely determined by the other, a student can still conceive of an inverse covariational relation. For example, imagine that the student had instead considered the volume of the water in the pool as time varied, with the pool first being drained and then refilled. In this case, each value of the volume of water in the pool would correspond to (at least) two different times, one as the pool is emptied and one as it is filled. The student can still consider volume varying first with the associated time values varying with it, despite the lack of uniqueness. For example, the student can imagine that as the volume of water in the pool increases, one of the corresponding time values also increases (becoming closer to the time when the pool is completely refilled) while the other decreases (becoming closer to the time when the pool started to drain). The mental actions that a student engages in when constructing an inverse relation and an inverse function are the same from this perspective, and an inverse function can be seen as just a type of relation that happens to fulfill the uniqueness property. This flexibility in imagining inverse covariational relations that are not necessarily functions is in direct contrast to the way that inverse functions are typically taught, in which students learn that functions are not invertible if they are not injective. This may lead students to believe that it is impossible to construct an inverse relation in these situations, or that these inverse relations would have no use.

A Mapping Approach to Inverse Function

Traditionally, functions are defined as mappings that take each element of a function's domain to exactly one element of the function's range. To construct the idea of function from a mapping perspective, a student must conceptualize two arbitrary sets with some sort of rule that takes elements from one set as inputs and gives elements of the other set as outputs, and where each possible input maps only to one output. To construct the idea of an inverse function, a student must then imagine another function that maps in the opposite direction, taking the original function's output values to the associated input values. To be able to define this map that reliably takes an output value to its associated input, the original function must satisfy the property that each input maps to a unique output value. A student must see that if this were not the case, there would be no way to determine which input value the inverse function should map the output value back to, and thus the injectivity of the original function becomes a requirement to define an inverse function. The student can then imagine elements from the function's domain being mapped to elements in the range by the function, and then these elements of the range being mapped back to the original elements in the domain when the inverse function is applied. The opposite order of applying these mappings will yield the same result, where elements of the range of the original function are mapped to elements of its domain by the inverse function, and then applying the original function will map these elements back to their "starting point".

With a covariation approach to function, the student imagines changes in each quantity occurring simultaneously and attends to these changes in relation to each other (e.g., as quantity A increases, quantity B decreases). With a mapping approach to function, the student instead imagines change from one set to the other (e.g., an element, *x,* being "transformed" by the function to an element, *y*). Both approaches have an imagined dynamic element, but the types of change that are emphasized are fundamentally different. In contexts where students are reasoning with actual quantities, often a transformation approach to understanding function is incompatible (at least from the perspective of the researcher) because one quantity cannot "transform" into the other. Recall the example above where the volume of water in a filling pool is seen as a function of time elapsed. This function cannot transform time into volume, but instead simply provides a relationship between the values of the two quantities. Additionally, in contrast to the covariation approach where the inverse is seen as maintaining the same relationship as the original function but from a different perspective, the mapping perspective instead views the inverse as a separate function in its own right. Instead of the emphasis being on the sameness of the relationship defined by the function, the emphasis is instead on the relationship between the function and its inverse, namely that the inverse undoes the action of the original function.

A Set Theoretic Approach to Inverse Function

Relations can also be defined from a set theoretic perspective, as a set of ordered pairs (a, b) where the first element in the pairs are elements of a set A and the second element in the pairs are elements of a set B. The set A is defined to be the relation's domain, and the set B is the relation's range. In contrast to the covariation and mapping approaches, there is no mental imagery of change or variation, only of static ordered pairs being elements of a relation. From this perspective, in order for a relation to be a function, the property must hold that if (a, b) and (a, c) are elements of the relation then b and c are the same element of the range. The inverse relation would then be the set of ordered pairs (b, a) obtained from reversing the order of the pairs in the original relation and thus interchanging the domain and range. This inverse relation would be a function if it satisfies the above property. Using this definition to think about functions and inverse functions leads to an understanding of inverse as a switching of coordinate points. This is useful for mathematical settings that do not involve continuity or that are in contexts outside of the real numbers but does not support the use of quantitative reasoning to imagine "real life" situations, particularly those in which images of (co)variation are critical. If functions are thought of solely as sets of ordered pairs, then there is not a natural way to envision how one quantity might change in relation to another.

Despite these limitations of a solely set-theoretic approach, this approach is in no way incompatible with a covariation or mapping perspective. All three perspectives foreground pairs of values, but the set theoretic approach does not necessitate the consideration of variation between the pairs of values or a transformation from one value to another. From a covariation perspective in which two quantities are varying simultaneously, one must keep in mind that for any given value of one quantity there is an associated value of the other quantity, and these pairs of values can be seen as the ordered pairs considered from the set theoretic perspective. Similarly, the mapping perspective emphasizes that values from the domain are mapped to values in the range of a function, and each input/output pair can be seen as the ordered pairs from the set theoretic perspective. The set theoretic approach assumes the existence of these pairs of values but does not support someone in producing the pairs. In order to produce the pairs, one must use either a covariation or mapping approach to function to determine the value of one quantity for a given value of another quantity.

Examples

Each of the three ways of reasoning described above provide useful ways for students to reason about inverse functions, and it is the goal-oriented activity that a student engages in that dictates which way of thinking is best suited for the situation. For example, consider a quantitative context where two quantities covary and in which it can be useful to think of either quantity as varying first. A covariation perspective can allow a student to reason flexibly about either quantity varying first, with the persistent realization that both ways of thinking about the context maintain the same relationship between the quantities. A student reasoning in this way can view the graph of a function as also representing the graph of the inverse by considering the quantity on the vertical axis as varying first instead of the standard view of values on the horizontal axis representing the independent variable. This can allow them to reason about the inverse relationship using only a graph of the original function without needing to construct a separate graph of the inverse. In this context, a mapping perspective is somewhat less natural, and in order to reason about the inverse of a given function a student would be more likely to have to construct an equation or graph for the inverse first.

However, in contexts involving composition of functions, thinking of inverse from the mapping perspective is more productive. From a covariation perspective there is usually no motivation to consider the inverse relationship as a separate function from the original, and so composition is not obvious. From the mapping perspective the fact that a function composed with its inverse should yield the identity function becomes clear. If the inverse of a function is a function that maps the outputs of a function back to the original inputs, then performing one mapping and then the other must always result in whatever the original input was. When students are asked to produce a graph of the inverse function based on the graph of a given function, set theoretic reasoning is often most efficient. The original graph of the function consists of the ordered pairs of that function, and thus a graph of the inverse function must consist of these same pairs but in the opposite order. This reasoning can allow a student to bypass the cognitive effort of the analogous reasoning from a mapping perspective, where a student might reason that if a point on the original graph (x, y) means that the function maps x to y, then the inverse function must map y back to x and thus the point (y, x) will be on the graph of the inverse.

Despite the varying degrees of utility of each way of reasoning in different contexts, the three ways of understanding inverse function are in no way incompatible. Consider the example of the relationship between temperature measured in degrees Fahrenheit (F) and Celsius (C): $C=5/9(F-32)$. From a covariation perspective, a student can reason that this equation gives a relationship between the measures of temperature in both units and can see a graph of this relationship as providing information about the relationship viewed with either quantity varying first. At the same time, that same student can know that, as written, the function takes values of temperature measured in Fahrenheit and maps them to the appropriate value measured in Celsius, but that they can also find a function that takes values of temperature in Celsius and maps the opposite way back to the appropriate value in Fahrenheit. Along with these ways of reasoning about the situation, the student can also understand that the original function can be thought of as ordered pairs (*F,C*) that are solutions to the equation above, and that the inverse relationship can be seen as the pairs in the reverse order (*C, F*). Depending on the student's goal when reasoning about this context, any of these three meanings could be drawn on productively.

Conclusion

The ways of reasoning described above provide a finer grained description of the mental actions involved in reasoning about inverse functions. Previous work has provided hierarchical frameworks for the development of an understanding of inverse functions (e.g., the action vs. process conceptions of inverse functions), but the details of what an understanding of inverse functions entails were underdeveloped. The action versus process dichotomy does not capture differences between a student reasoning from a set theoretic, mapping, or covariational perspective. While the description of the covariation perspective on inverse functions is not new (Paoletti, 2020), the juxtaposition of this perspective with the mapping and set theoretic perspectives offers a broader view of productive meanings for inverse function. In addition, the conceptions of inverse functions identified by Oehrtman et al. (2008) separated conceptions of inverse mostly by the procedures or products of reasoning associated with them (i.e., reflecting over a line to graph, switching variables and solving, or understanding inverse as the reversal of a process), without specifying the associated mental operations behind the three conceptions. This categorization gives us three ways that students must be able to engage with inverse functions but is of limited use in specifying how we would like students to think about inverse functions. The conceptual analysis conducted by Cook et al. (2023) also identified three ways of reasoning about inverse, but in the more general context of inverse not specific to inverse functions. I argue that, while there is overlap between the characterization presented in this paper and that provided by Cook et al. (2023), there are ways of reasoning that are important for productive engagement with inverse functions that are not captured in broader reasoning about inverse. For example, a covariational approach to inverse function relies on continuity, and while it is a useful way of reasoning for students in these contexts, there is not necessarily an analogous form of reasoning when considering inverse elsewhere in mathematics. In turn, the characterization of inverse function meanings provided here does not capture meanings about inverse functions as an instantiation of the larger idea of inverse, and I have not attended to how students understand functions and their inverses as elements of a set.

A significant limitation of the characterization of inverse function reasoning presented here is that it is a first order model (Thompson, 2000) that it is based on my own mathematical meanings. It is informed by the extant literature base on student understanding of inverse functions and by my interactions with students but has yet to be studied empirically. Future research should examine how these ways of reasoning emerge in practice and explore ways in which the coordination of these different ways of reasoning might support engagement with inverse function tasks. If these proposed ways of reasoning about inverse function maintain their viability after empirical research with students, then further research could begin to explore the development of these ways of reasoning. Reflected abstraction (Piaget, 2001) can help to connect previously disjoint mathematical meanings into one coherent scheme and brings these connections to the level of conscious awareness. Therefore, I hypothesize that reflected abstraction is a potential way for students to integrate these distinct ways of understanding inverse function, and that supporting students in engaging in reflected abstraction may encourage coherence in their meanings.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247-285.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In J. J. Kaput, E. Dubinsky, A. H. Schoenfeld, & T. P. Dick (Eds.), *Research in collegiate mathematics education III* (Vol. 3, pp. 114-163).

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Cook, J. P., Richardson, A., Reed, Z., & Lockwood, E. (2023). Using conceptual analyses to resolve the tension between advanced and secondary mathematics: The cases of equivalence and inverse. *ZDM – Mathematics Education*, *55*(4), 753-766. https://doi.org/10.1007/s11858-023-01495-2

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education*, *10*(6), 1243-1267. https://doi.org/10.1007/s10763-012-9329-0

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the jactus. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, *2*, 93-112.

Engelke, N., Oehrtman, M., & Carlson, M. (2005). Composition of functions: Precalculus students' understandings. In G. M. Lloyd, M. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the north american chapter of the international group for the psychology of mathematics education* (pp. 570-577).

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*(6), 521-544.

Even, R. (1992). The inverse function: Prospective teachers' use of "undoing". *International Journal of Mathematical Education in Science and Technology*, *23*(4), 557-562. https://doi.org/10.1080/0020739X.1992.10715689

Monk, S. (1992). Students' understanding of a function given by a physical model. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 175-194.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of zac. *Journal for Research in Mathematics Education*, *45*(1), 102-138.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-41).

Paoletti, T. (2020). Reasoning about relationships between quantities to reorganize inverse function meanings: The case of Arya. *The Journal of Mathematical Behavior*, *57*, 100741. https://doi.org/https://doi.org/10.1016/j.jmathb.2019.100741

Paoletti, T., Stevens, I. E., Hobson, N. L., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics*, *97*(1), 93-109.

Piaget, J. (2001). *Studies in reflecting abstraction*. Psychology Press.

Steffe, L. P. (2007). Radical constructivism and "school mathematics". In E. Von Glasersfeld & M. Larochelle (Eds.), *Key works in radical constructivism* (pp. 279-289). Brill Sense.
Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld* (pp. 412-448). Routledge.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, Cortina, J.L., Alatorre, S., Rojano, T., & Sepúlveda, A. (Ed.), *Proceedings of the joint meeting of pme 32 and pme-na xxx* (Vol. 1, pp. 31-49).
Thompson, P. W. (2013). In the absence of meaning. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57-93). Springer.
Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools*, *30*(1-2), 124-147.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). The National Council of Teachers of Mathematics, Inc.
Vidakovic, D. (1996). Learning the concept of inverse function. *Journal of Computers in Mathematics and Science Teaching*, *15*(3), 295-318.
Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356-366.
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Routledge Falmer.

Re-conceptualizing the Construct of Mathematical Autonomy: From Individual Trait to Quality of Action in Context

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In this paper, we motivate the need for a definition of mathematical autonomy that does not imply that autonomy is a trait of an individual. Rather, we demonstrate the usefulness of a lens on autonomy that captures the characteristics of actions taken in contexts that shape the space of sensible action. Shifting from trait to quality of action in context further allows us to analyze how actions of groups of individuals doing mathematical work can have qualities associated with mathematical autonomy. We ground our theoretical work in data from a longitudinal case study of an undergraduate student, Kaleb, whose orientation to mathematical sense-making changed sharply across multiple contexts of activity.

Keywords: Intellectual Autonomy, Mathematical Experience, Introduction to Proof, Upper-Division

Introduction

It has long been documented that U.S. students experience challenges of many sorts in moving from lower division courses to upper division courses (Moore, 1994). A challenge of particular interest is the movement from math coursework that is typically more algorithm and procedure oriented to upper division coursework that includes more focus on construction and validation of arguments and proofs (Selden, 2012). In light of the centrality of this challenge, a subset of the authors conducted a small-scale longitudinal study to investigate students' experiences as they navigated this challenge in the context of a large public U.S. university (Bae et al. 2018; Smith et al, 2017). This study focused on student experience and development of agency and autonomy through this transition to proof-intensive coursework in contrast to more traditional focus on mastery of proof methods and mathematical notation.

One way to capture an important facet of productive disciplinary engagement at the undergraduate level, and that is related in a key way to the transition from computationally-focused to proof-intensive work mentioned above, is that when students encounter a challenge in their mathematical work, they are able to consult perceived sources of mathematical authority (self, teachers, texts, internet resources) as appropriate and needed. This goal has been articulated in various ways in the literature discussing the need for students to be able to move away from appeals to external authority in determining validity of arguments (Harel & Sowder, 1998; Stylianides, 2007; Yackel & Cobb, 1996). Inglis & Mejia-Ramos (2009) caution that looking to authority figures, per se, is not a problem, but concur that students should be empowered to establish the validity of arguments through their own sense-making processes. That is, collegiate mathematics educators want students to come to see themselves as local disciplinary authorities that view the correctness of a proof as something they can assess via mathematical reasoning and mathematical logic, not something that they are dependent upon external sources of authority such as teachers or texts in determining.

In describing these goals for students and instruction, sometimes the term “autonomy” is invoked (NCTM, 2000) with the expectation that a goal for mathematics instruction is that students become increasingly autonomous as they move through their mathematics coursework and beyond. However, there are multiple perspectives on autonomy in the literature. Two prevalent perspectives relevant to students’ educational experiences include psychological and intellectual autonomy. Psychological autonomy (Deci & Ryan, 1980) is linked with self-determination, free-will, and independence (self-reliance). Intellectual autonomy, with roots in the work of Piaget, casts autonomy as a trait which individuals develop over time, in which individuals become increasingly independent in sense-making in the context of the ideas of others (Piaget, 1973). Later, Wood reformulated the notion of intellectual autonomy through a communicational lens, building upon Sfard, to focus more upon the individual in relation to sources of authority and whether individuals see themselves as sources of authority, capable of deciding whether a given mathematical inference is right or wrong (Wood, 2016).

In this paper, we broaden the original focus on individuals making determinations of right or wrong inferences to students remaining personally involved in processes of sense-making (e.g., making determinations of whether actions taken in response to mathematical challenges make personal sense or not). We further expand from individual sense-making to sense-making in group contexts and also consciously expand from other people as trusted external authorities to include other kinds of external sources of authority such as internet-based resources, including AI systems.

Research Focus and Question

The data for our study was collected at a large public U.S. university where students, both mathematics majors and minors, first encounter mathematical proof formally in a semester-long, problem-based ITP course. In the ITP course, project data collection involves a baseline interview, a task-based interview, and a series of homework reflection logs from each of the 15 participants in the first cohort. Both the mathematics majors and minors in our study subsequently enroll in two or more advanced, proof-based mathematics courses to satisfy their major or minor requirements. Our longitudinal study involves following students into these subsequent courses as well as collecting data from a second cohort of students. Participants were selected to represent a broad spectrum of gender, ethnicity, major/minor, and mathematical beliefs as measured by the Mathematics Attitudes and Perceptions Survey (Code, Merchant, Maciejewski, Thomas & Lo, 2016). In addition to the interview and reflection log data mentioned above, roughly a third of the ITP classes were observed by members of the project team to provide context for the research team in interpreting the experiences reported by students in the interviews.

The specific question that guided this inquiry in this paper is “Which view of mathematical autonomy [trait or quality of action] is more useful in understanding collegiate students’ progression through their upper-division mathematical coursework?”

Reconceptualizing Mathematical Autonomy: A New Definition

Our initial framing of mathematical autonomy followed Yackel and Cobb’s (1996) analysis in seeing intellectual autonomy as both a *state of awareness of one’s intellectual capabilities* and a *willingness to act on them in engaging in challenging mathematical work*. Where Yackel and

Cobb conceptualized intellectual autonomy solely within the social practices of mathematics classrooms, we needed our constructs to apply also to in–class and out–of–class mathematical activity, as college students have great freedom to act (or not) in ways they see as productive. Indeed, one key developmental task for college students is deciding how and why to deploy out–of–class time and activity productively.

Through our initial attempts to characterize the mathematical autonomy students were displaying at different points in time in their undergraduate experience, we became more attuned to the relationship between students and perceived sources of mathematical authority. That is, we came to appreciate how students' views of themselves as sources of mathematical authority was context-dependent, as well as students' views of themselves in relation to other sources of mathematical authority (instructors, other students, family members, or sources on the internet). In developing our definition, we previously analyzed students' shifting relationships to mathematical authorities such as family members (Castle et al., 2022) and students' usage and reflections on the usage of internet sources of authority (Levin et al., 2020). The working definition for mathematical autonomy that we settled upon is:

> *Mathematical Autonomy:* A quality of action that reflects the actor's active resistance to endorsing, following, or replicating the reasoning of mathematical authorities (e.g., texts, internet sources, instructors, peers) while engaging in a process of sense-making. The quality of action reflects the actor's judgment of using only the external resources necessary to continue a sense-making process when they meet a local obstacle and/or engaging in sufficient sense-making to transform the reasoning of others/other resources to their own.

Illustrating Features of Mathematical Autonomy: The Case of Kaleb

In this paper, we have chosen to illustrate the affordances of the new definition of mathematical autonomy using the data from a single case, Kaleb. Kaleb was a mathematics major, considering adding computer science as a second major or minor at the time of his Introduction to Proof course, when we began interviewing him. He had aspirations of graduate school in mathematics, and was possibly interested in teaching. By following Kaleb for multiple semesters, we found Kaleb displayed autonomy in different ways depending on mathematical context.

Kaleb's individual sense-making actions as autonomous in the ITP course

Early on in Kaleb's Introduction to Proof course, he showed more autonomous actions by actively engaging in sense-making and rejecting passively accepting others' mathematical authority. For example, in the ITP course, Kaleb thought the course was teaching them how students could think mathematically, and this aligned strongly with what he thought he needed to learn. Kaleb faced some challenges in doing homework, and his actions dealing with those challenges showed that Kaleb valued his work and tried to give him some time to think before he used other resources:

> Certain ones [tasks] took longer than others, but I found if I got stuck, I could get up, go get a drink, come back, play on my phone for a little bit, get my mind off of it. And then, looking at it with a fresh start, it would pop out.

He reported that if he had given enough thinking time for himself, he would reach out to other resources, such as peers or the course-provided online platform Piazza. In one instance, Kaleb reported in his homework log that he felt the instructor included more of a hint than what he thought necessary on Piazza. This led him to use only the first half of the hint and create his own outline for what he needed to do. After the whole process, he still spent another 40 minutes to an hour to write a proof. This excerpt shows Kaleb's autonomous action in this context in not accepting an authority figure's work when he was stuck, but rather trying to understand, evaluate, and use only the part he felt he needed to get unstuck. At this time, he expressed a strong view against copying solutions from other sources:

> If you are simply copying the answers down to get answers and submitting them as your own, then yes, that is cheating. However, looking for an answer and using it to get your own answer is fine… From what I can tell, one of the main themes about the class is teaching us how to think more mathematically. If you only copy someone, you let them do the thinking for you. You never learn anything and are wasting everyone's time.

Kaleb's joint work actions in ITP and Abstract Algebra as autonomous

When Kaleb could not solve problems, even though he gave himself some time to think about the problem, he consulted external resources. In the ITP course, the most used resource was a friend, Steven (pseudonym), whom he met in the ITP course and a statistics course. Kaleb and Steven regularly spoke on the phone and on Sunday night, and one explained to the other person as well as sometimes sending photos of their work when one of them was struggling.

Similarly, in Abstract Algebra, Kaleb worked with another friend who was also taking the same course in a different section but with the same instructor. Kaleb and the friend bounced off ideas with each other, and he described their contributions as 50-50. They sometimes had different ways of solving the same problem, and they examined each other's solutions and were content to submit their final work in different ways. We view this excerpt as showing that they did not just follow each other's work, but rather Kaleb's joint work with his friends in these contexts had features of mathematical autonomy.

Kaleb's use of internet resources as more and less autonomous, depending on context

In addition to human resources, Kaleb used internet resources, and his quality of actions shifted, depending on mathematical context. Above, we described how Kaleb resisted the reasoning of authorities when consulting internet-resources when stuck. However, Kaleb's actions towards internet resources changed in the following semesters. In the Linear Algebra course that Kaleb took concurrently with Abstract Algebra, he did not have a peer who could work together. He found himself consulting internet sources faster than in the ITP or the Abstract Algebra courses. He reported that once he spent 10-20 mins per question, he went to the internet to find some solutions. When he found a solution, he reported that 60% of the time he would try to make sense of what the found solution said and rewrite it in his own words. However, for the other 40% of cases, he reported that he copied the solution as he found it. Kaleb attributed the reasons for this action was that there were too many questions, each with little worth of his time because they were not a big part of his grade. In this course, homework was 8% of the final grade, and only some of the questions were graded in the course:

> I've definitely had it where I don't completely understand what's going on. […] It feels like cheating, or it feels…it doesn't feel good, but I know I'll get points and I'll get an okay grade.

Although there were external situations (i.e. having too many questions), it is important to note that in this context he started copying solutions from online sources. In the previous semester (in the ITP course), he viewed copying down others' solutions as relinquishing one's opportunity for sense-making. However, a less autonomous pattern of action emerged in Linear Algebra. His larger frame guiding action appeared to have shifted from understanding to earning points.

Later, in Spring 2020, when the COVID-19 pandemic erupted, Kaleb was taking Analysis I. Before COVID-19 changed the modality of the course, to do the homework in his analysis course, Kaleb said he spent 10 minutes thinking by himself and then left the problem to come back later. He said he usually repeated the cycle three times, then he tried to search online if he could still not solve the problem. Because the majority of the homework problems were from the textbook, Kaleb said it was very easy to find solutions on the internet. Also, he could find the solutions from other college courses or other professors. Even though he was using others' solutions, Kaleb reported that before COVID-19, he still valued understanding and sense-making because he knew that he would need them for the exams in the short term and for his career in the long term.

When COVID-19 happened, Kaleb had to go back to his home and had to go back to his parents' house, disrupting his sense of space and place (Küchle et al, 2023). His motivation for doing homework dropped significantly, which in turn influenced his actions, becoming less autonomous. Kaleb admitted that the quality of his actions before and after COVID-19 was different. Kaleb did not view his parents' house as a conducive workplace, and during this time he also suffered the loss of family. These changes made him focus on his personal life rather than schoolwork, which led him not to submit a few homework assignments, and to use the internet more on assignments he did complete.

Discussion

The data from the case of Kaleb illustrates how trait-based views of autonomy are insufficient for capturing the context-dependency of college students' actions in proof-based courses. As opposed to having a trait or characteristic of "being autonomous," Kaleb instead showed the actions he took could be cast as more or less autonomous depending on different contexts, such as his work with peers in his ITP and Abstract Algebra courses or the differences in his ways of using internet-based mathematics resources across his coursework. Further, as opposed to being a trait that increased in amount over time, we uncovered the way that circumstances and the constraints of his experience in different courses (e.g., different working conditions and course modality due to COVID-19) went hand in hand with him exhibiting less autonomous actions.

Thus, the data we present push the field to reconsider "autonomy" in a way that is not synonymous with a trait such as "individual independence." As opposed to framing autonomy as a trait that an individual may/may not possess, we are able to focus on qualities of actions taken during mathematical work. Our expanded way of framing autonomy focuses on qualities of action that align with being able to use resources (including social resources such as peers, and material resources such as the internet-based sources) appropriately. What is considered appropriate can align with current norms around collaboration and use of internet-based resources, both of which are features of modern work. Especially in the case of internet-based resources, the use of AI systems such as ChatGPT to assist in creative work, will only become a larger issue.

Methodologically, instead of assessing the amount of "autonomy" that an individual possesses, or even a Likert-based ranking, we can focus on records of specific interactions in specific contexts around mathematics. In conducting a case study of an individual over time, the focus shifts from differing amounts of "autonomy" across time to differences in the qualities of actions individuals take over time. One thing that complicates the tracing of qualities of action of individuals, is that the context in which actions are taken needs to be accounted for. The norms and practices in different course contexts can help one to explain how/why individuals take particular actions. For this reason, it is important for us to understand more about the constraints that participants perceive relative to their actions. We view this system of constraints as what constitutes the mathematical agency - the felt capacity for action - that students perceive (Levin, et al. 2020).

In terms of implications for instruction, our conceptualization of mathematical autonomy does not imply a restriction on the material and social resources available to students during their reasoning processes. In building the capacity to act with mathematical autonomy when faced with mathematical challenges, students need opportunities to engage deeply before abdicating their involvement in the resolution by, for example, calling over an instructor or other mathematical authority to direct them, or to search on the internet for the task they were working on and copying the argument. In undergraduate education, students' ability to navigate "stuck points" is linked with their ability to continue their engagement in mathematical challenges (Lu, 2021). Strategies for supporting productive struggle support the ability to act with mathematical autonomy (SanGiovanni et al, 2020; Warshauer, 2015).

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References

Bae, Y. Smith, J. Levin, M., Satyam, V. R., Voogt, K. (2018, February). *Stepping Through the Proof Door: Undergraduates' Experience One Year After an Introduction to Proof Course.* In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education, San Diego, California (pp. 102-114).

Castle, S., Smith, J., Levin, M., Hwang, J., Karunakaran, S., Küchle, V., & Elmore, R. (2022). *Shifts in External Authority and Resources for Sense-making in the Transition to Proof-Intensive Mathematics: The Case of Amelia.* In Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education, Boston, Massachusetts.

Code, W., Merchant, S., Maciejewski, W., Thomas, M., & Lo, J. (2016). The Mathematics Attitudes and Perceptions Survey: an instrument to assess expert-like views and dispositions among undergraduate mathematics students. *International Journal of Mathematical Education in Science and Technology*, *47*(6), 917-937.

Deci, E. L., & Ryan, R. M. (1980). Self-determination theory: When mind mediates behavior. *The Journal of Mind and Behavior*, 33-43.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, *7*, 234-283.

Inglis, M. & Mejia-Ramos, J-P. (2009) The Effect of Authority on the Persuasiveness of Mathematical Arguments, *Cognition and Instruction*, 27(1), 25-50, DOI: 10.1080/07370000802584513

Levin, M., Smith, J. P., Karunakaran, S., Küchle, V. A., & Castle, S. (2020). *Conceptualizing STEM Majors' Developing Agency and Autonomy in Undergraduate Mathematics.* In Gresalfi, M. and Horn, I. S. (Eds.), The Interdisciplinarity of the Learning Sciences, 14th International Conference of the Learning Sciences (ICLS) 2020, Volume 2 (pp. 887-888). Nashville, Tennessee: International Society of the Learning Sciences.

Levin, M., Smith, J., Karunakaran, S.S., Küchle, V. A.B., Castle, S., Hwang, J. Elmore, R. & Bae, Y. (2020). *Math and Moral Reasoning in the Age of the Internet: Undergraduate Students' Perspectives on the Line Between Acceptable Use of Resources and Cheating.* In Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education, Boston, Massachusetts.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Lu, Y. (2021). *Characterizing Undergraduate Students' Proving Processes around "Stuck Points"* Western Michigan University.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266.

Küchle, V., Karunakaran, S., Levin, M., Smith III., J.P., Castle, S., Hwang, J., Lu, Y. & Elmore, R. (2023). Collapsing spaces, colliding places: Leveraging constructs from humanistic geography to explore mathematics classes. *Journal of Humanistic Mathematics*, *13*(1), 54–69. Available at: https://scholarship.claremont.edu/jhm/vol13/iss1/5

Piaget, J. (1973). *To understand is to invent: The future of education.* Grossman Publishers.

SanGiovanni, J. J., Katt, S., & Dykema, K. J. (2020). *Productive math struggle: A 6-point action plan for fostering perseverance*. Corwin.

Selden, A. (2012). Transitions and Proof and Proving at Tertiary Level. In: Hanna, G., de Villiers, M. (Eds.) *Proof and Proving in Mathematics Education.* New ICMI Study Series, vol 15. Springer, Dordrecht. https://doi.org/10.1007/978-94-007-2129-6_17

Stylianides, A. L. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289-321.

Smith, J., Levin, M., Bae, Y., Satyam, V., Voogt, K. (2017, February) *Exploring Undergraduates' Experience of the Transition to Proof.* In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education, San Diego, California (pp. 920-927).

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

Warshauer, H. K. (2015). Strategies to support productive struggle. *Mathematics Teaching in the Middle School*, *20*(7), 390-393

Wood, M. B. (2016). Rituals and right answers: Barriers and supports to autonomous activity. *Educational Studies in Mathematics*, *91*, 327-348.

A Conceptual Analysis of Expressing Distances Algebraically within the Cartesian Plane

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This theoretical report offers a conceptual analysis of expressing distances on graphs of functions in the Cartesian plane algebraically. We frame this mental activity as a connection between the algebraic and graphical registers, and describe three key connections it comprises: (1) differences express distances between two positions, (2) points are ordered pairs of distances from the axes in the Cartesian plane, and (3) equations give the relation between x and y for every ordered pair on a graph. We detail the conceptual components of each connection, which involve a component from each of the graphical and algebraic registers and an underlying interpretation that forges the connection between the two. This conceptual analysis makes explicit the complex cognitive steps involved in algebraically expressing distances on graphs of functions, with implications for researchers and practitioners using algebraic and graphical representations together.

Keywords: Graphical representations, algebraic expressions, conceptual analysis, Cartesian plane

The central goal of this theoretical report is to offer a conceptual analysis (Thompson, 2008) of the mental activity involved in representing distances between functions graphed in the Cartesian plane using algebraic expressions. Many mathematical objects involved in key results of undergraduate mathematics are often represented graphically and expressed algebraically in textbooks and curricular materials, such as the difference quotient in the limit definition of the derivative. Specifically, students' ability to use algebraic expressions to represent distances within graphs may support them in making sense of such representations that are paired together. Further, this ability may also assist students in conceptualizing other key results, such as how integrals find area under and between curves, and volumes of solids of revolution. In this paper, we consider the mental activity involved in expressing distances algebraically by centering our discussion around the following task and asking: what are the conceptions involved in algebraically expressing the horizontal distance between any point on $y = \sqrt{x-1}$ and a point on $x = 2$ (for all x such that $1 < x < 2$) in terms of x and in terms of y? (Figure 1).

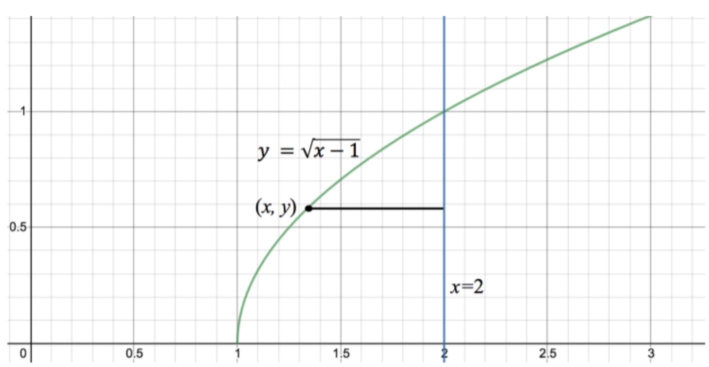

Figure 1. Graph showing a horizontal segment from a generic point (x, y) marked on $y = \sqrt{x-1}$ *to* $x = 2$*. The length of this segment may be expressed as* $2 - x$ *or equivalently as* $2 - (y^2 + 1)$*.*

Conceptual Analysis of Expressing Graphical Distances Algebraically

We engage in the process of conceptual analysis consistent with Thompson (2008) as describing "what students might understand when they know a particular idea in various ways" (p. 42) to detail the mental steps involved for students to conceptualize an algebraic difference expression as representing a horizontal (or vertical) distance within a graph in the Cartesian plane. Based on previous research and work with students on such tasks (e.g., Parr et al., 2021), we theorize that there are three main connections between the graphical register and algebraic register that underly this conception: (1) differences express distances between two positions, (2) points are ordered pairs of distances from the axes in the Cartesian plane and (3) equations give the relation between x and y for every ordered pair represented at a point on a graph. These three connections build upon each other to comprise the connection among *distances between functions in graphs* and *(algebraic) difference expressions*. We describe each of these connections between the graphical and algebraic register, as well as the underlying mechanism for how the connection is forged. We see potential obstacles for students within conceptualizing each component of the graphical register, algebraic register, as well as the mechanism of the connection. We will describe each of the three steps and previous research highlighting potential obstacles for students within each.

Connection 1: A difference represents a distance between two positions

From our perspective, the first main cognitive step to expressing distances on graphs of functions involves using a difference expression to represent a distance between two positions in a single dimension. Within the graphical register, this connection involves conceptualizing distance as a quantity, that is, a measurable attribute (Thompson, 1990, 2011) of the spatial arrangement of two positions within a number line or graph. Within the algebraic register, this connection involves operating from a "*determine the difference*," (Selter et al., 2012) also referred to as a "*comparison*" (Usiskin, 2008) model of subtraction, rather than a *takeaway* model. Connecting these conceptualized distances with difference operations relies on a magnitude interpretation of symbols (Parr, 2021). We summarize these components that result in this connection between differences and distances between two positions in Figure 2.

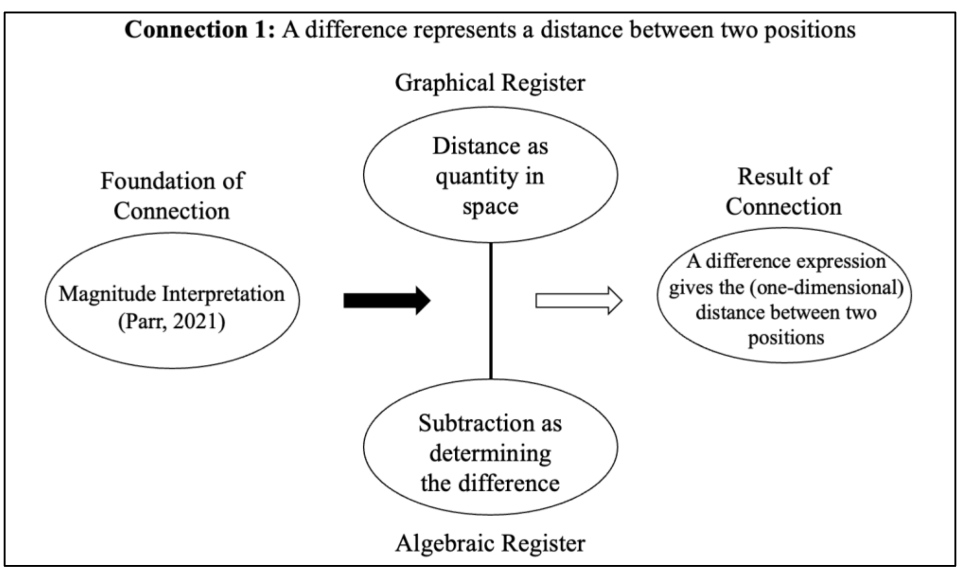

Figure 2. The components of connecting distances in graphs with difference expressions.

Previous research suggests that students may not use these ways of reasoning underlying the connection involved in representing distances with differences. Within the graphical register, conceptualizing a distance as a quantity in space is a non-trivial component of this connection. The Cartesian coordinate system relies on distances to relate points, yet students may not conceive of distance between points as relevant when viewing positions on a number line or graph. Alternatively, they may conceive of positions as labeled at arbitrary locations, or use some other comparison, such as relative location, without explicitly attending to or conceptualizing measurable distance between positions (Parr, 2021). At the elementary level, students may not necessarily place non-consecutive numerical values at appropriate distances apart from one another on a linear-scaled number line (Saxe et al., 2013). Instead of using a number line as a measurement model, some students may use it as a counting model (Diezmann & Lowrie, 2006), counting the number of tick marks or intervals (Mitchell & Horne, 2008). Within the algebraic register, conceiving of subtraction as an operation that determines the difference between two amounts that are compared is not a given for students. Previous research has found that the comparison operations are those that pose the most difficulty for students among types of subtraction problems in early grades (Stern, 1993). When asked to interpret an expression involving a difference, students may think exclusively of a counting down or takeaway model, rather than a comparison one (Figure 3).

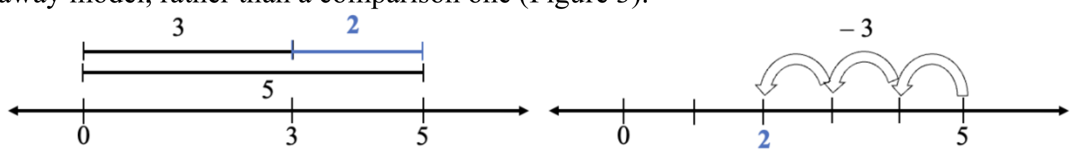

Figure 3. A determine the difference (comparison) model of 5-3 (left) vs. a takeaway model of 5-3 (right).

Students may also default to using the takeaway model for subtraction on a number line, and may not as readily use a determine the difference model, perhaps because the former is more commonly encountered in school mathematics and everyday situations (e.g., Selter et al., 2012). Even at the undergraduate level, students may default to using a takeaway model of subtraction. This was the case with a student, Peter who was at first unable to reconcile a difference expression in $|x–1| < \delta$ with a graph of a function showing a shaded vertical strip centered at $x = 1$ to represent all values of x within a given distance (δ) of 1. (David, 2018).

We view the magnitude interpretation (Parr, 2021) of a symbol (either a number or variable) or expression as the foundation of the connection between the graphical representation of distance and the algebraic operation of subtraction to determine the difference. A magnitude interpretation of a symbol recognizes that a symbol refers to both a position as well as a measure of a distance from 0 (Parr, 2021). To represent the "determining the difference" model of subtraction on a number line, one employs what Parr et al. (2021) refer to as a *composed* magnitude interpretation. For example, to understand conceptually why the difference expression $x_2 - x_1$ yields the distance between two positions, x_1 and x_2, in one-dimensional Cartesian space, one must first understand the positions x_1 and x_2 as distances from the origin themselves. Then, one can apply the operation of subtraction to find the difference between the length of x_2 and the length of x_1 to express the distance between position x_1 and x_2 as $x_2 - x_1$, as shown in Figure 4. Students may not readily use a composed magnitude interpretation of difference expressions, even in situations where it would support further mathematical activity. At the undergraduate level, students may still use a cardinal interpretation, counting tick marks or spaces to interpret a difference expression on an axis, rather than as a distance between two points, as in the case of Annie and Kate reported in Parr (2021).

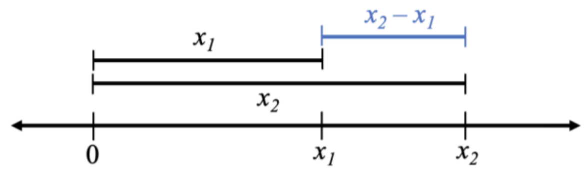

Figure 4. Composed magnitude interpretation of the expression $x_2 - x_1$ involves conceiving of x_1 and x_2 themselves as magnitudes, or distances measured from 0.

Connection 2: A point given by (x, y) represents an ordered pair of distances from axes in the Cartesian plane

The next connection involved in expressing distances in the Cartesian plane is the connection that an ordered pair of values, (x, y), that locate a point in the Cartesian plane represents an ordered pair of distances measured from the point to each axis. This connection builds on the prior one, such that x is a magnitude from the origin horizontally and y is a magnitude from the origin vertically. What is new in this step is the combination of the two magnitudes into two-dimensional space, so that a single entity given by an ordered pair is connected to the pair of distances. The foundation of this connection is value-thinking (David et al., 2019), in which one envisions a point as a multiplicative object (Saldanha & Thompson, 1998), at once a single entity that is comprised of two components conceived of simultaneously. When combined with a magnitude interpretation from the previous connection, one envisions a point as a multiplicative object of a pair of distances to the axes (Figure 5).

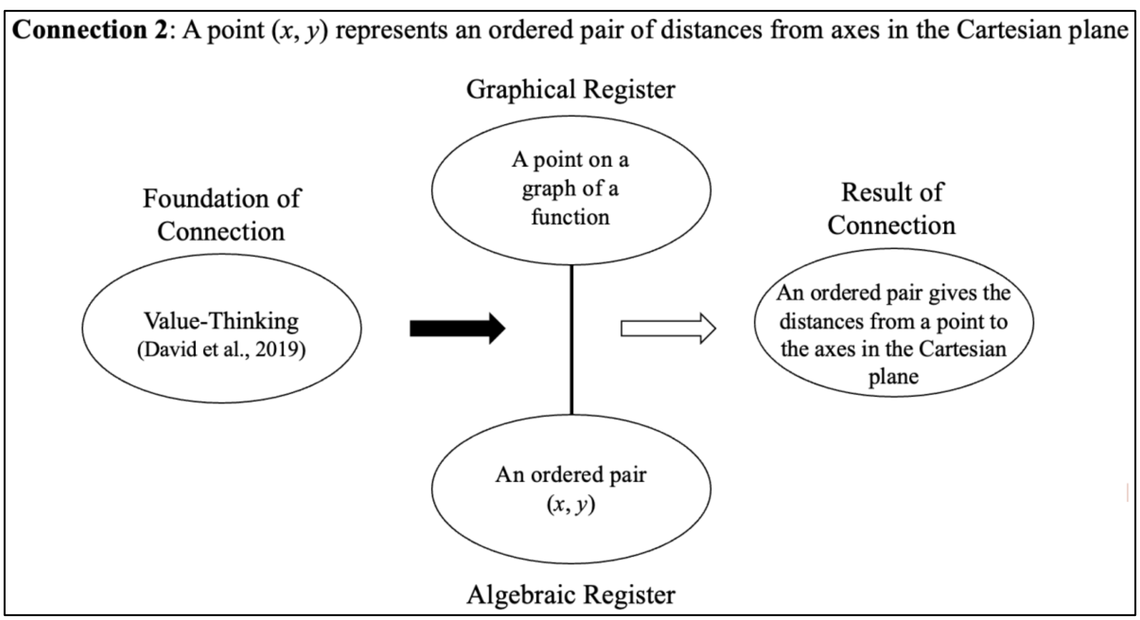

Figure 5. The components of connecting points on graphs with ordered pairs of variables.

Prior research suggests that students, both at the secondary and undergraduate level, may not connect a point on a graph of a function with an ordered pair of input and output values of the function via value-thinking. Students may instead associate a point solely with a single value- often the output of a function, rather than a pair of values, as in the case of location-thinking

(David et al., 2019). When students do connect a point with an ordered pair, they may conceive of the ordered pair purely as a directive for how to locate a point (Thompson et al., 2017)- right (or left) some amount *x* and then up (or down) some other amount *y*. Students who view ordered pairs this way may not be able to unite pairs of magnitudes represented on axes in a single point, even after appropriately plotting points themselves using the over and up technique (Frank, 2016). Thus, we view building the connection between ordered pairs and pairs of distances from axes as an essential part of supporting students in expressing distances between points on functions of graphs.

Connection 3: An algebraic relationship of *x* and *y* can be used to find equivalent expressions of distances

The third connection between the algebraic and graphical register that we view as essential to expressing distances is the Cartesian connection (Moschkovich et al., 1993). The Cartesian connection states that the set of all ordered pairs of points on a graph of an equation correspond to the complete set of pairs of values satisfying the equation. Using this connection with the prior two, in which ordered pairs for points give pairs of distances to the axes, allows one to conceive of the algebraic equation relating *x* and *y* as a way to express distances in the graph flexibly in terms of *x* or *y* as needed (Figure 6).

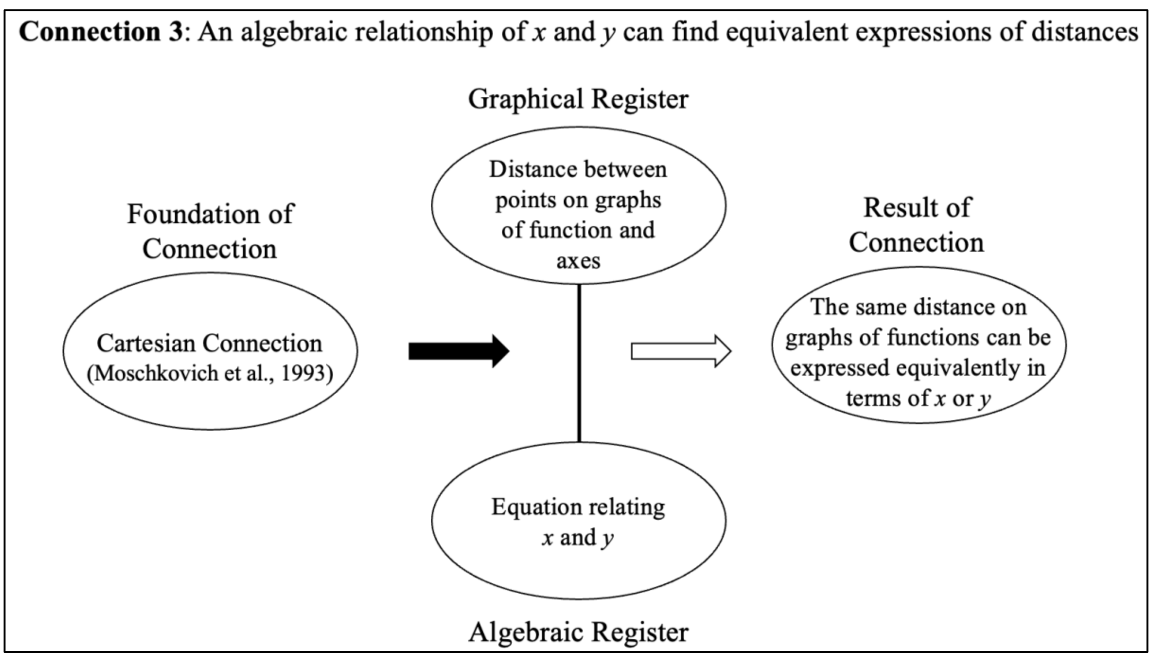

Figure 6. The components of connecting distances on graphs with equations of x and y.

Previous studies have shown that students may not recognize the Cartesian connection when working with graphs of functions and their algebraic relationships (Dufour-Janvier et al., 1987; Glen & Zazkis, 2021; Knuth, 2000; Moon, et al., 2013). Instead of viewing an equation for a linear function as comprising the set of all ordered pairs of values satisfying the equation, which can be plotted as points comprising the graph of the line, students may see a linear equation as a directive for how to draw a graph of a line using slope and intercept information (Parr et al., 2021). In the words of Todd, a Calculus student, "$y = 2x+1$… tells you how it's [the graph is] going to look, so the slope would be 2, the y-intercept would be 1" (Parr et al., 2021, p. 219).

Representing distances on two-dimensional graphs of functions using algebraic expressions is a complex cognitive activity. In order to conceptualize why expressions such as $2–x$ and $2–(y^2+1)$ in Figure 1 represent the distance depicted, students must connect the graphical and algebraic register in the three ways described in this section: 1) a distance between can be modeled using a difference, 2) an ordered pair of values gives the distances the associated point is located from the axes in the Cartesian plane, and 3) an algebraic relationship between x and y can be used to flexibly express distances within the graph of the relationship in terms of x or y.

In Figure 7, we show how these connections combine to connect difference expressions with distances between two points on functions in the Cartesian plane. Connection 1 supports students in expressing a straight-line distance between two positions x_1 and x_2 as $x_2–x_1$. Connection 2 supports students in expressing a straight-line distance from an axis in the Cartesian plane using the coordinates of the point, in this case, the horizontal distances as x_1. Combining these two connections supports students in describing the horizontal distance between two points in the Cartesian plane as $x_2–x_1$. Assuming the two points shown are located on functions f and g, respectively, Connection 3, the Cartesian connection, would support a student in expressing this same horizontal distance in terms of y, as $g^{-1}(y_2) – f^{-1}(y_1)$.

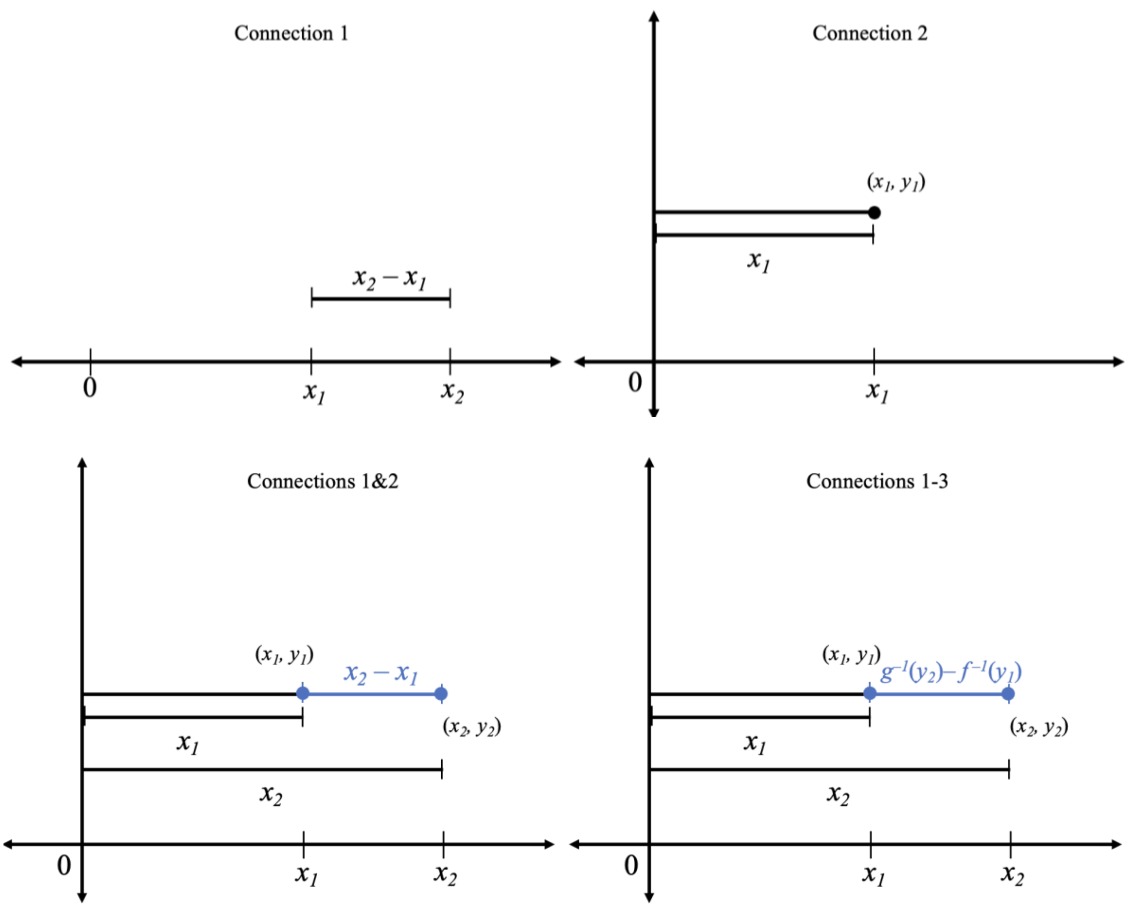

Figure 7. How Connections 1-3 combine to express horizontal distance between two points in the Cartesian plane.

Discussion

The conceptual analysis we present in this paper offers a detailed account of the cognitive connections need for students to meaningfully use an algebraic difference expression to represent distances between functions in graphs. This conceptual analysis has implications for both research and practice. First, the three connections we highlight between the algebraic and graphical register uncover the complex nature of the cognitive steps involved expressing distances algebraically. At the undergraduate level, researchers and instructors may take for granted students' fluency in moving between two representations. In fact, this connection is not included in typical calculus curricular materials and instruction. Yet, our research and this analysis suggest that students may benefit from more explicit instruction as to how these representations may be used together. In addition to drawing attention to its underlying complexity, this analysis may serve to support the development of learning trajectories and tasks to assist students in connecting graphs with algebraic expressions meaningfully. Further, this conceptual analysis may alert researchers and practitioners to potential issues in students' reasoning that they may anticipate when using subject matter represented both algebraically and graphically. These issues may arise in either conceptualizing the objects in the graphical register, the objects in the algebraic register, or the way in which these objects are connected.

More broadly, the structure of the conceptual analysis we offer here may serve as a model for others related to connections among registers. Many prior examples of conceptual analysis in the literature attend to students' understanding of mathematical ideas, but may not explicitly attend to representational structures involved in conceiving of or communicating these ideas. We add to the example of Lee et al.'s (2018) conceptual analysis of coordinate systems by attending to the representational structures involved in connecting graphical distances with algebraic expressions. We decompose the larger connection in question into sub-connections, which each include components in the algebraic register, graphical register, and an underlying interpretation. This structure may serve as a model for future conceptual analyses on other connections between the graphical and algebraic registers, or any other combinations among mathematical registers.

References

David, E. J., (2018). Peter's evoked concept images for absolute value in calculus contexts. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. (pp. 949-956). San Diego, CA.

David, E. J., Roh, K. H., & Sellers, M.E. (2019). Value-thinking and location-thinking: An empirical study of two ways students visualize points and reason about graphs in the Cartesian plane. *The Journal of Mathematical Behavior, 54. Article 100675.*

Diezmann, C. M., & Lowrie, T. (2006). Primary students' knowledge of and errors on number lines. In P. Grootenboer, R. Zevenbergen, & M. Chinnapan (Eds.), *Proceedings of the 29th Annual Conference of the Mathematical Education Research Group of Australasia* (Vol. 1, pp. 171–178). Sydney, Australia: Mathematics Education Research Group of Australasia.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning Mathematics* (pp. 109–122). Lawrence Erlbaum.

Frank, K. (2016). Plotting points: Implications of "over and up" on students' covariational reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the*

Psychology of Mathematics Education (pp. 573-580). Tucson, AZ: The University of Arizona: PMENA.
Glen, L. & Zazkis, R. (2021). On linear functions and their graphs: Refining the Cartesian connection. *International Journal of Science and Mathematics Education 19*, 1485-1504. https://doi.org/10.1007/s10763-020-10113-6
Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500–507.
Lee, H. Y., Hardison, H. L., & Paoletti, T. (2018). Uses of coordinate systems: A conceptual analysis with pedagogical implications. In Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Greenville, SC: University of South Carolina & Clemson University.
Mitchell, A., & Horne, M. (2008). Fraction number line tasks and additivity concept of length measurement. In M. Goos, R. Brown, & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia*, 353–360.
Moon, K., Brenner, M. E., Jacob, B., & Okamoto, Y. (2013). Prospective secondary mathematics teachers' understanding and cognitive difficulties in making connections among representations. *Mathematical Thinking and Learning*, *15*(3), 201–227.
Moschkovich, J., Schoenfeld, A., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representations of functions* (pp. 69–100). Hillside: Erlbaum.
Parr, E. D. (2021). Undergraduate students' interpretations of expressions from calculus statements within the graphical register. *Mathematical Thinking and Learning.* https://doi.org/10.1080/10986065.2021.1943608
Parr, E. D., Ely, R., & Piez, C. (2021). Conceptualizing and representing distance on graphs in Calculus: The case of Todd. *2021 Research in Undergraduate Mathematics Education* Reports. (pp. 214-221).
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America* (pp. 1–7). Raleigh, NC: North Carolina State University.
Saxe, G. B., Shaughnessy, M. M., Gearhart, M, & Haldar, L. C. (2013). Coordinating numeric and linear units: Elementary students' strategies for locating whole numbers on the number line. *Mathematical Thinking and Learning*, *15*(4) 235-258.
Selter, C., Prediger, S., Nührenbörger, M., Hußmann, S. (2011). Taking away and determining the difference—a longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79* (389-408).
Simon, M. A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104.
Stern, E. (1993). What makes certain arithmetic word problems involving the comparison of sets so difficult for children? *Journal of Educational Psychology, 85*(1), 7–23
Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Unpublished manuscript, Center for Research in Mathematics & Science Education. Available at http://pat-thompson.net/PDFversions/1990TheoryQuant.pdf

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 31-49). Morélia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95-111.

Usiskin, Z. (2008). The arithmetic curriculum and the real world. In D. de Bock, B. Dahl Søndergaard, B. Gómez Alfonso, & C. Litwin Cheng (Eds.), *Proceedings of ICME-11-Topic Study Group 10: Research and Development in the Teaching and Learning of Number Systems and Arithmetic,* pp. 11–16.

Inclusive and Equitable Mathematics Education: Active Learning is Necessary, but not Sufficient

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Numerous studies have demonstrated that active learning can increase student learning and reduce achievement gaps; research has also shown that active learning in undergraduate mathematics is not consistently equitable. These findings highlight a gap in what we know about active learning and indicate the need for a deeper understanding of the relationship between equity, inclusion, and active learning. Drawing on research about inclusive and equitable mathematics learning environments across secondary and postsecondary contexts, in concert with what is known about active learning in undergraduate mathematics classrooms, I present a theoretical argument that active learning is a necessary but insufficient condition for mathematics learning communities to be inclusive and equitable. I close by suggesting potential strategies for ensuring active learning is implemented in ways that are inclusive and equitable.

Keywords: active learning, inquiry, equitable, inclusive, critical

The use of active learning and inquiry-based mathematics education (IBME, a form of active learning) is becoming increasingly common in undergraduate mathematics courses (Stains, 2018). Active learning has been shown to result in improved student learning outcomes (Deslauriers et al., 2019; Freeman et al., 2014, Laursen et al. 2014) as well as significantly reduced achievement gaps between women and men (Laursen et al, 2014) and between students who are members of underrepresented and overrepresented identity groups (Theobald et al., 2019). Freeman et al. (2014) suggest that research supports "active learning as the preferred, empirically validated teaching practice in regular classrooms" (p. 8410), and Theobald et al. (2019) calls for evidence-based active-learning course designs to replace traditional lecturing across the STEM disciplines" (p. 6476). Active learning was found to be one of the common characteristics in a study of successful calculus programs at five doctoral degree-granting mathematics departments deemed to be exemplary based on persistence rates and students' reported enjoyment and confidence in mathematics (Rasmussen, Ellis, Zazkis & Bressoud, 2014; Rasmussen, Ellis & Zazkis, 2014). Abundant evidence of student-centered instruction supporting increased student success in mathematics at the primary and secondary levels (e.g., Boaler, 2006; Matthews et al., 2021) further reinforces the claim that active learning is a better way to teach mathematics than traditional lecture.

And yet, performance (Reinholz et al., 2022; Johnson et al., 2020) and participation (Reinholz et al., 2022) gaps between majority identity groups and those who are underrepresented in mathematics persist in some active learning classrooms. Johnson et al. (2020) showed that achievement gaps between women and men increased in a set of linear algebra classrooms using IBME. Reinholz et al. (2022) found performance differences and differences in participation rates between women and men in undergraduate mathematics classes taught using inquiry-oriented instruction and suggested that "simply implementing active learning is insufficient… for improving gender equity in mathematics" (p. 204). Research has shown that mathematics classrooms using active learning can be inclusive and equitable, while also demonstrating that not all active learning mathematics classrooms achieve this standard.

The relationship between equity, inclusion, active learning and IBME is under-researched, and scholars have not yet explained the distinctions between inclusive and equitable and exclusive and/or inequitable active learning. In this theoretical report I draw on research about inclusive and equitable mathematics learning environments in secondary and tertiary contexts, along with what is known about active learning and IBME in undergraduate mathematics classrooms, to answer the question: *What will it take to cultivate undergraduate mathematics learning environments that are reliably equitable and inclusive?* I present the argument that instructors' use of active learning in general, and IBME specifically, is a necessary but insufficient condition for mathematics learning to be inclusive and equitable. Then I propose additional criteria that may, when used in addition to IBME, contribute to creating reliably inclusive and equitable undergraduate mathematics classrooms.

Theoretical Perspectives: What is Inclusive and Equitable Mathematics?

Research showing the positive effects for students of learning mathematics through active learning or IBME is abundant and convincing. The strength of the case in favor of active learning can make it seem confusing or implausible when evidence is presented showing some settings exhibiting persistent inequity and/or exclusion. To explain how the extensive body of research supporting active learning can be correct while also failing to explain why some active learning settings are exclusive and/or inequitable, I leverage theoretical perspectives. A clear theoretical perspective names the researchers' commitments and perspectives; it provides a context that helps explain why certain decisions are made in conducting research, including the questions that are asked, the data that is collected and how that data is analyzed and interpreted. This section describes the dominant perspective that has shaped most of the existing research on active learning in undergraduate mathematics and the critical perspective that may help to usher in a new chapter focused on inclusion and equity in mathematics education. This is followed by a commitment to intersectionality and attending to the interplay of lived experiences across identity group memberships, and a description of the inclusive and equitable teaching practices that shape my thinking about the nature of inclusive and equitable undergraduate mathematics.

Critical Perspective

Gutiérrez (2007) and Martin (2003) called for mathematics education researchers to adopt a critical perspective, and yet much of the research on active learning in undergraduate mathematics continues to reflect a strongly dominant perspective. Gutiérrez's (2009) framework for understanding equity in mathematics provides a useful way to understand and interpret the body of research on active learning. This framework includes two axes. The dominant axis consists of the dimensions of access and achievement, and the critical axis consists of the dimensions of identity and power. The majority of the existing research on the effect of active learning in undergraduate STEM settings addresses questions about achievement – either learning outcomes or achievement gaps – or about access to learning opportunities that arise from engaging actively with course content.

Numerous mathematics education researchers across secondary and tertiary settings have called for paying greater attention to the role and impact of identity and power (Adiredja, 2015; Gutiérrez, 2009, 2013) – the critical axis of equity in mathematics – in mathematics learning spaces. Attending to the role of identity and power in mathematics classrooms means adopting a different vision if what it means for mathematics education, in this case specifically active learning, to be successful. Mathematics instruction that supports the "mathematical identities, excellence and literacies of marginalized students" (Gutiérrez, 2008, p. 357) may differ from that which leads to increased test scores and reduced participation gaps. Gutiérrez's framework helps to illuminate that the existing research on active learning has demonstrated active learning's success along the dominant axis. Active learning has been shown to increase *achievement* and *access* to learning opportunities. Gutiérrez's framework also highlights what remains to be investigated: by attending closely to *identity* and *power* in mathematics classrooms we may learn what helps students who are members of marginalized identity groups to succeed and thrive.

Framework for Understanding the Social Space of Mathematics

Leyva et al.'s (2022) proposed framework for understanding mathematics as a white cisheteropatriarchal space explains how sociomathematical (Yackel & Cobb, 1996; Leyva, 2017) and sociohistorical (Leyva, 2021) norms are at play in mathematics classrooms in ways that relate to identity including but not limited to gender, race and class. Leyva (2017) calls for researchers to "carefully attend to mathematics learning contexts and the interplay of students' multiple identities (including race or ethnicity, culture, class, gender, and sexuality)" (p. 406). Crenshaw (1991) suggests that intersectionality could provide the means for understanding and addressing experiences of marginalization that are shared across identity groups (p. 1299).

Inclusive and Equitable Instruction

Examples of instructional practices that have been shown to contribute to inclusive learning environments and equitable outcomes for students who are members of underrepresented or marginalized groups in mathematics include equitable teaching practices (Boaler, 2006) and belonging centered instruction (Matthews et al., 2021). Additionally, culturally relevant (LadsonBillings, 1995) and sustaining (Alim, Paris & Wong, 2020; Ladson-Billings, 2014; Paris, 2012; Paris & Alim, 2014) pedagogy supports inclusive and equitable learning environments that are not specific to mathematics. Describing each of these approaches to inclusive and equitable instruction in detail is beyond the scope of this paper. I will, instead, describe cross-cutting themes, since teaching strategies that appear in multiple of these instructional approaches are especially likely to be impactful across settings.

Teaching strategies that emerge as common across these instructional approaches include holding high expectations (Boaler, 2006; Ladson-Billings, 1995; Matthews et al., 2021), assigning competence (Boaler, 2006; Ladson-Billings, 1997) and decentering teacher authority (Matthews et al., 2021). Just as importantly, additional themes are present across these approaches that relate to interpersonal interactions, social dynamics, and community. Boaler (2006) described relational equity as a teaching approach that "valued different insights, methods, and perspectives in the collective solving of particular problems" (p. 45), an approach that supported students learning to "appreciate the contributions of different students, from many different cultural groups and with many different characteristics, and perspectives" (p. 45). Belonging centered instruction includes an interpersonal domain that attends to community, empathy, and social and emotional aspects of students' classroom experience (Matthews et al., 2021). Ladson-Billings (1997) emphasizes the "need to develop caring and compassionate relationships with students" (p. 707).

Review of the Literature: Three Plus Decades of Inquiry and Active Learning

The dominant perspective is evident in active learning and IBME in how researchers have sought to understand their impacts, and in the origins of these instructional practices. Active learning, inquiry-oriented instruction and inquiry-based learning all developed as strategies for increasing student engagement and student learning in undergraduate science, technology, engineering, and mathematics (STEM). While they each differ somewhat in their commitments and orientations, curriculum development and classroom instruction that adheres to principles of inquiry and/or active learning is consistently oriented toward a common goal of supporting students to understand STEM course content more deeply and to succeed in their undergraduate STEM courses. Over time there has been a trend toward coalescing around common definitions and principles, as described below.

Active Learning

Definitions and conceptions of active learning have consistently centered around the idea that students benefit from being actively involved and engaged in learning instead of passively listening to lecture. Prince (2004) defined "the core elements of active learning to be introducing activities into the traditional lecture and promoting student engagement" (p. 225). Bonwell and Eison (1991) suggested that students should be engaged "in such higher-order thinking tasks as analysis, synthesis, and evaluation" (p. *iii*), and they proposed defining active learning as "instructional activities involving students in doing things and thinking about what they are doing" (p. *iii*). In 2016, the Active Learning in Mathematics Research Action Cluster of the Mathematics Teacher Education Partnership synthesized the research on active and inquiry based learning available at that time and developed a set of five design principles for active learning that included 1) mathematical coherence, 2) instructional activities that promote "active construction of meaning" and "sense-making," 3) norms for classroom discourse that encourage students to share "reasoning in process," 4) an instructional environment that includes multiple modes of instruction, and 5) instructional decision-making in which "the choices made in lesson design and adaptation should favor the perspective of the learners." (Webb, 2016, p. 2).

Inquiry Based Mathematics Education

Laursen and Rasmussen (2019) named four pillars of Inquiry Based Mathematics Education (IBME), the term they proposed to unify inquiry-oriented instruction and inquiry-based learning, and to situate it as a specific form of active learning. The four pillars are as follows:

1. Students engage deeply with coherent and meaningful mathematical tasks.
2. Students collaboratively process mathematical ideas.
3. Instructors inquire into student thinking.
4. Instructors foster equity in their design and facilitation choices. (p. 138)

There is significant overlap between the four pillars of IBME as explicated by Laursen and Rasmussen (2019) and the design principles for active learning. One result has been that the four pillars have increasingly been taken up as a definition of active learning. A team of researchers from the Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) project analyzed data from interviews with 115 stakeholders (administrators, members of client disciplines, coordinators, leaders, instructors and learning assistants) who were asked about their conceptualizations of active learning. The research team compared stakeholders' responses to the four pillars of IBME and found that instructors' conceptualizations of active learning aligned closely with the first three of the four pillars (Williams et al., 2022). I conjecture that the inquiry and active learning instructional contexts for which research has shown increased student learning outcomes and reduced achievement gaps (Deslauriers et al., 2019; Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2020), particularly "high-intensity" (Theobold et al., 2020) forms of active learning, are closely aligned with the first three pillars of IBME.

Inclusive and Equitable Undergraduate Mathematics Education: What does it take?

The call for mathematics education researchers to investigate the nature and effect of inclusive and equitable instruction in undergraduate mathematics is becoming increasingly powerful. Hagman (2019) called for an eighth characteristic of successful calculus programs – diversity, equity, and inclusion practices. Laursen and Rasmussen (2019) included an aspirational fourth pillar specifying the need to foster equity in the design and facilitation of IBME and noted that it is not yet entirely clear how to accomplish this in inquiry-based classrooms. Theobald et al., (2020) posited that the large variation in efficacy observed across studies in their data set might result, in part, from variations in the culture of inclusion (p. 6479). Reinholz et al., (2022) suggested that implementing active learning alone is insufficient to improve gender equity in mathematics. And Leyva et al. (2022) analyzed thirty-four undergraduate Black and Latin* students' perceptions of "supportive-for-all practices" which include such active learning aligned practices as creating "space for questions and mistakes" (p. 339), discourse, discussion, and student-thinking centered instruction. They found that "supportive-for-all practices" were perceived by students of color to be "necessary yet insufficient to cultivate equitable opportunities for classroom participation and access to content" (p. 339).

Next I will describe the existing relationship between active learning and equity and argue that IBME is a necessary part of implementing inclusive and equitable instruction in undergraduate mathematics. I will propose instructional practices that are compatible with IBME, and which have potential for resulting in reliably inclusive and equitable IBME learning environments. Finally, I will close by calling for research that adopts a critical perspective to study the nature and effectiveness of inclusive and equitable active learning in mathematics.

Active Learning and Equity: An Under-Investigated Relationship

When researchers shared the seven characteristics of successful calculus programs (Rasmussen, Ellis, Zazkis & Bressoud, 2014; Rasmussen, Ellis & Zazkis, 2014), their definition of "successful" attended to students' experiences by considering reported levels of "enjoyment" and "confidence," but they did not consider how students who were members of underrepresented identity groups fared in the mathematics departments they described. Hagman (2019) identified this failure and called for the addition of an eighth characteristic – diversity, equity, and inclusion practices – not because it was found to have been present, but rather because that study had not been conducted with attention to the ways students' identities impacted their experiences of their calculus programs. The dominant perspective guiding that research project left the dimensions of identity and power uninvestigated.

The fourth pillar of IBME, "instructors foster equity in their design and facilitation choices" (Laursen & Rasmussen, 2019, p. 138), demonstrates commitment to currently under-investigated practices for supporting equity and inclusion. But it is presented with a caveat: "the research base in undergraduate mathematics education does not reveal just how to accomplish this in inquirybased college classrooms" (p. 138). The authors observe that, while research has shown the potential for inquiry classrooms to be equitable, "this is not automatic" (p. 138). They also point out that research on secondary contexts (e.g., Boaler 2006) provides potentially useful direction. In essence, Laursen and Rasmussen are not suggesting that IBME is equitable as it is typically enacted, but rather that IBME can and should be equitable when ideally implemented. Williams et al.'s (2022) findings show that instructors' conceptualizations of active learning align with the first three pillars of IBME but engage minimally, if at all, with the fourth pillar of fostering equity. These findings align with the aspirational nature of the fourth pillar of IBME. Active learning, as described by undergraduate mathematics faculty and instructors, appears to be accurately described by the first three pillars of IBME. For naming the critical goal of achieving equity and inclusion I appreciate Laursen and Rasmussen's inclusion of the fourth pillar that should motivate and inspire further research to describe exactly what is necessary to achieve inclusive and equitable undergraduate mathematics education. The remainder of this paper proposes what inclusive and equitable undergraduate mathematics education might require.

What Are Equitable Teaching Practices in Active Learning Classrooms?

First and foremost, I address the necessity of recognizing and including IBME as a central component, or perhaps a foundational building block, of inclusive and equitable mathematics education. I will then propose instructional practices that research across secondary and tertiary mathematics contexts suggests might be missing components of inclusive and equitable mathematics education. Finally, I call for research grounded in a critical perspective that increases our collective knowledge in the field related to creating the conditions necessary for students to succeed, thrive and to experience full membership in the mathematics community.

IBME: Is it necessary? Let's imagine mathematics instruction that *does not* meet any one of the first three pillars of IBME. What would this imply regarding equity and/or inclusion in such a classroom? I will take up each of the first three pillars individually.

1. A mathematics classroom in which students *do not* engage deeply with coherent and meaningful mathematical tasks denies students the experience of high expectations (Boaler, 2006, p. 44; Ladson-Billings, 1995; Matthews et al., 2021) and access to the rich learning experiences necessary to achieve high levels of learning. Mathematics instruction that fails to incorporate the first pillar of IBME fails to support the dominant axis dimensions of access and achievement and is not equitable.
2. A mathematics classroom in which students *do not* collaboratively process mathematical ideas misses crucial opportunities for students to experience equitable teaching practices of "assigning competence" (Boaler, 2006, p. 43) and decentering teacher authority (Matthews et al., 2021), and fails to support equitable access to learning opportunities.
3. A mathematics classroom in which instructors *do not* inquire into student thinking fails to exhibit equitable teaching practices of "assigning competence" (Boaler, 2006, p. 43), "student responsibility" (Boaler, 2006, p. 43-44) and decentering teacher authority (Matthews et al., 2021). Mathematics instruction that fails to incorporate the third pillar of IBME fails to provide access to learning opportunities. Failure to inquire into student thinking firmly upholds traditional power dynamics in which the instructor's ideas and thinking override other ways of thinking about or understanding mathematical content.

Such instruction fails to address the critical axis dimension of power.

This suggests that active learning that omits any of the first three pillars of IBME is inconsistent with inclusive and/or equitable mathematics instruction, demonstrating the necessity of including these practices in order to achieve inclusive and equitable mathematics classrooms. In fact, Mathematics instruction that results in students engaging deeply with coherent and meaningful mathematical tasks, collaboratively processing mathematical ideas, and in which instructors consistently inquire in student thinking is not only necessary, but it is an excellent place to start. Next, I will suggest what inclusive and equitable instructional practices could potentially be missing in an IBME classroom.

What else will it take? To consider what might need to be added to the first three pillars of IBME to cultivate reliably inclusive and equitable mathematics classrooms I return to the teaching strategies that I shared earlier in this paper. These included holding high expectations, assigning competence, and decentering teacher authority, all of which are somewhat aligned with the first three pillars of IBME. I also noted that inclusive and equitable teaching includes attention to interpersonal interactions, social dynamics, and community. In the socially interactive environments of active learning classrooms these social and community aspects of inclusive and equitable teaching are especially salient. Furthermore, the first three pillars of IBME do not explicitly call for attention to the community nature of active learning classrooms. The second pillar – students collaboratively process mathematical ideas – may imply the need for instructors to attend to social interactions; however, without careful and explicit attention to establishing norms of interaction, the group tasks that are often intended to foster collaboration are equally likely to reinforce preexisting and deeply established sociohistorical and socio-mathematical norms that are often marginalizing for students who are members of under-represented identity groups in mathematics. With a focus on the critical axis dimensions of identity and power as they are relevant in active learning mathematics classrooms, I suggest the following as potential strategies to support inclusive and equitable mathematics instruction:

1) co-development and ongoing revision of class norms of participation
2) opportunities for students to provide feedback about their experiences at regular intervals throughout the course
3) normalizing the experience of not knowing, particularly as it applies to “prerequisite” knowledge
4) flexible course structures that provide students multiple opportunities to demonstrate their learning.

Ultimately, applying a critical perspective and attaining the goal of supporting students to consistently develop positive mathematics identities that include a sense of belonging in mathematics and a sense of being a capable and skilled doer of mathematics should be central to achieving the aspirational fourth pillar of IBME: Instructors foster equity in their design and facilitation choices (Laursen & Rasmussen, 2019, p. 138).

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References

Alim, H. S., Paris, D., & Wong, C. P. (2020). Culturally Sustaining Pedagogy. In N. S. Nasir, C. D. Lee, R. Pea, M. M. de Royston, N. S. Nasir, C. D. Lee, R. Pea, & M. McKinney de Royston (Eds.), *Handbook of the Cultural Foundations of Learning* (1st ed., pp. 261–276). Routledge. https://doi.org/10.4324/9780203774977-18

Bonwell, C. C., & Eison, J. A. (1991). *Active learning: Creating excitement in the classroom. 1991 ASHE-ERIC Higher Education Report No. 1*. Washington D.C.: The George Washington University School of Education and Human Development.

Crenshaw, K. (1991). Mapping the margins: Intersectionality, identity politics, and violence against women of color. Stanford Law Review, 43(6), 1241–1299. doi:10.2307/1229039

Deslauriers, L., McCarty, L. S., Miller, K., Callaghan, K., Kestin, G. (2019). Measuring actual learning versus feeling of learning in response to being actively engaged in the classroom. *Proceedings of the National Academy of Sciences, 116(39),* 19251-19257.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, *111*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111 Gutiérrez, R. (2007).

(Re)Defining Equity: The Importance of a Critical Perspective. In N. S. Nasir & P. Cobb (Eds.), *Improving Access to Mathematics: Diversity and Equity in the Classroom* (pp. 37– 50). Teachers College Press.

Gutiérrez, R. (2009). Embracing the Inherent Tensions in Teaching Mathematics From an Equity Stance. *Democracy & Education*, *18*(3), 9–16.

Hagman, J. E. (2019). The Eighth Characteristic for Successful Calculus Programs: Diversity, Equity, & Inclusion Practices. *PRIMUS*, 1–24. https://doi.org/10.1080/10511970.2019.1629555

Ladson-Billings, G. (1995b). Toward a Theory of Culturally Relevant Pedagogy. *American Educational Research Journal*, *32*(3), 465–491. JSTOR. https://doi.org/10.2307/1163320

Ladson-Billings, G. (1997). It Doesn't Add up: African American Students' Mathematics Achievement. *Journal for Research in Mathematics Education*, *28*(6), 697–708. https://doi.org/10.2307/749638

Ladson-Billings, G. (2014). Culturally Relevant Pedagogy 2.0: A.k.a. the Remix. *Harvard Educational Review*, *84*(1), 74–84. https://doi.org/10.17763/haer.84.1.p2rj131485484751

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for Women and Men of Inquiry-Based Learning in College Mathematics: A Multi-Institution Study. *Journal for Research in Mathematics Education*, *45*(4), 406–418. https://doi.org/10.5951/jresematheduc.45.4.0406

Leyva, L. A. (2017). Unpacking the Male Superiority Myth and Masculinization of Mathematics at the Intersections: A Review of Research on Gender in Mathematics Education. *Journal for Research in Mathematics Education, 48(4),* 397-433.

Leyva, L. A., (2021). Black Women's Counter-Stories of Resilience and Within-Group Tensions in the White, Patriarchal Space of Mathematics Education. *Journal for Research in Mathematics Education, 52(2)*. 117-151.

Leyva, L. A., Amman, K., Wolf McMichael, E. A., Igbinosun, J., Khan, N. (2022). Support for All? Confronting Racism and Patriarchy to Promote Equitable Learning Opportunities through Undergraduate Calculus Instruction. *International Journal of Research in Undergraduate Mathematics Education, 8(2),* 339-364.

Leyva, L. A., McNeill, R. T., Balmer, B. R., Marshall, B. L., King, V. E., Alley, Z. D. (2022). Black Queer Students' Counter-Stories of Invisibility in Undergraduate STEM as a White, Cisheteropatriarchal Space. *American Education Research Journal. 59(5).* 863-904.

Paris, D. (2012). Culturally Sustaining Pedagogy: A Needed Change in Stance, Terminology, and Practice. *Educational Researcher*, *41*(3), 93–97. https://doi.org/10.3102/0013189X12441244

Paris, D., & Alim, H. S. (2014). What Are We Seeking to Sustain Through Culturally Sustaining Pedagogy? A Loving Critique Forward. *Harvard Educational Review*, *84*(1), 85–100. https://doi.org/10.17763/haer.84.1.982l873k2ht16m77

Prince, M. (2004). Does active learning work? A review of the research. *Journal of Engineering Education*, *93*(3), 223-231.

Rasmussen, C., Ellis, J., & Zazkis, D. (2014). Lessons Learned From Case Studies of Successful Calculus Programs at Five Doctoral Degree Granting Institutions. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (p. 8).

Rasmussen, C., Ellis, J., Zazkis, D., & Bressoud, D. (2014). Features of Successful Calculus Programs at Five Doctoral Granting Institutions. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 5, pp. 33–40). PME.

Stains, M., Harshman, J., Barker, M. K., Chasteen, S. V., Cole, R., DeChenne-Peters, S. E., Eagan Jr., M. K., Esson, J. M., Knight, J. K. , Laski, F. A., Levis-Fitzgerald, M., Lee, C. J., Lo, S. M., McDonnell, L. M., McKay, T. A., Michelotti, N., Musgrove, A., Palmer, M. S., Plank, K. M., Rodela, T. M., Sanders, E. R., Schimpf, N. G., Schulte, P. M., Smith, M. K., Stetzer, M., Van Valkenburgh, B., Vinson, E., Weir, L. K., Wendel, P. J., Wheeler, L. B., Young, A. M. (2018). Anatomy of STEM teaching in North American universities. *Science, 359(6383).* 1468-1470.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., … Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476–6483. https://doi.org/10.1073/pnas.1916903117

Webb, D. C. (2016). Applying Principles of Active Learning to Promote Student Engagement in Undergraduate Calculus. *School of Education Faculty Contributions*, *4*.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458–477. doi:10.2307/749877

Developing Personal Representations to Access a Set-Oriented Perspective on Counting

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In some classical semiotic perspectives, signs and sign systems can be viewed as either institutional or personal. This dichotomy can place personal representations as antecedent to institutional ones, sometimes with the explicit goal of transitioning away from the personal ones entirely. In this paper I draw from qualitative data to argue that personal representations are legitimate and sometimes necessary components of developing combinatorial facility, and they play an ongoing role in understanding that is not replaced by institutional representations. I frame this theoretical argument around a discussion of common institutional semiotic systems for combinatorics, which can insufficiently support a set-oriented perspective on counting. I discuss how to incorporate personal representations into analysis that uses a classical semiotic lens. Doing so includes attending to the motivations for creating (or refining) a sign system, the connections between those systems and others, and how the form and nature of the sign systems seem to impact or reflect student reasoning. I also discuss how some differences between combinatorics and other areas of mathematics might affect the personal/institutional dichotomy in semiotic analysis.

Keywords: combinatorics, semiotics, student reasoning, personal representations

Consider the counting problem *how many ways are there to flip a coin three times in a row?* This problem is an appropriate introduction to combinatorics because it invites solutions that draw from multiple modalities, including and certainly not limited to: flipping physical coins to exhaust outcomes (systematically or not), listing entire or partial sets of written outcomes such as HHT or 001, reasoning structurally about enumeration processes or representations of those processes such as H/T H/T H/T, using embodied and/or gestural reasoning with or without iconic imagery, and applying abstract counting principles formally or informally. Each of the possibilities mentioned here, and countless others, can find the same numerical solution—there are 8 possible ways to flip a coin three times in a row. And yet the approach might impact how the solution is expressed. While one approach might yield 2^3, others might yield $1 + 3 + 3 + 1$ or simply the integer 8. Institutional combinatorial texts (such as textbooks or lecture notes) might only present solutions within limited modalities, often confined to a natural language (e.g., English), symbolic mathematics, and figures (if the text is generous). I consider the limit of modalities to be a byproduct of the textual medium and not necessarily an indication that the solutions are more sophisticated than those drawing from other (often non-institutional) semiotic systems. Moreover, the limited presentations of solutions can perpetuate negative attitudes towards combinatorics when the singular way combinatorial reasoning is presented does not align with what is natural for learners. This is especially unfortunate considering that the abundance of solution approaches enrichens combinatorics.

The purpose of this paper is to legitimize personal representations in combinatorics by discussing why they have an ongoing role in the production and justification of solutions, and why institutional representations seem insufficient in satisfying this role. I apply a fairly narrow view on semiotics which examines the inscribed signs and sign systems students create and use as they solve counting problems. Although this view does not incorporate other semiotic resources such as rhythm and gesture, it is consistent with prior undergraduate combinatorics education literature while still offering additional contributions. My core argument is as follows. Semiotic systems are ubiquitous in mathematics in part because they facilitate mathematical production. Yet, common combinatorial notation is more conducive to expressing solutions than producing them. Students develop personal representations for combinatorics in order to produce mathematics externally because there are limited institutional means of doing so. Institutional and personal representations can cohabitate in combinatorics because they are used for different purposes. One is not transitional or auxiliary to the other, and personal representations are not antecedent to institutional ones. However, personal representations are not neutral mediators of combinatorial activity. Two different personal representations might lead to two different arguments and two different mathematical expressions of the same integer value, numerically equal but indicative of different structural reasoning. A semiotic lens on combinatorics must account for such differences.

Combinatorics and a Set-Oriented Perspective

Combinatorics is a broad area of mathematics that includes enumeration and existence. Most introductions are through enumerative combinatorics, more commonly called counting problems. From an expert's perspective counting problems task the counter (i.e., the person solving the counting problem) with determining the cardinality of a set of objects or determining the number of ways an event can be carried out. Lockwood (2013) characterizes combinatorial activity as occurring between three components: sets of outcomes, counting processes, and formulas/expressions. The set of outcomes is the collection of objects being counted in a problem, and the solution to the problem is typically the cardinality of that set. Counting processes are the real or imagined enumerative processes the counter engages in as they solve a problem. Formulas/expressions are the mathematical expressions that yield some numerical value, and solutions to counting problems are typically given as some expression. Lockwood (2014) describes a set-oriented perspective on counting as "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). At the core of the set-oriented perspective is the connection between mathematical expressions and the structure in a set of outcomes. A mathematical formula can be applied because it reflects structural elements of the set of outcomes. The set-oriented perspective contrasts other approaches to counting which involve identifying problems types and applying formulas associated with those problem types.

Lockwood's characterization of a set-oriented perspective is sufficiently broad so as to incorporate the manyfold ways of reasoning combinatorially, which inevitably raises the question of how the one engages in such a perspective. Antonides and Battista (2022a, b) captured certain cognitive processes for spatial-temporal-enactive structuring involved with geometric objects, which included recursive reasoning that ultimately led to closed-form solutions for permutation problems. Some work with younger students (e.g., English 1993; Speiser, 2010; Tarlow, 2010; Tillema, 2013, 2018) has examined how material resources, iconic imagery, and emergent representations are key components in developing and communicating combinatorial reasoning. Literature examining undergraduate students has often examined student inscriptions, such as lists of outcomes (Lockwood & Gibson, 2016), and how encoding outcomes impacts solutions and justifications (Lockwood et al., 2018; Wasserman & Galarza, 2019). This literature has all reported on various semiotic resources as important to student reasoning, even if the analysis was not focused on the mechanisms through which the semiotic resources mediated solutions. A further step towards understanding how students engage in a set-oriented perspective is to examine the role of the semiotic resources in solutions that use a set-oriented perspective.

Connecting Classical Semiotics to Combinatorics

Student use and development of personal semiotic representations in combinatorics are not new phenomena. They appear in literature spanning from early elementary to graduate school (e.g., Tillema, 2013; Wasserman & Galarza, 2019), and they are reported as indicative of or contributing to combinatorial facility. By using a semiotic lens to investigate personal representations I seek to better understand the representations themselves, why students create them, and how they contribute to solutions. I begin discussing semiotics by discussing semiotic systems, and how common combinatorial notation fits within broader semiotic systems. This will include a discussion of semiotic systems as communicative mediums as well as ways for producing new mathematics. Then, I will briefly discuss some insufficiencies of institutional combinatorial notation and why students create personal representations. In the following section I will discuss some personal representations in combinatorics, and use the examples to illustrate how the personal representations demonstrated sophisticated sign systems that contributed to mathematical production.

Semiotic Systems and the Combinatorics Register

In broad strokes, semiotic systems in mathematics are the systems of signs and symbols used to communicate and carry out mathematics. Common semiotic systems are fractions, decimals, set-builder notation, and function notation. Semiotic systems exist in many modalities (e.g., speech, manipulatives, imagery, pictures) but for the purpose of this paper I will focus solely on written (inscribed) and spoken signs and sign systems. Ernest (2008b) characterizes a semiotic system as being composed of three components: i) a set of signs, ii) a set of rules for sign use and production, and iii) an underlying meaning structure that incorporates the relationships between the signs and the rules for their use. For fractions we may all be aware that '0', '3', '8', and '/' are included in the set of signs, and that the set of rules for sign use says that '30/8' is a permissible sign whereas '038/' is not. The rules for sign use are a consequence of using signs as a communication vehicle for an underlying meaning structure. The rules ensure that there is a way of interpreting the signs, and that the interpretations of the signs are consistent with what they represent. The fraction 30/8 is permissible because it is interpreted as the ratio between the integers 30 and 8. The sign 038/ does not have such an interpretation, and so it is not permissible.

Sign systems are also used to produce new mathematics through rules that state how some signs can be transformed into others (Duval, 2006). That is, beginning with the expressions $\frac{d}{dx}3x^2$ we can carry out the derivative purely symbolically to find $\frac{d}{dx}3x^2 = 2 \cdot 3x^{2-1} = 6x$. Although we can relate this transformation back to the underlying meaning structure, in practice doing so is rare. In this instance the set of rules in the semiotic system allow for more efficient mathematical production because they allow for a temporary foregoing of the underlying meaning structure. Duval (1995, 2006) distinguishes between two types of transformations. Treatments are transformations that occur within the same semiotic system (e.g., a fraction to a fraction), and conversions are transformations that occur between different semiotic systems (e.g., a fraction to a decimal). Duval calls semiotic systems that permit both treatments and conversions *registers*. With these theoretical constructs in mind, we can characterize some mathematical activity as reasoning within or between various semiotic systems (registers), and we can also characterize some difficulties as difficulty with types of transformations or semiotic systems.

One of the difficulties within combinatorics is that there are limited institutional semiotic systems students can access as they solve problems. Many counting problems are given in a natural language (e.g., English), and they task the counter to determine the cardinality of a set of outcomes, or the number of ways in which an event can occur. Solutions to counting problems are integers or parametric expressions that can be transformed into integers, which may include signs and symbols associated with combinatorics, including $P(8,3)$ and $\binom{n}{k}$. I refer to the semiotic system used for combinatorics as the *combinatorics register*. Seen purely as signs that can be transformed into integer values, the combinatorics register might be considered an extension of the fraction register or algebraic register. However, the underlying meaning structure of the combinatorics register is quite different, which leads to a duality of the notation (Wasserman, 2019). Wasserman claims that the combinatorics register (which he described as combinatorial notation) can be interpreted in two ways, first as integer-valued expressions, but also as representations of enumerative or structural processes. Underlying any combinatorial expression is the need for that expression to be tied to a set of outcomes. Not only does the expression need to permit a transformation into an integer, but the expression also needs to permit an interpretation as an enumerative expression for a set (Wasserman, 2019). This connection has repercussions for the set of rules for sign use. From a purely algebraic/arithmetic standpoint it is difficult to connect the two signs 2^n and $\sum_{k=0}^{n}\binom{n}{k}$, and substituting one for the other (although correct) might be considered lacking justification. Interpreted from a combinatorics standpoint this substitution is justified because the two expressions represent different enumerative/structural arguments for the power set of a set of n. The underlying meaning structure reorganizes and rewrites the rules for sign use and production because it focuses interpretation on cardinalities and enumeration of sets rather than abstract integers.

The combinatorics register provides a means of communicating cardinality and enumerative processes simultaneously, but it does not provide a means of producing the mathematics. The interpretations of the combinatorics register are purely internal. This contrasts with the prior calculus example where the semiotic system also permitted treatments that facilitated producing mathematics purely symbolically. The combinatorics register thus conflicts with a set-oriented perspective because it does not provide notation specifically for the objects being counted, requiring students to either form the connection mentally or develop entirely new notation for the objects being counted. In practice students do both, and in the following section I present some examples of personal representations to illustrate how personal representations inform and impact solutions to counting problems.

Examples of Personal Representations

In this section I discuss some examples of personal representations students created over the course of several interviews. These data come from a larger study that aimed to examine the roles of representations in solutions to counting problems. I have chosen these particular examples because they exemplify the potential symbiosis between personal and institutional representations. Although the students expressed a 'final answer' using an institutional representation, the bulk of their work was done using personal representations. Moreover, the students developed and refined the personal systems of representations over the course of multiple interviews, and the systems indicate sophisticated attention to structural aspects of the sets of outcomes which impacted the form of their solutions.

Lists and listing processes: Hallie and Leah

The students Hallie and Leah used lists and listing processes repeatedly over the course of six 90-minute interviews. During the first few interviews Hallie and Leah used the lists to completely enumerate a set of outcomes, solving a counting problem by tallying the total number of outcomes in their list. This use of lists became increasingly intractable as counting problems became more complex. Hallie and Leah observed symmetric properties in their lists, and their subsequent uses of lists exploited symmetry to decrease the number of outcomes they needed to write down. The exploitation of symmetry was accompanied by newly introduced signs (such as arrows) to indicate continuations of patterns. In early problems Hallie and Leah would list half of the total outcomes and indicate a continued pattern. In later problems the listing process was increasingly segmented, and symmetry was exploited throughout. Figure 1 provides an example of a listing process represented through segmentation of symmetric components of the list. This example solves a combination whose solution can be written as $\binom{13}{3}$.

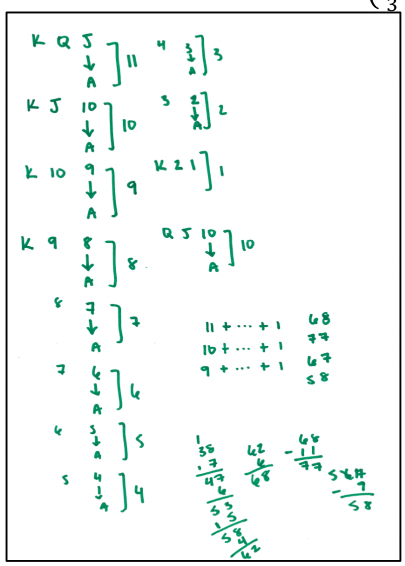

Figure 1: Representation of a listing process exploiting symmetry

In Figure 1 a listing process is represented using alpha-numerical characters, brackets, and arrows. The arrows are used to indicate a continuation of a pattern, and the brackets segment the list of outcomes into constituents. Hallie and Leah placed numbers next to the brackets to indicate the number of outcomes in each constituent piece. The list indicated in Figure 1 is only a partial list, but the sums in the middle of the figure illustrate how Hallie and Leah generalized the results of the partial listing to the remainder of the outcomes. The partial list in the example corresponds to the sum $11 + \cdots + 1$, and the remaining two sums $10 + \cdots + 1$ and $9 + \cdots + 1$ correspond to two other segments of the entire set of outcomes. Hallie and Leah used the partial sums to count the number of times each summand appeared in a partial sum. They found that 11 appears in one partial sum, 10 appears in two partial sums, and so forth, leading to their solution given in Figure 2. I present this example because it demonstrates that Hallie and Leah used personal representations (the lists and listing processes) to produce their solution, and their solution reflects the structure they observed in the outcomes.

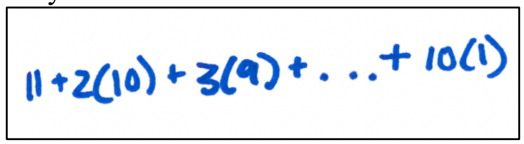

Figure 2: Solution to a combination problem

Sequences of characters: Pat and Damien

The students Pat and Damien repeatedly solved counting problems by representing outcomes as sequences of alphanumeric characters and leveraging know structural information about alphanumeric characters to solve their counting problems. This technique was challenged when they solved the road problem, which states “A certain map connecting locations A and B is given below. You are creating routes between A and B so that no roads are repeated. How many routes are there?”

Pat and Damien’s solution to this problem consisted of iteratively refining the method of encoding paths in the diagram as sequences of characters. The first method of encoding paths consisted of mapping the route of an imagined agent in terms of left and right path segments. This method of encoding was difficult to count because it resulted in paths of variable sizes with constraints that were challenging to characterize. Through several refinements consisting of reformulating how information was encoded or selectively choosing information to leave out of the outcomes, Pat and Damien found that each outcome could be encoded as a length-6 sequence of Ts and Bs. A T was used to indicate movement along the top of the diagram, a B was used to indicate movement along the bottom of the diagram, and vertical movement was omitted because it could be deduced by movement along the top or bottom of the diagram. After finding this method of encoding Pat and Damien concluded that there were 2^6 total paths along the diagram. I include this example because it illustrates that most of Pat’s and Damien’s solution consisted of using personal representations to characterize the outcomes in ways that were conducive to counting. Their final solution, which connected sequences of Ts and Bs to the expression 2^6, was merely one step in a larger process of iteratively refining representations of the outcomes, and the bulk of the mathematical production occurred in personal representational systems.

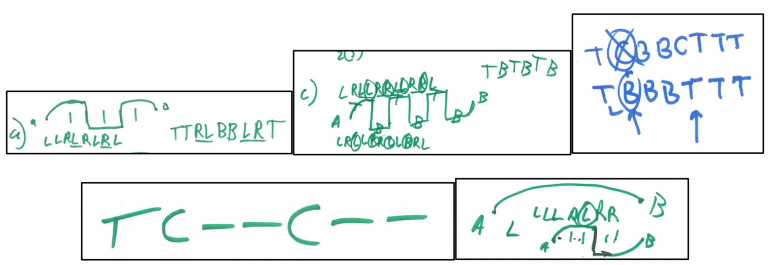

Figure 3: Multiple iterations of encoding paths as sequence of characters

Conclusions

In this paper I have argued that personal representations play a key role in developing combinatorial facility. In part, this role is necessitated by the combinatorial register (the common institutional semiotic system for combinatorics) insufficiently supporting mathematical production. Unlike semiotic systems common to calculus and algebra, the combinatorial register is most conducive to communication. The commonality of personal representations in combinatorics appears to be a result of this. Personal representations appear to facilitate mathematical production while reflecting student understanding.

Wasserman (2019) previously discussed the duality of combinatorial notation, and that the difficulty of learning to interpret combinatorial notation as representative of enumerative processes poses a pedagogical challenge. He proposed the development of institutional notation for sets of outcomes specifically in ways that align with the combinatorics register. In my view personal representations can fill the same role, and attending to personal representations can additionally increase student agency in solutions. This argument, of course, must weigh the possibility that students can still be challenged by creating personal representations, and that some personal representations might be more difficult to use in ways that are tractable. Further study is needed to determine how to leverage personal representations in a didactical setting, and potential tradeoffs between attending to personal representations and creating institutional representations that incorporate elements of observed personal representations.

Acknowledgments

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References

Antonides, J., & Battista, M. (2022). Spatial-temporal-enactive structuring in combinatorial enumeration. *ZDM—Mathematics Education, 51*(1), 795-807.

Antonides, J., & Battista, M. (2022). A learning trajectory for enumerating permutations: Applying and elaborating a theory of levels of abstraction. *Journal of Mathematical Behavior, 68*(1).

Duval, R. (1995). Sémiosis et Pensée: registres sémiotiques et apprentissages intellectuels [Semiosis and human thought: semiotic registers of intellectual learning]. Berne, Switzerland: Peter Lang.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics, 61*(1/2), 103-131.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22,* 451-47.

English, L. D. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal of Mathematical Behavior, 24*(3), 255-273.

Ernest, P. (2008). Towards a Semiotics of Mathematical Text (Part 2*). *For the Learning of Mathematics, 28*(2), 39-47.

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior, 32,* 251–265. doi:10.1016/j.jmathb.2013.02.008

Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics, 34*(2), 31-37.

Lockwood, E., & Gibson, B. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics, 91*(2), 247-270. doi:10.1007/s10649-015-9664-5

Lockwood, E., Wasserman, N. H., & McGuffey, W. (2018). Classifying combinations: Do students distinguish between different categories of combination problems? International Journal of Research in Undergraduate Mathematics Education, 4(2), 305–322. https://doi.org/10.1007/s40753-018-0073-x.

Speiser, R., (2010). Block Towers: From Concrete Objects to Conceptual Imagination. In Maher, C., Powell, A., Uptegrove, E., (1). *Combinatorics and Reasoning*. New York: Springer. DOI 10.1007/978-0-387-98132-1_3.

Tarlow, L. (2010). Pizzas, Towers, and Binomials. In Maher, C., Powell, A., Uptegrove, E., (1). *Combinatorics and Reasoning*. New York: Springer. DOI 10.1007/978-0-387-98132-1_3.

Tillema, E. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *The Journal of Mathematical Behavior, 32*(1), 331-352. http://dx.doi.org/10.1016/j.jmathb.2013.03.006

Tillema, E. (2018). An investigation of 6th graders' solutions of Cartesian product problems and representation of these problems using arrays. *The Journal of Mathematical Behavior, 52*(1), 1-20. https://doi.org/10.1016/j.jmathb.2018.03.009

Wasserman, N. H. (2019). Duality in combinatorial notation. For the Learning of Mathematics, 39(3), 16–21.

Wasserman, N., & Galarza, P. (2019). Conceptualizing and justifying sets of outcomes with combination problems, *Investigations in Mathematics Learning,* 11(2), 83-102, DOI: 10.1080/19477503.2017.1392208

Isomorphism across Courses: Students' Metaphors in Discrete Math, Linear Algebra, and Abstract Algebra

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Isomorphism appears across many courses in mathematics, including discrete mathematics, linear algebra, and abstract algebra. However, examinations of student understandings of isomorphism have mostly focused on group isomorphism in abstract algebra. In this paper, we compare survey responses from 49 discrete math, 19 linear algebra, and 27 abstract algebra students who were prompted to explain the concept of isomorphism to a child. Results include a cross-course emphasis on types of sameness but variations in the types of language used to characterize isomorphism in each course. Implications include the need for care among instructors and researchers when referring the concept of "isomorphism" without a context as well as the need for further work in understandings of graph theory.

Keywords: isomorphism, abstract algebra, linear algebra, discrete math

Introduction and Background Literature

Isomorphism, especially group isomorphism, in abstract algebra has received extensive attention in the RUME community. Group isomorphism was among the first topics studied in RUME (e.g., Dubinsky et al., 1994; Leron et al., 1995), and has received sustained attention for its connections to proof construction (e.g., Weber, 2001; Weber & Alcock, 2004), curriculum design (e.g., Larsen, 2013), understanding functions (e.g., Melhuish et al., 2020; Rupnow, 2017), and conceptions of sameness (e.g., Rupnow et al., 2022; Rupnow & Sassman, 2022). This focus is understandable, given its centrality to the abstract algebra curriculum (Melhuish, 2015).

However, isomorphism has not received comparable emphasis in discrete math or linear algebra. Like in abstract algebra, graph isomorphism is central to understanding graph theory and, more broadly, discrete math (Ebert et al., 2004). However, most research on discrete math understandings have focused on other areas, such as number theory (e.g., Zazkis & Campbell, 1996; Zazkis & Liljedahl, 2004) and combinatorics (e.g., Lockwood, 2013; Lockwood & Gibson, 2016). Research in linear algebra has also focused on other areas, such as bases (Serbin et al., 2021; Stewart & Thomas, 2010; Wawro et al., 2012), eigentheory (e.g., Serbin et al., 2020; Thomas & Stewart, 2011), and linear transformations (e.g., Andrews-Larson et al., 2017), though change of basis and linear transformations are relevant to understanding isomorphism in linear algebra. Moreover, comparative examinations of concepts appearing in multiple areas have focused on broad topics like functions (e.g., Oehrtman et al., 2008) and sameness (e.g., Rupnow et al., 2022) rather than concepts using the same term but with subtly different meanings in different contexts like isomorphism. Thus, we address two research questions:

1. How are characterizations of isomorphism similar across discrete mathematics, linear algebra, and abstract algebra students?
2. How are characterizations of isomorphism different across discrete mathematics, linear algebra, and abstract algebra students?

Theoretical Perspective

Conceptual metaphors is a theoretical lens that illuminates how individuals' thinking is structured based on their language choices (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 1997). Cross-domain conceptual mappings relate the cognitive structure of a target concept (e.g., isomorphism) to the more developed thoughts in source domains (e.g., same properties, transformation). For instance, “An isomorphism is a transformation” is a conceptual metaphor that gives information about a target domain (isomorphism) by relating it to a source domain that is already understood in some way (a transformation).

Previous work using conceptual metaphors include analyses of mathematicians’ views of isomorphism and homomorphism in abstract algebra (Rupnow, 2021; Rupnow & Randazzo, 2022; Rupnow & Sassman, 2022). Other work has examined undergraduate students’ understandings of various concepts, including bases and linear transformations in linear algebra (Zandieh et al., 2017; Adiredja & Zandieh, 2020) and isomorphism and homomorphism in abstract algebra (Melhuish et al., 2020; Rupnow, 2017). Here we extend the conceptual metaphors for isomorphism in abstract algebra identified in Rupnow (2021) and built upon in Rupnow and Randazzo (2022) to parallel uses in linear algebra and discrete mathematics.

Methods

Data was collected from surveys sent to six groups of students: two sections each of the first author’s discrete mathematics (DM) and linear algebra (LA) courses and two sections of other instructors’ abstract algebra (AA) courses. DM students were mostly computer science majors, LA students were STEM majors (including computer science, engineering, mathematics, and physics majors), and AA students were math majors or minors. The surveys were distributed after students had learned about isomorphism in all sections. Initial questions on the discrete math and linear algebra surveys asked about sameness in mathematics whereas initial questions on the abstract algebra survey assessed beliefs about learning (especially related to fixed or growth mindset). Here we analyze responses to one question from the middle of the survey: “How would you describe an “isomorphism” to a ten-year-old child?” (AA) or “How would you describe the concept of isomorphic/isomorphism to a ten-year-old?” (DM/LA). 21 and 28 students completed the DM survey in the two sections; 9 and 10 students completed the LA survey in the two sections; and 13 and 14 students completed the AA survey in the two sections. We did not observe major differences between the sections for each course and thus combine our reporting into a dataset of 49 DM responses, 19 LA responses, and 27 AA responses.

Data analysis was conducted by two researchers who coded independently, then discussed and came to consensus for each response. We recognize students were prompted to provide comparative explanations based on the task of explaining in a manner understandable by a child. Nevertheless, our goal was to classify the underlying meanings conveyed by these explanations according to our interpretations of the participants’ responses. As such, we used the metaphors for isomorphism in abstract algebra in Rupnow (2021) as an initial codebook, adding new metaphors as necessary. This process aligns with codebook thematic analysis (Braun et al., 2019), in which existing codes are used as a basis for coding but new codes are permitted to be added to adequately capture the nuances in data.

Results

We organize the results of our coding into three subsections, each corresponding to a different course. We also provide Table 1, which offers an initial overview of the frequencies of each metaphor by course. Metaphors are organized by cluster, with some belonging to sameness-based clusters such as sameness (*generic sameness*, *same properties*, *same but looks different*, *other branch analogues*), sameness/mapping (*renaming/relabeling*, *matching*), and sameness/formal definition (*structure-preservation, operation-preservation*) clusters. Other metaphors belong to the mapping cluster (*generic mapping/relation*, *morphing/transformation*, *invertible, journey*), the formal definition cluster (*literal formal definition*), or made *no attempt/unclear*.

Table 1. Frequencies of metaphors by course.

Metaphor Cluster	Metaphor Code	Discrete Math (n = 49)	Linear Algebra (n = 19)	Abstract Algebra (n = 27)
Sameness	Same but looks different	19	7	3
	Same properties	19	0	8
	Generic sameness	9	3	14
	Other branch analogues	0	0	1
Sameness/ mapping	Matching	4	4	3
	Renaming/relabeling	1	0	2
Sameness/formal definition	Structure-preservation	2	0	0
	Operation-preservation	0	1	0
Mapping	Morphing/transformation	2	9	2
	Invertible	4	2	0
	Generic mapping	1	0	5
	Journey	0	1	1
Formal definition	Literal formal definition	0	0	1
No attempt/ unclear	Unclear/no attempt	5	0	1

Discrete Math

We begin with metaphors used by Discrete Math (DM) students to describe isomorphism. The DM textbook defined isomorphism as: "A graph G_1 is isomorphic to a graph G_2 when there is a one-to-one correspondence f between the vertices of G_1 and G_2 such that the vertices U and W are adjacent in G_1 if and only if the vertices $f(U)$ and $f(W)$ are adjacent in G_2. The function f is called an isomorphism of G_1 with G_2" (Dossey et al., 2005, p. 159). Both *same properties* and *same but looks different* were commonly used metaphors, as was *generic sameness*.

Many of the metaphors discussed shared properties across groups of objects. As isomorphism in DM is centered on graphs, a large number of these metaphors identified the same properties across different graphs: "look and see if the two shapes of the graphs are identical, vertices, edges, weights." Other responses emphasized common properties across everyday objects, such as identifying similarities among houses or vehicles.

Another commonly used metaphor used to describe isomorphism was *same but looks different*. These types of instances place emphasis on the idea that while isomorphic objects have the same underlying structure or properties, they may appear different visually. The context of these metaphors varied; as with *same properties*, many discussed graphs: "we might have two different graphs…and even though they might look different, if we can rearrange the points and lines in one of the graphs to make it look exactly like the other graph, then we can say that the two graphs are isomorphic." Others emphasized the underlying action behind these surface differences through a variety of mediums, including sticks and puzzles, by demonstrating that physical manipulation does not change the inherent identity of the object being acted upon.

Still another metaphor characterizing isomorphism was *generic sameness.* The majority of metaphors in this category emphasized the underlying sameness of two graphs: "It's when something has a similar form or shape." In addition to more commonly utilized metaphors, *structure-preservation* was unique to DM in our data set. Two students chose to emphasize the *structure-preservation* property of isomorphism: "an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping."

Linear Algebra

We now move our focus to Linear Algebra (LA) students' depictions of isomorphism. The LA textbook's definition for isomorphism was: "Let V be a real vector space with operations $\oplus$ and $\odot$, and let W be a real vector space with operations $\boxplus$ and $\boxdot$. A one-to-one function L mapping V onto W is called an isomorphism (from the Greek *isos*, meaning "the same," and *morphos*, meaning "structure") of V onto W if (a) $L(\mathbf{v} \oplus \mathbf{w}) = L(\mathbf{v}) \boxplus L(\mathbf{w})$ for $\mathbf{v}$, $\mathbf{w}$ in V; (b) $L(c \odot \mathbf{v}) = c \boxdot L(\mathbf{v})$ for $\mathbf{v}$ in V, c a real number. In this case we say that V is isomorphic to W." (Kolman & Hill, 2007, p. 258). Among LA students, the most common metaphors were *morphing/transformation*, *same but looks different*, and *matching*.

Metaphors describing the act of *morphing* or *transformation* were the most common type used. Many such instances emphasized the ability of isomorphic objects to be physically manipulated into one another: "You can take one shape, move it, stretch it, or squish it into another shape, and be able to change it back to the original." This response also notes the *invertibility* of such transformations. Still others highlighted the underlying shared structure of isomorphic objects even under a transformation: "When you have a group of things and make the same change to everything in that group that is an isomorphism. So if I had a bucket of crayons and then shaved the tips off of all of them they would be isomorphic before and after."

Same but looks different instances emphasized that although an object or group of objects may look different, they still share core underlying properties: "I would describe isomorphism through the use of tiles or legos with numbers on it. The idea being that no matter how you rearrange the pieces, all of the numbers are still there, and you can trace the method used [to] mix up the shape."

Several students also discussed isomorphism in terms of *matching*. These examples often invoked non-mathematical objects such as dice or people, and emphasized an underlying correspondence or bijectivity:

> If we have 10 classmates on one side of a classroom, and 10 classmates on the other side of a classroom. If all 10 classmates went to the other side of the room, and found a friend. It is isomorphic if everyone has only one friend, and everyone has a friend.

In our data, Linear Algebra had the only use of *operation-preservation*: "if you were to change something in group one by adding or multiplying so there is more of it, there must be the same increase in the transformed group 2."

Abstract Algebra

Finally, we examine Abstract Algebra (AA) students' isomorphism descriptions. The two sections of AA used different textbooks. In one, the textbook's definition for isomorphism was: "Let G_1 and G_2 be groups. A bijective function f: $G_1 \to G_2$ with the property that for any two elements a and b in G_1, $f(ab) = f(a)f(b)$ is called an isomorphism from G_1 to G_2. If there exists an isomorphism from G_1 to G_2, we say that G_1 is isomorphic to G_2 (Pinter, 2010, p. 94)." In the other section, the definition given was: "Two groups $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a one-to-one and onto map ϕ: $G \to H$ such that the group operation is preserved; that is,

$\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all a, b in G. If G is isomorphic to H, we write $G \cong H$. The map ϕ is called an isomorphism (Judson 2019, p. 119)." Among these students, *generic sameness* and *same properties* were most frequently utilized, along with *generic mapping*.

Depictions invoking *generic sameness* were the most prominent among AA students, emphasizing some flavor of sameness that unites isomorphic objects: "It is when 2 things are similar in every way and act the same way when you do something to it." Also commonly utilized was *same properties*, which stresses specific shared properties across isomorphic objects: "Draw a group of triangles and a group of squares that have the same number of elements. The groups are similar because they are both shapes, or are of the same color, etc.". Several responses utilized both metaphors, such as the following which emphasizes both specific traits and generalized sameness: "With two bunches of objects, if they have the same number of things and each thing's traits are similar to one in the other bunch, they're probably isomorphic."

Generic mapping instances focused on the presence of some mapping between isomorphic objects: "It is a map that shows that two different things are the same." Unique to AA in our data were *other branch analogues* and *literal formal definition*, each used by a single student. *Other branch analogues* identified a parallel between isomorphism and equality: "It is similar to an equals sign except it can be between groups of numbers not just equations." An example of *literal formal definition* emphasized how elements are required to interact under isomorphism:

> When you have 2 groups A and B and all of the rules between the groups A and B are followed. There are also additional rules that A and B must follow. They must when given a number to plug in give you a number back, and if you put in the two numbers that are different they must give you different numbers back. For example if you put in the number 2 to A and put in the number 3 for A and get back 4 for both then it does not follow the rules. The same rules apply to group B.

Discussion

This study examined students' characterization of isomorphism in Discrete Math (DM), Linear Algebra (LA), and Abstract Algebra (AA) courses. Of note, each course had a different most-common metaphor. DM tied between *same properties* and *same but looks different*; LA's most common metaphor was *morphing/transformation*; and AA's most common was *generic sameness*. From this we note that while "isomorphism" is a shared name for concepts across courses, students still chose subtly different things to emphasize, indicating the need for care when referring to isomorphism and that isomorphism can mean different things in different contexts. This aligns with prior work with mathematicians, who carefully attended to the context-dependent nature of isomorphism (Rupnow & Sassman, 2022).

Despite these differences in emphasis, we note that metaphors from sameness-based clusters dominated the DM and AA students' descriptions of isomorphism and were a second emphasis behind the mapping cluster in LA. These differences do not appear to be related to the textbook's definition of the concepts given, as all definitions emphasized the function/mapping nature of isomorphism and the definition that directly used the word "same" was from LA, whose students had the least emphasis on sameness. However, the broader course contexts may have been impactful in terms of the objects to which isomorphism is applied, broader emphases of the course (i.e., proof), and whether isomorphism had been encountered in a previous course. Future research should compare the broader instructional contexts to better understand how students' perceptions of isomorphism were formed and what proved most impactful.

References

Adiredja, A. P., & Zandieh, M. (2020). The lived experience of linear algebra: A counter-story about women of color in mathematics. *Educational Studies in Mathematics, 104*, 239–260. https://doi.org/10.1007/s10649-020-09954-3

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology, 48*(6), 809–829.

Braun, V., Clarke, V., Hayfield, N., & Terry, G. (2019). Thematic analysis. In Liamputtong P. (Ed.) *Handbook of Research Methods in Health Social Sciences*. Springer: Singapore. https://doi.org/10.1007/978-981-10-5251-4_103

Dossey, J. A., Otto, A. D., Spence, L. E., & Vanden Eynden, C. (2005). *Discrete Mathematics* (5th edition). Pearson.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Ebert, C., Ebert, G., & Klin, M. (2004). From the principle of bijection to the isomorphism of structures: an analysis of some teaching paradigms in discrete mathematics. *ZDM, 36*(5), 172–183.

Judson, T. W. (2019). *Abstract algebra: Theory and applications.* http://abstract.ups.edu/download/aata-20190710-print.pdf.

Kolman, B., & Hill, D. (2007). *Elementary Linear Algebra with Applications* (9th edition). Pearson.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: The University of Chicago Press.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Lockwood, E. (2013). A model of students' combinatorial thinking. *The Journal of Mathematical Behavior, 32*(2), 251–265.

Lockwood, E., & Gibson, B. R. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics, 91*, 247–270.

Melhuish, K. (2015). Determining what to assess: A methodology for concept domain analysis as applied to group theory. In T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 736–744), Pittsburgh, PA: SIGMAA on RUME.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60,* 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Oehrtman, M., Carlson, M.P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 27–42). Washington, DC: Mathematical Association of America.

Pinter, C. C. (2010). *A book of Abstract Algebra* (2nd edition). Dover.
Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education*, (pp. 259–273), San Diego, CA.
Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*(2), 100867. https://doi.org/10.1016/j.jmathb.2021.100867
Rupnow, R., & Randazzo, B. (2022). Mathematicians' language for isomorphism and homomorphism. In *Proceedings of the 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 1099–1107). Nashville, TN. https://doi.org/10.51272/pmena.44.2022
Rupnow, R., Randazzo, B., Johnson, E., & Sassman, P. (2022). Sameness in mathematics: A unifying and dividing concept. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-022-00178-9
Rupnow, R., & Sassman, P. (2022). Sameness in algebra: Views of isomorphism and homomorphism. *Educational Studies in Mathematics, 111*(1), 109–126. https://doi.org/10.1007/s10649-022-10162-4
Rupnow, R., Uscanga, R., & Bergman, A. (2022). Connecting sameness in abstract algebra: The case of isomorphism and homomorphism. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1131–1136). Boston, MA.
Serbin, K. S., Robayo, B. J. S., Truman, J. V., Watson, K. L., & Wawro, M. (2020). Characterizing quantum physics students' conceptual and procedural knowledge of the characteristic equation. *The Journal of Mathematical Behavior, 58,* 100777.
Stewart, S., & Thomas, M. O. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology, 41*(2), 173-188.
Thomas, M. O., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal, 23*, 275-296.
Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS, 22*(8), 577-599.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209–234.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0
Zazkis, R., & Campbell, S. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education, 27*, 540–563.
Zazkis, R., & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education, 35*, 164–186.

A "Different Vibe": The Effect of Discrete Mathematics on Undergraduate Students' Mathematics Beliefs

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This paper presents emerging results from an investigation of undergraduate students' beliefs about mathematics (and themselves as learners) within the context of a discrete mathematics course. Our data sources include findings from a survey study with 11 participants and a follow-up interview study with 5 participants. Results suggest that students perceived discrete mathematics as being significantly different from their previous mathematics coursework, offering additional options for creativity and a shift in focus from computation to proof and logic. The impact of this shift on students' mathematical self-efficacy was mixed, with some students reporting unchanged self-efficacy but multiple students reporting decreased self-efficacy (either temporary or permanent). These results have implications for the teaching of discrete mathematics courses and shed additional light on our existing knowledge of students' belief change during the transition from K-12 school to undergraduate mathematics courses.

Keywords: Beliefs, Discrete mathematics, Mathematics self-efficacy

We present preliminary results of two studies examining undergraduate discrete mathematics students' beliefs about themselves as learners (their mathematics self-efficacy or confidence) and their beliefs about discrete mathematics. Discrete mathematics is a valuable site for study because it may allow for changes in students' beliefs. In particular, discrete mathematics courses typically include a variety of accessible topics that allow for deep mathematical thinking (Goldin, 2018; Sandefur et al., 2022). Particularly given recent increases in the number of undergraduate students majoring in the computer and information sciences, which typically require a course in discrete mathematics (Berg et al., 2023), it is timely to consider the impact of this course. However, little previous research has examined the effects of discrete mathematics on students' beliefs.

Conceptual Framework and Literature Review

In this study, we examine two types of beliefs held by undergraduate students enrolled in discrete mathematics courses. First, we examine students' mathematics self-efficacy (MSE), that is, their beliefs about their ability to do and learn mathematics (Bandura, 1986). The construct of MSE is related to mathematical confidence, although confidence is normally a broad measure whereas self-efficacy is task- or subject-specific (Morony et al., 2013). Self-efficacy beliefs are important because high MSE has been linked to higher achievement and improved learning, as well as non-academic skills such as perseverance and self-regulation (Hackett & Betz, 1989; Muenks et al., 2018; Usher et al., 2019). Generally, high MSE is seen as more beneficial than low MSE, though some researchers suggest that self-efficacy needs to align with actual ability (called calibration) to be fully beneficial (Russell & Phelps-Gregory, 2022). Thus, a goal of researchers and instructors should be, at the very least, to help students develop calibrated but confident self-efficacy.

Secondly, we examine students' beliefs about mathematics, including some of their generalized beliefs about the nature of mathematics (such as if math is creative) as well as specific beliefs about discrete mathematics. Unproductive beliefs (like the belief that mathematics is memorizing rules and is not creative) are commonplace among US students (Boaler, 1998; Kloosterman & Stage, 1992; Mann, 2006; Phelps-Gregory et al., 2020) and may harm students' learning because these beliefs correlate with higher anxiety and lower achievement (Geist, 2010; House, 2006).

We specifically examine students' beliefs (and possible beliefs changes) in discrete mathematics, a collegiate mathematics course typically taken by aspiring mathematicians, teachers, computer scientists and engineers, and information technology/information science majors. Discrete mathematics courses generally cover topics related to discrete objects and structures, though the exact topics covered are not always agreed upon. Sandefur and colleagues (2022) identified core topics and key themes of all discrete mathematics classes, including topics such as combinatorics and recursion. Previous research on discrete mathematics has examined student work (e.g., Greefrath et al., 2022) and teachers' pedagogical decisions and techniques in building student thinking (e.g., Alsardary & Blumberg, 2009; Love et al., 2006; Soto et al., 2022). However, one under-researched area is students' beliefs during and after discrete mathematics courses.

Discrete mathematics may be a good site to examine students' beliefs because researchers have argued that discrete students can engage in deep mathematical thinking regardless of their mathematical backgrounds and experiences (Colipan & Liendo, 2022; Goldin, 2018). This engagement might provide a site for changes in students' beliefs (Sandefur et al., 2022). In addition, discrete mathematics may also be a source of changes to MSE because mathematical transitions such as those required in discrete mathematics may be a site for disruption and change (Bandura, 1986; Gill, 2019). In particular, discrete mathematics courses often (but not always) include a focus on mathematical proof, thus prompting a transition to the so-called "axiomatic-formal" mode of mathematics (see Tall, 2008, for a fuller discussion of this transition). However, as noted by Stylianou and colleagues (2015), little research has examined undergraduate students' beliefs about proof, including their self-efficacy for learning proof. In general, few previous studies have examined students' beliefs in discrete mathematics. Our study sought to begin to address this. We examined the research questions:

1. How does discrete mathematics affect students' beliefs about their mathematics ability (their mathematics self-efficacy)?
2. What are students' beliefs about the nature of discrete mathematics?

Methods

We present here the related results of two studies of student beliefs. We first conducted a qualitative survey study examining undergraduate students enrolled in discrete mathematics at two regional universities. The survey used open-ended prompts to ask participants about their MSE and their beliefs about mathematics and discrete mathematics in particular. Eleven participants completed the survey, answering in their own words in sentences and phrases. We analyzed this data using an open coding procedure where codes were developed emergently from the data, following Campbell and colleagues (2013).

Once analysis of this data was complete, we conducted a second qualitative interview study to again examine undergraduate students' beliefs in discrete mathematics but with a new group of participants, enrolled in a different semester and with a different instructor. The goal of the interview was to explore the same questions as the survey, but in further depth and with the opportunity for follow-up questions. Questions included asking participants to describe their mathematics experiences before discrete mathematics, asking students to describe their discrete mathematics experiences, asking about the effect of discrete mathematics on their confidence, and asking belief questions (e.g., about the relevance and usefulness of mathematics). There were also questions asking participants to examine specific discrete mathematics questions and their confidence (MSE) in solving them. Five students completed interviews, lasting on average 29 minutes. Further interviews will be conducted in future semesters, but we present the results of the survey study and initial round of interviews here.

To analyze the interview data, we transcribed the interviews and then built cases for each participant by examining their data and highlighting important themes related to the research questions. Analysis then focused on categorizing participants by the effect of discrete mathematics on their MSE. We then compared and contrasted the groups of participants.

Results

Our preliminary results from both studies show that discrete mathematics may raise some students' MSE, temporarily or permanently, lower other students' MSE, and have no effect on the MSE of some students. This result is tied to students' beliefs about discrete mathematics, with multiple students mentioning their feeling that the course was qualitatively different compared to other mathematics classes. We will examine these findings in more detail below.

Findings of Survey Study

Of the eleven students who completed the survey, three participants reported that discrete mathematics did not affect their self-efficacy, three reported that it increased their self-efficacy and one reported that it temporarily lowered their MSE. Students' reasons for a change in self-efficacy included their course performance (i.e., grades), particularly given the reputation of the course as difficult and the nature of the mathematics in the course. For example, Noah[1] said, "I feel a little more confident because it isn't as scary as my dad made it out to be." Caleb reported that discrete mathematics temporarily lowered his MSE because the material was "different." When asked if discrete mathematics effected his confidence, Caleb said "Yes and no, it was harder for me to pick up the different topics because it's so different from, for example, a calculus class, but at the same time I just had to work harder to understand it and when I did my confidence was back."

Students' view of discrete mathematics as "different" could be seen in their other survey responses as well. Seven participants reported that discrete mathematics required more creativity than other mathematics subjects. Ava said, "The proofs in discrete math allow for slightly more creativity than in other math classes, because there is no one 'right' answer." However, not every participant agreed, with Noah saying, "Less creativity, [because] physics and calculus requires a lot of thinking outside of the box." Overall, the results of the survey study suggest that discrete mathematics could be a source of MSE change for some students due to the nature of the mathematics in the course.

Preliminary Findings of Interview Study

We are currently conducting an interview study to follow-up on the survey results. Based on preliminary analysis of data from five participants, one reported that their MSE was permanently lowered; one reported that, like Caleb, their MSE was temporarily lowered; and three reported no

[1] All names are pseudonyms to protect participants' identities.

change to their MSE. In contrast to the survey study, no students in the interview study reported that their MSE was raised as a result of discrete mathematics.

Jacob reported discrete mathematics permanently lowered his MSE, and Kaitlyn reported that discrete mathematics temporarily lowered her MSE. The major source of this lowered MSE for both students seemed to be their performance in discrete mathematics; both reported struggling some with the class material. Jacob said he was doing "not so hot [laughs]" in his discrete class. Kaitlyn said, "Usually math would come to me easy. Like I can pretty pick up pretty fast on it. But I just like for whatever reason cannot like wrap my head around some of the stuff that we're doing." Both attributed their performance and lowered confidence to the nature of the mathematics in the class being different from previous classes. Jacob said, "the material is like… logic based. So, it's really hard to like wrap your mind around sometimes." He also said:

> I guess there's not so many … like set formulas that I have to memorize. It's kind of just like applying all of the math that I've ever learned into one, without like formulas. And it's just kind of hard for me to … grasp my head around.

For Jacob, this lowered MSE seemed to be permanent; that is, he said overall now he believed he just was not as good at mathematics and would struggle in future classes. In contrast, for Kaitlyn, this change was temporary, and she assumed that she would be successful again in the future. The main difference between the two was that Jacob assumed that future classes would be like discrete mathematics. He said, "So I know a lot of like math from now on … basically builds off of discrete, like proofs and stuff like that." When asked to elaborate he said:

> Because I feel like the math before, … like calculus and stuff like that, it's all like here's the question and then there's an answer. But I feel like discrete and everything moving past this is more going to be like, why is this an answer? Or are there other answers? Or is there no answer? And like then explain why there isn't that and then prove it.

In contrast, Kaitlyn assumed that future mathematics classes would be similar to her previous classes and discrete mathematics was just an anomaly. She said, "I know it's just … one class. I know … it's like I'm good at math. I know I am, but it's just this one class. So, I guess short term [it affected my confidence], yes, but long term not really."

It is important to note that all five students reported that discrete mathematics was different from their previous classes, even students who reported that their MSE did not change. Four of the five said discrete mathematics required more creativity, with the fifth student reporting it was about the same. Makayla said:

> I would say it's more, more creative. I think there's definitely different ways you can go about getting the answer for some things. Which I know is true of like other math classes, but I think it's more, more true of this.

And Madison said it was more creative saying, "Probably more [strongly emphasized] because I would think the other classes are more just numbers and equations, whereas this one is thinking more outside the box with how they're related, I guess."

In addition, all participants implied that discrete mathematics had a "different vibe" from previous mathematics classes. Makayla said:

> I feel like it is just like a different overall like vibe, but I don't really know how to explain that. Like it just, it does kind of feel different in terms of math. It is more like logic and circuits and, and just kind of like… Um, I don't know. It's just … kind of like different.

Madison said, "How I described it to my friends is it's low-level math but use the most confusing words possible. [She laughs here.]" Tyler reported that discrete mathematics did not affect his MSE but did say the math was different and perhaps he was less confident in it. He said:

> I will say that I'm less confident about this class than most other math classes because with discrete math it's a bit different from other math. It's a lot less about computation and more about proofs and how the different concepts work together. And so I'm less confident in that than, say, solving equations. But … I would say it hasn't really affected my total confidence in math.

Overall, as with the survey study, the results of the interview study suggest that discrete mathematics could be a source of MSE change for some students due to the nature of the mathematics in the course.

Discussion

The results of this study add nuance to the field's understanding of how undergraduate mathematics courses such as discrete mathematics might influence students' beliefs. Previous research (e.g., Sandefur et al., 2022) suggests that the conceptual nature of discrete mathematics courses could be a source of increased MSE, because students might be able to engage more deeply with mathematical ideas compared with more computationally-focused undergraduate coursework which relies heavily on prerequisite knowledge and skills. In some ways, our results align with this result, in that nearly all our participants in both the interview and survey study described discrete math as being different from the math they have experienced in other coursework. Students described a "different vibe" including mathematics that seemed more creativity, had more options for completing problems, and had a focus on proof, logic, and concepts rather than computation. This aligns with existing research about students' perceptions of the transition to more formal mathematics at the undergraduate level (e.g., Tall, 2008).

However, in contrast with previous research and particularly within the interview study, we found that students reported that these perceived differences between their previous coursework and discrete math actually lowered their MSE (either permanently or temporarily). We might conjecture that this difference from previous research could be caused by the fact that the students in the interview study described themselves as having been successful in their prior mathematics coursework. Thus, rather than serving as a site where their insufficient prerequisite knowledge was not a barrier to achievement, for these students, discrete mathematics pushed them to think about math in unfamiliar and potentially uncomfortable ways. These results suggest that instructors of discrete mathematics courses should be aware that their students might be experiencing lowered self-efficacy, and this could cause students to disengage from the course. These students might need additional support to continue persevering despite the new challenges.

Questions for Further Discussion

We anticipate collecting further data for an additional two semesters, as well as finishing our analysis of existing data. Given this research is ongoing, we pose the following questions for the reader and the audience. First, the nature of interview work is that the questions you ask guide the interview. *What further questions should we ask during interviews to better explore students' beliefs?* In addition, as our analysis is ongoing, we are open to further interpretations of our data. *What analysis techniques and questions could help us better understand our data?* Finally, we pose a broader question to the field. *Given discrete mathematics, and possibly other advanced mathematics classes, may impact students' beliefs, what should be instructors' goals in these classes? How can instructors support students in positive belief change while also mitigating any effects of negative belief change (like permanently lowered self-efficacy)?*

References

Alsardary, S., & Blumberg, P. (2009). Interactive, learner-centered methods of teaching mathematics. *PRIMUS*, *19*(4), 401-416.

Bandura, A. (1986). Self-Efficacy. In *Social foundation of thought and action: A social-cognitive View* (pp. 290-453). *Englewood Cliffs*.

Berg, B., Lee, S., Randolph, B., Ryu, M., & Shapiro, D. (2023). *Current Term Enrollment Estimates: Spring 2023.* Herndon, VA: National Student Clearinghouse Research Center

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, 41-62.

Campbell, J. L., Quincy, C., Osserman, J., & Pedersen, O. K. (2013). Coding in-depth semistructured interviews: Problems of unitization and intercoder reliability and agreement. Sociological Methods & Research, 42(3), 294-320. DOI: 10.1177/0049124113500475

Colipan, X., & Liendo, A. (2022). Discrete mathematics as a resource for developing scientific activity in the classroom. *ZDM–Mathematics Education*, *54*(4), 909-923.

Goldin, G. A. (2018). Discrete mathematics and the affective dimension of mathematical learning and engagement. *Teaching and learning discrete mathematics worldwide: Curriculum and research*, 53-65.

Gill, J. (2019). *Mathematics Self-efficacy, Mathematical Mindset and Their Relationship to Study Habits and Perseverance* (Doctoral dissertation, Central Michigan University).

Geist, E. (2010). The anti-anxiety curriculum: Combating math anxiety in the classroom. *Journal of Instructional Psychology*, *37*(1), 24–31.

Greefrath, G., Siller, H. S., Vorhölter, K., & Kaiser, G. (2022). Mathematical modelling and discrete mathematics: opportunities for modern mathematics teaching. *ZDM–Mathematics Education, 54*(4), 865-879.

Hackett, G., & Betz, N. E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education*, *20*(3), 261-273.

House, J. D. (2006). Mathematics beliefs and achievement of elementary school students in Japan and the United States: Results from the Third International Mathematics and Science Study. *The Journal of Genetic Psychology*, *167*(1), 31-45.

Kloosterman, P., & Stage, F. K. (1992). Measuring beliefs about mathematical problem solving. *School Science and Mathematics*, *92*(3), 109-115.

Love, T., Keinert, F., & Shelley, M. (2006). Web-based implementation of discrete mathematics. *Journal of STEM Education*, *7*(3), 25-35.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236 – 260.

Morony, S., Kleitman, S., Lee, Y. P., & Stankov, L. (2013). Predicting achievement: confidence vs self-efficacy, anxiety, and self-concept in Confucian and European countries. *International Journal of Educational Research, 58*, 79-96.

Muenks, K., Yang, J. S., & Wigfield, A. (2018). Associations between grit, motivation, and achievement in high school students. *Motivation Science*, *4*(2), 158-176.

Phelps-Gregory, C. M., Frank, M., & Spitzer, S. M. (2020). Prospective elementary teachers' beliefs about mathematical myths: A historical and qualitative examination. *The Teacher Educator*, *55*(1), 6 – 27. DOI: 10.1080/08878730.2019.1618423

Russell, B., & Phelps-Gregory, C. M. (2022). Global And Task-Specific Mathematics Self-Efficacy and Performance: A Qualitative Study In Intermediate College Algebra. Poster

presentation. In *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Nashville, TN.
Sandefur, J., Lockwood, E., Hart, E., & Greefrath, G. (2022). Teaching and learning discrete mathematics. *ZDM–Mathematics Education*, *54*(4), 753-775.
Soto, O., Siy, K., & Harel, G. (2022). Promoting a set-oriented way of thinking in a US High School discrete mathematics class: a case study. *ZDM–Mathematics Education*, *54*(4), 809-827.
Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. *International Journal of Research in Undergraduate Mathematics Education*, *1*, 91-134.
Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5–24.
Usher, E. L., Li, C. R., Butz, A. R., & Rojas, J. P. (2019). Perseverant grit and self-efficacy: Are both essential for children's academic success? *Journal of Educational Psychology*, *111*(5), 877-902.

Case Studies of Undergraduate Students' Agentive Participation in the Parallel Spaces of Calculus I Coursework and Peer-Led, Inquiry-Oriented, Complementary Instruction

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Calculus has long been known as a "gateway course" to STEM fields in postsecondary education. To address this issue, researchers in the Math Department at Montclair State University designed a model of complementary instruction that features peer-facilitated workshops where Calculus I students work in groups on inquiry-oriented, groupworthy tasks. The purpose of this multiple-case study is to seek answers to the question, "How do undergraduate Calculus I students experience and navigate their learning of calculus in the parallel spaces of coursework and inquiry-oriented complementary instruction?" The findings of one case study are presented here and include characterizations of the different forms of agentive participation afforded to students in each of the two spaces, as well as their complementary nature relative to learning calculus with understanding. Implications for dismantling the persistent barriers imposed by calculus on access to postsecondary STEM fields are also discussed.

Keywords: Calculus, Complementary Instruction, Agency, Participation

Calculus has historically operated as a "gateway course" to STEM fields in postsecondary education (Hagman et al., 2017). In the hopes of transforming calculus education to be "lean and lively," the calculus reform movement in the 1990s called for a change in calculus instruction to include fewer topics and utilize an active and engaging approaches to teaching and learning (Johnson et al., 2014). Two decades later, the President's Council of Advisors on Science and Technology (2012) proposed a similar suggestion in an effort to provide students with the time necessary to develop deep and conceptual understandings of calculus. Unfortunately, despite the ongoing efforts to reform calculus education, calculus maintains its gate-keeping role.

The Insights and Recommendations from the Mathematical Association of America (MAA) (Bressoud et al., 2015) suggested seven essential practices for establishing a successful calculus program: (1) attention to the effectiveness of course placement procedures; (2) proactive student support services, including the fostering of student academic and social integration; (3) construction of challenging and engaging courses; (4) the use of student-centered pedagogies and active-learning strategies; (5) coordination of instruction, including the formation of communities of practice for instructor learning; (6) effective training of graduate teaching assistants; and (7) regular use of local data to guide curricular and structural modifications. Informed by these recommendations, researchers at Montclair State University designed a peer-led (Roth et al., 2001), inquiry-oriented *complementary* workshop that runs parallel to students' learning in class in order to address this pressing issue. Calculus I students in these workshops (Yu & Seventko, 2015) work collaboratively on deliberately designed groupworthy tasks (Buell et al., 2016; Cohen & Lotan, 2014) that address calculus concepts.

A review of the literature on peer-led cooperative learning models in postsecondary education confirms their effectiveness in various undergraduate mathematics courses in relation to students' academic achievement and other outcomes (Altomare & Moreno-Gongora, 2018; Liou-Mark et al., 2015; Trenshaw et al., 2019). As this body of literature evaluates effectiveness using quantitative methods, little is known as to how and why peer-led cooperative learning models yield the outcomes found in these studies. The study reported here aims to fill this research gap as it seeks to answer to the question, *How do undergraduate students experience their calculus learning in the parallel spaces of coursework and inquiry-oriented complementary instruction?*

Perspectives and Methods

This multiple-case study (Merriam, 1998) is framed from a situated perspective (Lave & Wenger, 1991), and uses Holland et al.'s (1998) concept of *figured worlds* to analyze changes in agentive participation and in relation to identity formation (Vågan, 2011). The unit of analysis is forms of agentive participation enacted by students in class and in workshop. A grounded theory analytical approach (Corbin & Strauss, 2014) is used to characterize students' participation in order to answer the research question. With the agentive participation codes as clusters, a word cloud for each class and workshop space was created for each case study participant to depict a summary overview of their enacted participation in each of the two spaces.

Two cohorts of Calculus I students who attended the workshops as part of their course requirements participated in the study. Each cohort consists of four participants from the same class taught by the same instructor. The data corpus consists of video recordings of 24 classes, six workshops, and three focus group interviews (Creswell, 2012), all of which were transcribed and subject to analysis.

Findings

The table in Figure 1 lists the various forms of participation that emerged from the grounded theory analyses of the two cohorts. These participation actions were further sorted into three categories of interactivity: *high*, *moderate*, and *nominal* activity. *Interactivity* describes students' interaction with their peers, tasks, or material resources. The *high interactivity* category describes agentive actions involving a high level of interaction among students, such as inquiring, sharing, and explaining. Agentive actions in this category involve the conceptual practices of making associations and connections among mathematical concepts (Pickering, 1995), which are the kinds of high cognitive demand (Stein et al., 2000) practices that support learning with understanding (Hiebert & Grouws, 2007). The *moderate interactivity* category describes agentive actions that entail independent work on executing procedures, providing brief responses to dichotomous questions with binary answers (e.g., yes/no or right/wrong), and seeking confirmation of ideas or the correctness of a solution. Lastly, agentive actions in the *nominal interactivity* category involve limited interactions with others and material resources, such as *note-taking* and *launching tasks*. Agentive actions in these latter two categories are considered low cognitive demand because they entail memorizing or carrying out procedures without making connections to facts, procedures, and ideas (Stein et al., 2000).

In addition to yielding the emergence of these codes and categories, the analysis also revealed periods of "integration" (assimilating norms and expectations) and "expansion" (growth in participation) in the students' participation, which I was able to discern using sequences of Venn diagrams that show the trajectory of students' participation over time. Given the space constraints, this phenomenon will be presented in-depth in my presentation should this proposal be accepted. In order to illustrate how participation codes and interactivity categories are used to address the research question, my analysis of Boris's case is presented next. I chose to present Boris's case because it accentuates the unique opportunities to enact different forms of agentive participation in each of the two spaces.

Categories of Interactivity	Level 1 Codes	Class A (Cohorts A & B)	Class B (Cohorts A & B)	Cohorts A & B Workshop
High	Sharing	(Voluntary *[Answer] [Idea] [Resources] [Work]*) (Upon request *[Answer] [Idea] [Resources] [Work]*) (Solicit *[Answer] [Resources] [Work]*)		(Voluntary *[Answer^{A}] [Idea^{A}] [Resources^{A}] [Work^{A}]*) (Upon request *[Answer^{A}] [Idea^{A}] [Resources^{A}] [Work^{A}]*) (Solicit *[Answer^{A}] [Idea] [Work^{A}]*) (Offer *[Work] [Idea]*)
	Inquiring	(Conceptual) (Procedure)	(Procedure)	(Conceptual^{A}) (Other mathematical) (Procedure^{A})
	Scaffolding			(Scaffolding)
	Explaining	(Concept *[Representation]*) (Mistake *[Peer's]*) (Procedure) (Struggle) (Task) (Technicality)	(Mistake *[Instructor's]*) (Procedure) (Reasoning)	(Concept *[Definition] [Representation^{A}]*) (Mistake *[Facilitator's] [Peer's^{A}] [Self]*) (Notation) (Procedure^{AB}) (Provide Example) (Reasoning^{B} *[Realistic]*) (Struggle^{A}) (Task^{A}) (Technicality^{A})
Moderate	Independent work	(Student initiated *[Task] [Review] [Homework]*) (Instructor initiated)	(Student initiated *[Task] [Review] [Homework]*) (Instructor initiated)	(Student initiated *[Task^{AB}]*) (Facilitator initiated)
	Seeking	(Confirmation) (Clarification *[About something] [For someone]*) (Help) (Resources) (Time)	(Confirmation) (Clarification *[About something]*) (Help)	(Confirmation^{AB}) (Help^{AB}) (Resources^{A}) (Time^{A}) (Clarification *[About something^{AB}] [For someone^{A}]*)
	Responding	(Agree/Disagree) (Answer) (Confirm) (Respond to help request) (Private) (Uncertain) (Unfamiliar)	(Agree/Disagree) (Answer) (Confirm) (Private) (Unfamiliar)	(Agree/Disagree^{AB}) (Answer^{AB}) (Confirm^{AB}) (Respond to help request ^{A}) (Uncertain^{A}) (Unfamiliar^{AB})
	Check-in	(Peer) (Self)	(Self)	(Peer^{AB}) (Self^{AB})
	Check (and revise)	(Compare) (Other's) (Self)	(Self)	(Compare^{A}) (Other's^{A}) (Self^{AB})
	Accessing resources	(Lesson) (Notes) (Online resources)	(Homework) (Notes) (Online resources) (Textbook)	(Homework^{B}) (Notes^{AB}) (Online resources^{AB})
Nominal	Agency request unfulfilled	(Public) (Private)	(Public)	(Private^{A})
	Refraining	(Refraining)	(Refraining)	
	(Re)launches task	(Read aloud)	(Read aloud)	(Read aloud^{AB}) (Recite info) (Invitation to work on problem)
	Emoting	(Affirmation) (Confusion) (Frustration) (Success)	(Affirmation) (Confusion) (Frustration) (Success)	(Affirmation^{AB}) (Confusion^{AB}) (Frustration^{AB}) (Relief) (Success^{AB})
	Note-taking	(Note-taking)	(Note-taking)	(Note-taking^{AB})
	General coursework	(Give) (Seeking)	(Seeking)	(Give^{AB}) (Seeking^{AB})
	Non-participation	(Non-participation)	(Non-participation)	(Non-participation^{AB})

(Lvl 2 code *[Lvl 3 code] [Lvl 3 code]*)
(Lvl 2 code *[Lvl 3 code^{ClassA}] [Lvl 3 code$^{Class B}$]*) = Occurred in both spaces

Figure 1. A table of participation actions in class and workshop across both cohorts.

Boris's Participation Profile

The instructor of Boris's class tended to teach through lectures and demonstrations of solving problems on the board. It was rare for the instructor to engage students by asking questions or providing problems for individual, in-class practice. Occasionally, however, the instructor implemented what he called a "homework active learning activity," where he would guide the class in solving a selected homework problem by having students take turns responding to his guiding questions. During this homework activity, the instructor would pose an assortment of moderate and higher interactivity questions to lead students through the problem-solving process.

Across 22 in-person class observations, when the instructor offered students an explicit opportunity to participate, Boris *refrained* from participating 136 times, averaging around six times per observation. [Note: Words in italics are participation codes.] He only *responded* to the instructor when the instructor directly asked him questions during a homework activity. In general, Boris was an *independent* and *resourceful* student in class and an *explainer* and problem-solving leader in workshop. Even though a comparison of his class and workshop word clouds (Figure 2 and Figure 3) shows that Boris *worked independently* in both spaces, his characteristic *independent work* was magnified in class. The *independent work* cluster supports this observation as it is the largest cluster in his word cloud, taking up nearly half the space. On the other hand, the *independent work* cluster in Boris's workshop word cloud is only the fifth largest cluster, which depicts his tendency to initiate *independent work* moments in class by working on homework assignments or on the instructor's examples on the board. Along with *working independently* on problems in class, Boris often *accessed resources* (e.g., textbook, notes, and other online resources) to support his sense making and problem solving. Hence, the *accessing resources* cluster is the second largest cluster in his class word cloud. Considering his active participation in workshop and the limited opportunities to enact agency or work on practice problems in class, it can be inferred that Boris was inclined to agentively pursue learning on his own, even when these opportunities were not explicitly presented to him.

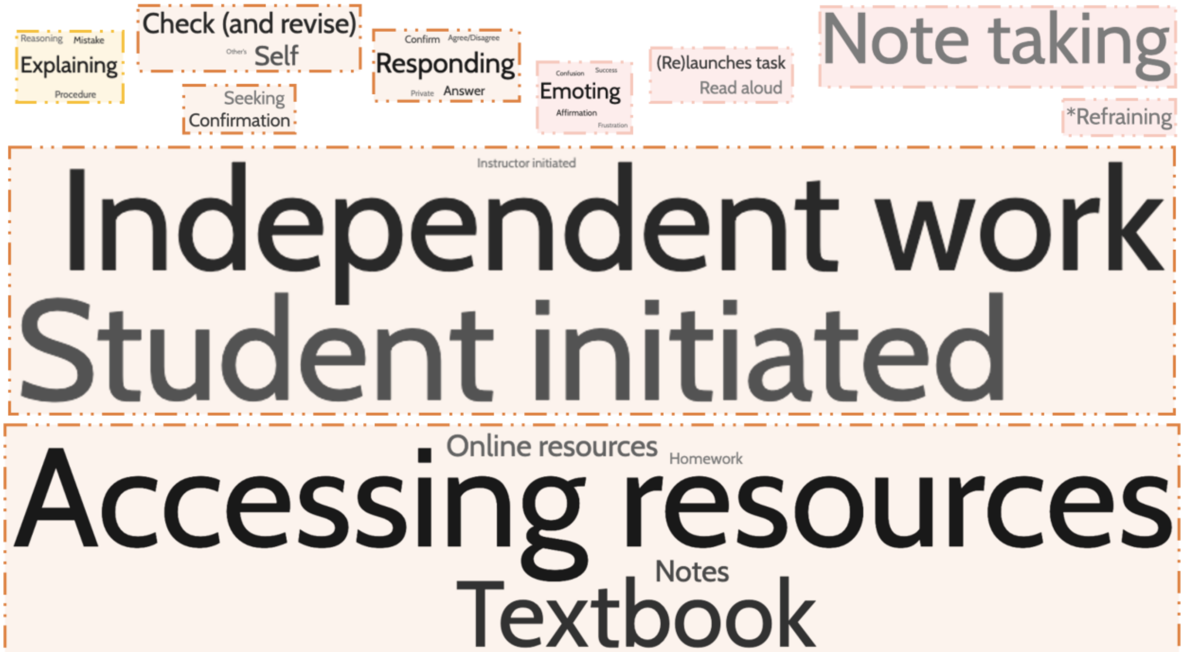

Figure 2. Boris's class word cloud.

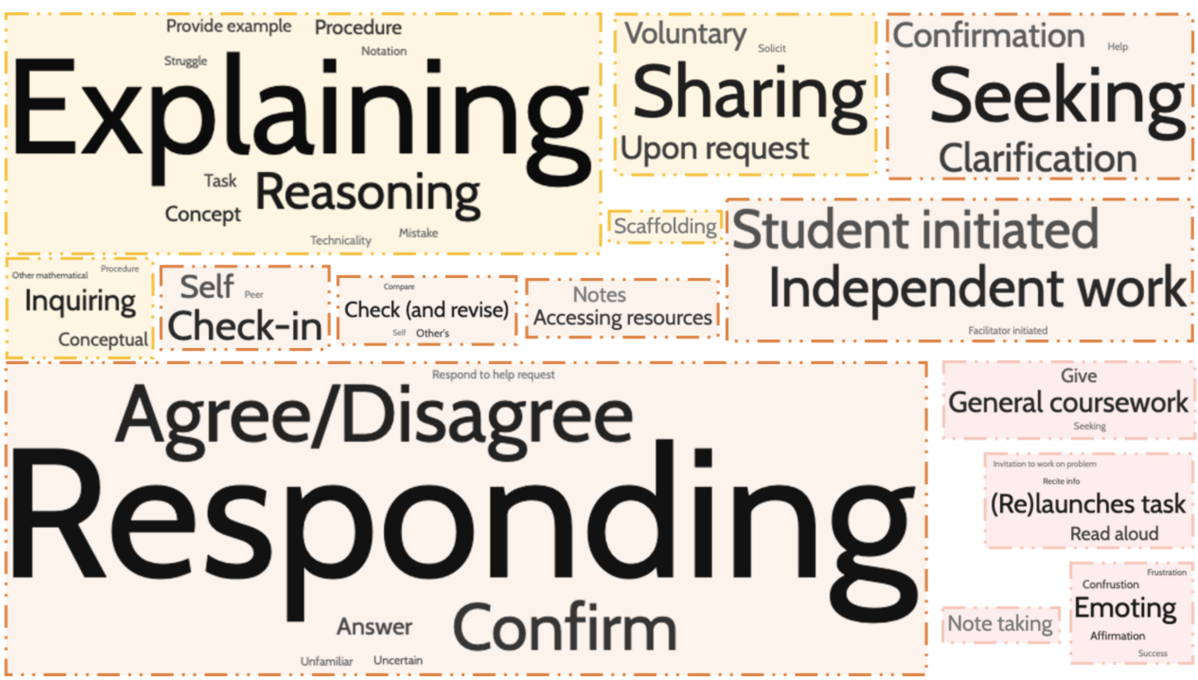

Figure 3. Boris's workshop word cloud.

Rather than choosing to *work independently*, as was his tendency in class, Boris tended to take on the highly interactive roles of a *responder*, *explainer*, and problem-solving leader in workshop. As a *responder*, his responses tended to articulate his *confirmation* and *agreement* with ideas shared by others. Compared to the rare occasions he enacted *explaining* in class, Boris had opportunities to enact *explaining* more frequently and extensively in workshop. This is evident from the *explaining* cluster as the second largest cluster in his workshop word cloud. Specifically, his workshop word cloud indicates his tendency to *explain concepts* and *procedures* and provide *reasonings* and *examples* in his explanations. Overall, Boris's role as a problem-solving leader in workshop was portrayed through his *explaining* actions and the occasional acts of *scaffolding* his cohort peer's problem solving.

Boris's case illustrates the differential forms of participation he enacted in class and in workshop. A review of his participation profile highlights these rather distinctive opportunities to enact agentive participation in each of the two spaces. In class, Boris tended to enact low-demand independent participation actions in moderate and nominal interactivity categories (i.e., *independent work*, *accessing resources*, and *note-taking*). In contrast, in workshop, Boris was more inclined to enact high demand participation actions in the high and moderate interactivity categories (i.e., *explaining*, *sharing*, and *seeking*).

As depicted in Figure 1, cohort A's participants had more opportunities to enact higher interactivity moves in class (i.e., explaining, inquiring, and sharing) than participants in cohort B, of which Boris was a member. Nonetheless, class and workshop were found to complement each other to offer all the participants a broad range of agentive actions. Given the value of highly interactive participation actions in particular, it is critical for students to have more of these in order to better support and enhance their learning.

Discussion and Conclusion

Reflecting on the MAA's seven recommendations for establishing a successful calculus program, the findings from this study can be used to inform calculus instruction by illustrating opportunities for high and moderate interactivity participation actions that can be enacted through student-centered pedagogies and active learning strategies (recommendation 4) in coursework or in complementary instructional workshop. Additionally, the participation codes observed in this study give a vision to calculus instructors of the kinds of interactive participation that are characteristic of challenging and engaging courses (recommendation 3). In turn, these findings can inform and guide the design and implementation of parallel spaces of coursework and complementary instruction, particularly when the realities of coursework alone impose constraints that do not allow for adequate opportunities for high and moderately interactive participation.

To summarize, this multiple-case study sought to address the research question, *How do undergraduate students experience their calculus learning in the parallel spaces of coursework and inquiry-oriented complementary instruction?* This study found a range of agentive participation actions that were further categorized into *high*, *moderate*, and *nominal interactivity* categories based on the quality of their interactions with others, tasks, or material resources. All in all, these findings would be of value to postsecondary calculus educators and program directors who are committed to offering students the kinds of participatory experiences that are productive for their learning of calculus. That way, they can be more mindful in planning, structuring, and designing their calculus programs so as to dismantle the persistent barriers imposed by calculus on access to postsecondary STEM fields.

References

Altomare, T. K., & Moreno-Gongora, A. N. (2018). The role and impact of supplemental instruction in accelerated developmental math courses. *Journal of College Academic Support Programs*, *1*(1), 19-24.

Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and recommendations from the MAA national study of college calculus*. The Mathematics Association of America.

Buell, C. A., Greenstein, S., & Wilstein, Z. (2016). Constructing an inquiry orientation from a learning theory perspective: Democratizing access through task design. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *27*(1), 75-95.

Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (3rd ed.). Teachers College Press.

Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.

Creswell, J. W. (2012). *Educational research: Planning conducting and evaluating quantitative and qualitative research* (4th ed.). Pearson.

Hagman, J. E., Estrella, J., & Fosdick, B. K. (2017). Factors contributing to students and instructors experiencing a lack of time in college calculus. *International Journal of STEM Education*, *4*(1), 15.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 371-404). Information Age Publishing.

Holland, D., Lachicotte Jr., W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.

Johnson, E., Ellis, J., & Rasmussen, C. (2014). It's about time: How instructors and students experience time constraints in calculus I. the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education, Vancouver, British Columbia: PME.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Liou-Mark, J., Dreyfuss, A. E., Han, S., Yuen-Lau, L., & Yu, K. (2015). Aim for success: Peer-Led Team Learning supports first-year transition to college-level mathematics. *Journal of Learning Development in Higher Education (Special Edition: Peer Assisted Learning)*, 1-24.

Merriam, S. B. (1998). *Qualitative research and case study applications in education: Revised and expanded from case study research in education*. Jossey-Bass Publishers.

Pickering, A. (1995). *The mangle of practice: Time, agency, and science*. University of Chicago Press.

President's Council of Advisors on Science and Technology (PCAST). (2012). *Report to the president—Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. Office of the President.

Roth, V., Goldstein, E., & Marcus, G. (2001). *Peer-led team learning: A handbook for team leaders*. Prentice Hall.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. Teachers College Press.

Trenshaw, K. F., Aish, N., Miskioglu, E. E., & Asare, P. (2019). *Leaders like me* The Collaborative Network for Engineering and Computing Diversity (CoNECD) Conference, Crystal City, VA.

Vågan, A. (2011). Towards a sociocultural perspective on identity formation in education. *Mind, Culture, and Activity*, *18*(1), 43-57.

Yu, K. T., & Seventko, J. (2015, August 7). *Inquiry Based Instructional Supplement (IBIS) for Calculus sequence.* Mathematical Association of America (MAA) Math Fest, Washington, DC.

Connecting Proof Comprehension with the Student Perspective: The Case of Christy

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The current state of proof comprehension research has focused on the instructor or researcher perspective. Either through assessing student proof comprehension (e.g., Mejia-Ramos, 2017) or developing interventions to improve student proof comprehension (e.g., Hodds et al., 2014). In this report, we discuss findings from one student from an exploratory study. We consider the influence of prior experiences on what it means to them to understand a proof and the reasoning behind the proof comprehension strategies used to understand a given proof.

Keywords: Proof comprehension strategies, student experiences, reading proof

Comprehending proof is one important task that a mathematician must do during their career. A goal for instructors of proof-based courses is to have their students develop the skills needed for mathematics research (Weber & Mejia-Ramos, 2011). One way of doing this is to have students participate in tasks done by mathematicians, such as comprehending proof. The existing literature has either focused on developing ways to assess student proof comprehension (e.g., Davies et al., 2020; Mejia-Ramos et al., 2017) or interventions to help improve student proof comprehension (e.g., Alcock et al., 2015; Hodds et al., 2014; Roy et al., 2010; Samkoff & Weber, 2015). Each of these studies take the perspective of researchers or instructors of proof-based courses and the student perspective is largely absent from the literature on proof comprehension (e.g., Weber, 2015). In this preliminary report, we begin to investigate the following research questions: (a) How do students' prior experiences inform their understanding of proof? (b) What reasons do students provide for using strategies to understand proof?

Relevant Literature

Proof Comprehension Assessment

Mejia-Ramos et al.'s model (2012) and tests (2017) have been used by multiple researchers to assess the impact of interventions. Alcock et al. (2015) used the model to create proof comprehension assessments to assess if workbooks given to students in an abstract algebra course improved their independent study of proofs. Zazkis and Zazkis (2016) used the model to identify what aspects (with respect to the seven question types) of a proof on the Pythagorean theorem preservice teachers focus on during a proof script task. As creating a valid and reliable proof comprehension test can be a daunting task (Mejia-Ramos et al., 2017), Davies et al. (2020) attempted to find another method to assess student proof comprehension by using student proof summaries. Investigating how student proof summaries could be used to assess student's comprehension of a given proof, the authors also hoped this would provide instructors with an alternative to the long process of creating a proof comprehension test. The authors found this method of comparative judgement to be reliable and valid, but the judgements from the experts correlated less with the final course grades in comparison to the comprehension test. The findings from Davies et al.'s study suggest that the proof comprehension tests created by Mejía-Ramos and colleagues may be more effective in assessing students' understanding with respect to course grades than the comparative judgements of student proof summaries.

Improving Student Proof Comprehension

Researchers have attempted to develop ways to present proofs to students so that the proofs can be easier for students to understand. Leron (1983) suggested an alternative to the linear presentation of proof by describing structuring mathematical proofs to highlight the overarching ideas of the argument. Rowland (2001) presented generic proofs for number theory, which used specific examples to highlight the general approach of the argument. Using the specific example sought to improve clarity and to remove abstractness for the reader.

More recently, researchers have also developed interventions to help with student proof understanding. These interventions attempted to help students in different ways, such as providing proofs that had visual and audio aspects to focus student attention (Roy et al., 2017), helping students learn how to productively study independently (Alcock et al., 2015), or providing students with self-explanation training (Hodds et al., 2014). These interventions attempted to support students in various aspects of reading proof. Either by providing students the chance to read the proof at their own pace and revisit instructor-provided explanations as many times as they would like (Roy et al., 2017), providing students opportunities to engage deeper with the concepts and proofs (Alcock et al., 2015), or by providing students with insight into how to productively summarize arguments (Hodds et al., 2014).

Weber (2015) and Samkoff and Weber (2015) offer seven strategies that successful students implement as they read proof for understanding. These authors implemented two instructional interventions, finding mixed results, to help students improve their proof comprehension skills.

Conceptual Framework

Since reading proof is an important part of a mathematician's career, it is equally an important task for students to take part in within their proof-based courses (Weber & Mejia-Ramos, 2011). Since students must read the proofs they encounter in class and textbooks, they are consumers of the presented proofs. As consumers, they are tasked (implicitly or explicitly) to understand a given proof. While we acknowledge Mejia-Ramos et al.'s (2012) model for assessing students' proof comprehension and the impact of this model on our own conceptions of proof comprehension, this study focuses on what it means to students to understand proof. When a student is given a proof with the goal of understanding, students may act to better understand particular points or aspects of the proof. That is, a student will employ a proof comprehension strategy.

Scholars have studied what students do when they are reading proof. For example, Weber (2015) investigated how successful mathematics majors attempt to understand a given proof through task-based interviews. Dawkins and Zazkis (2021) compared the reading behaviors between students who had little to no experience with reading proof to those who had experience. Additionally, Samkoff and Weber (2015) attempted to teach students seven productive strategies that can be employed to understand a given proof, such as dividing the given proof into independent sections.

Literacy education scholars have discussed student reading strategies and reading skills, with some scholars differentiating the constructs and others suggesting the two are the same. In both cases, scholars rarely (if at all) formally define them. Afflerbach et al. (2008) attempted to bring cohesion within the field by defining both constructs, offering the following definition of a reading strategy: "[r]eading strategies are deliberate, goal-directed attempts to control and modify the reader's efforts to decode text, understand words, and construct meanings of text" (pg. 368). With respect to reading proof, we define a strategy to be deliberate acts from the reader with the goal of understanding some subset of a given proof. To help in operationalizing this, we consider the ostensive strategies that a student enacts when attempting to understand a given proof. Thus, we define a proof comprehension strategy as any action of the reader that is done outside of the proof text with the goal to understand some portion of a given proof (inclusive of the whole proof). By outside of the proof, we mean the student engages in furthering their understanding through actions not done or recommended within the proof.

Meanwhile, the literature suggests that students' beliefs and prior experiences can have a significant impact on their actions within a classroom (Muis, 2004; Schoenfeld, 1989). Thus, as students are reading proofs to understand, they may rely on previous experiences with proofs or related topics. That is, they may reflect on past lived experiences, similar proofs they have seen, or on what proof comprehension strategies have previously been useful or productive. As such, we believe it is possible that students' beliefs and experiences may influence their own meanings of comprehension and the actions they take to understand a proof.

Theorem 1. *Suppose* $f : \mathbb{N} \to \mathbb{Z}$ *such that* $f(n) = \frac{(-1)^n(2n-1)+1}{4}$ *for* $n \in \mathbb{N}$. *Then* f *is a bijective function.*

Theorem 2. *Suppose* $f : A \to B$ *is a function. Then* f *is injective if and only if* $X = f^{-1}(f(X))$ *for all* $X \subseteq A$.

Theorem 3. *Suppose* A *is a set. Then,* $|A| < |\mathscr{P}(A)|$.

Figure 1: Three theorem statements

Methods

The data presented is from a basic qualitative study (Merriam & Tisdell, 2015) that investigated what students do as they attempt to understand given proofs on functions. Three participants were recruited from a large southern Hispanic-serving institution. Participants were interviewed during the winter break of 2022-2023 or during the Spring 2023 semester. Each participant had taken an introduction to proof course during the Fall 2023 semester. Participants were met with 1-3 times for hour-long clinical interviews (Ginsburg, 1997), based on their availability. For each interview, participants were given a theorem statement and its proof. Students were tasked to read the proof until they felt they understood the proof to the best of their ability. Each proof focused on functions, a topic covered in each of the participants' introduction to proof courses. Figure 1 shows the three theorem statements. Relevant definitions were available and provided if requested. After the participants indicated they understood the proof, interview protocol questions were asked. These questions focused on what it means to the participant to understand a proof, what the participants had done to understand the proof, and what they felt as they read the proof for understanding. All interviews were video and audio recorded. For this report, we focus on Christy[1], a computer science major at the time of the study (now a CS and mathematics double major). Christy successfully completed the course, continued in proof-based mathematics courses, and eventually engaged in undergraduate mathematics research. For the study, Christy met with the first author for three interviews. Data for this report was analyzed using open and axial coding by both authors (Corbin & Strauss, 2015). Both

[1] Pseudonym chosen by participant.

authors coded for student experiences and indications for how these experiences influenced what it meant for the student to understand a proof. Additionally, the authors coded for reasoning provided by the student for the use of identified proof comprehension strategies.

Preliminary Results

Christy's Definition of Understanding. In each interview, Christy was asked what it meant to them to understand a proof. Each time, Christy's response was similar: that understanding a proof means to understand the reasoning behind each action taken and decision made by the author throughout the proof. For instance, consider Christy's answer during the first interview:

> To understand, I think for me it's like understanding why people do things or like why certain actions we're taking. Like we are doing two parts cuz for proving it you need the onto and the one-[to]-one. ***But then like why do you have to assume - understanding why these assumptions are made. Because it's one thing to just do it and another thing to understand why it is*** […] ***[L]ike computer sciences, like CS. There's a difference between like having a program that works and actually knowing why it works. Because if you don't know why it works, then you could mess you up over later on when you're doing other things.*** […] Yeah, like why we make certain assumptions with things and like when we have to.

There are three key points to this quote. First, we note that Christy expressed the need to prove that a function is both one-to-one and onto to prove that it is a bijection – meaning that she is engaging in some proof comprehension strategies addressed by the literature, namely splitting the proof into various cases. Second, Christy emphasizes that simply identifying the modular structure (or cases) of the proof is insufficient. Rather, she states that it is also necessary to understand *why.* Why those two cases are made, why any assumptions can be made, and why the author is able to take different actions. Finally, Christy connects her need to consider both the cases of the proof and the reasoning behind these cases to her experiences in Computer Science. Throughout each of her interviews, Christy's focus on the "why" was consistent, as was her connection to computer science.

Christy's Comprehension Strategies. Christy was given a goal of understanding a given proof to the best of their ability. With their definition of what it means to understand a proof, Christy focused on various aspects of the proof to meet this goal. Christy used various proof comprehension strategies, some of which were previously identified by other researchers as strategies used by mathematicians. For example, for each of the given proofs, Christy segmented the proof into parts. Specifically, she marked each proof to indicate independent portions of the proof (such as differentiating the onto and one-to-one portions of the first proof). Christy also frequently used examples or drew diagrams to illustrate portions of a given proof. However, the reasons Christy used these strategies do not necessarily match the reasons the literature has indicated that mathematicians use and promote the use of these same strategies.

Consider partitioning a proof into independent portions. Samkoff and Weber (2015) state that identifying the modular structure of a given proof can be used to see overall structure of the proof. Additionally, Mejia-Ramos et al. (2012) discuss mathematicians views on the benefits of using this strategy as connections can be made between the portions identified. Yet, for Christy, this strategy held a different purpose, "I just need something there so I can see it. […] Like, yeah, block out what I don't need like sometimes I'll drive with the visor down even if it's there's no sun. Just so I can like focus on what's important." For Christy, portioning the proof was to allow them to focus on the independent pieces. Thus, we see Christy using productive comprehension strategies, but the personal reasons for implementing them differ from the motivation suggested by the literature.

Christy also used strategies not identified by Weber (2012). For example, one way Christy tried to understand certain portions of a given proof was to continue reading beyond the section she did not initially understand. In the second proof, Christy read the line "Since $f(x') = y$ and $f(x) = y$, then $f^{-1}(y) = \{x, x'\}$." Christy expressed she "didn't think functions could equal sets." When asked how she made sense of the line, Christy responded "mine was just read the next line." Similarly, while reading the third proof and attempting to make sense of the definition of a set (an element of the powerset and the mechanism by which the contradiction arises), Christy read over the definition multiple times before jumping forward in the proof saying "trying to see if the conclusion helps." In both of these instances, Christy continued to read further lines in the proof in an attempt to increase their understanding. This strategy was not identified by Weber in his 2015 study.

Connections to Computer Science. Christy made connections to their experiences with computer science in each of their interviews. Moreover, the connections varied in terms of scale. For instance, as discussed earlier, her experiences appear to strongly influence her personal definition of understanding a proof. Meanwhile, she also made connections between specific content in each field. For instance, during the first interview, Christy connected the concept of hash maps in computer science to the concept of a bijection:

> Computer science, there's like a hash map. So like to get values, you have to like, know the key for it […] There's just an infinite number of keys […] that's the definition of bijection is that you can get every single value. […] we just need to be able to get every single integer. Yeah, that makes sense in my head.

Discussion

Scholars have mainly focused on the instructor or researcher perspectives with proof comprehension research, though few have begun to investigate the student perspective. This study expands on Weber's (2015) findings that identified six strategies that successful students use to comprehend proof. Christy was a successful student in their introduction to proof course, yet her reasoning for using strategies previously identified by Weber are different than the motivation mathematicians might have for using these same strategies. Additionally, Christy was seen using strategies not reported by Weber. This study contributes to the field's knowledge about how students attempt to understand proofs in two ways. First, there may be additional strategies that successful students implement when reading proof. Additionally, we gain insight into the reasons students use different strategies. In this study, Christy did not necessarily use strategies for the same reasons a mathematician may, yet she productively used strategies that helped her understand the given proofs.

Investigating student perspectives on proof comprehension can inform continued and improved support to students. Additionally, investigating more students may provide deeper insight into ways to support students in comprehending proofs they see in their courses. Researchers can better identify which proof comprehension strategies are productive and when they are most productive to further support students in comprehending proofs they see in their courses. We aim to address this need in current and future work. We are investigating how more students attempt to make sense of given proofs. Further, we will be connecting student beliefs about proof and prior experiences with proof to what proof comprehension strategies they use with a given proof.

References

Afflerbach, P., Pearson, P. D., & Paris, S. G. (2008). Clarifying differences between reading skills and reading strategies. *The Reading Teacher*, 61(5), 364-373.

Alcock, L., Brown, G., & Dunning, C. (2015). Independent study workbooks for proofs in group theory. *International Journal of Research in Undergraduate Mathematics Education*, 1, 3-26.

Corbin, J., & Strauss, A. (2015). *Basics of qualitative research: Techniques and procedures for developing grounded theory.* Los Angeles: Sage.

Davies, B., Alcock, L., & Jones, I. (2020). Comparative judgement, proof summaries and proof comprehension. *Educational Studies in Mathematics*, 105(2), 181-197.

Dawkins, P.C., & Zazkis, D. (2021). Using moment-by-moment reading protocols to understand students' processes of reading mathematical proof. *Journal for Research in Mathematics Education*, 52(5), 510-538. Doi:10.5951/jresematheduc-2020-0151

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, 45(1), 62-101.

Mejia-Ramos, J.P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, 79, 3-18.

Mejia-Ramos, J.P., Lew, K., de la Torre, J., & Weber, K. (2017). Developing and validating proof comprehension tests in undergraduate mathematics. *Journal for Research in Mathematics Education*, 19(2), 130-146.

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: A guide to design and implementation*. John Wiley & Sons.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of educational research*, 74(3), 317-377.

Rowland, T. (2002). Generic proofs in number theory. Learning and teaching number theory: *Research in cognition and instruction*, 157-183.

Roy, S., Alcock, L.& Inglis, M. (2010).Undergraduates proof comprehension: A comparative study of three forms of proof presentation. In Proceedings of the 13th Conference for *Research in Undergraduate Mathematics Education*. Retrieved from. http://sigmaa.maa.org/rume/crume2010/Abstracts2010.htm.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, 20(4), 338-355.

Weber, K. (2015). Effective proof reading strategies for comprehending mathematical proofs. *International Journal for Research in Undergraduate Mathematics Education*, 1, 289-314.

Weber, K., & Mejía-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, 76(3), 329–344. doi:10.1007/s10649-010-9292-z

Zazkis, D. & Zazkis, R. (2016). Prospective teachers' conceptions of proof comprehension: Revisiting a proof of the Pythagorean Theorem. *International Journal of Science and Mathematics Education,* 14, 777-803.

Student Understanding of the Direction of Vector Dot Products Across Contexts and Levels

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As part of an effort to examine students' understanding of vector products, we present preliminary findings from student responses to survey questions asking whether the dot product has a direction in three different contexts and three different class levels. The contexts were two vectors with labels but no values, two vectors on a Cartesian grid with specified magnitudes and directions, and the flux of a uniform electric or magnetic field through a planar surface. The three courses were the second semester of an introductory calculus-based physics course, a sophomore-level mathematical methods in physics course, and a junior-level electricity and magnetism course. Depending on the context, between 25% and 30% of introductory students were successful at determining that the dot product has no direction. Performance increased with course level.

Keywords: Vector Products, Student Thinking, Pseudo-longitudinal

Introduction

Vectors are central to building a robust understanding of physics. Both conceptually and computationally, vectors play important roles through all levels of physics instruction. This has resulted in a significant body of literature addressing student understanding of vectors (e.g., Flores et al. (2004), Gire and Price (2014), and Nguyen and Meltzer (2003)). As is the case with the majority of physics education research, the majority of these studies are conducted in introductory calculus-based physics courses (Kanim & Cid, 2020). While some work has examined student understanding of vector fields (Bollen et al., 2017; Küchemann et al., 2021) and differential vector products (Topdemir et al., 2023; Walker & Dray, 2023) in the upper division, the evolution of students' ideas about vector products (dot and cross products) has not been thoroughly explored. This paper examines a preliminary study of the evolution of students' ability to identify properties of vector products at different levels and in different contexts. Data from a pseudo-longitudinal study is presented to demonstrate some aspects of how student performance is related to these factors.

Background

A significant amount of the work on student thinking about vectors has focused on student difficulties, with computational difficulties taking center stage. Mikula and Heckler (2013) examined students' ability to breakdown vectors into individual components by utilizing appropriate trigonometric operations. The decomposition of vectors into components has been shown to be challenging for students from a variety of introductory physics courses (algebra- and calculus-based, first and second semesters). Heckler and Scaife (2015) explored the effects of representation on student performance when adding and subtracting vectors, replicating previous findings on student difficulties adding vectors in the graphical (arrow) representation and identifying some novel difficulties. They also found that students performed significantly better using an algebraic representation than a graphical one. Bollen and colleagues (2017) examined student difficulties around the interpretation, construction, and translation between different representations of vector fields. They found that students' difficulties with addition and decomposition of vectors also impacted students' success transitioning between symbolic and graphical representations of vector fields and noted difficulties utilizing field line representations to appropriately represent field strength. A general lack of representational fluency was identified as a reason students may struggle with determining properties of fields such as the signs of the divergence and/or curl.

Zavala and Barniol (2013) investigated student understanding of the dot product in a graphical representation in introductory physics courses. This was done in three distinct contexts: mathematical (labeled "no context"), mechanical work, and electric flux. No significant difference was found between the two physical contexts, however there was lower performance on the no context problems than the others. Students also had significant difficulties connecting a conceptual understanding of the dot product to its formal representation. Barniol and Zavala (2014) subsequently produced the 20-item, multiple-choice Test of Understanding of Vectors (TUV), by considering difficulties reported by a large pool of previous studies and generating incorrect answers that tied directly to this literature. The TUV was found to track onto four primary factors for student difficulties: graphical properties, graphical procedures, geometric calculations, and unit vector notation calculations. However, the TUV did not include any physical contexts, and although the TUV did include some questions on the meaning of vector products, the majority of the items focused on procedures.

More recently, Carli and colleagues (2020) developed the Test of Calculus and Vectors in Mathematics and Physics (TCV-MP), examining differences in students' ability to answer questions in mathematical and physical contexts by asking 17 pairs of isomorphic questions. They found that students did not always use the same strategies to solve the isomorphic questions, implying that context influenced students' framing of the questions. Different difficulties were demonstrated by students between some question pairs. The researchers argue that differences in performance on some of the items suggest that success in physics is not solely dependent on mathematics preparation, but also necessitates students be able to appropriately blend the mathematical and physical reasoning in their problem solving. This is supported by other literature on student reasoning (e.g., Kuo et al. (2013) and Uhden et al. (2012)).

As part of a larger project to investigate student understanding of vector products at different levels and in different contexts, here we focus on recognition of features of the dot product. The research question we are pursuing here is: How, and in what ways, does student understanding of the direction of vector dot products differ between mathematics and physics contexts, and at different levels in the physics curriculum?

Methods

Data were collected via a four-item, online, multiple-choice survey, with three items focusing on dot products. While the result of the dot products are scalars and do not have a direction, it is not uncommon for physics students to ascribe directions to dot products and quantities described by dot products. Questions in the literature that ask students to calculate the dot product had both vector and scalars as part of the multiple-choice options (Carli et al., 2020). With common incorrect responses being vector options, it is unclear how students understand the nature of the operation of a dot product. Additionally, the selection of a calculated scalar response does not explicitly imply an understanding that the dot product has no direction. To address the vector versus scalar nature of vector products we asked about the magnitude and direction of the products in separate questions. Other questions from the literature include the words "vector" or "magnitude" in the answer (Barniol & Zavala, 2014). To avoid any cueing from specific vector-related vocabulary we asked, "Does the dot product $\vec{A} \cdot \vec{B}$ have a direction, and if so, what is it?"

Since the dot product results in a scalar quantity, the correct answer is, “It does not have a direction.” The answer choices were based on distractors from prior literature (Barniol & Zavala, 2014; Van Deventer, 2008). In all items students were presented with the appropriate formula for the dot product and asked whether it has a direction. A generalized list of the answer choices and abbreviated labels for presentation purposes can be found in Table 1. The same phrasing was used in all direction questions; only the vector labels were changed to reflect the specific context.

Table 1. Generalized multiple choice responses and abbreviated codes for the survey items.

Answer choices	Abbreviation
It does not have a direction	ND
In the direction of $\vec{A}$ (field vector $\vec{E}$ or $\vec{B}$ in the flux question)	A/FV
In the direction of $\vec{B}$ (area vector $\vec{A}$ in the flux question)	B/AV
In a direction between $\vec{A}$ and $\vec{B}$	BT
In the direction perpendicular to both $\vec{A}$ and $\vec{B}$ and into the page/screen	IP
In the direction perpendicular to both $\vec{A}$ and $\vec{B}$ and out of the page/screen	OP
Clockwise	CW
Counterclockwise	CCW
Not enough information to tell	NEI

The first two items used mathematical contexts. Students were provided with diagrams containing two vectors in space connected tail-to-tail, with labels $\vec{A}$ and $\vec{B}$. The vectors and labels were the only information provided in item 1 (Figure 1(a)); item 2 included a Cartesian coordinate grid and noted angles of each vector with respect to the positive *x* axis (Figure 1(b)). Item 1 asked about only the directions of the vector products $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$, and subsequent items asked about the direction and magnitudes of the vector products.

Item 3 used the physical context of electric or magnetic flux through a surface area (Figures 1(c) and 1(d)); the choice of field was course dependent. In either case the flux due to a uniform field through a planar surface is given by a dot product of the field and surface area vectors (e.g., $\Phi_E = \vec{E} \cdot \vec{A}$). The potential for both positive and negative flux through a surface can sometimes be conflated for flux having a direction rather than being indicative of source or sink of field.

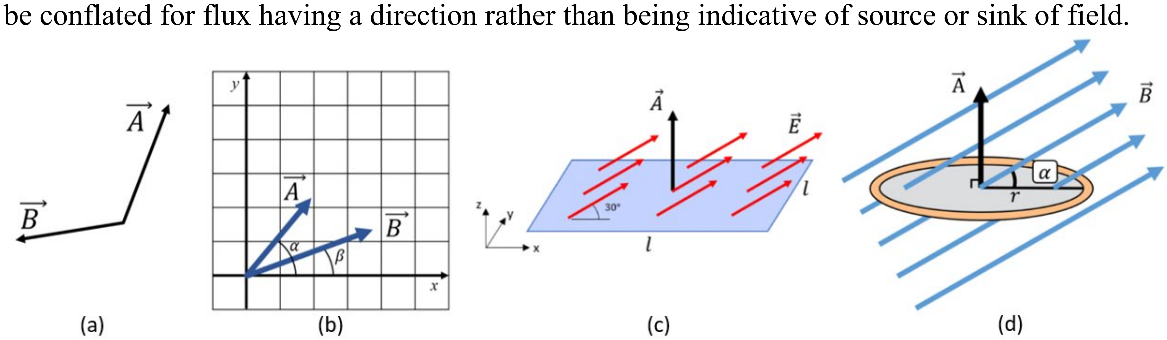


Figure 1. Diagrams provided in survey items asking about the direction of the dot product or the flux for the given situation. Diagrams for (a) item 1, (b) item 2, (c) item 3 for electric flux, and (d) item 3 for magnetic flux.

The survey was administered at a medium-sized public US university in three different physics courses: second-semester introductory calculus-based physics (N=153), sophomore-level mathematical methods in physics (math methods; N=14), and the second semester of junior-level electricity and magnetism (E&M; N=6). In each course, the survey was administered after instruction on the relevant mathematics or physics context(s). The context for item 3 for each course was most relevant to the context of instruction: magnetic flux in the introductory and E&M courses and electric flux in the math methods course.

To address the question of how student understanding of vector dot products differs at different levels of the physics curriculum we conducted a pseudo-longitudinal comparison between the three courses. For this comparison we limit the pool of introductory students to the physics and engineering physics majors who will go on to take the upper level courses (N=9).

Results

Performance across the three items related to dot products is presented for the introductory students before comparing the performance of students at different levels.

Introductory level

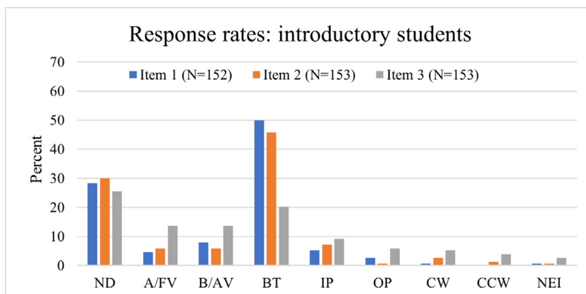


Figure 2. Introductory-level student performance on items 1, 2 and 3. See table 1 for answer choice descriptions.

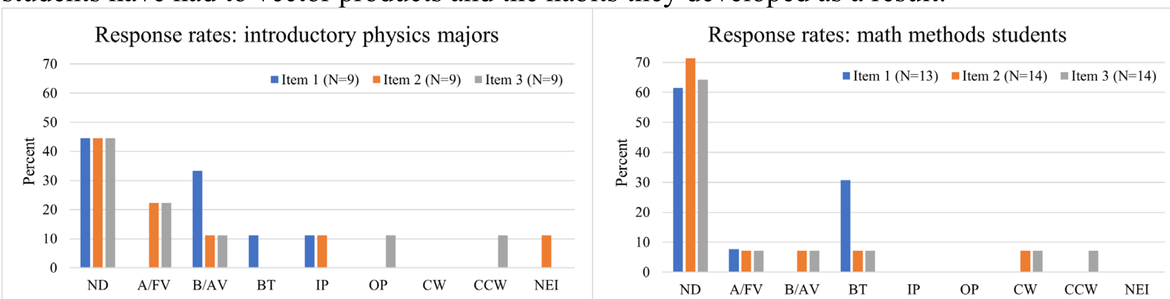
All three items had a similar percentage of introductory students selecting the correct response (Figure 2): 28% for item 1, 30% for item 2, and 25% for item 3. The most common choice for both items 1 and 2, 50% and 46% respectively, is that the direction of the dot product is between the two vectors (BT). This is consistent with the most common response seen by Van Deventer in a question asking which arrow, if any, describes the direction of the dot product of two given vectors (Van Deventer, 2008). Item 3, the flux question, had the most evenly distributed answer choices, suggesting less consistent understanding among students.

Similar percentages of introductory students selected the correct choice in each context, however only 16% of the introductory population selected the correct choice across all three items. Another 14% of students said the direction of the dot product is between the two vectors on all three items and 3% of students said that the direction of the dot product is in the direction of one of the two vectors (selecting either A/FV or B/AV) for all three questions. The remaining 67% of students did not answer consistently across all three contexts, which suggests that they do not have a coherent understanding of the dot product in multiple contexts.

Pseudo-longitudinal

The correct answer, "it does not have a direction," was the most common response on all questions through all three levels. At the introductory level, 44% of physics majors selected the correct answer on each item (Figure 3(a)). Between 62% and 71% of the math methods students chose the correct answer for each item (Figure 3(b)), and as expected, the junior E&M students had the highest performance of the three populations, with 100% correct in all three cases.

In all three levels, performance on the flux question, item 3, was similar to the performance on mathematics context questions, consistent with the similarities between isomorphic mathematics and physics questions in which mechanical work was used as the physics context conducted by Carli et al. (2020) and Van Deventer (2008). The most common incorrect answer in item 1 is different between the two lower courses: introductory students predominantly selected that the dot product is in the direction of $\vec{B}$, while math methods students selected that the dot product is between $\vec{A}$ and $\vec{B}$. The type of error may depend on how much exposure students have had to vector products and the habits they developed as a result.



(a) *(b)*

Figure 3. Performance by physics majors in (a) introductory and (b) math methods courses on items 1, 2 and 3. See Table 1 for answer choice descriptions.

Discussion and Conclusion

The performance of introductory students in this preliminary study (25-30%) is similar to end-of-semester introductory student performance on dot product direction questions in Van Deventer's study, with 25% of students in a mathematics context and 24% of students in a physics context selecting no direction (Van Deventer, 2008). The most common incorrect answer, a direction between $\vec{A}$ and $\vec{B}$, could be based on properties of vector addition. Some student explanations explicitly reference the tip-to-tail method, which is a strategy for vector addition not multiplication. For the introductory students, there seemed to be a wider variety of responses to the flux question than to the mathematics context questions. Asking these questions in other physics contexts (e.g., mechanical work) could document the relative difficulty of flux as a physics context for considering the dot product direction.

The trend of improved results as the course level increases is expected in the pseudo-longitudinal study, but the performance is not as good as one might expect, especially in the sophomore math methods course. Vectors and vector operations are introduced in the introductory sequence and proficiency in subsequent courses is almost assumed. A 100% correct response rate on all questions in junior E&M is a welcome result, but to promote student understanding of physics concepts in intermediate and upper division courses, vector operations need to be second nature as soon as possible, not after over half of the physics curriculum. While we see that student performance improves as we look at higher-level courses, a limitation of this study is the small sample size in each course.

Future work includes collecting more survey data, both internally and at other institutions, conducting interviews to elicit more detailed explanations of student reasoning, and connecting empirical data to existing frameworks.

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References

Barniol, P., & Zavala, G. (2014). Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics - Physics Education Research*, *10*(1), 1–14. https://doi.org/10.1103/PhysRevSTPER.10.010121

Bollen, L., Van Kampen, P., Baily, C., Kelly, M., & De Cock, M. (2017). Student difficulties regarding symbolic and graphical representations of vector fields. *Physical Review Physics Education Research*, *13*(2), 1–17. https://doi.org/10.1103/PhysRevPhysEducRes.13.020109

Carli, M., Lippiello, S., Pantano, O., Perona, M., & Tormen, G. (2020). Testing students ability to use derivatives, integrals, and vectors in a purely mathematical context and in a physical context. *Physical Review Physics Education Research*, *16*(1), 10111. https://doi.org/10.1103/PhysRevPhysEducRes.16.010111

Flores, S., Kanim, S. E., & Kautz, C. H. (2004). Student use of vectors in introductory mechanics. *American Journal of Physics*, *72*(4), 460–468. https://doi.org/10.1119/1.1648686

Gire, E., & Price, E. (2014). Arrows as anchors: An analysis of the material features of electric field vector arrows. *Physical Review Special Topics - Physics Education Research*, *10*(2), 1–11. https://doi.org/10.1103/PhysRevSTPER.10.020112

Heckler, A. F., & Scaife, T. M. (2015). Adding and subtracting vectors: The problem with the arrow representation. *Physical Review Special Topics - Physics Education Research*, *11*(1), 1–17. https://doi.org/10.1103/PhysRevSTPER.11.010101

Kanim, S., & Cid, X. C. (2020). Demographics of physics education research. *Physical Review Physics Education Research*, *16*(2).

Küchemann, S., Malone, S., Edelsbrunner, P., Lichtenberger, A., Stern, E., Schumacher, R., Brünken, R., Vaterlaus, A., & Kuhn, J. (2021). Inventory for the assessment of representational competence of vector fields. *Physical Review Physics Education Research*, *17*(2), 20126. https://doi.org/10.1103/PhysRevPhysEducRes.17.020126

Kuo, E., Hull, M. M., Gupta, A., & Elby, A. (2013). How students blend conceptual and formal mathematical reasoning in solving physics problems. *Science Education*, *97*(1), 32–57. https://doi.org/10.1002/sce.21043

Mikula, B. D., & Heckler, A. F. (2013). Student Difficulties with Trigonometric Vector Components Persist in Multiple Student Populations. *2013 PERC Proceedings [Portland, OR, July 17-18, 2013], Edited by P. V. Engelhardt, A. D. Churukian, and D. L. Jones*. https://doi.org/10.1119/perc.2013.pr.051

Nguyen, N.-L., & Meltzer, D. E. (2003). Initial understanding of vector concepts among students in introductory physics courses. *American Journal of Physics*, *71*(6), 630–638. https://doi.org/10.1119/1.1571831

Topdemir, Z., Thompson, J. R., & Loverude, M. E. (2023). How students reason with derivatives of vector field diagrams. *The Learning and Teaching of Calculus Across Disciplines Conference Proceedings*. https://doi.org/10.1111/j.1365-201X.1994.tb10660.x

Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G. (2012). Modelling Mathematical Reasoning in Physics Education. *Science and Education*, *21*(4), 485–506. https://doi.org/10.1007/s11191-011-9396-6

Van Deventer, J. (2008). Comparing student performance on isomorphic math and physics vector representations. *Electronic Theses and Dissertations.*, *1348*. https://doi.org/https://digitalcommons.library.umaine.edu/etd/1348

Walker, M., & Dray, T. (2023). Instances of confounding when differentiating vector fields. *The*

Learning and Teaching of Calculus Across Disciplines Conference Proceedings.
Zavala, G., & Barniol, P. (2013). Students' understanding of dot product as a projection in no-context, work and electric flux problems. *2012 PERC Proceedings [Philadelphia, PA, August 1-2, 2012], Edited by P. V. Engelhardt, A. D. Churukian, and N. S. Rebello*, *1513*, 438–441. https://doi.org/10.1063/1.4789746

Multivariational Reasoning in Linear Algebra:
How Mathematicians Reason about Linear Transformations

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Student difficulties in introductory linear algebra courses are often attributed to the novelty of the concepts in the course and the disconnectedness of these concepts to students' prior mathematical experiences. However, researchers have stated that a strong prerequisite understanding of the function concept is essential to students' understanding of linear transformations in linear algebra. In calculus and related courses, covariational and multivariational reasoning has been determined to be necessary for students to appropriately reason about functions in two or more variables. However, research on multivariational reasoning in linear algebra, especially with respect to linear transformations, is scarce. To contribute to the literature, we designed a study exploring a hypothetical model of how students might come to reason multivariationally about linear transformations. In this preliminary report, I discuss a part of the study – interviews with seven mathematicians who have either taught linear algebra or used linear algebra in their research.

Keywords: Linear Algebra, Linear Transformations, Multivariational Reasoning

Linear algebra is an important topic for many STEM students, including those studying computer science, data science, engineering, mathematics, and physics (Stewart et al., 2022). Recently, the second Linear Algebra Curriculum Study Group (LACSG 2.0) published recommendations for linear algebra instruction, taking into account advances in the field, industry, and technology over the last 30 years. Among the recommendations was the removal of calculus as a prerequisite for introductory linear algebra courses (Stewart et al., 2022).

In response to recommendations put forth by the first Linear Algebra Curriculum Study Group in 1993, Dubinsky (1997) claimed that students' struggles with mathematical concepts that are not necessarily part of linear algebra curricula but that are foundational to understanding linear algebra topics are one of the primary sources of students' difficulties in the course. In particular, Dubinsky noted that a strong prerequisite understanding of the function concept is essential to students' understandings of linear transformations. Additionally, Dorier et al. (2000) attributed students' conceptual difficulties in linear algebra to their difficulties with the "avalanche of new words, new symbols, new definitions, and new theorems" (p. 95) and the lack of connection to what students have already encountered in their mathematical experiences. Considering these views, it is important to connect students' learning of the linear transformation concept to students' prior experiences with the function concept, especially if linear algebra precedes calculus in the college curriculum.

There exist studies exploring the connection between students' understandings of the function concept and their learning and understanding of linear transformations (Andrews-Larson et al., 2017; Bagley et al., 2015; Zandieh et al., 2012; Zandieh et al., 2017). Much of this work has focused on a comparison of the concept images that students come to possess for functions and linear transformations. As part of a larger study, Zandieh et al. (2017) characterized student responses into three categories of mathematical structures (computations, properties, and clusters of metaphorical expressions) to compare their concept images of these concepts. In their work, the researchers found that "a reliance on properties appeared to impede students' development of a unified concept image of function while an ability to draw on metaphors facilitated such a development" (p. 36). That is, when students leveraged metaphorical linguistic expressions in their reasonings they were more successful in understanding various functions and transformations as examples of the same phenomenon.

In calculus and related coursework, covariational reasoning has been determined to be a vital way of thinking for understanding of functions (e.g., Castillo-Garsow, 2012; Oehrtman et al., 2008; Tallman et al., 2021; Thompson & Carlson, 2017; Thompson & Harel, 2021). Covariational reasoning encompasses the mental actions involved in reasoning about how two related quantities change in tandem with respect to perceptual time. When able to engage at the highest levels of covariational reasoning, a student can envision changes in one quantity's value as happening simultaneously with changes in another quantity's value and envision both quantities varying smoothly and continuously. Being able to coordinate the dynamic relationship between two quantities as they covary is crucial for students to be able to reason about concepts relating to functions, including derivative and accumulation (Silverman, 2017).

In linear algebra, Turgut (2019) utilized a dynamic geometry environment (DGE) to promote student understanding of matrix representations of geometric transformations as representing co-variation of inputs and outputs, consistent with a covariational view of functions. A review of the literature reveals that this study is the only one to-date that focuses on conceptualizing co-variation of two mathematical objects in linear algebra. However, the covariational reasoning framework was not utilized as a lens through which the author analyzed the data. That is, the study was not concerned with *how* or *to what extent* the students reasoned covariationally. Rather, Turgut (2019) employed the use of a semiotic mediation lens to investigate students' development of a conceptualization of matrices as representing geometric transformations, where treating a function as co-variation of independent and dependent variables was the byproduct of the use of the DGE.

More recently, there has been a greater focus on students' reasoning about the simultaneous change in more than two variables, excluding conceptual time as mediator, in calculus and other STEM subjects; this reasoning has aptly been referred to as *multivariational reasoning*. The development and evolution of multivariational reasoning as a framework of mental actions is rooted in the conceptual analyses reported on by Jones (2018). This theoretical report was "meant to form the basis of future empirical work" (Jones, 2018) centered around multivariational reasoning. Jones's first conceptual analysis was focused on analyzing a large set of functions and formulas derived from a collection of STEM textbooks. In particular, Jones considered how the variables in these functions and formulas "could be conceptualized as changing with respect to one another" (p. 1111) simultaneously and interdependently (Castillo-Garsow, 2012; Jones, 2018). The analysis included an identification of four potential types of multivariation inherent to mathematical functions and formulas: *independent multivariation*; *dependent multivariation*; *nested multivariation*; and *vector multivariation*.

To date, no studies have employed use of the multivariational reasoning frameworks to linear algebra concepts outside of vector field tasks presented in the context of differential equations (Jones & Kuster, 2021; Kuster & Jones, 2019). However, there are implications for the teaching and learning of linear algebra concepts when considering vector multivariational reasoning, even when viewed as the composition of parallel independent multivariations. Jones and Jeppson (2020), in particular, noted many similarities in the various types of reasoning within multivariation contexts. And "that it might not be necessary for students to learn about each type of multivariation separate from the others. By learning to reason about one type, they may simultaneously be developing reasoning abilities that transfers to other types" (p. 1146).

If it is indeed the case that the mental actions involved in multivariational reasoning are transferrable across the different types, then it may be that multivariational reasoning could serve as the catalyst for connecting students mathematical reasoning across courses and disciplines. In particular, the teaching of linear algebra concepts, such as linear transformations, from a multivariational reasoning lens might provide additional support and opportunities to make connections to prior mathematical experiences and concepts, such as reasoning covariationally about the function concept. In this study, I aim to explore this idea by addressing the research question: *What would be a hypothetical model of how students might come to reason multivariationally about linear transformations?*

Theoretical Underpinnings

Quantitative reasoning, "conceptualizing a situation in terms of quantities and relationships among quantities" (Carlson et al., 2002, p. 425), is an important component of students' mathematical learning. In this framework, a student has conceived of a quantity when they have conceived of a measurable attribute of some object; in this way, the quantity is idiosyncratic to the student conceiving of it (Thompson et al., 2017). When a student mentally unites the attributes of two or more quantities to make a new conceptual object that simultaneously represents both original quantities, this new object is referred to as a *multiplicative object*. When the variations of two quantities, coordinated to form a multiplicative object, are conceptualized in tandem, we call this *covariational reasoning*.

The idiosyncratic nature of quantity under these framings has allowed for the identification of mental actions pertaining to increasingly sophisticated levels of covariational reasoning. For example, the first identifiable mental action in the covariational reasoning framework is "coordinating the value of one variable with changes in another" (Carlson et al., 2002). Similarly, researchers have begun identifying the mental actions pertaining to increasingly sophisticated levels of multivariational reasoning (e.g., Jones, 2018). In working to identify potential types of multivariation and associated mental actions, Jones (2018) conducted two conceptual analyses.

Conceptual analysis is a tool that leverages the tenet of radical constructivism that knowledge persists because it has proven viable in the knower's experiences. Because, as mathematics education researchers, we seek to improve the learning attained by all who study mathematics, the purpose of a conceptual analysis is to develop a model of how a concept may be structured, ways of knowing the concept that may be favorable to learners, or to develop models of what learners actually know at a moment in time and understand in specific situations (Glasersfeld, 1995; Thompson, 2008). In discussing the role of conceptual analyses in developing hypothetical models of a concept and how students might come to reason about it, von Glasersfeld (1995) stated:

> [I]t is indispensable to have a fairly explicit model of what these concepts might be in the adult. Mathematics textbooks are not very illuminating in that regard and philosophers of mathematics rarely stoop to say anything about the conceptual raw material of their construction. (p. 161)

Accordingly, I designed a study to investigate a hypothetical model of how students might come to reason multivariationally about linear transformations by first performing an initial conceptual analysis leveraging several undergraduate linear algebra textbooks. This analysis guided my work on designing interview protocols to elicit multivariational mental actions of mathematicians.

Methods

In this report, I share a part of a larger study in which I investigate mathematicians and students' multivariational reasoning in linear algebra, with an aim of developing a hypothetical model for how students might come to reason about linear transformations. To achieve this goal, I first conducted interviews with mathematicians. The purpose of the interviews with mathematicians was to identify the mental actions that they call upon when reasoning multivariationally about linear transformations. Then, I plan to refine the interview protocols to conduct interviews with students to again identify the mental actions that students call upon when reasoning multivariationally about linear transformations. With the results of from these data sets, I aim to develop a hypothetical model for how students might come to reason about linear transformations.

At this stage in the project, seven mathematicians each participated in two 60-minute, one-on-one, semi-structured interviews centered around the concept of linear transformation. Mathematicians were considered suitable for participation in the study provided they had previously taught a course in linear algebra, or their research had leveraged linear algebra knowledge.

The first interview was used to familiarize the researcher with each mathematicians' background in linear algebra and determine how the mathematician participants reasoned multivariationally about linear transformations represented algebraically. These algebraically represented transformations were determined to be more similar in appearance to functions that a student might have prior exposure to before taking an introductory linear algebra course. For example, after being asked which given transformations could be linear, mathematicians were each asked to describe the relationship between x and $T(x)$ for the transformation $T(x) = (x, -3x)$. If there was no discussion of varying quantities, then a follow-up question at this point was "What would happen to $T(x)$ as you changed/varied the value of x?". This prompting was used to elicit multivariational reasoning, so that the mental actions of multivariational reasoning that mathematicians engaged in could be identified during data analysis and used to inform the hypothetical model for students' multivariational reasoning.

In the subsequent second interview for each mathematician, participants were similarly asked to reason about matrix transformations and geometric transformations; an example of each is shown in Figure 1. For each type of representation, mathematicians were first asked which given transformations could represent linear transformations and were then asked to reason about the relationships between inputs and outputs for particular transformations. During task design, it was hypothesized that reasoning about the images of the standard basis vectors and the domain and range of a linear transformation might help students reason more powerfully about the relationship between the inputs and outputs of the transformation. This led to development of sub-questions ii. and iii. for the matrix transformation tasks (Figure 1). These sub-questions were given iteratively to the mathematicians, provided the ideas had not been brought up authentically by the participant while engaging in previous parts of the task. Throughout both interviews, each mathematician was frequently asked to reflect on anticipated student difficulties with each task.

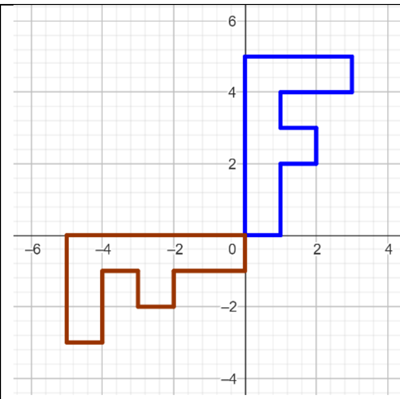
Given the matrix $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$, determine whether the transformation given by $T(x) = Ax$ could be a linear transformation.

i. What happens to the output vector as you vary the input vector?

ii. What does A do to a vector?

   a. What does A do to the basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? To $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

iii. What is the domain and range of the transformation?



What effect does the transformation have on an arbitrary vector?

Figure 1. An example of a matrix transformation task (left) and a geometric transformation task (right) given to mathematician participants during their second interviews.

During each interview, an over-the-shoulder camera was used to collect data relating to mathematicians' physical cues (e.g., hand gestures, motions, and written work). Mathematicians' written works were captured in real time using screen recording on the interviewer's iPad Pro®; data from the camera and screen recording for each interview was then superimposed for the proposed of data analysis.

In the initial coding of the data, I am identifying and categorizing mathematicians' reasonings (e.g., reasoned geometrically) and potential related mental actions (e.g., recognized one quantity's value as being dependent on another) from the overlayed video and transcribed audio. Next, I will identify properties of these categories (e.g., whether a mental action was task or representation specific). Following this initial analysis and coding of mathematicians' reasonings, I will perform a member check with each participant to ensure that their views and reasonings are accurately reflected. During axial coding, subcategories of mental actions will be related to categories of mental actions to which they are subordinate to or supporting. I will also examine subcategories of mathematicians' mental actions to determine whether any categories are repetitive and can be removed, condensed, or combined.

Mathematicians' responses to anticipated student difficulties for each task will also be coded thematically and compared to existing literature on students' difficulties. Insight from mathematician responses and results from data analysis will be used to modify interview tasks and protocols ahead of student interviews.

Discussion

I will present data analysis results from the interviews at the conference. In preliminary initial coding, I observed that some mathematicians focused on varying the input of a transformation component-wise by small arbitrary changes to determine the effect this had on the corresponding output vector. It seems that their reasonings included recognizing in/dependence and coordination mental actions. The audience will be asked to discuss some interview questions to elicit additional mental actions that I may have not observed in my data set.

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References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, *48*(6), 809–829. https://doi.org/10.1080/0020739X.2016.1276225

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: The case of function and linear transformation. *The Journal of Mathematical Behavior*, *37*, 36-47.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context*, *2*, 55–73.

Dubinsky, E. (1997). Some Thoughts on a First Course in Linear Algebra at the College Level. In D. Carlson, C. Johnson, D. Lay, A. Porter, & A. Watkins, *Resources for teaching linear algebra* (pp. 85–106). Mathematical Association of America.

Jones, S. R. (2018). Building on covariation: Making explicit four types of “multivariation.” *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1110–1118.

Jones, S. R., & Jeppson, H. P. (2020). Students’ reasoning about multivariational structures. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1139–1147). Cinvestav / AMIUTEM / PME-NA.

Jones, S. R., & Kuster, G. E. (2021). Examining students’ variational reasoning in differential equations. *The Journal of Mathematical Behavior*, *64*, 100899. https://doi.org/10.1016/j.jmathb.2021.100899

Kuster, G. E., & Jones, S. R. (2019). *Variational Reasoning Used by Students While Discussing Differential Equations*.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational Reasoning Abilities that Promote Coherence in Students’ Function Understanding. *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, *73*, 27.

Silverman, J. (2017). Supporting Teachers’ Understandings of Function through Online Professional Development. *Journal of Computers in Mathematics and Science Teaching*, *36*(1), 17–39.

Stewart, S., Axler, S., Beezer, R., Boman, E., Catral, M., Harel, G., McDonald, J., Strong, D., & Wawro, M. (2022). The Linear Algebra Curriculum Study Group (LACSG 2.0) Recommendations. *Notices of the American Mathematical Society*, *69*(05), 1. https://doi.org/10.1090/noti2479

Tallman, M. A., Reed, Z., Oehrtman, M., & Carlson, M. P. (2021). What meanings are assessed in collegiate calculus in the United States? *ZDM – Mathematics Education*, *53*(3), 577–589. https://doi.org/10.1007/s11858-020-01212-3

Thompson, P. W. (2008). Conceptual Analysis of Mathematical Ideas: Some Spadework at the Foundation of Mathematics Education. *Plenary Paper Presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education*, *1*, 31–49.

Thompson, P., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM – Mathematics Education*, *53*(3), 507–519. https://doi.org/10.1007/s11858-021-01270-1

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational Reasoning Among U.S. and South Korean Secondary Mathematics Teachers. *Journal of Mathematical Behavior*, *48*, 95–111.

Turgut, M. (2019). Sense-making regarding matrix representation of geometric transformations in R 2: A semiotic mediation perspective in a dynamic geometry environment. *Zdm*, *51*(7), 1199-1214.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Zandieh, M., Ellis, J., & Rasmussen, C. (2012). Student concept images of function and linear transformation. In *Proceedings of the 15th annual conference on research in undergraduate mathematics education* (Vol. 1, pp. 524-532).

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, *95*(1), 21-38.

Learning to Teach Teachers: Community College Faculty Explore Fraction Tasks for Teaching

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Teaching mathematics for future elementary teachers is fundamentally different from other forms of mathematics and thus requires different knowledge. As community colleges become increasingly involved in the process of training future teachers, it is essential to explore how instructors at these institutions develop as mathematics teacher educators. This paper reports on a preliminary exploration of how community college faculty grappled with teaching-oriented mathematical tasks involving fractions. Choices of mathematical representation, selection of answer before and after discussion, and overall themes are discussed, with a focus on development of mathematical content knowledge for teaching.

Keywords: Professional Development, Math Teacher Educators, Preservice Elementary Teachers

Mathematics teacher educators (MTEs) carefully consider the preparation and development of preservice elementary teachers (PTs). Though four-year institutions have predominantly prepared students for teaching and provide certification/licensure for elementary teachers through credential programs (Masingila et al., 2012), community colleges have begun to focus attention on widening the teacher preparation pipeline "as more students [turn] to them to take required mathematics and education courses" (Blair et al., 2018, p. 185). Masingila et al. (2012) found in a survey of 207 two-year college math departments that over 80% offered math content courses for PTs, implying that "two-year schools play a key role in the mathematical preparation of teachers" (p. 352).

Our study is framed by the perspective that teaching math for future teachers entails a fundamentally different approach than teaching other math courses, as learning to teach math requires different and complex ways of understanding (Ball et al., 2008). Just as teachers of math require a knowledge of math different than those *not* engaged in teaching, MTEs require knowledge of teaching mathematics that is developed and held in a way different than how teachers know it (Beswick & Goos, 2018). While the content being taught in elementary math content courses may appear simple, conceptual meaning underlying topics, addressed at both the level of the PT and the future elementary school student, is deceptively complex. Masingila et al. (2012) argue that "instructors teaching mathematics content courses designed for [PTs] may not be prepared to teach those courses in ways that will provide the type of mathematical support needed by [PTs]" (p. 355). While faculty may hold strong mathematical knowledge, many have not had extensive pedagogical training nor training for how to be a MTE. This paper focuses on the following question: How do community college math faculty reason through teaching-oriented mathematical tasks involving fractions?

Methods

This paper focuses on the mathematical work collected from a one-week professional development (PD) of 15 math faculty who are developing MTEs. None of the faculty had specific training as MTEs prior to the PD. Ten of the faculty were full-time math instructors from three community colleges, while six were part-time math instructors at a university and/or community college. Teaching experience ranged from two years to over 20 years, with over half of participants having some kind of K-12 teaching experience. Five instructors had taught a math course for PTs at least once before, while ten instructors had never taught a math course for PTs and therefore had never engaged with the ways that PTs think mathematically. All participants expressed a desire to develop their understanding of how to teach mathematics content in a first mathematics course for PTs.

Each morning, faculty engaged in a selected task from the Learning for Mathematics Teaching (LMT) Project from the University of Michigan (Hill et al., 2004). These LMT tasks were designed to be used in many different contexts. For purposes of the PD, we used tasks as "open-ended prompts which allow for the exploration of teachers' reasoning about mathematics and student thinking." (Hill et al., 2004, p. 2). We utilized a total of five tasks, each focusing on a mathematical topic related to the PD activities for that day. This paper highlights participants' responses for two of these tasks, shown in Figure 1. Task 1 targets the knowledge needed by a teacher to develop children's reasoning about comparing and ordering fractions. Task 2 demonstrates a task related to fractions, providing sequences of questions that may help a child determine how many 4s are in 3.

Task 1: Comparing and Ordering Fractions

Mr. Foster's class is learning to compare and order fractions. While his students know how to compare fractions using common denominators, Mr. Foster also wants them to develop a variety of other intuitive methods.

Which of the following lists of fractions would be best for helping students learn to develop <u>several</u> different strategies for comparing fractions?

a) $\frac{1}{4}$ $\frac{1}{20}$ $\frac{1}{19}$ $\frac{1}{2}$ $\frac{1}{10}$

b) $\frac{4}{13}$ $\frac{3}{11}$ $\frac{6}{20}$ $\frac{1}{3}$ $\frac{2}{5}$

c) $\frac{5}{6}$ $\frac{3}{8}$ $\frac{2}{3}$ $\frac{3}{7}$ $\frac{1}{12}$

d) Any of these would work equally well for this purpose

Task 2: How many 4s in 3?

Mrs. Brockton assigned the following problem to her students: How many 4s are there in 3?

When her students struggled to find a solution, she decided to use a sequence of examples to help them understand how to solve this problem. Which of the following sequences of examples would be <u>best</u> to use to help her students understand how to solve the original problem?

a) How many:
4s in 6?
4s in 5?
4s in 4?
4s in 3?

b) How many:
4s in 8?
4s in 6?
4s in 1?
4s in 3?

c) How many:
4s in 1?
4s in 2?
4s in 4?
4s in 3?

d) How many:
4s in 12?
4s in 8?
4s in 4?
4s in 3?

Figure 1: Task 1 and Task 2 from the LMT sample tasks.

Participants were first given five minutes individually to review the task, select a response, and explain their reasoning. In groups of four, participants were then given eight minutes to discuss the question with their peers before we discussed as a whole group. Finally, we asked the participants to reflect on their thinking after group discussion. Fifteen people responded to Task 1 and 13 responded to Task 2. This paper discusses the written reflections from participants. Data were analyzed through constant comparison analysis (Corbin & Strauss, 2008). All authors first read through each individual response, noting which choices were selected initially and after group discussion and notable themes from their reasoning. This guided our second read which focused on all participants' initial choice reasoning. Our third read focused on all participants' final choice reasoning. During the second and third read, we specifically focused on participants' use of language, inclusion of visual representations, and overarching themes in their reasoning.

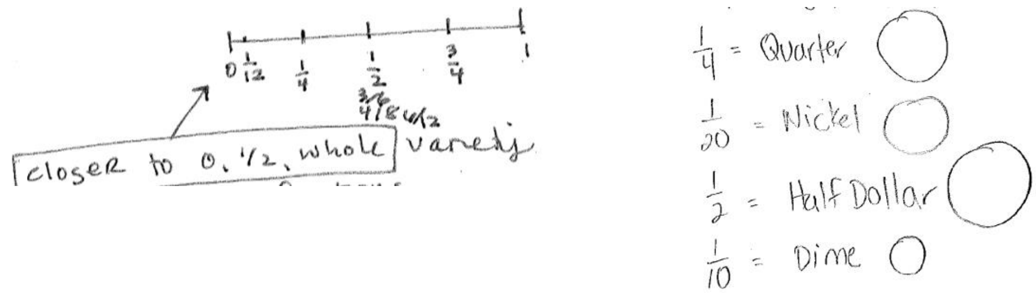
Findings

Findings are discussed for each task, with a focus on language/vocabulary participants used to explain their reasoning, types of visual mathematical representations provided in the response, and participants' initial and final answer selection.

We first discuss Task 1. Some language trends used by the participants were the words: *unit fraction*, *size*, *easiest*, and *variety*. Initially, option A was the most selected answer. Eight of the 15 participants identified (in some way) the list of fractions as unit fractions. Nine participants referred to the size of the fractions in option A, indicating that option A could help students focus on the value in the denominator and its meaning. One participant, Heidi, noted that "it is important for students to understand the 'size' of fractions first and also determine how the denominator affects the size of a fraction before comparing them." Across participants, there was significant overlap between thinking in terms of *unit fraction* and *size*, implying that the only difference in the fractions in option A was the denominator, which may help children determine the size of the fraction. Four participants labeled option A as *easy,* mentioning that the values were "easiest to compare." For example, Stephanie wrote, "Beginning by understanding fractions with 1 in the numerator makes comparing them easier to access," further sharing that most of the fractions "can be re-written with a common denominator of 20 fairly easily." The idea of *variety* was also something participants discussed, although it was not clear what they meant by the word. Four participants selected option C and one participant option B because of variety in the numerators and denominators. One participant initially liked option C because of the variety of prime and composite values in the denominators. The participant who initially chose option B changed to option C, this time mentioning variety to describe difficulty. Variety was also used to describe the many strategies that could be used to help students compare fractions.

Four participants drew diagrams in their response, while three provided real-life contexts related to the fractions in the options. The types of diagrams drawn included a tape diagram, fraction circles, a number line, and a coin model. For example, one participant drew a number line from 0 to 1, showing tick marks for 1/12, 1/4, 1/2, and 3/4 (see Figure 2a). Another bridged the idea of student knowledge and fractions in option C, stating that the denominator in 3/7 could represent days in a week or the denominator in 3/8 could represent the number of slices in a medium pizza. Courtney drew a diagram of coins to connect the denominators in option A (Figure 2b), which included different American coins in relation to a one dollar whole, drawing them to their relative size, noting that 1/19 caused a challenge for this model, but that students could compare it to a nickel.

There were substantial changes to the final selection in Task 1, with 12 participants changing their selection after group discussion. Initially, eight participants selected option A, three selected option B, two selected option C, three selected a combination of options, and one was unclear on their selection. After discussion, many participants selected a different option, and many struggled to select one option. Overall, 11 participants chose option C in some way, and *no* participants selected option A. Most participants' choices included some type of conditional statement, which indicated that they liked an option, but with some adjustments. One participant stated that she would pick options A, B, and C, sharing she would start with option A, move to option C, and then depending on class time, also use option B. It appeared that participants were highly open to hearing the perspective of others and multiple ways of thinking of the same problem. Participants noted that option C included many interesting values, could highlight the concept of benchmark fractions, and how the values could be organized in relation to closeness to 0, 1/2, and 1, as shown in Figure 2a.

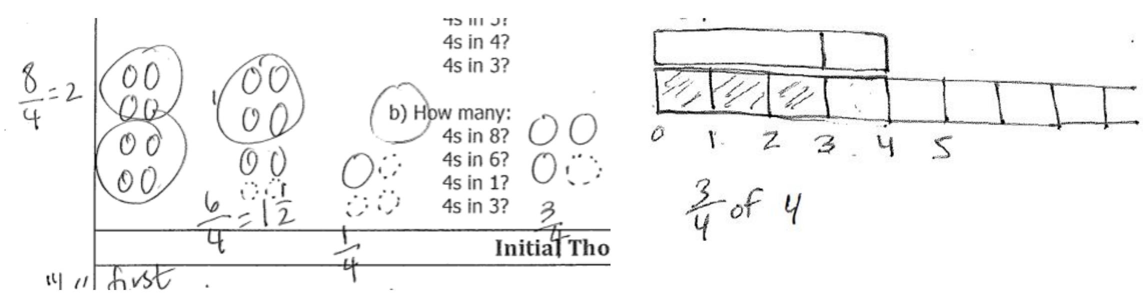


(a) (b)

Figure 2. (a) Diagram of Jane's number line. (b) Diagram of Courtney's thinking.

Next, we discuss Task 2, which asked for the best sequence to help students understand the concept of "4s in 3." Participants responding to Task 2 differed in how they conceptualized the problem, either as a division problem, fraction problem, or both. From the 13 participants, only one viewed the problem as solely a *division* problem, stating that option D led students "to recognize the use of division." Eight participants had writing indicating that they viewed this problem as a *fraction* problem, using language like whole, mixed number, or unit fraction, or by writing fractions. Four participants used language that indicated thinking of the problem as pertaining to both division and fractions.

Six participants utilized diagrams in Task 2, three of which had also drawn a diagram in the first task. One participant, Mara, had a visually distinct drawing for Task 2 involving a discrete model for her initial choice, option B, showing circles in groups of four with dotted lines to indicate fractional parts (see Figure 3a). The other four participants, in contrast, used number lines and tape diagrams. Figure 3b shows Patricia's use of a tape diagram to visually demonstrate that there should be less than one 4 in 3.



(a) (b)

Figure 3. (a) Mara's discrete drawing representing option B. (b) Patricia's tape diagram showing 4's in 3.

Task 2 also had a high number of people unable to choose one option after group discussion. Eight people had more than one answer listed, with six settling on options B and D. Reasoning included comments like "I think both explanations are valid" and "Perhaps I would use a combination of B and D," often showing slight preference for one or the other but not making a clear decision. Furthermore, one person had no answer listed but instead wrote, "context matters," with a list of the different factors that would affect a teacher's decision for which response to select, such as grade level and whether the teacher used discovery learning. Two more participants also did not choose an answer after group discussion, with one participant only including their ideal list of nine examples that would help students to understand the concept. In total, there were 11 participants who struggled to pick an answer in some way.

Due to the general indecisiveness of the group, participants were given the opportunity to create an ideal list of four example problems to lead up to the "4s in 3" question, creating a list of five examples. Twelve participants had a mixture of examples from options B and D, with six people from this group adding in the example "4s in 2" into the list. One person had a visually distinct list that was sourced from option C, using the same problems but instead listed as 4s in 4, 4s in 1, 4s in 2, then 4s in 3. Three people had lists that were longer than five examples, and after some large group discussion one participant remarked that in his classroom, he could give as many problems as he wanted. Rather than settling on a specific answer, the group felt relieved to create their own lists, agreeing that no option in Task 2 gave the "best" sequence of examples.

Discussion

The findings described above highlight both the challenges and affordances that may be leveraged in training community college faculty to become effective MTEs. Participants' work with two LMT tasks revealed differences in how they described and understood fractions and division, choices of visualization, and how they incorporated their experience into the work.

One of the most surprising features highlighted across both tasks was participants' limited use of visualizations. Very few participants provided number lines, tape diagrams, or other visuals in their work. Within their explanations and in discussions, many participants noted the importance of visualization when working with fraction ideas yet did not include a diagram themselves. It remains unclear whether this was due to a lack of need for diagrams personally or a perception that diagrams were not required when explaining to their peers. It has been shown that MTEs struggle to know when and how to incorporate visual fractional representations (Petit et al., 2016); without evidence of visualizations in the responses, it was not yet clear whether the MTEs had developed knowledge on how visualizations may support student thinking.

Across both tasks, it became clear that participants had a difficult time selecting a single answer after the whole group discussions. Many participants placed a strong emphasis on instructional context, providing qualifiers next to multiple choices. This indecisiveness and need for additional information were likely influenced by the background of the participants. As experienced teachers, they were acutely aware of the need to adapt materials to the specific class. They were also willing to engage in group discussions with colleagues and subsequently adapt or modify their choices - a necessary component of learning new ideas and building a community amongst their peers. These changes support the idea that participants were beginning to reshape their knowledge of teaching PTs, especially in the context of fractions and division, and were open to substantive change in their practice as future MTEs.

Questions for further discussion: (1) How have others transitioned from mathematicians to MTEs, specifically around elementary-school mathematics content? (2) How might lessons learned in this space transfer to teaching and learning in other mathematics courses?

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References

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. Journal of Teacher Education, 59(5), 389-407.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here?. *Journal of Mathematics Teacher Education, 21,* 417–427.

Blair, R. M., Kirkman, E. E., & Maxwell, J. W. (2018). S*tatistical abstract undergraduate programs in the mathematical sciences in the United States: 2018 CBMS survey.* American Mathematical Society.

Corbin, J., & Strauss, A. (2008). Basics of qualitative research: Techniques and procedures for developing grounded theory (3rd ed.). Thousand Oaks, CA: Sage Publications.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11–30.

Masingila, J. O., Olanoff, D. E., & Kwaka, D. K. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education*, *15*(5), 347–358.

Petit, M. M., Laird, R. E., Ebby, C. B., & Marsden, E. L. (2016). A focus on fractions: Bringing research to the classroom. Routledge.

Barriers and Drivers to Implementing GTA Professional Development focused on Active Learning, Equity, and Inclusivity

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Mathematics Graduate Teaching Assistants (MGTAs) play a significant role in undergraduate mathematics education, as they are often in a teaching-related position for courses in the College Algebra through Calculus II sequence. Further, research shows that teaching practices that promote active learning, equity, and inclusion lead to improved student outcomes (e.g., Freeman et al., 2014; Laursen & Rasmussen, 2019; Mulnix, Vandegrift, and Chaudhury, 2016). However, MGTA PD that reflects these practices may be limited due to a variety of reasons. In this manuscript we discuss barriers and drivers to implementing PD that reflects active learning and equity-focused values. The goal is to understand ways that we can utilize institutional-specific drivers and navigate barriers to support the implementation of more PD programs like the one in this study.

Keywords: Mathematics Graduate Teaching Assistant, Professional Development, Active Learning, Equity, Barrier

Mathematics graduate student professional development for teaching has become increasingly important in the Mathematics Education community. However, what that professional development looks like depends on local institutional contexts. A department's culture can impact what classroom practices graduate students implement as well as whether equity and inclusion are identified as being an important element for instruction. Department leaders who work closely with graduate students who have teaching roles, such as department chairs, course coordinators, and professional development providers, have an understanding of that culture and recognize existing barriers and drivers to implementing professional development for mathematics graduate teaching assistants (MGTAs).

The work presented here is part of a larger NSF-funded project focused on creating, implementing, and studying a multi-term professional development program for MGTAs across three different mathematics departments. The program titled, ELITE PD, has two primary foci: 1) active learning and 2) equity and inclusivity. However, how departments think about and approach these ideas can vary. Thus, we sought to understand institutional-specific barriers and drivers for implementing a MGTA professional development program focused on active, equitable, and inclusive teaching practices. Specifically, we ask: What barriers and drivers do department leaders identify for implementing an MGTA PD program focused on active learning, equity, and inclusivity?

Theoretical Frameworks

We sought to identify analytical frameworks that reflected the ELITE PD program foci. In the work by Shadle et al. (2017), researchers sought to understand STEM faculty perspectives with regard to shifting teaching practices from instructor-centered approaches to more student-centered approaches. Additionally, Harris and Wood have done extensive work in understanding barriers to advancing equity in instruction in higher education (e.g., The Community College Equity Assessment Lab (CCEAL), 2017; Harris & Wood, 2018; Wood & Harris, 2015). Together, the work of Shadle and colleagues and Harris and Wood provide us with analytical frameworks that allow us to understand the barriers and drivers to implementing an MGTA PD program focused on active learning, equity, and inclusivity.

Shadle et al. (2017) examined STEM faculty response to a vision for teaching as proposed by university leadership. The vision proposed a shift towards student-centered instruction and leadership solicited input from faculty regarding the proposed vision, and faculty-perceived barriers and drivers toward this vision were then identified. Identifying barriers allowed leaders to better understand aspects and characteristics that might prevent change from occurring (e.g., university reward structures, individual beliefs) and drivers that could be leveraged to support change (e.g., collaboration in teaching, personal satisfaction). We drew upon Shadle et al.'s barriers and drivers as a starting point for our analytical framework for understanding barriers and drivers to implementing active learning in MGTAs' instruction.

While Shadle et al. (2017) address barriers and drivers associated with active learning and student-centered instruction, Harris and Wood (e.g., CCEAL, 2017; Harris & Wood, 2018; Wood & Harris, 2015) examine the barriers for doing equity-related work in higher education. Several of the barriers overlap with those identified by Shadle and colleagues. For example, both Shadel et al. and Harris and Wood highlight individual attitudes as playing a role in resistance to change efforts. Similarly, both groups of scholars emphasize institutional culture as potentially impacting change (e.g., institutional policies or processes). Harris and Wood additionally identify institutional "politics and power dynamics" as a potential barrier for equity-related work (2018, Barriers to Actually Achieving Equity section). Combined, both analytical frameworks provided us with a starting point for understanding department leadership perspectives on an MGTA PD program focused on active, equitable, and inclusive teaching practices.

Methods

A previous RUME paper with this project explored barriers and drivers tied to implementing an MGTA PD program focused on active learning, equity, and inclusivity (Fifty et al, 2022). That work examined a single institution through thematic analysis. Here we expand upon this work by looking across three institutions and implementing an a priori codebook built from the codes established by Shadle et al. (2017) and the work of Harris and Wood (e.g., CCEAL, 2017; Harris & Wood, 2018; Wood & Harris, 2015). We discuss our methods in the following sections. While we are building upon this work, this work is still preliminary, with only initial findings from coding being presented.

Data

ELITE PD is being implemented at three large, public universities: Beta University, Epsilon University, and Gamma University. Each university offers graduate programs in mathematics. Epsilon University primarily grants master's degrees in mathematics and Beta and Gamma Universities primarily grant PhD degrees in mathematics. At each institution there is a structure within the department around MGTA professional development and interviews were conducted with the leaders in these structures. Table 1 lists the titles of the leaders at each institution. In some cases, more than one person carried the same title, and those instances are documented by the number in parenthesis.

Table 1. MGTA Professional Development Leaders at each institution.

Epsilon University	Gamma University	Beta University
Incoming Dept Chair Outgoing Dept Chair Course Coordinators (3) PD Facilitators (1) Graduate Advisor	Dept Chair Associate Chair Graduate Program Chair PD Facilitator	Dept Chair PhD Director Course Coordinators (2) PD Facilitator

These interviews were semi-structured conversations around ELITE PD practices and implementation, which is described in more detail below. Interviews were conducted at the beginning of the study, before ELITE PD had been implemented at any of the institutions.

Coding

Analysis of these interviews was done through open coding. The initial codebook was taken from Shadle et al. (2017) and supplemented by the work done by Harris and Wood (e.g., CCEAL, 2017; Harris & Wood, 2018; Wood & Harris, 2015). As the interviews were coded, these codes were then refined, and other codes were added based on what was uncovered in the data. Codes were identified as being in one of three main categories: barrier, driver, or tension. Barriers were those codes that impeded ELITE PD practices, drivers supported ELITE PD practices, and tension codes were not necessarily barriers or drivers but emphasized local context or values that impacted ELITE PD practices. ELITE PD practices are:

- Active Engagement
- DEI practices in and out of the classroom
- Assessment (or feedback) of teaching practices aligned with active learning and/or DEI practices
- Faculty or peer MGTA mentoring practices
- Fostering MGTA autonomy/agency
- Awareness of and attending to power differentials (how MGTAs are treated, classroom practices, etc.)

Examples of barriers to ELITE PD practices include *Unclear Implementation* and *Student Resistance*, and *Improves Teaching and Assessment* is an example driver. Examples of tensions are *MGTA Workload* and *Student Preparedness*.

An utterance was coded only if it could be related to at least one of the ELITE PD practices listed above. If the utterance was not clearly related to one of those practices, it was not coded. For example, discussions around professional development for MGTAs were not coded unless it was clear that there was a connection to ELITE PD practices; general discussions about professional development were not coded.

Once a codebook had been formed, three researchers coded all of the interview. Each interview was coded by at least two researchers. For each interview, the researchers coded individually and then came together to discuss any differences in coding until there was 100% agreement.

Initial Findings

Some initial findings have included the most common barriers and drivers found across the three institutions. Each of these are described in more detail in the following sections.

Drivers

The three most common drivers were *Supportive Attitude*, *Current Practices*, and *Department Support*. A description of each of these codes is given in Table 2.

Table 2. Common drivers

Supportive Attitude	The interview participant expresses (personal) view in line with ELITE PD practices such as being supportive of efforts that align with ELITE PD values.
Current Practices	● Some MGTAs and/or faculty participate in PD that reflect ELITE PD practices ● MGTAs and/or faculty member(s) have already adopted ELITE PD practices
Department Support	In reference to the department as a collective (use of "we") in relation to efforts associated with ELITE PD practices. May be part of department-wide culture, expectations, vision statement, and/or department mission.

The drivers described in Table 2 were found to be the most common among the three institutions, with *Supportive Attitude* being the most common driver across all three institutions. These drivers did not vary much between the three institutions. For all three institutions, these were the most commonly discussed drivers, which was not the case for barriers as described below. Other drivers included *Institutional Support* and *Aligns With Existing Resources*.

One limitation with this finding is that all three of these drivers are quite broad and encompass many different beliefs and ideas. We are currently working on a secondary code system for each of these drivers, as well as for *Institutional Support*, so as to gain a better understanding of what the specific drivers are within each of these categories.

Barriers

The three most common barriers were *Unclear Implementation*, *Lack of Awareness*, and *Faculty Divisions*. Each of these codes is described in Table 3.

Table 3. Common barriers

Unclear Implementation	Don't know how to assess if doing ELITE PD practices, their appropriateness, or how to implement them effectively
Lack of Awareness	Participant indicates they are unaware of policies, structures, happenings/efforts in the department/institution related to or aligned with ELITE PD practices
Department Support	Disagreement and/or unwillingness among faculty/lecturer groups with regard to implementing ELITE PD practices. (e.g., tenured vs. un-tenured faculty; lecturers vs. faculty; etc.)

An interesting difference between the most common barriers and drivers was that the barriers varied across the institutions. For Beta University, the top three common barriers were Unclear Implementation, Lack of Awareness, and Faculty Divisions, which is consistent with the overall list described above. However, for Epsilon University, the top three common barriers were Unclear Implementation, Faculty Divisions, and Not-My-Job, and for Gamma University, the top three common barriers were Lack of Awareness, Focused on Metrics, and Misinformed/Theorizing, which includes codes not on the overall common barriers list above. It was important to delineate by institution, as the number of interviews were not the same for each institution. Further, examining barriers by institution allows us to better understand the role that local contexts can play when engaging in PD work focused on active learning, equity, and inclusive teaching strategies.

In addition, we found that the Not-My-Job code was used to note when the interviewee stated that something focused on ELITE PD practices was not part of their job. We found that this was not necessarily a strict barrier as we had originally thought it would be. Rather, it tended to mark instances the interviewee recognized that there were other people more qualified to address specific aspects or tasks in the department. In contrast, Focused on Metrics was a code used to note times the interviewee was more focused on the metrics of the department over the implementation of ELITE PD practices and Misinformed/Theorizing was used to note times the interviewee made a statement about ELITE PD practices that was not correct (e.g., active learning is not possible in large classes).

Conclusion

Understanding the barriers and drivers that can support or impede the implementation of an MGTA PD program focused on active learning, equity, and inclusion is important because they can be used to encourage the adoption of such a program. While drivers did not vary much across the three institutions, barriers did. As barriers can play an important role in hindering implementation of a PD program, understanding what those barriers are can help in knowing how to best leverage drivers in order to implement change. For example, the focus on metrics at Epsilon University could be leveraged in such a way that data collected for assessing impact of a PD program could include metrics that 'speak' to administrators and leaders. Lack of awareness and/or unclear implementation, on the other hand, may require a focus in communication and enhanced training to support PD efforts. Such a focus could also address the barrier of misinformed/theorizing. Moving forward, we aim to understand how institutional-specific barriers can be addressed through existing drivers.

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References

The Community College Equity Assessment Lab. (2017). *Faculty Perspectives on Student Equity and Success for Students of Color at Palomar College* [White paper]. https://www.palomar.edu/sse/wp-content/uploads/sites/158/2018/04/CCEAL-Faculty-One-on-One-Findings.pdf

Fifty, D., Hill-Lindsay, S., Zimmerman, S., Beisiegel, M., Pilgrim, M., & Miller, E. (2022, February). Potential barriers to a sustainable MGTA PD program focused on equity and inclusivity. In S. S. Karunakaran, & A. Higgins (Eds.), *Proceedings of the Conference on Research in Undergraduate Mathematics Education*. (pp. 973-978). Boston, MA.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111*(23), 8410-8415.

Harris, F. III. & Wood, L. (2018, October 17-18). *Actually achieving equity*. In California State University Office of the Chancellor, 2018 Graduation Initiative 2025 Symposium. [Virtual Symposium]. https://www.calstate.edu/csu-system/why-the-csu-matters/graduation-initiative-2025/symposium/2018-symposium/Documents/BreakoutSession-AchievingEquity.pdf

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129-146.

Mulnix, A. B., Vandegrift, E., & Chaudhury, S. R. (2016). How important is achieving equity in undergraduate STEM education to you? *Journal of College Science Teaching, 45*(4), 8-11.

Shadle, S. E., Marker, A., & Earl, B. (2017). Faculty drivers and barriers: Laying the groundwork for undergraduate STEM education reform in academic departments. *International Journal of STEM Education, 4*(1), 1-13.

Wood, J.L., & Harris III, F. (2015). Large scale faculty development: A taxonomy of faculty perspectives on teaching men of color. *NISOD Innovation Abstracts, 37* (23). https://www.nisod.org/archive_files/abstracts/pdf/XXXVII_23.pdf

Instructional Characterizations of Foundational Math Coordinators with Attention to Instructor-Student Interactions

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We aimed to get a better understanding of participants' (eight foundational math course [FMC] coordinators') teaching approaches. In the first year of this grant project, we primarily gathered data (through surveys, self-reflections, and class observations) on these individuals as instructors. These data were compiled into narrative summaries for each participant and analyzed and compared. We discuss our findings from this analysis, using the instructional triangle as a framework, and particularly focusing on instructor-student interactions. This project aims to develop an understanding of what is needed to support instructional change in FMCs by evaluating how math-specific professional development (PD) cycles affect FMC coordinators' teaching practices and perspectives. We seek audience feedback on potential next steps towards fostering effective instructor-student interactions and future PD cycles.

Keywords: Foundational Math, Course Coordinators, Professional Development

Introduction

College math instructors typically work in isolation, negatively affecting quality of instruction and sustainability of evidence-based teaching practices in college math classrooms (Bressoud et al., 2015). Few instructors have access to or utilize explicit communities of support (Reinholz, 2017). This paper reports on data from the first year of a 3-year grant project (Rogers, 2022-2025) that ultimately aims to test and refine a community of practice (CoP; Wenger et al., 2002) among eight faculty members who coordinate and teach foundational math courses (FMCs, e.g., College Algebra, Precalculus, Quantitative Reasoning, Introductory Statistics, Math for Elementary Teachers, & Calculus). Course coordinators are a key population within college math education because they teach and supervise instruction of thousands of undergraduate students each semester. A CoP can provide a means for them to share and manage professional knowledge, in this case about math-specific professional development (PD) opportunities.

The overarching objectives of this grant project evaluate how math-specific PD cycles affect FMC coordinators' teaching practices and perspectives and contribute to theory about supporting meaningful instructional change in FMCs. The hypothesis is that by designing and implementing PD opportunities about teaching undergraduate math using active-learning (AL) strategies (Freeman et al., 2014; Laursen & Rasmussen, 2019) with FMC coordinators, they will develop their mathematical knowledge for teaching and become more effective instructors. Ultimately, this project will develop an understanding of what is needed to support instructional change in FMCs. Our first step toward these goals is to understand the current beliefs, practices, and competencies of the FMC coordinators in our study, use these findings to inform future PD decisions, and elicit feedback from the RUME community to consider moving forward. In this paper, we aim to answer, what are the prevailing characteristics (e.g., beliefs, competencies, classroom practices) of these FMC instructors?

Literature Review

Over a decade ago researchers (Speer et al., 2010) conducted a literature review on collegiate math teaching and concluded that instructors' practice (i.e., their pedagogical actions and reasoning related to those actions in the context of instructional activities) remains largely unexamined in the research literature. Research on PD in college math education is still sparse, especially when the focus is on instructors and tenure-track faculty (Florensa et al., 2017). Within the context of our larger research project, we first seek to better understand who these participants are as instructors and as a community of instructors. When we say community of instructors, we are examining this group of instructors as a CoP.

In a 97-paper meta-analysis of change theories in STEM higher education, the most prevalent change theory was CoPs (Reinholz et al., 2021). A CoP strives to create, expand, and exchange domain knowledge to develop individual capabilities and cohesion dependent upon passion, identity, and commitment (Wenger et al., 2002). These cohesive properties align with the model used in this study because all participants share the FMC instructor and coordinator identities, a commitment to teaching (per employment), and passion to support student learning outcomes.

To consider these CoP components, we note how the MAA Instructional Practices Guide (MAA, 2018) emphasizes student-centered teaching practices and pushes us to move away from direct instruction. This push is due to research findings that highlight how student-centered practices help increase access to learning opportunities for diverse learners (e.g., Laursen & Rasmussen, 2019). We specifically conceptualize student-centered instruction through the instructional triangle. Instead of solely emphasizing the teacher, Cohen et al. (2003) defined teaching as "what teachers do, say, and think with learners, concerning content, in particular organizations and other environments, in time" (p. 124). In their definition, four critical aspects of teaching become apparent- teachers, students, content, and environment- which are situated in a model that represents instruction as interaction where teacher, learners, and content create a triangle of interaction, existing within the environment (i.e., the instructional triangle). This framework for instruction is appropriate for our study because it allows us to look at participants' self-report data and classroom observation data and consider how their responses and practices emphasize components of the instructional triangle.

Method

Context: University and Course Coordinator Backgrounds

At a rural, public, liberal arts college in the Midwest, participants are eight FMC coordinators. Course coordination duties include (but are not limited to) deciding on, designing, and distributing course materials (i.e., syllabi, lecture notes, pacing calendars, example assessments, and activities) to maintain consistency across sections of the course that semester. They also include facilitating meetings with all course instructors, adjusting assessments so they align with learning outcomes, providing observation feedback for novice college math instructors (i.e., graduate student instructors and faculty), and addressing student concerns. Our campus has a main (4-year) location and secondary location less than an hour away. The secondary location focuses on the first 2-years of undergraduate course work, and providing explicit pathway supports for students. Table 1 lists participants' course names, years coordinating (self-reported), and campus location. For College Algebra, both campus locations utilize a math emporium instructional style[1], a computer-based opportunity to fill in gaps for content knowledge, while the other FMCs are taught in in-person lecture formats (pseudonyms are used throughout this paper).

[1] Math emporiums use adaptive learning systems to individualize student pathways (Cousins-Cooper et al., 2017).

Table 1. Participants' backgrounds and course coordination details.

Pseudonym	Course(s) Coordinating	Yrs Coordinating	Campus
Alaina	College Algebra	9	Main
Alexis	College Algebra	0	Main
Camille	(1) Calculus I, (2) Calculus II, & (3) Calculus III	15	Main
Madeline	(1-3) Math Ed Elementary Math Content Courses	4	Main
Patrick	(1) College Algebra & (2) Pre-Calculus	3	Secondary
Pricilla	Pre-Calculus	4	Main
Reema	(1) Math for Architectures & (2) Quantitative Reasoning	12-15	Main
Stella	Introduction to Statistics	4	Main

Community of Practice Context

Before this grant project began, Author 1 met with the FMC coordinators at least twice a month for course coordination meetings to address administrative and policy needs of the FMCs, department, and university. By utilizing these pre-existing meetings, we implement PD activities, learn about one another's teaching practices, and collect self-report and survey data without adding additional time commitments.

Data Collection and Analytical Approach

During project year 1, FMC coordinators completed surveys[2], answered reflection questions, completed an empathy map, and were observed teaching. These data points informed our understanding of their professional history, beliefs both as an individual instructor and as part of the FMC coordinator group, and typical classroom practices. We created narrative summaries of each participant's quantitative and qualitative responses. We analyzed the narratives to examine participants' beliefs and classroom practices, specifically regarding participants' teaching approaches. We conceptualize "teaching approaches as actions and strategies described and enacted by instructors when they talk about teaching mathematics or when they actually teach mathematics (Mesa et al., 2014, p. 122). We coded the data for student-, content-, or instructor-centered approaches, defined as instructor descriptions and strategies that are driven by…

1. …"instructors' interest in attending to students' cognitive, social, and emotional needs, seeking to give students a more prominent role in classroom activities" (Mesa et al., 2014, p. 122): Student-centered.
2. …"instructors' interest in emphasizing the content over students' cognitive, social, or emotional needs and involvement" (Mesa et al, 2014, p. 123): Content-centered.
3. …instructors' teaching or learning plans, goals, or decisions that do not explicitly attend to student- or content-centered aspects: Instructor-centered.

[2] The surveys included items from the Collective Teacher Efficacy Instrument by Goddard et. al. (2000), and the (self-) Efficacy Instrument by Enochs & Riggs (1990)

Findings: Characterization as Instructors

After analyzing each individual case study and comparing them across the group, we report on our initial findings of the identities of these FMC coordinators.

Beliefs

Based on their survey responses, we examined the participants' beliefs about self-efficacy and collective teacher efficacy. Each coordinator's response indicated a high level of self-efficacy in terms of their ability to teach mathematics. For most, this ability included self-assessed strength in terms of their content knowledge (content-centered). The self-efficacy instrument also highlighted the coordinators' beliefs on how influential instructors are to the learning and success of math students. Specifically, all but one coordinator said the instructor (instructor-centered) significantly impacts student learning. Only Alaina's survey responses indicated she believes classroom practices need to be organized to help students be the mechanism for their success in learning (student-centered).

Every participant also indicated a high level of collective teacher efficacy, with survey responses that suggest they believe the other individuals in their FMC CoP are effective in their teaching of mathematics (instructor-centered). Even so, the coordinators were not in consensus as to whether further training was needed for the group to "know how to deal with undergraduate students" (Enochs & Riggs, 1990). When responding to this Likert-scale question, Patrick, Madeline, and Stella all somewhat agreed more training was needed, Alexis somewhat disagreed, Alaina, Camille, and Reema disagreed, and Pricilla was not sure. Interestingly, those who somewhat agreed more training was needed have only been in their roles for 3-4 years, however those who disagreed with the statement have between 9-15 years of experience as a coordinator. We interpret that result to mean that individuals with more course coordination experience feel like they, along with their CoP, do not require further training.

Competencies

Self-reported data was collected during two different group meetings. In one meeting, participants reflected on their areas of strength and struggle as instructors. They described some strengths/struggles that were more instructor- or content-centered, however, most strengths/struggles described were student-centered. All but one participant explicitly discussed strengths related to their students and interactions with students (e.g., engaging students in discussion), and every coordinator described a student-centered area of struggle (e.g., difficulty motivating students to come to class). These student-centered aspects indicate that as a group, the coordinators place value on their students' role in the classroom and interactions with them.

We also asked the coordinators to reflect on what a typical day in their class is like. Alaina stated practices that are primarily content-centered but are implicitly student-centered by design of the emporium class she coordinated (e.g., creating problems for certain students based on their current content). We are gathering this data from the recently appointed emporium coordinator, Alexis, on her intended practices to compare with Alaina's (emporium director and former coordinator). Camille, Madeline, and Stella all described student-centered approaches including dedicating class time for students to engage in the content and interact with classmates. Stella, specifically, reports having a mostly student-centered class, while Camille and Madeline spoke to both instructor- and student-centered aspects of their class time. Patrick, Pricilla, and Reema all described a lecture-based environment (instructor-centered) and focused somewhat on the content aspect of their classes (content-centered). We therefore conducted observations to investigate how these reported practices aligned with what happens in their classrooms.

Classroom Practices and Observed Interactions

Each coordinator was observed twice in the same semester, except for Stella who was observed once (due to scheduling constraints). When comparing the coordinators' stated practices to what was observed, their reported practices mostly aligned with what they did during observations of their class. Instances where there were discrepancies between the stated and observed practices were mainly things that they articulated did not occur every class session. For example, Pricilla stated she occasionally gives students an exit slip or a quiz, however, neither of these were observed.

The observations also allowed us to analyze instructor-student interactions. For some coordinators, particularly those with a more student-centered classroom environment, students tended to give affirming body language (e.g., nodding their head, actively writing) during these interactions and appeared engaged in the math. However, in other coordinators' classrooms, students were observed having negative body language (e.g., turning away from the instructor, dissociating) and appeared to disengage with the conversation or content. These instructors did not react in a way that would indicate their acknowledgement of the students' negative body language. This observation shows a disconnect between the interpretation of these interactions by the instructors and the students we intend to investigate further.

Discussion and Questions for the RUME Audience

From their survey responses and self-reflections, it is clear this group of coordinators care about the quality of their interactions with students. During committee meetings, some of the coordinators discussed not being able to get through to some students, or not understanding why their interactions with students were not as fruitful as they wanted them to be. After considering their observations, however, we gained insight into the differences in reported perceptions between students and instructors.

With this finding, we are interested in hearing any ideas from the RUME audience about how to illuminate this discrepancy to the group as well as what interventions or PDs could be used to promote affirming instructor-student interactions. We attempted to make some initial headway in this area in year 1 when we guided the coordinators through developing *empathy maps* (Aldrup et al., 2022; Gibbons, 2018) about personas of students in the FMCs they teach. Given the student-centered approaches documented in this study, we were surprised to be met with an unwillingness to participate by a few participants who felt the empathy map was not appropriate in the educational setting. Discussing this pushback with the RUME audience and exploring factors underlying this pushback could provide avenues for future PD with this group.

Additional specific questions we plan to ask the audience during our presentation include:

- What research-based, but practitioner-focused papers or resources should we be aware of that could inform how we help instructors consider students' perceptions about instructor-student interactions?
- This paper focuses on the participants as *instructors* of FMCs. During this second year of the project, what suggestions do audience members have regarding data we should prioritize gathering about these participants' perspectives as coordinators? And why?

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References

Aldrup, K., Carstensen, B., & Klusmann, U. (2022). Is empathy the key to effective teaching? A systematic review of its association with teacher-student interactions and student outcomes. *Educational Psychology Review, 34*, 1177–1216. https://doi.org/10.1007/s10648-021-09649-y

Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and Recommendations from the MAA National Study of College Calculus.* MAA Press. Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2015). *Learning to improve: How America's schools can get better at getting better*. Harvard Education Press. Cambridge, MA.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25,* 119–142.

Cousins-Cooper, K., Staley, K. N., Kim, S., & Luke, N. S. (2017). The effect of the math emporium instructional method on students' performance in college algebra. *European Journal of Science and Mathematics Education, 5(1),* 1–13.

Enochs, L., G. & Riggs, I., M. (1990). Further development of an elementary science teaching efficacy belief instrument: A preservice elementary scale. *School Science and Mathematics, 90(8)*, 694-706.

Florensa, I., Bosch, M., Gascón, J., & Ruiz-Munzon, N. (2017). Teaching didactics to lecturers: A challenging field. Proceedings of the *11th Congress of European Research in Mathematics Education*. https://hal.science/hal-01941653

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111(23)*, 8410–8415. https://doi.org/10.1073/pnas.1319030111

Gibbons, S. (2018, January 14). *Empathy Mapping: The First Step in Design Thinking*. Nielsen Norman Group. Retrieved September 1, 2023, from https://www.nngroup.com/articles/empathy-mapping/

Goddard, R. D., Hoy, W. K., & Hoy, A. W. (2000). Collective teacher efficacy: Its meaning, measure, and impact on student achievement. *American Educational Research Journal, 37(2),* 479-507.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5,* 129–146.

Mathematical Association of America. (2018). *Instructional Practices Guide.* Washington, DC. Retrieved August 12, 2023, from https://www.maa.org/sites/default/files/InstructPracGuide_web.Pdf

Mesa, V., Celis, S., & Lande, E. (2013). Teaching approaches of community college mathematics faculty: Do they related to classroom practices? *American Educational Research Journal, 51(1),* 117-151.

Reinholz, D. L. (2017). Not-so-critical friends: Graduate student instructors and peer feedback. *International Journal for the Scholarship of Teaching and Learning,* 11(2). https://doi.org/10.20429/ijsotl.2017.110210

Reinholz, D. L., White, I., & Andrews, T. (2021). Change theory in STEM higher education: a systematic review. *International Journal of STEM Education, 8(1),* 1-22.

Rogers, K. C. (2022-2025). *Adapting Problem-Solving Cycles in Professional Development with Foundational Mathematics Course Coordinators: A Potential Gateway for Instructional Change*, NSF grant project funded August 2022. Award No. 2225351.

Speer, N. M., Smith, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior, 29,* 99-114.

Wenger, E., McDermott, R. A., & Snyder, W. (2002). *Cultivating communities of practice: A guide to managing knowledge.* Harvard Business Press.

Invoking Conceptual Change in Adult Learners: "Seeing" Fractions Differently

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I was charged with developing and implementing a course redesign to improve low success rates in the lowest university-level developmental mathematics course (median ACT score = 17). As part of the initial implementation of the course redesign a diagnostic assessment was given to understand the nature of students' conceptions of foundational concepts. Student responses on several items indicated that over 25% of students had a dominant part-whole conception of fractions; not seeing fractions in terms of measures. This dominant part-whole conception of fractions also influenced how students reasoned about items involving other concepts (e.g., addition of fractions and ratio comparisons). In this preliminary report the results that led to our conclusions, the instructional intervention we used to move students' conceptions of fractions forward, and the impact of the intervention, based on the results of the post-assessment, on students' conceptions of fractions will be shared.

Keywords: developmental mathematics, conceptual change, fraction conceptions

Introduction

At the university in which I teach the five-year success rates in the lowest developmental mathematics (DM) course has consistently hovered between 50% and 60%. Despite these low success rates, not much has been known about the nature of these students' conceptions of foundational concepts besides their over reliance on procedures without knowing the mathematical "why" (Stigler et al., 2010). Charged with improving success rates in this course it was important to first understand the "roots" of their struggle with mathematics. A thirteen-item multiple-choice diagnostic assessment (n = 230) was given to assess students' current conception of place-value, fractions, area/perimeter, and proportional relationships. While the results suggested that students' faced challenges within each of these constructs the most striking and revealing were those related to their understanding of fraction.

Existing research with students at various levels of academic experience suggested that students' knowledge of fraction magnitude, or as a number, is highly associated with their computational skills (Schneider & Siegler, 2010), algebraic knowledge (Booth et al., 2014), and general mathematics achievement (Torbeyns et al., 2015). Seeing fractions as a magnitude requires students to engage in multiple levels of unit coordination that involve relationships between quantities and various size of units. We had previously conducted research with 4^{th} through 7^{th} graders and were curious as to whether DM students' conception of fractions was similar. We had previously found that elementary and middle school students often struggled with fraction magnitude concepts because their dominant conception of fractions was as a part-whole relationship between two quantities without regard for the size of the unit represented by the denominator of the fraction. An example of student reasoning reflecting this part-whole relationship is shown in figure 1. When asked to compare the fractions 5/6 and 7/8 the student partitioned the whole unit into a quantity of equally sized pieces represented by the denominator of the fraction (i.e., 6 and 8) and shaded the quantity of pieces represented by the numerator (i.e., 5 and 7). A student with a dominant part-whole conception of fractions will reason that the two fractions are equal because the quantity of missing pieces is the same (i.e., 1); not considering the size of the unit attached to each of those quantities, sixths and eighths, respectively. Students who do reason about the size of the units demonstrate elements of fraction-as-measure conception (Wilkins & Norton, 2018). We were curious as to whether DM students' conception of fractions was similar. We hypothesized that if DM students had a dominant part-whole conception of fractions it may also influence how these same students reason about operations involving fractions and proportional relationships.

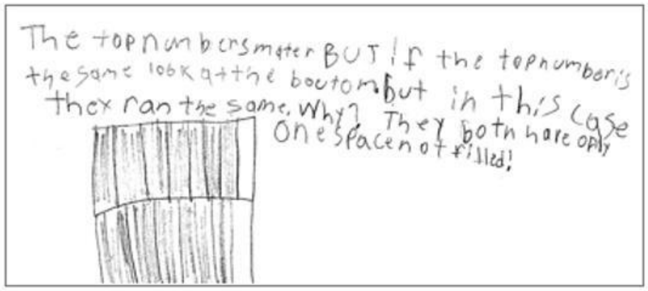

Figure 1. Student reasoning to compare 5/6 and 7/8.

To explore the nature of DM students' current conceptions a thirteen-question multiple-choice diagnostic assessment was given at the beginning of the DM course The results of three items from the diagnostic is shown in Table 1. Q4 is an item that was used to determine whether students had a dominant part-whole conception of fractions. Q9 and Q12 are also shared because it was hypothesized that student with a dominant part-whole conception of fractions would reason about these two tasks in a particular manner. The notation (C) in Table 1 represents the correct answer while the bolded response indicates the response in which a student with a dominant part-whole conception of fractions would most likely choose. It is important to note that given that Q4 was a multiple-choice we do not know whether the students who chose the correct answer had a meaningful conception of fractions or simply relied upon a whole-number relationship (e.g.,7 > 5). However, we believe we can infer that if a student chose "they ate the same amount" they were at best utilizing a part-whole relationship.

Table 1. Sample diagnostic questions (n = 230)

#	Questions			
Q4	Two pizzas are the same size. Carlos ate 5/6 of one of the pizzas and Terrell ate 7/8 of the other pizza. Who ate more?			
	Carlos	Terrell (C)	**They ate same amount**	Impossible to know
	30.43%	41.30%	**26.52%**	1.74%
Q9	Thomas ate 3/4 of a whole medium pizza and Lydia ate 5/8 of a whole medium pizza. Together they ate how much of a whole medium pizza?			
	8/12	11/8 (C)	8/8 (C)	Cannot determine
	40%	45.65%	9.13%	5.22%
Q12	Mix A: 3 cups of OJ to 4 cups of water Mix B: 6 cups of OJ to 8 cups of water Which mixture (A or B) will be juicier?			
	Mix A	Mix B	They will be the same (C)	Impossible to determine
	34.44%	14.50%	49.85%	1.21%

As part of our preliminary analysis of the diagnostic results a chi-square test for independence was conducted to examine our hypothesis that students answering Q4 in a way suggesting a dominant part-whole conception (i.e., two fractions are equal) would also answer Q9 (i.e., 8/12) and Q12 (i.e., Mix A) in a predictable manner. The results of this brief analysis confirmed our initial hypothesis.

Table 2. Chi-square analysis. X^2 *(230,1)* * $p<.05$

	Q9	Q12
Q4	6.6868	16.4698
	$p = 0.0097$*	$p = 0$*

Interventions

As the preliminary analysis of the diagnostic assessment revealed many of the DM students relied on a part-whole conception of fractions to reason about fraction comparisons. The first challenge of the course redesign was to develop learning experiences that confronted the limitations of a part-whole conception while also moving them towards fraction-as-measure conceptions. Guided by two frameworks, Conceptual Change (Vosniadou, 2013) and Progression of Fraction Schemes (Wilkins & Norton, 2018) a set of instructional activities were designed. One of the elements of the conceptual change framework is that "knowledge acquisition is not always a process of enriching conceptual structures. Sometimes the acquisition of new information requires the radical reorganization of what is already known" (Stafylidou & Vosniadou, 2004, p. 504).

A radical reorganization of DM students' fraction conceptions required explicit attention to the meaning and role of the unit. This was addressed in two ways. First, tasks were designed to engage students in mental activities that supported the development of fraction-as-measure conceptions (Wilkins & Norton, 2018). These tasks engaged students in the mental activities of partitioning, iterating, and disembedding. For example, as part of one the tasks we asked students to fold (partitioning) fraction strips representing the whole unit into equally-sized pieces such that only the unit fraction was showing. Then we asked them to iterate the unit fraction to determine the length of copies of the unit fraction. For example, as shown in figure 2, consider a paper strip representing 1 whole unit of length.

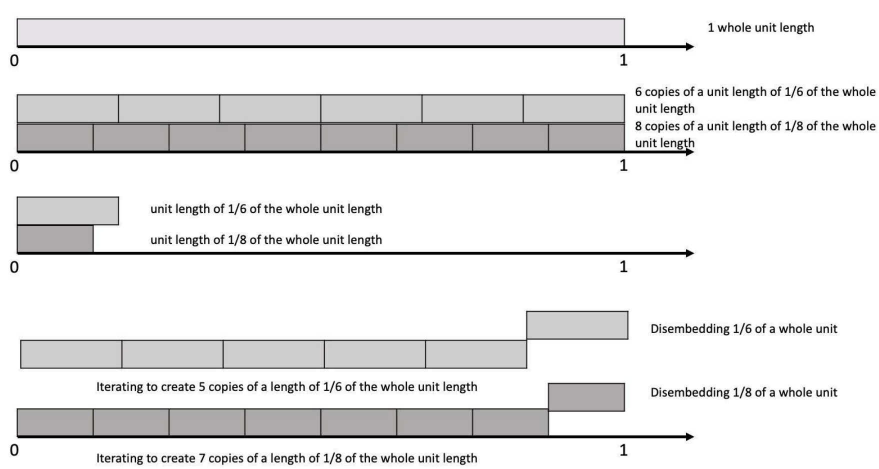

Figure 2. Coordination of units to engage in a measurement scheme for proper fractions.

The first action involves *partitioning* the whole into equally sized lengths and recognizing that each of those lengths represents a unit fraction (e.g., 1/8). *Iterating* the unit fraction, a given number of times represents a length that is multiple copies of the length of the unit fraction. For example, a length of 7/8 of a whole unit is 7 copies of the length of the unit fraction 1/8.The *disembedding* action happened to determine the "missing part" from the whole. For example, as shown in figure 2, the missing part of 7/8 of a whole unit is a unit fraction of 1/8 because 8 copies of the unit fraction 1/8 is the same length as the whole unit.

The second way in which the unit was made explicit is that fractions were represented using a novel notation, numeral-unit-name notation. For example, the fraction 7/8 was written as 7 eighths using the numeral to represent the quantity and the unit-name to represent the unit. This decision supported a learning acquisition goal and a connection goal. Students were introduced to the idea that the morpheme "ths" indicated that the whole unit is partitioned into a quantity of equally-sized lengths represented by the word preceding the morpheme (e.g., eighths). It also supported efforts to utilize overarching principles to connect operations across numerical and algebraic expressions. For example, instead of seeing an addition problem such as 3/4 and 5/8 in terms of "getting a common denominator" our goal was for students to see adding fractions as combining quantities of the same size of unit. Thus, 3 fourths and 5 eighths cannot be combined because the size of the units are not the same. However, utilizing an equal exchange of 6 eighths for 3 fourths the numbers 6 eighths and 5 eighths can be combined because the size of the units are the same (6 eighths + 5 eighths = 11 eighths).

While not a significant part of this preliminary report, it is important to note that making the units explicit was also an important aspect of the development of understandings related to equivalent ratios and proportional relationships in the DM class. For example, consider the ratios represented in Q12 of the diagnostic test. The ratio 6 cups of OJ to 8 cups of water is the same juiciness as 3 cups of OJ to 4 cups of water because it represents 2 copies of the ratio 3 cups of OJ to 4 cups of water [2 x (3:4) = (6:8)]. These multiplicative relationships between ratios were made explicit using ratio tables and double number lines.

Results

After one iteration of the course redesign the success rate in the DM course increased from 56% to 80%. Despite a significant shift of the DM content to more arithmetic concepts, as opposed to algebraic concepts, the failure rate in the ensuing general education mathematics course also decreased slightly, 31.40% to 28.74%.

A post-assessment involving the same questions as the diagnostic assessment was given ten weeks after the fraction concept module. Students were not given advanced notice that they would be re-taking the diagnostic assessment. The percentage of students that correctly answered each question on both the pre-assessment and same post-assessment was computed. Only the results of students ($n = 117$) who completed both the pre- and post-assessment are included to remove potential variation in percentage correct resulting from differences in pre- and post-assessment participants. A difference of proportions z-test was conducted to measure whether the change in percentage correct from pre- to post-assessment was statistically significant. Level of significance was set at $p < .05$. Results of this analysis are shown in Table 3.

Table 3. Comparing pre- and post-test results.

#	Question	% correct (pre)	% correct (post)	p
Q4	Two pizzas are the same size. Carlos ate 5/6 of one of the pizzas and Terrell ate 7/8 of the other pizza. Who ate more pizza?	47.86	61.54	.035*
Q9	Thomas ate 3/4 of a whole medium pizza and Lydia ate 5/8 of a whole medium pizza. Together they ate how much of a whole medium pizza.	52.14	76.92	.000*

The results suggest that there was growth in students' fraction-as-measure conceptions. We acknowledge, as discussed earlier, that comparing the fractions 5/6 and 7/8 requires a high degree of unit coordination sophistication (i.e., disembedding 1/8 and 1/6 to describe the measure of the "missing piece") and reasoning. We were specifically curious as to the change in percentage of students who indicated that the two fractions were equal. This percentage shifted from 26.5% on the pre-assessment to 13.68% on the post-assessment. We also conducted the same chi test of independence on the results of the post-assessment as we did on the pre-assessment and found the same relation between those who answered Q2 as equal and Q9 as 8/12, $X^2(1,117) = 7.87$, $p = .005$*.

Three other questions showed a statistically significant relation between indicating that 5/6 and 7/8 were equal and an incorrect response. These questions included Q8 (identifying measure on a tape measure), $X^2(1,117) = 24.159$, p = .0012* , Q11 (proportional reasoning involving distance traveled and a fraction quantity), $X^2(1,117) = 7.57$, $p = .006$*, and Q13 (proportional reasoning requiring unitizing), $X^2(1,117) = 5.258$, $p = .022$*.

Discussion

Consistent with what others have found (Pesek & Kirshner, 2000) as well as elements of Vosniadou's (2003) Conceptual Change Framework, it is challenging to move conceptions forward once students have procedures and conceptions already in place that interfere with the intended concept advancements. Even after the intervention 13.68% of DM students still indicated that the fractions 5/6 and 7/8 were equal and many still used this part-whole fraction relationship to add fractions.

This study gave us an opportunity to reflect on our teaching practices. We found that we need to spend more time engaging students in iterating unit fractions. From the analysis of the pre-and post-assessment we concluded that we need to begin these actions with whole numbers while also making more explicit the multiplicative relationship that exists between quantities and size of units of decimal numbers. Inherently numbers have an implied multiplicative relationship. For example, 0.5, is 5 x (1/10). Same is true with fractions, 4/8 or 4 eighth is also 4 x (1/8). Writing out the multiplicative relationship sheds light on the meaning of the unit fraction in relation to the whole unit and that 4/8 is 4 times greater than 1/8.

References

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118.

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education*, *31*(5), 524–540.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, *36*(5), 1227.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). Developing Effective Fractions Instruction for Kindergarten through 8th Grade. IES Practice Guide. NCEE 2010-4039. *What Works Clearinghouse*.

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science*, *19*(3), 341–361.

Stigler, J. W., Givvin, K. B., & Thompson, B. J. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator*, *1*(3), 4–16.

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13.

Vosniadou, S. (2013). Conceptual change in learning and instruction: The framework theory approach. In *International handbook of research on conceptual change* (pp. 11–30). Routledge.

Wilkins, J. L., & Norton, A. (2018). Learning progression toward a measurement concept of fractions. *International Journal of STEM Education*, *5*(1), 1–11.

Dimensions of Mathematics Graduate Students' Professional Identities as Prospective Faculty

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Graduate education in mathematics is instrumental to the socialization of prospective mathematics faculty, however, our understanding of how graduate students develop their professional identities is still limited. This preliminary study examines and compares how two doctoral students at different stages of their graduate programs reflect on and understand their professional identities. I highlight qualitative similarities and differences in how each participant identified as a teacher, as a researcher, and as a mathematician. The structure of the program and progression through degree milestones reflected how strongly participants were anchored onto dimensions of their professional identities. Both students strongly identified as teachers and both described how their gender identity as women negatively impacted dimensions of their professional identities, especially their mathematics identities.

Keywords: Graduate Students, Professional Identities, Socialization, Doctoral Education

Introduction

Many mathematics graduate students pursue faculty positions, however, our understanding of how they are socialized into the profession (Tierney & Rhoads, 1993) and develop their professional identities as faculty is still limited (Clarke et al., 2013), likely contributing to the minimal professional development opportunities that aim to develop strong professional identities (Austin & McDaniels, 2006; Jensen, 2011). I draw on Beijaard et al.'s (2000) conception of teachers' professional identity as "how they perceive themselves as teachers and what factors contribute to these perceptions" (p.751). Although there are similarities between K-12 teachers and higher education faculty, the differences in the preparation, institutional contexts, societal pressures, roles and expectations, and the discipline-specific cultures constitute a starkly different backdrop through which professional identities are developed (Nyquist, 1999; Van Lankveld et al. 2016). The experiences in graduate education are instrumental to students' professional identity development as future mathematics professors, and it is imperative that we understand how they make sense of their professional identities and identify the experiences that are formative or destructive to their sense of selves as faculty.

Additionally, it is important to examine the professional identities of prospective mathematics faculty as unique cases because the mathematics discipline has norms, beliefs, and values that set it apart from other disciplines. Clarke et al. (2013) noted:

> Discipline-based cultures are the primary source of faculty members' identity and expertise and include assumptions about what is to be known and how tasks [are] to be performed, standards for effective performance, patterns of publication, professional interaction and social and political status (p.7).

Examining the professional identity development of mathematics graduate students affords insight into the experiences and features of mathematics graduate education that prepare students for faculty positions. For this preliminary study, I examined the professional identities of two graduate students at different stages of their doctoral programs. The research questions that guided this study were: (1) How did mathematics graduate students reflect on their professional identity development as prospective faculty? (2) What experiences were formative to their professional identity development?

Framing

I drew on a sociocultural perspective of professional identity development (Solari & Ortega, 2020) to frame this study. Specifically, I consider the process of socialization into the profession (Tierney & Rhoads, 1993; Weidman et al., 2001) to be a mechanism of professional identity development. This view on identity development recognizes the weight and consequences of the context and the social interactions with others (Gee, 1996), and posits that the transmission of culture and the acquisition of the knowledge, dispositions, and skills necessary to perform in a profession are the product of the everyday social interactions that an individual has with others in a situated context (Clarke et al., 2013; Solari & Ortega, 2020). This can involve observing more expert others, learning by doing, interacting with others, and reconciling personal ideologies and beliefs about the profession with the everyday realities (Pillen et al., 2013; Tateo, 2012).

Within mathematics graduate education, I consider experiences such as serving as teaching assistants, engaging in research, attending conferences, receiving mentorship, and interacting with others can shape and inform graduate students' professional identities. Additionally, members of their profession and the mathematics community attach and attribute meanings to graduate students, prompting them to reconcile these various meanings and discourses (Gee, 1996; Sachs, 2001) to construct their professional identities. Lastly, I acknowledge that mathematics graduate students are embedded within institutional contexts, power structures, societal and political perspectives, and mathematics-specific cultures, values, beliefs, and norms. These factors, along with graduate students' individual social and cultural identities (Hayley et al., 2014; Trepte, 2013) interact to uniquely inform their professional identities as prospective faculty.

Method

This study is part of a larger dissertation project focused on the pathways to the mathematics professoriate and was conducted at a Minority-Serving Institution in California. Purposeful sampling (Miles et al., 2020) was used to recruit mathematics doctoral students pursuing faculty positions. They participated in 90-minute interviews over Zoom that included responding to a semi-structured interview protocol (Rubin & Rubin, 2011), writing a letter to the mathematics discipline, and graphing a visual representation of their graduate education experiences. For this preliminary study, I focus on two graduate students, Kayla and Morgan (pseudonyms), because they were revelatory (Yin, 2016) about the dimensions of their professional identities and the disparity in the time spent in graduate school afforded a comparison of the experiences that seemed to most impact their professional identity development. At the time of the data collection, Kayla had completed the first year of the doctoral program and Morgan had completed the fourth year (after being previously enrolled in a PhD program at another institution). Both graduate students self-identified as White/Caucasian, straight or heterosexual, female/feminine, and reported that they are actively pursuing careers as mathematics faculty.

The data for this study were the participants' responses to the semi-structured interview protocol centered around their experiences in graduate school and perceptions of the mathematics professoriate. I open coded (Miles et al., 2020) each response to make sense of how participants described themselves as future mathematics faculty, their understanding of the profession, and how they hope to be recognized as such by their prospective professional community. The initial codes that I conceptualized as dimensions of professional identity, included: (1) teacher identity, (2) researcher identity, and (3) mathematics identity. I analyzed themes across their responses and made sense of how these dimensions reflected similarities and differences in Kayla's and Morgan's identities and graduate education experiences.

Findings

I highlight the dimensions of participants' professional identities that most emerged when they reflected on their experiences and considered the ways they have seen themselves as faculty and how they hope to be recognized as such in the future. For this preliminary analysis, I focus on how Kayla's and Morgan's developing professional identities as faculty encompassed their teacher identities, researcher identities, and mathematics identities.

Teacher Identity

Kayla's and Morgan's professional identities as mathematics faculty were primarily anchored onto their identities as teachers (Berger & Lê Van, 2019; Mockler, 2011). They both described how an early passion for teaching, experiences with tutoring and helping others, and serving as teaching assistants or instructors of record during graduate school were formative and positively contributed to their professional identities. However, there were also differences in how each student recognized themselves as faculty or described what they would emphasize as a teacher or instructor.

Kayla said, "I've always known I wanted to teach." and mentioned that her decision to pursue a career as a mathematics faculty was primarily motivated by her desire to teach. She considers herself as an educator who is uniquely positioned to teach and support students in a variety of ways. As she reflected on her identity as a teacher, Kayla emphasized dimensions of teaching that went beyond subject matter expertise and the mathematics content. She expressed that getting to know students on a more personal level, connecting with them, and sharing about her own experiences are practices that she hopes to engage in. This was most influenced by her own experiences as a student when she learned the most from faculty who were more personable – explicitly articulating how she does not want to be like the professors she did not enjoy. An important aspect of Kayla's professional identity was that she saw herself as someone who can reframe dominant discourses and narratives around mathematics through teaching. She hopes to "reconfigure the narrative that mathematicians are these godlike people you can't talk to" and just make mathematics "more human" to the students. To Kayla, the experience that most formed her identity as a teacher was serving as an instructor of record, when she was afforded more ownership over the courses she was teaching, rather than just taking up a more supportive role as a teaching assistant to a faculty member.

Morgan shared a similar passion and early desire to teach students, however, given her more extensive experiences in graduate school, she recognized herself more jointly as a teacher and researcher in more evident ways than Kayla. To describe what being a mathematics faculty meant to her, she said "It means you get to teach students during the whole school year…You get to contribute to the field by research and publishing papers." Like Kayla, Morgan most recognized herself as a teacher when she was serving as an instructor of record during the summer terms – again emphasizing that employment positions that provide more ownership and agency of a course significantly contributes to professional socialization. Additionally, Morgan saw her identity as a teacher connected to her desire to mentor students and "foster collaborative environments" in mathematics. As she reflected on her future students and colleagues, she said, "I would like to be seen as someone that students want as a professor." We can see how Kayla and Morgan described their professional identities as teachers with direct consideration of how they would like to engage with and be perceived by students. Their teacher identities are consistent with literature on women faculty in STEM and how they tend to describe needing to be more caring and approachable (e.g., Guarino & Borden, 2017; Hart, 2016) and exhibit more student-centered pedagogies (e.g., McMinn et al., 2022).

Researcher Identity

Although both students acknowledged their identities as researchers (Castelló et al., 2020), Morgan identified more strongly as a researcher than Kayla, likely due to her more extensive research experiences, professional experiences (e.g., presenting conferences), mentorship from her advisor, and more time spent in graduate school. In the structure of the focal PhD program, students are not assigned an advisor until they pass their qualifying exams (i.e., advance to candidacy), and spend time reading scholarly works assigned by a potential research advisor. The mathematics department's norms with progressing through the degree milestones reflected the extent to which Kayla and Morgan disparately recognized themselves as researchers.

At the time of data collection, Kayla had not yet passed all her qualifying exams nor was in a position to complete readings and prepare for research with an advisor. Thus, Kayla seemed to identify less as a researcher when prompted to describe how she saw herself as a prospective mathematics faculty. She said, "I do think I'll probably end up doing research." which reveals an early understanding of what the profession may entail. Prior to graduate school, Kayla originally thought that mathematics faculty were solely focused on teaching. She said that the research aspect of the profession is something that she is still getting accustomed to.

On the other hand, Morgan explicitly said, "There's two aspects of being a math professor." – referring to her understanding of the mathematics professoriate as encompassing both teaching and researching responsibilities. She saw herself as someone who can contribute to the field by researching and publishing works, in addition to teaching mathematics to students. Morgan expressed that she wanted to be recognized as "someone who is known to be good at teaching but at the same time produces high quality mathematics." which reveals how her professional identity as a mathematics professor deeply intertwines her sense of self as a teacher and as a researcher. Morgan described that working with her advisor and completing her thesis were the most formative experiences to her identity and preparation as a researcher. Morgan more strongly identified as a researcher because she was in the latter stages of her program, whereas Kayla had only completed her first year. Within this particular institutional context, this revealed that graduate students are socialized as researchers later in the doctoral program.

Mathematics Identity

Lastly, Kayla and Morgan revealed that their mathematics identities (Cribbs et al., 2015; Voigt, 2020) were important dimensions of their professional identities as faculty. Despite recognizing themselves as math persons and being positioned as competent from an early age, graduate school marked a time when their mathematics identities were negatively affected. For Kayla and Morgan, adverse interactions and interpersonal relationships with members of the mathematics community damaged their self-concepts and identities. Kayla said, "I'm not *the* math person, I'm *a* math person." to describe how arriving in graduate school and seeing herself in relation to other students caused her to feel "less special" and made her question her place in mathematics. Additionally, Kayla articulated how her qualifying exams experience – particularly how she perceived the exams to be unnecessarily difficult and how she felt unsupported by faculty – caused her to doubt her career pursuits and sense of self as a mathematician.

It is important to note that Kayla and Morgan explicitly connected their mathematics identities with their gender identity and highlighted how navigating graduate school as women involved facing gender-based bias and microaggressions (Wilkins-Yel et al., 2019). Morgan was previously enrolled in another PhD program prior to her current program, and the experiences at that prior institution caused her to leave academia for a few years. She cited challenges such as the low salary to have contributed to her exiting this program, however, her identity as a mathematician and a prospective faculty was most negatively affected by how she was treated as a woman. She described the culture at the previous institution as "sexist" and said, "I felt like I had to work twice as hard to be taken seriously." To illustrate an instance of some of the microaggressions she experienced, she described an encounter with another graduate student:

> My first year, I won an award for best incoming, or best first year [student]. And then another grad student was like, "Oh, you just won that because you're a woman," or something like that. I felt like I wasn't really listened to as much and there, wasn't really supported.

Unfortunately, this is consistent with our contemporary understanding of the experiences of women-identifying graduate students or faculty in STEM (e.g., Herzig, 2004). The patriarchal and masculine discourses that have historically dominated the field (Leyva, 2021) fostered a climate where Morgan felt like her successes do not belong or are simply handed to her because of her gender identity, which in turn challenged her mathematics identity. After leaving that program, she spent a few years in industry before deciding to pursue a doctorate at the current institution. She has described feeling a sense of belonging in the current program and has more strongly identified as a mathematician because of the support she has received.

Discussion and Conclusion

The professional identity development of mathematics graduate students has been an often-overlooked dimension of the pathways to becoming mathematics faculty. The present work is aligned with scholars who have studied the socialization of graduate students (e.g., Clarke et al., 2013; Solari & Ortega, 2020), and have illustrated the complexity and multidimensionality of professional identities (e.g., Trede et al., 2011). I extend that work and highlight how dimensions of participants' professional identities (i.e., identities as teachers, as researchers, and as mathematicians) developed within higher education mathematics contexts. Kayla's and Morgan's professional identities reveal how formative the experiences of graduate education can be, and how students are more strongly anchored onto certain dimensions of their professional identities at different stages of their doctoral programs. Using professional identities as a lens also afforded insight into the experiences that seemed to drive or hinder professional identity development. For example, serving as instructors of record, where graduate students are given more agency and ownership, most developed their identities as teachers. Also, graduate students may identify more strongly as researchers later in their graduate education, which reflects the structure of the degree milestones. This suggests that mathematics departments can better support students' development as researchers at earlier stages of their graduate education or provide them more opportunities to serve as instructors of record.

Moreover, this preliminary study briefly illustrated how graduate students' professional identities, especially their mathematics identities, are influenced by their social and cultural identities. The tensions between participants' gender identities and their sense of selves as mathematics faculty demonstrates how mathematics can still be a patriarchal and masculine space (Herzig, 2004; Leyva, 2021). This suggests that for women-identifying graduate students, the self-recognition as mathematics faculty may be more difficult and challenging. Future research can more closely examine the gendered experiences of mathematics graduate students pursuing faculty positions and continue to make sense of how students' intersectional identities (e.g., race, sexual orientation, etc.) inform their professional identity development. In all, a more nuanced understanding of these dimensions of their identities allows higher education institutions and mathematics departments to broaden the access and reimagine systems of support that can better prepare graduate students to enter the mathematics professoriate.

References

Austin, A. E., & McDaniels, M. (2006). Preparing the professoriate of the future: Graduate student socialization for faculty roles. *Higher Education, 21,* 397-456. Springer. https://doi.org/10.1007/1-4020-4512-3_8

Berger, J. L., & Lê Van, K. (2019). Teacher professional identity as multidimensional: Mapping its components and examining their associations with general pedagogical beliefs. *Educational Studies*, *45*(2), 163-181. https://doi.org/10.1080/03055698.2018.1446324

Clarke, M., Hyde, A., & Drennan, J. (2013). Professional identity in higher education. In: Kehm, B., Teichler, U. (eds), *The academic profession in Europe: New tasks and new challenges* (pp. 7-21). Springer. https://doi.org/10.1007/978-94-007-4614-5_2

Castelló, M., McAlpine, L., Sala-Bubaré, A., Inouye, K., & Skakni, I. (2021). c *Higher Education*, *81*, 567-590. https://doi.org/10.1007/s10734-020-00557-8

Cribbs, J. D., Hazari, Z., Sonnert, G., & Sadler, P. M. (2015). Establishing an explanatory model for mathematics identity. *Child Development*, *86*(4), 1048-1062. https://doi.org/10.1111/cdev.12363

Gee, J. P. (1996). *Social linguistics and literacies: Ideology in discourses.* Routledge.

Guarino, C. M., & Borden, V. M. (2017). Faculty service loads and gender: Are women taking care of the academic family?. *Research in Higher Education, 58*, 672-694. https://doi.org/10.1007/s11162-017-9454-2

Haley, K. J., Jaeger, A. J., & Levin, J. S. (2014). The influence of cultural social identity on graduate student career choice. *Journal of College Student Development, 55*(2), 101-119. 10.1353/csd.2014.0017

Hart, J. (2016). Dissecting a gendered organization: Implications for career trajectories for mid-career faculty women in STEM. *The Journal of Higher Education, 87*(5), 605-634. https://doi.org/10.1080/00221546.2016.11777416

Herzig, A. H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research*, *74*(2), 171-214. https://doi.org/10.3102/00346543074002171

Jensen, J. L. (2011). Higher education faculty versus high school teacher: Does pedagogical preparation make a difference?. *Bioscene: Journal of College Biology Teaching*, *37*(2), 30-36. http://www.acube.org/files/6813/5328/1114/2011_2.pdf.

Leyva, L. A. (2021). Black women's counter-stories of resilience and within-group tensions in the white, patriarchal space of mathematics education. *Journal for Research in Mathematics Education*, *52*(2), 117-151. https://doi.org/10.5951/jresematheduc-2020-0027

McMinn, M., Dickson, M., & Areepattamannil, S. (2022). Reported pedagogical practices of faculty in higher education in the UAE. *Higher Education*, *83*(2), 395-410. https://doi.org/10.1007/s10734-020-00663-7

Miles, M. B., Huberman, A. M., & Saldaña, J. (2020). *Qualitative data analysis: A methods sourcebook.* SAGE.

Mockler, N. (2011). Becoming and 'being' a teacher: Understanding teacher professional identity. In Mockler, N., Sachs, J. (Eds.) *Rethinking educational practice through reflexive inquiry* (pp. 123-138). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-0805-1_9

Nyquist, J. D., Manning, L., Wulff, D. H., Austin, A. E., Sprague, J., Fraser, P. K., ... & Woodford, B. (1999). On the road to becoming a professor: The graduate student experience. *Change: The Magazine of Higher Learning*, *31*(3), 18-27. https://doi.org/10.1080/00091389909602686

Pillen, M., Beijaard, D., & Den Brok, P. (2013). Professional identity tensions of beginning teachers. *Teachers and Teaching*, *19*(6), 660-678. https://doi.org/10.1080/13540602.2013.827455
Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data*. Sage.
Solari, M., & Martín Ortega, E. (2020). Teachers' professional identity construction: A sociocultural approach to its definition and research. *Journal of Constructivist Psychology*, *35*(2), 1-30. https://doi.org/10.1080/10720537.2020.1852987
Tateo, L. (2012). What do you mean by "teacher"? Psychological research on teacher professional identity. *Psicologia & Sociedade*, *24*(2), 344-353. https://doi.org/10.1590/S0102-71822012000200012
Tierney, W. G., & Rhoads, R. A. (1993). Enhancing promotion, tenure and beyond: Faculty socialization as a cultural process. *ASHE-ERIC Higher Education Report No. 6*. ASHE-ERIC Higher Education Reports, The George Washington University. https://eric.ed.gov/?id=ED368321
Trede, F., Macklin, R., & Bridges, D. (2012). Professional identity development: a review of the higher education literature. *Studies in Higher Education*, *37*(3), 365-384. https://doi.org/10.1080/03075079.2010.521237
Trepte, S. (2013). Social identity theory. In *Psychology of Entertainment* (pp. 273-290). Routledge.
Van Lankveld, T., Schoonenboom, J., Volman, M., Croiset, G., & Beishuizen, J. (2017). Developing a teacher identity in the university context: A systematic review of the literature. *Higher Education Research & Development*, *36*(2), 325-342. https://doi.org/10.1080/07294360.2016.1208154
Voigt, M. (2020). *Queer-spectrum student experiences and resources in undergraduate mathematics* (Order No. 27745009) [Doctoral dissertation, University of California, San Diego]. ProQuest Dissertations & Theses Global. http://www.proquest.com/docview/2435858844/abstract/608F986CC12E4095PQ/1
Weidman, J. C., & Stein, E. L. (2003). Socialization of doctoral students to academic norms. *Research in Higher Education*, *44*(6), 641-656. https://doi-org/10.1023/A:1026123508335
Wilkins-Yel, K. G., Hyman, J., & Zounlome, N. O. (2019). Linking intersectional invisibility and hypervisibility to experiences of microaggressions among graduate women of color in STEM. *Journal of Vocational Behavior*, *113*, 51-61. https://doi.org/10.1016/j.jvb.2018.10.018
Yin, R.K. (2016). *Qualitative research from start to finish*, Second Edition. New York: The Guilford Press.

Estudiante Perceptions of Classroom Inclusion: A Pilot Study

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Feelings of inclusion in the classroom are one contributor to students' decisions to pursue a particular degree pathway, and students from marginalized groups are more likely to experience feelings of exclusion in these spaces. These feelings may be exacerbated by particular classroom actions and interactions, including microaggressions based on race and/or gender as well as being ignored or talked over; these actions are often compounded by instructor and student biases. Being able to observe such events is key to instructors' ability to disrupt those moments and develop more inclusive spaces, but many observation protocols are built from an external researcher perspective about practices that support equity. We report on a pilot study at an HSI, comparing what Hispanic and Latine students report makes them feel more or less included to what is typically captured by observation protocols.

Keywords: Estudiantes, Inclusion, Instructional Practice, Observation Protocols

Over the past decade, Estudiantes (students who identify as Hispanic, Latine/o/a/x, Chicane/o/a/x, Nuyorican, etc.) have experienced rapid growth (an increase of 53%) as a share of students enrolled in higher education (NCES, 2011; 2021). Likewise, the proportion of STEM degrees awarded to Estudiante undergraduates increased from 8.5% in 2008 to 15.1% in 2018 (NCSES, 2021). This growth has coincided with increased attention to STEM equity and inclusion, but Estudiantes in STEM fields continue to have negative experiences, such as encountering negative assumptions about their group, stereotype threat, and "scientific disidentification" during STEM coursework (Ong, Smith, & Ko, 2018; Woodcock et al., 2012). The knowledge base for creating equitable undergraduate classroom learning environments is growing yet remains underdeveloped. Scholars have called for postsecondary research to adopt a sociopolitical lens in ways that parallel the work done at the K-12 level (Adiredja & Andrews-Larson, 2017), supported by recent research illustrating that the links between active learning and equity (e.g., Tang et al., 2017; Laursen et al., 2014) are not consistent across contexts, and in fact, active learning has the potential to amplify or introduce new inequities (Johnson et al., 2020). Mathematical classrooms are not neutral spaces; they can, and often do, serve to perpetuate systemic inequities regardless of modality. Assessing the ways in which a class perpetuates or disrupts inequities is therefore valuable, however the majority of our field's instructional observation tools do not attend to equity, nor the perspectives of students in the classroom (Yee et al., 2022). We have both a moral and practical imperative to better understand inclusive instruction in partnership with students most affected by systematic racism. To further this goal, we report on a pilot study at an HSI, comparing what Estudiantes report makes them feel more or less included to what is typically captured by observation protocols.

Literature Review and Theoretical Perspective

This pilot project is part of a larger project aimed at developing new observation protocols for assessing the inclusivity of undergraduate mathematics classrooms for Estudiantes. We approach this broader work through a critical lens, drawing on tenets of LatCrit and QuantCrit. LatCrit is an extension and specification of Critical Race Theory (CRT) focused on the lived experiences of Hispanic and Latine people in the United States (Solorzano & Yosso, 2002). Consistent with CRT, LatCrit seeks to challenge the dominant ideologies of society that continue to propagate White patriarchal hegemony inside and outside the classroom (Battey & Leyva, 2016). A central tenet of QuantCrit is to bolster quantitative data with the experiential knowledge of people belonging to marginalized groups. This pilot is the start of an investigation of the lived experiences of Estudiantes in undergraduate mathematics classrooms that will be expanded and, in partnership with Estudiantes, inform the development of quantitative instruments that center their lived experiences.

In keeping with our critical lens, we incorporate the idea of *intersectionality*, which acknowledges and accounts for intersecting systems of oppression/marginalization and identities, highlighting the heterogeneity of experiences rather than homogenizing the lives of people sharing a particular identity characteristic (Collins & Bilge, 2016; Crenshaw, 1991). For example, the higher education experiences of Hispanic and Latine students vary significantly based on their economic class, gender, sexuality, and disabilities, among other factors (Byrd, Brunn-Bevel, & Ovink, 2019; Ovink, 2014). There is no singular "classroom experience" that is shared by all students in a classroom: students have multiple and differing experiences that are inflected by their identity and social location with reference to those of peers and instructors (Mohajeri et al., 2019). Applying an intersectional theoretical lens also focuses our attention on intragroup power dynamics: e.g., classroom experiences of Estudiantes will additionally vary because of power differentials between genders, upper- and lower-income status, other sources of social inequality, and histories of advantage and disadvantage, the legacies of which still reverberate in contemporary higher education environments (Collins, 1971; Karabel, 2005). We leverage intersectionality as we strive to develop local theories of how particular instructional approaches differentially affect students.

We see instructional practice—that is, what instructors actually do during instruction—as an essential element of promoting equitable classrooms, and an entry point where transformation can occur (Gutiérrez, 2012). Higher education literature also identifies the undergraduate classroom as critical to the experiences of minoritized students in terms of racial climates and the prevalence of microaggressions (Suárez-Orozco et al., 2015; Solorzano et al., 2000; Yosso et al., 2009). In the undergraduate mathematics education literature base, there are not many studies of classrooms and equity, and many of these focus on achievement or persistence outcomes (e.g., Laursen et al., 2014; Johnson et al., 2020) or instructor bias (e.g., Reinholz et al., 2019); these have primarily focused on inquiry-based learning (IBL) as the key to equitable instruction. We suggest two limitations of this: inquiry instruction is not a panacea for equity (e.g., Johnson et al., 2020) and most undergraduate instruction does not use inquiry (Larsen et al., 2015).

Recently, Leyva et al. (2020; 2021) have explored the ways that precalculus and calculus course instruction may reflect racialized and gendered mechanisms. Students' journaling revealed several mechanisms at play that reinforce broader discourses and stereotypes. For example, the "ignored student" event: students reported instructors more often ignored Black or Latinx students, potentially reinforcing "troublemaker" stereotypes about Black and Latinx students who spoke up and discouraging others of similar background from asking questions in class. A related finding is that the management of stereotype threat (particularly in relation to "who" is capable of doing mathematics) is significant to the STEM experience for undergraduates of color (e.g., McGee & Martin, 2011). Additional narrative, such as the myth of meritocracy, and advocacy for constructs like "grit" take a substantial toll on Latinx and Black students when societal racism is deemphasized, while individual perseverance is celebrated. These studies begin to unpack the experiences of minoritized students, but outside of a few cases and institutions, we still know little about these experiences in critical courses like precalculus and calculus (Leyva et al., 2021). We position our work as part of that which is needed "to document if, when, and how [inquiry] instructional approaches are equitable for all students" (Adiredja & Andrews-Larson, 2017, p. 459).

Methodology

Pilot data was collected in Summer 2023 at a large HSI in a state which shares a border with Mexico and has a high proportion of Estudiantes in their student body (40%); the proportion of bachelor's degrees awarded to Estudiantes is slightly lower (37%) and the proportion of STEM bachelor's degrees awarded to Estudiantes is slightly lower still (34%). Data comes from two courses taught in different modalities: Precalculus (flipped), Calculus 2 (in-person). Data collection for each class was conducted in the span of a single week. For each class, two sessions were video and audio-recorded on consecutive days, and interviews with students from that class were conducted on the following day. This will allow us to compare what is accessible to an observer with the moments that students found impactful. At the end of the second observation day, the observer/researcher (a Hispanic man) invited students who self-identify as Hispanic/Latine to participate in a group interview about their experiences. The two semi-structured interviews involved 2 men (Precalculus) and 1 woman (Calculus 2). Each lasted approximately one hour, and were also audio and video-recorded. Participants were asked to reflect on moments in which they felt included/welcome or excluded/unwelcome in the class, as well as specific actions that led them to believe that their teacher cared about them and other students (or did not); with an eye toward intersectionality, they were asked if they felt that their Estudiante and/or gender identities impacted their experience and/or treatment in the course.

Interviews were transcribed using Adobe Premiere Pro and manually checked for accuracy. Three members of the authorship team conducted the qualitative analysis of these transcripts, and that analysis was reviewed by the other authors. Analysis proceeded according to the stages of thematic analysis (Braun & Clarke, 2006). In the first phase, data familiarization, transcripts were developed and read independently by the researchers. In the second phase, the team independently made research memos (Maxwell, 2013) and generated initial codes of interest. We take a data-driven approach as part of our commitment to understanding what is salient to students themselves, while retaining a focus on students' feelings of inclusion/exclusion, perceptions of the impact of race/gender on their experiences, and aspects of classroom practice that students connect to those feelings. We then progressed to the interpretive phases, beginning with *thematizing*, in which individual codes are organized into broader umbrella themes and sub-themes. We are currently engaged in the fourth and fifth phases, in which the themes are reviewed in connection with the original data, edited, and finally named. In this report, we present the current state of our work, in which codes have been identified and themes have provisional names. Our next steps include analyzing the classroom data using three existing observation protocols and comparing those reports to the student interviews.

Preliminary Findings

All three students who were interviewed for this project report generally positive feelings about their current mathematics course, though they mention having negative experiences elsewhere. These were often brought up as a foil, in order to explain what *wasn't* wrong with their current class. As such, most themes emerged with both positive (inclusive, encouraging) and negative (exclusive, discouraging) codes; the exception being the *microaggression* theme. We present these themes and some of their constituent codes in Table 1.

Table 1. Themes and examples of codes within each theme from the interview analysis.

Theme	Positive Codes	Negative Codes
Involvement in daily classroom activities	Working in groups Going to the board Involved in in-class Q&A	Sitting and watching lecture
Personal relationship with instructor	Instructor shares personal information (e.g., hobbies) Instructor is fun / funny / personable	No insight into instructors' life Feeling like "a number"
Perception of inst. care [human]	Instructor knows name(s) Flexible deadlines / extensions	Instructor doesn't know name(s)
Perception of inst. care [math/learning]	Instructor's goal is everyone passes Instructor notices when Ss are confused Instructor tries to understand what Ss are thinking before helping Extra help sessions	Instructor states that many students will fail Instructor gives "how" solutions without "why"
Community [Class]	Knowing people in the class Having identity peers in class Small classes	Having trouble finding a group Large class / online class
Community [Uni]	Joining affinity groups Having identity peers on campus	Fewer Hispanic people than high school Seeing fewer Catholic people than home
Microaggressions	"How could you not understand this" Pretty girls aren't smart People assuming they're not smart	Being ignored in group projects Being told that engineering (STEM) is a "man's" major/field

From the themes that emerge, we note an emphasis on interpersonal relationships and community. These three students report that opportunities to connect on a quasi-personal level with an instructor who they believe cares about them and their learning, and enjoys being in the classroom with them, help them feel like they belong and are being included in a classroom community; similar things were reported regarding interactions with their peers and classmates. They also report experiences with distant instructors where they felt like "a number" and believed that the instructors were concerned only with doing the bare minimum with regards to interacting with them - those experiences left them feeling like they didn't belong or were unwanted in those academic spaces; again, there were similar comments regarding interactions with peers and classmates. These students also point to in-class instructional practices which made them feel more included and a part of the learning experience, such as working with other students (particularly when they were able to answer other students' questions), while large lectures felt impersonal and uninviting. All three students speak of a desire to be in community with others who shared identity markers, evincing clear awareness of the demographic makeup of their campus, majors, and classes. These ideas are evinced in the following exchange from the Precalculus interview:

Interviewer: If you could suggest any improvements or changes to this precalculus class to better support Hispanic/Latinx students, what would they be?

Student: Other than adding more Hispanic people? There's only like two or three in our class that are, like, Hispanic.
Interviewer: Well, how, what, how would that make it better?
Student: Um, because I think, like, because I'm going back to culture. Um, so like, where I'm from, like, everybody learns, you know, like the professor talks. And maybe that's why I'm so receptive to the professor, right, because he talks, like, on a personal level [...] And other professors here [...] were more impersonal, and maybe that's why the retention rate, like, for Hispanics is, like, lower, because they're just not, like, not relating to the professor. It's like a white, smart man who's like, you know, obviously he's worked hard for it, but maybe they just don't relate to it. I don't blame them, you know, but maybe it's just harder for them to understand or even want to be there, because they're uncomfortable [...] Nothing's familiar to them.

One theme did not fit in with the positive/negative coding, that of *microaggressions*, as they are inherently negative. We used this term to capture codes which referred to specific comments and interactions which negatively affected students' sense of inclusion. These include events that indicated others assumed they were not smart, although students could not always pinpoint *why* that assumption was being made. The two men in Precalculus were not specific about the root of the assumption, but they reported incidents which made them feel that way - including interactions with instructors and teaching assistants. The woman in Calculus 2 reported events which led her to wonder if people believed that she was "too pretty to be smart," compounded by others in her life making comments that "engineering is a man's major."

Discussion and Next Steps

All three students interviewed in this pilot study report feelings of inclusion in their current class, compared to prior experiences where they felt less sense of belonging. Common themes across the students included a recognition of the value of being in a community space and human interactions with peers and instructors. There were, of course, some differences in what they said. Only the woman noted gendered microaggressions and stereotypes, although the men recognized that women were a minority in their classes. The men emphasized the personal, humanized relationship they had with their instructor as valuable, while the woman did not mention this. Further work is needed to uncover the extent to which these distinctions are related to intersectional heterogeneity of identity, instructor, class modality, and/or course level. In this pilot, we inquired about students' ethnoracial and gender identity, and one student volunteered religious identity (Catholic) as a salient identity characteristic. Parsing heterogeneous intersectional experiences, while avoiding determinism, is one of the goals of this work.

The immediate next steps of this work are the application of three existing observation protocols to the class recordings, and comparison of the resulting reports to students' interviews. The three protocols are *Classroom Interpersonal MicroAggression* (CIMA; Suárez-Orozco et al., 2015), *Equity and Access Rubrics for Mathematics Instruction* (EAR-MI; Wilson, 2022), *Equity QUantified In Participation* (EQUIP; Reinholz & Shah, 2018). A cursory comparison of the themes and codes from the interview analysis with the constructs targeted by these protocols suggests the potential for considerable overlap with CIMA, some overlap with practices outlined in EAR-MI, but no overlap with the EQUIP constructs of participatory equity. Full analysis will reveal the extent to which these protocols capture these students' experiences in these particular classes. Future work will scale up this research to many more classrooms across multiple sites, building toward more generalizable findings.

References

Adiredja, A. P., & Andrews-Larson, C. (2017). Taking the sociopolitical turn in postsecondary mathematics education research. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 444–465. https://doi.org/10.1007/s40753-017-0054-5

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, *9*(2), 49–80.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Byrd, W. C., Brunn-Bevel, R. J., & Ovink, S. M. (Eds.). (2019). *Intersectionality and higher education: Identity and inequality on college campuses*. Rutgers University Press.

Collins, R. (1971). Functional and conflict theories of educational stratification. *American Sociological Review*, 1002–1019.

Collins, P. H., & Bilge, S. (2016). *Intersectionality*. John Wiley & Sons.

Crenshaw, K. (1991). Mapping the margins: Intersectionality, identity politics, and violence against women of color. *Stanford Law Review, 43*, 1241–1299.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in Discourse for Mathematics Education* (pp. 17–33). Springer Netherlands. https://doi.org/10.1007/978-94-007-2813-4_2

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504. https://doi.org/10.5951/jresematheduc-2020-0043

Karabel, J. (2005). *The chosen: The hidden history of admission and exclusion at Harvard, Yale, and Princeton*. Houghton-Mifflin Harcourt.

Larsen, S., Glover, E., & Melhuish, K. (2015). Beyond good teaching. *insights*, 93.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406-418.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2020). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. 1–34. https://doi.org/10.1080/07370008.2020.1849218

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). “It seems like they purposefully try to make as many kids drop”: An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *The Journal of Higher Education*, *92*(5), 784–814. https://doi.org/10.1080/00221546.2021.1879586

Maxwell, J. A. (2013). *Qualitative research design* (3rd ed.). SAGE.

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture? *American Educational Research Journal, 53*(6), 1626–1662. doi:10.3102/0002831216676572

McGee, E. O. (2020). Interrogating structural racism in STEM higher education. *Educational Researcher, 49*(9), 633-644.

McGee, E. O., & Martin, D. B. (2011). “You would not believe what I have to go through to prove my intellectual value!” Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal, 48*(6), 1347–1389. https://doi.org/10.3102/0002831211423972

Mohajeri, O., Rodriguez, F., & schneider, finn. (2019). Pursuing intersectionality as a pedagogical tool in the higher education classroom. In W. C. Byrd, R. J. Brunn-Bevel, & S. M. Ovink (Eds.), *Intersectionality and Higher Education: Identity and Inequality on College Campuses*. Rutgers University Press.

NCES (National Center for Education Statistics), Integrated Postsecondary Education Data System (IPEDS). (2011). Enrollment component. Retrieved September 2023.

NCES (National Center for Education Statistics), Integrated Postsecondary Education Data System (IPEDS). (2021). Enrollment component. Retrieved September 2023.

NCSES, (National Center for Science and Engineering Statistics). (2021). *Women, minorities, and persons with disabilities in science and engineering* (Special Report NSF 21-321). National Science Foundation. https://ncses.nsf.gov/wmpd

Ong, M., Smith, J.M., & Ko, L.T. (2018). Counterspaces for women of color in STEM higher education: Marginal and central spaces for persistence and success. *Journal of Research in Science Teaching 55*(2):206–45. https://doi.org/10.1002/tea.21417.

Ovink, S. M. (2014). "They always call me an investment": Gendered familism and Latino/a college pathways. *Gender & Society*, *28*(2), 265–288. https://doi.org/10.1177/0891243213508308

Reinholz, D. L., Pilgrim, M. E., Corbo, J. C., & Finkelstein, N. (2019). Transforming undergraduate education from the middle out with Departmental Action Teams. *Change: The Magazine of Higher Learning*, *51*(5), 64–70. https://doi.org/10.1080/00091383.2019.1652078

Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education, 49*(2), 140-177.

Suárez-Orozco, C., Casanova, S., Martin, M., Katsiaficas, D., Cuellar, V., Smith, N. A., & Dias, S. I. (2015). Toxic rain in class: Classroom interpersonal microaggressions. *Educational Researcher, 44*(3), 151-160.

Solorzano, D., Ceja, M., & Yosso, T. (2000). Critical Race Theory, Racial Microaggressions, and Campus Racial Climate: The experiences of African American college students. *The Journal of Negro Education, 69*(1/2), 60–73. http://www.jstor.org/stable/2696265

Solórzano, & Yosso, T. J. (2002). Critical Race Methodology: Counter-Storytelling as an analytical framework for education research. *Qualitative Inquiry, 8*(1), 23–44. https://doi.org/10.1177/107780040200800103

Tang, G., El Turkey, H., Cilli-Turner, E., Savic, M., Karakok, G., & Plaxco, D. (2017). Inquiry as an entry point to equity in the classroom. *International Journal of Mathematical Education in Science and Technology*, *48*(sup1), S4–S15. https://doi.org/10.1080/0020739X.2017.1352045

Wilson, J. (2022). Initial steps in developing classroom observation rubrics designed around instructional practices that support equity and access in classrooms with potential for "success." *Teachers College Record: The Voice of Scholarship in Education, 124*(11), 179–217. https://doi.org/10.1177/01614681221140963

Woodcock, A., Hernandez, P. R., Estrada, M., & Schultz, P. W. (2012). The consequences of chronic stereotype threat: Domain disidentification and abandonment. *Journal of Personality and Social Psychology, 103*(4), 635–646. https://doi.org/10.1037/a0029120

Yee, S. P., Rogers, K. C., Miller, E., & Galvin, T. (2022). Collegiate mathematics observation protocols: Current characteristics and desired foci. *International Journal of Research in Undergraduate Mathematics Education*, 1-32.

Yosso, T. J., Smith, W., Ceja, M., & Solórzano, D. (2009). Critical race theory, racial microaggressions, and campus racial climate for Latina/o undergraduates. *Harvard Educational Review*, *79*(4), 659–691. https://doi.org/10.17763/haer.79.4.m6867014157m707l

What Mathematicians' Emotions Can Teach Us About Groupwork and Emotion Regulation

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Two mathematics professors at a large university were interviewed about how their emotions impact their research work. Joy and frustration were the emotions they reported experiencing most frequently. Both reported the benefits of collaboration and the importance of regulating both positive and negative emotions. Implications on collaborative learning in the classroom (e.g. groupwork, pair work) and implications for emotions in the classroom are discussed.

Keywords: emotions, collaborative learning, emotion regulation

This paper brings together research on emotions in mathematics education, collaborative learning, and the professional work of mathematicians.

Literature Review

Emotions and Mathematics

There has been an increased emphasis on research on emotions and affect in mathematics education (McLeod, 1992; Middleton et al., 2017). In recent years, there have been more publications in mathematics education journals focused on emotions and affect, and the number of publications appears to be continuing to increase (Schukajlow et al., 2023). Both negative emotions (D'Mello et al., 2014; Schukajlow et al., 2021) and positive emotions (Camacho-Morles et al., 2021) have been shown to be beneficial to student learning. However, both negative and positive emotions have been shown to be detrimental to student learning (Barenes, 2021; Villavicencio et al., 2016). Villavicencio et al. (2016) found that pride had a negative association with final course grades among engineering students in the Philippines. Barnes (2021) found that joy was associated with decreased focus and self regulation.

Collaborative Learning

Research on collaborative learning tends to focus on social components or the connections between collaborative learning and mathematical ideas (e.g. Langer-Osuna et al., 2020; Dekker et al., 2006; Fujita et al., 2019; Sjöblom et al., 2021).

Additionally, research on collaborative learning in classrooms includes work on collaboration promoting students building off of each other's ideas which in turn promotes individual learning (Francisco, 2013; Schindler & Bakker, 2020).

Some work has been done demonstrating emotional benefits of collaborative learning for students. Group work can have a positive impact on student problem posing (Schindler & Bakker, 2020). Students also report higher levels of perceived control (Pekrun, 2006) when working in pair or small groups which is associated with higher reported positive emotions (Bieg et al., 2017).

Mathematicians as Research Participants

Research on mathematicians has focused mainly on the cognitive aspects of their work. Researchers have explored mathematicians' thinking and problem solving abilities (Burton, 1998; Carlson & Bloom, 2005). Other work has shown that mathematicians enjoy collaboration with others, and that part of this enjoyment is through getting the intellectual input from colleagues on their work (Burton, 1998). Mathematicians' negative feelings of being stuck on solving a problem and potential implications on students' negative emotions towards mathematics have also been discussed in the literature with regards to choosing particular problems to do (Misfeldt & Johansen, 2015).

However, research focused specifically on mathematician's emotions appears to be a new area of study. The previously mentioned literature is focused on the cognitive side of mathematics rather than emotional components of doing and learning mathematics.

Theoretical Perspective

Two areas of psychological research provide a background for this study: the Broaden and Build Theory of Positive Emotions and The Cognitive Theory of Emotions. In relation to the mathematicians in this study, these theories imply that a) positive emotions provide two distinct benefits and b) emotions are the result of cognitive appraisals of events.

Broaden and Build Theory of Positive Emotions

The Broaden and Build Theory describes two distinct benefits of positive emotions beyond just feeling good (Fredrickson, 2001). First, positive emotions broaden awareness. When people experience positive emotions, they are more aware of "the bigger picture" and notice more than when they are feeling negative emotions. Second, this broadened state of awareness created by positive emotions allows individuals to build more resources for themselves. The explanatory power of this theory has been demonstrated by multiple authors (e.g. Fredrickson, 2004; Garland et al., 2010; Vacharkulksemsuk & Fredrickson, 2013).

For this study, this creates the perspective that positive emotions could give mathematicians benefits beyond just feeling good. Positive emotions could allow them to potentially work more creatively or be interested in solving new problems.

Cognitive Theory of Emotions

The Cognitive Theory of Emotions posits that cognitive appraisals of a situation are what triggers an emotional response to that situation (Ortony, Clore, & Collins, 2022). For example, a student taking a test realizes they do not know how to answer the questions. The student wants to do well on the test, but now knows that they will fail. Thus, the student feels anxious because of their appraisals that a) the test is important and b) they are unable to succeed. It is not the test itself which makes the student anxious. It is the student's interpretation of the situation that leads to the emotions.

Research Questions

The focus of this research was investigating the emotions experienced by mathematicians, and their interpretations of significance of these emotions. The guiding research questions were: (1) How can the emotions experienced by mathematicians be characterized?, (2) In what ways do mathematicians feel that emotions impact their work?, (3) In what ways does collaboration with other mathematicians relate to these emotions?

Methods and Analysis

Two active research mathematicians at a large research university in the Southwestern United States participated in interviews over Zoom. Professors were recruited via an email sent to all full time faculty members of the mathematics department. The participants will be referred to as Professor 1 and Professor 2.

Both participants were full research professors at the university, and they both described themselves as theoretical mathematicians and both were male.

The interviews lasted between 30 and 40 minutes. Participants were asked about their current mathematical research interests and the role that their emotions play in their work. They were asked to describe positive and negative emotions felt while doing their research. The interviews were loosely structured around the impact of the participants' emotions on their research, as well as emotions related to collaboration with other researchers. Interviews were recorded over Zoom then transcribed.

Due to the limited research on mathematicians' emotional experiences, the interviews were coded using open coding to identify common themes of mathematicians' emotions (Strauss & Corbin, 1990). After completing the interviews, field notes were taken by the researcher on any details from the interviews which felt relevant in the moment. Then, the interviews were transcribed. The transcribed interviews were read with the goal of identifying specific emotions experienced and how the participants felt the emotions impact their work. Preliminary categories of emotions were grouped together. Afterwards, the interviews were reread multiple times to compare the categories found with the interviews.

Results

Joy and frustration were the most common emotions reported. Without being prompted, both participants discussed the importance of emotion regulation of both joy and frustration in order to do their work effectively. Both participants also discussed the impact of collaboration on their emotions. These results are discussed in more detail below.

Joy: A Common Positive Emotion

Joy was the positive emotion that both participants described as feeling most often while doing their research work.

Collaboration Enhances Joy Both participants felt that working with others increased their joy in research. Professor 1 said, "I think it [collaboration] enhances the joy. So when you prove something, you're happy to prove it. But, you're even more happy if you could show it to someone." Being able to share their positive feelings with others amplifies the joy that these mathematicians feel from their work. Professor 2 said, "If I'm working with a co-author, then it's nice to have it [an idea] affirmed by the co-author. And then, when that happens, that feels really good."

Emotional Regulation of Joy However, these mathematicians did not feel that joy and positive emotions should be fully embraced. When discussing joy, both participants felt that joy should be treated with some caution and distrust while doing their work. After a probing question on Aha! moments of sudden realization (Liljedahl, 2005), Professor 2 said, "I always have to temper my enthusiasm… you get a stroke of brilliance…want to come back and write it down… Oh no, that's not gonna work. So I'm careful to not get too excited until I write it down, and see if it all works."

Professor 1 shared a story of another mathematician who would just lay down in bed and enjoy the feeling of believing he had figured something out because, in Professor 1's words, "9 times out of 10 - what you've done is wrong".

Frustration: A Common Negative Emotion

Both professors described frustration as their most commonly experienced negative emotion. Again, both spoke of the importance of regulating that frustration.

Collaboration Helps Regulate Frustration Both professors discussed collaboration as a method of dealing with the inevitable frustrations of research. Co-authors were mentioned by both participants as a useful tool to offset feelings of frustration.

Professor 1 said, “It [collaboration] may actually spread it [frustration] out so that it’s more tolerable. It’s more tolerable to be frustrated if you’re frustrated at the same thing that somebody else is frustrated at.” After a slight pause, he continued and said, “Isn’t that the reason for help groups?” in reference to group therapy. Professor 1 felt strongly that collaborative work helped him deal with the frustration of being stuck on a problem. Being able to share the load of frustration with another made the frustration easier to bear.

Regulation of Frustration in Other Ways - Taking Breaks Professor 2 spoke frequently of the importance of regulating frustration through taking breaks from research work. Specifically, he said “the other thing I've always done throughout my career is, I do a lot more than just research. Like, I teach classes. I get involved with some administrative work and I think for me that was always important. I didn't really like to have a job where 100% of the time is spent doing research”. He also discussed that stepping back from a problem and looking at examples is a “first way of relieving the frustration” for him.

Professor 2 actively works to keep frustration at bay and prevent burnout in his work.

Another Benefit to Collaboration: A New Perspective

Both professors described the benefits of being able to use a colleague’s perspective to benefit their work.The perspective of others helps them get through when they are stuck and having trouble making progress.

Professor 1 describes it as, “You try to prove things, and you know at least 9 tenths of the time. You can't do it. So yeah, you have to keep trying, and that's another reason why it's worthwhile to work with other people. Because you have to have lots of ideas to try it. I mean if you can't do something… You never know whether that’s because you're being stupid, or because it’s not true or really hard.”.

Professor 2 says, “ We can share what we're stuck on with the other. And you know, maybe the coauthor oftentimes sees something that you don't see And then that's a a way to make a forward progress”.

Discussion

Both collaboration and emotion regulation appear to play an important role in the work of mathematicians. Limitations and future directions will be discussed.

Limitations

Emotions are a vulnerable topic to discuss, thus the topic itself creates many limitations. Validity of self-reported data on emotions depends on many factors. First, participants may not be fully honest with their self-reported emotions (Rasinski et al., 2005). Also, social pressure exists against reporting negative emotions (Tourangeu & Yan, 2007). Social pressure may have presented itself in this study through participants focusing on overcoming and regulating the negative emotion of frustration while not discussing the actual experience of frustration in much detail. In other words, social pressure against negative emotions may have led the participants to focus the discussion on ending negative emotions rather than discussing the experience of negative emotions. An additional limitation is the possibility that participants may not have the emotional intelligence (Salovey & Mayer, 1990) to recall and name their emotions.

All of the limitations discussed above are out of the control of the researcher, but should be considered when interpreting the results.

Future Directions

Emotions and Collaboration in the Classroom As discussed in the literature review above, most research on collaborative learning focuses on cognitive rather than emotional factors in the classroom. Considering the results of this study, future research could examine how students can provide emotional support for each other while working through problems.

Collaboration is an important part of their work for these mathematicians. Yet group and pair work in the classroom is not frequently used, while lecture style direct instruction is likely the most common form of instruction (Bieg et al., 2017; Givvin et al., 2005). Both positive and negative emotions were shown to be an important part of mathematicians' work, and collaboration has a positive effect on both the positive and negative emotions. Confusion, an emotion similar to frustration, has been shown to be beneficial for learning in adults if the confusion is resolved (D'Mello et al., 2014). Thus, lecture style teaching in which confusion and frustration are avoided may be detrimental to student learning. Science education research has seen a push to engage students in ways that mimic the feelings of scientists (Jaber & Hammer, 2016). Perhaps, a similar movement can occur in mathematics education.

Emotion Regulation for Students With the discussed importance of emotion regulation of both positive and negative emotions, future research can examine ways of promoting this regulation among students. Some existing work provides promising results from an intervention on helping students cope with confusion in mathematics at the elementary level (Di Leo & Muis, 2020). This work can be extended to help students regulate other negative emotions. Students tend to report more negative emotions related to mathematics as they progress through the school system (Brown et al., 2008; Frenzel et al., 2009; Zazkis, 2015). Thus, it is crucial to help students to regulate these increasing levels of negative emotions at the undergraduate level. Other interventions can also be developed to help students regulate positive emotions. A study with elementary students has shown that positive emotions can be a barrier to perseverance in mathematical reasoning (Barnes, 2021), so similar interventions promoting regulation of positive emotions could also provide fruitful and beneficial results.

References

Barnes, A. Enjoyment in learning mathematics: its role as a potential barrier to children's perseverance in mathematical reasoning. *Educ Stud Math* 106, 45–63 (2021). https://doi.org/10.1007/s10649-020-09992-x

Bieg, M., Goetz, T., Sticca, F., Brunner, E., Becker, E., Morger, V., & Hubbard, K. (2017). Teaching methods and their impact on students' emotions in mathematics: An experience-sampling approach. *ZDM*, 49, 411-422.

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics?. *Educational studies in mathematics*, 37(2), 121-143.

Brown, M., Brown, P., & Bibby, T. (2008). "I would rather die": Reasons given by 16-year-olds for not continuing their study of mathematics. *Research in mathematics education*, 10(1), 3-18.

Camacho-Morles, J., Slemp, G. R., Pekrun, R., Loderer, K., Hou, H., & Oades, L. G. (2021). Activity achievement emotions and academic performance: A meta-analysis. *Educational Psychology Review*, 33(3), 1051-1095.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational studies in Mathematics*, 58, 45-75.
Dekker, R., Elshout-Mohr, M., & Wood, T. (2006). How children regulate their own collaborative learning. *Educational Studies in Mathematics*, 62, 57-79.
Di Leo, I., & Muis, K. R. (2020). Confused, now what? A Cognitive-Emotional Strategy Training (CEST) intervention for elementary students during mathematics problem solving. *Contemporary Educational Psychology*, 62, 101879.
D'Mello, S., Lehman, B., Pekrun, R., & Graesser, A. (2014). Confusion can be beneficial for learning. *Learning and Instruction*, 29, 153-170.
Fujita, T., Doney, J., & Wegerif, R. (2019). Students' collaborative decision-making processes in defining and classifying quadrilaterals: a semiotic/dialogic approach. *Educational Studies in Mathematics*, 101, 341-356.
Francisco, J. M. (2013). Learning in collaborative settings: Students building on each other's ideas to promote their mathematical understanding. *Educational Studies in Mathematics*, 82(3), 417-438.
Fredrickson, B. L. (2001). The role of positive emotions in positive psychology: The broaden-and-build theory of positive emotions. *American psychologist*, 56(3), 218.
Fredrickson, B. L. (2004). The broaden–and–build theory of positive emotions. Philosophical transactions of the royal society of London. Series B: Biological Sciences, 359(1449), 1367-1377.
Frenzel, A. C., Goetz, T., Lüdtke, O., Pekrun, R., & Sutton, R. E. (2009). Emotional transmission in the classroom: Exploring the relationship between teacher and student enjoyment. *Journal of educational psychology*, 101(3), 705.
Garland, E. L., Fredrickson, B., Kring, A. M., Johnson, D. P., Meyer, P. S., & Penn, D. L. (2010). Upward spirals of positive emotions counter downward spirals of negativity: Insights from the broaden-and-build theory and affective neuroscience on the treatment of emotion dysfunctions and deficits in psychopathology. *Clinical psychology review*, 30(7), 849-864.
Givvin, K. B., Hiebert, J., Jacobs, J. K., Hollingsworth, H., & Gallimore, R. (2005). Are there national patterns of teaching? Evidence from the TIMSS 1999 video study. Comparative Education Review, 49(3), 311-343.
Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational studies in Mathematics*, 49, 193-223.
Jaber, L. Z., & Hammer, D. (2016). Learning to feel like a scientist. *Science Education*, 100(2), 189-220.
Langer-Osuna, J., Munson, J., Gargroetzi, E., Williams, I., & Chavez, R. (2020). "So what are we working on?": How student authority relations shift during collaborative mathematics activity. *Educational Studies in Mathematics*, 104, 333-349.
McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. Handbook of research on mathematics teaching and learning, 1, 575-596.
Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In J. Cai (Ed.), Compendium for Research in Mathematics Education (pp. 87-119). Reston, VA: National Council of Teachers of Mathematics.

Misfeldt, M., Johansen, M.W. Research mathematicians' practices in selecting mathematical problems. Educ Stud Math 89, 357–373 (2015). https://doi.org/10.1007/s10649-015-9605-3
Mueller, M., Yankelewitz, D., & Maher, C. (2012). A framework for analyzing the collaborative construction of arguments and its interplay with agency. *Educational Studies in Mathematics*, 80, 369-387.
Ortony, A., Clore, G. L., & Collins, A. (2022). The cognitive structure of emotions. Cambridge university press.
Pekrun, R. (2006). The control-value theory of achievement emotions: Assumptions, corollaries, and implications for educational research and practice. *Educational psychology review*, 18, 315-341.
Rasinski, K. A., Visser, P. S., Zagatsky, M., & Rickett, E. M. (2005). Using implicit goal priming to improve the quality of self-report data. *Journal of Experimental Social Psychology*, 41(3), 321-327.
Salovey, P., & Mayer, J. D. (1990). Emotional intelligence. Imagination, cognition and personality, 9(3), 185-211.
Schukajlow, S., Rakoczy, K. & Pekrun, R. Emotions and motivation in mathematics education: Where we are today and where we need to go. *ZDM Mathematics Education* 55, 249–267 (2023). https://doi.org/10.1007/s11858-022-01463-2
Schukajlow, S., Blomberg, J., Rellensmann, J., & Leopold, C. (2021). Do emotions and prior performance facilitate the use of the learner-generated drawing strategy? Effects of enjoyment, anxiety, and intramathematical performance on the use of the drawing strategy and modelling performance. *Contemporary Educational Psychology*, 65, 101967.
Sjöblom, M., & Meaney, T. (2021). "I am part of the group, the others listen to me": theorising productive listening in mathematical group work. *Educational Studies in Mathematics*, 107(3), 565-581.
Strauss, A., & Corbin, J. (1990). Basics of qualitative research. Sage publications.
Tourangeau, R., & Yan, T. (2007). Sensitive questions in surveys. *Psychological bulletin*, 133(5), 859.
Vacharkulksemsuk, T., & Fredrickson, B. L. (2013). Looking back and glimpsing forward: The broaden-and-build theory of positive emotions as applied to organizations. In Advances in positive organizational psychology (Vol. 1, pp. 45-60). Emerald Group Publishing Limited.
Villavicencio, F.T., Bernardo, A.B.I. Beyond Math Anxiety: Positive Emotions Predict Mathematics Achievement, Self-Regulation, and Self-Efficacy. *Asia-Pacific Edu Res* 25, 415–422 (2016). https://doi.org/10.1007/s40299-015-0251-4
Zazkis, D. (2015). Monsters, lovers and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics*, 35(1), 33-38.

Where is Calculus in Chemistry? Determining Content Alignment Between Calculus and Chemistry

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Previous research indicates that mathematics ability predicts student success in chemistry courses. Not surprisingly, chemistry students are required to take courses in mathematics as prerequisites and co-requisites, including calculus. In this work, we seek to identify the ways calculus skills and concepts appear in the undergraduate chemistry curriculum, focusing on the introductory general chemistry sequence. To this end, we combined relevant disciplinary frameworks and looked for alignment: the Calculus Content Framework (CCF) and the Anchoring Concepts Content Map (ACCM) for general chemistry. Preliminary results indicate explicit use of calculus is not common in introductory chemistry, although there are opportunities for instructors to use the general chemistry topic of chemical kinetics as a context for the application of differentiation and integration. Future work will involve mapping out the landscape of the remainder of the undergraduate chemistry curriculum and as an expected outcome of this work we seek to provide tangible resources to facilitate cross-disciplinary discussions between instructors in calculus and chemistry.

Keywords: Calculus, Chemistry, Curriculum Analysis, Interdisciplinary

Introduction

There is a large body of literature that indicates students' mathematical ability correlates with success in chemistry courses (Bain et al., 2014; Becker & Towns, 2012; Derrick & Derrick, 2002; Hahn & Polik, 2004; House, 1995; Nicoll & Francisco, 2001; Spencer, 1996; Tsaparlis, 2007; Wagner et al., 2002), which is not surprising given the large role mathematics plays in typical chemistry courses (Stowe et al., 2021). That said, student performance on standardized mathematics assessments (e.g., SAT) is often used to identify at-risk students and identify particularly challenging chemistry topics (Lewis & Lewis, 2007; Ralph & Lewis, 2018). Based on the emphasis placed on mathematics as a tool to explain and predict in chemistry, it is not uncommon to have prerequisite and co-requisite calculus courses for undergraduate students taking chemistry. For example, at the University of Wisconsin – Milwaukee, chemistry majors are required to take three semesters of calculus (covering topics including limits, derivatives, integrals, partial derivatives, and vectors), with the recommendation (not requirement) to take a course focusing on linear algebra and differential equations("Chemistry Major," 2024). However, there is a need to evaluate the alignment between the mathematics used in chemistry courses and the mathematics presented in calculus courses. To this end, we are interested in mapping the calculus concepts (Sofronas et al., 2011) on to chemistry concepts (Murphy et al., 2012).

In broad strokes, this project is the result of an interest in developing resources that further interdisciplinary connections and continue the on-going dialogue regarding how disciplines such as chemistry and mathematics can support one another to improve teaching and learning. Recently the authors attended *The Learning and Teaching of Calculus Across Disciplines* conference in Bergen, Norway (*Welcome*, 2023). The aim of this interdisciplinary conference was to bring researchers from multiple fields (biology, chemistry, economics, engineers, physicists) together to discuss how each discipline uses calculus concepts. The current preliminary report builds on a plenary talk given by Dr. Brian Faulkner at this conference that discussed prior work involving analyzing where calculus concepts are used in an engineering course (Faulkner et al., 2020), with Faulkner's work mirroring a study in mathematics that analyzed the role of calculus concepts in an advanced mathematics course (Czocher et al., 2013). Focusing on connections between chemistry and calculus communities, the intended outcome of this work is a practical tool for calculus instructors to see the contexts and extent calculus is used across the undergraduate chemistry curriculum to inform instructional choices. The guiding research question addressed in this report is, *Which concepts and skills learned in calculus are applied in general chemistry?*

Disciplinary Frameworks

The scope of this study is supported by readily accessible disciplinary-specific frameworks that outline the target content goals for chemistry (Murphy et al., 2012) and calculus (Sofronas et al., 2011). In the case of chemistry, the American Chemical Society Exams Institute developed the Anchoring Concepts Content Map (ACCM), which was developed through workshops with content experts (Murphy et al., 2012). There is currently an ACCM for the following chemistry course sequences in the undergraduate chemistry curriculum: general chemistry (Holme et al., 2015); organic chemistry (Raker et al., 2013); inorganic chemistry (Marek et al., 2018); physical chemistry (Holme et al., 2018); and analytical chemistry (Holme et al., 2020). The general multi-tiered structure for each ACCM involves a series of nested concepts that become increasingly more specific: big idea (also called anchoring concept), enduring understanding, subdisciplinary articulation, and content detail. For each ACCM, the big idea and enduring understanding levels reflect broad concepts that are relevant throughout the chemistry curriculum, whereas the subdisciplinary articulation and content detail levels are specific to individual courses (or course sequences), such as general chemistry or organic chemistry. Within the ACCM framework, there are ten big ideas, one of which is *Kinetics*. To illustrate the structure of the ACCM, the levels from the general chemistry ACCM are provided related to the *Kinetics* big idea (Holme et al., 2015):

1. Big Idea. Kinetics: Chemical changes have a time scale over which they occur.
2. Enduring Understanding. Chemical change can be measured as a function of time and occurs over a wide range of time scales.
3. Subdisciplinary Articulation. The rate of the reaction must be defined in a manner that is not dependent on which reactant or product is used to measure it.
4. Content Detail. The reaction rate should incorporate reaction stoichiometry when it is defined.

In the case of describing the calculus concepts, this study utilizes the Calculus Content Framework (CCF) (Sofronas et al., 2011). The CCF was developed through interviewing experts in the calculus field, including prominent textbook authors, experienced educators, and researchers. The focus of the interview questions was to elicit instructional goals for students in a first-year calculus sequence. To this end, the resulting CCF is organized based on four broad categories: (a) mastery of the fundamental concepts and/or skills of the first-year calculus; (b) construction of connections and relationships between and among concepts and skills; (c) the ability to use the ideas of the first-year calculus, and (d) a deep sense of the context and purpose of the calculus. Prior work has illustrated the utility of the CCF, with Czocher et al. (2013) applying the framework to investigate the necessary calculus content and skills in an advanced mathematics course and Faulkner et al. (2020) doing the same for an engineering course. In both cases, the researchers found a lack of alignment across courses, suggesting the need to consider not just whether calculus is used in future coursework, but also how calculus is used.

Methods

Data Collection & Analysis

To map calculus topics to chemistry, we focused on the *subdisciplinary articulation* level within the general chemistry ACCM, which typically represents a two-semester course sequence for first-year students. This was compared to the calculus concepts and skills developed in the CCF (*mastery of the fundamental concepts and/or skills of the first-year calculus*, discussed above). Our approach for this report is based on the methods described by (Czocher et al., 2013) and (Faulkner et al., 2020), which involved attending to mathematics-in-use to assess where and how the calculus appears in general chemistry. Using the mathematics-in-use technique requires analyzing the chemistry content (as described in the subdisciplinary articulation of the ACCM) and noting the ways calculus concepts and/or skills are involved by solving problems associated with the subdisciplinary articulation. As part of this, it is helpful to assess secondary routes that are available to solve the problem which often results in situations where calculus skills and concepts are an available route to successfully solve the problem, but another route is favored. To this end, the subdisciplinary articulations were evaluated based on whether: (0) there was no calculus associated with the related concepts and skills; (1) calculus was present, but it was not the preferred method to solve related problems and reason through and the concepts; (2) the application of calculus was the preferred approach toward solving problems and reasoning through the concepts. For this work, we report on our analysis related to the subdisciplinary articulations that fall nested under the ten big ideas in the general chemistry ACCM.

Preliminary Results

Our preliminary analysis, shown that the only calculus concept or skill present in general chemistry is derivatives. The subdisciplinary articulations that involve this concept are shown in Table 1. The only big idea from the ACCM that involved calculus concepts and skills (derivative) was chemical kinetics, which had three subdisciplinary articulations characterized as having calculus present, but not as the preferred approach. The remainder of the subdisciplinary articulations across the ten big ideas in the general chemistry ACCM did not involve calculus concepts and skills.

Table 1. ACCM subdisciplinary articulations involving derivatives.

The rate of the reaction must be defined in a manner that is not dependent on which reactant or product is used to measure it.
Rate is generally defined as the change in concentration of a reactant or product as a function of time.
When solids are included in reactions, surface area is an important factor in the rate of reaction.
Laboratory observation of reaction rates helps to establish the concept of reaction time scales empirically.

Chemical kinetics involves modeling the rate of a reaction to track changes in the concentration of reactants as a function of time. Thus, although chemical kinetics could readily be modeled using the derivative, this perspective is often circumvented in a first-year chemistry course. In the chemistry curriculum, topics are often explained with less rigorous mathematics detail, and subsequently revisited in upper-level courses. For example, one subdisciplinary articulation that was coded as having an alternative calculus explanation available was: “Rate is generally defined as the change in concentration of a reactant or product as a function of time.” In first-year general chemistry courses, the rate of a simple reaction ($A + B \rightarrow AB$) is typically defined as:

$$rate = k[A]^m[B]^n \quad (1)$$

Here, *rate* is the rate of the reaction, *k* is the rate constant, *m* and *n* are empirically determined reaction orders that describe the concentration dependence of rate, and the final equation is known as the *rate law*. This is the most common way to discuss chemical kinetics with introductory students, without associating the equation with derivatives. With this version of the rate law, students are expected to determine the reaction order, solve for the value of *k,* and generally, use algebra computational skills to manipulate variables and compute values.

In contrast, in upper-level courses such as physical chemistry, the rate law would be discussed using the differential form:

$$-\frac{d[AB]}{dt} = k[A]^m[B]^n \quad (2)$$

In the case, we make use of calculus to describe the rate of the reaction to afford a more complicated analysis of chemical systems. Often, the rate law for a reaction is integrated to further describe reactions (Table 2). The integrated rate law is helpful because the rearranged linear form of the integrated rate law can be used in laboratory contexts to determine reaction order by fitting kinetics data and assessing its agreement with the equations for each reaction order.

Table 2. Summary of equations corresponding to each reaction order.

	First-Order Reaction	**Second-Order Reaction**	**Zero-Order Reaction**
Chemical Reaction	$A \rightarrow B$	$2A \rightarrow B$	$A \xrightarrow{catalyst} B$
Rate Law	$rate = \frac{-d[A]}{dt} = k[A]^1$	$rate = \frac{-d[A]}{dt} = k[A]^2$	$rate = \frac{-d[A]}{dt} = k[A]^0 = k$
Integrated Rate Law	$-\int_{[A]_0}^{[A]} \frac{d[A]}{[A]} = k\int_0^t dt$	$-\int_{[A]_0}^{[A]} \frac{d[A]}{[A]^2} = k\int_0^t dt$	$-\int_{[A]_0}^{[A]} d[A] = k\int_0^t dt$
	$\ln\frac{[A]_0}{[A]} = kt$	$\frac{1}{[A]} - \frac{1}{[A]_0} = kt$	$[A] - [A]_0 = -kt$
Linear	$\ln[A] = -kt + \ln[A]_0$	$\frac{1}{[A]} = kt + \frac{1}{[A]_0}$	$[A] = -kt + [A]_0$

Conclusion and Questions

Within the first-year general chemistry course, calculus concepts and skills are not prominent, with the relevant mathematics required being related to other mathematics courses (e.g., algebra). This is not to suggest that calculus concepts and skills are not relevant for modeling phenomena in chemistry, but that calculus becomes more important in undergraduate chemistry courses as students advance to higher coursework. As an implication, we highlight that although calculus is often a prerequisite or corequisite for students in introductory chemistry courses, students struggling with the mathematics in chemistry need additional support and practice with mathematics at the pre-calculus level. Regarding future work, as part of the ongoing analysis, we are interested in expanding our focus toward the ways calculus concepts and skills are used across the rest of the undergraduate chemistry curriculum. Especially in upper-level chemistry courses such as physical chemistry in which calculus is heavily integrated through the curriculum. Moreover, our goal is for this work to be helpful to connect the mathematics and chemistry communities by discussing how we use the shared language of calculus. For the scope of this conference report, we have some general questions for the mathematics community:

1. What are some of the ways chemistry instructors and chemistry education researchers can support mathematics instructors and researchers interested in undergraduate mathematics education?
2. What suggestions do you have regarding the methodological approach and scope of this work?

References

Bain, K., Moon, A., Mack, M. R., & Towns, M. H. (2014). A review of research on the teaching and learning of thermodynamics at the university level. *Chem. Educ. Res. Pract.*, *15*(3), 320–335. https://doi.org/10.1039/C4RP00011K

Becker, N. M., & Towns, M. (2012). Students' understanding of mathematical expressions in physical chemistry contexts: An analysis using Sherin's symbolic forms. *Chemistry Education Research and Practice*, *13*(3), 209–220. https://doi.org/10.1039/C2RP00003B

Chemistry Major. (2024). *Chemistry & Biochemistry*. Retrieved January 4, 2024, from https://uwm.edu/chemistry/undergraduate/chemistry/

Czocher, J. A., Tague, J., & Baker, G. (2013). Where does the calculus go? An investigation of how calculus ideas are used in later coursework. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 673–684. https://doi.org/10.1080/0020739X.2013.780215

Derrick, M. E., & Derrick, F. W. (2002). Predictors of success in physical chemistry. *Journal of Chemical Education*, *79*(8), 1013–1016. https://doi.org/10.1021/ed079p1013

Faulkner, B., Johnson-Glauch, N., Choi, D., & Herman, G. L. (2020). When am I ever going to use this? An investigation of the calculus content of core engineering courses. *Journal of Engineering Education*, *109*(3), 402–423. https://doi.org/10.1002/jee.20344

Hahn, K. E., & Polik, W. F. (2004). Factors influencing success in physical chemistry. *Journal of Chemical Education*, *81*(4), 567–572. https://doi.org/10.1021/ed081p567

Holme, T. A., Bauer, C., Trate, J. M., Reed, J. J., Raker, J. R., & Murphy, K. L. (2020). The American Chemical Society exams institute undergraduate chemistry anchoring concepts content map V: Analytical chemistry. *Journal of Chemical Education*, *97*(6), 1530–1535. https://doi.org/10.1021/acs.jchemed.9b00856

Holme, T. A., Luxford, C., & Murphy, K. (2015). Updating the general chemistry anchoring concepts content map. *Journal of Chemical Education*, *92*(6), 1115–1116. https://doi.org/10.1021/ed500712k

Holme, T. A., Reed, J. J., Raker, J. R., & Murphy, K. L. (2018). The ACS exams institute undergraduate chemistry anchoring concepts content map IV: Physical chemistry. *Journal of Chemical Education*, *95*(2), 238–241. https://doi.org/10.1021/acs.jchemed.7b00531

House, D. J. (1995). Noncognitive predictors of achievement in introductory college chemistry. *Research in Higher Education*, *36*(4), 473–490. https://doi.org/10.1007/BF02207907

Lewis, S. E., & Lewis, J. E. (2007). Predicting at-risk students in general chemistry: Comparing formal thought to a general achievement measure. *Chem. Educ. Res. Pract.*, *8*(1), 32–51. https://doi.org/10.1039/B6RP90018F

Marek, K. A., Raker, J. R., Holme, T. A., & Murphy, K. L. (2018). The ACS exams Institute undergraduate chemistry anchoring concepts content map III: Inorganic chemistry. *Journal of Chemical Education*, *95*(2), 233–237. https://doi.org/10.1021/acs.jchemed.7b00498

Murphy, K., Holme, T. A., Zenisky, A., Caruthers, H., & Knaus, K. (2012). Building the ACS exams anchoring concept content map for undergraduate chemistry. *Journal of Chemical Education*, *89*(6), 715–720. https://doi.org/10.1021/ed300049w

Nicoll, G., & Francisco, J. S. (2001). An investigation of the factors influencing student performance in physical chemistry. *Journal of Chemical Education*, *78*(1), 99–102. https://doi.org/10.1021/ed078p99

Raker, J., Holme, T. A., & Murphy, K. (2013). The ACS exams institute undergraduate chemistry anchoring concepts content map II: Organic chemistry. *Journal of Chemical Education*, *90*(11), 1443–1445. https://doi.org/10.1021/ed400175w

Ralph, V. R., & Lewis, S. E. (2018). Chemistry topics posing incommensurate difficulty to students with low math aptitude scores. *Chemistry Education Research and Practice*, *19*(3), 867–884. https://doi.org/10.1039/C8RP00115D

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, *30*(2), 131–148. https://doi.org/10.1016/j.jmathb.2011.02.001

Spencer, H. E. (1996). Mathematical SAT test scores and college chemistry grades. *Journal of Chemical Education*, *73*(12), 1150–1153. https://doi.org/10.1021/ed073p1150

Stowe, R. L., Scharlott, L. J., Ralph, V. R., Becker, N. M., & Cooper, M. M. (2021). You are what you assess: The case for emphasizing chemistry on chemistry assessments. *Journal of Chemical Education*, *98*, 2490–2495. https://doi.org/10.1021/acs.jchemed.1c00532

Tsaparlis, G. (2007). Teaching and learning physical chemistry: A review of educational research. In M. D. Ellison & T. A. Schoolcraft (Eds.), *Advances in Teaching Physical Chemistry* (Vol. 973, pp. 75–112). American Chemical Society. https://doi.org/10.1021/bk-2008-0973.ch007

Wagner, E. P., Sasser, H., & DiBiase, W. J. (2002). Predicting students at risk in general chemistry using pre-semester assessments and demographic information. *Journal of Chemical Education*, *79*(6), 749. https://doi.org/10.1021/ed079p749

Welcome. (2023). The Learning and Teaching of Calculus Across Disciplines. https://matriccalcconf2.sciencesconf.org/

Professional Obligations of Graduate Student Instructors and Undergraduate Learning Assistants in Partnership within Active Learning Mathematics Classrooms

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Learning Assistants (LAs) are undergraduate students who support active learning classrooms by acting as a second instructional figure in the classroom. Although research has shown a variety of benefits from the integration of LAs, little is known about the nature of the LA-instructor relationship and its impact on instruction. We present preliminary findings from a comparative case study of how graduate student instructors of record (GSIs) and undergraduate learning assistants (LAs) work together within an active learning classroom. Specifically, we focus on how the interactions between GSIs and LAs relate to the professional obligations felt by GSIs and LAs, both of whom are positioned as mathematics instructors, albeit in different ways. This preliminary report highlights initial findings and suggestions for further investigation.

Keywords: Learning Assistants, Graduate Student Instructors, Active Learning, Obligations

Research suggests that undergraduate students who are hired as near-peers can support students' academic, social, and emotional well-being (Barrasso & Spilios, 2021; Dawson et al., 2014; Whitman & Fife, 1988). There are many different near-peer models, including supplemental instruction, tutoring, peer-led team learning, and the learning assistant (LA) model. However, institutions are increasingly relying on the LA model, or variations of it, to support change efforts centered on the use of active learning pedagogies in STEM classrooms (Learning Assistant Alliance, 2023; Otero et al., 2006). Although there has been much research conducted about the LA model in other STEM disciplines, there is a dearth of research on the integration of LAs in mathematics classrooms (Barrasso & Spilios, 2021). Furthermore, existing literature suggests that implementation of the LA model varies considerably across different institutions and contexts, such as lectures, recitations, and online courses (Hill et al, 2023). Literature is also scant on the nature of the partnerships between LAs and instructors (Barrasso & Spilios, 2021). Although some researchers are investigating these relationships (Davenport et al., 2017; Jardine 2020; McHenry et al., 2010; Sabella et al., 2016), instructors are typically faculty. Some research has compared self-perceptions of the roles of graduate teaching assistants, learning assistants, and professors (Becker et al., 2016), but there is little known about LAs working with graduate students acting as instructors of record (GSIs). Given the increasing prevalence of LA programs and the large proportion of mathematics departments that hire GSIs to teach mathematics courses, it would benefit our community to investigate the nature of this relationship and its influence on instruction.

To that end, we investigated how GSIs and LAs described their interactions in and outside of an active learning mathematics classroom, focusing on the conditions (e.g., professional obligations) that influence those interactions. We asked the following research question:

> *RQ: To what extent did GSIs and LAs of an active learning mathematics classroom interact with each other in and outside of the classroom, and how does this relate to the professional obligations they felt in their roles?*

[1] Williams and Funk contributed equally to this work.

Theoretical Background

Instruction involves interactions between students, teacher(s), and content, but is also constrained by the environmental conditions in which these interactions take place (Cohen et al., 2003; Hawkins, 1974). These environments act as a source for the professional obligations felt by those positioned as mathematics teachers, which are in turn used to justify what teachers do in mathematics classrooms. Herbst & Chazan's (2011, 2012) theorize that four professional obligations influence the decision making by individuals who take on the position of mathematics teacher: disciplinary, individual, interpersonal, and institutional. See Figure 1 for definitions of these obligations.

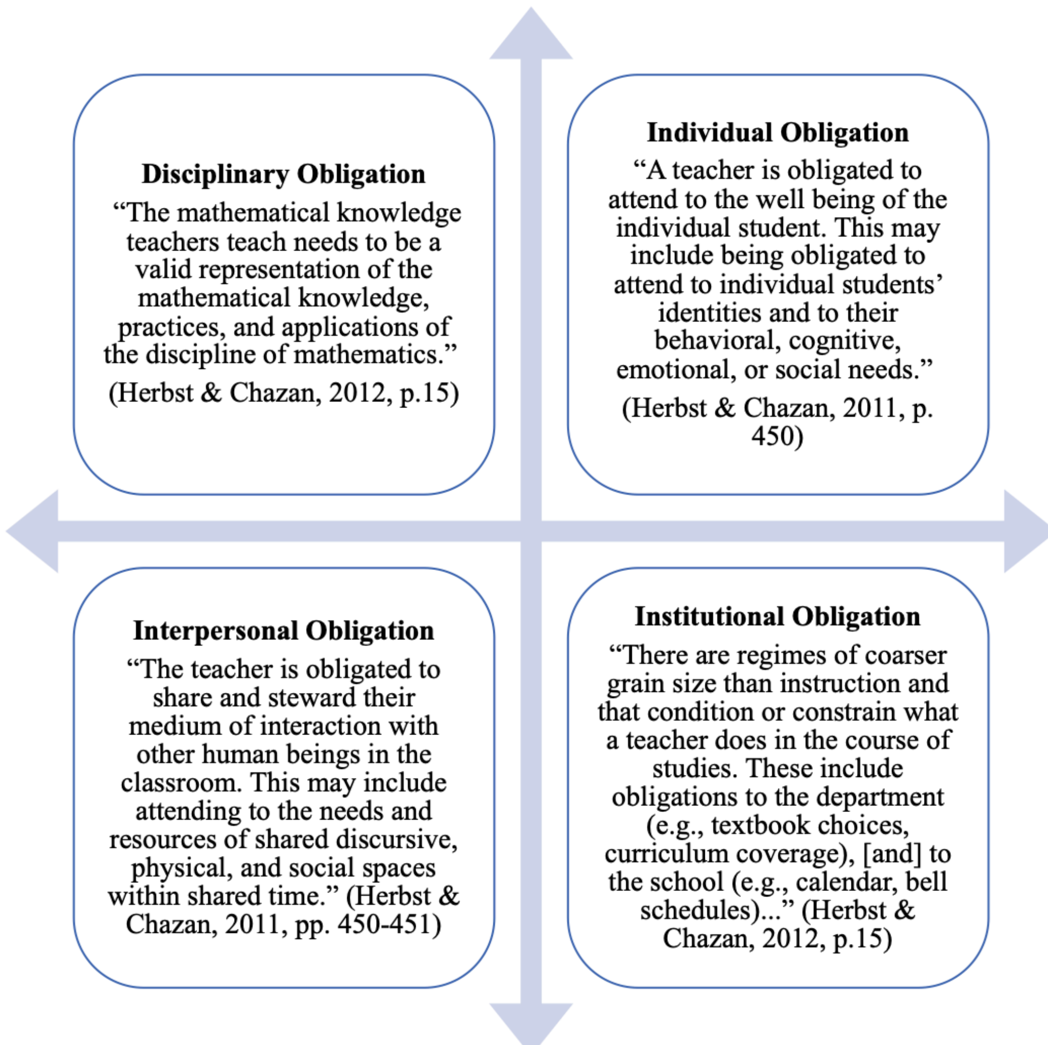

Figure 1. Herbst & Chazan's (2011, 2012) professional obligations

Instructors may feel these obligations acutely or be relatively unaware of them, nevertheless these obligations exist for every mathematics teacher. Furthermore, instructors who use active learning may become more aware of these obligations as obligations come into conflict. For example, instructors may feel an obligation to cover prescribed content (an institutional obligation) competing with an obligation to let students engage deeply with mathematics (a disciplinary obligation) (Mesa et al., 2020). Using the LA model could change what these obligations look like. Specifically, the obligations felt by GSIs and LAs may shift depending upon how GSIs and LAs are positioned in the classroom and in relation to one another. Although at times LAs may be positioned as mathematics teachers, and thus are subject to the same four professional obligations as that of an instructor, LAs take on additional positions in the classroom given their status as undergraduate students (Funk, 2023; Jardine, 2020). As such they may feel different obligations or feel these obligations differently. For example, LAs and GSIs may feel an individual obligation to students, but LAs might feel a stronger obligation to connect to individual students on a personal, friendly level, whereas the GSI may not feel that similar obligation because they are assigning grades.

These professional obligations are not the only obligations that teachers may feel in a classroom, however they provide a basis for understanding the decisions that teachers make in a classroom (Herbst & Chazan, 2012). We contend that integrating LAs into a classroom adds additional complexity related to the interpersonal obligation - how are LAs and GSIs sharing discursive, physical, and social spaces? Do instructors feel obligated to interact with one another? These are important questions to consider, as they influence how GSIs and LAs navigate their roles and relationship within an active learning environment, ultimately influencing the decisions they make as instructors.

Methods

This report draws from a larger multiple case study (Kaarbo & Beasley, 1999; Stake, 2006) to examine how the roles of GSIs and LAs may shift in a mathematics department involved in change efforts to improve their courses. The larger study includes data collected from two separate dissertation studies, both occurring within the same mathematics department but at different points in time. The first study occurred in Fall 2015, at the start of the mathematics department's efforts to transform their precalculus courses using an active learning, coordinated model. This study focused on the relationship and evolving beliefs of a GSI-LA pair across a semester, who had previously worked together in Spring 2015. The second study was conducted once the department's change efforts had been sustained for multiple years, but involved changes to instruction due to the COVID-19 pandemic. For this study, data were collected from Summer 2020-Fall 2021, and involved a total of 18 GSIs and 9 LAs, including 4 GSI-LA pairs.

Although multiple forms of data were collected in these dissertations, for this report we draw on interview data to examine how GSIs and LAs described their relationship. Despite being drawn from two separate studies, interview questions focused on similar concepts, including beliefs about the role of mathematics instructors and the GSI-LA relationship and changes to that relationship (e.g., "Did you notice any changes in [LA/GSI] over the course of the semester?" -Fall 2015 Interview Question; "What are the major roles and responsibilities for LAs that support your class? How do these roles and responsibilities differ from yours?" -Fall 2020 Interview Question).

We began analysis by reading the dissertations to identify specific instances of GSIs and LAs describing their interactions. Importantly, we operationalized interactions as encompassing direct conversations, as well as observations or noticings of each other's teaching. We then traced these pieces of evidence back to individual transcripts to examine the raw data. These instances were open coded (Miles et. al, 2014) by each author individually, then they were compared and discussed to generate preliminary themes. Through this analysis we identified variations in how participants interacted with one another in and outside of class, noting differences in the obligations felt by participants in these interactions. We use the findings sections of this proposal to elaborate on this theme. In future data analysis, we plan to analyze the entire corpus of interview data for evidence of professional obligations using Herbst and Chazan's framework as an a priori analytic framework.

Preliminary Findings

We found evidence of three different levels of obligation felt by the GSI and LA to interact: a definite sense of obligation, some obligation, and no obligation to interact. These different levels sometimes depended on whether interactions occurred inside or outside the classroom. In the following paragraphs we share evidence from each level of obligation.

Some GSIs felt a definite sense of obligation to interact with their LAs. This was evident from their insistence on the importance of communicating with or observing their LA in or outside of class. For many, this sense of obligation was tied to an obligation to ensure active learning methods were being used by LAs. GSI Jay said that for them to be an effective teacher, their LA's teaching "approach and philosophy" had to align with their own "at least in practice, if not necessarily in belief." Jay intentionally observed LA Jessie early in the semester to ensure that Jessie was using active learning. They said, "I know that my LA is not going into rooms and just giving students answers, right? That my LA is not undermining the structure [of the active learning course]." GSI Jay also hoped to treat LA Jessie as a "colleague" and a "peer" outside of class; they met frequently outside of class time to discuss student dynamics and approaches to teaching content. This related to their desire that during class the LA and themselves presented a "united front." Similarly, GSI Sally intentionally had "several quite long conversations about what it means to teach, what's the most effective way to teach, and what I want to happen in the classroom" with her LA. Sally admitted, "when we first started teaching together last semester, LA Cara stressed me out more than the students did" but Sally had come to trust her. Both GSI Jay and GSI Sally proactively interacted with their LAs to fulfill an obligation that the course be taught using active learning methods. Furthermore, their LAs sought-out interactions with the GSI to learn how to teach the course. For LA Jessie, this was in part inspired by an obligation to ensure they were presenting mathematics in a way that was aligned with the instructor. They described observing GSI Jay so that they were using "the same wording that [they do] in class." Further, they said it was important to communicate with the GSI to make sure "you're always on the same page, cause you don't want to split the class into two."

GSIs and LAs who had a less definite obligation to interact often described feeling constraints on their ability to interact, particularly in class. GSI River identified being a novice teacher as one such constraint, saying that "there is a cognitive overload" when one first starts teaching that prevents an instructor from paying attention to the LA. They elaborated by saying that "paying attention to what my LA was doing was a lower priority" for them, as they were trying to make sure they did "a good job" teaching. River believed that, as a more experienced instructor, they would now be able to "manage paying attention to what [their] LA is doing." However, other more experienced GSIs described feeling constrained by the dynamics of an active learning classroom. As GSI Aiden stated:

> I always had a really hard time knowing whether my LA was doing a good job because I was just very focused on teaching and trying to go to all of the groups and get a sense of how class is doing. So I just completely ignored my LA basically all the time until we reconvened at the end of class.

For GSI Aiden, the obligation to interact with each student group took precedence over an obligation to make sure the LA "was doing a good job." They did, however, reconvene with their

LA at the end of class, suggesting that they felt an obligation to interact with their LA beyond class time. Similarly, Instructor Blake felt an obligation to "hit every group so I don't miss anyone" and described not knowing what their LA was doing because they were "focused on the student." Unlike GSI Aiden, Blake interacted very minimally with their LA outside of class time, suggesting that Blake operated relatively autonomously from the LA.

Like Blake, GSI Poppy talked about ignoring their LA because they were busy "personally knowing the students." However, they also questioned the appropriateness of monitoring LAs during group work, suggesting that LAs are meant to "support us [GSIs] monitoring the [group] work." GSI Harper shared similar sentiments, but with a stronger statement. They felt no obligation to their LA during the class session and further felt this was in line with the views of the department. They said:

> I guess just sort of the general advice, or attitude I inherited from the department…they're [LAs] there to help you, they're there to help students. I don't know if this is advice or just a statement about how my classroom works, but don't focus on the LA. Assume that they're off doing good work on the other side of the classroom and instead just work with the students.

Thus, Harper did not feel the need to have any focus on the LA beyond assuming the LA was there to support the GSI and the active learning environment.

Discussion & Questions for Audience

Our findings highlight that participants felt different levels of obligation to interact with one another, and further that these differences can in part be accounted for by the context of the interaction (in or outside the classroom) and other professional obligations. For example, some GSIs prioritized having conversations with their LAs outside of class time to ensure that LAs explained content in ways faithful to the discipline (from their perspective) or to support LAs in using active learning methods. Likewise, one LA felt an interpersonal obligation to keep the classroom from splitting into two, and thus prioritized being on the "same page" as the GSI through conversations outside of class time. Some GSIs used in-class observations to ensure that LAs were using active learning strategies to engage students. However, one GSI felt no obligation to interact with their LA in the classroom, and furthermore described their view as being "inherited" from the department, suggesting that they felt an institutional obligation to not "focus on the LA" during class time. Several GSIs remarked on the difficulty of attending to students and observing the LA, suggesting that while they felt an interpersonal obligation to interact, this took less precedence than their obligation to interact with students. This brings up several questions: How do GSIs and LAs prioritize competing professional obligations? How does department culture influence this prioritization? Do GSIs and LAs prioritize competing obligations differently? Should GSIs feel an obligation to interact (or notice) LAs? Should LAs feel an obligation to interact (or notice) GSIs?

More work needs to be done to consider how Herbst and Chazan's (2011, 2012) framework supports understanding of the GSI-LA relationship, particularly in active learning classrooms. For example, White and Smith (2023) suggest that the interpersonal obligation should be magnified to support inclusion in active learning classrooms; they argue that instructors should explicitly be encouraged to support student-student interactions. In what ways does integrating LAs into a classroom add additional possibilities for magnification of the interpersonal obligation?

References

Funk, R. (2023). *Positioning undergraduate learning assistants in instruction: A case study of the LA role in active learning mathematics classrooms at the University of Nebraska-Lincoln.* (Doctoral dissertation). https://digitalcommons.unl.edu/mathstudent/119/

Barrasso, A. P., Spilios, K. E. (2021). A scoping review of literature assessing the impact of the learning assistant model. *International Journal of STEM Education, 8*(12). https://doi.org/10.1186/s40594-020-00267-8

Becker, A. P., Goldberg, B., & Jariwala, M. (2016). Self-perception of teaching fellows and learning assistants in introductory physics classes. *Physics Education Research Conference Proceedings*, 48–51. https://doi.org/10.1119/perc.2016.pr.007

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142. https://doi.org/10.3102/01623737025002119

Davenport, F., Amezcua, F., Sabella, M. S., Van Duzor, A. G. (2017). Exploring the underlying factors in learning assistant—faculty partnerships. *Cincinnati: Proceedings of the Physics Education Research Conference*. Retrieved June 5, 2023, from https://www.compadre.org/Repository/document/ServeFile.cfm?ID=14579&DocID=4756

Dawson, P., van der Meer, J., Skalicky, J., & Cowley, K. (2014). On the effectiveness of supplemental instruction: A systematic review of supplemental instruction and peer-assisted study sessions literature between 2001 and 2010. *Review of Educational Research, 84*(4), 609-639. https://doi.org/10.3102/0034654314540007

Hawkins, D. (1974). I, thou, and it. In *The informed vision: Essays on learning and human nature* (pp. 48-62). Agathon Press.

Herbst, P., & Chazan, D. (2011). Research on Practical Rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405-462. https://doi.org/10.54870/1551-3440.1225

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM, 44*(5), 601–612. https://doi.org/10.1007/s11858-012- 0438-6

Hill, C.J., Barrasso, A.P., & Spilios, K.E. (2023). A mixed-methods analysis of perspectives toward learning assistant-faculty relationships. *Journal of College Science Teaching, 52*(3). 75-83.

Jardine, H. E. (2020). Positioning undergraduate teaching and learning assistants as instructional partners. *International Journal for Students as Partners, 4*(1), 48–65. https://doi.org/10.15173/ijsap.v4i1.4032

Kaarbo, J., & Beasley, R. K. (1999). A practical guide to the comparative case study method in political psychology. *Political psychology*, *20*(2), 369-391.

Learning Assistant Alliance. (2023). Retrieved May 25, 2023, from www.learningassistantalliance.org

McHenry, N., Martin, A., Castaldo, A., & Ziegenfuss, D. (2010). Learning assistants program: Faculty development for conceptual change. *International Journal of Teaching and Learning in Higher Education, 22*(3), 258–268.

Mesa, V., Shultz, M., & Jackson, A. (2020). Moving away from lecture in undergraduate mathematics: Managing tensions within a coordinated inquiry-based linear algebra

course. *International Journal of Research in Undergraduate Mathematics Education, 6*(2), 245–278. https://doi.org/10.1007/s40753-019-00109-1
Miles, M.B., Huberman, A.M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook*. (3rd ed.). Sage.
Otero, V., Finkelstein, N., McCray, R., & Pollock, S. (2006). Who is responsible for preparing science teachers? *Science, 313*(5786), 445–446. https://doi.org/10.1126/science.1129648
Sabella, M. S., Van Duzor, A. G., & Davenport, F. (2016). Leveraging the expertise of the urban STEM student in developing an effective LA Program: LA and instructor partnerships. *2016 Physics Education Research Conference Proceedings*, 288–291. Retrieved June 5, 2023, from https://www.compadre.org/Repository/document/ServeFile.cfm?ID=14250&DocID=4604
Stake, R. (2006). *Multiple Case Study Analysis*. The Guilford Press.
White, N. & Smith, B. A. (2023). Understanding instructors' beliefs and obligations within cultures of exclusion. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 217-225). Omaha, NE.
Whitman, N.A & Fife J.D. (1988). *Peer teaching: to teach is to learn twice.* ASHE-ERIC Higher Education Report No. 4. Association for the Study of Higher Education.

Exploring the Interplay Between Preservice Teachers' Mathematical Beliefs and EB Students' Mathematics Education

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We explored the intersections between two sets of pre-service teachers' beliefs, revealing potential relationships between beliefs about mathematics and beliefs about EB mathematics education. PSTs whose beliefs about mathematics were consistent with reform-oriented mathematics were identified as social constructivists, and they were more likely to express equitable beliefs about EB mathematics education. However, we find a more mixed pattern regarding beliefs about EB mathematics education among PSTs identified as moderately-traditionalists. Specifically, we find that the association between PSTs' demographic backgrounds and engagement with EB issues and their endorsement of more moderate beliefs may differ across different groups of PSTs. This research contributes to understanding the complex interplay of teacher beliefs in mathematics and EB education, offering insights that can inform more effective teacher-preparation programs to enhance PSTs' ability to teach EB students effectively.

Keywords: Emergent Bilinguals, Pre-Service Teachers, Beliefs about Mathematics

While teacher-preparation programs have made strides in adopting asset-oriented perspectives regarding what Emergent Bilinguals (EB) students can achieve, research reveals that many pre-service teachers (PSTs) still graduate underprepared to meet the mathematical needs of EB students, including prevalent deficit-oriented views about how EB students learn mathematics (e.g., Fernandes, 2007; National Center for Education Statistics, 2013; Pappamihiel et al., 2017). These beliefs strongly influence the instructional practices that PSTs enact in their mathematics classrooms (Akinsola, 2008; Beswick, 2005; De Araujo, 2017; Wilkins, 2008).

Recent research has started exploring the experiences shaping PSTs' beliefs about EB students' mathematics education, including PSTs' beliefs about mathematics, its teaching, and learning and how these appear to influence their views on how EB students should learn mathematics (e.g., De Araujo, 2017; Janzen, 2008; McLeman & Fernandes, 2012). For that reason, identifying relationships between PSTs' beliefs about mathematics and EB students can further inform the development of more effective teacher-preparation programs. By addressing PSTs' preconceived notions about mathematics learning and teaching, these programs can enhance PSTs' capabilities to teach EB students. The study aims to answer two research questions:

1. What beliefs do PSTs seem to hold about the nature of mathematics, its teaching and learning, and the mathematics education of EB students?
2. Are there any relationships between PSTs' beliefs about the nature of mathematics, its teaching and learning, and the mathematics education of EB students?

The study utilizes a quantitative approach to collect data and situates its work within the broader context of research on teacher beliefs, encompassing beliefs about mathematics and the education of EB students. The hypothesis is that PSTs who share beliefs about mathematics that are aligned with reform-oriented efforts of mathematics teaching and learning will also share equity-oriented beliefs towards EB learners and mathematics.

Literature Review and Theoretical Background

Teacher Beliefs about the Nature, Teaching, and Learning of Mathematics

According to Borg's theoretical framework (1997, 2003, 2005) on teacher cognition, teachers enter teaching with existing mathematics beliefs, knowledge, and attitudes categorized as either instrumentalist, Platonist, or problem-solving orientations (Ernest, 1991; Raymond, 1997). These orientations represent distinct views of mathematics: instrumentalists see it as facts and rules, Platonists as static interconnected knowledge, and problem-solvers as dynamic and inquiry-oriented. This has spurred extensive research into exploring such teachers' beliefs about mathematics through various survey instruments, including the Mathematics Belief Instrument (Hart, 2002), the Mathematics Beliefs Questionnaire (Adnan & Zakaria, 2010), and the Teachers Beliefs Survey (Beswick, 2005; Hughes, 2016). These studies consistently reveal a divide between teachers who view mathematics as interconnected knowledge best acquired through student-guided experiences and inquiry-based learning and those who see mathematics as a finite set of facts and procedures conveyed directly to students, with limited room for exploration.

More recently, Hughes (2016) adapted Ernest's framework, categorizing instrumentalist and problem-solving/Platonist beliefs as traditional and constructivist beliefs, respectively. Research finds that teachers tend to align with either traditional or constructivist beliefs (Hughes, 2016; Paolucci, 2015). However, recently among newer teachers, another group has emerged whose mixed beliefs combine elements of both traditions. However, the literature on teacher beliefs rarely delves into the factors influencing these beliefs about mathematics' nature, teaching, and learning. Limited studies suggest a positive link between teachers' mathematical backgrounds (including coursework and teaching knowledge) and constructivist beliefs about mathematics (Ogan-Bekiroglu & Akkoç, 2009; Kutaka et al., 2018). Yet, the evidence in this area is inconclusive regarding the role of other factors, such as experiences with linguistically or culturally diverse students, on teachers' beliefs about mathematics.

Teacher Beliefs about the Mathematics Education of EB Students

In recent decades, studies exploring teachers' broad beliefs regarding EB students, extending beyond mathematics education, have surged. This body of literature finds that teachers often exhibit deficit views, perceiving their role mainly as content instructors (e.g., Tan, 2011; Garmon, 2005; Mellom et al., 2018). De Araujo (2017) observed that secondary mathematics teachers often chose procedure-focused tasks for EB students, reflecting their deficit-oriented beliefs about EB students' ability to engage with mathematics. Additionally, McLeman and Fernandes (2012) noted that even PSTs with overall positive beliefs about EB students also endorsed the commonly held belief that lack of parental and family involvement, potentially contributed to EB students' mathematics underachievement.

An intriguing aspect of this research is the exploration of factors shaping teachers' beliefs about EB students' mathematics education. Training in EB instruction appears to lead to more positive beliefs about teaching EB students (Pettit, 2011b; Polat & Mahalingappa, 2013). Similarly, PSTs who engaged in multicultural courses during their teacher preparation, specifically addressing EB student issues, tended to hold more positive beliefs about EB students and their mathematics performance. These PSTs also had opportunities to interact with EB students during their student-teaching experiences (McLeman, Fernandes, & McNuttly, 2012; McLeman & Fernandes, 2012).

Methods

This study aims to investigate a diverse array of PSTs beliefs, encompassing perceptions, attitudes, conceptions, and beyond. Therefore, the study adopts Borg's (2003) comprehensive conceptualization of teacher cognitions as its main theoretical framework, integrating various facets of teacher beliefs, particularly those related to the nature of mathematics and the mathematics education of EB students. This framework aligns with the study's perspective of teachers as active agents who base their instructional decisions on their situated understanding and therefore, frames PSTs' beliefs as a flexible yet robust collection of "what teachers know, believe, and think" (Borg, 2003, p. 81), particularly in the context of mathematics and the education of EB students.

We collected survey data from 84 PSTs attending a southwestern US public research university. Specifically, we recruited PSTs enrolled in an elementary teacher preparation program and pursuing either an ESL Generalist Certification (*n* = 68) or a Bilingual Generalist Certification (*n* = 16). As our intent in this study is to focus on views held by PSTs about the mathematics education of EB students as well as their beliefs about the nature of mathematics, we selected Bilingual and ESL elementary PSTs who had either completed (*n* = 53) or were currently enrolled (*n* = 31) in the mathematics pedagogy course (i.e., Mathematics Methods) for the elementary teacher preparation program. Through this course, PSTs were exposed to research-based instructional strategies for teaching mathematics in elementary grades.

To capture beliefs held by PSTs regarding the nature of mathematics, teaching mathematics, learning mathematics, and the mathematics education of EB students, we utilized two different sets of survey items. The first set, or Survey A, (Cronbach's alpha=0.77) describes beliefs about mathematics and mathematics education and was comprised of 28 previously validated items (Hart, 2002). A sample item includes "A demonstration of good reasoning should be regarded even more than students' ability to find correct answers." Response categories were 5-point Likert-scales, ranging from 1=*Strongly Disagree* to 5=*Strongly Agree,* with higher scores indicating more alignment with reform-oriented efforts. The second set, or Survey B, consisted of 30 previously validated items (Cronbach's alpha=0.87) describing beliefs about the mathematics education of EB students. One item from this set is "The inclusion of EB students in classes of mathematics creates a positive educational atmosphere." Response categories followed the same 5-point Likert scale described earlier, with higher scores indicating more asset-oriented views towards the mathematics education of EB students. Survey data collected also included PSTs' demographic background information, including gender, race/ethnicity, and previous college course experience.

As this study seeks to explore the potential relationship between PSTs' beliefs about mathematics and the mathematics education of EB students, we utilized cluster analyses, which is a descriptive statistical method appropriate for this exploratory study about PSTs' beliefs (e.g., Beswick, 2005; Rodriguez-Muñiz et al., 2022). For each set of beliefs (Surveys A & B), we utilized hierarchical cluster analysis to identify and categorize latent types of common beliefs shared by PSTs. To analyze the intersection between beliefs about mathematics (Survey A) and mathematics education of EB students (Survey B), we used a cross-tabulation analysis and identified unique relationships between these two sets of beliefs.

Results

Teacher Beliefs about the Nature, Teaching, and Learning of Mathematics, and the Mathematics Education of EB Students

Results from the hierarchal cluster analysis on Survey A responses provided a solution of three clusters (i.e., CA1, CA2, & CA3). On the other hand, results from the hierarchal cluster analysis on the responses of Survey B yielded two clusters that reflected the PSTs' beliefs pertaining as it relates to EB students' mathematics education (i.e., CB1 & CB2). These results were further validated by splitting the data into two equal sets and confirming the persistence of the same solution for the split data, respectively (Garson, 2012; Hair & Black, 2000; Sarstedt & Mooi; 2014). Furthermore, given that significant differences are to be expected among clusters (Hair & Black, 2000), a comparison of means for all survey items incorporated ANOVA and post-hoc tests among CA1, CA2, CA3 and CB1, CB2. Survey items that presented significant differences (i.e., $p < 0.05$) among all the two groups of clusters were used to create cluster descriptions and labels.

Focusing on Survey A clusters, Traditionalist PSTs (CA1; n=29) shared more traditionalist orientations towards mathematics. This includes beliefs that mathematics is a collection of isolated strands of knowledge best taught through repetition and reinforcement. Moderately-Traditionalist PSTs (CA2; n=34) shared similar beliefs about mathematics, such as the view that there exists a right or wrong answer in math, strongly contradicting the beliefs expressed in mathematics reform efforts. However, these and other similar traditionalist beliefs are not as strongly believed as those from CA1, and therefore the "Moderately" in the label. Social Constructivist PSTs (CA3; n=21), on the other hand, shared beliefs about mathematics, including its teaching and its learning, that strongly aligned with those of reform-oriented efforts. This includes beliefs about mathematics being best taught through a student-centered approach that values peer discussions, guided explorations, and self-discovery.

Regarding the beliefs about the mathematics education of EBs, or Survey B, Moderately-Equitable PSTs (CB1; n=56) expressed beliefs that align well with recommendations found in the literature on the learning and instruction of mathematics to EBs. However, beliefs in CB1 are "weaker" than those believed in CB2 (n=28). That is, PSTs in CB1 were more moderate compared to CB2 PSTs in how much they agreed and disagreed with research-based recommendations for EBs' mathematics education, and therefore the "Moderately" in the label for CB1 and "Equitable PSTs" for CB2, respectively. Demographic and other background information were also considered when exploring the composition of each cluster. However, results from the Chi-square and ANOVA tests did not reveal any significant associations.

Relationships between PSTs' Beliefs about the Nature of Mathematics, its Teaching and Learning, and The Mathematics Education of EB Students

The percentages from the crosstabulation analysis between the two sets of clusters of PSTs are shown in Table 1. We focus on the interesting yet important intersections formed by Moderately-Traditionalists and Moderately-Equitable (i.e., CA2 $\times$ CB1) and Moderately-Traditionalists and Equitable (i.e., CA2 $\times$ CB2) while recognizing that we also observed other relationships, such as between PSTs labeled as Social Constructivists and Equitable (i.e., CA3 $\times$ CB2).

Table 1. Crosstabulation Between MBI Clusters and MEELBI Clusters

	Moderately-Equitable (CB1)	Equitable (CB2)
Traditionalists (CA1)	28 (49.6%)	1 (4.3%)
Moderately-Traditionalists (CA2)	23 (40.6%)	11 (40.2%)
Social Constructivists (CA3)	5 (9.8%)	16 (55.5%)
Total	56 (100%)	28 (100%)

Specifically, as seen in Table 1, both CB1 and CB2 shared a large proportion of PSTs that were labeled as Moderately-Traditionalists. This finding suggests the possibility that Moderately-Traditionalists' beliefs might be more susceptible to other factors, and these factors may contribute to these beliefs differently according to PSTs' beliefs about the mathematics education of EB students. In fact, looking closely at each proportion's composition revealed differences between Moderately-Traditionalists and Moderately-Equitable PSTs and Moderately-Traditionalists and Equitable PSTs, respectively. For example, PSTs within the Moderately-Traditionalists and Moderately-Equitable intersection were most likely to identify as White (72.7%) and be pursuing a Generalist/ESL certification (100%) whereas PSTs from Moderately-Traditionalists and Equitable intersection share more diversity in both ethnicity and degree plan. PSTs from the Moderately-Traditionalists and Equitable intersection were also more likely to be bilingual (52.6%) than those in the Moderately-Traditionalists and Moderately-Equitable intersection (37.35%). In terms of previous course enrollment, Moderately-Traditionalists and Moderately-Equitable PTSs reported the lowest average number of courses that focused on EB issues (M=2.55; SD=2.16) and the highest rate of interaction with EB students during their teacher education program (M=2.57; SD=.731). Moderately-Traditionalists and Equitable PSTs, on the other hand, reported a higher average number of EB-focused courses taken (M=5.75, SD=3.36) and the lowest rate of interaction with EB students during their teacher education program (M=1.42; SD=.729).

Discussion

As hypothesized, evidence suggests that PSTs who embrace progressive, reform-oriented approaches to mathematics education appear to be more inclined to endorse empirically-based strategies for effectively teaching EB students in mathematics. Conversely, PSTs with more traditional beliefs about mathematics seem to exhibit a degree of caution or reluctance when it comes to fully embracing research-based recommendations for the education of EB students in mathematics. However, there also existed a group of PSTs that, while adhering to traditional views on mathematics, paradoxically champion equitable practices when it comes to teaching mathematics to EB students. Several additional factors may also come into play, potentially shaping and influencing the beliefs of PSTs. Elements such as the ethnic and cultural backgrounds of PSTs, the extent of their exposure to EB-focused coursework during their educational journey, and the depth of their interactions with EB students during their student-teaching placements are all critical factors that warrant exploration. These variables suggest that the development of beliefs about mathematics education and equity for EB students is a nuanced process influenced by a multitude of contextual and experiential factors. Further exploring this holds the potential to better inform teacher preparation programs.

References

Adams, M. (2018). I can solve all the problems: LatinX students (re)write their mathematics stories. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Rehumanizing Mathematics for Black, Indigenous, and LatinX students* (pp. 121-134). The National Council of Teachers of Mathematics.

Adnan, M., & Zakaria, E. (2010). Exploring beliefs of pre-service mathematics teachers: A Malaysian perspective. Asian Social Science, 6(10), 152–159. https://doi.org/10.5539/ass.v6n10p152

Akinsola, M. K. (2008). Relationship of some psychological variables in predicting problem-solving ability of in-service mathematics teachers. The Montana Mathematics Enthusiast, 5(1), 79–100.

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. Mathematics Education Research Journal, 17(2), 39-68.

Borg, S. (1997). Unifying concepts in the study of teachers' cognitive structures. Unpublished manuscript.

Borg, S. (2003). Teacher cognition in language teaching: A review of research on what language teachers think, know, believe, and do. Language Teaching, 36(2), 81–109. https://doi.org/10.1017/S0261444803001903

Borg, I. (2005). Facet Theory. In Encyclopedia of Statistics in Behavioral Science (eds B.S. Everitt and D.C. Howell).

De Araujo, Z. (2017). Connections between secondary mathematics teachers' beliefs and their selection of tasks for English language learners. Curriculum Inquiry, 47(4), 363–389. https://doi.org/10.1080/03626784.2017.1368351

Ernest, P. (1991). The philosophy of mathematics education. London, United Kingdom: Falmer Press.

Fernandes, A. (2007). Mathematics preservice teachers learning about English Language Learners through task-based interviews and noticing. Mathematics Teacher Educator, 1(1), 10–22. https://doi.org/10.5951/mathteaceduc.1.1.0010

Garmon, M. A. (2005). Six key factors for changing preservice teachers' attitudes/beliefs about diversity. *Educational Studies*, *38*(3), 275-286.

Hart, L. C. (2002). Preservice teachers' beliefs and practice after participating in an integrated content/methods course. School Science and Mathematics, 102(1), 4–14. https://doi.org/10.1111/j.1949-8594.2002.tb18191.x

Hughes, P.T. (2016). The Relationship of Mathematics Anxiety, Mathematical Beliefs, and Instructional Practices of Elementary School Teachers. Dissertation, Georgia State University.

Janzen, J. (2008). Teaching English Language Learners in the content areas. Review of Educational Research, 78(4), 1010–1038. https://doi.org/10.3102/0034654308325580

Kutaka, T. S., Ren, L., Smith, W. M., Beattie, H. L., Edwards, C. P., Green, J. L., ... & Lewis, W. J. (2018). Examining change in K-3 teachers' mathematical knowledge, attitudes, and beliefs: The case of primarily math. *Journal of Mathematics Teacher Education*, *21*, 147-177.

McLeman, L., & Fernandes, A. (2012). Unpacking preservice teachers' beliefs: A look at language and culture in the context of the mathematics education of English Learners. Journal of Mathematics Education, 5(1), 121–135.

Mcleman, L., Fernandes, A., & McNuttly, M. (2012). Regarding the mathematics education of English Learners: Clustering the conceptions of preservice teachers. Journal of Urban Mathematics Education, 5(2), 112–132.

Mellom, P. J., Straubhaar, R., Balderas, C., Ariail, M., & Portes, P. R. (2018). "They come with nothing:" How professional development in a culturally responsive pedagogy shapes teacher attitudes towards Latino/a English language learners. *Teaching and Teacher Education*, *71*, 98-107.

National Center for Education Statistics. (2013). The nation's report card: A first look—2013 mathematics and reading (NCES 2014-451). Washington, DC: Institute of Education Sciences, U.S. Department of Education.

Ogan-Bekiroglu, F., & Akkoç, H. (2009). PRESERVICE TEACHERS'INSTRUCTIONAL BELIEFS AND EXAMINATION OF CONSISTENCY BETWEEN BELIEFS AND PRACTICES. *International journal of science and mathematics education*, *7*, 1173-1199.

Paolucci, C. (2015). Changing perspectives: Examining the potential for advanced mathematical studies to influence pre-service teachers' beliefs about mathematics. *Teaching and Teacher Education*, *49*, 97-107.

Pappamihiel, N. E., Ousley-Exum, D., & Ritzhaupt, A. (2017). The impact of digital stories on preservice teacher beliefs about English language learners. *Teaching and Teacher Education*, *67*, 171-178.

Pettit, S. K. (2011b). Factors influencing middle school mathematics teachers' beliefs about ELLs in mainstream classrooms. IUMPST: The Journal, 5, 1–6.

Polat, N., & Mahalingappa, L. (2013). Pre-and in-service teachers' beliefs about ELLs in content area classes: A case for inclusion, responsibility, and instructional support. *Teaching Education*, *24*(1), 58-83.

Raymond, A. M. (1997). Inconsistency between beginning elementary school teachers' mathematical beliefs and teaching practice. Journal for Research in Mathematics Education, 28, 550–576.

Rodríguez-Muñiz, L. J., Aguilar-González, Á., Lindorff, A., & Muñiz-Rodríguez, L. (2022). Undergraduates' conceptions of mathematics teaching and learning: an empirical study. *Educational Studies in Mathematics*, *109*(3), 523-547.

Tan, M. (2011). Mathematics and science teachers' beliefs and practices regarding the teaching of language in content learning. Language Teaching Research, 15(3), 325–342. https://doi.org/10.1177/1362168811401153

Wilkins, J. L. M. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. Journal of Mathematics Teacher Education, 11(2), 139–164.

Beyond Correctness: Characterizing Authority to Assess in an Abstract Algebra Classroom

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In what ways do students bear mathematical authority in advanced mathematics classrooms? What authority relations arise within inquiry-oriented classrooms? Answering these questions may give insight into the inequities that exist within undergraduate mathematics education. We share findings from our initial investigation of 7 lessons from one inquiry-oriented abstract algebra (IOAA) classroom to determine who gets to have authority to do certain activities. Taking an action-based approach to parsing authority relations, we attended to who bears authority for the activities of authorship, communication of ideas, and assessment of ideas. Our preliminary findings propose five categories of activities for assessment as well as several observations about the nature of authority relations within an IOAA classroom.

Keywords: Abstract algebra, authority, assessment

Increased attention on student-centered instruction eschewing traditional lecture formats in undergraduate mathematics demands better understanding of the complex dynamics by which students participate in mathematics discourse, especially when they are expected to engage in activities like conjecturing, proving and justifying mathematical ideas. Investigating the authority relations created in such courses is one avenue for interpreting both productive and unproductive mathematics discourse. For example, it has been argued that inequitable learning and participation outcomes (Johnson et al., 2020) in inquiry-oriented classrooms arise from students' differing experiences in leveraging their authority in small-group and whole-class discussions, especially in terms of whose ideas are taken up, and how (Hicks, Tucci, Koehne, Melhuish & Bishop, 2021). In this paper, we extend an existing framework for interpreting authority relations by further parsing another relevant activity found within classroom authority dynamics: assessment of mathematical ideas.

Authority is a multi-dimensional construct, and as such, there are multiple approaches to parsing authority structures. Weber's (1947) power-oriented perspectives on authority have been used extensively, often to make observations about the nature of inequitable classroom interactions (e.g., Langer-Osuna, 2017; Esmonde & Langer-Osuna, 2013), or discursive positioning moves (e.g., Wagner & Herbel-Eisenmann, 2009). Despite the prominence of Weber's power-oriented authority in sociological research, concerns have been raised over the negative attention attributed to authority as a necessary evil to be resisted rather than as a positive force for social change. In the context of educational philosophy, Benne (1970) espoused a perspective that characterized authority in education with greater optimism, one which treated *bearers* of authority as a source of assistance to *receivers* of authority who willfully seek help from an authority and may be provided opportunities to bear authority for themselves. This type of authority is known as *anthropogogical* authority. The reform-oriented nature of Benne's contributions established a natural dichotomy between the Weberian and anthropogogical perspectives wherein the first is treated as typical of traditional classrooms and persists in most situations, while the second is thought to be an ideal form of authority to be strived for (Amit & Fried, 2005).

In contrast to the mathematics education literature as a whole, authority relations in undergraduate mathematics education have been understudied. This research has largely focused on *sources* of authority. Gerson and Bateman (2010) conducted a grounded theory investigation of an inquiry-based calculus classroom and developed a framework for identifying the many types of authority that can exist in the university setting, including hierarchal, mathematical, and pedagogical authorities. Other research has focused on authority for specific activities relevant to undergraduate advanced mathematics classrooms. For example, Bleiler-Baxter, Kirby and Reed (2023) investigated ways in which authority manifested in situations related to proof and proving activity, finding that students attended to course syllabi, their peers, and logical structure as sources of authority. In this paper, we focus specifically on investigating *mathematical* authority in the context of an inquiry-oriented abstract algebra class. We pose the following questions:

RQ1. What are the authoritative activities associated with assessment in an inquiry-oriented abstract algebra classroom?

RQ2. What are the mathematical authority relations in this classroom?

Theoretical Framing

We define authority as a dynamic and negotiated relationship between people (or groups, or organizations) in which one party agrees to lead and another party agrees to follow in a given situation. Thus, authority is transient and can shift between parties from one moment to the next rather than remain as a permanent fixture within a single person.

In contrast to Weber's power-oriented and Benne's reform-oriented definitions of authority, we adopt an *action-oriented* approach to parsing mathematical authority (Bishop, Koehne & Hicks, 2022). In particular, we contend that within any given *field* (i.e., the specific context in which bearers and receivers negotiate relevant authority relations), a member of the field becomes a bearer of authority by actively participating in one or more activities. These activities are situated discursively within a community of practice, either by recognizing that the community at large views the activity as pertinent to the field (e.g., a medical doctor bears authority by providing a patient with a diagnosis), or through localized negotiation that might be exclusive to a subset of the community (e.g., students might become bearers of mathematical authority through public *validation* of proof in one class, but never have such an opportunity to publicly validate proof in another.)

Table 1. The AAA Authority Framework for Authority Relations

Authorship	Refers to the significant contribution to the mathematical ideas under consideration.	Who is the primary source, or author, of the mathematical ideas?
Animation	Refers to who is publicly communicating or extending mathematical ideas.	Who is communicating mathematical ideas in the classroom?
Assessment	Refers to judgements/evaluations made about an idea.	Who is assessing mathematical ideas in the classroom and who is being assessed?

While several researchers have investigated authority relations between individuals, we examine the relations between students (as a collective party) and their teacher in this study. Thus, when one student partakes in an activity and bears authority, they bear authority for the students as a whole. This allows for a broadscale analysis of authority relations across several days of class, a scale of analysis that is absent from current literature. To characterize mathematical authority, we attend to the activities of *authoring* ideas*, animating* (or communicating) ideas, and *assessing* ideas. See Table 1 for a description of these broad activities.

Methods

The preliminary data consist of seven 1.25-hour long recordings of whole class activity from an introductory abstract algebra class at a large, land-grant university in the eastern United States, taught using the Inquiry-Oriented Abstract Algebra (IOAA) curriculum materials (Larsen et al., 2013). The class was composed of 15-20 students and covered the group isomorphism units from the curriculum materials in the 7 classes that were analyzed. Analysis began with segmenting classroom video, which indicated a stretch of classroom activity associated with a focal mathematical idea or task, or consistent participation structure, ranging from 30 seconds to 5 minutes in length. Four types of segments were identified within each video: whole class (WC) activity, small group activity (SG), independent activity (i.e., students worked by themselves rather than in groups), and segments that were not considered mathematical in nature (i.e., task set-up or returning graded work). Each segment that was labelled as WC was assigned codes using a version of the AAA framework modified to capture assessment authority.

Because the AAA framework was originally developed to capture the authority of whole-class interactions, the authority of students (versus the authority of the teacher) was documented collectively, rather than attending to the authority of individual students. In other words, if a student authors an idea in a segment, then students are credited collectively for that authorship. Codes were assigned based on which group (teacher, students, or both) held authority for authorship, animate-speak, animate-represent, and several types of assessment. In our initial analysis, we viewed assessment as any explicit statement that indicates judgement or evaluation of a mathematical idea. That is, utterances that could be implicit assessments were not coded.

The first two classroom videos were coded as a team to establish an initial coding scheme. The third lesson was coded independently by the first two authors, and then disagreements were resolved as a group. After that, the first author coded the fourth and sixth lessons, and the second coded the fifth and seventh lessons, with each author bringing difficult segments to the group to get consensus. The codebook was then refined over the course of analysis resulting in five types of mathematical assessment. These types are exemplified below in our findings, along with some general trends in authority relations captured by preliminary counts of the current coding.

Findings

In order to parse the authoritative activity of assessing, we attended to five different categories related to how students or teachers assess mathematical ideas within public classroom discourse: (1) correctness focused assessments, (2) assessments focused on comparison of ideas, (3) assessments focused on a particular quality of an idea, (4) justification of assessments, and finally (5) reflective or metacognitive assessments. Below we share examples of a subset of these activities identified within two interactions.

We view correctness focused assessments as largely related to the traditional discursive structure of initiate-respond-evaluate (IRE) (Mehan, 1979), although correctness focused assessments can appear more productively as well. Consider the following excerpt from one lesson in which the instructor sets up the class for a new phase of investigation into subgroups, beginning first by establishing collective understanding of some possibly unfamiliar notation:

Teacher:	We're going to go into our next reinvention phase now. And to get there, we need to know what 5Z is. Do you guys know what 5Z is?
Student A:	Z like integers?
Teacher:	Yes, Z as in integers (*points and nods at student while speaking.)*

The above interaction showcases an example of a correctness-focused assessment where the teacher was a bearer of authority for assessing a mathematical contribution. To capture the wide range of productive assessment activity within an inquiry-oriented course, attention to other forms of assessment was required. Consider the following excerpt:

Teacher:	*(Revoices student's idea at the board proving an implication about abelian groups.)*
Student B:	Is it not simpler than that?
Teacher:	Can I make it even simpler than that?
Student B:	*(Describes a proposed alternative proof.)*

In this interaction, Student B claims to have a simpler alternative proof than the previous one, and thus bears authority for comparing two mathematical ideas. In the discussion that followed, the teacher commented, "I think they are equally slick," giving her authority for comparison in addition to the students. Because each of students and the teacher was a bearer of authority for comparing ideas, we assigned a code of Both to this segment. Such interactions are evidence that students are engaging in mathematical practices valued by the field as a whole. Here, the student is attending not only to the correctness of a given proof, but also to the comparison of multiple ideas in an effort to find a more efficient approach. This is just one of five types of assessment indicative of mathematical community practices that arose, the remainder of which are summarized in Table 2 below.

Table 2. Five Authoritative Activities for Assessment

Correctness Focused	Assessments that indicate correctness/incorrectness of a mathematical contribution.
Comparison Focused	Refers to the comparison of two or more mathematical contributions.
Quality Focused	Assessments that are focused on a particular quality of a contribution, such as efficiency or clarity.
Justification Focused	Providing evidence or rationale to support a previously made assessment. Often paired with another assessment type.
Reflective/Metacognitive Focused	Assessments that are reflective of one's own thinking, and typically appear as an assessment of one's own contributions.

General Trends and Observations on Authority Relations

Authority was shared by teacher and students across the activities of authoring, animating, and assessing. Out of 150 total segments, students had or shared authority to author ideas in 101 segments. This preliminary finding count may be an indicator of the increased attention to student thinking within the inquiry-oriented setting. Inquiry-oriented classrooms often leverage tasks designed for students to inquire into mathematical ideas with their peers in small group settings, but allowing students to work in small groups generates a certain level of risk. Pimm (2019) identified these risks as *gambits.* One trend we have observed within this IOAA classroom is that by allowing students to engage in group work, students may bear either more authority for themselves or share more authority with the teacher. Small group work was followed by student authorship in nearly all transitional segments across all seven lessons.

Furthermore, authority for speaking was nearly always shared between teacher and students, with speech coded as both in 131 segments. Taken together with the high level of student and shared authorship, we view this as an indicator that the class did not engage in teacher-driven lecture very often. However, students were also provided little opportunity to completely control the spoken public discourse without teacher intervention. In contrast to speaking, students rarely wrote on the board during whole class discussion, with examples of students bearing authority for written representations occurring in only 17 segments. Thus, the teacher had significantly more authority to animate ideas through representation.

Students bore authority to make assessments in all 7 classes, most often focused on correctness, justification, or metacognitive aspects. Strikingly, students engaged in justification assessment nearly as often as in correctness assessment, and they made assessments that reflected on their own thinking far more than the teacher (see Table 3). Furthermore, students also always had authority to assess when assessment was the focus of a segment. In terms of the teacher's authority for assessment, the teacher bore authority for corrective assessment most often, while authority for justification was the second highest. Note that due to varying segment lengths, these percentages are proportions of segments containing assessment activity, of which there were 143. Because segments could receive multiple codes, the totals do not add to 100%.

Table 3. Number of Segments Containing Assessment by Type

Type	**Student**	**Teacher**	**Both**
Correctness	11 (~8%)	69 (~48%)	7 (~5%)
Comparison	6 (~4%)	11 (~8%)	3 (~2%)
Quality	2 (~1%)	19 (~13%)	2 (~1%)
Justification	10 (~7%)	21 (~15%)	9 (~6%)
Reflective	8 (~5%)	2 (~1%)	0 (0%)

Discussion

As this work is preliminary, more analysis is needed to establish claims about trends throughout the course of the semester. In particular, as the content within abstract algebra increases in difficulty (e.g., quotient groups are commonly identified as a difficult topic), authority relations may shift more toward one of student, shared, or teacher. In particular, we wonder whether teachers are more likely to assess as content difficulty increases. In addition to the success of groupwork gambits at promoting student authority in whole class discussion, the very act of engaging in groupwork yields students some private authority to do mathematics, free of assessment from the teacher. While prior research indicates inequities in small group interactions due to social factors, such as influence (Engle et al. 2014), examining assessments on a small group level within an inquiry-oriented course could explore inequities related to mathematical authority, such as parsing whose ideas are assessed and how, as well as who bears authority to make such assessments. Future work may consider the following questions: Are marginalized students' ideas assessed only for correctness, as opposed to more mathematically demanding assessments for justification or quality? Do students of higher social status assess more often? More generally, what other activities can lead to students bearing authority?

References

Amit, M., & Fried, M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th grade classroom. *Educational studies in Mathematics*, *58*(2), 145-168.

Benne, K. D. (1970). Authority in education. *Harvard Educational Review, 40*, 385–410.

Bishop, J. P., Koehne, C., & Hicks, M. D. (2022). By Whose Authority? Negotiating Authority for Modes of Activity in Mathematics Classrooms. *For the Learning of Mathematics*, *42*(2), 35-41.

Bleiler-Baxter, S., Kirby, J. & Reed, S. (2023). Understanding authority in small-group co-constructions of mathematical proof, *The Journal of Mathematical Behavior, 70,* 1-20.

Engle, R., Langer-Osuna, J. & McKinney de Royston, M. (2014). Toward a model of influence in persuasive discussions: Negotiating quality, authority, privilege, and access within a student-led argument, *Journal of the Learning Sciences, 23*(2), 245-268.

Esmonde, I. & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education*, *44*(1), 288-315.

Gerson, H., & Bateman, E. (2010). Authority in an agency-centered, inquiry-based university calculus classroom. *The Journal of Mathematical Behavior*, *29*(4), 195-206.

Hicks, M. D., Tucci, A. A., Koehne, C. R., Melhuish, K. M., & Bishop, J. L. (2021). Examining the distribution of authority in an inquiry-oriented abstract algebra environment. In S. S. Karunakaran, & A. Higgins (Eds.), *2021 Research in Undergraduate Mathematics Education Reports*.

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504-516.

Langer-Osuna, J. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education, 48(3)*, 237-247.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *The Journal of Mathematical Behavior*, *32*(4), 693-711.

Mehan, H. (1979). 'What time is it, Denise?": Asking known information questions in classroom discourse. *Theory into practice*, *18*(4), 285-294.

Pimm, D. (2019). *Routledge Revivals: Speaking Mathematically (1987): Communication in Mathematics Clasrooms* (Vol. 4). Routledge.

Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention to classroom positioning. *Educational Studies in Mathematics*, *72*(1), 1-15.

Weber, M. (1947). *The theory of economic and social organization* (A.M. Henderson & T. Parsons, Trans.). New York: Oxford University Press.

Investigating the Resources Framework through the Lens of Analytic Autoethnography: Analyzing Research Questions and "Resource" Grain Size

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There has been a recent trend in the application of fine-grained constructivist frameworks such as the resources framework; however, the resources framework has been operationalized in a variety of ways, particularly regarding analytic choices. In this work I seek to further discussions surrounding theory use and the ways frameworks may inform (1) research questions and (2) data analysis. Using an analytic autoethnographic approach, I characterize the variation in the application of the resources framework in my own work (n=10). This allows me to provide specific examples without evaluating the quality of colleagues' research, as well as contextualize the research within my experiences and within the broader norms of the chemistry education research community. To this end, I discuss alignment between the research questions and the resources framework and demonstrate variation in how "cognitive units" were operationalized. As an implication, suggestions are made regarding future work using the resources framework.

Keywords: theory; cross-disciplinary; research questions; methodologies

Background

Rationale and Scope

This work is part of a broader interest in furthering a dialogue related to methods and theory use in education research, which is of interest to the RUME community (Haas et al., 2022; Melhuish & Czocher, 2022). Last year at RUME I presented on a preliminary report that focused on the presence and use of frameworks across discipline-based education research (DBER) articles (Rodriguez & Nardo, 2023); this work was recently published as a systematic review in *Chemistry Education Research and Practice* (Rodriguez et al., 2023). As part of this review, we noted the potential for frameworks to connect DBER fields, "Therefore, it is more than shared disciplinary skills, language, and concepts that connect DBER communities. We are connected by theories and frameworks related to concerns such as how students learn and how to promote conceptual change." (Rodriguez et al., 2023). Illustrating the potential bridging connection afforded by theory, we noted research articles in the review sample across chemistry (Kelly et al., 2021; Parobek et al., 2021; Watts et al., 2021), mathematics (Abu-Ghalyoun, 2021), and physics (Barth-Cohen et al., 2021; Goodhew et al., 2021; Robertson et al., 2021) that used fine-grained constructivism as the theoretical framework. Nevertheless, although frameworks may be used across fields, there may be variation in their application. This work focuses on variation in theory use *within* a field (chemistry education research, CER), focusing specifically on analyzing my work that utilized the resources framework.

To this end, this work is guided by the research question: Across n=10 research articles involving the application of the resources framework to investigate undergraduate chemistry students' reasoning, what trends emerge related to the ways the resources framework informs: (1) the research question(s) and (2) analytic decisions?

Resources Framework

In brief, the resources framework is a constructivist-based theory used to model the nature and structure of knowledge, emphasizing the "fine-grained" and context-dependent nature of knowledge and describing the mechanism of conceptual change involving the gradual restructuring of a network of cognitive units ("resources") (diSessa & Wagner, 2005; Elby, 2000). For a theory of learning to be useful for education researchers it must: support prediction, possess explanatory power, be applicable to a broad range of phenomena, help organize thinking about phenomena, serve as a tool for analysis, and provide a language for communicating about learning (Dubinsky, 2001; Schoenfeld, 1998). I have found the resources framework to meet these criteria, using the framework to predict and explain patterns in student responses across contexts, informing how I think about students' reasoning, and influencing how I analyze data. Nevertheless, to improve teaching and learning we need to further refine the methods and theories that inform how we conduct research, employing an analytic agenda that emphasizes theory-building and the evaluation of theory.

Analytic autoethnography

Generally, ethnography is a field of study that emphasizes interpreting culture and its products, with autoethnography highlighting and leveraging the experiences of the researcher in this process (Adams et al., 2017). Acknowledging variation in the goals for studies within autoethnography, Anderson (2006) highlights a subgenre called analytic autoethnography. Here, the modifier *analytic* is used to draw attention to research aims focused on the analysis of empirical data that contributes toward theory development. To this end, the intended output of this work is a refined understanding of how to model cognition with a practical concern related to the ways it informs project scope and its application in data analysis. I view this as personally relevant in my application of theory but also seek to provide support for other researchers, particularly new and emerging researchers.

In the current study, the culture that serves as context for this work is the CER community of practice. The boundaries of a community of practice are defined through shared goals and norms, but these boundaries can often be fluid and complex (Wenger, 1998). This is especially true given the connections made between communities through boundary objects (e.g., journal articles) and brokers (e.g., researchers attending conferences), which could result in a "constellation" of interconnected communities of practice (p.127).

Data Collection & Analysis

For the purposes of this report, I narrowed the scope of the sample to include research articles involving the analysis of undergraduate chemistry students' reasoning where the resources framework was explicitly discussed as the theoretical framework (n=10) (Bain et al., 2018, 2019; Rodriguez et al., 2018; Rodriguez, Bain, Hux, et al., 2019; Rodriguez, Bain, & Towns, 2019; Rodriguez, Bain, Towns, et al., 2019; Rodriguez, Hux, et al., 2019; Rodriguez, Bain, et al., 2020; Rodriguez et al., 2021; Rodriguez & Towns, 2019). Analysis was guided by the research question presented above emphasizing the research question(s) and analytic decisions of the articles in the sample.

With regard to analyzing the research questions in the sample, I applied a framework that was developed in the context of ethnography (Spradley, 1980). The framework describes nine dimensions an ethnographer should attend to when collecting observational data: *space* (physical location), *object* (physical or cognitive entities), *actor* (individuals involved), *action* (behaviors and activities carried out by individuals), *time* (sequencing in a process), and *goal* (individuals' aims). Here, *object* has been adapted to include not just the physical entities but also cognitive or mental entities to afford a focus on students' knowledge and reasoning, as suggested by Melhuish and Czocher (2022); also, *act*, a*ctivity*, and e*vent* will be collectively described as *actions* for this work because the distinction between these types of observable behaviors is less relevant here. As discussed by Melhuish and Czocher (2022), this framework serves as a useful tool for discussing qualitative research questions, and in this work, analysis focused on the presence of different dimensions in the phrasing of the research questions, as well as the nature of the research question phrasing in relation to exploring relationships between dimensions and its connection to the resources framework. In terms of the analytic decisions expressed in the sample, analysis focused on the ways the framework was operationalized with an emphasis on the "size" of the identified resources. Additional analysis involved inductively grouping articles based on patterns related to the research questions and grain size. Throughout this process I engaged in reflexivity, which contributes to trustworthiness and validity of claims (Vagle, 2009, 2018). As a member of the CER community of practice, I'm familiar with expectations related to theory use in research. I implicitly interact with these norms as a producer (when publishing CER articles, presenting at CER conferences, etc.) and consumer (when reading CER articles, listening to presentations at CER conferences, etc.) but I also explicitly communicate these norms as an evaluator (when reviewing articles for a journal). The specific examples of articles discussed in the next section were selected to maximize variation, and I encourage interested readers to reference the articles if additional context is needed.

Preliminary Findings

Research Questions

Across the sample, the research questions tended to involve slight variations of the *actor-action-object* triad, where the stated goal was to elicit students' (*actor)* chemistry knowledge (*object*) using an interview task (*action*). References were not made to the other dimensions (*space*, *time*, *goal*). For example, in Rodriguez et al. (2021), our research question used problem-solving (*action)* as way to characterize students' (*actor*) understanding related to emergent properties and molecular behavior in a system (*object)*, *How do students use assumptions regarding system ontology to solve enzyme kinetics problems on an exam?* According to Melhuish and Czocher (2022), research questions that follow this pattern tend to have a singular focus and suggest a list as the research outcome. Although there was some variation with the research questions, this involved: an implicit reference to the *action* (the task is unspecified) – e.g., *How do students reason about rate constants in chemical kinetics?* (Bain et al., 2019); and multiple *objects* identified related to an interest in students' blending of chemistry and mathematics knowledge – e.g., *In what ways do students use mathematics in combination with their knowledge of chemistry and chemical kinetics to interpret concentration versus time graphs?* (Rodriguez, Bain, Towns, et al., 2019). The latter example moves closer to a relational focus that emphasizes connections between dimensions in the research question (Melhuish & Czocher, 2022).

Grain Size

As discussed by Wittmann (2006), resources can be described as fractal in nature: a cognitive unit described as a *resource* could be divided into a cluster of smaller resources. This is consistent with how I've observed the resources framework used in my work and others' broadly. Although this provides flexibility related to analytic decisions, this flexibility contributes to challenges with its application and defines the focus of this work. In the current work, thematic grouping resulted in the following grain sizes depicted in Figure 1: *p-prim*, *distributed element*, *concept projection*, and *frame*. Apart from *frame*, these terms are taken directly from the class of theoretical models outlined within knowledge-in-pieces (diSessa et al., 2016), the implications of which are discussed below.

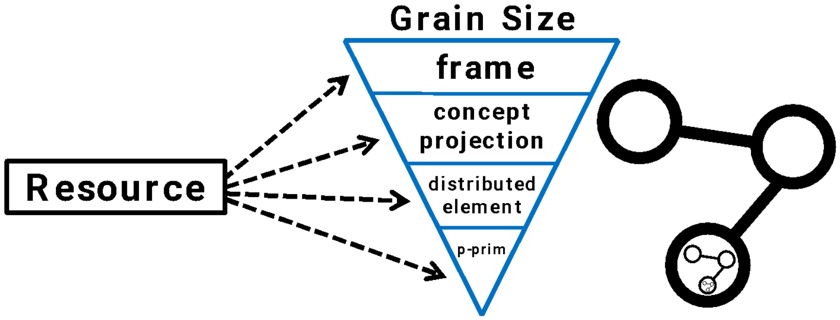

Figure 1. Variation in grain size observed in the articles.

Frames describe the expectations individuals have about a situation, which is important because of the ways epistemological frames may mediate the activation and use of knowledge (Redish, 2004). *Concept projection* is a reference to coordination class theory, which defines a *concept* and the mechanism of conceptual change within the knowledge-in-pieces perspective (diSessa et al., 2016; diSessa & Sherin, 1998; diSessa & Wagner, 2005). Here, *concept* (or *coordination class)* is defined as the combination of an inferential net (network of knowledge elements) and extractions (features attended to); it is the coordination of the knowledge component and perceptual component that allows students to “see” a *concept*. In practice, a *coordination class* would be a large cognitive structure and individuals use a subset of that knowledge when solving a task, a *concept projection*. Additionally, *distributed element* is a reference to specific knowledge elements within a *concept projection*. Lastly, *p-prims* (phenomenological primitives) are the smallest functional unit within a fine-grain constructivist view, reflecting intuitive knowledge that is self-explanatory and developed based on experiences (e.g., *dying away* is abstracted based on the observation that physical phenomena such as motion or sound tend to dissipate over time). In the next sections I will provide examples, focusing on distinguishing between *distributed element* and *concept projection*.

In Bain, Rodriguez, and Towns (2019) we analyzed students’ reasoning related to the rate constant (*k*, a parameter relevant when discussing equations related to chemical kinetics). Some examples of the resources identified are *rate and rate constant as directly proportional* and *k is defined by the Arrhenius equation*. What I would like to draw attention to here is that these cognitive units are all narrowly focused on the rate constant and are phrased as statements (inferential or propositional statements), which provide a certain level of specificity, negating the need for a formal code description. This focus and specificity make the resources too small to be a larger cognitive structure like a *concept projection,* but are larger than *p-prims*, thus, described here as *distributed elements*.

In Rodriguez and Towns (2019), we analyzed how students reason about competitive, non-competitive, and uncompetitive enzyme inhibition. In this case, the inductive codes reflect cognitive units that are broader and larger than the *distributed elements* discussed above: *competitive inhibition, mechanism*; *non-competitive inhibition, mechanism*; *uncompetitive inhibition, mechanism*. Each of these codes has a general description: “Student discusses what physically happens with this inhibitor in a general sense or at the molecular level". Here, the codes are not statements and phrases related to a topic, but rather terms referencing broad discussions. Other examples might be codes such as *equilibrium, acid-base reactions*, *chemical kinetics*, etc., where the code description emphasizes that students "discussed" this topic, as opposed to stating the specific details students articulated about the topic. In the study referenced here, each of the different inhibition types can be described as a different *concept projection.*

Discussion

Reflecting on the connection between the research questions in the sample and the resources framework, only listing resources does not leverage the potential power of fine-grained constructivist frameworks in making claims related to the nature and structure of students' knowledge and its dependence on context (i.e., span and alignment) and time (i.e., conceptual change) (diSessa et al., 2016). As part of this, for researchers interested in applying the resources framework, there is room for incorporating more considerations related to other dimensions and relational aims across these dimensions, such as emphasizing how students' knowledge structures change over time (e.g., *How do chemistry majors' knowledge structures related to reaction rate change as they move through the undergraduate chemistry curriculum?).*

Regarding resource grain size, my plan for this analysis was to provide theoretical clarity related to the resources framework. I initially planned on avoiding making claims related to discussing the relationship between the knowledge-in-pieces and resources frameworks, but I found the language from knowledge-in-pieces to be helpful for describing the variation within the resources framework. According to diSessa (2016), the resources framework builds on knowledge-in-pieces providing language to discuss general nodes with varying structure and function within a knowledge system, but "resources are not particular kinds of knowledge elements" (p.58). Aligned with this, I discussed above that a resource could be different types of knowledge elements: a *p-prim*, *distributed element*, *concept projection*, or *frame*. Given a resource can vary in, for clarity, it is important that researchers are very clear in describing the scale of the knowledge they are analyzing. This does not necessarily have to involve the use of the terms discussed above, but detailed descriptions are important, such as "We are identifying resources at the level of evaluating students' inferential and propositional statements". As a disclaimer, these discrete resource sizes are just a few observable sizes within the data analyzed, and although they have a specific rank order, these are not all the possible grain sizes and the scale or distance between these levels are not necessarily equal. Moreover, there could also be dynamic interplay as multiple scales are considered, such as analyzing *frames* of individuals and groups (Conlin & Hammer, 2016).

Lastly, as a practical consideration, depending on choices related to the grain size in analysis there may be some drawbacks. For example, the resolution provided by coding at the *concept projection* level is too coarse to allow discussions regarding the connections students' make within a concept, with additional analysis needed (e.g., constructing resource graphs from a line-by-line analysis of an interview since the coded resources are too broad to afford claims about the structure of students' knowledge). On the other hand, the challenge with analysis at the *distributed element* level is that coding individual students' responses may be too idiosyncratic, resulting in concerns related to saturation of themes and workload when looking at large samples related to student responses on open-ended surveys and exam questions. Nevertheless, narrative coding and smaller samples involving selected episodes and cases (Rodriguez, Stricker, et al., 2020) aligns more with the approach toward knowledge analysis described by diSessa et al. (2016). Balancing the analytic tensions with the scope reflects on area warranting further inquiry.

References

Abu-Ghalyoun, O. (2021). Pre-service teachers' difficulties in reasoning about sampling variability. *Educational Studies in Mathematics*, *108*(3), 553–577. https://doi.org/10.1007/s10649-021-10067-8

Adams, T. E., Ellis, C., & Jones, S. H. (2017). Autoethnography. In J. Matthes, C. S. Davis, & R. F. Potter (Eds.), *The International Encyclopedia of Communication Research Methods* (1st ed., pp. 1–11). Wiley. https://doi.org/10.1002/9781118901731.iecrm0011

Anderson, L. (2006). Analytic autoethnography. *Journal of Contemporary Ethnography*, *35*(4), 373–395. https://doi.org/10.1177/0891241605280449

Bain, K., Rodriguez, J.-M. G., & Towns, M. H. (2018). Zero-order chemical kinetics as a context to investigate student understanding of catalysts and half-life. *Journal of Chemical Education*, *95*(5), 716–725. https://doi.org/10.1021/acs.jchemed.7b00974

Bain, K., Rodriguez, J.-M. G., & Towns, M. H. (2019). Investigating student understanding of rate constants: When is a constant "constant"? *Journal of Chemical Education*, *96*(8), 1571–1577. https://doi.org/10.1021/acs.jchemed.9b00005

Barth-Cohen, L. A., Braden, S. K., Young, T. G., & Gailey, S. (2021). Reasoning with evidence while modeling: Successes at the middle school level. *Physical Review Physics Education Research*, *17*(2), 020106. https://doi.org/10.1103/PhysRevPhysEducRes.17.020106

Conlin, L. D., & Hammer, D. (2016). Commentary: From the individual to the ensemble and back again. In A. A. diSessa, M. Levin, & N. Brown (Eds.), *Knowledge and interaction: A synthetic agenda for the learning sciences* (pp. 311–325). Routledge.

diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, *20*(10), 1155–1191. https://doi.org/10.1080/0950069980201002

diSessa, A. A., Sherin, B., & Levin, M. (2016). Knowledge analysis: An introduction. In A. A. diSessa, M. Levin, & N. Brown (Eds.), *Knowledge and interaction: A synthetic agenda for the learning sciences* (pp. 30–71). Routledge.

diSessa, A. A., & Wagner, F. (2005). What coordination has to say about transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 121–154). IAP.

Dubinsky, E. (2001). Using a theory of learning in college mathematics courses. *MSOR Connections*, *1*, 10–15. https://doi.org/10.11120/msor.2001.01020010

Elby, A. (2000). What students' learning of representations tells us about constructivism. *The Journal of Mathematical Behavior*, *19*(4), 481–502. https://doi.org/10.1016/S0732-3123(01)00054-2

Goodhew, L. M., Robertson, A. D., Heron, P. R. L., & Scherr, R. E. (2021). Students' context-sensitive use of conceptual resources: A pattern across different styles of question about mechanical waves. *Physical Review Physics Education Research*, *17*(1), 010137. https://doi.org/10.1103/PhysRevPhysEducRes.17.010137

Haas, C. A. F., El-Adawy, S., Hancock, E., Sayre, E. C., & Savić, M. (2022). Emerging mathematics education researchers' conception of theory in education research. *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, 1009–1014.

Kelly, R. M., Akaygun, S., Hansen, S. J. R., Villalta-Cerdas, A., & Adam, J. (2021). Examining learning of atomic level ideas about precipitation reactions with a resources framework. *Chemistry Education Research and Practice*, *22*(4), 886–904. https://doi.org/10.1039/D0RP00071J

Melhuish, K., & Czocher, J. A. (2022). Strengthening the RUME field: Substantive research questions and suitable claims. In S. S. Karumakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 834–841).

Parobek, A. P., Chaffin, P. M., & Towns, M. H. (2021). Location-thinking, value-thinking, and graphical forms: Combining analytical frameworks to analyze inferences made by students when interpreting the points and trends on a reaction coordinate diagram. *Chemistry Education Research and Practice*, *22*(3), 697–714. https://doi.org/10.1039/D1RP00037C

Redish, E. F. (2004). A theoretical framework for physics education research: Modeling student thinking. *The Proceedings of the Enrico Fermi Summer School in Physics*, 1–50.

Robertson, A. D., Goodhew, L. M., Scherr, R. E., & Heron, P. R. L. (2021). University student conceptual resources for understanding forces. *Physical Review Physics Education Research*, *17*(1), 010121. https://doi.org/10.1103/PhysRevPhysEducRes.17.010121

Rodriguez, J.-M. G., Bain, K., Hux, N. P., & Towns, M. H. (2019). Productive features of problem solving in chemical kinetics: More than just algorithmic manipulation of variables. *Chemistry Education Research and Practice*, *20*(1), 175–186. https://doi.org/10.1039/C8RP00202A

Rodriguez, J.-M. G., Bain, K., & Towns, M. H. (2019). Graphs as objects: Mathematical resources used by undergraduate biochemistry students to reason about enzyme kinetics. In M. H. Towns, K. Bain, & J.-M. G. Rodriguez (Eds.), *It's just math: Research on students' understanding of chemistry and mathematics* (Vols. 1316, Chapter 5, pp. 69–80). American Chemical Society. https://doi.org/10.1021/bk-2019-1316.ch005

Rodriguez, J.-M. G., Bain, K., & Towns, M. H. (2020). The role of epistemology and epistemic games in mediating the use of mathematics in chemistry: Implications for mathematics instruction and research on undergraduate mathematics education. *International Journal of Research in Undergraduate Mathematics Education*, *6*, 279–301. https://doi.org/10.1007/s40753-019-00110-8

Rodriguez, J.-M. G., Bain, K., Towns, M. H., Elmgren, M., & Ho, F. M. (2019). Covariational reasoning and mathematical narratives: Investigating students' understanding of graphs in chemical kinetics. *Chemistry Education Research and Practice*, *20*(1), 107–119. https://doi.org/10.1039/C8RP00156A

Rodriguez, J.-M. G., Hux, N. P., Philips, S. J., & Towns, M. H. (2019). Michaelis–Menten graphs, Lineweaver–Burk plots, and reaction schemes: Investigating introductory biochemistry students' conceptions of representations in enzyme kinetics. *Journal of Chemical Education*, *96*(9), 1833–1845. https://doi.org/10.1021/acs.jchemed.9b00396

Rodriguez, J.-M. G., & Nardo, J. E. (2023). A literature review across discipline-based education research: What is a theoretical framework and how is it used in education research? In S. S. Karumakaran (Ed.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 953–959).

Rodriguez, J.-M. G., Nardo, J. E., Finkenstaedt-Quinn, S. A., & Watts, F. M. (2023). The use of frameworks in chemistry education research. *Chemistry Education Research and Practice, https://doi.org/10.1039/D3RP00149K.*

Rodriguez, J.-M. G., Philips, S. J., Hux, N. P., & Towns, M. H. (2021). Assessing system ontology in biochemistry: Analysis of students' problem solving in enzyme kinetics. In G. Tsaparlis (Ed.), *Problems and problem solving in chemistry education: Analysing data,*

looking for patterns and making deductions (pp. 199–216). Royal Society of Chemistry. https://doi.org/10.1039/9781839163586-00199
Rodriguez, J.-M. G., Santos-Diaz, S., Bain, K., & Towns, M. H. (2018). Using symbolic and graphical forms to analyze students' mathematical reasoning in chemical kinetics. *Journal of Chemical Education*, *95*(12), 2114–2125. https://doi.org/10.1021/acs.jchemed.8b00584
Rodriguez, J.-M. G., Stricker, A. R., & Becker, N. M. (2020). Exploring the productive use of metonymy: Applying coordination class theory to investigate student conceptions of rate in relation to reaction coordinate diagrams. *Journal of Chemical Education*, *97*(8), 2065–2077. https://doi.org/10.1021/acs.jchemed.0c00496
Rodriguez, J.-M. G., & Towns, M. H. (2019). Analysis of student reasoning about Michaelis–Menten enzyme kinetics: Mixed conceptions of enzyme inhibition. *Chemistry Education Research and Practice*, *20*(2), 428–442. https://doi.org/10.1039/C8RP00276B
Schoenfeld, A. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94. https://doi.org/10.1016/S1080-9724(99)80076-7
Spradley, J. P. (1980). Step four: Making descriptive observations. In *Participant Observation* (pp. 73–84). Holt, Rinehart and Winston.
Vagle, M. D. (2009). Validity as intended: 'Bursting forth toward' bridling in phenomenological research. *International Journal of Qualitative Studies in Education*, *22*(5), 585–605. https://doi.org/10.1080/09518390903048784
Vagle, M. D. (2018). Phenomenological research methodology. In *Crafting phenomenological research* (2nd Edition, pp. 71–84). Routledge.
Watts, F. M., Zaimi, I., Kranz, D., Graulich, N., & Shultz, G. V. (2021). Investigating students' reasoning over time for case comparisons of acyl transfer reaction mechanisms. *Chemistry Education Research and Practice*, *22*(2), 364–381. https://doi.org/10.1039/D0RP00298D
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.
Wittmann, M. C. (2006). Using resource graphs to represent conceptual change. *Physical Review Special Topics - Physics Education Research*, *2*(2), 020105. https://doi.org/10.1103/PhysRevSTPER.2.020105

Student Interpretations of Conditional Statements in Probability: The Cases of Byron and Sophie

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Students often have to interpret conditional statements in mathematics classes without prior training in logic. In a previous study, we investigated how students interpreted variants of the conditional statement: If two random variables are independent, their covariance is 0. The purpose of the current study is to explore surprising results and apparent inconsistencies in student interpretations using clinical interviews after a lecture introducing the conditional statement. In this paper we contrast the experiences of two students, Byron and Sophie. Preliminary results suggest that Byron relies both on his logic training and his domain knowledge to solve problems about independence and covariance. Sophie's incorrect domain knowledge led to her reversing the implication. Once this was resolved, she was not confident in assigning truth values to statement variants. We discuss teaching implications that arise from issues of both mathematical efficiency and accuracy in presenting this theorem.

Keywords: Probability, Conditional Statements, Logic, Covariance, Independence

Undergraduate math courses typically present multiple definitions and theorems in the form of conditional statements. We know students often do not interpret these statements in the mathematically normative manner, and learning how to interpret conditional statements is particularly challenging due to epistemic issues related to set-theoretic components of conditional statements (Case, 2015; Dawkins & Cook, 2017; Durst & Kaschner, 2020; Sellers, 2020). In a previous study (Ryals, et al., 2023), we analyzed Probability students' written work to determine how they interpreted and used the following theorem: "If X and Y are independent random variables, then $\text{Cov}(X, Y) = 0$" (Casella & Berger, 2002, p. 171). We also considered whether students' prior logic training, or lack thereof, would impact their interpretations. Approximately 1/3 of the participants incorrectly used covariance being 0 as justification for two random variables being independent, and 10% of students failed to use covariance being nonzero as justification for variables being dependent. The results did not significantly differ for students who had and had not previously received logic training in a prior course.

In Spring 2023, we began conducting clinical interviews (Clement, 2000) with students in the same course in an effort to investigate results from our prior study. Specifically, we address:

1. How do students view the relationship between covariance and independence?

2. How does their view of this relationship influence how they use the value of covariance to determine and justify whether two variables are independent?

In this report, we present preliminary findings from interviews with two students, Byron and Sophie. We discuss and compare their conceptions of covariance and independence and the relationship between the two as well as how and why they used covariance values to determine independence of two random variables.

Review of Literature

For multiple reasons, students often do not interpret conditional statements in conventional ways and the same student can interpret logically equivalent statements differently in different contexts (Dawkins & Cook, 2017; Durand-Guerrier, 2003). Students often interpret conditional statements as biconditionals (Hoyles & Küchemann, 2002; O'Brien et al., 1971; Wason, 1968) which is often *intended* in common language (Epp, 2003). Moreover, a statement indicating P implies Q, devoid of quantified variables, is an open sentence, which could be considered true for one case and false for another (Durand-Guerrier, 2003). Yet, mathematicians typically read these statements as generalized conditional statements. This means we interpret our theorem as saying "*For all pairs* of random variables, if X, Y are independent, then $\text{Cov}[X, Y] = 0$." However, the implicit "for all" quantifier is not typically stated explicitly, which understandably leads to non-normative evaluations for some students (Durst and Kaschner, 2020).

Efforts have been made to improve teaching students how to interpret conditional statements appropriately within the mathematics register (Dawkins & Norton, 2022; Dawkins, et al., 2023). Evidence suggests students have an easier time interpreting verbal and mathematical statements than abstract or symbolic ones. Consequently, one approach has been to use colloquial statements whose inverse, converse, and contrapositive are obviously true or false to students through context or familiarity as an introduction to the topic (Case, 2015; Epp, 2003, Stylianides et al., 2004). However, training students to generally place a label of "true" or "false" on specific variants such as the converse has not proven to transfer to using those variants appropriately in novel situations in mathematics (Sellers et al., 2017; Attridge et. al, 2016; Cheng et. al, 1986; Inglis & Simpson, 2008). This implies that many students have not fully abstracted logic rules with conditional statements across different mathematical domains. Thus, there is a need to study how students interpret conditional statements in different mathematical contexts and determine how mathematical content of a particular statement may impact students' logical inference.

Methods

Data Collection

This study was conducted in Spring 2023 in a semester-long Applied Probability course designed for engineering and computer science majors. The first author taught the lesson introducing the theorem to four sections of the course. After presenting the theorem, students were asked to individually determine whether each of the three variants listed below was true or false in a pre-class quiz. In this paper, we subsequently refer to these statements as the inverse, converse, and contrapositive, respectively, though we did not use this terminology with students.

1. *Theorem: If X and Y are independent random variables, then $Cov(X, Y) = 0$.*
2. *Inverse: If X and Y are not independent, then Cov (X,Y) ≠ 0.*
3. *Converse: If Cov (X,Y) = 0, then X and Y are independent.*
4. *Contrapositive: If Cov (X,Y) ≠ 0, then X and Y are not independent.*

After the quiz, the instructor clarified the fourth statement was true and the second and third statements were false and explained it is possible for two random variables to have a nonlinear relationship and have a covariance of 0. The instructor then worked through two examples related to the theorem. Each required first determining whether variables were independent and then finding the value of covariance. In one example, the variables were independent, so the theorem could be invoked to conclude the covariance was 0. In the other, the variables were not independent, so the covariance had to be calculated. These examples were chosen to demonstrate the theorem's utility in specific situations and limitations in others.

Following a brief discussion, students individually solved two similar problems. In each problem, they were given a joint probability mass function (Figure 1) and first asked to compute covariance and then determine whether X and Y were independent. In the first example, covariance was 0 so independence had to be determined using the definition of independence. In the second example, the covariance was not 0 so dependence was guaranteed. We will subsequently refer to these examples as the Converse problem and the Contrapositive problem, respectively.

X / Y	-1	0	1	$P_Y(y)$
-1	0.2	0	0.2	0.4
0	0	0.2	0	0.2
1	0.2	0	0.2	0.4
$P_X(x)$	0.4	0.2	0.4	

X / Y	0	1	2	$P_Y(y)$
0	0.1	0.2	0.2	0.5
2	0.1	0.25	0.15	0.5
$P_X(x)$	0.2	0.45	0.35	

Figure 1. Probability Distributions used in the Converse and Contrapositive Problems

We invited all students who had either used the converse on the Converse problem or neglected to use the contrapositive on the Contrapositive problem to participate in a 30-60 minute semi-structured interview. Three students agreed to be interviewed one week after instruction and prior to any formal assessments on the topic. The first author was the primary interviewer and the second author prompted the first with additional questions. We used a prepared protocol but allowed for both follow-up questions and probes (Rubin & Rubin, 2005). During the interview, students were asked to provide definitions of covariance and independence and justify their in-class work. We then presented several colloquial statements and asked the students to provide truth values for those statements and their variants to check for consistency or inconsistencies in logic across different mathematical domains.

Analytic Framework and Analysis

During the interviews and throughout analysis we attempted to construct second-order models of students' thinking about their domain knowledge and logic; "those the observer constructs of the subject's knowledge in order to explain their observations or experience of the [student's statements] and activities" (Steffe et al., 1983, p xvi). To achieve this goal, we asked multiple follow-up questions during the interview and the authors discussed multiple possible interpretations of student claims to come to consensus.

We developed several categories of codes related to specific interview questions. Our first group of codes described the participant's conceptions of independence and covariance. We paid particular attention to whether these were formalized or intuitive notions, whether they were normative, and whether linearity was part of the covariance description. We asked them to describe any mental images that they may have had in relation to covariance, in particular. Our second group of codes addressed their beliefs about the relationship between independence and covariance. Finally, a third group of codes attended primarily to responses to questions involving everyday statements to describe the participant's approach to determining truth values of conditional statements in various contexts. In the subsequent section, we present results for two participants, Byron and Sophie, which includes their conceptions of independence and covariance and the relationship between them as well as their level of comfort with logical equivalence. We then draw comparisons between the two and discuss implications for practice.

Results

Byron was invited to participate in the interviews because he did not use the fact that covariance was nonzero to conclude variables were dependent on the end-of-class exercise.

Instead, Byron correctly applied the definition of independence. Byron's approach was particularly interesting since he had identified the contrapositive as true on the pre-class quiz and he had previously taken Discrete Math.

The interview revealed that Byron held normative and robust mental images related to both independence and covariance. He provided both the formal definition of independence as well as an explanation, saying "I think that means the observed value of one of them will not affect what you'll observe for the other." Byron stated that covariance was a measure of a linear relationship and stated that variables could have a relationship that was not linear. He provided an example, saying "…if all of your data fell on a circle, [the two variables] would have a relationship. But I think…the covariance would come out to exactly 0 if it was exactly on a circle." Additionally, Byron was clear about the claims and limits of the theorem. He stated that independence guarantees a covariance of 0 but a covariance of 0 does not guarantee independence.

Byron had not made use of the theorem on the end-of-class Contrapositive problem, which seems surprising in light of the evidence above. However, when he was asked if there was an alternative way he could have justified X and Y being dependent, he immediately said, "Oh yeah, I do remember now. If two random variables are independent, their covariance is 0, is equivalent to saying if their covariance is not 0, they are not independent. So I guess you can tell from [the fact the covariance is not 0] that they're also not independent."

First, we hypothesize that Byron may have been influenced to use the definition of independence on the Contrapositive problem, rather than invoking the theorem, by the examples worked by the instructor just moments before. In the interview, when asked about these two approaches, Byron said they are equally valid, and that invoking the theorem would make more sense in the case where a nonzero covariance has already been established. Second, we note that Byron introduced the word "equivalent" to explain the relationship between the statement and its contrapositive. He made the same argument to justify his choice of truth value for a variant of a colloquial statement, this time introducing the word "contrapositive," which we had not used in the interview. With this example, he used logical equivalence of the contrapositive as his primary justification for his provided truth value.

Sophie, who had not previously taken Discrete Math, was invited for an interview because she stated that X and Y must be independent because $\mathrm{E}[XY] = \mathrm{E}[X]E[Y]$ on the Converse problem. We coded this response as Applying the Converse. Consistently, Sophie had classified the converse statement as true on the pre-class quiz.

Sophie demonstrated a conceptual understanding of independence by comparing selection with and without replacement. However, Sophie's concept of covariance was influenced and limited by a graphic presented in lecture. When asked what it means to say $\mathrm{Cov}[X, Y] = 0$, she replied, "it means that X and Y are not correlated, so the graph would be more like no pattern, a bunch of scattered dots everywhere." A graphical depiction of dependent variables with a covariance of 0 had not been presented. Additionally, when asked what it meant for two random variables to be independent, she responded, "When the expected value of both of them equals the expected value of each of them multiplied." That is, Sophie believed that two variables are independent *if and only if* $E[XY] = E[X]E[Y]$. This was further clarified when we presented Sophie with a situation where $\mathrm{Cov}[X, Y] = 0$ and she concluded X and Y must be independent. Sophie was not intentionally applying the converse. Rather, Sophie had believed she was checking independence *directly using the definition*. Accepting Sophie's definition of independence would also mean believing the theorem is a *biconditional* statement. Yet, Sophie was unsure about this claim because of what she remembered from class.

Interviewer: What is [covariance] comes out to 0; what will that tell you, if anything?
Participant: I think what I wanna say is that they're independent, but (*pause*)
Interviewer: How confident would you be on a scale of 1-10?
Participant: Probably like 5. I don't know. I remember you saying something in class, like there's a caveat.

At this point Sophie believed a covariance of 0 guarantees independence because of her definition of independence, but she did not believe independence guarantees a covariance of 0 because she remembered from lecture there was a "caveat." Sophie was then prompted to review the definition of independence as well as the theorem written in her notes. She quickly revised her claim about the relationship between independence and covariance.

> "Now I'm thinking if they're independent based on that equation, then covariance should equal 0, but then if they're not independent, covariance can still equal 0. But they're not related inextricably. It's like a weird relationship."

At this point Sophie believed independence guaranteed a covariance of 0 but not the reverse. However, when prompted, she could not produce an example of two dependent variables with a covariance of 0. Moreover, she was still hesitant to provide truth values of the statement variants. When Sophie was asked about her pre-class quiz response of false to the contrapositive, she first claimed that response was incorrect, but then hesitated and ultimately did not provide a definitive answer. She responded similarly to a prompt about the contrapositive of one of the everyday statements later in the interview and relied primarily on context rather than logical equivalence.

Discussion

Byron has two distinct advantages over Sophie. First, since his concept of covariance involves linearity, he can conceive of dependent pairs of random variables with a covariance of 0. He can reference his domain knowledge to determine truth values of variants of the theorem. Secondly, he can avoid referencing his domain knowledge when expedient because he is assured of which statements are logically equivalent. Our analysis of Byron suggests that Probability students may rely on their logic training to quickly assess the truth value of variants of conditional statements, regardless of whether they are focused on the mathematical content in the predicate of a conditional statement. On the other hand, a student who avoids using the contrapositive could simply be relying on their knowledge of the mathematics of the statement.

Sophie demonstrates how gaps in domain knowledge, and specifically lack of counterexamples, can prevent students from determining which conditions imply others and lead them to rely on memorization. The power of examples in justification has been noted already in the literature (Zazkis, et al., 2008). We strongly suspect that instructors including a graphical depiction of dependent variables with a covariance of 0 when introducing covariance would have established for Sophie that the theorem's converse was false. Even so, without a strong notion of logical equivalence, truth values of the inverse and contrapositive would still not be obvious. We do not know at this point whether prior logic training gives students an advantage in Probability courses, especially in light of our previous study (Ryals, et al., 2023). However, we certainly value the flexibility Byron had in using both his domain knowledge and logical inference to answer questions confidently. We need to conduct further interviews or larger-scale surveys with more students who have taken Discrete Math and those who have not to make appropriate conclusions regarding logic training. Also, absent from our participants thus far were students who stated the converse was false on the pre-class quiz but then used it during the exercises. We plan to repeat the in-class instruction and invite more interview participants.

References

Attridge, N., Aberdein, A., & Inglis, M. (2016). Does studying logic improve logical reasoning? 2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 27–34. Szeged, Hungary: PME

Case, J. (2015). *Calculus students' understanding of logical implication and its relationship to their understanding of calculus theorems*. [Unpublished master's thesis]. University of Maine. https://digitalcommons.library.umaine.edu/etd/2254

Casella, G. & Berger, R. L. (2002). *Statistical inference* (2nd ed.). Duxbury.

Cheng, P. W., Holyoak, K. J., Nisbett, R. E. & Oliver, L. M. (1986). Pragmatic versus syntactic approaches to training deductive reasoning. *Cognitive Psychology, 18*, 293-328. https://doi.org/10.1016/0010-0285(86)90002-2

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, *547*, 341-385.

Dawkins, P. C., & Cook, J. P. (2017). Guiding reinvention of conventional tools of mathematical logic: students' reasoning about mathematical disjunctions. *Educational Studies in Mathematics, 94*(3), 241-256. https://doi.org/10.1007/s10649-016-9722-7

Dawkins, P. C., & Norton, A. (2022). Identifying mental actions for abstracting the logic of conditional statements. *Journal of Mathematical Behavior, 66*(100954). https://doi.org/10.1016/j.jmathb.2022.100954

Dawkins, P. C., Roh, K. H., & Eckman, D. (2023). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. *Journal of Mathematical Behavior, 70*(101043). https://doi.org/10.1016/j.jmathb.2023.101043

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, *53*(1), 5-34. https://doi.org/10.1023/A:1024661004375

Durst, S., & Kaschner, S. R. (2020). Logical Misconceptions with Conditional Statements in Early Undergraduate Mathematics. *Primus*, *30*(4), 367-378. https://doi.org/10.1080/10511970.2019.1597000

Epp, S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly*, *110*(10), 886-899. https://doi.org/10.2307/3647960

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*(3), 193-223. https://doi.org/10.1023/A:1023629608614

Inglis, M., & Simpson, A. (2008). Conditional inference and advanced mathematical study. *Educational Studies in Mathematics*, *67*(3), 187-204. https://doi.org/10.1007/s10649-007-9098-9

O'Brien, T. C., Shapiro, B. J., & Reali, N. C. (1971). Logical thinking-language and context. *Educational studies in mathematics*, *4*, 201-219. https://doi.org/10.1007/BF00303385

Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data*. Sage.

Ryals, M., Hellings, C., & Ma, H. (2023). Students' Inferences Using a Conditional Statement in Probability: The Relationship between Independence and Covariance of Random Variables. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Conference on Research in Undergraduate Mathematics Education.* (pp. 702-710). Omaha, NE

Sellers, M. (2020). *Students' quantifications, interpretations, and negations of complex* *mathematical statements from calculus* [Unpublished doctoral dissertation]. Arizona State University. https://keep.lib.asu.edu/items/158397

Sellers, M. E., Roh, K. H., & David, E. J. (2017). A comparison of calculus, transition-to-proof, and advanced calculus student quantifications in complex mathematical statements. *Proceedings of the 20th annual conference on Research in Undergraduate Mathematics Education*. (pp. 285-297). San Diego, CA.

Steffe, L. P., Glasersfeld, E. v., Richards, J., & Cobb, P. (1983). Children's counting types: Philosophy, theory, and application. New York: Praeger Scientific

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*(1), 133-162. https://doi.org/10.1023/B:EDUC.0000017671.47700.0b

Wason, P. C. (1968). Reasoning about a rule. *Quarterly Journal of Experimental Psychology*, *20*(3), 273–281. https://doi.org/10.1080/14640746808400

Zazkis, R., Liljedahl, P., & Chernoff, E. J. (2008). The role of examples in forming and refuting generalizations. *ZDM – The International Journal on Mathematics Education, 40*, 131- 141. https://doi.org/10.1007/s11858-007-0065-9

Promoting College Algebra Instructors' Adoption of Evidence-Based Teaching Practices

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While there is evidence that college students benefit from classroom activities that engage them, a shift in instruction from the lecture style of teaching to methods where students are actively involved in their learning is difficult to generate. This study focuses on providing structure and support for college algebra instructors to implement evidence-based instructional practices. We interviewed participants each semester they were involved with this project to gain insight into their perspectives on their teaching practices. We also video-recorded three lessons each semester when participants were implementing co-constructed in-class materials. Initial results showed how collaboration empowered instructors to implement evidence-based instructional practices in their class. Additionally, even though instructors collaborated to co-create in-class materials, results showed variation in their implementation styles as they moved towards more student-centered practices.

Key words: Professional Development, Active Learning, Instructional Change

College students' learning and conceptual understandings of mathematics benefit from instruction that actively engages students in their learning (e.g., Kogan & Laursen, 2014). Although there have been numerous calls to increase student engagement in college mathematics classrooms (CBMS, 2016; PCAST, 2012), lecture (where instructors present information and students receive it) remains the most prominent form of instruction (Stains et al., 2018). Notably, instructional change is difficult to catalyze, especially amongst college-level instructors (Fairweather, 2008; Henderson & Dancy, 2007). Shadle et al. (2017) identified several barriers to implementing practices that actively engage students and drivers that foster instructional change and support college instructors in adopting evidence based instructional practices (EBIPs). The project described in this report aims to better understand how a particular form of a professional learning community (PLC) can encourage the adoption of EBIPs by college mathematics instructors. We address the research question: In what ways does the structure and goals of the PLC support college algebra instructors in making changes in their teaching?

This study is part of a larger ongoing funded project of collaborative instructional improvement at the university level, aiming to support the implementation of EBIPs that actively engage students with the course content (NSF IUSE: HSI #2116187). The project supports the development of a PLC of instructors of all sections of a university-level college algebra course at a single large, southwestern, Hispanic-serving institution. Before the start of the project, the mathematics department had recently implemented a strong effort of course coordination, including a leadership structure with a coordinator for every multi-section course, weekly or bi-weekly meetings of all the instructors of the course, and aligned assessment procedures, instruments, and structures. Additionally, the college algebra team chose to implement innovative assessment techniques in the semester before this project began.

Theoretical Framework

The theoretical framing that guided the design of this project is the Ethic of Practicality (Doyle & Ponder, 1977), which describes the factors teachers consider when deciding whether innovative curricula is practical or realistic for implementation in the context of an actual classroom. Doyle and Ponder (1977) argue that in order for curriculum to be considered practical, it must: (a) Be compatible with the instructor's classroom, setting, and instructional goals (congruence); (b) Have potential benefits (e.g., student outcomes, student attitudes) that outweigh the effort and other costs of implementation (cost); and (c) Consist of clearly articulated procedures for ease of implementation in the instructor's classroom (instrumentality).

The Continuous Improvement (CI; Berk & Hiebert, 2009) framework began as a roadmap for instructional improvement within courses for preservice teachers, but its alignment with the backwards design principles for lessons are broadly applicable (Wiggins & McTighe, 2005). These principles advocate for planning lessons by first defining the student learning outcomes and the aligned assessment and then designing the in-class task. As part of this project, instructors worked as a group to implement CI cycles (Berk & Hiebert, 2009) to develop and facilitate lessons on particular course topics. In this model, each semester participants isolated specific lessons and implemented the following cycle: 1) *Design a task* that targets a particular student misconception or deepens understanding of a particular mathematical idea based on existing research on student learning about that concept. 2) Develop hypotheses about *anticipated student responses*. 3) *Collect data* in the form of student work and classroom recordings and *analyze the data* for evidence of the desired student learning outcomes. 4) *Record this information* and use it to revise the task for use in subsequent iterations of the course.

Instructors chose to create and use Desmos Classroom activities as a way to incorporate EBIPs in their teaching. Although instructors differed in how they used these activities (with some soliciting and leveraging student thinking more than others), they mostly gave students time to work individually or in small groups before discussing the activities in class. The CI model was a natural fit for a project guided by the Practicality Ethic. The model empowers instructors to make choices about curricular revisions and drive decisions about implementation and the CI framework leverages the knowledge, experience and priorities of instructors to guide these changes; thus, it maximizes congruence. Additionally, in order to maximize instrumentality and minimize cost, we used project funds to secure course releases for college algebra instructors so that they would have time and space to implement the CI cycles and participate in the PLC.

Methods

This report focuses on two participants who were full-time fixed-term faculty participating in the PLC instructional improvement project: Alex and Ivy. While Alex and Ivy were involved in the project, the course coordinators, Nicholas and Shay, also participated in the project. In the final semester of Alex's and Ivy's participation, two more instructors joined. Data were collected over the first three full academic semesters that these instructors were involved, which were also within the first two years they taught the course at this institution. Instructors were interviewed each semester. Interviews were semi-structured and included questions that prompted instructors to describe their teaching practices and participation in the project activities. We also collected class video data. In the first semester one lesson was filmed, and this recording served as a snapshot for what instruction looked like before implementing lessons co-created as part of the project. During the second semester, we filmed three lessons when the instructors taught lessons that were co-created by all the members of the instructional / PLC team.

Interviews were analyzed using thematic analysis (Braun & Clarke, 2006). We focused on coding segments that shed light on instructional practices in the context of the collaboration and coordination within the project. Multiple people from the research team reviewed the interview transcripts and identified representative quotes. Video from the filmed lessons were coded by at least two people from the research team using the College Observation Protocol for Undergraduate STEM (COPUS; Smith et al., 2013). Researchers coded the recordings individually and then met to discuss coding decisions and to reach consensus. After the filmed lessons were coded, we used the collapsed codes of Instructor Presenting, Instructor Guiding, Student Receiving, Student Working, and Students Talking in order to create radar plots of the percentage of class time where students and the instructor were engaged in these activities (Reisner et al., 2020). Collapsed codes grouped together codes that captured similar behaviors. For a detailed description of the codes grouped in these collapsed codes see Smith et al. (2014).

Results

It was clear from the interviews with instructors that the collaboration and support in the context of this project enabled them to try new things in their classes. For example, Alex described how watching other instructors implement different lessons gave him ideas of what to try in his class. He also highlighted how he was motivated to continue trying new things as the semester progressed instead of primarily lecturing. He said:

> Because of the grant I'm refreshing and recycling through different strategies much more than I think I would. I think normally I would just settle into what's easy, which might end up looking like semi-lecture style … But this time I randomly was pushing my students to do more group work. And I wouldn't have been doing that if it wasn't for the grant and seeing all the [other instructors use] group work.

This quote helps illustrate that there was a sense of accountability to keep implementing EBIPs amongst the instructors, as well as support. Alex emphasized, “I don’t have to worry if what I’m doing in the classroom is good enough because it’s what everyone else is doing and we’re all trying it together.”

Ivy also commented on how she tried new things because of the collaboration and coordination of the course. She said, “I've always wanted to do the flipped classroom, and I'm getting the benefit of seeing how it works right now.” In another part of the interview, Ivy said that “peer pressure” enabled her to try more EBIPs in her classes. She went on to emphasize how having access to the co-created resources was important in her decision to implement more student-centered activities. She said:

> I have … more freedom to - I can explore this stuff. So the fact that Shay and Alex are doing all these activities, and they're amazing, so I've got all these resources. My experience was lecture, and now I get to try the things I've always wanted to try. So I'm loving it. I love the activities, and I'll be honest, at the start of the semester, I was like, ‘Oh, we're going to do this now?’ because I didn't know how powerful it was. … And yeah, it's just cool to see everything that we can do with [Desmos], and it really helps the students understand.

In addition to instructors discussing how participating in this project enabled them to implement more EBIPs in their teaching, the analysis of what happened in class (as measured by the COPUS; Smith et al., 2013) indicated that there was some instructional change that occurred. Before being involved with this project, Ivy spent most of class time lecturing while students listened and took notes (Instructor Presenting and Student Receiving), see Fall 2021 semester in Figure 1. As Ivy participated in co-creating Desmos activities to use during class time, her instruction shifted to include more time where students were working and she was guiding them, see Spring 2022 and Fall 2022 radar plots for the Algebraic Properties, Fractions, and Factoring lessons in Figure 1. In the following quote Ivy describes how she used the Desmos activities in her class.

> I think Nicholas and I take a more laid-back approach to the Desmos and let the students fight through it and ask us questions. Whereas the other four instructors this semester seem to do the pacing a whole lot more. And especially Shay, Shay is … really talking through every slide … Whereas Nicholas and I, instead of pacing them, … we let them pace themselves. … we take the more passive approach and answer individual questions.

Alex also demonstrated a shift in how he spent class time before and during the project. Before the project, he tended to spend a large percentage of class time presenting material (Instructor Presenting), see Fall 2021 in Figure 2, but then as he taught the lessons with the co-created instructional materials, he spent less time presenting. In the interview, Alex shared how he started to create more space for students to work through problems themselves or share their answers with the class. He said:

> From the interactive style that we would do with … the Desmos [activities], and then talking about it, I think that's how I was able to tie in with my lectures. Like even just having those long pauses to let the students answer I think I got better at doing all of that with the Desmos. And that helps with breaking it up. ... Even though it's like a lecture, I try to have some moments where we just really stop and think about a question.

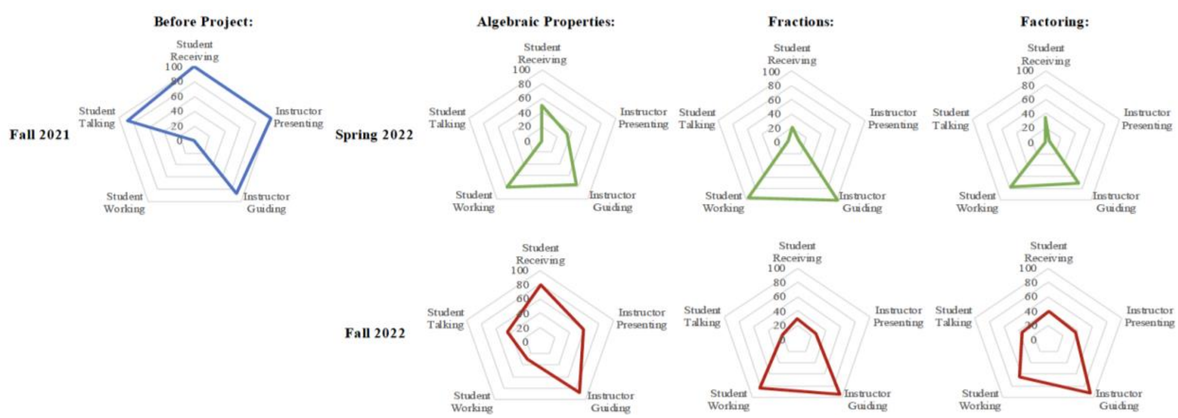

Figure 1. Radar plots by semester / lesson of the percentage of class time spent with Students Receiving, Talking, Working and the Instructor Presenting or Guiding for Ivy's recorded classes.

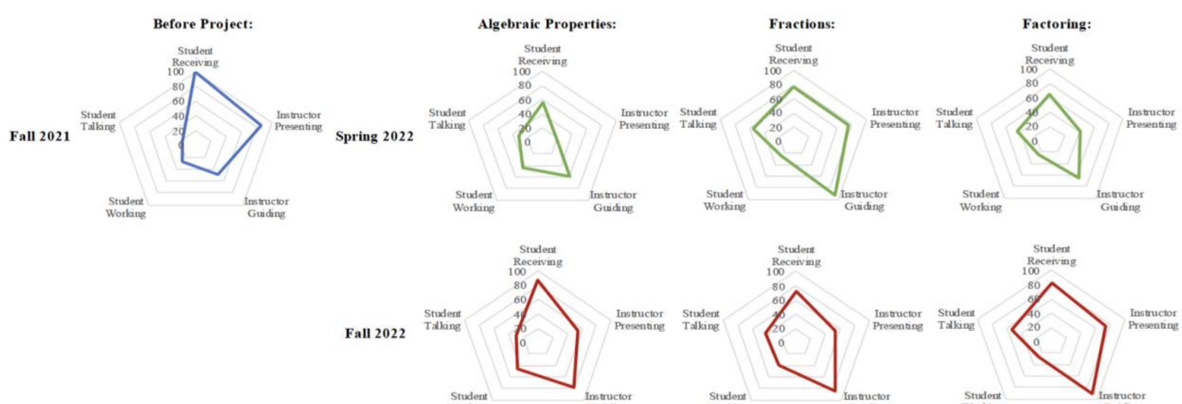

Figure 2. Radar plots by semester / lesson of the percentage of class time spent with Students Receiving, Talking, Working and the Instructor Presenting or Guiding for Alex's recorded classes.

Although the instructors co-created the Desmos materials as part of this project, it is important to notice that their implementation of the lessons varied (see Figures 1 and 2). That is, the analysis video data showed that the instructors (and their students) spent different amounts of class time with the instructor presenting or guiding and the students receiving, talking, or working. Each instructor had their own way of incorporating the materials into their instruction, retaining independence and control over what happened in their section(s). Ivy's quote above also points to these different approaches to implementing the co-created lessons, with her and Nicholas taking a more "passive approach" and Shay providing pacing and leading class discussions on nearly every slide of the Desmos activity.

Discussion

Instructors discussed how the project impacted the activities they did in class and their pedagogical decisions. The analysis of class video shows some differences in instructor and student activities over the course of three semesters. We hypothesize that the course coordination system in place allowed for "coordinated independence" (Rasmussen & Ellis, 2015), where instructors were encouraged to implement more evidence-based and student-centered activities while also retaining independence with their pedagogical and instructional decisions. In an interview that was conducted in the semester after her involvement with the project, Ivy said:

> For me the flipped classroom, it didn't work for my teaching style. … That was one of the reasons actually, that I asked to try to teach something else … so the flipped classroom was just... It just didn't work for me because I preferred to be more 'do it in the moment.'

This quote highlights that while participating in the project and teaching in this coordinated course, Ivy still used this style of teaching even though the "flipped classroom" did not align with her teaching preferences. The coordination and collaboration are also what empowered Alex to try different practices throughout the semester. Often, instructors may be isolated when attempting to adopt EBIPs, but the experiences of Alex and Ivy show how institutionally supported collaboration structures (such as course coordination) can be a powerful force in changing instruction.

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References

Berk, D., & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. *Teachers and teaching: Theory and practice, 15*(3), 337-356.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology, 3*(2), 77-101.

Conference Board of the Mathematical Sciences. (2016). *Active learning in post-secondary mathematics education.* Washington, DC: Conference Board of the Mathematical Sciences.

Doyle, W., & Ponder, G. A. (1977). The practicality ethic in teacher decision-making. *Interchange, 8*(3), 1-12.

Fairweather, J. (2008). Linking evidence and promising practices in science, technology, engineering, and mathematics (STEM) undergraduate education. *Board of Science Education, National Research Council, The National Academies, Washington, DC*.

Henderson, C., & Dancy, M. H. (2007). Barriers to the use of research-based instructional strategies: The influence of both individual and situational characteristics. *Physical Review Special Topics-Physics Education Research*, *3*(2), 020102.

Kogan, M., & Laursen, S. L. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education, 39*(3), 183-199.

President's Council of Advisors on Science and Technology (2012). *Report to the president: engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, DC: Executive Office of the President.

Rasmussen, C., & Ellis, J. (2015). Calculus Coordination at PhD-granting Universities: More than Just Using the Same Syllabus, Textbook, and Final Exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and Recommendations from the MAA National Study of College Calculus* (pp. 107-115. MAA Press.

Reisner, B. A., Pate, C. L., Kinkaid, M. M., Paunovic, D. M., Pratt, J. M., Stewart, J. L., ... Smith, S. R. (2020). I've been given copus (classroom observation protocol for undergraduate stem) data on my chemistry class... now what?. *Journal of Chemical Education*, *97*(4), 1181-1189.

Shadle, S. E., Marker, A., & Earl, B. (2017). Faculty drivers and barriers: laying the groundwork for undergraduate STEM education reform in academic departments. *International Journal of STEM Education, 4*(1), 1-13.

Smith, M. K., Jones, F. H., Gilbert, S. L., & Wieman, C. E. (2013). The Classroom Observation Protocol for Undergraduate STEM (COPUS): A new instrument to characterize university STEM classroom practices. *CBE—Life Sciences Education*, *12*(4), 618-627.

Smith, M. K., Vinson, E. L., Smith, J. A., Lewin, J. D., & Stetzer, M. R. (2014). A campus-wide study of STEM courses: New perspectives on teaching practices and perceptions. *CBE—Life Sciences Education*, *13*(4), 624-635.

Stains, M., Harshman, J., Barker, M. K., Chasteen, S. V., Cole, R., DeChenne-Peters, S. E., Eagan, M. K., Esson, J. M., Knight, J. K., Laski, F. A., Levis-Fitzgerald, M., Lee, C. J., Lo, S. M., McDonnell, L. M., McKay, T. A., Michelotti, N., Palmer, M. S., Plank, K. M., Rodela, T. M., … & Young, A. M. (2018). Anatomy of STEM teaching in North American universities. *Science, 359*(6383), 1468-1470.

Wiggins, G. P., & McTighe, J. (2005). *Understanding by design*. Alexandria, VA: Association for Supervision & Curriculum Development.

The State of Research on Proof and Proving at RUME Conferences from 2018-2023: A Systematic Literature Review

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Proof and proving are critical to the work of mathematicians, and in turn research on the teaching and learning of proof has historically been one of the larger strands of research presented at annual RUME conferences. In this project we systematically reviewed the RUME conference papers on mathematical proof from 2018-2023 to provide a picture of (a) the most prevalent topics of proof research in RUME, and (b) the data sources drawn upon in proof research methodologies in RUME. This review revealed the most frequent primary topics of RUME proof research to be proof instruction, assessment and feedback, and proof conceptions. We also noticed an overwhelming proportion of RUME proof research papers rely on qualitative data sources. Our continued work on this project will allow us to examine the impact research RUME has on the larger body of knowledge of undergraduate proof research.

Keywords: proof, research, literature review, methodology

Within the past six years, two major reviews of research on the teaching and learning of proof have been published (i.e., Stylianides et al., 2017; Stylianides et al., 2023), both casting a wide net and synthesizing research across K-16 mathematical contexts. In the first review, a contributed chapter in the Compendium for Research in Mathematics Education, Stylianides et al. (2017) examined the field of research on the teaching and learning of proof across K-16 mathematics education and classified the proof literature into three research perspectives: proof as problem solving, proof as convincing, and proof as a socially-embedded activity. In the second review, in a recent issue of ZDM-Mathematics Education, Stylianides et al. (2023) again examined the field of research on teaching and learning of proof across K-16 mathematics education, but this time classified the research according to the various didactic relationships emphasized between student, teacher, and mathematical content. We aim to add to the literature review work of these authors by (a) focusing exclusively on recent research accepted and published in the RUME proceedings, (b) taking a more emergent approach to identifying topics studied by proof researchers, and (c) attending to the methods utilized by proof researchers.

The reasons for isolating our literature search to RUME conference proceedings are threefold. First, by focusing on the RUME conference proceedings, we can better understand how this community of scholars is approaching proof research and empower the community to recognize its strengths and areas for growth. Second, we argue the RUME conference proceedings can indicate the most up-to-date trends in research, as the conference accepts contributed, preliminary, and theoretical proposals, providing a space for researchers to share early ideas and research-in-progress. Third, confining our literature search to RUME conference papers will enable us to later observe patterns in how proof research presented at the RUME conference evolves (or fails to evolve) into eventual journal publication, providing greater insight into the impact of the community's work.

Our goals in this review are to, in RUME, (1) identify trends in topics of proof research and (2) identify trends in methods of proof research. Specifically, our research questions are:

1. When RUME researchers study proof, what topics are they investigating? (In other words, what are they studying as it relates to proof?)
2. When RUME researchers study proof, what data sources are they drawing upon? (In other words, through what mechanism(s) are they telling the story of their research?)

Theoretical Framing: Advocating for Variability of Topics and Methods

In this research, we come from a pragmatic perspective, which assumes "knowledge guides method while method also guides knowledge" (Paul, 2005, p. 46). In this way, we seek to understand both the topics proof researchers are attending to in their research and the methods they are using to do so. We believe that variability in both the topics of investigation and methods of investigation can lead to a field of research that is more robust and has greater impact on practice. We draw from Stoecker and Avila (2021), who describe the benefits of addressing complexity of topics through variability of methods:

> From [a mixed-methods research] perspective, quantitative and qualitative methods are complementary, each compensating for the other's weaknesses in studying complexity. […] Greene (2007) and Clark and Ivankova (2016) believe that different methods can produce different kinds of information, such that one method can initiate research questions for another method, or expand a research project and lead to broader conceptual development [… and] even advocate that mixed methods research can promote social justice. (Stoecker & Avila, 2021, pp. 627-628)

In particular, we recognize different data sources have a different ability to tell a story. If we, as RUME researchers, have a preference toward certain data sources we use to make conclusions, we may be getting a limited perspective on a given proof topic. For example, consider video data from a classroom setting versus in a task-based interview setting. The classroom video data can help us better understand the proof learning contexts we are most interested in influencing, while the interview data provides a potentially cleaner and more focused picture of proof topics. Similarly, survey data has a potentially wide (but not deep) reach while interview data offers a potentially deep (but not wide) reach. Variety of methods is critical to the impact, robustness, and balanced perspective of a field of study (Stoecker & Avila, 2021).

Moreover, exploring the topics of study within a field (i.e., proof research), and in this case within a professional community (i.e., RUME), will allow us to identify areas of proof research that are over- or under-represented. Given the recent reviews of research on proof and proving we summarized above, we believe it will benefit the RUME community to compare the results of this literature review to those of other reviews. Previous proof reviews (Stylianides et al., 2017; Stylianides et al., 2023) were focused on proving literature related to teaching and learning in K-16 broadly, or on the teaching proof in undergraduate settings (Melhuish et al., 2022) and thus future research could benefit from comparing the RUME-specific topics of study from this review to the findings from previous broader reviews.

Methods

In this study examined proof research published in the RUME conference proceedings during 2018-2023. After collecting all contributed, preliminary, and theoretical reports from these years, we identified reports related to proof by a word search for "proof" or "proving" in both the paper titles and abstracts. The four authors sorted these reports into three categories: Proof Inquiries, Proof Settings, and Unrelated. Proof Inquiries include research about the teaching, learning, or professional practices of proof or research regarding proof-related concepts/topics conducted in a proof-based course context. Proof Settings refers to research that occurs in a proof course but was not investigating topics specific to proof. Finally, reports classified as Unrelated are those reports with the words "proof" or "proving" in the text of the title or abstract, yet these words were not used in the sense of mathematical proof. All reports were sorted by at least two authors who met to reach consensus on any disagreements. We excluded the 2021 RUME Reports due to the difference in that year's call for research to be conducted primarily by early career scholars. Moreover, our analysis of the 2023 proceedings is ongoing. We describe the methodological approach used to answer each research question below.

RQ1 Methods: Topics in RUME Proof Research

To identify the topics of RUME Proof research, we took an emergent approach to analysis. First, all four authors independently described the study focus of each report in their own words. We collapsed our individual descriptions into a list of topic codes through a card sort activity. We then each independently coded reports using these topic codes by assigning weights to the most relevant topics (3: Most Relevant Topic; 2: Very Relevant Topic; 1: Relevant Topic). Although each researcher was required to assign a most relevant (3) code to each study, topics weighted with a (2) or (1) were optional. Each proceedings year was coded by two authors, and although we did not seek to reach exact agreement, we discussed our codes in pairs and sought to understand each other's rankings and potentially altered our codes following further discussion.

We analyzed our topic codes by first summing the weighted topic codes of both coders on each article, so the topic code for each report ranged from 0 to 6. Our analysis proceeded in two ways: (1) we considered the topic codes with maximum sum value as the "primary topic" for each report, and (2) we observed trends in overall topic prevalence, regardless of weight, by considering the frequency of topics by code sum greater than or equal to one.

RQ2 Methods: Data Sources in RUME Proof Research

We first developed a list of data source codes by consulting textbooks and publications regarding educational research methods (e.g., Creswell, 2013; Ottinger, 2019; Merriam & Tisdell, 2016). Although our research team originally set out to classify research reports according to a set list of categories, after many efforts to precisely define our data sources, we determined this goal was much more complicated than expected. Upon further reading, we learned we are not the first mathematics education researchers who have encountered such difficulties. Schoenfeld (2010) described his efforts to create a taxonomy of research methods,

> My original intention […] was to provide a selective overview of some relevant categories of research methods, and to raise some issues about their use. This is by no means a straightforward task. (p. 495).

After reaching a conclusion that, like Schoenfeld (2010), we could not create a list of well-defined, mutually exclusive sources of data, we decided to proceed with our list of codes, treating each code as independent despite potential overlap. The categories for analysis were theoretical reports, literature reviews, qualitative data sources in the form of interviews, observations, artifacts, and surveys, and quantitative data sources in the form of surveys and assessments. The four researchers met in teams of two to discuss and resolve any disagreements.

Preliminary Findings

Across the RUME conference proceedings in the last five years (2018-2023), there have been 736 preliminary, contributed, and theoretical reports. Our search for reports regarding proof yielded 88 total reports: 18 reports from 2018, 15 reports from 2019, 20 reports from 2020, 18 reports from 2022, and 18 reports from 2023. Broadly, proof papers have composed about 12.3% of preliminary, contributed, and theoretical reports published in the RUME proceedings in the last five years (excluding 2021). We report preliminary findings from the 70 proof-focused reports from 2018-2022, as analysis of the 2023 proceedings is ongoing.

RQ1 Findings: Topics in RUME Proof Research

Across the RUME proceedings for which our research team has completed analysis, the most frequent primary topic in the RUME proof reports was proof instruction and feedback (12/70, 17.1%), followed closely by proof conceptions (views/beliefs about proof; 11/70, 15.7%). In contrast, the topic least prevalent as a primary focus was the transition to proof (2/70, 2.9%). Analysis of the primary topics of each report also revealed the, unsurprising, emergence of investigations of proof and online learning and technology in 2022, likely due to the transition to online instruction of many university courses during the COVID-19 pandemic. Figure 1 illustrates the prevalence of the primary topics of RUME proof research across 2018-2022.

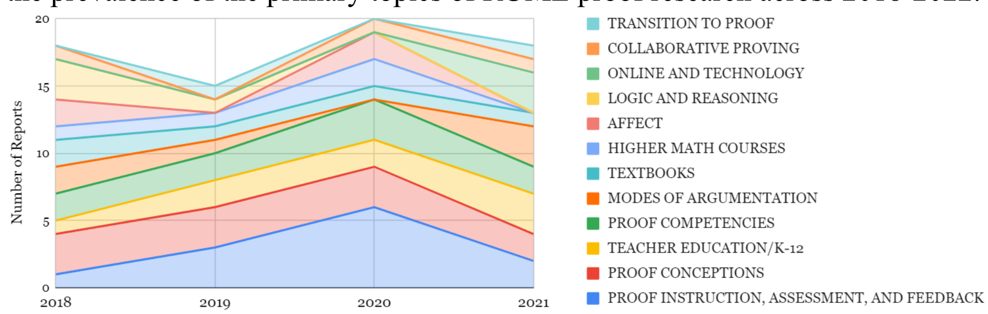

Figure 1: Primary Topics of RUME Conference Proof Research 2018-2022

By viewing the data, instead, with respect to the topics labeled any level of relevancy for each report rather than only the primary topic, we also observe trends in potential secondary topics and/or contexts of the RUME proof research. For example, although the transition to proof was least frequently the primary topic of investigation for the proof reports 2018-2022, it was the topic most frequently marked as relevant to the study in some capacity (i.e., at least one of the coders marked Transition to Proof as a 3, 2, or 1). Meanwhile, the investigation of textbooks appeared less frequently as a topic across all codes, yet when it was labeled as relevant this was often the primary topic for a report, so we can deduce textbooks are rarely a secondary or tertiary topic focus for proof research in RUME.

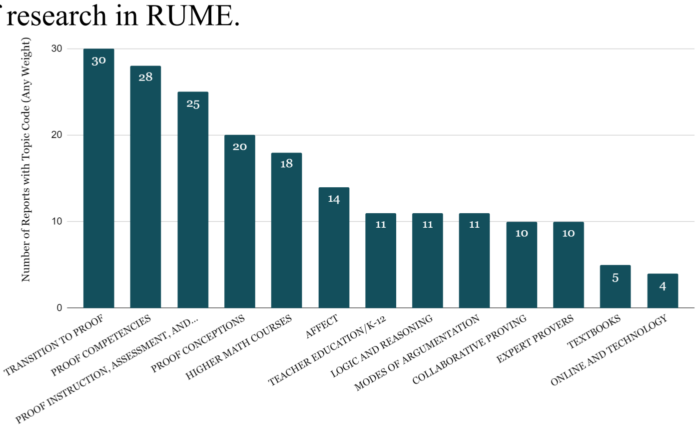

Figure 2: All Topics Marked Relevant (Any Weight) in RUME Proof Research 2018-2022

RQ2 Findings: Data Sources in RUME Proof Research

Because we encountered difficulties developing precise code definitions for data sources, we took suggestions from Schoenfeld (2010) in conjunction with methodological precedence from DiMartino and colleagues (2022) and classified RUME reports on proof as having drawn upon qualitative data, quantitative data, or both. This simple classification of reports revealed a drastic disparity between qualitative and quantitative data in RUME proof research. Two relevant theoretical reports did not employ any empirical data, so leaving 68 reports to consider data sources. Of these reports, 67 (98.5%) used some source of qualitative data, whereas only 7 (10.3%) used some form of quantitative data. Six reports used both qualitative and quantitative data, and therefore there was only one report which drew upon quantitative data alone. A final notable finding is the prevalence of interview research. Thirty-four (50%) of the proof reports using empirical data included interview data.

Discussion and Conclusion

After analyzing the 736 preliminary, contributed, and theoretical reports published in RUME conference proceedings in 2018-2023, we identified some of the most common topics of and data sources used in proof research within the RUME community. These preliminary findings indicate there is a strong presence of research regarding proof instruction, assessment, and feedback as well as proof conceptions, but there is much less research regarding collaborative proving, online learning and technology, and logic and reasoning. Much of the proof research (n=30, 42.8%) concerned the transition to proof, yet only two of these reports (2.8%) had the transition to proof as its primary topic. This finding indicates that there are many researchers conducting research within transition-to-proof courses but are not as frequently foregrounding students' transition to proof-based mathematics as their research focus.

With respect to methodologies and data used in RUME proof research, it is clear an overwhelming majority of reports used some form of qualitative data, and only one report used purely quantitative data. We speculate this absence of quantitative data may be due to the lack of reliable measurement tools relevant to the research of mathematical proof. Although we have not yet explored the reports in each topic in depth, we are interested to discover if RUME research regarding proof instruction, assessment, and feedback might be working toward developing measures which would allow for the collection of quantitative data.

In conclusion, we see several areas in which the RUME proof community could grow to assure there is a diverse presentation of topics and methodologies used to investigate proof at our conferences. In particular, there is room for more research on the transition to proof as a moment of mathematical development, and research on collaborative proving and proof learning and instruction through online mediums and technology. We also recognize the need for more quantitative data in order to complement the depth of qualitative data with widespread, generalizable claims. In moving this work forward, we intend to complete analysis on the 2023 proceedings, and then shift our focus to research journal publications on proof at the undergraduate level. This shift to journal publications will allow us to compare the distribution of research topics at RUME to the larger body of work in undergraduate proof research as well as track how research presented at RUME conferences evolves (or fails to evolve) into eventual journal publication. We pose the following questions for discussion with our audience:

1. What strategies for classifying proof research by methodology could best inform the field?
2. What research journals does our audience either seek publication in for proof research or look to for research on proving at the undergraduate level?

References

Clark, V. L. P., & Ivankova, N. V. (2016). *Mixed methods research: A guide to the field.* SAGE Publications, Inc.

Creswell, J.W. (2013) *Research design: Qualitative, quantitative, and mixed methods approaches.* 4th Edition, SAGE Publications, Inc., London.

Greene, J. C. (2007). *Mixed methods in social inquiry.* Jossey-Bass.

Karunakaran, S. S., & Higgins, A. (Eds.). (2022). *Proceedings of the 24th annual conference on research in undergraduate Mathematics Education.* Boston, MA.

Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). (2020). Proceedings of the 23rd annual conference on research in undergraduate mathematics education. Boston, MA.

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior, 67,* 100986.

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative Research: A Guide to Design and Implementation (4th ed.).* San Francisco, CA: Jossey Bass.

Ottinger, S. (2019). *Mathematical conjecturing and proving: the structure and effects of process characteristics from an individual and social-discursive perspective* [Doctoral Dissertation]. LMU München.

Paul, J. L. (2005). *Introduction to the philosophies of research and criticism in education and the social sciences*. Upper Saddle River, NJ: Prentice Hall.

Schoenfeld, A. H. (2010). Research methods in (mathematics) education. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 481-533). Routledge.

Stoecker, R., & Avila, E. (2021). From mixed methods to strategic research design. *International Journal of Social Research Methodology, 24*(6), 627-640.

Stylianides, G. J., Stylianides, A. J., & Moutsios-Rentzos, A. (2023). Proof and proving in school and university mathematics education research: a systematic review. *ZDM–Mathematics Education*, 1-13.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). Reston, VA: National Council of Teachers of Mathematics.

Weinberg, A., Moore-Russo, D., Soto, H., & Wawro, M. (Eds.). (2019). *Proceedings of the 22nd annual conference on research in undergraduate mathematics education*. Oklahoma City, Oklahoma.

Weinberg, A., Rasmussen, C., Rabin, J., Wawro, M., & Brown, S. (Eds.). (2018). *Proceedings of the 21st annual conference on research in undergraduate mathematics education.* San Diego, California

Supporting Instructors in Implementing Team-Based Inquiry Learning

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Team-Based Inquiry Learning (TBIL) is a novel active learning pedagogy designed to facilitate the use of inquiry-based learning in lower division courses. This preliminary report examines supports provided by the TBIL project to instructors, as well as the fidelity of implementation of TBIL by participants of the project. Initial findings suggest that classroom-ready materials and ongoing support, both synchronous and asynchronous, were most helpful to faculty in their TBIL implementations.

Keywords: inquiry-based learning, team-based learning, professional development

Introduction

Inquiry-Based Learning (IBL) is a well-established collection of pedagogies with many documented benefits for students (Laursen et al., 2011; Laursen & Rasmussen, 2019). Despite its benefits, IBL is more likely to be implemented in upper level courses, smaller courses, and courses for mathematics majors (Ernst et al., 2017). Team-Based Inquiry Learning (TBIL) is a novel active learning pedagogy implementing Team-Based Learning in an effort to facilitate the use of IBL in lower division courses (Lewis et al., 2021). TBIL was initially studied by the authors in the context of a linear algebra course at a single institution, where it was shown to improve students' content mastery, grades, and procedural flexibility (Lewis & Estis, 2020).

This paper reports preliminary findings from ongoing work to study the effectiveness of TBIL across varied instructional contexts. The authors conducted faculty development workshops (described below) to train interested instructors, who then implemented TBIL in their Calculus I, Calculus II, or Linear Algebra course. These instructors were invited to participate in the present study aimed at addressing the following research questions.

RQ1: (A) Which of the supports provided to faculty led to a successful implementation of TBIL in various instructional contexts? (B) Which additional supports would aid faculty in their implementation of TBIL in various instructional contexts?

RQ2: How faithfully do faculty implement TBIL after participating in the training workshops?

Team-Based Inquiry Learning

Team-Based Learning (TBL) is a highly structured active learning pedagogy that focuses on application of course content through collaborative problem-solving. Each module, or unit of instruction, consists of three phases: Preparation, Readiness Assurance, and Application of Course Concepts (Michaelsen & Sweet, 2008). TBL balances individual preparation and responsibility with the benefits gained from working together as a team to solve problems. Students receive frequent and timely feedback, and assignments are designed to promote team development, as well as learning. This is typically operationalized through the usage of four practical components: permanent teams; a readiness assurance process; so-called '4-S' application activities (in which students work on the **S**ame problem, which is to be a **S**ignificant problem, and make a **S**pecific choice that is **S**imultaneously reported); and peer evaluations (Michaelsen et al., 2004).

Team-Based Inquiry Learning (TBIL) utilizes the structure, flow, and principles of TBL to implement IBL in lower division mathematics courses. The 4-S application activities are designed to allow students to engage deeply with coherent and meaningful mathematical tasks, while the simultaneous reporting structure helps the instructor to inquire into students' thinking. The readiness assurance process is designed to remind students of prerequisite knowledge needed to fully engage with the challenging 4-S inquiry tasks by reducing extraneous cognitive load. A further explanation of how TBIL fulfills the four pillars of IBL is found in (Lewis et al., 2021).

The TBIL project, led by the authors, began in 2021 with goals to (1) Determine the extent to which TBIL is effective across differing instructional contexts; (2) Create and publish a library of accessible, classroom-ready, open-source TBIL materials for lower division courses; and (3) Train and support faculty as they implement TBIL at a diverse group of institutions and instructional contexts.

Supporting TBIL Instructors

The TBIL project provided a number of supports to instructors, beginning with an intensive faculty development workshop. Two cohorts of 13 instructors each participated in these workshops in the summers of 2021 and 2022, respectively. The first workshop was held in a hybrid format to maximize participation (5 faculty participated in person, with 8 connecting remotely), while the second workshop was entirely in person. The first workshop was five days in duration, while the second was three days; this was done to allow time for the first cohort to contribute to development of the curricular materials (described below). Both trainings included sessions on the fundamentals of team-based learning, integrating IBL into team-based learning, and mock teaching activities.

Instructors were also provided with a set of curricular materials prior to implementing TBIL. The linear algebra materials were initially written by the authors, and then revised by participants in the first cohort; while the single variable calculus materials (both Calculus I and Calculus II) were developed by participants in the first cohort, building on existing open-source calculus active learning materials such as *Active Calculus* (Boelkins et al., 2018). These materials included a full set of classroom-ready, student-facing activities, in addition to other support materials such as banks of exercises for practice and assessment, readiness assurance resources and quizzes, and videos.

In addition to initial training, previous work on adoption of IBL has shown that ongoing support during implementation is crucial (Hayward et al., 2016). Thus, the project team provided instructors with Online Working Groups (Fortune & Keene, 2021; Wawro et al., 2023), which were synchronous meetings of the instructors and a project team leader to have informal conversations about challenges and success in their TBIL implementations. Additionally, the project team created and maintained a Slack community devoted to TBIL. This ongoing resource serves to provide instructors with asynchronous support, again both from peers and from the project leaders.

Methods

We employed a survey methodology with the 26 instructors who completed the provided TBIL training. These instructors were located at 23 different institutions, 17 classified as predominantly white institutions and six classified as minority-serving institutions. Of the 23 institutions, 21 were four-year colleges and two were two-year colleges. To answer the first research question, this paper reports on a single survey administered at mid- and end-of-semester for each semester in which participants implemented TBIL. This survey contained three open-ended questions about instructor supports ("Of all the provided support, which has been most useful and why?", "Are there additional supports that have aided in your implementation of TBIL?", and "What additional supports would aid you in your implementation of TBIL?"), as well as four Likert-scale questions asking the frequency at which the instructor implemented the four practical components of TBIL (Permanent Teams, Readiness Assurance Process, 4-S Application Activities, and Peer Evaluations). Twenty-five of the 26 participants completed the faculty support survey at least once, with a total of 54 responses across all participants and semesters of data collection--Fall 2021 through Spring 2023.The three open-ended questions about supports were coded by the first author using open coding to identify the kind(s) of support described in each response.

In addition to the surveys, the 26 instructors were also invited to submit a syllabus from their TBIL course. 12 responses were received, which were coded by the first author for the presence or absence of the same four practical components of TBIL (Permanent Teams, Readiness Assurance Process, 4-S Application Activities, and Peer Evaluations).

Results

When determining which supports faculty identified as either leading to a successful implementation of TBIL, or which additional supports would be helpful in implementing TBIL, five codes emerged, with their frequencies presented in Figure 1. The most frequently mentioned support was 'Materials' in reference to the curricular materials provided. In discussing materials as a helpful support, one participant said, "Ready-made materials for the obvious reasons: I've really been able to focus my time on my students and facilitation, rather than higher level course design work." In contrast, when materials were described as a needed support, instructors desired either additional ancillary student resources (such as videos), or a desire to customize the materials to meet their unique needs. It should also be noted that 13 of the 14 respondents who identified materials as a needed support also identified elements of the materials as a helpful support. The next most frequent codes were 'Online Working Groups' and 'Slack,' referring to the synchronous and asynchronous informal support from the project team and their peers that extended through the academic semesters. The code 'Training' was unique in that there were more codes specific to additional supports needed (n=8) versus the training being helpful in implementation (n=7). Those instructors with a desire for additional support related to training either wanted a more nuanced understanding of creating TBIL activities, "how-to-guides" related to generating additional fluency-building or assessment questions, or desired to have additional training specific to TBIL implementation, possibly a refresher on in-class implementation or facilitation strategies. The last code was 'Peer', which referred to support from peer instructors, either through the project or at their institution.

To address the second research question, we first considered the frequency with which participants reported they implemented various components of TBIL in two ways. Since instructors were invited to respond to the survey in several semesters, we considered their initial response (n=25), as well as the most recent response from those who responded multiple times (n=19). We see a high implementation rate of permanent teams and the readiness assurance process in the initial response, with the latter waning somewhat in the final response. Implementation of the 4-S application activities was somewhat lower; only 72% of initial responses indicated that they were implemented 'Most of the time' or 'Always,' though this rose to 95% in the final response. Peer evaluations were the least used component; notably 47% of the final responses indicated that they were never used.

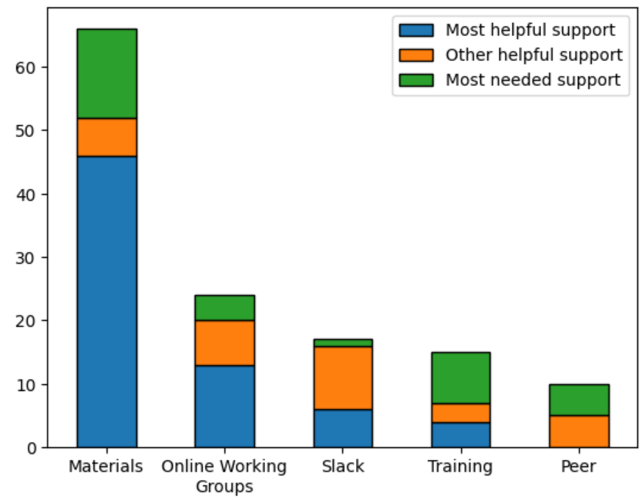

Figure 1. Supports identified as the "Most helpful support" provided, "Other helpful support" provided, or an "Additional needed support".

As noted above, 12 participants provided a syllabus for analysis. All of them indicated the use of 4-S application activities, with 92% indicating a Readiness Assurance Process and 83% indicating the usage of permanent teams. Only 67% of syllabi indicated that peer evaluations would be used.

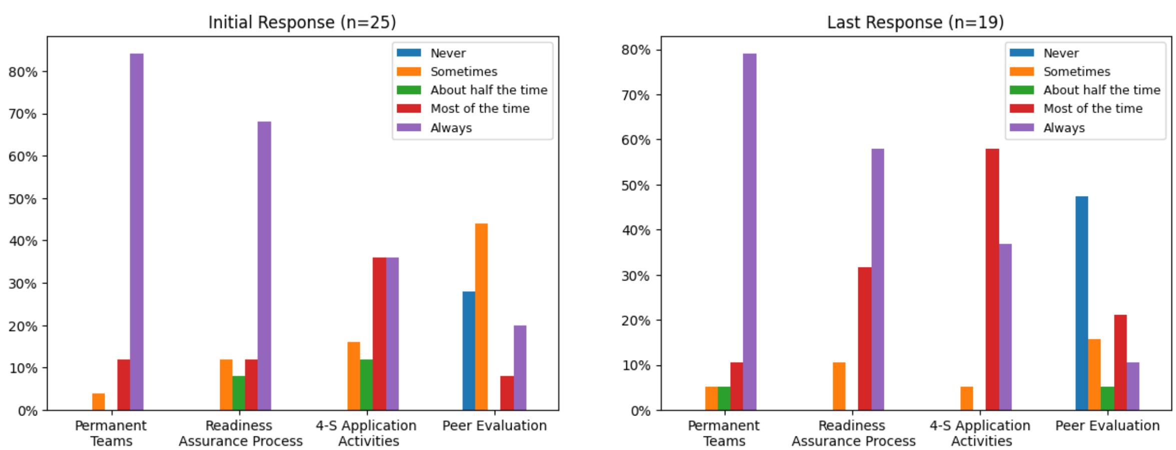

Figure 2. Initial and final responses of the frequency at which participants reported implementing components of TBIL

Discussion

Overall, we found participants identified two key supports in their implementation, namely the provided curricular materials, and ongoing support from peers and project leaders. Several participants specifically noted that having the provided curricular materials reduced the planning time necessary for the course, allowing them to focus on the novel aspects of the TBIL pedagogy such as classroom facilitation. We interpret this as the provided materials serving to reduce the cognitive load on participants as they learned to implement the other aspects of TBIL. We also note that one participant observed that the choice of the project to distribute the materials without requiring a login made it easier to share the materials directly with students.

We note this latter ongoing support was described by participants and coded separately as referring to either the synchronous online working groups or the asynchronous Slack community. The prevalence of both suggests that each communication modality offers something valuable to instructors that the other does not. The implications of these results are that leaders of similar pedagogical and curricular reform projects need to design ongoing faculty development efforts that accommodate these support preferences. While some aspects like the Slack channel almost maintain themselves after the initial setup, the synchronous online working groups are resource-intensive in terms of investigator time, which needs to be taken into consideration in the project design. An additional observation: the project's financial investment was greatest in materials development and initial training workshops. While the curricular materials were viewed as very valuable by our participants, the training was not. In fact, the online working groups and the Slack are both mentioned as more important than the initial training. However, we expect the community-building aspects of the intensive workshops, which led to participants' participation in the online working groups and Slack, were quite important second order benefits that do not appear in participants' responses.

Regarding the second research question on fidelity of implementation, we observed very high usage of permanent teams and the Readiness Assurance Process across both the self-report and syllabi. Usage of 4-S application activities was also quite high on both measures, but the self-report showed many instructors only used them most of the time. Usage of peer evaluations was much lower. We suspect one aspect contributing to this is the growing movement in mathematics (and other disciplines) to use alternative assessment and grading practices. Indeed, we (Lewis et al., 2021) specifically advocate for the use of Standards-Based Grading (e.g. Elsinger & Lewis, 2020) in our paper describing the TBIL pedagogy. We assume this stance implicitly permeated our trainings and support structure. We also note that Lewis and Estis (2020) reported that peer evaluations had no correlation with content mastery. It seems likely that participants may similarly have not found value in the evaluative nature of the peer evaluations, and instead opted for other peer feedback and team-building pedagogical moves.

Future Work

While this preliminary report represents partial progress towards addressing the research questions, we believe additional qualitative work is needed to fully answer these questions. In particular, we have collected (but not yet analyzed) video recordings of classroom sessions to further address the second research question regarding fidelity of implementation, particularly with a view toward facilitation moves and the usage of the 4-S application activities. We intend to conduct follow-up interviews with some of the participants at the end of the academic year to try to paint a fuller picture of the utility of various supports (in regards to the first research question), as well as to try to understand why various aspects of TBIL (such as peer evaluations) were implemented less frequently.

Acknowledgements

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References

Boelkins, M., Austin, D., & Schlicker, S. (2018). *Active Calculus*. http://activecalculus.org

Elsinger, J., & Lewis, D. (2020). Applying a standards-based grading framework across lower level mathematics courses. *PRIMUS*, 30(8-10), 885-907.

Ernst, D. C., Hitchman, T., & Hodge, A. (2017). Bringing inquiry to the first two years of college mathematics. *PRIMUS*, *27*(7), 641-645.

Fortune, N., & Keene, K. A. (2021). Participating in an online working group and reforming instruction: The case of Dr. DM. *International Journal of Research in Undergraduate Mathematics Education*, *7*, 107-139.

Hayward, C. N., Kogan, M., & Laursen, S. L. (2016). Facilitating instructor adoption of inquiry-based learning in college mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 59-82.

Laursen, S., Hassi, M. L., Kogan, M., Hunter, A. B., & Weston, T. (2011). Evaluation of the IBL mathematics project: Student and instructor outcomes of inquiry-based learning in college mathematics. *Colorado University*.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 5, 129-146.

Lewis, D., Clontz, S., & Estis, J. (2021). Team-based inquiry learning. *PRIMUS*, 31(2), 223-238.

Lewis, D., & Estis, J. (2020). Improving mathematics content mastery and enhancing flexible problem solving through team-based inquiry learning. *Teaching and Learning Inquiry*, *8*(2), 165-183.

Michaelsen, L. K., Knight, A. B., & Fink, L. D. (Eds.). (2004). T*eam-based learning: A transformative use of small groups in college teaching*. Taylor & Francis.

Michaelsen, L. K., & Sweet, M. (2008). The essential elements of team-based learning. *New Directions for Teaching and Learning*, 116, 7-27.

Wawro, M., Andrews-Larson, C., Zandieh, M., & Plaxco, D. (2023). Inquiry-Oriented Linear Algebra: Connecting Design-Based Research and Instructional Change Research in Curriculum Design. In *Practice-Oriented Research in Tertiary Mathematics Education* (pp. 329-348). Cham: Springer International Publishing.

A Preliminary Investigation of the Language of Proof in Mathematical Textbooks

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Mathematical proof writing is known to follow various norms and conventions set by the mathematical community. Yet, recent scholars have questioned if these norms and conventions hinder the progression of the field to be inclusive in proof courses. In this report, we discuss preliminary findings of a linguistic analysis on the degree of gendered language used in eight undergraduate textbook proofs ranging in content. We report a mixture of gendered language use in mathematical proof writing.

Keywords: proof, language, gendered communication

It is largely accepted that the language of mathematical proof is genre-specific and follows particular norms and conventions (e.g., Burton & Morgan, 2000; Dawkins & Weber, 2017; Lew & Mejia-Ramos, 2020). A small body of research has attempted to empirically investigate these norms such as Burton and Morgan's work exploring mathematical texts and interviewing mathematicians related to elements of proof related to authority or Lew and Mejia-Ramos's investigation of a set of conventional breaches and mathematicians' evaluations of them. Such work points to the fact that there are observable linguistic features of mathematical proofs that can be studied. Further, these features do not only involve the structure and nature of argumentation; they primarily concern language in which proof is conveyed.

Recently, Weber and Melhuish (2022) argued that the adherence to language conventions may be in tension with critical and inclusive aims in mathematical proof courses. Indeed, scholars such as Brown (2018) have illustrated the ways that students who are first generation and urban may use more "varied and at times colorful dialects" (p. 7), losing advanced mathematical contributions because they are not communicated in a standard way. It is possible that the conventions of proof writing are not due to epistemic aims entirely and might systematically exclude certain individuals. To make this point, Keith (1998) argued that proof characteristics (such as being impersonal and directive) may align with typical ways men communicate rather than women. However, to our knowledge, there has been little empirical work on how proof texts may be masculine or genderized. With this background in mind, the aim of this preliminary report is to build on the work identifying language features of mathematical proof and to begin an exploration related to Keith's (1998) observation. We focus on two research questions: (1) What are the language features of proof found in undergraduate textbooks? (2) To what degree do these features reflect types of communication associated with gendered communication?

The Norms and Language of Proof

A number of scholars have pointed to the language and norms of proofs being distinct from other types of language. Selden and Selden (2014) refer to the distinct stylistic and linguistic features as part of the genre of proof. Recently, Dawkins and Weber (2017) reflected on the values and norms of proofs both held by professional mathematicians and in mathematics classrooms. Of particular relevance to our examination of proofs, they identified, "Mathematical knowledge is justified by a priori arguments" and "Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author." Upholding these values is then reflected in linguistic choices in proofs. For example, they identified that proofs are written in such a way to reference the author or reader's agency. So, a proof would not contain an "I" statement, but rather impersonal "we." Such choices reflect the value of a priori arguments and that mathematical proof is meant to be universal and dependent on a particular author.

Perhaps in contrast, other scholars, such as Burton and Morgan (2000) argue that, "Conventions of mathematical writing are neither necessary nor natural consequences of the nature of the subject matter" (p. 450). In fact, in Burton and Morgan's analysis of published proofs and mathematician interviews, they identified similar conventions as Dawkins and Weber (2017), but also noted that authority can be found in proofs. For example, "terms such as clearly and obvious are relative to the individuals using them" and Burton and Morgan further suggest, "the extent or absence of such words is one of the interpersonal aspects of the writing that will influence the ways in which the readers of the text will construct an image of the author and will consequently judge the worth of the text itself."

The fact that there are some conventions that are not necessary to the subject matter is unlikely to be contested. For example, Lew and Mejía-Ramos's (2019) norm breaching study identified a number of breaches that mathematicians found unacceptable, especially in the context of textbooks. These include, "using non-statements, overusing variable names, lacking punctuation and capitalization, mixing mathematical notation and text, lacking verbal connectives, using formal propositional language, using unclear referents, and using an unspecified variable with an existential quantifier" (p. 55). While some of these breaches may hamper interpretability or validity of a proof, others are merely stylistic.

The imposition of linguistic and stylistic features that are not necessary to the validity of mathematical arguments has led several researchers to wonder about this impact for students (e.g., Brown, 2018; Tanswell & Rittberg, 2020; Weber & Melhuish, 2022). In Tanswell and Rittberg's reflection, they explain the concern with the particularities of mathematical language in proof (e.g., the interweaving of symbols and words, avoiding the first-person "I", the unusual "Let n be a number" style constructions, specialized use of imperatives and instructions, and being devoid of many features of natural language):

> This has the effect of elevating the regimented mathematical language that has grown up over the centuries to the level of being the sole vessel of objective mathematics. ... Once we realise that learners do not come from uniform linguistic backgrounds it quickly becomes apparent that the allegedly impartial language of mathematics is easier to access for some students than others (p.1202).

Brown (2018) illustrated this point sharing data from an inquiry class where majority of students were first generation and urban. She pointed to the "colorful dialects" they used and the ways they stand in contrast to standard ways of communicating math in proofs. Students' rich mathematical arguments may not been seen as important contributions if they are not structured in a way to align with conventional preferences. Overall, the literature base points to the fact that there are strong conventions in how proofs are written and that this is an important element to study because they have the potential to disenfranchise some students more than others.

Theoretical Framing

We take on the perspective that language reflects culture, and more broadly that language can be treated as a cultural artifact itself (Maltz & Borker, 1982). The culture in which one develops their language shapes their communication choices and how they interpret others. As noted by Maltz and Borker, these communication differences are salient across different ethnic backgrounds. Furthermore, the culture of language is not just developed at a global level (such as American English), but within gender, class, and other subgroups. For the scope of this paper, we provide particular attention to gender, and adopt Mulac et al.'s (2001) view of gender-as-culture. They suggest that, "much of what people know about interpersonal communication is learned from same-sex peers during the ages of 5 to 15 years." This then leads to communication differences across gender-lines. We note that this sociological interpretation does not mean that power dynamics are not at play for how and why there are communication differences, rather that it provides a mechanism for their existence.

This allows for identifying distinct language and communication aspects that may reflect different subcultures. For example, women may be associated with communicating in ways that involve more uncertainty and hedging while men may be associated in communicating in more direct ways. We further make the assumption that the language of proof reflects a distinct culture that may or may not align with communication elements salient in how different genders communication. While there are some that conjecture proof is more masculine in communication features, this remains an empirically open question.

Methods

In order to begin to analyze language features of proof, we began with the framework developed by Mulac et al. (2001). This framework stemmed from the synthesis of a number of studies identifying differences in how men/boys and women/girls communicate in both written and orally. The features can be found in Table 1. We note they align with four verbal communication style dimensions: direct versus indirect, elaborate versus succinct, personal versus contextual, and instrumental versus affective (Gudykunst & Ting-Toomey, 1998). The direct-indirect style captures the degree to which one explicates their intention through verbal communication. Elaborate-succinct style describes the richness and expressiveness of the language used, ranging from the most expressive (i.e., elaborate) to the most understated (i.e., succinct). Along the dimension of personal-contextual, a personal verbal style heavily rests on the use of personal pronouns (e.g., we, us) and locatives (e.g., then, there) whereas a contextual style deemphasizes the use of these language features. Instrumental communication style is goal-oriented and relies on the speaker (sender-oriented) to make the message clear. Alternatively, an affective communication style is process-oriented and the responsibility falls on the listener (receiver-oriented) to read contextual cues to decipher the message.

Table 1. Features of Language Found in Empirical Studies (Mulac et al., 2001) where M: male features, F: female features; X: mixed results from empirical studies.

Language Variable	Definition
Elliptical sentences (M)	A unit beginning with a capital letter and ending with a period in which either the subject or predicate is understood
Questions (X)	But not including directives in question form
Tag questions (F)	An assertion that is followed by a question asking for support
Directives (M)	Telling another person what to do

Negations (F)	A statement of what something is not
Sentence initial adverbials (F)	Answers the questions: how, when, or where? regarding the main clause [why and how long]
Dependent clauses (F)	A clause that serves to specify or qualify the words that convey primary meaning
Oppositions (F)	Retracting a statement and posing one with an opposite meaning
Judgmental adjectives (M)	Indicating personal evaluations rather than merely description
Uncertainty verbs (F)	Verb phrases indicating lack of certainty or assuredness
Progressive verbs (X)	Verbs presented in the "-ing" form
Intensive adverbs (F)	Adverbs such as "really" or "so"
Hedges (F)	Modifiers that indicate lack of confidence in, or diminished assuredness of, the statement
Justifiers (X)	A reason is given for a previous statement
References to emotion (F)	Any mention of an emotion or feeling
References to quantity (M)	References to an amount or quantity
Locatives (M)	Usually indicating the location or position of objects
Personal pronouns (X)	Words that stand for beings
"I" references (M)	First person singular pronoun in the subjective case
Fillers (X)	Words or phrases used without apparent semantic intent

We selected four common undergraduate textbooks across different subject areas: Smith et al.'s (2004) *A Transition to Advanced Mathematics*, Gallian's (2012) *Contemporary Abstract Algebra*, Fitzpatrick's (2009) *Advanced Calculus*, and Munkres' (2000) *Topology*. We then randomly selected a number between 1 and 10 (4), and selected the fourth proof in each text and the fourth proof after the halfway point in each text. Each of the four members of the research team applied the framework from Table 1. We met to reach consensus on the meanings of codes, discuss borderline cases, and ultimately arrive at finalized coded versions of the textbook proofs.

Results

We begin by providing some of the expected results in terms of features of proof language. None of the proofs contained: *elliptical sentences* (M), *questions* (F), *tag questions* (F), *uncertainty verbs* (F), *reference to emotion* (F), *"I" references* (M), *fillers* (X), *intensive adverbs* (F), or *hedges* (F). We note that this evidence points to a particular way of communicating in proof that does not contain many features of communication in other contexts. We also note that proofs do not contain some common masculine ways of speaking (elliptical sentences and I statements) nor contain many feminine ways of speaking such as reflecting questions and uncertainty or references to emotions.

We now consider the language codes that did appear in proofs beginning with the most universally found. All proofs had initial adverbials (F). These are sentences beginning with words or phrases serving the function of adverbs. "Thus" or "Therefore" are initial adverbials, as are clauses such as, "By Theorem 16.1," or "If I = {0}." On average, each proof had 6 initial adverbials (ranging from 4 to 10). Thus, we suggest that the use of initial adverbials is standard feature of proof and seems to serve the purpose of chaining one logical claim to the next. We contrast these with *justifiers* (X). Justifiers, as defined in the literature, provide a reason for something *after* it is stated. Only one of the eight proofs had any justifiers with the first of the topology statements having two sentences containing "for" statements. The first was, "The case which B=∅ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set {1, ...,n}." Otherwise, any explicit warrants were found before the inference being warranted. That is, they were found in initial adverbials with language such as, "Since X is normal,..." or "Because the theorem..." So, while justification is common in proofs, it is not provided as a justifier after a statement, but rather anticipates the statement that will be justified.

Other features found across all proofs were *dependent clauses* (F) (average of 3) and the *personal pronoun* (X) "we." Dependent clauses are more frequently found in the communication of women than men, but it is unclear whether the average of 3 is more or less than a baseline. The existence of "we" across all the proofs is hardly surprising as the literature has pointed to the commonality of the royal or impersonal we. We do know that "we" and one instance of "us" were the only personal pronouns in the proofs. We found no uses of the pronoun "I".

We also found that all, but one proof (7 of 8) contained *directives* (M). These directives begin with, "Let," "Suppose," "Assume," "Show," "Define," "Note," and "Choose." We note 23 such instances indicating that the commonality of directives mentioned in literature seems to bear out our analysis of the undergraduate proofs.

We quickly note the remaining features' prevalence. *Quantity* (M) was found in all eight proofs (usually references counts for abstract quantities such as "infinite"), but this seems like a contextual feature rather than a convention feature. *Location* (M) was not found in the proofs (although reference set membership was common). Again, we hesitate to say this is anything but a context feature. We also did not observe any *oppositions* (F) occurring within sentences as described in the framework, but we note there existence as complete new sentences in proofs by contradiction. We found that half of the proofs we analyzed had *negations* (F), that is defining something by what it is not such as "b-1/2 is not an upper bound." Three of the eight proofs contained progressive verbs (e.g., "consisting", "covering") reflecting a use, but not frequent use of these. Finally, we note the use of *judgmental adjectives.* One proof contained "trivial" while another contained "it is clear" reflecting at least the existence of these authority-driven language even within undergraduate texts.

Discussion

Our preliminary analysis explores the extent that language features from the linguistic literature are present in proofs. In the eight proofs we explored, we found that the impersonal "we" and directives were common in these proofs. We did not find any instances of uncertainty, including hedges, qualifiers, and leaving questions, nor did we find any references to emotion, oppositions, or the personal pronoun "I". While justification was common, *justifiers* (i.e., phrases used to justify prior statements) were rare. It is interesting that some features of proof align with masculine ways of communicating, such as the giving of directives, the lack of references to emotions, and the lack of hedges and oppositions. However, other features, such as the absence of "I" statements, are associated with the literature as feminine ways of speaking. At this point, it would be premature to make any claims about the degree to which mathematical proofs reflect masculine ways of speaking. Future research may involve a corpus linguistics analysis to document more accurate prevalence of these features, a convention breach analysis to see which of these features are considered salient to mathematicians, and comparisons to other texts both academic and non-academic. This is especially important when documenting prevalence of features such as dependent clauses. Dependent clauses are found more often in communication by women, but that is a comparison and not a claim that men do not use dependent clauses. It would be useful to consider how these features in proof compare to some baseline texts that are not proofs.

References

Brown S (2018). E-IBL: An exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. In Weinberg, A., Rasmussen, C., Rabin, J., Wawro, M., and Brown, S. (eds.), Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education. 1–15.

Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education*, *31*(4), 429-453.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics,* 95, 123-142.

Gudykunst, W. B., & Ting-Toomey, S. (with Chua, E.) (1988). Culture and interpersonal communication. Newbury Park, CA: Sage.

Keith, S. (1988, June). Women and communication in mathematics: One woman's viewpoint. Paper presented at the 10th Annual Meeting of the National Women's Studies Association, Minneapolis, MN.

Lew, K., & Mejía-Ramos, J. P. (2019). Linguistic conventions of mathematical proof writing at the undergraduate level: Mathematicians' and students' perspectives. *Journal for Research in Mathematics Education*, *50*(2), 121-155. **AND** Lew, K., & Mejía Ramos, J. P. (2020). Linguistic conventions of mathematical proof writing across pedagogical contexts. *Educational Studies in Mathematics*, *103*(1), 43-62.

Maltz, D., & Borker, R. (1982). A cultural approach to male-female misunderstanding. Language and social identity, 198-215.

Mulac, A., Bradac, J. J., & Gibbons, P. (2001). Empirical support for the gender-as-culture hypothesis: An intercultural analysis of male/female language differences. *Human Communication Research*, *27*(1), 121-152.

Selden, A., & Selden, J. (2014). The Genre of Proof. In K. Weber (Ed.), Reflections on Justification and Proof. In T. Dreyfus (Ed.), Mathematics & Mathematics Education: Searching for Common Ground (pp. 248–251). Dordrecht, the Netherlands: Springer.

Tanswell, F. S., & Rittberg, C. J. (2020). Epistemic injustice in mathematics education. *ZDM*, *52*(6), 1199-1210.

Weber, K., & Melhuish, K. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education*, *22*(2), 305-314.

Using CAS to Promote Students' Ways of Thinking Through Observation and Conjectures: The Case of Eigenvalues and Eigenvectors

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Linear algebra is an important topic in mathematics and many other disciplines. In this paper, we consider a set of digital interactive figures (I-figs) using Mathematica created for linear algebra students in an introductory course. The figures were designed to facilitate students' ability to visualize and work with eigenvalues and eigenvectors while minimizing computation for the benefit of conceptual focus while looking at an unlimited number of examples. The worksheets provide a foundation for motivating students to participate in a system of observation, conjecture, proof, and theorem. Based on our preliminary analysis of students' reflections on the I-figs, we found that students were confident in making and believing example-based conjectures.

Keywords: eigenvalues and eigenvectors, technology, observation, and conjecture

Literature Review

Already an important topic in mathematics, science, and engineering, the relevance of linear algebra has expanded into other areas as it provides the foundation for a wide range of data-driven and AI-related techniques. Linear algebra plays a central role in almost any area that uses quantitative information and methods to process data, hence the way linear algebra is taught is of paramount importance.

In the early 1990s, the Linear Algebra Curriculum Study Group (LACSG) recommended that "faculty should be encouraged to utilize technology in the first linear algebra course" (Carlson et al., 1993). As a part of this initiative, David Lay implemented the use of computer algebra systems (CAS) educational support such as MATLAB, Maple, and Mathematica in his book *Linear Algebra and its Application* (Lay, 1994). For example, in working with matrices and vectors, solving systems, matrix multiplications, finding inverses, and reducing a matrix to a reduced echelon form, technology can help with accurate and fast calculations as well as allow one to concentrate on the deeper conceptual aspects of the discipline that are often difficult for students to grasp. In addition, technology can be employed to build a visual image of the details, as well as the big picture, and allows for predictions, investigations, and producing conjectures.

Despite national initiatives in the US on teaching linear algebra with MATLAB, for example, the ATLAST project initiated by Steven Leon and his colleagues (Leon, Herman, & Faulkenberry, 2002) in the early 90s, systematic studies of linear algebra instructors' thought processes and their students' feedback on the effect of technology on their understanding are lacking. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors looked at linear algebra education papers from 2008-2017 in mathematics education journals and found almost no systematic classroom studies on the effectiveness of the use of Mathematica, Maple, Python, or MATLAB in the classroom, with the exception of a study of using MATLAB in mathematical modeling (Dominiques-Garcia, Garcia-Plana, & Taberna, 2016). However, the survey paper found studies on eigentheory using Geometer's Sketchpad (e.g., Gol Tabaghi, 2014; Caglayan, 2015), and Geogebra (Beltran-Meneu, Murillo-Arcila, & Albarracin, 2016) focused on the geometric aspects of the topic. The linear algebra recommendations by LACSG

2.0 also emphasized the importance of technology in teaching linear algebra (Stewart et al., 2022).

The second author has developed a series of digital interactive figures (I-figs) that aim to provide students with an opportunity to play with linear algebraic objects in an embodied format meant to be easy-to-access and requiring minimal time investment. Linear algebra students' conceptual understanding of eigenvalues and eigenvectors have been a topic of study in some studies (e.g., Thomas & Stewart, 2011; Salgada & Trigueros, 2015; Wawro, Watson, & Zandieh, 2019), largely focused on student thought processes. We also wish to study student ways of thinking, with the added structure of a series of I-figs targeting properties of eigenvalues and eigenvectors.

Theoretical Perspective

The theoretical framework for this study will utilize Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2010). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, justifying, and problem solving, which are not necessarily unique to mathematics. These several types of mental acts are the foundation of ways of thinking defined as "a cognitive characteristic of a mental act" (p. 269). These ways of thinking have the characteristics of being able to abstract, generalize, structure, visualize, and reason logically. Tall (2010) describes the embodied world as "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (2010, p. 22). In his view, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as a number)" (2010, p. 22). The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (2010, p. 22).

Our research questions are: What are the effects of the use of these I-figs on students' ways of thinking involving a system of observation and conjecture and how does that influence their understanding of proof and theorem? How did the use of the I-figs influence learning of eigentheory?

Methods

In this case study, 16 participating students were enrolled in an honors section of introductory linear algebra at a large public land-grant R1 university in the USA. These students were first and second-year students, with most declared or interested in degrees in mathematics, science and engineering, or medicine.

Five I-figs related to properties of eigenvalues were collected as one digital worksheet for students to explore. The worksheet was given to the students over a weekend's time to complete, immediately following the first class on eigenvalues and eigenvectors. In this lesson, students were shown the definition of eigenvalue and eigenvector with necessary conditions such as non-zero eigenvectors and uniqueness of eigenvalues for an eigenvector. By the time of worksheet participation, students had not practiced or discussed in class how to compute eigenvalues or eigenvectors, nor had any of the eigenvalue properties displayed in the worksheet been demonstrated. Students were instructed to spend approximately 15 minutes manipulating the digital figures and look for any observable patterns or trends.

Three forms of written data were collected at different times following the students' participation. Immediately after their time with the worksheet, students were asked to write a few paragraphs describing anything that they found interesting or noteworthy while working in the digital worksheet. Second, a few days later in the following in-person class time, students completed a short quiz asking if they could recall the definition of eigenvalue and eigenvector, and again to recall anything that stood out to them about the worksheet. Lastly, two weeks after the worksheet was administered when the class had concluded their study of eigenvalues and eigenvectors, students were asked to complete a final exit survey consisting of seven short answer response questions: (1) In what ways, if any, did you find the digital worksheet useful in understanding the concept of eigenvalues and eigenvectors? (2) What pros/cons do you see with this activity being a digital worksheet versus a pen-and-paper worksheet? (3) Describe, if any, the connections you see between the digital worksheet and the following week's content on eigenvalues and eigenvectors. (4) Do you like the idea of these exploratory activities before the content is introduced in class? (5) Would you have preferred a different sequencing? Please share your reasoning. (6) In our modern data-driven society, how do you see the relationship between linear algebra (matrices as data) and technology? (7) What part of the digital worksheet stood out to you as most important? Why?

As another data perspective, the instructor of the course (the first author) logged teaching reflections after each class meeting. These reflections included a summary of the day's class content, delivery, and notes about what did and did not go well, especially including important (from the instructor's perspective) student interactions. The instructor had weekly meetings with the rest of the research team and discussed their perspectives about the class further. The research team helped to design the study and research tools. Open coding by Strauss and Corbin (1998) was performed to analyze the data. In this paper, we will only analyze students' responses to their immediate post-worksheet reflection.

Preliminary Analysis

Some of the common themes in students' data so far have been as follows: play and exploration, pattern/non-pattern seeking, desire for relevance, and mathematical (formal) conjecture. These reflections ranged from one to three paragraphs, typically no more than half a page of writing.

We will begin by analyzing student responses relevant to each I-fig. The first I-fig (Figure 1) treats eigenvectors geometrically. Students may adjust the second component of v manually or use presets to snap to two eigenvector relations. Only four of the thirteen respondents addressed this geometric figure in their reflection. Those that did unilaterally employed physically embodied language surrounding their work, such as "It was interesting to scale the vector with my hand and check the gradual change of the values on my own" (S09). However, only one student explicitly named those special vector equations as demonstrating an eigenpair relationship.

For the second and third I-figs (Figures 2), eight students were able to correctly comment on the intended patterns in these I-figs (the same students for each). From this activity, these students make assertive statements (conjectures) such as "when you scale a matrix, the eigenvalues are scaled the same amount" (S03) or "When the matrices were put to a specific power, the eigenvalues were also just put to that power" (S05). From these notes we observe that those eight students all communicated a confidence that their perceived relationship is true (without supplying or referencing any justification), and also were able to supply a productive phrasing of the appropriate eigenvalue property, without having seen or otherwise been exposed to those properties in the class. As mathematics instructors, we can capture this as students' displaying an emergent way of thinking of the form (way of thinking: I can build a conjecture with a finite set of examples) and (way of thinking: I feel strongly that my conjecture is true). Notably, in the second way of thinking, students are unbothered by the absence of any supporting proof and have yet to become skeptical that a small set of examples can be used to identify a general trend.

Only one student who addressed these two I-figs proposed related though fundamentally incorrect statements such as "when scaling a eigenvalue, it affects the matrix" (S07). Nevertheless, we see the fundamental idea (of matrices and their eigenvalues sharing a scalar relationship) communicated, though this student's implication is reversed.

Next, Figure 3 shows an I-fig meant to display eigenvalue properties (or lack thereof) in relation to matrix operations on two matrices A and B. Seven students remarked upon this I-fig. Of those, five students were able to either state that the eigenvalues of A and B do not simply determine the eigenvalues of AB or A+B ("adding two matrices together does not necessarily add the matrices' eigenvalues, nor does multiplying two matrices together multiply the eigenvalues" (S06)), or stated that they were not able to find those patterns ("I wasn't sure how there is a correlation between eigenvalues of matrixes and their sums and products" (S12)). We note that these students leave open the possibility of there existing some pattern or rule that they simply were unable to find.

The final I-fig (Figure 4) concerns the relationship between eigenvalues, trace, and determinant of a matrix. Six of the thirteen respondents were able to conjecture a relationship, with five of those being correct ("I realized that adding the eigenvalues gives you the trace, while multiplying the eigenvalues gives you the determinant" (S07)). The final student observed "With the matrix provided, if I took the sum of all the eigenvalues, I received the trace of the matrix. Then, if I took that sum and multiplied by negative two, I got the determinant of the matrix" (S12). We see this student extending a previously identified way of thinking (I can build a conjecture with a finite set of examples) to now creating an incorrect conjecture apparently from a single viewed example. Again, showing no hesitation in endorsing this conjecture, indicating a lack of a way of thinking such as a mathematical conjecture requires a proof to be accepted.

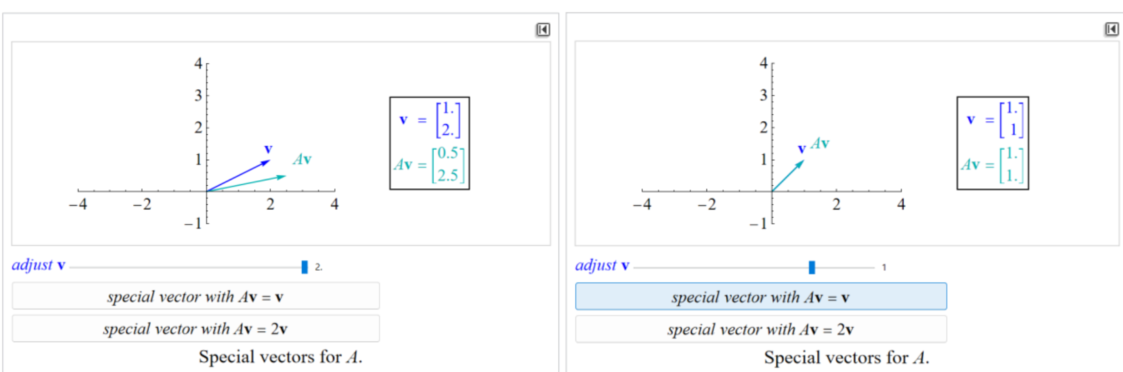

Figure 1. Geometric and algebraic connection of eigenvalues and eigenvectors.

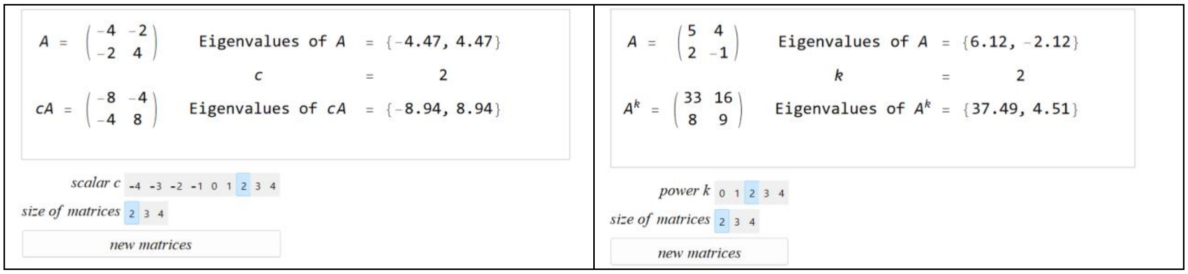

Figure 2. Effect of scalar multiplication and matrix exponentiation on eigenvalues.

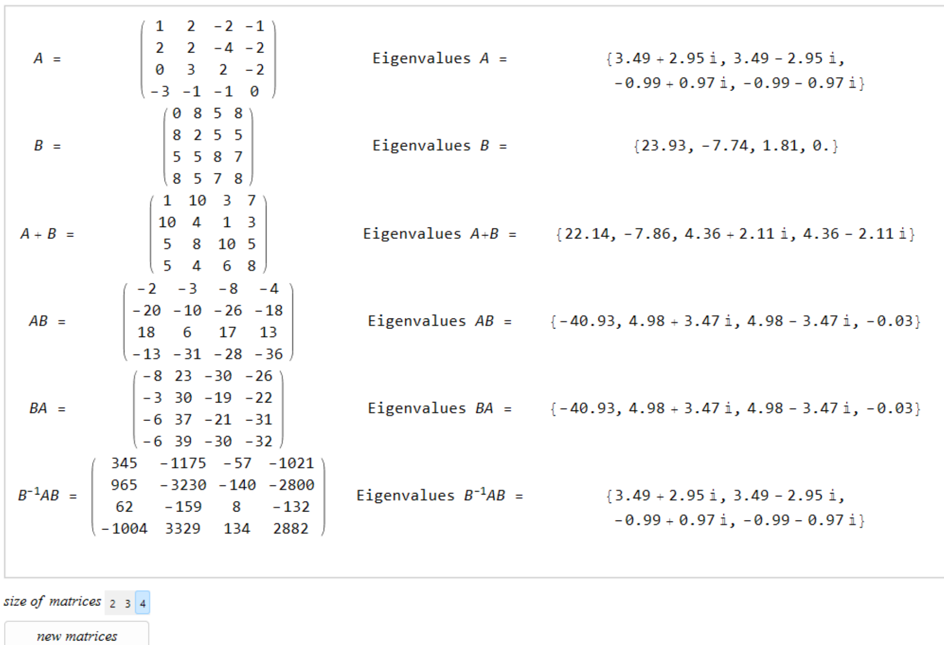

Figure 3. Eigenvalues and sums and products of matrices.

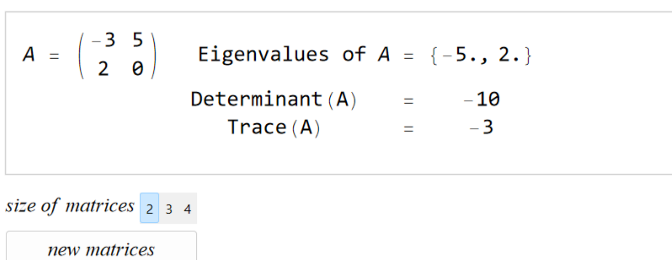

Figure 4. Relationship between eigenvalues, determinant, and trace.

Concluding Remarks

As we continue working on this research study, we plan to develop the theoretical framework further, complete the coding, and analyze the data by employing our theoretical framework hybridizing ideas of Tall's (2010) three worlds of mathematics and Harel's (2008) ways of thinking and understanding. Our analysis will also include some recommendations for teaching linear algebra using and developing CAS worksheets, and to target future iterations and additional worksheets.

References

Beltran-Meneu, M. J., Murillo-Arcila, M., & Albarracin, L. (2016). Emphasizing visualization and physical applications in the study of eigenvectors and eigenvalues. *Teaching Mathematics and Its Applications: An International Journal of the IMA*, *36*(3), 123–135. https://doi.org/10.1093/teamat/hrw018.

Caglayan, G. (2015). Making sense of eigenvalue–eigenvector relationships: Math majors' linear algebra–geometry connections in a dynamic environment. *The Journal of Mathematical Behavior*, *40*, 131–153.

Carlson, D., Johnson, C., Lay, D., & Porter, A. D. (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *College Mathematics Journal*, *24*, 41-46.

Dominguez-Garcia, S., Garcia-Plana, M. I., & Taberna, J. (2016). Mathematical modelling in engineering: An alternative way to teach linear algebra. *International Journal of Mathematical Education in Science and Technology*, *47*(7), 1076–1086. https://doi.org/10.1080/0020739X.2016.1153736.

Gol Tabaghi, S. (2014). How dragging changes students' awareness: Developing meanings for eigenvector and eigenvalue. *Canadian Journal of Science, Mathematics and Technology Education*, *14*(3), 223–237.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In B. Gold & R. Simons (Eds.) *Proof and other dilemmas: Mathematics and Philosophy*, Washington, DC: Mathematical Association of America, pp. 265-290.

Harel, G. (2019). Varieties in the use of geometry in the teaching of linear algebra. *ZDM Mathematics Education*, *51*(7), 1031-1042.

Lay, D. (1994). *Linear Algebra and Its Applications*. Addison-Wesley.

Leon, S., Herman, E., & Faulkenberry, R. (2002). *ATLAst Computer Exercises for Linear Algebra* (2nd ed.) Pearson.

Salgado, H., & Trigueros, M. (2015). Teaching eigenvalues and eigenvectors using models and APOS Theory. *The Journal of Mathematical Behavior,* 39, 100–120.

Stewart, S. Andrews-Larson, C.,& Zandieh, M. (2019). Linear algebra teaching and learning: Themes from recent research and evolving research priorities, *ZDM Mathematics Education*, *51*(7), 1017-1030.

Stewart, S., Axler, S., Beezer, R., Boman, E., Catral, M., Harel, G., McDonald, J., Strong, D., & Wawro, M. (2022). The linear algebra curriculum study group (LACSG 2.0) recommendations. *The Notices of American Mathematical Society*, *69*(5), 813-819.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed.). Newbury Park, CA: Sage.

Tall, D. O. (2010). Perceptions Operations and Proof in Undergraduate Mathematics, Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter, 2, 21-28.

Thomas, M., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic, and formal thinking. *Mathematics Education Research Journal*, *23*, 275–296.

Wawro, M., Watson, K., & Zandieh, M. (2019). Student understanding of linear combinations of eigenvectors. *ZDM Mathematics Education*, *51* (7), 1111-1124.

Academic Resources and Precalculus Students: Knowledge and Use of Tutoring Services, Skill Sessions, and Academic Coaching

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Discussion of a study that examines data from a single large, public university to determine how and when precalculus students elect to use learning center resources and their experiences using these resources. Data collected and to be analyzed includes survey and interview data from the precalculus students, interview data from instructors, class and learning center artifacts, and interview data from learning center leaders.

Keywords: learning centers; mathematics; precalculus

Colleges and universities provide many supplementary academic resources to support students outside of the classroom in their learning. Designated spaces that provide many of these supplementary resources in one space are often referred to as learning centers and can be found at the majority of colleges and universities (Bhaird et al., 2009; Byerley et al., 2019; Gilbert et al., 2021; Grove et al., 2020; Lawson et al., 2020; Matthews et al., 2013; Rickard & Mills, 2018). However, a large number of factors may influence a student's decision to use these resources, including the difficulty of finding new sources of help when first entering college, their desire to learn the course material, and their prior knowledge (Giblin et al., 2021). While a variety of research on learning centers has examined the impact of learning centers on students' success (e.g., Wurtz, 2015) or the various roles of learning centers on a campus (e.g., Solomon et al, 2010), one of the gaps in learning center research is how students make the decision to seek out help and their experiences of the provided resources. This study intends to approach that gap using a multiphase mixed methods design study to explore what precalculus students experience concerning the phenomenon of using free supplemental learning resources. Data from the students, instructors, and learning centers at a single university will be analyzed to look at how and when students elect to use learning center resources and their experiences using them once there.

Given the importance of learning centers to postsecondary institutions and of knowing more about how students decide to use these sites, this study will look at how, in past, present, and future, precalculus students learn about, decide to use, and experience using academic resources external to their introductory mathematics course at a large, southern, public research university, following the proposed theory of source selection by Giblin and colleagues (2021). The resources focused on in the study will be the mathematics content tutoring offered by the campus learning center (LC), mathematics content tutoring offered by the campus student success center (SSC), academic coaching offered by the SSC, and skill sessions offered by the SSC. In particular, I ask:

1. What opportunities do precalculus students at a large, southern, public research university have to learn about each of the four resources being studied, and to what extent do students identify as being aware of each resource?
2. What factors relate to patterns or changes in use by students over time?

I will answer these questions using a variety of data sources, including site observations, student surveys and interviews, instructors and center coordinators interviews, and center and course artifacts. These data and research questions are part of a larger study that also incorporates learning center usage and demographic data from students in the precalculus course.

Literature Review and Theoretical Framework

Postsecondary institutions provide a variety of resources to support students in their learning of course material outside of classroom and lab time. However, in order to take full advantage of these resources, students must learn that the resources are available, decide to use those resources, and then, decide whether to continue or discontinue use for the duration of the academic term. Access to these resources is often found in central locations on campus, called learning centers, but places that provide academic supports also often go by the name of tutoring centers or subject specific centers.

Postsecondary Academic Support

In his glossary of developmental education and learning assistance terms, Arendale (2007) defines support areas as, "Institutional services, other than regularly scheduled classes and labs, designed to assess and improve the academic and emotional well-being of students" (p. 29). These services include a number of common support areas, including tutoring and advising services (Parker, 2020; Truschel & Reedy, 2009). Institutions also frequently offer academic skill sessions, which are usually defined to have the purpose of supporting students in developing skills that will transfer to future courses, such as how to study or manage time (Munley, 2010; Parkinson, 2009; Wurtz, 2015).

Learning Centers

When an institution creates a central location where multiple of the previously discussed supports are located, they are usually referred to in the literature as either learning assistance centers or learning centers (Arendale, 2007). As previously discussed, learning centers often are called by a variety of names, including tutoring centers and mathematics support centers.

Mathematics course support. Students seeking success in their mathematics courses frequently find their way to learning centers. In fact, in a study surveying 61 learning centers, the only service that every center reported offering was mathematics assistance/tutoring (Franklin & Blankenberger, 2016). Learning centers can provide a variety of services, but the courses a student is enrolled in determine which of the previously discussed resources are available to them at their institution. As this study explores student resource use in relation to a mathematics course, special attention is on the relationships that the learning center structure and resources have with mathematics content and coursework.

Learning center structure. The structure of learning centers varies widely across institutions (Byerley et al., 2019; Johnson & Hanson, 2015; Perin, 2004; Truschel & Reedy, 2009). Three of the most common services offered in learning centers are tutoring, academic coaching, and skill workshops (Truschel & Reedy, 2009). Structural differences include drop-in versus appointment only assistance, online versus in-person support, the qualifications of center staff, and the relationship, or lack thereof, between mathematics departments and learning centers.

Student decisions surrounding learning center use. This study explores precalculus students' experiences concerning the phenomena of using free supplemental learning resources. I look at this phenomena using Giblin and colleagues' proposed theory of source selection of (a) narrowing of sources, (b) evaluation, (c) solicitation, (d) evaluation of presented help, and (e) use of source as a theoretical framework (2021). In other words, students must find the resource, decide to try the resource, and then, after using the resource, decide whether or not to continue use.

Methods

Seven main types of data will be used to answer the presented research questions: class artifacts, interviews of course instructors, center artifacts, interviews of learning center supervisors, center observations, student surveys, and student interviews.

Class Artifacts

The course syllabus and information shared on the Blackboard home page were gathered. Information from these sources have been reviewed and will be coded for any evidence of information of any of the four free academic resources being considered in this study.

Interviews of Course Instructors

Once during the semester, I interviewed the course instructors. Questions focused on the nature and frequency of instructor-classroom communication concerning academic resources available to students as well as the instructors' knowledge of these resources. Interviews of course instructors used a semi-structured format, meaning that the interview protocol is used as a guide and questions and probes may or may not be used as well as may be presented in a different order than written (Roulston, 2013).

Center Artifacts

Center artifacts in the form of data presented on resource webpages and distributed advertisements for services sent to students were collected. These data were collected because direct contact from the centers may be a way that students learn about these available academic resources outside of the classroom. Center artifacts will be reviewed for any information that may support student survey and interview responses.

Interviews of Learning Center Supervisors

Once during the semester, I interviewed the learning center supervisors. The questions asked during the interview focused on the center's communications with students concerning available academic resources, including any intrusive messaging, and what they perceived as the typical experiences of students when using each of the resources being studied. Interviews of learning center supervisors used a semi-structured format described previously (Roulston, 2013). The questions and guidelines provided in the initial protocol were framed by the six significant dimensions of undergraduate mathematics tutoring centers described by Byerley and colleagues (2019): "(1) Specialist versus Generalist Math Tutor Models, (2) Strength of Relationship between Center and Math Instructors, (3) Type and Extent of Tutor Training, (4) Types of Tutoring Services, (5) Physical Layout and Location, and (6) Tutoring Capacity." The SSC leader interviews also contained questions about the skill sessions and academic coaching appointments available.

Center Observations

Twice during the semester, I observed both the campus LC and SSC. The observation protocol focused on gathering the nature of student-computer, student-student, and student-staff interactions in the centers. The questions in the observation protocol were framed by the six significant dimensions of undergraduate mathematics tutoring centers described by Byerley and colleagues (2019), described previously, as both centers provide tutoring services.

Student Surveys

Twice during the semester, I surveyed students about their use or plans to use the four resources being considered across the two centers. The first survey took place immediately following the end of the drop/add period at the university, in the second week of classes. The second survey was given near the end of the semester. Both surveys asked about each of the four resources being studied, with students reporting (1) the extent to which they have heard about or seen advertisements for that resource, (2) how much they have used the resource in the past, and (3) how much they intend to use the resource in the future. Questions concerning knowledge of, past use, and intended future use of resources were asked using scales increasing in frequency (e.g., never; happens on occasion, happens at least once a month, happens at least once a week, happens daily) along with clear layout and other recommendations for survey creation (Walston et al., 2017). The second survey also included questions about help-seeking behaviors and attitudes towards mathematics.

Student recruitment. Students were recruited into the study during a classroom visit that occurred immediately following the drop/add period of classes. During this classroom visit, I described the survey including risks and benefits and then provide a link to the first survey using an electronic QR code that was projected during class. In addition to asking about students' knowledge and use of the academic resources, the linked survey also contained a question asking students if they have interest in participating in the interviews.

Student Interviews

During the semester, I interviewed five different student participants, with three student participants completing two interviews. The first interview took place within the first month of the course, and the second interview took place following the second survey distribution. Interview questions focused on students' decisions to seek help, what informed their decisions to use any combination (including none) of resources offered by the two centers, and their experiences while using the resources.

Student interviews were conducted using semi-structured format as described previously (Roulston, 2013). The first interview focused on collecting information about the student, their course history, and other prior experiences related to seeking help in a mathematics course followed by the tailored questions concerning resource use experiences. The second interview followed a similar format. However, there were data points from two surveys as well as the first interview to guide questions. For example, if a student in the second survey indicated that they were using a resource after indicating they did not intend to use that resource during the first survey, I asked a question that aimed to dive into how that decision was made.

Planned Analysis

As I have just begun my analysis, the following descriptions of planned analysis are in future tense. However, all data have been collected at the time of this proposal submission.

Research Question 1

What opportunities do precalculus students at a large, southern, public research university have to learn about each of the four resources being studied, and to what extent do students identify as being aware of each resource? Using appropriate psychometric analysis on the student survey data, the frequency that precalculus students hear about resources available to them will be determined. Class artifacts, interviews of course instructors, center artifacts, interviews of center supervisors, center observations, and student interviews will first be reviewed for mention of different ways that resource availability may be advertised or discussed. The analysis of these data sources will flag any mention of these services, which could come from a variety of sources, including verbal or written communications, posted information in course Blackboard, or physical flyers. Finally, this information on the different methods of exposure to the resource availability will then be used to explain the student survey results related to whether students report being aware of those resources. These all point to the first step of Giblin and colleagues' theory of source selection, narrowing of sources (2021).

Research Question 2

What factors relate to patterns or changes in use by students over time? Using appropriate psychometric analysis on the student survey data, separately, the changes over time in survey responses will be analyzed. Analysis of student interviews, center supervisor interviews, and instructor interviews using a phenomenological line-by-line approach will point towards factors related to certain combinations of resources being used. The analysis of these data sources will flag any mention of decision to use or stop use of a resource, which could present in a variety of forms. Possibilities regarding decision of use include mentions of failing a test, struggling with homework, being referred to the resource by their instructor, or being referred to the resource by a friend. Possibilities regarding the decision to stop the use of a resource include expectations of using resources were not met, description of services by source they heard about resource from turned out not to be accurate, by their perception, or mention of not having time. Any mention of deciding to use no resources, a single resource, multiple resources, or all resources will be coded and considered. These data will point toward the other steps of Giblin and colleagues' theory of source selection including evaluation, solicitation, evaluation of presented help, and use of source (2021).

Next Steps and Questions

Next steps include completing the analysis; although all data have been collected, analysis has just begun at the time of submitting this proposal. Thus, there are not yet any results or conclusions to share. The analysis will continue until the conference in February, at which time I expect to provide results and conclusions.

Audience Questions

1. Can you think of any other frameworks that I should look at this data with?
2. Any experiences at your institutions/in your learning centers that might shed light on this study that you would like to share?

References

Arendale, D. R. (2007). A glossary of developmental education and learning assistance terms. *Journal of College Reading and Learning*, *38*(1), 10-34.

Bhaird, C. M. A., Morgan, T., & O'Shea, A. (2009). The impact of the mathematics support centre on the grades of first year students at the National University of Ireland Maynooth. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *28*(3), 117-122.

Byerley, C., Moore-Russo, D., James, C., Johns, C., Rickard, B., & Mills, M. (2019, February 28-March2). *Defining the varied structures of tutoring centers: Laying the foundation for future research* [Paper presentation]. Twenty-second Annual Conference on Research in Undergraduate Mathematics Education, Oklahoma City, OK, United States.

Franklin, D., & Blankenberger, B. (2016). Program Evaluation of Community College Learning Assistance Centers: What Do LAC Directors Think?. *Community College Review*, *44*(1), 3-25.

Gilbert, H., Hodds, M., & Lawson, D. (2021). 'Everyone seems to be agreeing at the minute that face-to-face is the way forward': practitioners' perspectives on post-pandemic mathematics and statistics support. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *40*(4), 296-316.

Giblin, J., Stefaniak, J., Eckhoff, A., & Luo, T. (2021). An exploration of factors influencing the decision-making process and selection of academic help sources. *Journal of Computing in Higher Education*, *33*(1), 1-18.

Grove, M., Croft, T., & Lawson, D. (2020). The extent and uptake of mathematics support in higher education: results from the 2018 survey. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *39*(2), 86-104.

Johnson, E., & Hanson, K. (2015). Academic and social supports. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus* (pp. 69-82). MAA Press.

Lawson, D., Grove, M., & Croft, T. (2020). The evolution of mathematics support: a literature review. *International Journal of Mathematical Education in Science and Technology*, *51*(8), 1224-1254.

Matthews, J., Croft, T., Lawson, D., & Waller, D. (2013). Evaluation of mathematics support centres: a literature review. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *32*(4), 173-190.

Munley, V. G., Garvey, E., & McConnell, M. J. (2010). The effectiveness of peer tutoring on student achievement at the university level. *American Economic Review*, *100*(2), 277-282.

Parker, S. (2020). From roadblock to gateway: Improving developmental education for student success. In *Grantmakers for Education* (pp. 1–11). Portland, OR.

Parkinson, M. (2009). The effect of peer assisted learning support (PALS) on performance in mathematics and chemistry. *Innovations in Education and Teaching International*, *46*(4), 381-392.

Perin, D. (2004). Remediation beyond developmental education: The use of learning assistance centers to increase academic preparedness in community colleges. *Community College Journal of Research & Practice*, *28*(7), 559-582.

Rickard, B., & Mills, M. (2018). The effect of attending tutoring on course grades in Calculus I. *International Journal of Mathematical Education in Science and Technology*, *49*(3), 341-354.

Roulston, K. (2010). *Reflective interviewing: A guide to theory and practice*. Sage.
Solomon, Y., Croft, T., & Lawson, D. (2010). Safety in numbers: mathematics support centres and their derivatives as social learning spaces. *Studies in Higher Education*, *35*(4), 421-431.
Truschel, J., & Reedy, D. L. (2009). National Survey--What Is a Learning Center in the 21st Century?. *Learning Assistance Review*, *14*(1), 9-22.
Walston, J., Redford, J., & Bhatt, M. P. (2017). *Workshop on survey methods in education research: Facilitator's guide and resources.* Regional Educational Laboratory Midwest.
Wurtz, K. A. (2015). Impact of learning assistance center utilization on success. *Journal of Developmental Education*, *38*(3), 2-10.

Student Perceptions of Individual and Group Creativity in Proving

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Student described experiences in creativity can provide insight into instruction to foster creativity. In this report, I describe the emergent themes among student narratives describing when and why they felt creative during a collaborative proving activity. Continued work to refine these themes and their prevalence in student reflections can provide implications for how to best structure collaborative work to foster creativity in proof-based courses.

Keywords: Creativity, proof, collaboration

Both collaboration and creativity are central to the work of mathematicians (Karakok et al., 2015; Sriraman, 2004). Professional mathematicians have indicated collaboration is an important feature of their work (Sriraman, 2004), and the Conference Board of Mathematical Sciences (2016) has called for university mathematics classrooms to incorporate more active, collaborative learning. Therefore, there is a substantial need for K-16 education to focus on developing creative mathematicians and problem solvers with strong communication and collaboration abilities. In mathematics, students follow a journey in which there is an eventual transition from computational to proof-based mathematics (Civian & Schley, 1996). During this transition to proof, point students are often expected to become producers of mathematical ideas on their own for perhaps the first time (Boyle et al., 2015). For this reason, it is critical to study mathematical creativity and collaboration within the context of undergraduate mathematics courses and proof in order to cultivate the expertise needed by future mathematicians.

The purpose of this study is to investigate the creativity at work within an individual during creative collaborative proving. Although this study is being conducted in the larger context of a dissertation study, for this report I focus on the following research question: What actions or moments of the collaborative proving process do individuals report fostered their creativity? In this report, I first provide background literature on undergraduate mathematical creativity. I then describe the definitions and theoretical framings adopted in this study, outline the context and methods of the study, describe some preliminary results, and discuss the potential implications and future directions of this research.

Definitions and Theoretical Framing

Mathematical Creativity

Creativity has been notoriously difficult to define. Mann (2005) claimed there are over 100 existing definitions of creativity in the mathematics education literature, and this number has only grown with the volume of research on creativity published in the last 17 years (Savic et al., 2022). In this study, I adopt a perspective on mathematical creativity that describes creativity as *domain-specific*, meaning specific to the context of mathematics and mathematical proof, *a process*, meaning I will consider the motivation, perception, learning, thinking, and communicating involved in creativity, and *relative*, meaning considered within the context of the knowledge, abilities, and experiences of an individual. Put succinctly, ***mathematical creativity*** is defined to be the processes of creating, constructing, or implementing mathematical ideas, strategies, or processes, which are perceived as non-routine by the individual.

In this study I use the terms ***collaborative creativity in proving**,* or creative collaborative proving, to describe the process of approaching a shared goal in proving with significant contributions from two or more people, in which the proving task is either (a) assumed to be non-routine for all group members based on context or (b) established to be non-routine for all group members by asking them for their perspective.

Proof

In this study, situated within a university Introduction-to-Proof course, I adopt the definition of ***proof*** given by A. J. Stylianides, (2007):

> Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification.
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291, emphasis in original)

This definition allows for flexibility in the growth of a classroom community and the tools (both accepted statements and modes of argumentation) valid within this community.

Collaboration

One way to highlight the inner workings of students' approaches to proof initiation, proof construction, and proof validation is to engage students in collaborative proving. Collaboration is a tool for revealing the deeper processes behind proving and reflects the work of mathematicians (Grossman, 2002). To define collaboration beyond social interaction (Sriraman, 2004), I emphasize two elements of collaboration as a process: (1) contribution from two or more parties and (2) a common goal. This definition of collaboration can be applied to a classroom setting to describe collaborative learning, wherein typically student groups of two or more gain mutual understanding or create a product (Smith & MacGregor, 1992)**.**

Residue

Taking mathematical creativity as a relative construct requires asking students what they recall as fostering creativity for them, and these recollections will describe "what actually comes to the fore of [students'] attention" (Marton et al., 2004) regarding their creativity during collaborative proving as well as other activities in the Introduction-to-Proof course. Hiebert et al. (1996) described the importance of remembered experiences and influences as residue. Residue provides a way for thinking about what students take with them from classroom experiences. Therefore, residue for students with respect to creativity can give insight into the experiences, activities, and interactions that were meaningful for the student and inform how instructors can best shape classroom environments and activities to foster collaborative creativity in proving.

Background

It is undisputed that mathematical creativity has grown immensely as a research domain in the last ten years (Savic et al., 2022; Sriraman, 2017), yet research on mathematical creativity has been slow to extend to the tertiary setting (Savic et al., 2022). In this report, I aim to describe the actions or moments students experienced during in-class collaborative proving that fostered creativity for either themselves or their group. It is generally accepted that mathematical creativity can be developed and enhanced in students (Sriraman & Haavold, 2017; Zazkis & Holton, 2009), and mathematicians believe that mathematical creativity can and should be fostered in undergraduate mathematics courses (Karakok et al., 2015). Despite this consensus, there is little empirical research on what teaching strategies develop mathematical creativity among undergraduate students (Savic et al., 2022). No extant research has explicitly considered the impact collaborative work or collaborative proving may have on creativity. Investigating student perceptions of how collaborating with their peers on proving tasks fosters (or does not foster) creativity for them can inform how to best facilitate collaborative activities to promote creativity in proving.

In extant literature, the teacher actions most supported as effectively promoting mathematical creativity are choosing appropriate tasks (Satyam et al., 2022; Savic, El Turkey, et al., 2017a; Tang et al., 2022), allowing time for incubation (e.g., Savic, El Turkey, et al., 2017b), demonstrating different ways to solve problems and illustrating the mathematical process for students (Satyam et al., 2022; Tang et al., 2022), making the classroom a safe place to take risks (Satyam et al., 2022; Savic, El Turkey, et al., 2017b; Tang et al., 2022), attending to the emotions of students (Satyam et al., 2022; Tang et al., 2022), and providing space for discussion and disagreement (Satyam et al., 2022; Savic, El Turkey, et al., 2017a, 2017b; Tang et al., 2022).

Both Cilli-Turner et al. (2023) and Satyam et al. (2022) investigated how to foster creativity in undergraduate settings by considering student perceptions of creativity. Both of these studies, conducted in the context of Calculus I, reported six themes of student views of creativity: Actions and Attitudes, Application, Different Ways, Originality, Against Authority, and Understanding. Although these themes can help describe the ways in which students in an Introduction-to-Proof setting may view creativity, I seek to understand the specific ways in which students perceive themselves (or their group) as creative in a collaborative proving setting.

Methods

Context and Participants

Data were collected from an undergraduate Introduction-to-Proof course at a large public southeastern university in the United States. This course was facilitated in a collaborative, inquiry-based learning environment in which small groups of 3-4 students worked together to prove instructor-provided mathematical conjectures. The study discussed in this report is a small portion of a dissertation study on collaboration and creativity in mathematical proving. This report focuses on undergraduate students' impressions of their individual and group creativity after working in groups in class to collaboratively prove the statement: The product of consecutive twin primes is one less than a perfect square. In this report, I draw upon available written narrative data from eight participants.

Data Collection and Analysis

Students were instructed to remain seated at their group tables for five minutes of individual think time and then move to a whiteboard space to share their ideas with their peers and construct a proof. Students were each given a different color whiteboard marker to track their ideas on the collective dry erase board space. After the class session, students completed a retrospective writing assignment in which they were tasked with reflecting upon their experience working collaboratively in class and describing the times in which they felt (or did not feel) creative during the activity. These assignments were due at 11:59pm the day of the collaborative proving activity to mitigate delayed recall (Gass & Macky, 2000), in which a person creates a new mental process instead of recalling their original ideas (Lyle, 2003). Students were asked to (I) provide a sequential narrative of their group proving experience and (II) answer the question "When during the collaboration did you feel like you or your group were creative? If you did not feel creative, explain why not. Include as many details as you can."

To determine the actions or moments students perceived as influential on their individual and group creativity during the in-class activity, I narrowed my data source to only include written responses to task (II) of the writing prompt. First, I conducted an inductive, in-vivo coding strategy (Saldaña, 2016). The in-vivo codes were grouped into themes, and I conducted a second round of coding using this list of themes to identify the most frequently cited moments and incidents that initiated creativity for students. All sentences of responses to (II) containing reasons for feeling (or not feeling) creative were given at least one thematic code.

Preliminary Results

The inductive analysis of participant retrospective writing assignment submissions revealed 11 themes in their reflection on their individual creativity: Brainstorming, mistakes, noticing patterns/making conjectures, task novelty, suspense/pressure, teamwork, trying different ways/seeing things in different ways, writing an equation/formula, connecting mathematics and words, uncreative, and other. The most common themes were teamwork/collaboration, noticing patterns/making conjectures, uncreative, and writing an equation. Table 1 provides the frequency of sentences, proportion of participant responses, and a participant quotation illustrating the code theme for the themes reported by at least three participants, excluding the "other" category.

Table 1. Themes to Answer RQ1: What actions or moments of the collaborative proving process do individuals report stimulated creativity for them?

Theme	Frequency	Proportion of Students	Participant Quotation
Teamwork	13	5/8	"I believe that my group was even more creative than we could have been individually."
Noticing Patterns/ Conjectures	10	6/8	"We also felt creative in noticing the pattern with every pair of numbers we used to verify the proof."
Uncreative	10	5/8	"I did not feel very creative at the beginning of the group exercise, as I was sure that every group was thinking about writing out examples for the first step."
Writing an Equation/ Formula	9	6/8	"In our group work I felt that we were able to be the most creative when we were figuring out our initial equation."

Trying Different Ways/ Seeing Things in Different Ways	7	5/8	"Thinking outside of the box and using a variety of methods to solve a proof is what creativity in mathematics is all about."
Brainstorming	4	3/8	"We all began throwing out ideas on how we could solve this problem."

Discussion and Conclusion

In this report, I have characterized the types of actions or moments students reported as causing them to feel creative during an in-class collaborative proving exercise. A first observation is that students rarely emphasized themselves as feeling individually creative in their narratives (e.g., "I felt creative when…"), but rather emphasized the creative actions of the group as a whole (e.g., "my group was creative when…"). This distinction, in part, may be attributed to feeling as though the group's capacity for acting creatively was greater than capacity of the sum of the constituent members, as one participant indicated, "I believe that my group was even more creative than we could have been individually." Further, students may feel uncomfortable taking individual ownership in their narrative writing reflecting upon a collaborative task.

Regarding the themes present in student narratives, many align with the teacher actions previously indicated to foster creativity in different undergraduate mathematical contexts (e.g., Tang et al., 2022). For example, five students recognized themselves or their groups as creative when they tried to approach the proof using a different strategy or look at the problem in a new way, which corresponds to Tang et al.'s "encourage mathematical behavior" (p. 541) teaching action to foster creativity. Moreover, the most frequently mentioned moment, engaging in teamwork (13 mentions), corresponds to "allow for discussions" (Tang et al., 2022, p. 541), yet participants in this study attributed their feeling creative to positive teamwork, rather than simply the opportunity to engage with their peers. One student said,

> I think a vital part of collaboration is the group's overall willingness to actively collaborate. In this way uncooperative group members stifle the creativity of the group. I really appreciated that no one in my group ignored the ideas of others.

These preliminary findings indicate students may recognize opportunities to be creative in mathematical proving by collaborating with their peers in specific ways. Data used in this report is limited to eight student narratives following a single in-class collaborative proving activity; however, data collection and analysis will continue throughout the Fall 2023 semester to provide a more holistic image of the ways in which students perceive themselves and their teams to be creative in proving. The results of this study may be able to both verify the results from previous research on student perceptions of creativity in Calculus I (Satyam et al., 2022; Cilli-Turner et al., 2023) as applicable also to an Introduction-to-Proof setting, but also provide a guide for facilitating creativity-fostering collaborative work in undergraduate proof-based mathematics courses.

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References

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y. Y. (2015). Transforming perceptions of proof: A four-part instructional sequence. *Mathematics Teacher Educator, 4*(1), 32–70. https://doi.org/10.5951/mathteaceduc.4.1.0032

Cilli-Turner, E., Satyam, V. R., Savić, M., Tang, G., El Turkey, H., & Karakok, G. (2023). Broadening views of mathematical creativity: Inclusion of the undergraduate student perspective. *Journal of Creativity, 33*(1), 100036.

Civian, J., & Schley, S. (1996). *Pathways for women in the sciences II: Retention in math and science at the college level.* Annual Meeting of the American Educational Research Association.

Conference Board of the Mathematical Sciences. (2016). *Active learning in post-secondary mathematics education.* https://www.cbmsweb.org/wp-content/uploads/2016/07/active_learning_statement.pdf

Gass, S. M., & Mackey, A. (2000). *Stimulated recall methodology in second language research.* Routledge.

Grossman, J. W. (2002). Patterns of collaboration in mathematical research. *SIAM News*, *35*(9), 8–9.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12–21. https://doi.org/10.3102/0013189X025004012

Karakok, G., Savic, M., Tang, G., & El Turkey, H. (2015). Mathematicians' views on undergraduate students' creativity. In K. Krainer & N. Vondrová (Eds.), *Ninth congress of the European society for research in mathematics education* (pp. 1003–1009). ERME. https://hal.archives-ouvertes.fr/hal-01287301

Lyle, J. (2003). Stimulated recall: A report on its use in naturalistic research. *British Educational Research Journal*, *29*(6), 861–878. https://doi.org/10.1080/0141192032000137349

Mann, E. L. (2005). *Mathematical creativity and school mathematics: Indicators of mathematical creativity in middle school students* [Ph.D., University of Connecticut]. https://www.proquest.com/docview/305010927/abstract/83AA9C2DEED34EEEPQ/1

Marton, F., Tsui, A. B. M., Chik, P. P. M., Ko, P. Y., & Lo, M. L. (2004). *Classroom discourse and the space of learning*. Routledge. https://doi.org/10.4324/9781410609762

Saldaña, J. (2016). *The coding manual for qualitative researchers*. SAGE.

Satyam, V. R., Savic, M., Cilli-Turner, E., El Turkey, H., & Karakok, G. (2022). Exploring the role of students' views of creativity on feeling creative. *International Journal of Mathematical Education in Science and Technology*, *53*(1), 151–164. https://doi.org/10.1080/0020739X.2021.1961032

Savic, M., El Turkey, H., Tang, G., Karakok, G., Cilli-Turner, E., Plaxco, D., & Omar, M. (2017a). Pedagogical practices for fostering mathematical creativity in proof-based courses: Three case studies. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*.

Savic, M., El Turkey, H., Tang, G., Karakok, G., Cilli-Turner, E., Plaxco, D., & Omar, M. (2017b). Pedagogical practices that foster mathematical creativity at tertiary-level proof-based courses. In D. Pitta-Pantazi (Ed.), *Proceedings of the 10th International Mathematical Creativity and Giftedness Conference*. Department of Education, University of Cyprus.

Savic, M., Satyam, V. R., Turkey, H. E., & Tang, G. (2022). Mathematical creativity at the tertiary level: A systematic review of the literature. In *Mathematical Creativity* (pp. 105–120). Springer.

Smith, B. L., & MacGregor, J. (1992). What is collaborative learning? In A. S. Goodsell, M. R. Maher, V. Tinto, B. L. Smith, & J. MacGregor (Eds.), *Collaborative learning: A sourcebook for higher education*. National Center on Postsecondary Teaching, Learning, and Assessment.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator*, *14*(1), 19–34. https://doi.org/10.1007/s11858-008-0114-z

Sriraman, B. (2017). Mathematical creativity: Psychology, progress and caveats. *ZDM - Mathematics Education*, *49*(7), 971–975. https://doi.org/10.1007/s11858-017-0886-0

Sriraman, B., & Haavold, P. (2017). Creativity and giftedness in mathematics education: A pragmatic view. In J. Cai (Ed.), *First compendium for research in mathematics education* (pp. 908–916). National Council of Teachers of Mathematics. https://doi.org/10.1007/BF00553919

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Tang, G., Savic, M., Satyam, V. R., El Turkey, H., & Karakok, G. (2022). “The reason why I didn’t like [math] before is because I never felt creative”: Affective outcomes from teaching actions to foster mathematical creativity in Calculus 1. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*.

Zazkis, R., & Holton, D. (2009). Snapshots of creativity in undergraduate mathematics education. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 345–365). Sense. https://doi.org/10.1163/9789087909352_022

Motivations for Grading Among Math and Physics Faculty

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A student's course grade is typically an overall representation of multiple graded categories chosen by faculty. This study uses interviews with math and physics faculty to determine their rationale for selecting grading and assessment systems. We hope by identifying common faculty motivations, we can provide a baseline for future discussions on the implementation of alternative grading practices. Semi-structured interviews were conducted to present math and physics faculty with questions regarding enrollment size, length of implementation, and feedback processes. Preliminary analysis of faculty interviews demonstrates significant similarities in rationale across departments. All faculty considered enrollment size and course content when selecting systems. Findings indicated two central student learning goals and the emergence of four dominant motivations.

Keywords: Faculty Rationale, Goals for Student Learning, Teaching Practice, Grading Practice

Background

Teaching practices are at the heart of transforming STEM education in U.S. colleges and universities. More specifically, students' experiences in introductory mathematics courses play a crucial role in their overall retention in STEM fields. Both high and low-performing students have frequently cited teaching in their introductory courses as a factor in their choice to switch majors (PCAST, 2012). Members of groups underrepresented in STEM fields have also experienced significantly lower retention rates (Brainard & Carlin, 1998; Bressoud, 2011; NASEM, 2016). Recent calls for improving mathematics education have focused on the implementation of instructional activities that engage students in active learning and promote higher order thinking (CBMS, 2016; Freeman, 2014; Saxe and Braddy, 2015). This study seeks to expand upon prior research by focusing on teaching practices, specifically grading practices, rather than instructional activities.

To distinguish between instructional activities and teaching/grading practices, we employ the definitions provided by Speer, Smith, & Horvath (2010). Instructional activities refer to "the organized and regularly practiced routines for bringing together students and instructional materials" (p. 101). In contrast, teaching practices encompass the thinking, judgments, and decision-making teachers use to prepare for and teach their courses (Speer et al., 2010, p. 101). Research in this underexamined area of mathematics education will be a valuable catalysis for the widespread adoption of empirically validated teaching practices—a key recommendation advised by PCAST (2012). Furthermore, this research will benefit the expansion of professional development opportunities for faculty as prior research has been limited in scope (Crooks-Monastra, 2021; Deshler, Hauk, & Speer, 2015; Speer, Guttmann, & Murphy, 2009). This study focuses on tenured and tenure-track faculty, as opposed to graduate teaching assistants, because these faculty are more independent and less transient. Concerns expressed by faculty may also be useful for designing professional development materials appropriate to this demographic.

Within the scope of faculty's teaching practices, this study examines the role of beliefs, orientations, and resources in the selection of grading and assessment systems. Assessing student learning is a core component of collegiate teaching and varies greatly by instructor, discipline, and institution type (Lipnevich et al., 2020). Variations of grading scales have been proven to amplify racial or ethnic inequities in introductory STEM courses (Paul & Webb, 2022). Moreover, arbitrary depression of grading scales has contributed to poor student perceptions of STEM courses and decreased persistence in STEM majors (PCAST, 2012). While a large and growing body of research has indicated a need to move toward alternative learning environments, traditional teaching practices have been resistant to change (PCAST, 2012). We believe that understanding the current landscape of collegiate teaching will improve strategies for implementing alternative grading practices. To address these concerns, we sought to answer the question: *What considerations do faculty make when selecting grading and assessment systems for their classes?*

Theoretical Framework

This study utilizes a post-facto implementation of Schoenfeld's goal-oriented decision-making theory. At the core of this theory is the idea that "one's decisions about what goals to pursue and how to pursue them, are made on the basis of one's current resources, goals, and orientations" (Schoenfeld, 2011, p. 8). This theory is especially prevalent in "well practiced" domains, like teaching, where individuals are engaged in established patterns of behavior. Instructors enter the classroom with an array of knowledge and resources that may be conceptual, social, or material. These knowledge bases and resources are then called upon to achieve goals at the macro and micro level of classroom instruction. In this study, we rely upon Schoenfeld's (2011) definition of goals as "something that an individual wants to achieve, even if simply in the service of other goals" (p. 20). In applying this theory to the study of college faculty, we seek to connect motivations for selecting grading and assessment systems to overarching goals for student learning. We rely solely on interviews for this study because assessment practices are generally not accessible to classroom observation (Speer et al., 2010). A comparative study of these interviews will provide insight on the rationale of faculty across the disciplines of math and physics.

Methods

Participants

An invitation to participate in the study was sent to all tenured and tenure-track faculty in the math and physics departments at a public mid-sized research university in the Midwestern United States. Two math and two physics faculty volunteered for the study. Including graduate school experience, the faculty averaged 22 years of teaching. Expanding the study's participant pool to include physics faculty provides greater context for drawing comparisons across and within disciplines.

Data Collection

The first author conducted in-person, 30–60-minute, semi-structured interviews (Browner, 1988) with each faculty member in the summer of 2023. Zoom was used to audio record and auto-generate transcripts. Each interview was followed by a short-written survey to collect demographic information about the participants. All transcriptions were cleaned for accuracy and anonymity by the first author, before being shared with the second author for data analysis. Transcripts were labeled "M#" and "P#" to represent participants' respective disciplines. The numbers indicate the order in which the interviews were collected.

The interview protocol focused on the implementation of grading systems in small and large enrollment courses. For the purposes of the study, small enrollment courses were defined as having 15 or less students and large enrollment courses were defined as having 150 or more students. To distinguish between the discussion of these courses in interviews, participants were asked to answer separate sets of questions about courses of each enrollment size. This section of the interview protocol included the following questions:

1. How do you determine grades for a course with an enrollment of 15/150 students?
    a. Could you describe the percentile breakdown of those different graded tasks?
    b. Why have you chosen these percentile breakdowns?
    c. What course are you using as a point of reference?
2. How long have you been using this grading system?
    a. What do you value about this system?
    b. Some past participants have said they value "ease" in their grading system. Is this something you consider?

Other interview questions included follow-ups about enrollment size that were asked if participants did not distinguish between these courses in their initial response. The interview protocol concluded with questions concerning faculty's consideration of alternative ways of determining a grade. When asked for the definition of "alternative" participants were told that "alternative" referred to systems different from their own.

Data Analysis

We employed generative coding (Otero & Harlow, 2009) to identify themes in participant's responses. Together, both authors read the transcripts and highlighted text segments that characterized participants' motivations for implementing grading and assessment systems within their classes. The highlighted segments were used to find emergent themes across participants. The first author then went through and compiled these motivations, generating a list of 17 items.

Preliminary Results

Our analysis found that faculty across the disciplines of math and physics were influenced by four dominant motivations: (a) grades reflect learning, (b) assumed maturity, (c) clear feedback, and (d) ease of grading. While these motivations were similar across faculty, their manifestations within individual faculty's courses differed greatly by course content and enrollment size. Table 1 summarizes these findings with examples from faculty interviews. We note that the motivations are not ordered hierarchically and were prevalent in all faculty's discussion of grading and assessment systems.

Table 1. Dominant motivations expressed by faculty when discussing grading and assessment systems.

Motivation	Definition	Example
Grades Reflect Learning	Faculty believe grades should be an accurate reflection of student learning.	Participant M2: So, if A is excellent, B is very good, C is average and D is just passing, then how can it be that a whole class gets just A's? **I don't know what the purpose of grades are if everybody's excellent.** How can everybody be above average? That makes no sense to me.
Assumed Maturity	Faculty assume students in upper	Participant M1: Whereas for the **higher-level class**, it was like there are things that I'm not measuring exactly

	division courses have developed more “mature” learning processes.	like are you reading the textbook and, uh, like are you actively trying problems outside of the ones that I specifically asked you? And so yeah, I hope that **there's some level of maturity** that is reflected in the, uh, grading breakdown.
Clear Feedback	Faculty desire systems that allow for clear and timely feedback.	Participant P2: Usually for [a] 20-minute exam, **I use at least the same time**, probably more, to just **discuss the solutions**. To discuss what are possible approaches to solve it and what were the difficulties and challenges and all these things.
Ease of Grading	Faculty rely on systems that simplify grading processes and optimize time spent grading.	Participant P1: We have so few, uh, graduate teaching assistants that we've been forced, and some of us at least. I know that others, I commend them for the effort they put in to, you know, give direct feedback to students. But, um, some of us have resorted, you know, **out of necessity**, to an online homework system.

Grades Reflect Learning reflects faculty’s desire for grading systems that accurately measure student learning. In our interviews, this perspective was captured with the question: “What does an ‘A’ grade represent in your class?” All participants expressed that an “A” student is someone who has a strong understanding of the course content and performs well in multiple graded categories. While the graded categories varied between courses, all participants emphasized the importance of grading systems that accounted for student performance on assessments and engagement in the learning process, e.g., completing and submitting homework on time.

Assumed Maturity reflects instances in which faculty’s perception of student maturity influenced their grading system. The “maturity” referred to by faculty is regarding students' approach to the learning process. Across all participants, we found that the inclusion of graded categories for procedural tasks—pre- and post-lecture assignments, clicker questions, and reading quizzes—was primarily done in lower-division courses. Whereas in upper-division courses, grading systems consisted almost exclusively of homework and exams. We found that this was because faculty believed that lower-division students were in the early stages of developing strong study habits and needed incentives to engage in these learning processes.

Clear Feedback reflects faculty’s concerns regarding clear and timely feedback processes. We found that participants used a variety of feedback systems in their courses from personalized written notes on assignments to whole-class discussions of exam solutions. Some participants emphasized the importance of “grade transparency,” e.g., always having overall course grades available to students through an online gradebook. All participants discussed difficulty with offering detailed feedback to students in large enrollment courses.

Ease of Grading reflects faculty’s consideration of time spent grading when selecting systems for their courses. When asked to discuss their grading procedures, all participants commented on having a reliance on grading assistance for large enrollment courses. When graders and teaching assistants were not available, participants utilized online homework systems to provide students with quick and direct feedback. For small enrollment courses, we found that faculty preferred to do all the grading because it gave them a better snapshot of student understanding.

Discussion / Conclusion

We initially approached this study with inductive analysis to determine whether broad themes would emerge in faculty's discussion of grading and assessment systems. Our study found that faculty consider four dominant motivations when making these choices for their courses. We believe the discussion around these motivations could additionally provide insight into the goals faculty aim to accomplish in the classroom. Employing Schoenfeld's goal-oriented decision-making theory, the motivations we found in our initial analysis can be mapped to overarching goals for student learning.

An example goal could be, "Create Timely Feedback Systems." This goal directly connects two of our previously identified motivations: *Clear Feedback* and *Ease of Grading*, (see Table 1). By considering faculty's beliefs, orientations, and resources, we can explain the stark differences between the implementation of grading and assessment systems in large and small enrollment courses. Faculty's lack of material resources—graders and teaching assistants—has led to the use of online, auto-graded homework systems in large enrollment courses. Whereas in small enrollment courses, material resources played less of a role in faculty's decision-making processes.

A second example goal could be, "Balance Process and Product Grading Criteria." This goal refers to faculty's decision making in determining graded categories for their courses and makes use of one previously identified motivation, *Assumed Maturity*. Process criteria reflect student behavior and/or the procedural tasks students complete throughout the learning process (Lipnevich et al., 2020). Product criteria reflect student performance at a point in time, assessed by written and oral exams (Lipnevich et al., 2020). Past research suggests that the most important faculty beliefs are those that concern faculty's perception of the causes of student behavior (Clark & Peterson, 1984). Our research found that faculty's perception of student maturity resulted in variations between grading systems in lower and upper division courses. For example, all participants believed that active student engagement in the learning process could only be accomplished by incentivizing lower division students with points for completing procedural tasks. While in their upper division courses, there were few or no points awarded for these tasks because faculty believed students had developed more mature learning processes.

As we continue to apply Schoenfeld's theory to our preliminary findings, we will build a deeper understanding of how faculty's beliefs, orientations, and resources shape their goals for student learning. These findings will shape the approach to wider implementations of alternative grading practices among tenured and tenure-track faculty.

Questions for the Audience

Is Schoenfeld's goal-oriented decision-making theory an appropriate tool for understanding faculty motivations, or are there other theoretical frameworks that would provide greater explanatory power?

We foresee multiple avenues for this research going forward. What next-step questions do you most want answered?

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References

Brainard, S. G. & Carlin, L. (1998). A six-year longitudinal study of undergraduate women in engineering and science. *Journal of Engineering Education,* 87(4), 369-369.

Browner, C. H., Bernard R. Ortiz de Montellano, Rubel, A. J., Benoist, J., Cerroni-Long, E. L., Charzewska, J., Colby, B. N., Garro, L. C., Gonzalez, N. L., Good, B., Hall, R. L., Janzen, J. M., Kleinman, A., & Pollak-Eltz, A. (1988). A methodology for cross-cultural ethnomedical research [and Comments and Reply]. *Current Anthropology,* 29(5), 681–702. http://www.jstor.org/stable/2743606

Bressoud, D. (2014, November 1). MAA calculus study: Women are different. *MAA Launchings*. Retrieved from: https://launchings.blogspot.com/2014/11/maa-calculus-study-women-are-different.html

Clark, C., & Peterson, P. (1986). Teachers' thought processes. In M.C. Wittrock (Ed.), *Handbook of Research on Teaching* (3rd ed; pp.255-296). New York, NY: Macmillan.

Conference Board of the Mathematical Sciences (2016). Active Learning in Post-Secondary Mathematics Education. Retrieved from http://www.cbmsweb.org/Statements/Active_Learning_Statement.pdf

Crooks-Monastra, J. S. (2021). Planning and teaching for student learning in mathematics: How graduate student instructors develop and implement instruction. (Doctoral dissertation). Retrieved from https://scholarcommons.sc.edu/etd/6664

Deshler, J., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. *Notices of the AMS*, 62(6), 638-643.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. Proceedings of the National Academy of Sciences, 111(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Lipnevich, A. A., Guskey, T. R., Murano, D. M., & Smith, J. K. (2020). What do grades mean? Variation in grading criteria in American college and university courses. *Assessment in Education: Principles, Policy & Practice,* 27(5), 480–500. https://doi.org/10.1080/0969594X.2020.1799190

National Academies of Sciences, Engineering, and Medicine. (2016). Barriers and opportunities for 2-year and 4-year STEM degrees: Systemic change to support students' diverse pathways. Washington, DC: The National Academies Press. https://doi.org/10.17226/21739

Otero, V. K., & Harlow, D. B. (2009). Getting started in qualitative physics education research. In Henderson, C., & Harper, K. (Eds.), *Getting Started in PER* (1, 2). College Park: American Association of Physics Teachers. Retrieved from https://www.compadre.org/Repository/document/ServeFile.cfm?ID=9122&DocID=1218

Paul, C. A., & Webb, D. J. (2022). Percent grade scale amplifies racial or ethnic inequities in introductory physics. *Physical Review Physics Education Research*, 18(2), 020103. https://doi.org/10.1103/PhysRevPhysEducRes.18.020103

President's Council of Advisors on Science and Technology (U.S.), & United States. Executive Office of the President [PCAST]. (2012). Report to the president, engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics. Washington, D.C.: Executive Office of the President, President's Council of Advisors on Science and Technology

Saxe, K., & Braddy, L. (2015). A common vision for undergraduate mathematical sciences programs in 2025. Retrieved from https://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf
Schoenfeld, A.H. (2011). How we think. A theory of goal-oriented decision making and its educational applications. New York: Routledge.
Speer, N., Murphy, T., & Gutmann, T. (2009). Educational research on mathematics graduate student teaching assistants: A decade of substantial progress. *Studies in Graduate and Professional Student Development*, 12, 1-10.
Speer, N. M., Smith, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, 29(2), 99–114. https://doi.org/10.1016/j.jmathb.2010.02.001

Conceptualizations of Equity in Mathematics Education Research: 2000-2022

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We present work in progress that addresses the question: How do researchers in mathematics education define and operationalize the concept of equity in their work? Using a wide search of articles published in mathematics education research journals over the 2000-2022 period, we identified an initial sample of 47 empirical and theoretical articles, which was then reduced to 28 articles that operationalized or defined equity. In 17 of the 28 articles that included a definition, we found that equity definitions encompassed mainly notions of fairness, access, participation, and demanding mathematics, and we observed several references to bridging, identity, agency, and power. We summarize the scholars' rationales, purposes, and motivations for addressing equity across the 28 articles and suggest that a more critical stance is needed in the operationalization of a definition of equity.

Keywords: equity, mathematics education research, literature review, instruction

Addressing equity in mathematics classrooms has been a concern for mathematics education researchers since the 1980s (Secada, 1989; Stanic, 1989) and for the RUME community at least since 2018, when the following Equity Statement was issued:

> The SIGMAA on RUME recognizes that equity issues are present and relevant in our research and practice. The SIGMAA on RUME affirms that as an organization we are committed to being critical and introspective about the ways that equity can be more meaningfully integrated into our discipline (http://sigmaa.maa.org/rume/Site/About.html)

The statement addressed "ways in which equitable practice can be furthered in the context of mathematics instruction" and "ways in which the SIGMAA on RUME aims to increasingly value and cultivate research on issues of equity in undergraduate mathematics" issuing "calls of action to the community to begin to remove or circumnavigate (…) barriers [to equity]." However, our perusal of the literature yielded very few research mathematics education studies addressing equity in undergraduate education and moreover, we found a multiplicity of definitions and in some cases, studies without a definition, perhaps suggesting that there is a common understanding of the construct. We believe that for advancing equity work through research, understanding how the community defines *equity* is critical. Most of the information, however, comes from work on K-12 contexts.

Prior to the 2000s, much work on equity was driven by significant differences in student outcomes by student characteristics such as gender, race, ethnicity, and socio-economic (e.g., Lee et al., 1997; Secada, 1995; Warren & Roseberry, 1995). The publication of the *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (NCTM) in 2000 marked the first time in which equity was explicitly addressed as a principle[1] for teaching and learning in schools, paired with a notion of ambitious mathematics (Lampert, 2001) that was expected to be made available to all students. From such work, we have learned that equity is not equality (Secada, 1989) and that equity and social justice are intertwined terms. Yet, to inform research that improves practice, the research community needs a definition that can be operationalized, so that equitable teaching can be systematically studied. For this reason, we engaged in a literature review process that would allow us to answer the following questions as related to the US context: (1) What are the definitions or operationalizations of equity used in articles published in mathematics education journals in the period 2000-2022? and (2) How do researchers justify and motivate their equity work?

Methods

We searched for empirical or theoretical articles in peer-reviewed scholarly journals under the assumption that such work would have been vetted by the mathematics education community. The journals included: *Cognition and Instruction*; *Educational Researcher*; *Educational Studies in Mathematics*; *Journal for Research in Mathematics Education*; *Journal of Mathematics Teacher Education*; *Journal of the Learning Sciences*; *Journal of Urban Mathematics Education*; *Mathematical Thinking and Learning*; *Race Ethnicity and Education*; *and Review of Educational Research*. We chose 2000 as the starting year for the review, when the *Principles and Standards of School Mathematics* (NCTM, 2000) were published. This was necessary because most of the work on equity has been advanced with the K-12 context and has been used to investigate tertiary contexts (e.g., Leyva et al. 2021).

We searched various databases (e.g., ERIC, ProQuest, PsychInfo) for articles that dealt with equity and mathematics instruction ("equit*" AND "instruct*", "equit*" AND "teach*", and "equit*" AND "classroom"; added "math*" for non-math specific journals). We only selected articles that collected United States data. This was done to provide a common historical, cultural, political, and social context (e.g., the legacy of slavery, the Civil Rights and women's movements, or immigration policies) for framing equity which is harder to assume for studies conducted in other countries. The process of identifying the analytical sample involved two steps (see Figure 1), screening (to eliminate articles that did not address equity) and focused reading (to identify key elements in the article related to equity). Eight researchers were involved in the process.

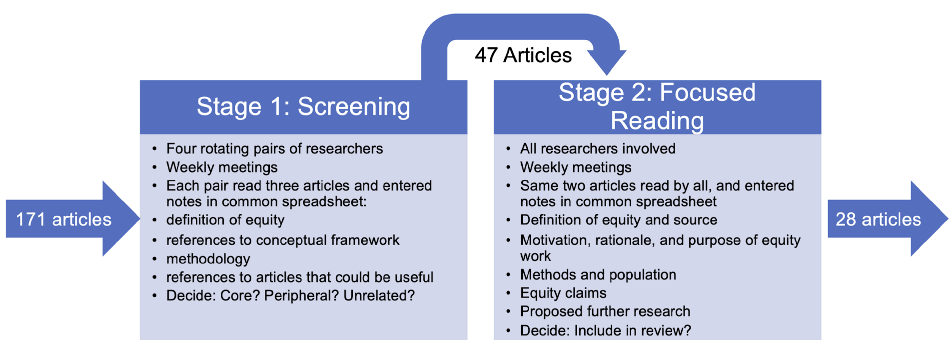

Figure 1. Article reviewing process. Note: Core: equity is central to the article. Peripheral: relevant topic for contextualizing equity. Unrelated: no mention of equity beyond reference list.

[1] **Equity.** Excellence in mathematics education requires equity—high expectations and strong support for all students. (NCTM, 2000, p. iv)

Preliminary Findings

The 28 articles included in this report were published in eight of the 10 journals chosen (see Table 1). Of these, five were theoretical papers and 23 were empirical studies. Twelve of the 23 empirical studies targeted students at various levels from elementary school through undergraduate students at university as the research population, eight on teachers, and three on both. Of the 12 articles that target students, three focused on elementary grades, six on middle school, six on high school mathematics, four combined data from middle school and high school, and four on undergraduate mathematics. Five articles did not specify a grade level. Sample sizes were relatively small: 1- 4 teachers for studies that focused on teachers and less than three classes for studies that focused on students. Two studies of students had large samples (288 and 522); two studies of teachers had over 20 participants (23 and 39). The theoretical papers did not specify a target population.

Table 1: Distribution of the 28 papers across the research journals

Journals	# Articles per Journal
Journal for Research in Mathematics Education, Journal of Urban Mathematics Education	6
Mathematical Thinking and Learning, Race Ethnicity and Education	4
Educational Studies in Mathematics, Journal of Mathematics Teacher Education	3
Review of Educational Research, The Journal of the Learning Sciences	1

In what follows, we present our findings on the definitions and operationalizations of equity and the rationales, motivations, and purposes for conducting equity research.

Definitions of Equity

Of the 28 articles included in this analysis, only 17 either defined or operationalized equity. The earliest two pieces, by Cobb and Hodge (2002) and Gutiérrez (2002), proposed two distinct conceptualizations of equity. Cobb and Hodge defined equity and diversity as correlated terms, with diversity referring to the different students' (in and out of school) practices that they participate in and with equity referring to how, once the common practices and less common practices are recognized, students can successfully access them, through mathematical reasoning that has "clout" (Bruner, 2009). Gutiérrez (2002) proposed that equity can be recognized when researchers can't predict inequities (in outcomes or in achievement) by individual traits. Her definition has three elements related to the practices enacted by teachers, students, and mathematics to (1) develop proficiency in dominant mathematics; (2) be critical of math and its role in society; and (3) improve the relationship between people, mathematics, and society to erase inequity (that requires the cultivation of individual agency and the recognition of power) (p. 174). These two definitions made references to mathematics tangentially, Cobb and Hodge in terms of classroom practices and discourse and Gutiérrez in terms of "dominant" mathematics.

Esmonde (2009a, b) published two articles in 2009, both of which addressed equity. In her conceptualization of equity, two key distinctions are made—first, that equity relates to *fairness* ("a qualitative sense of fairness", 2009b, p. 1010) and second, that equity and equality are not the same. Esmonde used a definition of equity that refers to a fair distribution of opportunity to learn and a fair distribution of opportunities to participate in ambitious mathematics. In the set of articles we reviewed, this is the first time we observed an explicit reference to the *kind* of mathematical content and instruction that teachers should be striving to offer, embracing students' contributions, beyond a general description of *dominant* mathematics.

The definitions present in the 12 articles published after 2009 have elements that strongly resonate with these four and in some cases advance the attention that researchers put on studying equity in classrooms. There are several constant features; for example, references to access to worthwhile, high level, or meaningful, rigorous, mathematics experiences and inquiry (e.g., Cross et al., 2012; Dunleavy, 2015; Felton-Koestler, 2019; Louie, 2018, Munter et al., 2019) or that all students should be offered such practices (Cho et al., 2022; Cross et al., 2012; Dunleavy, 2015; Hand 2012). But there are also differences, for example defining equity as a *process* (Gregson, 2013) with two components – first, giving students the opportunity to learn mathematics content, its processes and norms, and second, supporting those who do have not had access to those, in bridging their own cultural knowledge to get to the dominant one (e.g., Felton-Koestler, 2019; Lo & Ruef, 2020). The goal of this type of equity in mathematics is to reduce social inequity. This definition resonates with Gutiérrez's 2002 definition, but also with her 2012 article that framed equity as learning to "play the game" to being able to "change the game" (p. 11).

We observed other differences in how authors further specified the object of equity or how equity was to be realized; for example, in terms of how students relate to each other in classrooms and the recognizable actions that can lead to democratic learning communities (Sengupta-Irving, 2014), recognizing "competent sense-makers" (Dunlevy, 2015, p. 62), and acknowledging that racialized experiences can burden students when doing highly cognitive demand work (Munter, 2019); these scholars address agency, power, and identity, even when this is not explicitly stated. These definitions can be seen as influenced by Gutiérrez's (2012) chapter that defined equity along two axes, a dominant one that related access and achievement and a critical one that related to identity and power. The influence of this piece is seen throughout several of the articles, notably Mintos et al. (2019), who used this framing in their study of how equity was taught in secondary teacher education programs.

One final element of the definitions from the later articles refers to how researchers and teachers notice, become aware, or recognize equity or inequity, and indicate that such noticing is fundamental to social justice work. We note that while all 17 articles made some reference to fairness in access and opportunity to participate and learn, very few assumed a critical stance regarding the role of mathematics in maintaining the status quo or addressed specific individual identities. Almost all the articles emphasized that equity work should be for *all* (e.g., "broad range of learners from dominant and nondominant ethnic, racial, and linguistic backgrounds in rigorous mathematical inquiry", Hand, 2012, p. 237) which erases the need to attend to identity.

Purpose, Motivations, and Rationale for Conducting Equity Research

In our analyses, we defined *purpose* as the objective of the article, specifically the questions the authors wanted to answer or the argument they wanted to make; *motivation* as the reason why authors identified the purpose, or the broader context that made the purpose worth pursuing; and *rationale* as the authors' reasons for why equity is worth researching; the rationale signals macro-level aspects (e.g., society) whereas purpose and motivation are specific to the article. Empirical studies and theoretical articles had different purposes. The most common purposes in empirical studies related to analyzing teachers' practices and implications for equity (e.g., Battey, 2013; Gregson, 2013), teachers' learning about and understanding of equitable instructional practices (e.g., Felton Koestler, 2019; Mintos et al., 2019), understanding what specific curriculum or pedagogy works for whom (e.g., IBL for women, Ernest et al., 2019), and understanding the relationship between equity and how students work in groups (e.g., Esmonde, 2009a; Lo & Ruef, 2020). The most common purposes of theoretical pieces were to offer a new perspective on equity research, for example that equity research needs a critical perspective (Gutiérrez, 2002) or that equity is "a delusion" and antiblackness needs to be addressed (Martin, 2019). Motivations included (1) responding to or testing out a reform curriculum or pedagogy (e.g., Lubienski, 2000; Johnson et al., 2020), (2) addressing gaps in understanding of teacher practice and preparation (e.g., Gregson, 2013; Hunt et al., 2022), and (3) arguing that current perspectives on equity are incomplete (Gutiérrez, 2002; Louie, 2018). The latter was present in theoretical and empirical articles and includes motivations for countering a deficit-based lens (e.g., Berry et al., 2014; Boaler, 2002b). Rationales included the unequal access to learning mathematics and achievement gaps by social identity groups (race, gender, or SES). In this case, the rationale for researching equity was that students need or should be able to participate and succeed in mathematics, and that this is not currently happening. In contrast to unequal access, Martin (2019) and Gutiérrez (2002) centered the field of mathematics, not the students, as the rationale for why equity needs to be attended to. Gutiérrez (2002) noted that the field of mathematics could benefit from perspectives from marginalized groups and Martin (2019) pointed out that mathematics needs fundamental change to support the liberation of Black learners. Though all the articles argued for change in some respect, Gutiérrez and Martin are notable because they shifted attention to mathematics away from students' success and reminded researchers that students have agency in this process.

Discussion

Across the articles analyzed, we saw definitions and purposes that directly responded to, or challenged, reform practices and definitions of equity in the wake of NCTM's (2000) *Principles and Standards*. In post-secondary education, the two studies that investigated equity did so by identifying *inequities* (in performance, e.g., Johnson et al., 2020; in participation, Ernest & Reinholz, 2019), so equity is only possible to identify when it does not occur. We recognize the need for equity after inequity has been measured. Such approach is aligned with Gutierrez's 2002 proposal that we will be able to say that equity is achieved when we can't predict performance based on identity traits. Our findings suggest that such a strong focus on responding to calls for reform might prevent giving greater attention to taking a critical perspective on equity. The scholars' focus on the dominant axis of access and achievement (Gutierrez, 2012) may unintentionally disengage with the critical access that requires attention to both identity and power. Finally, we noted that the varied definitions are symptomatic of a field that has not grappled with unifying forces to collectively investigate equity so that it can further inform practice across contexts.

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References

The * indicates an article included in the review.

*Battey, D. (2013). "Good" mathematics teaching for students of color and those in poverty: The importance of relational interactions within instruction. *Educational Studies in Mathematics*, *82*, 125-144.
*Berry III, R. Q., Ellis, M., & Hughes, S. (2014). Examining a history of failed reforms and recent stories of success: Mathematics education and Black learners of mathematics in the United States. *Race Ethnicity and Education*, *17*(4), 540-568.
*Boaler, J. (2002a). Paying the price for "sugar and spice": Shifting the analytical lens in equity research. *Mathematical Thinking and Learning*, *4*(2-3), 127-144.
*Boaler, J. (2002b). Learning from teaching: Exploring the relationship between reform curriculum and equity. *Journal for Research in Mathematics Education*, *33*(4), 239-258.
Bruner, J. S. (2009). *Actual minds, possible worlds*. Harvard University Press.
*Cho, S. (2022). Mathematic lesson design for English learners versus Non-English learners from perspectives of equity and intersection. *Journal of Urban Mathematics Education*, *15*(1), 31-53.
*Cobb, P., & Hodge, L. L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning*, *4*(2-3), 249-284.
*Cross, D. I., Hudson, R. A., Adefope, O., Lee, M. Y., Rapacki, L., & Perez, A. (2012). Success made probable: Creating equitable mathematical experiences through project-based learning. *Journal of Urban Mathematics Education*, *5*(2), 55-86.
*Dunleavy, T. K. (2015). Delegating mathematical authority as a means to strive toward equity. *Journal of Urban Mathematics Education*, *8*(1), 62-82.
*Ernest, J. B., Reinholz, D. L., & Shah, N. (2019). Hidden competence: Women's mathematical participation in public and private classroom spaces. *Educational Studies in Mathematics*, *102*, 153-172.
*Esmonde, I. (2009a). Ideas and identities: Supporting equity in cooperative mathematics learning. *Review of Educational Research*, *79*(2), 1008-1043.
*Esmonde, I. (2009b). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *The Journal of the Learning Sciences*, *18*(2), 247-284.
*Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education*, *44*(1), 288-315.
*Felton-Koestler, M. D. (2019). "Children know more than I think they do": The evolution of one teacher's views about equitable mathematics teaching. *Journal of Mathematics Teacher Education*, *22*, 153-177.
Frankenstein, M. (1995). Equity in mathematics education: Class in the world outside the class. In W. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 165-190). Cambridge University Press.
*Gregson, S. A. (2013). Negotiating social justice teaching: One full-time teacher's practice viewed from the trenches. *Journal for Research in Mathematics Education*, *44*(1), 164-198.
*Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, *4*(2-3), 145-187.
Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education: theories, practices, and policies* (pp. 17-33). Springer Netherlands. https://doi.org/10.1007/978-94-007-2813-4_2

*Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. *Educational Studies in Mathematics*, *80*, 233-247.
*Hunt, J. H., Martin, K., Patterson, B., & Khounmeuang, A. (2022). Special educators' knowledge of student mathematical thinking. *Journal of Mathematics Teacher Education*, *25*, 581-598.
*Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504-516.
Lampert, M. (2001). *Teaching problems and the problems of teaching.* Yale University Press.
Lee, V. E., Croninger, R. G., & Smith, J. B. (1997). Course-taking, equity, and mathematics learning: Testing the constrained curriculum hypothesis in US secondary schools. *Educational Evaluation and Policy Analysis, 19*, 99-121.
Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *The Journal of Higher Education, 92*(5), 784-814. https://doi.org/https://doi.org/10.1080/00221546.2021.1879586
*Lo, M. T., & Ruef, J. L. (2020). Student or teacher? A look at how students facilitate public sensemaking during collaborative groupwork. *Journal of Urban Mathematics Education*, *13*(1), 15-33.
*Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*, 55-69.
*Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, *48*(5), 488-519.
*Lubienski, S. T. (2002). Research, reform, and equity in US mathematics education. *Mathematical Thinking and Learning*, *4*(2-3), 103-125.
*Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, *31*(4), 454-482.
*Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459-478.
*Mintos, A., Hoffman, A. J., Kersey, E., Newton, J., & Smith, D. (2019). Learning about issues of equity in secondary mathematics teacher education programs. *Journal of Mathematics Teacher Education*, *22*, 433-458.
*Munter, C., & Haines, C. (2019). Mathematics teachers' enactment of cognitively demanding tasks and students' perception of racial differences in opportunity. *Mathematical Thinking and Learning*, *21*(3), 155-177.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Author.
*Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education*, *10*(2), 66-105.
Secada, W. G. (1989). Agenda setting, enlightened self-interest, and equity in mathematics education. *Peabody Journal of Education, 66*(2), 22-56.
Secada, W. G. (1995). Social and critical dimensions for equity in mathematics. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 146-164). Cambridge University Press.

*Sengupta-Irving, T. (2014). Affinity through mathematical activity: Cultivating democratic learning communities. *Journal of Urban Mathematics Education*, *7*(2), 31-54.
Stanic, G. M. A. (1989). Social inequality, cultural discontinuity, and equity in school mathematics. *Peabody Journal of Education, 66*(2), 57-71.
Warren, B., & Roseberry, A. S. (1995). Equity in the future tense: Redefining relationships among teachers, students, and science in linguistic minority classrooms. In W. G. Secada, E. Fenemma, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 298-328). Cambridge University Press.

Predictive Effects of Students' Grades in Prior Mathematics Courses

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DFW rates in first-year mathematics courses are a major concern for undergraduate institutions. In this study, we explored statistical trends in grades for students who repeat mathematics courses or transition to subsequent courses. We analyzed trends among final grades in courses from Intermediate Algebra through Calculus II over the past five years at a major midwestern metropolitan university. Our results show that students are statistically more likely to pass a class on their first attempt, and that DFW rates among students who repeat a course hover around 50% for subsequent repetitions of the same course. We also found that final grades are a strong predictor of grades in subsequent courses, with students who earn an A in the previous course being 3 and 4 times more likely to pass the subsequent course than students who earn a B or C, respectively. We conclude by discussing the implications of our results.

Keywords: DFW rates, First-year mathematics courses, Course trajectories

DFW rates in first-year mathematics courses are a concern for universities, four-year colleges, and community colleges alike. Many factors contribute to student success in a course, including past mathematical experiences and prior mathematical knowledge. In this preliminary study, we explore trends in student grades as students transition from prerequisite mathematics courses into subsequent courses along a standard course trajectory. In particular, we focus on a sequence of first-year courses ranging from Intermediate Algebra to Calculus II and analyze the relationship between students' grades in prior courses and DFW rates in subsequent courses.

Background

Enrollment in first-year mathematics courses is increasing. The College Board of Mathematical Sciences reported that estimated enrollments increased by 21% for precollege level mathematics courses, by 16% for introductory level courses, and by 8% for calculus level courses between fall 2010 and fall 2015 at four-year institutions (Blair, Kirkman, & Maxwell, 2018, p. 1). Over the same time period, Bressoud et al. (2015) found that DFW rates in Calculus 1 courses ranged from 22% to 38% among two-year and four-year institutions in the USA. Notably, no systematic data collection has yet taken place to provide statistics on DFW rates in precollege level mathematics courses or introductory level mathematics courses at the college or university level in the United States. Yet, administrators at many institutions cite their college algebra courses as significant challenges to student retention (e.g., Callahan & Belcheir, 2017; Gordon, 2008).

Success in a student's first college mathematics course is especially important for retention and persistence. In one study, Callahan and Belcheir (2015) found that only 55% of students earning a D, F, or W in their first mathematics course were still enrolled in college one year later (p. 168). To ensure that students succeed in passing their first college mathematics course, appropriate placement is critical (Bressoud et al., 2015; Harrell & Lazari, 2020). In their report on characteristics of successful calculus programs, Bressoud et al. (2015) highlighted the importance of effective placement procedures on student success and, in particular, the attention paid to students near the cutoff. It was found to be essential that students who were identified as "ready for calculus" but more at-risk than their peers, as well as students who did not place into calculus, were given special attention and placed into programs that suited their individual needs.

Our work adds to the body of research on factors affecting success in first-year mathematics courses. Rather than focusing on a single course in isolation, we instead examine a sequence of courses that STEM-intending students typically take as they progress through lower-level undergraduate mathematics courses. While we acknowledge that student "success" in mathematics encompasses many variables beyond just a student's final grade in the course, our preliminary work focuses on two primary research questions:

1. How does a student's grade in a prior course affect their likelihood of earning a passing grade in a subsequent course?
2. For students who do not pass a course, what is their likelihood of passing the course in a repeated attempt?

Methods

For our study, we gathered institutional data for all undergraduate students enrolled in any mathematics course at a major midwestern metropolitan university from Fall 2018 to Spring 2023, along with the enrollment records of any previous mathematics courses students had taken for a period of at least two years prior. Individual students were referenced within our dataset by anonymized identification numbers (pseudo-ID) along with the term, course, section, and registrar grade for all enrollments.

The first step of our analysis was to take this list of student enrollments and use it to build a data structure which captured the variety of paths students may take when progressing through courses. To begin, we loaded the enrollment data into a relational database and cleaned entries to ensure that the three-tuple of pseudo-ID, course name, and term of enrollment was unique. This cleaning step made it possible to accurately account for students repeating courses by computing the instance number of each student's enrollment in any given course. That is, for a student with a given pseudo-id enrolling into course X in term Y, we counted the number of times that this same student enrolled in course X in a term chronologically before term Y, then added 1. This yielded a natural way to find students taking a given course for the first, second, or *n*th time by searching for instance numbers of 1, 2, or *n*, respectively within those course enrollments and forms the basis for the analysis necessary to answer RQ2.

We then used these data to create a course transition table to track the history of what mathematics courses a student had taken prior to any given enrollment by identifying pairs of prior and subsequent courses. This transition table forms the basis of the analysis of student grades in subsequent courses necessary to answer RQ1.

The DFW rates we reference in our Results section are obtained by dividing the number of grades of D, F, W, and NC (No Credit) out of the total number of grades consisting of A, B, C, D, F, W, NC (No Credit) and CR (Credit). This excludes grades such as Incomplete or Audit which have no clear interpretation. Our analysis in this paper focuses on a sequence of courses typically taken by STEM-intending students within their first year or two of entering college: Intermediate Algebra, College Algebra, Pre-Calculus Algebra, Trigonometry, Precalculus, Calculus I, and Calculus II.

Results

Effect of Grades in Prior Courses

To analyze pass rates in subsequent courses, we first searched within the transition table for pairs of prior/subsequent courses associated with the sequence of STEM-intending courses described above. The data was then filtered to look only for cases where the student had passed (C- or above) the previous course within 24 months and was taking the subsequent course for the first time. Prior courses from the Fall 2018 semester onward were considered. The resulting DFW rates in subsequent courses are reported in the table below grouped by the grade the student achieved in the prior course. We group together +/- grades (i.e., earning an A means that a student earned an A+, A, or A- in the prior course). As an example, of the 2144 students who took Intermediate Algebra followed by College Algebra, students who earned an A in Intermediate Algebra went on to earn a grade of DFW or NC in College Algebra at a rate of 13.29%. Row and column totals refer to specific course parings and grades, respectively. That is, 31.11% of the 2144 students who took Intermediate Algebra followed by College Algebra received a grade of DFW or NC in College Algebra, whereas 12.64% represents the aggregate DFW rate of enrollments with a grade of A in the prior course.

Table 1. DFW Rates in Subsequent Courses

		Grade in Prior Course			
Sequence	*N*	A	B	C	Total
Int. Alg. → College Alg.	2144	13.29%	43.72%	66.47%	31.11%
College Alg. → Precalc. Alg.	679	3.30%	18.57%	37.14%	13.25%
Precalc. Alg. → Trig.	416	14.40%	36.67%	66.67%	25.72%
Trig. → Calculus I	339	22.06%	50.74%	61.19%	41.30%
Precalc. → Calculus I	435	25.37%	58.22%	72.62%	45.52%
Calculus I → Calculus II	936	7.99%	32.36%	43.18%	25.96%
Totals	4949	12.64%	39.61%	56.29%	29.20%

Looking over longer timescales, we can also see how DFW rates in courses compare with student grades in a prerequisite course several courses prior in the usual sequence. Table 2 provides a few examples of these course sequences with intermediate courses omitted. It is evident that far fewer students progress through the entire sequence of these courses – especially in the groups with lower grades in prerequisite courses. As such, data entries for groups with fewer than 15 students are left blank.

Table 2. DFW Rates for Sequential Courses with Further Separation.

		Grade in Prior Course			
Sequence	*N*	A	B	C	Total
Int. Alg. → Precalc. Alg.	285	5.53%	30.65%	45.83%	14.39%
Precalc. Alg. → Calculus I	261	34.24%	54.84%	66.67%	41.00%
Trig. → Calculus II	118	27.42%	36.59%	40.00%	32.20%
Precalc. → Calculus II	159	22.34%	48.89%	55.00%	33.96%
College Alg. → Calculus II	83	29.85%	46.67%		32.53%
Totals	906	21.78%	43.11%	50.67%	29.47%

Analysis of DFW Rates by Course Repetition

Students taking courses from Fall 2018 through the Spring 2023 semester were grouped by their instance taking (or re-taking) given courses. The values reported in the Table 3 are presented in the form "DFW Rate (*N*)," where *N* is the total number of students taking the course for the *n*th time. As with the prior analysis, entries with fewer than 15 students are left blank to avoid reporting on exceptionally small populations.

Table 3. DFW Rates for Students Retaking Courses

Course	1st Instance	2nd Instance	3rd Instance	4th Instance
Intermediate Algebra	32.06% (3656)	52.40% (521)	46.67% (60)	
College Algebra	30.32% (5301)	53.54% (693)	50.00% (148)	23.33% (30)
Precalculus Algebra	16.81% (2046)	42.64% (197)	52.27% (44)	64.71% (17)
Trigonometry	27.58% (689)	50.00% (106)	68.57% (35)	66.67% (18)
Precalculus	19.58% (715)	41.67% (72)		
Calculus I	34.10% (2217)	46.51% (458)	53.77% (106)	62.86% (35)
Total	28.78% (14624)	50.02% (2047)	52.22% (406)	52.83% (106)

As is the case for the preceding analysis of moving from one course to the next, one limitation of the method employed here it that it is not possible to determine if a student took any other prerequisite courses between instances of retaking the course in question.

Conclusions and Future Work

Our findings have implications for instructors at all levels as well as for course conveners and departmental administration. Our results seem to confirm that grades in previous courses are a strong predictor of whether a student will pass a subsequent course. To an instructor of a subsequent course, our results suggest that particular attention should be paid to students who passed their previous course with a final letter grade of B or C, as these students are at roughly three- and four-times higher risk, respectively, of not passing their present course as compared with students who earned an A in their previous course on the basis of Table 1. To the instructor of a student's prior course, our results underscore the importance of supporting and encouraging students to gain a deep understanding of what they are learning – and doing so the first time to avoid failing and repeating a course. It also brings into focus the reality that students leaving a given course with a grade in the C range have less than a 50/50 chance of passing their next math class on the first attempt. Preliminary results from future work confirm the fact that student grades decrease on average when students move between any of the course pairs detailed in our results section and emphasize the need to focus especially on STEM-intending students who begin the sequence in courses below Calculus. Instructors should consider how to incorporate specific interventions to better support these students to help them persist in their STEM majors.

Our preliminary results also provide a method whereby particularly troublesome pairs of course transitions could be systematically identified and investigated further. In future work, we hope to strengthen this case for efficacy of grades in predicting outcomes with a more rigorous statistical analysis and expound further on some of the other aspects of student demographics which may factor into these results. It is further worth mentioning that some of the most "difficult" pairs of course transitions (viewed either in terms of DFW rate or an average decrease in grade) may not be responsible for the greatest numbers of DFW grades issued within a given department. Future work will aim to further quantify the numbers of students transitioning between various course pairs with the aim of suggesting where limited resources to help students could be most effectively distributed.

The results for students who end up having to repeat a class are stark where, in aggregate, students are 1.75 times less likely to pass a course when repeating for the second, third, or fourth time as compared to taking it for the first time. Some courses (e.g. Calculus I) exhibit a nearly linear increase in DFW rates with each subsequent retake ($R^2 = 0.9883$) whereby a student is nearly 10% less likely to pass the course on each subsequent retake.

This somewhat surprising statistic raises questions about why students who repeat a course may be less likely to pass it. On one hand, one might expect that retaking a course would confer certain advantages to students such as familiarity with the content of assessments and insight into what aspects of the material are most relevant, in addition to having recently seen the material. On the other hand, if the student was not successful on a first attempt, then it may be unlikely that they will succeed on a second attempt if the instruction is not significantly different. Furthermore, students who repeat a course may feel more confident than their peers due to having seen material before, and as a result, they may be less likely to regularly attend class or study as much as their classmates who are taking the course for the first time.

Moreover, viewing DFW rates grouped by the instance in which a student is taking the course can provide a much more detailed picture than simply considering the aggregate DFW rate alone. That is, it allows the assessment of how well a given course is performing for students who are seeing it for the first time as opposed to those that are repeating it.

Another area of future work is investigating how students are distributed across sections of multi-section courses. Coordinators of larger courses should be aware that the mathematical background of students may not be comparable between different sections, and that the DFW rates alone may not give a complete picture of the success or failure of an instructor's or team's teaching; one must consider factors such as total enrollment across all sections, class sizes within individual sections, the mathematical backgrounds of the students enrolled, and other factors. This is especially important at any institution where faculty may have DFW rates included as part of their evaluation process.

Some technical limitations of this analysis were expounded upon in the methodology and results section: most notably the inability to exclude intervening classes in many instances. While there is reason to believe that such events may be comparatively rare, we plan to undertake a more thorough study of this in future work and switch to a more robust method of recording student progress through courses that could account for these factors if warranted.

Questions for the Audience

- Are there researchers or practitioners at other institutions who are performing similar analyses? Have you identified similar groups of students who might need additional supports and what interventions have you studied or tried?
- What are reasonable metrics to "assess" a course by? Is it useful to partition DFW rates by metrics like "instance re-taking a course" to better reflect course outcomes? What are the most representative DFW rates to consider?
- What do our results say about prerequisite knowledge and coursework? How might we use our results to ensure students are on a successful trajectory of courses?
- To help refine the range of factors which might affect students' grades, we are interested in looking at other data such as ACT scores, high school courses, placement exams, etc. What other factors would you recommend investigating more thoroughly?

References

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2015 CBMS survey*. American Mathematical Society. https://doi.org/10.1090/cbmssurvey/2015

Bressoud, D. (2015). Insights from the MAA national study of college calculus. *The Mathematics Teacher, 109*(3), 178-185. https://doi.org/10.5951/mathteacher.109.3.0178

Callahan, J., & Belcheir, M. (2017). Testing our assumptions: The role of first course grade and course level in mathematics and english. *Journal of College Student Retention: Research, Theory & Practice, 19*(2), 161-175. https://doi.org/10.1177/1521025115611620

Gordon, S. P. (2008). What's wrong with college algebra? *PRIMUS, 18*(6), 516-541. https://doi.org/10.1080/10511970701598752

Harrell, G., & Lazari, A. (2020). Results of a placement system for the first college mathematics course. *Georgia Journal of Science, 78*(2), 3. Available at: https://digitalcommons.gaacademy.org/gjs/vol78/iss2/3

Blending Office Hours into Scheduled Class Time in Gateway Math

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A blended course model uses both classroom learning and online learning to create flexibility in the weekly schedule for students. In this paper, we describe a blended course model and the rationale for implementing it in Calculus I. We report on student responses to the model and the behaviors it encouraged. The design of the blended course model created the opportunity for office-hours-like interactions in scheduled class time to help reveal the hidden curriculum of office hours. Two goals the model strived to achieve for students were (1) normalizing help-seeking behaviors and (2) normalizing working with peers. Data collected through questionnaires and focus group interviews suggest that the blended course model achieved these goals and more.

Keywords: office hours, blended classroom

The term *hidden curriculum* was first introduced by Jackson (1968) to describe implied social norms of classrooms and schools. Today we commonly refer to the *hidden curriculum* of modern academic institutions as the norms, strategies, and expectations that students need to navigate their learning and networking on college campuses. Office hours are a component of the hidden curriculum; many teaching and learning centers reinforce the need for instructors and teaching assistants to explain office hours (*Using Office Hours Effectively,* n.d.; CTEI, n.d.; Office Hours/Helprooms, n.d.; CRTL, 2023) and invite and encourage students to attend. Jack (2020) highlights that professors at elite institutions expect students to attend office hours to make connections needed to request letters of recommendation, and to enhance their learning. Jack states that some students are ready to tackle these interactions upon arrival at college, while other students are less comfortable initiating these connections.

This paper describes a course model that created the opportunity for office-hours-like interactions during scheduled class time and the rationale for implementing it in Calculus I. The model uses both classroom learning and online learning to create flexibility mid-week for students. There is some debate about whether such a course structure should be considered a "partial-flip" (Lax, et al., 2017; Urquiza-Fuentes, 2020) or a "blended model" (Webster, et al., 2020); and we settled on Johnson's (2012) description of a "blended model" where students use "class-time to discuss, apply and clarify" the content and interact with online course components. The blended course model (BCM) described here was designed to provide accessibility to office hours and (1) normalize help-seeking behavior and (2) normalize working with peers. Data collected through questionnaires and focus group interviews support that the model achieved these goals and more.

Rationale for the Blended Course Model

The blended course model used in gateway mathematics builds on a typical course meeting structure of 3 class meetings per week in a large (n=150) lecture meeting with an instructor, plus a small (n=24) recitation meeting with a team of graduate student and undergraduate student teaching assistants (TAs). In the blended course model, interactive lectures (using think-pair-share, polling and exit tickets) are delivered in person on Mondays and Fridays. Recitation time (on Tuesday or Thursday) is used for an alternating schedule of group work, quizzes, and review. The Wednesday class meeting is repurposed for an office-hours-like large group meeting where instructors and TAs work together to facilitate the class. During this

50-minute class, students are prompted to prioritize what they need to work on or study, and the time is dedicated for students to do that work with opportunities to ask and answer questions. This prioritization is intended to help the class, as a community, identify difficulties that are arising in real-time and validate students' need for support. In this blended course model (BCM), the lecture content for the Wednesday classes is replaced by asynchronous video lectures created by the instructional staff and published on the learning management system. The schedule is depicted in Figure 1. Students watch the videos on their own time. The BCM pushes one hour of lecture into unscheduled time and replaces one hour of scheduled contact time with high-impact in-person office-hours-like interactions.

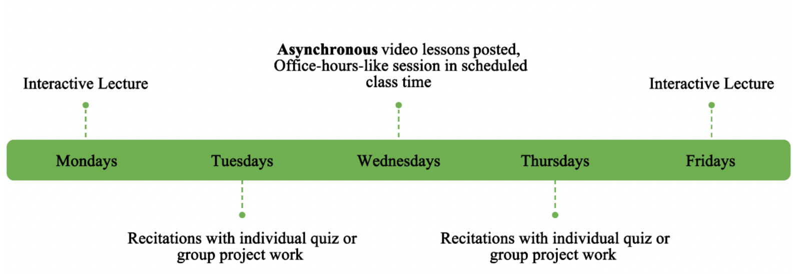

Figure 1. The weekly schedule in the blended course model.

The design goals of the BCM aimed to create access to office hours in scheduled class time to address two key goals:

(1) Normalize help-seeking behaviors
(2) Normalize working with peers and provide a scheduled meeting time for peer meetings

Data Collection

Methods

This study aimed to build a case study of a blended course model and used a convergent parallel mixed methods design in which a questionnaire and focus group interviews were used to collect data. In convergent parallel design, both qualitative and quantitative data were collected and analyzed to triangulate results (Creswell & Creswell, 2018).

This study was conducted at a private institution in the Mid-Atlantic region of the United States. The institution is a "very high research university," and we will refer to it as VHRU in this paper.

Data Sources. I. Questionnaire: The questionnaire collected enrollment information, demographic information (major, gender identity, racial/ethnic identities), and the option to self-identify as a first-generation student, limited-income student, person with a disability, international student, immigrant, etc. This paper will focus on student rankings and responses about the instructional components of the course that contributed most to their learning.

Respondents were asked to rank instructional components according to what worked best for them: blended course model, UPT (Undergraduate peer tutors) tutoring space, office hours in-person, office hours via Zoom, supplemental instruction sessions, recitations with TAs and UPTs, textbook, and in-class worksheets. Additionally, a five-point Likert scale portion asked the level of agreement on questions related to comfort interacting with instructors, TAs and peers, as well as feelings of support and workload.

II. Focus Groups. Dr. Yap interviewed 11 focus groups; 4 in person with 23 participants and 7 online via Zoom with 20 participants. The interviewer referenced trends in the instructional component ranking and responses from the questionnaire in interviews and encouraged participants to elaborate on their written responses.

Participants. In February 2023, first-year students from the 1,310-student incoming class in Fall 2022 who were enrolled in Calculus I were identified as the population of interest. From 161 randomly selected students from the top, middle, and bottom thirds of the final exam grade distributions, 43 students responded to the questionnaire; the response rate for recruitment was 26.71%. The 43 students participated in 11 focus groups. Participants came from a variety of majors, with the most represented as Computer Science (23.26%), Mechanical Engineering (16.28%), Pre-Major (16.28%), and Chemical and Biomolecular Engineering (13.95%). Just over half, 51.16% of participants identified as female, while 48.84% of participants identified as male.

Results and Discussion

Overall class experience. In general, participants had a positive experience in Calculus I. Think-pair-share, in-class activities and group projects contributed to participants studying with their classmates (76.75% of participants) and forming meaningful connections with Calculus I peers (88.37% of participants). One student reported that they "*ended up becoming good friends of [theirs].*" They continued, "*One of the TAs walked up to us and asked if Calculus I turned us all into friends, and we said, 'Yeah.' We had fun.*" 86.04% of participants felt encouraged by their Calculus I professor, and 72.09% felt comfortable participating in class discussions.
Ranking Instructional Components. Students identified the two most helpful instructional components to be the in-class worksheets and the blended course model (BCM). Students described the worksheets as "very structured", "easy to follow", and "organized." The in-class worksheets average ranking was 2.72 out of 10 with a standard deviation of 1.76. The BCM's average ranking was 2.84 out of ten with a standard deviation of 1.90.

The BCM created access to Office Hours (OHs) with instructors and TAs in scheduled class time. This addressed student feedback that OHs were not always scheduled conveniently and students had inequitable access to the instructional staff. Comments, like the two below, state explicitly that this was helpful for students and addressed issues with time-conflicts:

> I really like the Wednesday sessions where you could come in and then ask questions and I liked that it was the time of a class time because sometimes I wouldn't be able to go to office hours because of schedule conflicts. So I really liked starting the homework early and then coming up with questions that I would ask on Wednesdays, and I felt like that was really helpful.

> *Well, because you have more than one class. All of those teachers have different office hours. All of those TAs have different office hours in different buildings. It sometimes you're just too tired to go… you can't go. Also, clubs, organizations, the fact that you have to eat. It's kind of just the volley of being a human being who has to take care of his or her body, which gets annoying.*

In the sections below we will discuss evidence that the BCM addressed the intended design goals and provided flexibility for students. We will also comment on how students used the asynchronous videos and opportunities for improvement.

Achieving the Blended Course Model Design Goals. The first goal of the OH-like session was to normalize help-seeking behaviors. We wanted to make visible needing help from instructional staff on homework, project work and quiz/exam preparation. This strategically addressed previous student comments that some students are accustomed to working on their own and not confident asking for help. These sessions helped reveal some of the *hidden curriculum* of OH:

> Having Wednesdays as an in-person office hours when I couldn't usually attend [instructor]'s office hours was really nice for me because I would just work on homework and then have her... At my disposal is kind of a bad way to say it, but I would have her there to help me if I needed help, and that was nice.

> So a few of my friends and I would just take the homework problem sets to the in-class office hours and just kind of work through them as we were sitting there and if we ever just ran into a problems doing the homework, we can just ask conceptual questions or even just ask the professor or the TAs to go over the sample problems that we went over in class again. And that really helped me shorten the amount of time I needed to put into studying.

The OH-like session normalized working together to do homework or study in the class, which was the second goal the design hoped to address. It guaranteed scheduled time that students could collaborate on homework or project work. This strategically addressed previous complaints that scheduling group meetings outside of class time was a challenge for students:

> … you could work on projects or do homework or something. So I ended up not having to meet outside of class with my group other than once or twice, because we got most of it done on Wednesdays.

> And so on Wednesdays, I got together with my teammates and we worked on the project sometimes… So it prevented me from procrastinating, I'll say, the work session on Wednesday was most helpful.

Providing Flexibility and Support. Students reported that the BCM provided flexibility that they appreciated. Students were comfortable navigating the digital learning tools embedded in the model and liked the mixed mode of content delivery. Students reported that the asynchronous videos in the BCM provided an opportunity to pace their learning:

> I think the [BCM] was super helpful for me, especially first semester of college, learning how to study, learning how to keep up in all your classes, just having a little bit of flexibility in the middle of the week was super helpful.

> ...it also allows me to go home, study on my own time, especially with those videos, which I can pause, play, replay, pause, take notes, do whatever, and then study on my own and be able to go to a class and ask about the things I have trouble with and have more time dedicated to that.

> I like the fact that I could [watch videos] on my own time and because the concepts on the Wednesday videos were obscure, I liked having the ability to rewind and look over them again.

Many students used the videos as a reference and review while studying for quizzes and exams:

> The Wednesday videos… I would stay and watch the videos and they would be more or less of add-ons to the lectures. So for some of the quizzes, I would go back and watch some of the videos to review.

> "Oh, there's these videos I can watch." …if I was struggling on a topic, I would go to that video… So I used them as a textbook in a way or a resource.

A few participants felt that the videos were not well enough aligned with what happened in the Monday and Friday lessons, but still reported watching and using the videos as part of the BCM:

> When I started watching the hybrid videos, I was like, "These are off-topic." It just didn't feel like it was super relatable to the other content. There's a gap between what we were learning in class and what the videos were showing.

This is consistent with feedback from students about flipped-classrooms (Tague & Czocher, 2016) in other contexts. Messaging the planned cohesion of the asynchronous videos and the interactive lectures on Mondays and Fridays is an area for reconsideration and improvement in future implementations of the model.

Conclusions and Questions

In this blended course model, students are provided an office-hours-like learning environment during scheduled class time. Students can work independently and collaboratively and at the same time access help from instructors, TAs, UPTs, and classmates. Participants ranked the blended model, with the in-class-office hour, as one of the most valuable course components that contributed to their learning. This study has implications for math departments acknowledging that the evolving student population expects easy access to instructional staff and flexibility in their courses. This model helps reveal the hidden curriculum of office hours and makes accessible the valuable resource of attending office hours weekly.

References

Bluic, M., Goodyear, P., & Ellis, R. A. (2007). Research focus and methodological choices in studies into students' experiences of blended learning in high education. *Internet and Higher Education*, 10, 231–244. https://www.sciencedirect.com/science/article/pii/S1096751607000516

Bowers, J., & Zazkis, D. (2012). Do students flip over the flipped classroom model for learning college calculus? *Proceedings of the 34th annual meeting of the North American chapter of the international group for the psychology of mathematics education.* Kalamazoo, MI: Western Michigan University.

Center for Teaching Excellence and Innovation (CTEI), J. H. U. (n.d.). *Equitable and Inclusive Teaching Practices*. Ctei.jhu.edu. Retrieved September 12, 2023, from https://ctei.jhu.edu/teaching/equitable-and-inclusive-teaching-practices

Creswell, J.W., & Creswell, J.D. (2018). Mixed methods procedures. In, *Research design: Qualitative, quantitative, and mixed methods approaches* (5th ed., pp. 213-246). Los Angeles, CA: SAGE Publications, Inc.

CRTL. (2023, February 24). *Reframing Rigor to Promote Equity in Teaching and Learning | CRLT*. Crlt.umich.edu. https://crlt.umich.edu/blog/reframing-rigor-promote-equity-teaching-and-learning

Jackson, P. (1968). Life in Classrooms. Rinehart and Winston: New York.

Johnson, D. (2012). Power up!: Taking charge of online learning. Educational Leadership, November, 84-85.

Tague, J., & Czocher, J. (2016) A Theoretical Approach to Ensuring Instructional and Curricular Coherence in the Flipped Classroom Model of a Differential Equations Course. *Int. J. Res. Undergrad. Math.* 2, 23–245. https://doi.org/10.1007/s40753-016-0028-z

Lage, M. J., Platt, G. J., & Treglia, M. (2000). Inverting the classroom: a gateway to creating an inclusive learning environment. Journal of Economic Education, 31, 30–43.

Lax, N., Morris, J., & Kolber, B.J. (2017) A partial flip classroom exercise in a large introductory general biology course increases performance at multiple levels. *Journal of Biological Education*, 51:4, 412-426, DOI: 10.1080/00219266.2016.1257503

Love, B., Hodge, A., Grandgenett, N., & Swift, A. W. (2014). Student learning and perceptions in a flipped linear algebra course. International Journal of Mathematical Education in Science and Technology, 45(3), 317–324.

Office Hours / Helprooms. (n.d.). Bokcenter.harvard.edu; The Derek Bok Center for Teaching and Learning. Retrieved September 13, 2023, from https://bokcenter.harvard.edu/remote-office-hours

Strayer, J. F. (2007). The effects of the classroom flip on the learning environment: A comparison of learning activity in a traditional classroom and a flip classroom that used an intelligent tutoring system. (Doctoral dissertation).

Urquiza-Fuentes, J. (2020) Increasing Students' Responsibility and Learning Outcomes Using Partial Flipped Classroom in a Language Processors Course, *IEEE Access. 8,* 211211-211223.

Using Office Hours Effectively. (n.d.). Learning Center. https://learningcenter.unc.edu/tips-and-tools/using-office-hours-effectively/

Webster, D.R., Kadel, R.S., Newstetter, W.C. (2020). What do we gain by a Blended Classroom? A comparative study of student performance and perceptions in a Fluid Mechanics course. *International Journal of Engineering Education.* 36(1), 2–17.

Using Theory to Scale Innovation in Higher Education: Supporting the Implementation of Multiple Mathematics Pathways and Corequisite Support in Arkansas

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The Charles A. Dana Center worked across all public colleges and universities in Arkansas through a project called Strong Start to Finish (SStF) Arkansas to support the implementation of corequisite support for underprepared students in mathematics and multiple mathematics pathways for all. Consistent with Reinholz, White, and Andrews' (2021) call to more effectively ground change efforts in change theories, the Dana Center designed its innovation using a change theory-based theory of change. This preliminary report describes this project's outcomes and how the projects' results further the field's understanding of Rogers' Diffusion of Innovations theory, one of the change theories on which the project's theory of change was based.

Keywords: Change Theory, Diffusion of Innovation, Corequisite Support, Multiple Mathematics Pathways, Higher Education

Introduction

STEM higher educational change is complex, multi-faceted, and a necessary constant to improve access, teaching and learning, and outcomes for diverse educational stakeholders. Although long-term research has documented innovation and outcomes for STEM projects emphasizing STEM higher educational change, fully scaled sustained change in these projects is less known. Furthermore, change efforts have rarely used change theory to directly inform their work, making it difficult to build knowledge across the field (Reinholz et al., 2021).

Reinholz, White, and Andrews (2021) have called for change efforts to more effectively rely on theory by using both *change theories* and *theories of change*. Reinholz and Andrews (2020) defined a change theory as a broad framework of ideas that employs theoretical and empirical knowledge to explain some aspect of change. They defined a theory of change as project-specific and connected to evaluation; it explicitly promotes underlying assumptions about how and what change will occur in a defined project and then uses desired outcomes to guide project planning, implementation, and evaluation. A single change theory can support multiple components of a theory of change, and a single theory of change can draw from many change theories. The call by Reinholz, et. al. urged change efforts to create theories of change that are directly informed by change theories so that the results of projects can be used to build fieldwide knowledge.

The Charles A. Dana Center at The University of Texas at Austin's Mathematics Pathways Project (DCMP) is a national STEM higher educational change initiative that has employed a theory of change to create broad, deep, and sustainable change in undergraduate mathematics pathways. The Dana Center based the projects' theory of change on four change theories and used the theory of change to design and implement this national change effort. In this preliminary research report, we will describe one element of this theory of change (the part based

Rogers' (1983) Diffusion of Innovation Theory) and how the interventions based on this theory of change impacted state-, region-, and institution-level reform across Arkansas public colleges and universities. We will show how the results of this intervention can be used to inform thinking about the change theory on which it is partially based. We will investigate the research question: *What can be learned about Rogers' Diffusion of Innovation Theory for STEM higher educational change by looking at the results of the DCMP project in Arkansas*. Future publication by the authors will address all four change theories used to create the DCMP theory of change.

Theoretical Framework

Change Theory

The DCMP project drew on four change theories to build their theory of change (Ortiz & Cook, 2019). In this paper, we will focus on the aspects of the theory that utilized Rogers' Diffusion of Innovation theory (DoI). In this section, we will discuss the aspects of Rogers' (1983) theory that are relevant to the theory of change.

Rogers described classifications of potential adopters of a new idea by how quickly they adopt innovation — a characteristic Rogers referred to as *innovativeness*. The quickest and slowest groups (*innovators* and *laggards*) are the smallest—estimated at 2.5% and 16% of the population respectively. The middle three groups (*early adopters, early majority,* and *late majority*) comprise the bulk of the population—estimated at 13.5%, 34%, and 34% of the population respectively. He characterized innovators as able to tolerate a high degree of uncertainty about an innovation at the time they adopt it. In contrast, even early adopters require higher amounts of legitimization to adopt a new idea.

Rogers described two different structures of diffusion systems: systems in which innovations originate from experts or other authoritative sources and then spread, or do not spread, to potential users without alteration (*centralized*), and systems in which innovations are created by users and spread to other users who may adopt the innovation as is or may alter it to suit their circumstances (*decentralized*). He pointed out that decentralized systems are likely to create innovations that better fit with the needs of the users and that users are more likely to take ownership of the innovation – creating deeper and more sustainable change. However, he argued that a centralized aspect of a system can be important if users are not aware of a need for innovation or if there are larger centralized structures that need to be modified to enable the innovation to operate effectively in the larger system.

Lastly, Rogers described how the nature of relationships between individuals in a social system impacts how innovations spread. When most communication in a system is between individuals that are similar in attributes such as beliefs and social status, etc. (homophilic communication), diffusion is hampered. This is because earlier adopters tend to have different beliefs, communication styles, economic status, and social status than later adopters. If there is little communication between people who are different (heterophilic communication), there will be no route for innovation to spread beyond the early adopters.

DCMP Theory of Change

Following Reinholz (2020), we organize our description of the DCMP theory of change by first discussing the context of the project, then the assumptions that connect the DCMP theory of change to DoI, then the intervention, and then indicators that were used to judge the success of the intervention and the validity of the change theory.

Context. Following a national call to action to modernize undergraduate mathematics, the state of Arkansas joined the Dana Center in 2015 in a multi-state initiative to lead a mathematics faculty task force to develop and publish recommendations that addressed barriers to equitable access and success in undergraduate mathematics. In 2019, the Dana Center expanded this work in Arkansas through the Strong Start to Finish Arkansas (SStF) initiative, which strategically coordinated three state-level agencies and two national advocacy organizations to collectively lead the work. The multi-prong leadership approach intentionally pulled together change agents within the state of Arkansas to help set the pace, process, and conditions for implementing and scaling mathematics pathways[1] and corequisite supports[2], creating a 'coalition of the willing' across state-, region-, and campus-level communities and actors.

Assumptions. Three assumptions of the DCMP theory of change were drawn from Rogers' DoI. The first is that *high-profile legitimization of change is an important factor in the decision-making of early and late majority adopter types.* This assumption is based on the relative conservatism that Rogers associates with all adopters aside from innovators. If an innovation is to spread past the 2.5% of the population that are innovators, it will be helpful if said innovations have support from influential regional or national stakeholders. The second assumption of the DCMP theory of change drawn from Rogers' DoI is that *deep, broad, and sustainable scaling requires a combination of top-down and bottom-up approaches.* This is based on Rogers' description of the advantages and disadvantages of centralized and decentralized diffusion systems. The third assumption is that *broad scaling requires initial change efforts to be distributed across the higher education ecosystem.* This is based on Rogers' contention that change does not diffuse well between different types of institutions (Rogers, 1983); an initial focus on any particular type of institution will limit the prospects for widespread dissemination.

Intervention. To achieve the SStF project outcomes, guided by the DCMP theory of change and its underlying change theories, The Dana Center worked across state, region, and institution levels. At the state level, interventions focused on two separate policy areas: (1) support state-level transfer and applicability work to expand implementation of Quantitative Literacy (QL) recommendations, and (2) establish a new task force on secondary-postsecondary mathematics alignment. At the region level, Dana Center interventions leveraged a previously established system of regional coordinators who strengthened their college and university partnerships. Each regional coordinator supported both breadth and depth across their assigned institutions through coordinated project activities that focused on transfer and applicability, secondary-postsecondary alignment, mathematics pathways and corequisite support implementation, local leadership capacity, and data-driven decision-making. Last, institution-level interventions focused on building depth of knowledge and leadership capacity within institutions. The Dana Center supported both individuals and institutions through multi-institution professional learning opportunities for mathematics faculty, academic advisors, and mid-level administrators. This professional development increased capacity in key roles, so practitioners were better positioned to lead and serve as models for others.

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[1] Mathematics pathways are a mathematics course or sequence of courses that students take to meet the requirements of their programs of study. This concept applies to college-ready and underprepared students.

[2] Corequisite support refers to placing students who have been designated as underprepared directly into college-level courses and providing necessary additional support to help them effectively engage with the college-level coursework.

Indicators. Broadly speaking, the SStF-identified indicator of success was that all Arkansas public institutions of higher education would implement at least two mathematics pathways and 75% of underprepared students would be enrolled in corequisite support. For this analysis, we adopted three additional indicators to examine the extent to which the assumptions of the DCMP theory of change held. These were: (1) whether high-profile legitimization occurred and, if so, whether it was an important factor in the decision-making of potential adopters, (2) whether both top-down and bottom-up approaches were used and, if so, whether the combination impacted the depth, breadth, and sustainability of change, and (3) whether change efforts were distributed across a variety of institutions and if so did this enable broader scaling or is there evidence that heterophilic communication occurred, reducing the need for distributed initial change efforts.

Methods

Quantitative data was collected annually in 2020, 2021, and 2022. All public colleges and universities in Arkansas submitted a SStF Arkansas Data Collection Workbook which gathered data about degree-seeking students relevant to academic placement for mathematics, student enrollment and success in remedial mathematics in students' first year of college, and student enrollment and success in introductory college-level mathematics in students' first year of college.

Qualitative data was collected through ten 45- to 60-minute conversations each with between one and eight faculty and/or administrators. Fourteen institutions were represented, seven two-year and seven four-year institutions. Participating institutions were chosen in consultation with the regional coordinators to represent a range of experiences implementing mathematics pathways and corequisite support. Participants were asked about the extent to which multiple mathematics pathways and corequisite support had been implemented in their institutions, and about supportive and inhibiting factors in the implementation. Data addressing the second question was analyzed using thematic analysis (Braun & Clarke, 2006).

Findings

In order to assess the validity of the theory of change we must first look at whether change happened. The data suggests that mathematics pathways and corequisite support were implemented widely but, in many cases, not deeply. Every one of the fourteen institutions that participated in qualitative data collection offered corequisite support for college-level mathematics classes and had at least one pathway in addition to College Algebra, most commonly a quantitative literacy (QL) pathway. Statewide progress is being made in increasing enrollment in corequisite support and shifting student enrollment to multiple pathways. The statewide percentage of underprepared students enrolled in mathematics corequisite support increased from 55% in 2020 to 72% in 2022. The percentage of students in college algebra decreased from 69% in 2021 to 66% in 2022. It was lower in 2020 (59%) due to many community college students taking non-transferable classes – an issue that began to be addressed in 2021. Despite the prevalence of QL pathways and corequisite support, about half of the institutions represented in the conversations reported that the QL pathway was underutilized, and most institutions offered only corequisite support for QL classes but still required prerequisite developmental classes for some students taking College Algebra. Most of the institutions that participated in conversations reported that about 30% of students who required academic remediation for College Algebra took prerequisite classes.

Participants in the conversations reported that high profile legitimization did occur and was an important factor in stakeholder decision making. This legitimization took a variety of forms including (1) state education department expectations, (2) Dana Center advocacy, (3) example setting by peer institutions, and (4) the legitimacy of four-year institutions' adoption of mathematics pathways and corequisite support for two-year institutions.

Participants also reported advantages of the top-down, bottom-up intervention approach by the Dana Center. Advantages of the top-down approach were (1) the power of the state to push for change, (2) supportive accountability from the Dana Center in the form of data provision and check-ins from regional coordinators, (3) the ability of the Dana Center to spread information about the innovation including basic concepts, data that supported its feasibility, and technical assistance to support implementation, (4) logistical support from the Dana Center such as small amounts of funding and protected meeting time, and (5) the importance of four-year adoption in assuaging transfer concerns by two-year institutions. Advantages of the bottom-up aspects of the approach were (1) much of the persuading at the local level came from colleagues citing local data, and (2) local advocates could work out local issues such as strengthening communication between faculty and advisors, initiating curriculum review to establish need, creating course teams to support implementation, and navigating institutional politics.

Finally, participants described the advantages of distributing change efforts across institutions with some qualifications. Working with a variety of institutions ensured that the top-down aspects of the effort reached everyone. Furthermore, multi-institutional convenings enabled heterophilic communication. However, there is evidence that some heterophilic communication would have occurred in the absence of these convenings. Transfer issues require two-year institutions to pay close attention to what is happening in four-year institutions. It is unclear, however, whether an innovation that starts with two-year institutions would migrate to four-year schools.

Discussion

Thus, although SStF successfully supported change in Arkansas, its full-scale goals have not yet been completely realized. Multiple mathematics pathways and corequisite support have spread broadly in the state but could still be deeper within the institutions. The parts of its theory of change based on Rogers' DoI theory (1983) were largely shown to be valid with some exception. High-level legitimization was, indeed, important as was the combination of the top-down, bottom-up intervention design. It is less clear that initiating change across the institutional landscape was the only way to spread the innovations broadly. This suggests that Rogers' characterization of most potential adopters as conservative and needing external legitimization has merit as does his description of the affordances and constraints of centralized and decentralized diffusion systems. It is less clear that his concern about the diffusion-preventing character of homophilic communication systems applies to higher education in Arkansas. More research will be needed to illuminate this last question. What is clear is that Reinholz, White, and Andrews' (2021) call has merit. Creating a theory of change based on change theory(ies) and then building an intervention using the theory of change can enable a change effort to efficiently inform theory and provide usable information to the field about how change happens.

Questions for the Audience

1. What do you believe it means to have a change theory directly or rigorously inform a theory of change?
2. Please give us feedback on our effort to inform change theory.
3. What do you think of the call by Reinholz et al. for STEM higher education researchers and do you structure your change efforts in this way?

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Ortiz, H., & Cook, H. (2019). Dana Center Mathematics Pathways (DCMP) Theory of Scale: Exploring State-Level Implementation Successes and Challenges. In R. Hartzler & R. Blair (Eds.), *Emerging Issues in Mathematics Pathways* (pp. 65–76). the Charles A. Dana Center at The University of Texas at Austin.

Reinholz, D. L., & Andrews, T. C. (2020). Change theory and theory of change: What's the difference anyway? *International Journal of STEM Education*, *7*(1), 2. https://doi.org/10.1186/s40594-020-0202-3

Reinholz, D. L., White, I., & Andrews, T. (2021). Change theory in STEM higher education: A systematic review. *International Journal of STEM Education*, *8*(1), 37. https://doi.org/10.1186/s40594-021-00291-2

Rogers, E. M. (1983). *Diffusion of innovations* (3rd ed). Free Press ; Collier Macmillan.

Confidence and Sense of Belonging: Undergraduate Mathematics Student Experiences with Calculus Support Course

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Studying student experiences in undergraduate mathematics education serves as a fundamental aspect of understanding how mathematics departments can best serve the most at-risk populations of students who are often harmed by racial and gender inequities present in student success rates in undergraduate mathematics courses. In this study, we focus on support courses for Precalculus and Calculus I, where students engage in frequent Supplemental Instruction (SI) sessions and an online course run by the university Math Equity Coordinator. In analysis of a mixed-methods approach consisting of interviews and a survey, we examine the impact of these supports on students' sense of belonging and confidence in mathematics. Preliminary results from survey analysis show that students in the support course report higher levels of belonging and confidence in mathematics. Furthermore, interviews highlight the importance of instructor mentorship in enhancing students' confidence and sense of belonging.

Keywords: identity, belonging, calculus, student success

Introductory undergraduate mathematics, particularly the study of Calculus, often functions as a filtering and access-controlling mechanism that either impedes students' progress in STEM degrees or discourages them from continuing altogether (PCAST, 2012; Weston et al., 2019). A comprehensive examination conducted by Koch and Drake (2018) encompassing 36 higher education institutions throughout the U.S. revealed that a substantial 34.3% of students received a D, F, Withdraw, or Incomplete (DFWI) in Calculus. Even more unsettling were the statistics for Black or African American students and Hispanic or Latinx students, who faced even higher DFWI rates of 47.8% and 47.9% respectively. Furthermore, women are known to leave STEM fields 1.5 times more often than their male counterparts (Ellis et al., 2016). If the rightfully due pressure put on mathematics departments to close these racial and gender equity gaps is to be taken seriously, mathematics educators must dedicate time, money, and labor towards listening to student voices and understanding how we can support them through these historically difficult and harmful experiences.

The large, Southwestern University (SWU) that is the setting for our study has not been immune to the sustained racial and gender equity gaps in Calculus success observed across the U.S. In response, we are implementing a 2-year holistic approach to closing equity gaps present in our precalculus through calculus courses. One of the arms of this multi-faceted approach was to create support courses for Precalculus and Calculus I. The unique features of these courses are the nature of instruction and student engagement in that students opt-in to a 1-unit course that requires frequent attendance to university-provided Supplemental Instruction (SI) sessions and regular engagement with an online Canvas course hub run by a coordinator for the Math Equity Initiative. We see this definition of support course as fitting somewhere between a corequisite course that takes place in a classroom with a dedicated instructor and the regularly offered SI sessions.

In this preliminary report, we focus on the voices and experiences of students who opted-in to the new SI sessions in Precalculus and Calculus I through a mixed-methods study that utilizes semi-structured interviews and a comprehensive survey sent to all students enrolled in the precalculus and calculus courses in the spring semester of 2023. These experiences include their perception of the inclusivity of teaching practices, their praise of and suggestions for improvement to the SI sessions, and the impact on their confidence, enjoyment, interest, and sense of belonging in mathematics. Focusing on the experiences of the students who enrolled in the support course, this study is guided by the following research question: How has enrollment in the support course impacted students' sense of belonging in the math community and their confidence in mathematics courses?

Relevant Literature and Theoretical Perspective

Investigating the experiences of students in our courses requires an acknowledgment of the unique identities and voices they offer to our research. Understanding that student's identities are potentially redefined and reshaped every day, we view identities as dynamically designed by a variety of factors and constantly in flux (Collinson, 2006). More specifically focused on their identities within a mathematics context, Martin (2012) defines mathematical identity as the ways that students enact their own identities as "doers" and "learners" of mathematics. We extend mathematical identity to include students' self-reported interest, enjoyment, and confidence in mathematics. Since a sense of belonging is so closely tied with the inclusivity a student perceives, students' multiple identities, including their mathematical identity, are intertwined with their sense of belonging in mathematics. We believe that with a focus on identity and sense of belonging in mathematics we may begin to understand one of the myriad reasons why in the U.S. roughly half of the students who pursue a degree in science, technology, engineering, or mathematics (STEM) never complete it (National Center for Education Statistics, 2018).

Supplemental Instruction originated at the University of Missouri-Kansas City in 1973 and has since evolved into a widely adopted pedagogical approach practiced at over 1,500 education institutions across 29 countries. As explained by Arendale (1994), SI is designed to address historically challenging academic courses by providing free, voluntary sessions to all students. These sessions held several times throughout the week are led by SI Leaders, students who have previously excelled in the targeted course. The hallmark of SI is its collaborative nature, where students actively engage with course content and hone essential study skills. SI Leaders draw upon insights gained from attending all class lectures to facilitate active learning techniques, which include interactive board diagramming and group work.

While active pedagogical strategies have been found to be effective for learning and narrowing opportunity gaps (Freeman et al., 2014; Theobald, 2020), research has also demonstrated that the situations that transpire in active learning classrooms put students at most risk for marginalizing and anxiety-producing experiences (Aguillon et al., 2020; Cooper et al., 2018; Cooper & Brownell, 2018; Shah et al., 2020). Since our students experience a mostly traditional large lecture atmosphere when they meet with their instructors, we acknowledge that the effect of an active learning classroom only applies when students are in their twice-weekly discussion sections run by graduate teaching assistants and the SI sessions. It is important to note that the participants of our study experienced a variety of classroom atmospheres that all impact their overall experience in mathematics and more specifically their own sense of self as they interact, problem solve, and learn with groups of their peers.

Methods

SWU is a large, public 4-year HSI research university. The focus of this study are 26 students (gender and race/ethnicity shown in Table 1) who opted-in to enroll in the newly developed

Precalculus and Calculus I SI-based support course that also responded to a survey sent at the end of spring 2023 semester. Acknowledging our own identities of two cisgender, white, male researchers (one heterosexual and one gay) at this institution is important to note, because we aim to push back against and rectify the long history of the dominant race and gender performing studies that have often taken from underrepresented communities and not given back.

Table 1. Students self-reported identity. This table shows a count of each gender present by race/ethnicity.

Support Course Student Demographics						
Gender	Alaskan Native or Native American	Black or African American	Hispanic/ Latinx	Middle Eastern or North African	East Asian	White
Man	1	1	7	1	1	3
Woman	0	1	4	0	0	2

Three of these 26 students participated in a one-hour long, semi-structured individual interview that focused on a variety of topics related to their experiences in their current mathematics courses as well as their own histories of experiences with mathematics in general. Topics that were discussed included how their identity has impacted their learning of mathematics, how their confidence in their mathematics ability has changed, and the extent to which they have utilized university-offered support structures such as SI.

In order to develop a broader sense of experiences across all precalculus and calculus courses, the survey we administered was an amalgamation of items built from subsets of two existing surveys. One of these surveys is the student postsecondary instructional practice survey (SPIPS-M), which specifically targets students' interpretations of instructional practices, changes in attitudes towards learning and doing mathematics, and their perceptions of the climate in the classroom (Apkarian et al., 2019). Supplemental to the focus of this survey, the second survey developed by Brown and colleagues at Penn State University called the Inclusive Instructor Behaviors (IIB) survey was included to leverage student experience to determine various aspects of the perceived inclusivity of teaching behaviors and sense of belonging in mathematics.

Results

Comparing the 26 students who enrolled in the support course and the 700 students enrolled in Precalculus and Calculus I combined, the most notable statistical differences emerged among the questions that focus on a sense of belonging. In order to compare two independent groups with such variance in population size, a non-parametric statistical test called the Mann-Whitney U test was performed to determine effect size of agreement to a variety of statements. The resulting U statistic was then transformed into Cohen's d, which is a real number between 0 and 1 that has been stratified and defined by Cohen (1985) into categories of strength. An effect size less than 0.2 is "little-to-no effect", between 0.2 and 0.5 is a "small effect", between 0.5 and 0.7 is a "medium effect", and greater than 0.7 is a "large effect".

Students in the support course agreed with the statement, "I feel like I fit in" more than students who did not take the support course with an effect size of $d=0.301$ ($p<0.001$). More specifically attending to their sense of belonging in mathematics, students in these students also agreed more with the statements, "I feel like I am part of the math community" and, "I consider myself a member of the math world" more than students who did not take the support course with effect sizes of $d=0.246$ ($p<0.001$) and $d=0.242$ ($p<0.001$) respectively. Furthermore, these students also agreed with the statements, "I feel a connection with the math community" and, "I feel that I belong to the math community" more than students who did not take the support course with effect sizes of $d=0.237$ ($p<0.001$) and $d=0.221$ ($p<0.001$) respectively. Finally, students in the support course agreed with the statement "I feel valued" more than students who did not take the support course with an effect size of $d=0.219$ ($p=0.002$).

Regarding students' confidence in their mathematical ability, students who enrolled in the support course overall were more confident entering the course. When prompted to consider their confidence at the beginning of the semester, students who enrolled in the support course agreed with the statement, "I am confident in my mathematics ability" more than students who did not enroll in the support course with an effect size of $d=0.26$ ($p<0.001$). However, students were actually not confident in their mathematics ability, so the difference between the groups can be translated as the students who enrolled in the support course disagreed less with the statement, "I am confident in my mathematics ability" than the students who were not enrolled with the support course. In terms of the Likert scale range on the survey, students who enrolled in the support course "slightly disagreed" on average while the students who did not enroll in the support course "disagreed". This question was paired with a follow-up question that asked about their confidence in mathematics at the time of taking the survey, which was in the last 3 weeks of the semester. While both groups showed a statistically equivalent increase in confidence in mathematics ability between the beginning and end of the semester, the students who were enrolled in the support course felt a higher confidence overall.

Acknowledging that responses to a survey do not provide a complete picture of students' experiences and reasoning for their responses, three students interviewed near the end of the semester supplemented these findings with their own lived experience. The first student, Silvia, is an Italian female immigrant who went to primary and secondary schools in Italy before coming to SWU as a Biology major. She plans to attend medical school to become a doctor, with a passion for helping people. Before she attended SWU, she was not confident in her mathematics ability because of the teaching practices of her high school instructors. Specifically, she was made to feel that women do not belong in the mathematics community, which she attributed to a cultural difference between the old-time ways of her hometown and the liberal feeling of California. However, she speaks to how her confidence has changed since she came here:

> So I feel that I feel like more confident after taking this class. Probably in my math abilities. I work for the math department because I do like tutoring for college algebra. So I kind of start liking the math environment way more since I came here. So I think it has like changed my way to like see math and the fact that if I really understand I can like help other people to have success in the class.

Another student interviewed named Santiago is a male Mexican immigrant with the goal of becoming a computer scientist. He is a first-generation college student whose parents completed schooling through primaria, which is equivalent to 6th grade in the U.S. Due to the teaching practices he experienced in secundaria and preparatoria, equivalent to 7-9th grade and 10-12th grade in the U.S. respectively, he felt at a disadvantage to his peers at SWU. Not only because of the way mathematics was taught, but also the general lack of resources he found there to support him in his educational goals. When he approached his lecture instructor during office hours to comment on how he felt, he expressed his relief in their response by claiming, "I entered the room with insecurity, and I left the room feeling more confident about myself."

The third student who was interviewed named Bianca is a female Mexican immigrant who is a first-generation college student with the goal of going to medical school. She also spoke to the lack of resources she had going to school in Mexico, particularly in the teaching practices that left her feeling inadequately prepared for higher education. However, she felt that instruction at SWU changed her confidence in mathematics when she said, "the instructor knew how to explain it better. Like, I felt more confident here in college than in high school, with the problems."

All three of these students felt that their confidence was increased due to experiences with their instructors at SWU. While these experiences were not directly related to enrollment in the support course specifically, their experiences impacted their mathematical identities. Specific to the support course, Silvia and Santiago spoke to the canvas course instructor that served as a mentor to them and the benefits they gained from her being in that role. Silvia said,

> The professor that led the course, she was like, not like a math professor, but she worked more as a mentor… so [the support course] was more like a class to support myself, like 360 degree, like a total thing. Not just within my math, but also like enrolling into classes, understanding what to do next, what classes to take. So, I think it was really good.

Santiago also felt that a having this person in the role of a mentor was beneficial to his experience:

> So, it isn't like she's only the instructor of the class, but she's also like a huge mentor for me. And I think that it is very connected with the [support course] programs. So they've been super helpful.

In summary, we have found that well-guided mentorship and engagement from instructors are two emergent themes from the interviews that stand to bring some reason as to why students in the support course had higher confidence and more sense of belonging in mathematics. Experiencing support from the SI instructors, their GTA's and their course instructors that form the mathematical community on campus and one key instructor that assumed the role of a mentor for them made a powerful difference in their experience in introductory undergraduate mathematics courses. Future research and effort are necessary to understand how we may broaden our reach to the many students who are often failed by our institutional systems. However, one key take-away is that support courses like the one we implemented are a cost-effective way to provide much-needed support for our most at-risk students.

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References

Aguillon, S. M., Siegmund, G. F., Petipas, R. H., Drake, A. G., Cotner, S., & Ballen, C. J. (2020). Gender differences in student participation in an active-learning classroom. *CBE—Life Sciences Education*, 19(2), 1-10.

Apkarian, N., Smith, W., Vroom, K., Voigt, M., Gehrtz, J., PtC Project Team, & SEMINAL project team. (2019). X-PIPS-M survey suite. Retrieved from: https://www.maa.org/sites/default/files/XPIPSM%20Summary%20Document.pdf

Arendale, D. R. (1994). *Understanding the supplemental instruction model*. Josey-Bass.

Collinson, J. A. (2006). Just 'non-academics'? Research administrators and contested occupational identity. *Work, Employment and Society*, *20*(2), 267-288.

Cooper, K. M., Downing, V. R., & Brownell, S. E. (2018). The influence of active learning practices on student anxiety in large-enrollment college science classrooms. *International Journal of STEM Education*, 5(1), 1-18.

Cooper, K. M., & Brownell, S. E. (2016). Coming out in class: Challenges and benefits of active learning in a biology classroom for LGBTQIA students. *CBE—Life Sciences Education*, 15(3), 1-19.

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PloS one*, *11*(7), e0157447.Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the national academy of sciences, 111*(23), 8410-8415.

Koch, A. K., & Drake, B. (2018). Digging into the disciplines: The impact of gateway courses in accounting, calculus, and chemistry on student success. *Chemistry*, *34*, 29-4.

Martin, D. B. (2012). Learning mathematics while Black. *Educational Foundations*, 26, 47-66.

McFarland, J., Hussar, B., Wang, X., Zhang, J., Wang, K., Rathbun, A., ... & Mann, F. B. (2018). The condition of education 2018. NCES 2018-144. *National Center for Education Statistics*.

President's Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. The White House.

Shah, N., Christensen, J. A., Ortiz, N. A., Nguyen, A. K., Byun, S., Stroupe, D., & Reinholz, D. L. (2020). Racial hierarchy and masculine space: Participatory in/equity in computational physics classrooms. *Computer Science Education*, *30*(3), 254-278.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., ... & Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences, 117*(12), 6476-6483.

Weston, T. J., Seymour, E., Koch, A. K., & Drake, B. M. (2019). Weed-out classes and their consequences. *Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education*, 197-243.

White Male Allies in STEM Diversity, Equity, and Inclusion Faculty Service

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I describe findings from a hermeneutic phenomenological study of white male engagement in DEI faculty service. I frame the phenomenon of interest using the DEI Institutional Activism Framework, and questions of structure and agency through Strong Structuration Theory. Two major findings are reported here. First, that DEI service is disincentivized by contradictory institutional structures that devalue engagement. Second, that individual faculty agentically engage in DEI service because of relational connections to students, but also to women they know personally (daughters, wives, friends, colleagues). These preliminary findings are based on semi-structured interviews with four STEM faculty.

Keywords: diversity, equity, inclusion, structure, agency

Women faculty of all races and male faculty of color are more likely to engage in STEM diversity, equity, and inclusion (DEI) initiatives than white men (Jimenez et al., 2019; National Science Foundation, 2019).[1] This disproportionate engagement highlights how underrepresented faculty bear the burden of DEI service in STEM. This burden of equity service silos gender-based DEI initiatives, prevents critical mass from forming, and stifles progress in STEM departments where white men still compose a majority of tenure-line faculty (Jauchen, 2023). White men could contribute to equity progress in STEM fields and relieve some of the burden of DEI service that men of color and women currently bear. The research question in this study was identified through a review of gender-based work in other institutional spaces and centers on the experiences of faculty who have historically not been involved in gender-based DEI service: white non-Latino men. The goal of the study is to understand how individual agency and institutional structures impact white non-Latino male engagement in gender-based DEI service. I hope to understand how current structures are impacting white male engagement in equity work and how individual white men can agentically contest those structures. I focus on one major research question:

> What can we understand about structure and agency in DEI service through the lived experiences of white men?

Theoretical Framework

Service is "the catchall name for everything that is neither teaching, research, nor scholarship" which is still generally relevant for promotion, tenure, or evaluation (Blackburn & Lawrence, 1995, p. 22). DEI service is service faculty perform to encourage historically excluded students to enter, persist, and succeed in STEM fields. I have previously argued that DEI faculty service (the phenomenon of interest) is a form of institutional activism and can be

[1] The word "white," when not understood to be complex, socially constructed, and changing over time, can homogenize all people who are perceived to be white into a singular, hegemonic group. Other aspects of identity (gender, sexuality, class, education, and nationality) impact how white identities are experienced and performed (Crenshaw, 1995). Similarly, I use the phrase "people of color" to refer to people who are not white or who are perceived to be non-white. This phrase was originally embraced as a phrase of collective solidarity for oppressed groups, but it too can essentialize and homogenize a large, diverse group of people. I intend it to be a descriptor of a small part of a multiple, changing identity that is impacted by intersecting systems of privilege and oppression.

framed through the DEI Institutional Activism Framework (Jauchen, 2023). When engaging in DEI service, faculty develop a collective identify which includes: (1) Cognitive definitions regarding the environment, aims, and means of the engagement; (2) a network of relationships between individuals engaged; and (3) emotional investment in feeling a part of the engaged group (Jauchen, 2022, 2023; Melucci, 1996, p. 71). This framework was instrumental in developing the interview protocol for the study.

Questions of structure and agency are framed through Stones' Theory of Strong Structuration (2005). Based in Giddens' Structuration theory, strong structuration asserts that all social interaction is made up of moments where agency and structure are intertwined, what Giddens called the "duality of structure." Under this duality, social interactions are a result of external structures, internally held structures, and human agency. For this study, I focused on components of agency that included critical distance, motivations, hierarchy of purpose, horizon of action, and creativity. To analyze structure, I focused on "conjuncturally-specific structures" which are internal structures based in the immediate context of action and on the agent's position within that context (Stones, 2005, p. 85). These internal structures are "memory traces" within agents that help agents understand how things are supposed to be done. In this context, it is helpful to think of these conjuncturally-specific structures as participant understandings of "the way faculty are supposed to act" or "things faculty are supposed to do." These include position-practices (interdependencies, rights, and obligations), asymmetries of power, and social conditions (Cohen, 1989; Stones, 2005, p. 122).

Methods

Through a critical, intersectional, Western feminist lens, I utilize a structural-hermeneutic design based on Strong Structuration theory (Stones, 2005). Van Manen's (2016) approach to hermeneutic phenomenology guided me throughout. I pursue understanding of structure and agency in DEI service through agent conduct analysis and agent context analysis. In agent conduct analysis, I intentionally orient myself to the agent's knowledgeability, turning away (bridling away) from structures and toward the agent's understanding of the space and their hermeneutic interpretation of the actions available to them. In context analysis, I intentionally orient myself to conjuncturally-specific structures.

Through purposeful selection, I recruited seven white, non-Latino men who held PhDs in a STEM discipline, were employed full-time as faculty at primarily white institutions, and who self-described as being involved in DEI service. Because I am interested in the hermeneutic meaning that white men attach to their own DEI service, self-selection is appropriate as it foregrounds the men's perceptions of their commitment to DEI. I intentionally obscured most information about the participants (discipline, institution type, rank) to protect participant confidentiality. I interviewed men who worked at small regional public schools, large public schools, and small private schools. Four STEM disciplines were represented. I report findings from the first four participants only (some analysis is ongoing).

Each faculty participated in three 90-minute interviews following Seidman's (2019) phenomenological interview protocol. All interviews were recorded, transcribed, and coded according to the Structuration framework. Once that coding was complete, I engaged in phenomenological reflection, rereading the coded sections, asking "What statement(s) or phrase(s) seem particularly essential or revealing about the aspect of agency/structure?" and "What is the main significance of this passage as related to agency/structure and DEI service?" These reflections were then organized into broad themes, of which I present two here.

Findings

Contradictory Structures: DEI Service as Everyone's Work and No One's Work

While every participant said their university and departments wanted to support students and faculty from diverse backgrounds, they also told me that they understood DEI service was not highly valued in promotion and tenure. Faculty felt free to engage in DEI service but also didn't consider DEI service a formal part of their jobs. For example, Daniel, who had earlier argued that DEI service was primarily faculty responsibility, also knew that he would face little negative feedback if he avoided DEI work completely:

> Most faculty are on board with diversifying our university and our fields. But I'm not getting any pressure from the department or from the university more broadly to do these sorts of things. I have been told many times, especially pre-tenure, that things like this are not a formal part of our expectations for you. You have specific things you need to do, like research. And if you need to say, "Sorry student, I can't do this for you," that's okay. So, I have been told that and I hopefully have internalized it. I could decide to ignore these issues completely. I'm not really gaining a whole lot by doing extra stuff on the diversity and equity inclusion front.[2]

Similarly, John said that his university supported DEI service, but only cared about research during promotion and tenure evaluations:

> While institutions claim that they care about teaching, at tenure time they're only evaluating the research as long as you're not awful at the teaching, which is a mixed message, I would say, right? So, on paper, they'll tell you it's teaching, research, and service, right? But I think the people evaluating the tenure case in the end are looking at research the most.

While participants knew their DEI efforts counted broadly as service, they understood service to be undervalued by the university reward structure. In fact, participants were warned not to spend too much time on DEI service because it could impact research productivity.

These contradictions impacted the support universities provided participants. Participants told me that they had not been trained to engage productively in DEI service. Here's Josh speaking about how his graduate school training provided no information about DEI service.

> Outside of my certificate, I have no formal training in what to do. I took a bunch of STEM classes. They didn't really talk about any of this. So, it's really great to be able to go into Research Gate and enter in a few words and get like a view of literature.

Similarly, Grant developed some of his understanding of DEI through his own personal reading.

> I mean, I mostly set up training at the different places where I worked. I was the one who was instituting those sorts of things. And they're really difficult because most people think they're just blow off: "What am I gonna learn from this?" I think all my theoretical learning and training and understanding about this happened when I was in graduate school. I read widely. I would read business books. I would read Dale Carnegies, How to Win Friends, influence people. I did a lot of this stuff.

When university training was offered, it was not STEM specific (as with Josh above), or not practical. John expressed frustration because he attended DEI training to better understand race

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[2] All quotes are phenomenological anecdotes (van Manen, 2016). Anecdotes are edited versions of participant quotes. In addition to extraneous words being omitted (um, and, like), participant stories were edited for clarity and space, but remain true to the themes identified by the phenomenological analysis.

and gender and those trainings morphed into conversations about how "people learn differently" that were not practical for his class size.

> On paper, I feel like the intention of some of these trainings was to discuss race and gender, but then they turn into this general, "Oh, well, the point is people learn differently." So, I attended one that was focused on how to assess student thinking. So, you give a student an exam and they don't put the right thing on the piece of paper, and you don't give them any points. But did you really understand what they were thinking? And ideally, I'd sit down with the student and say, "Explain to me what you're thinking here" but I can't do that for 300 students. So that bothered me a little bit that we're learning these ideas and trying to think about these things, which was great. But at the same time, is this really practical?

These contradictions in structures – that discursively the university voiced support for DEI initiatives, while functionally not providing resources is an indication of structural inequity.

Agency Amid the Chaos: Navigating Contradictions

In spite of these contradictory structures, participants cultivated a commitment to DEI service. While all participants described their engagement to the belief that DEI service was "the right thing to do," they also were motivated to engage in DEI service by relational commitments they held toward their students and/or toward women they knew personally (wives, daughters, friends, colleagues). For example, Daniel described a deeply held ethical commitment to know students as individuals and to communicate their inherent value as humans:

> I talked about this once with a woman who had worked with college students for a long time. She said that students would tell her all the time that "I just don't know if anyone at this university cares about me for myself and not me for my ability to do X, Y, or Z." That really struck me. I try to communicate to students that their performance in my class is not a make-or-break moment for who you are as a person or for your worth. Their value to the university community is not just about their ability to do well in specific classes. It's not at all about what they're able to do or not do, or what they're able to contribute or not contribute. And I see that belief as very much in opposition to a lot of things in the academy.

Josh was motivated to pursue grant funding for a DEI initiative because of his relational commitment as a dad to two daughters:

> So, I remember thinking, my youngest daughter was expressing interest in going into video game design or something like that. And I remember thinking like, "This grant wouldn't impact her life specifically, but what kind of environment would I want her to be in, or how could she be best supported?" Then I had this real desire to do that for our students: think of them like they are someone's daughters. Someone else feels this way about their daughter. I can care about them too.

These relational connections motivated participants to agentically engage in DEI service, even in the midst of contradictory messages from the university. In addition, participants told me they learned a lot about women as a result of these relationships. Here is Josh:

> I was talking to a fellow soccer dad. We played sports growing up, so we've been coached. We know about the culture and how it works, at least on the men's side. I learned very quickly the way we were coached as kids does not work at all on young girls' teams. Everyone that coaches girls every day understands this phenomenon either going in or right away. They don't question it. They just treat these groups a little bit differently. Still trying to promote the same skills. Both groups are equally good skill

wise. But then it's weird when you get to an academic setting, it's not really as accepted. It's like if you say something like that, the response is "We'll just treat everyone equally and kinda move on." I don't really understand that now, looking back on it.

And Grant, despite having committed a lot of time to understanding race and gender, told me that he learned so much when he met his partner, a Black woman.

> When I met my partner and I got to know her girls and I started raising her girls, I learned so much more about women of color, about all those extra things that I never had to think about growing up. And I was very sympathetic to it, but I mean, it was just like little dumb things. Like we'd be in a grocery store, and she'd be at the counter, and then I'd walk up and afterwards she said, "Did you notice the way the cashier's whole demeanor changed when you walked up to the counter? That's white versus black." But it's hard for people to really appreciate if they haven't experienced it.

Discussion

Universities have established a system that disincentivizes faculty from participating in DEI service while also discursively positioning diversity as a common good. DEI work is structured as an optional, individual pursuit. Individual faculty can engage, but it is not in their interest to do so. In fact, it is explicitly against their interest as an individual, working in a system that disincentivizes DEI service. This establishes a structure under which faculty are encouraged to engage in either/or thinking about their time – that they can do DEI work *or* work on their research. Since the tenure clock is ticking, the research part of this is considered urgent with DEI goals falling to the wayside. But faculty rarely make decisions based purely on university reward structure. Faculty weave together a sense of who they are, what they "should do," through their hermeneutic interpretations of their DEI engagement. In spite of little support from universities, men relied on personal relationships with women to develop understanding of DEI service and to motivate their engagement. This weaving is an agentic practice itself, of course. But it also creates new structures that guide future action, resist harmful university contradictions, and support students who have historically been marginalized. This early study raises many more questions about white male DEI engagement, but also supports so much of the research based on the experiences of marginalized faculty.

Positionality

I am a white, queer, cisgender woman (BS/MS Mathematics), currently teaching mathematics at George Mason University, a public, four-year university. I am part of a community of STEM faculty engaged in DEI service. I study DEI service through the lens of institutional activism, hoping to understand who is involved in DEI work in STEM, who is not involved, and how universities can strategically increase faculty engagement in DEI while retaining, valuing, and centering the work that underrepresented faculty have been doing.

References

Blackburn, R. T., & Lawrence, J. H. (1995). *Faculty at work: Motivation, expectation, satisfaction*. Johns Hopkins University Press.

Cohen, I. J. (1989). *Structuration theory: Anthony Giddens and the constitution of social life*. StMartin's Press. https://doi.org/10.1007/978-1-349-20255-3_2

Crenshaw, K. W. (1995). Mapping the Margins: Intersectionality, Identity, Politics, and Violence Against Women of Color. In K. Crenshaw, N. Gotanda, & G. Peller (Eds.), *Critical Race Theory: The Key writings that Formed the Movement* (pp. 357–383). Routledge New York.

Jauchen, J. G. (2022). Everyday Activism: Gender-Based Service in STEM. *PRIMUS*, *0*(ja), 1–0. https://doi.org/10.1080/10511970.2022.2059729

Jauchen, J. G. (2023). Institutional activism in diversity, equity, and inclusion faculty service in STEM. *Journal of Women and Minorities in Science and Engineering*, *29*(2), 51–73. https://doi.org/10.1615/JWomenMinorScienEng.2022036614

Jimenez, M. F., Laverty, T. M., Bombaci, S. P., Wilkins, K., Bennett, D. E., & Pejchar, L. (2019). Underrepresented faculty play a disproportionate role in advancing diversity and inclusion. *Nature Ecology & Evolution*, *3*(7), 1030–1033. https://doi.org/10.1038/s41559-019-0911-5

Melucci, A. (1996). *Challenging codes: Collective action in the information age*. Cambridge University Press. ttps://doi.org/10.1017/CBO9780511520891

National Science Foundation. (2019). *NSF Advance Grant Awards, 2001 – 2018*. https://www.nsf.gov/awardsearch/advancedSearchResult?ProgEleCode=1738,%207568,1739,%20016Y,%207650,%207690,%20011Z&BooleanElement=ANY&BooleanRef=ANY&#results

Seidman, I. (2019). *Interviewing as qualitative research: A guide for researchers in education and the social sciences* (Fifth edition.). Teachers College Press.

Stones, R. (2005). *Structuration theory*. Palgrave Macmillan.

van Manen, M. (2016). *Phenomenology of practice: Meaning-giving methods in phenomenological research and writing*. Routledge. https://doi.org/10.4324/9781315422657

Linear Algebra Students' Reasoning with Compositions of Linear Transformations

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We report on preliminary findings from a study of the ways in which linear algebra students used function composition to describe the result of applying two linear transformations, defined using symbolic vector notation, to a graphical region. Our current results from even a small set of data suggest that the students engage in function composition in a variety of ways. These results suggest areas for future exploration regarding how students engage in these forms of composition with linear transformations, and how their conceptualizations of individual transformations inform their understandings of the composition.

Linear transformation is an important idea within linear algebra. Furthermore, several areas of linear algebra rely on the idea of composing transformations, particularly in relation to multiplying matrices. These include the invertible matrix theorem (e.g., Wawro, 2014), and the diagonalization equation (e.g., Zandieh, Wawro, et al., 2017). Currently a few studies have investigated students' understandings of linear transformations, a small subset of which have included composition as one of the focal ideas (e.g., Bagley et al., 2015). Thus, a study of how linear algebra students conceptualize compositions of linear transformations as a form of function composition could have useful implications for the teaching and learning of linear algebra. This paper reports on our initial findings from pursuing this line of research.

Background and Literature Review

A large body of research has investigated students' understandings of function (early examples include Breidenbach et al., 1992; Carlson, 1998; Sfard, 1992). In contrast, few studies have had an explicit focus on function composition (e.g., Bagley et al., 2015; Bowling, 2014; Chen et al., 2023; Engelke et al., 2005; Headrick, 2023a; Kimani, 2008; Modabbernia et al., 2023). Most of these studies have focused on high-school or early college students (Bagley et al., 2015 being an exception). Furthermore, these studies have mainly used students' reasoning in the context of function composition to draw conclusions about their understandings of function. Some research has revealed evidence that students' reasoning with function composition extends beyond their reasoning with individual functions, suggesting a need for research that focuses on what is unique and important about students' reasoning with function composition specifically (e.g., Headrick, 2023a; Bowling, 2014).

Function composition has hardly been studied in linear algebra students' applications of linear transformations. A few studies have investigated students' understandings of linear transformations as functions (e.g., Andrews-Larson et al., 2017; Bagley et al., 2015; Oktaç, 2019; Turgut, 2019; Zandieh, Ellis, et al., 2017). One such study with a focus on composition of transformations (Bagley et al., 2015) found that students constructed the idea of an identity transformation as a "do-nothing function" and composing a transformation with its inverse as "doing" and "undoing." Our study will add to this research by investigating the reasoning linear algebra students use to describe the result of composing two distinct linear transformations.

Theoretical Framework

The theoretical framework currently guiding our data analysis in this study has two components. The first and primary component, which we refer to as *function composition* *reasoning*, is intended to explain students' reasoning unique to function composition specifically, accounting for ways students conceptualize multiple linear transformations and consider the result of applying them in sequence or in some other combination (Headrick, 2023b). The second component, called *clusters of metaphorical expressions*, focuses on the language students use to describe how they imagine individual transformations occurring (Zandieh, Ellis, et al., 2017).

Function Composition Reasoning

By function composition reasoning, we refer not only to previously described conceptualizations of function composition in research (e.g., Ayers et al., 1988; Breidenbach et al., 1992), but also other forms of reasoning students use to combine or apply multiple functions in response to a situation. Reviewing prior research led to identifying four distinct types of function composition reasoning high school and early college students engage in (Headrick, 2023b). Our current data suggests that these four types could also explain linear algebra students' reasoning with compositions of transformations. An example of each type of reasoning from our data is presented in the preliminary results.

The first type, *modifying a function with another function* (in short, *modifying*), involves conceptualizing a specific transformation and subsequently applying small tweaks or changes to this transformation via another, modifying transformation. The second type, *applying an operation on two functions* (in short, *operation)*, involves conceptualizing multiple transformations individually, and subsequently imagining the result of applying all of them together. The third type, *chaining input/output relationships*, involves taking some starting point (or input), applying one transformation to it, producing a particular result (or output), applying another transformation to the output of the first transformation, producing another output, and so forth. The fourth type, *chaining relationships between variables*, is an application of the third type in which the 'input' and 'output' of each transformation being composed is a variable.

Clusters of Metaphorical Expressions

Zandieh, Ellis, and Rasmussen (2017) presented five types of metaphorical expressions they found students to use when reasoning with linear transformations. Our current data suggests that students used these metaphors when composing transformations. The first, *input/output*, involves a transformation taking in or accepting some initial input, and giving some output in return. The second, *morphing*, involves an entity changing from one form to another through the transformation. The third, *machine*, involves the transformation doing something or acting on an initial entity to produce a result. The fourth, *traveling*, involves an entity moving from one location to another through the transformation. The fifth, *mapping*, involves an assignment of a one entity or value to another through some rule of correspondence.

Research Question

In light of existing research, we consider the question: In what ways do linear algebra students use function composition reasoning when composing two vector-defined linear transformations in a graphical context?

Method

Data Collection

The first author conducted one-on-one task-based clinical interviews with six students. The students had recently finished a Linear Algebra course taught by the second author and in which the first author served as a teaching assistant, at a large, public university in the United States. The Linear Algebra course was designed from a research-based curriculum (Wawro et al., 2013). The interviews were audio and video-recorded, and all written work was retained for additional evidence of the students' thinking.

Each task involved describing the result of applying a transformation to a 1-unit by 1-unit square region in the standard Cartesian plane, with its bottom-left vertex at the origin. Each task was structured identically with a three-fold structure: (i) predicting the result of applying a particular transformation, (ii) explaining the role of the notation used in the task to define the transformation in making the prediction, and (iii) sketching the transformed region, which for some (but not all) students, involved more precise calculations for determining points in the transformed region. Students applied these prompts to a single transformation T, another single transformation U, and the results of composing T and U: T(U(original square)), and U(T(original square)). Transformations in the tasks were defined using two different notations: vector notation, and matrix notation (different transformations were defined for each notation). Our preliminary results are from compositions of transformations T and U from the vector notation tasks.

Data Analysis

Since there are few prior studies on how students make sense of compositions of linear transformations, we began with an open-ended analysis in which we examined the video data to determine what themes might emerge from the students' reasoning. Upon further examination, we determined that existing frameworks for function composition reasoning (Headrick, 2023b) and for students' reasoning with transformations as clusters of metaphorical expressions (Zandieh, Ellis, et al., 2017) could enable us to develop a useful coding scheme.

Preliminary Results

In this section we present three distinct examples from our data to illustrate how the four types of function composition reasoning (underlined and italicized) appear to have emerged so far. Within each of these examples, the students appeared to use at least one cluster of metaphorical expressions (underlined). These examples suggest that linear algebra students could imagine composing linear transformations from a single problem context in a variety of ways. The data presented in this section are from students' predictions of how a composition of two transformations, defined as $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \end{bmatrix}$ and $U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ y \end{bmatrix}$, will transform a 1-unit by 1-unit square region drawn in the standard Cartesian plane with its bottom-left vertex at (0, 0).

Luna: Chaining Input/Output Relationships with Input/Output Transformation Metaphor

To consider the potential result of T(U(original square)), Luna evaluated the vector-components definition of U, then T at the original square's top-right vertex: 1, 1.

Luna: Yeah, the T, U. Then, yeah, it would be--you would use the U which would turn it--the 1, 1 into 3, 1 and then you plug 3, 1 into T, which would give you 6, 1…

Luna described first applying U to the point, getting particular coordinates as a result, and then applying T to the resulting coordinates, suggesting she was *chaining input/output relationships*. She appeared to use the input/output metaphor when describing how she would apply individual transformations ("then you plug 3, 1 into T, which would give you…"), and perhaps a morphing metaphor ("turn it—the 1, 1 into 3, 1").

James: Modifying and Operation Function Composition Reasoning with Morphing

James appeared to focus on describing the entire square as a completed shape. When initially describing his prediction for the result of T(U(original square)), James said,

James: So if I do U on the original square that's going to turn it into the--that parallelogram that I--I drew. But if I do the T after that, that's just gonna stretch that parallelogram twice its x-value. So if I would just like to take that side and just pull it 'til it's twice it's x-value, I think that's what would--oh, wait, is that what would happen? 'Cause that 2 (pauses) Oo, it's also italicizing it further.

After further consideration, James concluded that:

James: …It basically doubles the size, but it also becomes twice as italicized.

James' overarching reasoning, including a description of applying U "to the original square", producing a result ("that parallelogram"), and applying T to the result ("if I do the T after that") suggest he imagined the composition as a *chain of input/output relationships*. He appeared to engage in *modifying* reasoning when he initially predicted the result of T(U(original square)). He began by focusing on the parallelogram he described as resulting from applying U to the original square ("that parallelogram I drew"). Then, he described how applying T would change the shape of the parallelogram ("stretch…twice its x-value"; "italicizing it further"). Thus, when forming his initial prediction, James appeared to think of applying U as the emphasis and applying T as tweaking the results of applying U. However, when drawing a final conclusion about the transformed region, James also appeared to put greater emphasis on how T as an individual transformation would transform the parallelogram resulting from applying U to the square ("doubles the size"; "twice as italicized"), suggesting potential *operation* reasoning.

Throughout his description, James' language suggests he imagined a region of interest changing from one shape to another through the transformations ("turn...into"; "stretch"; "italicizing"; etc.). Thus, James appeared to use predominantly morphing metaphors.

Olivia: All Four Types of Function Composition Reasoning with Machine and Morphing

Olivia seemed to imagine applying transformations to a collection of points that she said comprised the square. She illustrated this idea by using vector notation to denote the points being transformed (later she said, "x, y is really just the same as the whole bunch of points that make up our square"). Olivia's orientation to the task and written work (Figure 1) are given below.

Olivia: So T of U of--we're gonna call it x y, 'cause the original square freaked me out. So that means U is gonna transform x y first, and then T will do it.

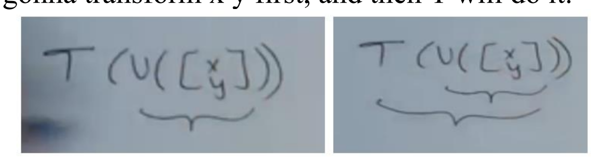

Figure 1. Olivia's Illustration of Applying U, then T to a Set of Points in the Cartesian Plane

Olivia's initial description of applying U, then T to the set of points suggests she imagined a *chain of input/output relationships*. When predicting the result of T(U(original square)) in more detail, she drew diagrams for the individual result of applying each transformation to the square and the combined result of applying both (see Figure 2). Meanwhile, she said,

Olivia: So this is our whatever x y is we call the original shape. This is my square. U took the square and made it like a parallelogram. That's what U did. And if we remember what T did, T took it and made it like, boop, it stretched it. So then, T would take this U thing, and stretch that to be like twice as long…

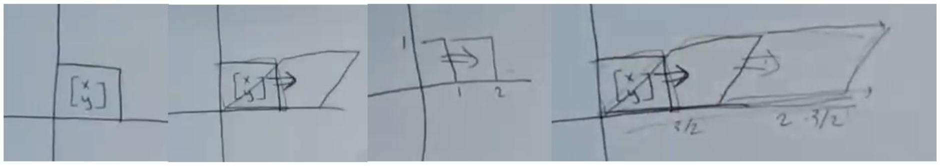

Figure 2. Olivia's Diagrams to Illustrate (left to right) the Original Square as a Set of Points, Applying U to the Square, Applying T to the Square, and Applying U, then T to the Square

Olivia appeared to imagine both U and T as having some individual effect on the square and T(U(original square)) as applying a combination of their individual effects (suggesting *operation* reasoning). There is also subtle evidence of her engaging in *modifying* when she described how T would change the result of U ("take this U thing, and stretch that…"). When asked what would happen to the corner points of the square "when you do T of U", Olivia described the transformation affecting the x-coordinates of multiple points. This reasoning, along with her earlier references to the square as a set of points suggests Olivia conceptualized T(U(original square)) as a *chain of relationships between variables*.

Olivia: …you stretch everything, all the x-coordinates by 2. So I guess that would include these, like, these points [points to the edges of parallelogram for T(U(original square))]. Their x-coordinates have to get shifted by 2 too, so then the diagonals would have to.

Throughout her description, Olivia used machine metaphors, describing transformations as doing particular actions to points or figures ("U took…and made it"; "U did"; ect.), and morphing metaphors ("made it…a parallelogram"; stretched it; etc.).

Discussion

All three students in the results presented above appeared to use chaining of input/output relationships in some form. They all described some starting shape or value, applying the inner transformation to that starting shape or value to produce a result, and then applying the outer transformation to the result of applying the inner transformation. The pervasiveness of this reasoning in the data thus far could be partly due to the nature of the tasks posed; the two transformations were pre-defined and the task prompts involved the notation T(U(original square)) and U(T(original square)). This is an area for further exploration.

On the other hand, each student described the chain of input/output relationships they appeared to conceptualize quite differently. Chaining input/output relationships as a form of function composition reasoning was most prominent for Luna, and her reasoning with each individual transformations appeared to be largely connected to input/output metaphors. James' focus on transforming entire shapes and morphing seemed to naturally give rise to his modification reasoning. Olivia's focus on transformations as machines performing actions on a set of points seemed to inform her operation reasoning (i.e., first describing how each transformation would act on points individually and then constructing the composition), and chaining relationships between variables (i.e., describing how multiple points were transformed). Our results thus far lead us to consider the following questions: (a) What contexts for linear transformation problems might lead to specific types of function composition reasoning; and (b) How might different types of function composition reasoning with linear transformations support students in studying related ideas in linear algebra, such as inverses or diagonalization?

References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, *48*(6), 809–829.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education*, *19*(3), 246–259.

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: The case of function and linear transformation. *The Journal of Mathematical Behavior*, *37*, 36–47.

Bowling, S. (2014). *Conceptions of Function Composition in College Precalculus Students* (Publication No. 3619278) [Doctoral Dissertation, Arizona State University]. ProQuest Dissertations Publishing.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Carlson, M. P. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. III. CBMS issues in mathematics education* (Vol. 7, pp. 114–162).

Chen, Y., Wasserman, N. H., & Paoletti, T. (2023). Exploring Geometric Reasoning with Function Composition. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. SIGMAA on RUME, Omaha, NE.

Engelke, N., Oehrtman, M., Carlson, M., & Carlson, M. (2005). Composition of functions: Precalculus students' understandings. *Proceedings of the 27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 20–23.

Headrick, L. (2023a). A Student's Application of Function Composition when Solving Unfamiliar Problems: The Case of Zander. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 744–751.

Headrick, L. (2023b). *High School Students' Early Function Composition Reasoning in the Context of Transformations of Polynomial Functions* [Unpublished manuscript]. School of Mathematical and Statistical Sciences, Arizona State University.

Kimani, P. M. (2008). *Calculus students' understandings of the concepts of function transformation, function composition, function inverse and the relationships among the three concepts* (Publication No. 3333570) [Doctoral Dissertation, Syracuse University]. ProQuest Dissertations Publishing.

Modabbernia, N., Yan, X., & Zazkis, R. (2023). When algebra is not enough: A dialogue on the composition of even and odd functions. *Educational Studies in Mathematics*, *112*(3), 397–414.

Oktaç, A. (2019). Mental constructions in linear algebra. *ZDM*, *51*(7), 1043–1054.

Sfard, A. (1992). Operational origins of mathematical notions and the quandary of reification—The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. Mathematical Association of America.

Turgut, M. (2019). Sense-making regarding matrix representation of geometric transformations in $\mathbb{R}^2$: A semiotic mediation perspective in a dynamic geometry environment. *ZDM*, *51*(7), 1199–1214.

Wawro, M. (2014). Student reasoning about the invertible matrix theorem in linear algebra. *ZDM*, *46*(3), 389–406.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A Characterization of a Unified Notion of Mathematical Function: The Case of High School Function and Linear Transformation. *Educational Studies in Mathematics*, *95*(1), 21–38.
Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An Example of Inquiry in Linear Algebra: The Roles of Symbolizing and Brokering. *PRIMUS*, *27*(1), 96–124.

Assessing Student Learning Experiences in a Corequisite Calculus Course

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The corequisite model of academic support has been touted as an efficient way of supporting student learning while decreasing the time to degree completion for students needing academic support. With the learning losses that have occurred globally as a result of the global coronavirus pandemic, the Department of Mathematics at California State University, Fullerton piloted a corequisite course to support one section of Calculus I, a course historically riddled with high DFW rates. In this paper we share preliminary results of our department's implementation of the corequisite model. While corequisite students from the first iteration of the course earned higher GPAs than their non-corequisite peers, more data (including longitudinal data) is needed to determine the lasting impact from the Calculus I corequisite experience.

Keywords: Corequisite mathematics, academic support, curriculum development

Background

Introductory mathematics courses have historically been a roadblock for students intending to major in the Science, Technology, Engineering, and Mathematics (STEM) fields. For most STEM majors, students must complete a sequence of calculus courses before commencing their major coursework. Historically Calculus I and Calculus II have been identified as bottleneck courses at California State University, Fullerton (CSUF). From 2012-2022, the Calculus I course was ranked eighth highest in terms of DFW (grades of D, Fail, or Withdraw) with a rate of 33%, while Calculus II ranked sixth highest with a 38% DFW rate. Given this institutional data, the authors developed an academic support course to try to combat this systemic issue through the creation of a Calculus I corequisite, and piloted the course with first-time-freshmen beginning their academic journey at CSUF with a declared Mathematics major.

The primary goal of the Calculus corequisite course at CSUF was to reinforce student conceptual understanding of prerequisite mathematical content knowledge, provide students with additional opportunities to engage with course content, and to help students develop the required skills to perform and succeed in college. Research (e.g., Hancock et al., 2021; Kashyap & Matthew, 2017; Logue, 2014; Richardson, 2021; Stone-Johnstone, 2023) on corequisite courses like the one created here at CSUF, have demonstrated benefits to student learning, as well as supported the placement of students directly into gateway courses (with academic support) as opposed to a long sequence of prerequisite courses.

The primary goal of this work is to increase access to Calculus I and Calculus II. We do this by creating a course with the aim of helping students better communicate mathematically (in writing and in speech), supporting students in problem solving, and developing their skills in using technology when doing mathematics. We hypothesize, based on previously research, that this corequisite course will help decrease the time to graduation at CSUF by providing students with targeted academic support. The driving research questions in this study are:

1. What effect does a Calculus I corequisite course have on student success in Calculus II?
2. What effect does a Calculus I corequisite have on students intending on majoring in mathematics?

Methods

This study employs a convergent mixed methods approach (Creswell, 2012) where quantitative and qualitative data were collected to assess the efficacy of the Calculus corequisite. Quantitative data in the form of Calculus I course outcomes of all course sections from the Fall 2021 semester and the Calculus II course outcomes from a special section of Calculus II composed primarily of mathematics majors in the Spring 2022 semester were collected and analyzed. Comparisons were made between the subgroups (corequisite and non-corequisite students) by performing a series of one and two-sample independent t-tests.

Qualitative data in the form of student focus groups and classroom observations of the corequisite course during the Fall 2021 semester were collected and subsequently analyzed. Fifteen corequisite students were interviewed during the data collection period, and from interview data, structural codes (Saldaña, 2021) based on the focused research questions were used to categorize and interpret their narratives around their experiences in mathematics during and after participating the Calculus corequisite course.

Preliminary Findings

The Calculus corequisite course was piloted during the Fall 2021 semester with only mathematics majors. Incoming students who intended to major in mathematics were advised to take the course in conjunction with their regular Calculus I course. Certain students who would have normally been placed into Pre-Calculus were also invited to take Calculus I with a corequisite support course during their first semester at CSUF. In this section we report both preliminary quantitative and qualitative results from the first iteration of the corequisite course at CSUF.

Quantitative Results

In Fall 2021, 38 math majors were enrolled in the corequisite course. Preliminary results from the quantitative analysis of the Calculus 1 course data shows that students who participated in the corequisite (versus all other students in other sections of Calculus 1) had higher overall course GPAs (grade point averages) and lower DFW rates. In addition, when only considering those students who received a non-DFW grade, the corequisite students outperformed the non-corequisite students (see Table 1). This latter category is infrequently reported, but at CSUF we have noticed that students who pass with lower grades in Calculus I tended to struggle more in their subsequent calculus and upper-division STEM courses. Out of the 38 corequisite students, 23 enrolled in the special section of Calculus II that was composed of only mathematics majors in the semester that followed. Nine of the other students did not pass Calculus I with a C or better (a requirement for STEM majors at CSUF), and the six others changed to a major that did not require any additional mathematics coursework.

Table 1. Calculus I Course Data from Fall 2021 across all sections

	Corequisite Students	Non-Corequisite Students
n	38	410
GPA	2.66*	1.88
DFW	24%**	41.8%
GPA (excluding DFW)	3.36*	2.93

*Significant at the .001 level
** Significant at the .007 level

Course data from the Spring 2022 Calculus II special course shows that students who were enrolled in the corequisite during the previous semester had slightly higher course GPAs and lower DFW rates in that course than those students who did not take Calculus I with a corequisite support course. When comparing all students that did not receive a DFW in Calculus II, the corequisite students earned higher GPAs relative to their non-corequisite classmates (see Table 2). It is important to note, of the 21 non-corequisite students, only two were first-time-freshmen and the rest were either upper-classmen or students repeating the course for credit.

Table 2. Calculus II Special Course Data from Spring 2022

	Corequisite Students	Non-Corequisite Students
n	23	21
GPA	2.178	1.952
DFW	26%	33%
GPA (excluding DFW)	2.88	2.786

*Significant at the .001 level
** Significant at the .007 level

Qualitative Results

There were two dominant themes that arose from the qualitative analysis: *Students valued the course structure of the corequisite course*, and *Students valued opportunities for community-building*. During the Fall 2021 semester, the corequisite students received more than six hours per week of course contact with their course instructor and peers. They were scheduled to attend their Calculus I course in person on Mondays and Wednesdays for 1 hour and 50 minutes each, and their corequisite course virtually on Tuesdays and Thursdays for 1 hour and 15 minutes each. During the corequisite sessions, students spent their time actively engaging in either Calculus I review or Pre-Calculus and Trigonometry activities that prepared them for the new Calculus content they would cover during the following day. In contrast, the instructor focused Calculus I class time on presenting new course material through a mixture of traditional lecture and active learning methods.

The first theme, *Students valued the course structure of the corequisite course*, arose as students were describing the nature of the corequisite course. One student shared that,

> [the corequisite course] helps us refresh and review … like with radicals and radical numbers and rationalizing numbers and stuff like that, so I feel like that was really helpful … I feel like it's pretty helpful and backing up like you're just reviewing - to help support what you're learning in calculus …"

Students recognized that the purpose of the corequisite course was for review and developing a deep conceptual understanding of their Calculus material. This course was a low-stakes support course where students could seek and receive help from their instructor, teaching assistants, and peers. Another student elaborated,

> So [the corequisite course] kinda introduced to us what we're going to do the next day. So it's not as hard the next day we're kind of like we know what we're about to do, like we have a taste of it. And it was really nice seeing all my classmates, all my groupmates day in, day out and seeing [my instructor] four days a week. And yeah … it wasn't that hard - it's like I wouldn't go stressing … like I have this class, like, I look forward to it. Because the problems aren't that hard and they're a good introduction to the next day and yeah it was fun.

The second theme, *Students valued opportunities for community-building*, showed up in all the interview data. For instance, when asked what they have gotten out of the corequisite course, one student shared,

> I mean, uh, I mean as cheesy as it might sound … like a community. I feel comfortable talking in front of those people, there's like 30 of them. I asked the teacher questions. In a class last year I wouldn't be doing that because I wouldn't be comfortable. We would be on ourselves most of the time. Now, we're like actually like you have people that you know and that you trust and that are actually helping you and want to help you too. If you're actually struggling and you wanna help them if they're struggling, it's more like personable I guess. You know everyone I guess too. Like even if I haven't talked to someone, I feel comfortable talking to them, which is odd because I've never really done that before, but yeah this class has definitely helped that.

Another student added,

> We engaged a lot, as a group ... We sat in groups of four and we engaged a lot … and I feel like that also helped us understand the material more because it's just you're figuring it out not by yourself, this time, but with, with the group, and I feel like, figuring it out together also helps us understand the material.

The corequisite students engaged in active learning and discovery within both their Calculus I and corequisite course four days per week. The early and consistent engagement among the students helped foster relationships that spanned past the Fall 2021 semester and into their Calculus II course during the following semester. When asked again, during the Spring 2022, to reflect on their corequisite course, several students reiterated the benefits of the collaborative learning experience. From their experience in the corequisite, students developed friendships and study groups that have helped them navigate Calculus II.

Discussion

The first driving research question for this work was, what effect does a calculus corequisite course have on student success in Calculus II? The preliminary data demonstrates that the corequisite course has the potential to support student learning in their Calculus I course as well as equip them with the tools for success in Calculus II. The Fall 2021 corequisite students attributed their success to the greater mathematical foundation they obtained from their corequisite course as well as the community they were able to build to help support them in their studies in Calculus II.

The second driving question was, what effect does a calculus corequisite have on students intending on majoring in Mathematics? While some students who intended to major in Mathematics ended up switching majors by the end of the semester, many others persisted into Calculus II. However, one student that ended up switching majors shared,

> I was struggling, I think more than I needed to. Or, more than like most of my peers. And, when I did my major advising, I was telling her that I was struggling and she was like "It's only going to get harder" and I was like "Hmm" and I was like "I don't know how much more of this I can take" so I did change my major.

This student was discouraged by an academic adviser to continue her mathematics pursuit based on her present academic struggles in Calculus I. It is unknown whether this student could have passed the course should they have stayed the course and sought extra help outside of what was offered in the corequisite course and the instructor's office hours.

While 23 of the original 38 students persisted to Calculus II, it is too early to firmly measure the effect a Calculus I corequisite will have on student persistence in STEM in general. A full assessment of the benefits of the corequisite requires a longitudinal study, where we can track student progress throughout the tenure of their college experience. Based on the prior research and the preliminary findings reported here, we hypothesize that a Calculus I corequisite course has the potential to increase student success in Calculus I and Calculus II. The findings from this study will be used to optimize future iterations of the corequisite course to support student learning and achievement. As we collect more data, we will better understand the potential of the corequisite model in supporting our student learning and academic success in STEM.

1. What kinds of data would convince you that a corequisite course is doing its intended job, of academically supporting students and helping them to persist in mathematics?
2. Does it make sense for an institution to adopt a corequisite model when they also have a Supplemental Instruction program?

References

Creswell, J. W. (2012). *Educational research: Planning, conducting and evaluating quantitative and qualitative research* (P. A. Smith, C. Robb, M. Buchholtz, & K. Mason (Eds.); 4th ed.). Pearson.

Hancock, E., Franco, L., Bagley, S., & Karakok, G. (2021) A Holistic Approach to Supporting Student-Centered Pedagogy: Navigating Co-Requisite Calculus I, *PRIMUS*, *31*:3-5, 608-626, DOI: 10.1080/10511970.2020.1802794

Kashyap, U., & Mathew, S. (2017). Corequisite model: An effective strategy for remediation in freshmen level quantitative reasoning course. *Journal of STEM Education: Innovations & Research, 18*(2), 23–29.

Logue, A. W. (2014). College-level statistics trumps remedial algebra in CUNY study. *Inside Higher Ed*.

Richardson, C. (2021). *Corequisite Mathematics Toolkit: Tools and resources for the design and implementation of equitable and effective support courses*. Charles A. Dana Center.

Saldaña, J. (2021). *The Coding Manual for Qualitative Researchers* (J. Seaman (ed.); 4E ed.). SAGE Publications Inc.

Stone-Johnstone, A. (2023). Exploring the impact of the corequisite climate on students' attitudes toward mathematics. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-023-00226-y

Exploring How Undergraduate Students Engage in Computational Thinking with Data

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Modern statistics education requires that we support students in building powerful and productive ways of computational thinking. In this paper, we seek to understand the ways of computational thinking that undergraduate students employ as they engage with data. To address this question, we administered task-based interviews with three participants using the R programming language. Problem-solving approaches focusing on formatting data and efficient coding emerged as early aspects of student thinking. We are still reviewing interview transcripts and intend to compare our findings with existing frameworks from the literature, working towards a framework highlighting beneficial ways of thinking. This work is part of a larger study with additional tasks and additional individuals with varying levels of experience and expertise.

Keywords: computational thinking, data, statistics education, student thinking

The Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report endorsed several recommendations for introductory statistics courses, such as “use technology to explore concepts and analyze data”, “integrate data with a context and purpose”, and “teach statistics as an investigative process of problem-solving and decision-making” (Carver et al., 2016). These require that instructors support students in building powerful and productive ways of computational thinking.

Wing (2006)’s seminal paper on computational thinking emphasized that it is a “fundamental skill for everyone” that “involves solving problems, designing systems, and understanding human behavior by drawing on the concepts fundamental to computer science” (p. 33). Aho (2012) defined computational thinking as “the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms” (p. 832). Lockwood et al. (2016) built upon Aho’s work, by studying the role of such thinking in mathematics through interviews with mathematicians. They arrived at the working definition “a logical, organized way of thinking used to break down complicated goals into a series of (ordered) steps using available tools”, which they termed “algorithmic thinking” (Lockwood et al., 2016, p. 1591). Shute et al. (2017) considered research in K-12 and higher education settings to define computational thinking as “the conceptual foundation required to solve problems effectively and efficiently (i.e., algorithmically, with or without the assistance of computers) with solutions that are reusable in different contexts” (p. 151). These definitions may appear overly broad, so we discuss existing frameworks to ground them.

Background

Computational thinking cuts across multiple disciplines including computer science, statistics, and mathematics; further, computational thinking appears at all levels of education. Working from a computer science education perspective, Barr and Stephenson (2011) discussed the core computational thinking elements that emerged during a Thought Leaders meeting for K-12 implementation organized by the Computer Science Teachers Association and the International Society for Technology in Education. Focusing on a particular environment, Brennan and Resnick (2012) studied young programmers (ages 8-17) through workshops and online project portfolios. To study the effect of age and gender on the development of computational thinking skills, Atmatzidou and Demetriadis (2016) conducted robotics training seminars in Greek junior high (age 15) and high school (age 18) classrooms. Weintrop et al. (2016) focused on creating a taxonomy of computational thinking practices to incorporate in math and science curriculums at the high school level. Lastly, Shute et al. (2017) proposed their own model after a thorough literature review and extensive discussion of the four previously mentioned works.

Despite these different contexts and methods, common themes emerged among each set of researchers' frameworks, speaking to the widespread relevance and applicability of computational thinking. One such common theme is the role of data. Barr and Stephenson (2011) discuss data collection, data analysis, and data representation as part of the core elements and capabilities of computational thinking. Brennan and Resnick (2012) refer to data as a key concept with regard to the storage, retrieval, and updating of values within computing environments. Within their taxonomy, Weintrop et al. (2016) consider five data practices– collecting data, creating data, manipulating data, analyzing data, and visualizing data. Shute et al. (2017) referenced data collection and analysis. Thus, we see that anticipating and planning data wrangling as early and vital components to computational thinking. Additional commonalities appear in Table 1.

Table 1. Additional commonalities with accompanying references.

Themes	References
Decomposition	Barr and Stephenson (2011); Atmatzidou and Demetriadis (2016); Shute et al. (2017)
Algorithms	Barr and Stephenson (2011); Atmatzidou and Demetriadis (2016); Shute et al. (2017)
Parallelization	Barr and Stephenson (2011); Brennan and Resnick (2012)
Simulation	Barr and Stephenson (2011); Weintrop et al. (2016)
Generalization	Atmatzidou and Demetriadis (2016); Shute et al. (2017)
Modularity	Brennan and Resnick (2012); Atmatzidou and Demetriadis (2016); Weintrop et al. (2016)
Debugging	Brennan and Resnick (2012); Shute et al. (2017)
Iteration	Brennan and Resnick (2012); Shute et al. (2017)
Abstraction	All frameworks listed

Theoretical Perspective

When discussing these five computational thinking frameworks, we find utility in thinking about them through the lens of Harel's (2007) DNR determinants. Harel (2007) proposed the DNR system for curriculum development and instruction which consist of premises, determinants, and instructional principles (specifically: Duality, Necessity, and Repeated reasoning). Harel describes the determinants of *mental act* (actions performed in constructing knowledge), *way of understanding* (product or outcome of a mental act), *and way of thinking* (character or feature of a mental act). An example provided in the paper illustrates how proving is a mental act, a proof is a way of understanding, and a proof scheme is a way of thinking. Within the present study, the code that a participant writes to import data would reflect their way of understanding data importing while their imagery for and anticipations of loading data into an environment would reflect their way of thinking about data importing. With this backdrop in mind, we define computational thinking as *the ways of thinking that individuals employ when using computational tools to problem-solve efficiently and effectively.*

Many of the common themes (see Table 1) and our own definition involve the mental act of problem-solving. Barr and Stephenson (2011) also discuss some dispositions and pre-dispositions (e.g., "confidence in dealing with complexity") and classroom culture (e.g., "increased use of computational vocabulary") that would facilitate productive ways of computational thinking. These appear to be outcomes of problem-solving. Similarly, Brennan and Resnick's (2012) computational thinking concepts (e.g., "loops") and computational thinking perspectives (e.g., "questioning") appear to be products of problem-solving. We consider these to be ways of understanding. Our research focuses on the question: **What are ways in which computational thinking appears as part of undergraduate students' thinking as they work with data?**

Methodology

This project is part of a larger study regarding computational thinking for individuals of varying levels of experience and expertise. In this report, we narrow our focus to two tasks designed to probe attributes of computational thinking while participants engage and wrangle with complex data to achieve a defined purpose. While many computational tools exist, we focused on the R programming language and its related software (e.g., R Studio).

Participants

We interviewed three undergraduate students enrolled in a data science course focusing on statistical reasoning and computation, which has a pre-requisite of an introductory R course. The task-based interviews (Goldin, 2000) were recorded over Zoom. Students also submitted their R file in which they worked. We refer to the students as S1, S2, and S3.

Tasks

The two tasks are based on the 2021 American Time Use Survey dataset (ATUS; U.S. Bureau of Labor Statistics, 2022). Participants are provided with: (a) the raw Activity Summary data file from the 2021 ATUS, (b) a simplified data dictionary we created for demographic variables in the data file, (c) an activity lexicon to describe the activity variables in the data file, and (d) task prompts. The ATUS data was selected due to the topic requiring minimal background knowledge and its relevance as real (not simulated) data with practical applications. Both tasks follow a similar format, where we present a data visualization, ask participants regarding their observations of the chart, and then observe the participants as they explore and analyze the data to calculate a particular estimate that is displayed using the ATUS data.

Task 1 prompt. Consider the visualization below (Figure 1) made by the U.S. Bureau of Labor Statistics which tells us the average hours per day spent in selected activities. "Average per day, total" is selected. What does this chart tell you about the average hours per day, total, spent on personal care, including sleep? Would you walk me through how you would recreate this estimate using the atussum_2021.dat file?

Task 2 prompt. We maintain the same format of Task 1, but the context of the problem is modified. In Task 2, both "Average per day, men" and "Average per day, women" are selected and compared in a data visualization. We are interested in how participants address the coding for sex in the dataset and how they adapt or modify their previous solution to Task 1. (We omit the full prompt and figure in this report for brevity.)

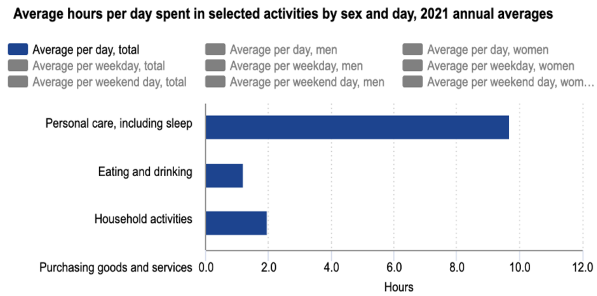

Figure 1: Task 1 Data Visualization from U.S. Bureau of Labor Statistics (truncated for space constraints)

Results

Preliminary analyses are currently underway. Below, we share some excerpts from participants and a couple of potential ways of thinking (pertaining to problem-solving approaches) that could be emerging.

Problem-Solving Approach: Ensure Proper Format of Data

All participants considered how to load data into RStudio. They used different coding environments; S1 used an R script, S2 used an R Markdown file, and S3 opted for an R Notebook file. S1 proceeded to directly write a *read.csv*() command and ran the line of code which provided a preview of the data frame in the console. S2 and S3 both used the data import wizard (GUI) of the RStudio IDE. Despite the two different ways of data importation, all students performed a visual inspection of the data frame; S1 checked post-import while S2 and S3 checked during import. They all commented on what they perceived to be correct format:

S1: So everything is already separated in columns.

S2: Let's see, and I see that there is a heading so "Yes", import that.

S3: The variable column headings are stuck as V1, V2, it goes on. So what I did was switch Heading to "Yes" and this makes it the proper headings.

In contrast to S2 and S3, S1 added in a couple of lines of code using the *complete.cases*() and *unique*() commands. It appeared that S1 had additional data cleaning procedures to apply regardless of the data context, based on the following conversation:

S1: That's usually what I do first, I think.

Interviewer: And can you remind me again, what these 2 lines of code will do?

S1: So, this, the first one, is to make sure all, everything is completed. There's nothing empty, nothing like the rows. There's no one with missing information. And then, the second one is to make sure there's nothing repeated.

Interviewer: Gotcha. And you mentioned that you, you said that this is sort of what you normally do. So are these things that you check every time you code, normally?

S1: I think so, to make sure everything is more clean.

Problem-Solving Approach: Find Efficient Way to Write Code

When the interviewer asked participants to talk about their approach for Task 2, all participants attempted to find the average hours for men and average hours for women using two distinct code blocks. They all expressed a desire for alternative ways to write the code:

S1: Maybe there is way to do female on the same dataset as the male and then write a code to do the rest. I'm just not sure how I'd write that, but I think, the easiest way for me to do [it] would be [to] do separately, but I'm sure there's probably a way to do it together.

S2: If I want to do it together, how would that work? So then I'm thinking, like, okay, I, the very beginning part's the same. Uh, I'm just thinking about when I plot it, how am I gonna be able to distinguish between male and female and how I want it, I guess. Like, your code running is very important but also I think having it be very concise and clean is also pretty important.

S3: So, there probably is a way to do it within one data set, but it's just the first thing that came to mind.

Based on their terminology ("do it together" and "do it within one"), all three participants had considerations of reducing redundancy and increasing efficiency. S2 specifically verbalized the reason for doing so, including anticipation of future analysis (plotting) and what they considered to be best practices of coding.

Discussion

We have outlined some initial ways of thinking that undergraduate students possess and will continue to analyze our data. Our next steps will be to consider the points of commonality with (and departure from) existing frameworks. This would involve additional coding of the data based on these frameworks, and it may be useful to note the frequency and length of time that participants spend in each unique code.

We note some planned future work as this work is part of a larger study. One direction is collecting data from individuals with additional experience and expertise, including graduate students, faculty, and industry professionals. This would allow us to consider a new research question: What similarities and differences do we observe among individuals who self-identify along the novice-expert continuum? We will also consider how an individual may move along the expert-novice continuum in terms of their data-ing and computational thinking skills. Another direction is analyzing participant responses to several additional tasks, including: (a) creating a visualization, (b) brainstorming their own question about the dataset, (c) discuss coding challenges and review code, and (d) verbalize how they would approach a new prompt about the dataset. The purpose of (a) and (b) are to highlight how and what we can communicate with data. We hope to spark a discussion regarding effective problem-solving with (c) and how computational thinking transfers across different contexts with (d).

A key outcome from our study will be understanding the role of computational thinking, in conjunction with interrogating data sources, in reshaping the statistics curriculum. In creating our framework, we will highlight concepts and skills that we might want to foster and support as educators. This would support the generation of hypothetical learning trajectories, which can then inform classroom interventions and expectations while engaged in data exploration, analysis, and communication. In addition, we can reflect and learn from participant feedback regarding the tasks we created, which can inform the way we design future tasks and instruction in statistics, data science, and related courses. A possible extension in the future would be to develop an assessment to measure student development of computational thinking skills.

Intended Questions for Audience

1. What similarities or differences in students' thinking have you noticed in your classroom experience compared to our (preliminary) results?
2. What are your thoughts on or suggestions regarding our planned future work?
3. What suggestions do you have for supporting students in further developing their computational thinking?

References

Aho, A. V. (2012). Computation and computational thinking. *The computer journal*, *55*(7), 832-835.

Atmatzidou, S., & Demetriadis, S. (2016). Advancing students' computational thinking skills through educational robotics: A study on age and gender relevant differences. *Robotics and Autonomous Systems*, *75*, 661-670.

Barr, V., & Stephenson, C. (2011). Bringing computational thinking to K-12: What is involved and what is the role of the computer science education community?. *ACM Inroads*, *2*(1), 48-54.

Brennan, K., & Resnick, M. (2012). New frameworks for studying and assessing the development of computational thinking. In *Proceedings of the 2012 annual meeting of the American educational research association, Vancouver, Canada* (Vol. 1, pp. 25).

Carver, R., Everson, M., Gabrosek, J., Horton, N., Lock, R., Mocko, M., Rossman, A., Roswell, G. H., Velleman, P., Witmer, J., & Wood, B. (2016). Guidelines for assessment and instruction in statistics education (GAISE) college report 2016.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In *Handbook of research design in mathematics and science education* (pp. 517-545).

Harel, G. (2007). The DNR system as a conceptual framework for curriculum development and instruction. In R. A. Lesh, J. J. Kaput, & E. Hamilton (Eds.), *Foundations for the Future in Mathematics Education* (pp. 21). Erlbaum.

Lockwood, E., DeJarnette, A. F., Asay, A., & Thomas, M. (2016). Algorithmic Thinking: An Initial Characterization of Computational Thinking in Mathematics. *North American Chapter of the International Group for the Psychology of Mathematics Education.*

Shute, V. J., Sun, C., & Asbell-Clarke, J. (2017). Demystifying computational thinking. *Educational research review*, *22*, 142-158.

U.S. Bureau of Labor Statistics. (2022). *American Time Use Survey—2021 Microdata Files* [Data file and code book]. https://www.bls.gov/tus/data/datafiles-2021.htm

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of science education and technology*, *25*, 127-147.

Wing, J. M. (2006). Computational thinking. *Communications of the ACM, 49*(3), 33-35.

Abstract Mathematics as Perceived by Pre-Service Mathematics Teachers: "It is Not Gonna Benefit Me in Teaching"

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The secondary-tertiary transition (STT) in mathematics presents university students with multiple cognitive-epistemological, didactical, sociocultural, and affective challenges. This qualitative study explores the perceptions of two pre-service mathematics teachers regarding the relevance of abstract mathematics to their future teaching practices. Using three interviews and eleven reflections, the investigation yields in-depth insights. In addition to highlighting the affective aspects of the STT in mathematics, the study addressed certain difficulties pre-service teachers encountered while dealing with advanced mathematics courses.

Keywords: affect, perception of abstract mathematics, pre-service mathematics teachers, secondary-tertiary transition

The secondary-tertiary transition (STT) in mathematics represents a significant milestone in the academic journey of pre-service mathematics teachers as they progress through university. While these individuals possess a deep passion for mathematics and aspire to become educators, they often encounter formidable challenges as they navigate the shift from concrete, school-level (i.e., empirical) mathematics to the more abstract and theoretical realms of university mathematics. This transition is of paramount importance, as it not only influences their own understanding of the subject but also shapes their ability to effectively teach complex mathematical concepts to future students. This research study seeks to explore the multifaceted challenges that pre-service mathematics teachers encounter during this pivotal transition to abstract mathematics in university settings. By examining these challenges in-depth, I aim to further understand the pertinent issues and identify potential strategies to better prepare pre-service mathematics teachers for the unique demands of abstract mathematics, ultimately enhancing the quality of mathematics instruction in the classroom.

Theoretical Perspectives

The secondary-tertiary transition (STT) in mathematics education has been extensively explored in the literature, with scholars identifying various categories of challenges. De Guzman et al. (1998) categorized these challenges into three main areas: cognitive-epistemological, didactical, and sociocultural, with recent studies adding affective factors as a fourth component (Di Martino & Gregorio, 2019). The nature of mathematics changes as students transition to university, becoming more formal and abstract (Tall, 2008). This shift requires advanced thinking and can lead to cognitive and affective challenges when there is a lack of alignment between the mathematical expectations of secondary and tertiary institutions (Pepin, 2014). Affect in mathematics learning, including beliefs, attitudes, and emotions, plays a pivotal role in shaping students' engagement and performance in the subject (McLeod, 1998). Understanding and addressing the affective aspects of mathematics education is crucial for fostering a positive and productive learning environment.

The Study

The current study focuses on exploring the secondary-tertiary transition experiences in university mathematics from affective perspectives. Five women and/or racially ethnically minoritized women participated in this research who were majoring in mathematics at a large university in the southeastern United States. The semi-structured interviews and reflection videos were used as data collection method focusing on students' experiences in demographic information, secondary school mathematics, decision to major in mathematics, current experiences in the major, vision of mathematics, perceived competence in mathematics, approaches to and assessments of learning mathematics, and mode of belonging.

In this study, I focus on Nia and Bethany, both pre-service mathematics teachers who frequently brought up the disparities between university-level mathematics and secondary school mathematics. This emphasis on their experiences arises from the study's primary focus on the differences between secondary school and university mathematics, which is recognized as a "knowledge gap," or "discontinuities" between two institutional settings (Gueudet, 2008). It is noteworthy to highlight that both Nia and Bethany were first-generation college students who also openly discussed their financial challenges in pursuing a college education. Nia identified as a biracial female with Black and White heritage, while Bethany was a White female student. Nia and Bethany participated in three monthly interviews and recorded eleven weekly reflections in the semester of Spring 2022. Following stages of open, axial, and selective coding, the grounded theory method guided data analysis and enabled the identification of themes that emerged from student responses (Charmaz, 2006).

Preliminary Findings

I explore the specific beliefs held by Nia and Bethany about abstract mathematics, which arose as they reflected on their mathematical skills and the nature of advanced mathematics at the university level. These beliefs contribute to their skepticism about the relevance of advanced math to their future careers. Using several direct quotes, I delve deeper into Nia's and Bethany's views on abstract mathematics, which are rooted in their perceived ability to grasp abstract concepts in proof-based courses. The following excerpt illuminated Nia's first proof experiences in the Introduction to Advanced Mathematics (IAM) class:

> I didn't like that [IAM] course mainly because the proofs weren't really interesting to me. Because it was me memorizing how to do the proofs and really understanding how to prove something in math, which is something that I couldn't kind of wrap my brain around like, 'okay, where do you start?' It wasn't concrete. These are always like, 'do the proof this way' it was like well, 'maybe the proof is this way, maybe it's that way.' So, I have to assume a lot of information. It was pretty hard to keep up. I mean, I ended up with a B in there. But it was not my favorite math course. (Nia, Interview 1, February 2022)

Nia shared her experience in the Introduction to Advanced Mathematics (IAM) course, primarily expressing her dislike for it. She found the memorization of proofs uninteresting and struggled to grasp the abstract nature of mathematical proofs. This aligns with her preference for mathematics that involves practical application and processes. Nia's frustration stemmed from not being able to determine the correctness of her proofs, as she lacked clear expectations and the means to verify her work:

> When I don't know the expectations or when I am not able to check my work, that's when I am unsure, as I will be 'is this right thing?' is my problem with the proofs as well. Well, I took the [Introduction to] Advanced Math, and I couldn't determine for myself, like 'is this right, is this the answer?' I don't understand how I know if I'm writing this proof right and if it's an accepted answer. (Nia, Interview 1, February 2022)

Similarly, Bethany had her own perspective on proof-based classes. She believed that exam questions should closely resemble the material covered in class, and when they didn't, she considered the course difficult. Bethany didn't enjoy proofs, especially in courses like Abstract Algebra and Analysis, where the abstract nature of the subject made it challenging for her to connect with the content. She believed that her future teaching career wouldn't require knowledge of such abstract proofs and preferred mathematics with practical applications:

> I don't like the proofs as much. Well, I guess it depends on what it's about, because I think groups in Geometry were really fun because I could do it. It was really tangible, I could understand angles and shapes. But in Abstract Algebra and Analysis, I definitely struggled more because it was more abstract. I think for teaching, like, even in high school, I don't need to know proofs for abstract things. It's not gonna benefit me in teaching, really. So I don't think it'll be a problem in that context. (Bethany, Interview 2, March 2022)

Bethany's responses revealed a mix of emotions, including both satisfaction and aversion, when she contemplated abstract mathematics. She provided several explanations for her lack of enthusiasm towards mathematical proofs. Initially, Bethany linked this sentiment to the inherent abstractness of proofs, where the concepts lacked a tangible quality. In her first interview, she also conveyed that the majority of her university mathematics courses were formal and did not align with her aspirations in teaching. Bethany's perspective on mathematics in her initial and subsequent interviews remained consistent. Specifically, she conveyed a preference for mathematics with practical applications, believing it would have greater relevance to her future career in teaching. Bethany also mentioned her preference for what she termed a "straightforward course," where lectures were followed by problem-solving sessions, and exams focused on class exercises. In contrast, she perceived proof-based classes as less straightforward because they required students to think independently rather than follow step-by-step instructions:

> I guess the courses are just lectures, and then you do the problems. Like in Linear Algebra, you just write on the doc camera, doing the problems, and then you'd go home and do the same problems. The tests were about the exercise problems from class. So, I thought that was pretty straightforward, versus proof-based classes, they don't tell you how to do the problems. You have to figure it out. In Calc II, we also had some group projects which required more thinking on our part and not just doing what we were told to do. (Bethany, Interview 3, April 2022)

Bethany's perception of proof-based classes as challenging due to their requirement for independent thinking rather than following explicit instructions reflected her initial beliefs. These beliefs highlighted a mismatch between in Bethany's expectations of upper-division mathematics courses and the actual classroom experience in secondary school teaching. Consequently, both Nia and Bethany viewed abstract mathematics as less relevant to their career aspirations. This misalignment led to their disengagement from proof-based courses. They did not see a connection between their mathematical abilities and the demands of these courses. Additionally, both students expressed little interest in pursuing careers that involve abstract mathematics, possibly due to the perceived gap in their mathematical background and competency required for such careers.

Significance and Implications

When analyzing Nia's and Bethany's cases, several noteworthy patterns surfaced in their experiences with university-level mathematics, with a specific focus on abstract mathematics. Both Bethany and Nia raised doubts about the relevance of abstract mathematics, as they believe it is disconnected from their future teaching practices (Wasserman et al., 2019). Some of their concerns stemmed from instructional practices encountered in university, which predominantly adhered to traditional lecture formats. Additionally, their doubts were further fueled by the perceived difficulty in discerning the validity of mathematical proofs, shedding light on challenges related to mathematical competence. Furthermore, Nia and Bethany commented on the abstract nature of mathematics at the university level, contrasting it with the more tangible mathematics of secondary education, which also influenced their attitudes toward learning abstract mathematics. Schoenfeld's assertion that "formal mathematics has little or nothing to do with real thinking or problem solving" (1985, p. 43) seemed to resonate with the beliefs held by Bethany and Nia about the nature of mathematics. Their reactions to the changing nature of mathematics towards upper-division courses corroborated the findings of Geisler and Rolka (2021), which highlighted that students anticipate university-level mathematics to align with the practical and applied aspects of the discipline. This observation underscores the disparities in students' perceptions of mathematics, particularly in the context of secondary school mathematics and university-level mathematics courses (Rach & Heinze, 2017).

Solomon and Croft (2016) proposed that when undergraduate students feel a challenge to their sense of ownership of mathematics, they tend to disengage from the subject. This disengagement often stems from mathematics either not meeting their high expectations for success or failing to reveal its inner workings while imposing strict adherence to its rules. The experiences of Nia and Bethany, who both excelled in high school mathematics and currently perceive themselves as rule followers in college mathematics, align with these ideas.

High school preparation significantly impacts students' experiences in university mathematics courses. Nia and Bethany, who are first-generation students appeared to be less prepared and struggling with advanced mathematics courses. Both students highlighted their difficulties in keeping up with their peers, attributing it to their limited background in advanced mathematics or the instructional style and expressing a need for additional resources to grasp the course materials effectively. Future research should investigate the alignment between pre-service mathematics teachers' teaching goals and the content of the advanced mathematics courses, with the purpose of tailoring these classes to their specific needs. Furthermore, future research should investigate the relationship between the experiences of first-generation college students and their views on abstract mathematics. Investigating the requisite support systems necessary for their success and their development into proficient mathematics educators should also be a priority.

References

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative research*. SAGE.

De Guzman, M., Hodgson, B., Robert, A., & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. *Proceedings of the International Congress of Mathematicians* (Vol. 3, pp. 747–762). Documenta Mathematica.

Di Martino, P., & Gregorio, F. (2019). The mathematical crisis in secondary – tertiary transition. *International Journal of Science and Mathematics Education*, *17*(4), 825–843. https://doi.org/https://doi.org/10.1007/s10763-018-9894-y

Geisler, S., & Rolka, K. (2021). "That wasn't the math I wanted to do!"—Students' beliefs during the transition from school to university mathematics. *International Journal of Science and Mathematics Education*, *19*, 599–618. https://doi.org/10.1007/s10763-020-10072-y

Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, *67*(3), 237–254.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In Grows, D. A. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). Macmillan Publishing Company.

Pepin, B. (2014). Using the construct of the didactic contract to understand student transition into university mathematics education. *Policy Futures in Education*, *12*(5), 646–657. https://doi.org/10.2304/pfie.2014.12.5.646

Rach, S., & Heinze, A. (2017). The transition from school to university in mathematics: Which influence do school-related variables have? *International Journal of Science and Mathematics Education*, *15*(7), 1343–1363. https://doi.org/10.1007/s10763-016-9744-8

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.

Solomon, Y., & Croft, T. (2016). Understanding undergraduate disengagement from mathematics: Addressing alienation. *International Journal of Educational Research*, *79*, 267–276. https://doi.org/10.1016/j.ijer.2015.10.006

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5–24. https://doi.org/10.1007/BF03217474

Wasserman, N. H., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*, *22*(4), 379–406. https://doi.org/10.1007/s10857-019-09431-6

The Meta-Narratives about Function Conveyed by a Commonly Used Multivariable Calculus Textbook

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Function is a unifying, cross-curricular theme that plays a central role in nearly every subdiscipline of mathematics. Yet, this cultural meta-narrative that is widely accepted by mathematicians is not one students readily adopt. In this report, I share a preliminary analysis of the stories told about three different types of multivariable functions in a commonly used calculus textbook. In doing so, I juxtapose these stories and consider how this textbook—a cultural artifact of the discipline of mathematics—might support or hinder the transmittal of desirable cultural meta-narratives about the role of function. Ultimately, I find that real-valued functions are consistently positioned as functions, while the function properties of both types of vector-valued functions are de-emphasized. As only one of the three function types are positioned as functions, this suggests that the stories in the textbook could hinder students' adoption of the meta-narrative of function as a unifying mathematical theme.

Keywords: Function, Multivariable Calculus, Curriculum as Story, Textbook Analysis.

Function is a crucial recurring theme across the story of the K-16 mathematics curriculum. (e.g., CCSSM, 2010; Zorn, 2015). Some mathematics education researchers even go as far as to contend that the concept of function is "the single most important mathematical concept studied from kindergarten to graduate school" (Harel & Dubinsky, 1992, p. vii). Mathematicians have also spoken about the centrality of the function concept. For instance, Gowers et al. (2008) stated how "One of the most basic activities of mathematics is to take a mathematical object and transform it into another one" (p. 10). In this sense, function is a unifying, cross-curricular theme that occurs across nearly every subdiscipline of mathematics.

This cultural meta-narrative about the role of function has endured for decades. Yet, this is not a narrative that students readily adopt, as a large body of research suggests (e.g., Martínez-Planell & Trigueros, 2021; Melhuish, 2020; Zandieh et al., 2017). While there are several reasons for this, in this preliminary report, I examine the role that the stories told in a commonly adopted calculus textbook (Stewart et al., 2021)—a cultural artifact of the discipline of mathematics (Plut & Plesic, 2003)—might play in transmitting or failing to transmit this cultural meta-narrative to students. Specifically, I examine the stories told about the three different types of multivariable functions featured in multivariable calculus (MVC)—parametric, vector-valued functions; multivariable, real-valued functions; and vector fields—in the chapter they are first introduced. In this arts-based textbook analysis, I lean into the metaphor of curriculum as story (Dietiker, 2015) and treat each of these function types as mathematical characters. I investigate two questions: *(1) How are these characters portrayed in the chapters they are introduced and how do these portrayals compare with one another? (2) What meta-narratives about function(s) are conveyed collectively by these character introductions?*

Background

MVC textbook analyses that focus on the concept of function are admittedly limited. Notably, McGee et al. (2015) observed that in commonly used calculus textbooks, explicit conversations linking representations of single-variable and multivariable functions are virtually nonexistent, even though students may not spontaneously draw these connections on their own (Martínez-Planell & Gaisman, 2012). More generally, Harel (2021) noted that traditional MVC textbooks tend to introduce key concepts without proper motivation and introduce computational shortcuts prematurely. While other analyses attend to the meta-narratives conveyed by textbooks, they tend to be with an eye to a particular topic, like line integrals (Dray & Manogue, 2023).

Textbooks as Cultural Artifacts & Curriculum as Story

In undergraduate education, many instructors use textbooks as a primary curricular guide (Fraser & Bosanquet, 2006). So, even though mathematics students do not read them cover to cover (Weinberg et al., 2012), the stories conveyed (or not) in textbooks play a powerful role in the reproduction of mathematical culture and specifically the meta-narratives that are valued by the discipline (Plut & Plesic, 2003). By a *meta-narrative*, I mean a "cultural narrative schema which orders and explains knowledge and experience" (Stephens & McCallum, 1998, p. 6). In other words, meta-narratives are narratives that recur (implicitly or explicitly) across cultural artifacts which individuals subsequently leverage to frame and explain their past and subsequent experiences. For example, the meta-narrative that "good always conquers evil" is common across children's books. That said, as Brown (2022) demonstrated, the meta-narratives perpetuated by undergraduate mathematics textbooks do not always align with those held by members of the discipline of mathematics. In these cases, textbooks might actually convey messages that serve a counterproductive role toward enculturating students into the discipline.

I adopt the perspective that mathematics curriculum (and therefore textbooks) can be conceptualized as a story (Dietiker, 2015; Gadanidis & Hoogland, 2003). Like a story, curricula features (mathematical) characters inhabiting settings and engaging in actions that constitute the plot. A curriculum also features literary themes and morals in the same way a good story might. As textbooks are used for enculturating students into the discipline of mathematics, the stories contained within them should not merely be an afterthought. After all, the stories we are told and then re-tell influence how we organize our experiences (Clark & Rossiter, 2008). Indeed, the power of story has been acknowledged repeatedly by mathematicians (Doxiadis & Mazur, 2012) as well as mathematics educators (Burton, 1999), but analyses of the stories conveyed in textbooks from a literary and aesthetic angle remain rare (e.g., Dietiker & Richman, 2021).

I employ Dietiker's (2015) framework for analyzing mathematical stories based on the narratological work of Bal (2017). Specifically, I attend to the characters, action, settings, and plot. Mathematical characters are those concepts that get objectified in the text, including functions, numbers, etc. They are often imbued with *character traits* (e.g., a function may be 1-1, even, etc.). I attend specifically to which mathematical characters are positioned as *protagonists* and *side characters*, as well as the *relationships* between characters. Mathematical action refers to moments where a new character is created or introduced as well as when characters are manipulated mathematically (e.g., the characters "2" and "3," might be added to form a new character, "5"). The setting of mathematical stories includes different representations (tables, graphs, etc.) as well as physical contexts. Finally, the mathematical plot includes a reader's aesthetic reactions to the unfolding story as they attempt to discern its structure.

Data & Methods

I analyzed the character introductions of three different multivariable function types in Stewart et al.'s (2021) *Calculus*, a commonly adopted textbook in U.S. undergraduate classrooms for teaching the calculus series (Mesa, 2010; Mkhatshwa, 2022). By "character introduction", I mean the first section within the unit in which each function type is introduced (e.g., 13.1). Hereafter, I refer to these sections as "chapters" as a reminder that I am conceptualizing the textbook as a literary story. The three chapters analyzed were: (1) 13.1: Vector Functions and Space Curves, within the Vector Functions unit, in which parametric, vector-valued functions are introduced; (2) 14.1: Functions of Several Variables, within the Partial Derivatives unit, in which multivariable, real-valued functions are introduced; and (3) 16.1: Vector Fields, within the Vector Calculus unit, in which vector fields are introduced. I also included the paragraph overview of each unit because they immediately precede each chapter and briefly introduce each function type.

Methods

I first read the story of each chapter three times while making marginal notes in the same way someone might while trying to interpret literary fiction. Following an arts-based approach (Leavy, 2018; McNiff, 2018), I did not restrict the structure or modality of these notes. Instead, I focused on recording my thoughts, feelings, and observations in whatever modality was most appropriate. Sometimes, I would write a comment or pose a question; other times, I would draw a sketch or write a quick poem to artistically convey my reactions as a reader.

To add focus to my analysis, I attended primarily to the story elements outlined in Dietiker's (2015) framework (character, action, setting, plot). However, I did not shy away from attending to other dimensions that were salient in my reading of each story, especially those related to my aesthetic interpretations of how the story was told or the overall moral of each story. My intent was to lean into my subjectivity (Tremaine & Hagman, 2023) and unique positionality as a reader who is a disciplinary expert and has taught MVC multiple times. This involved engaging my full senses and emotions: I did not restrict myself to purely logical analysis of structure, mathematical content, etc. I allowed myself to wonder about what came next, why the textbook was or was not introducing certain elements, and to make aesthetic judgements.

Following this open-ended analysis, I re-read and reflected across my marginal notes and conducted secondary analyses to follow up on recurring themes and questions that had appeared across all the character introductions. In this report, I share the preliminary results from two such analyses: (1) a "nickname analysis" of the changing names and phrases used to refer to each character (e.g., function, equation, curve) paired with (2) a plot structure analysis of each story.

Story Analyses

Parametric, Vector-Valued Functions (PVVFs)

As soon as the unit introduction, the purpose of introducing PVVFs is clearly signposted: "We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space" (p. 927). While PVVFs are introduced formally as "a function whose domain is a set of real numbers and whose range is a set of vectors" (p. 928), they are soon relegated to a side character used primarily to introduce the main character of space curves. In fact, once the relationship between these characters has been established, the word function only appears two more times. Additionally, across the entire chapter, "function" is mentioned only about half as frequently as "curves" and about the same number of times as (parametric) "equation". The set theoretic functional properties (e.g., domain and range) of PVVFs are effectively backgrounded in favor of sketching curves and writing parametric equations in the subsequent examples.

Ultimately, the character of PVVFs appears to have been introduced as a technicality: even before space curves are introduced, the (unboxed) function definition is followed quickly by the introduction of component functions. After this, the primary action involving PVVFs is to quickly reduce them to their component single-variable functions for further computation. This backgrounding of PVVFs as functions sends a strong message about the defining traits of such functions: they decompose into multiple other single-variable functions and *these* functions are the most important.

Multivariable, Real-Valued Functions (MRVFs)

The character of MRVFs, on the other hand, is treated as a function from beginning to end. Unlike both other chapters analyzed, this one starts by outlining four points of view for studying MRVFs: verbally, numerically, algebraically, and visually (by a graph or level curves). This phrasing positions the words, tables, formulas, graphs, and level curves as in service of MRVFs, rather than the other way around (as was the case with PVVFs and geometric space curves). Additionally, this is the same framing used to introduce the function concept in the first (single-variable) chapter of the textbook, further affirming that MRVFs are, by their nature, functions.

Unlike with PVVFs, the formal set-theoretic definition of MRVFs of two variables is boxed and even "domain" and "range" are bolded, further emphasizing their functional characteristics. Next, the independent and dependent variables of a MRVF are defined and followed immediately by an aside to the reader that draws a comparison between single- and multivariable RVFs: "Compare this with the notation y = f(x) for functions of a single variable" (p. 927). Interleaved throughout the remaining pages are recurring explicit references to the domain and range in text and in the examples. Finally, the chapter ends by introducing the formal definition of MRVFs of three or more variables, returning to highlight that these characters are functions.

"Function" is far and away the most common name used to refer to MRVFs (87 instances). This use of this name does not subside, even as "level surfaces" (51 instances) and geometric names (such as "graph" or "surface", 44 instances) are introduced as alternatives.

Vector Fields

The caption for the image featured in the vector calculus unit introduction immediately states how "vector fields can be used to model such diverse phenomena as gravity, electricity and magnetism, and fluid flow" (p. 1161). Nearby, the first sentence of the main text reads, "In this chapter we study the calculus of vector fields (These are functions that assign vectors to points in space)" (p. 1161). The use of parentheses to convey the technical details persists throughout this paragraph and feels almost like the narrator is whispering to the reader, implicitly conveying that perhaps the function definition of vector fields is a sidenote.

The emphasis on modeling realistic phenomena carries into the beginning of the chapter introducing vector fields, which features nearly a full page of four example velocity fields depicted visually alongside written interpretations of what meaning the plotted vectors convey. Afterwards, the general definition of vector field is introduced (unboxed) as "a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3)" (p. 1163) followed by separate boxed definitions of a vector field on $\mathbb{R}^2$ and on $\mathbb{R}^3$. Curiously, the language of domain and range is less explicit in the boxed definitions: "Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function **F** that assigns to each point (x,y) in D a two-dimensional vector **F**(x,y)" (p. 1163). After each definition, component functions are introduced, with a brief mention that the reader should already be familiar with these characters from reading about them alongside PVVFs.

After two examples concerning how to sketch some specific vector fields, the story shifts primarily to what can best be described as extended vignettes of notable types of vector fields (e.g., a gravitational field, electric field, gradient fields, etc.). Aside from using function and vector notation as a symbolic necessity, once technology is introduced as a way of plotting vector fields, the functional aspects of vector fields go almost entirely unmentioned. Indeed, there are only 5 explicit references to vector fields as "functions" across this entire chapter, most of which occur near the formal definitions.

Discussion

These story analyses emphasize key differences between the character introductions of these three types of multivariable functions in one commonly used MVC textbook. While real-valued functions are clearly positioned as a main character and repeatedly portrayed as a type of function, the same cannot be said about both types of vector-valued functions. The difference is most extreme in the case of parametric, vector-valued functions, which play the role of a secondary character meant to introduce space curves and whose function character traits are quickly backgrounded. Vector fields remain the main character of their story; however, their functional character traits are mostly de-emphasized across a series of vignettes that focus on introducing and interpreting the physical meanings of specific vector fields (e.g., velocity fields, gravitational fields, etc.). Considered collectively, these stories perpetuate the meta-narrative that the most important type of multivariable functions in MVC are the real-valued ones. Vector-valued ones are also functions—at least formally—but this is portrayed as a mere technicality and not a fact used frequently in practice. Readers (our students) are more likely to come away with alternative messages about vector-valued functions that are foregrounded in these stories, such as how readily they can be decomposed into component functions ("the truly important functions") for the purposes of further computation. Or, in the case of vector fields, the importance of physical interpretation of vector outputs.

Whether these alternative messages are consistent with the curricular goals of MVC is beyond the scope of this preliminary report. However, it is worth noting that the dominance of such messages may hinder students from picking up on the cultural importance of function as a unifying, cross-curricular theme that includes both RVFs and VVFs. Rather, readers may be led to believe the meta-narrative that function is an unnecessary boondoggle, "extra", or formalism that only plays a passing role in the overarching story of MVC and, more generally, mathematics. Already, research suggests MVC students often need support in making sense of MRVFs as a generalization of single-variable real-valued functions. The same may well be true for making sense of vector-valued functions as multivariable functions, *especially* if there are even fewer textbook supports for making this generalization.

There is emerging empirical evidence suggesting students benefit from reasoning about different function types as instantiations of the same overarching function concept (Melhuish et al., 2020; Zandieh et al., 2017); however, none of these studies are in the context of MVC. I hypothesize that MVC students may similarly benefit from recognizing PVVFs, MRVFs, and vector fields as functions, as this would better position them to recognize structural similarity across the different types of calculus required for each function type (e.g., the Chop, Multiply, Add conceptual pattern for integration, Dray & Manogue, 2023). Simultaneously, I recognize that while an overarching story for MVC centered on function may support students' enculturation into the discipline of mathematics, it may have the opposite effect on students' enculturation into other STEM disciplines, given the differing ways that scientists and mathematicians conceptualize functions (e.g., Dray & Manogue, 2004). This suggests further cross-disciplinary research collaboration is needed to ensure that our MVC curricula feature coherent stories which set up our STEM students to be successful, regardless of their discipline.

References

Bal, M. (2017). *Narratology: Introduction to the theory of narrative* (4th ed.). University of Toronto Press.

Brown, S. A. (2022). The staging of proof by contradiction in texts: An exploration of disciplinary artifacts. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 752–759).

Burton, L. (1999). The implications of a narrative approach to the learning of mathematics. In B. Leone (Ed.), *Learning from mathematics: From hierarchies to networks* (pp. 21–35). Falmer Press.

Clark, M. C., & Rossiter, M. (2008). Narrative learning in adulthood. *New Directions for Adult and Continuing Education*, 119, 61–70. https://doi.org/10.1002/ace.306

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, 90, 285–302. https://doi.org/10.1007/s10649-015-9627-x

Dietiker, L., & Richman, A. S. (2021). How textbooks can promote inquiry: Using a narrative framework to investigate the design of mathematical content in a lesson. *Journal for Research in Mathematics Education*, 52(3), 301–331. https://doi.org/10.5951/jresematheduc-2020-0318

Doxiadis, A., & Mazur, B. (Eds.). (2012). *Circles disturbed: The interplay of mathematics and narrative*. Princeton University Press.

Dray, T., & Manogue, C. A. (2004). Bridging the gap between mathematics and the physical sciences. In D. Smith & E. Swanson (Eds.), *Preparing future science and mathematics teachers* (pp. 39–41). https://bridge.math.oregonstate.edu/papers/bridge.pdf

Dray, T., & Manogue, C. A. (2023). Vector line integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education*, 9, 95–120. https://doi.org/10.1007/s40753-022-00206-8

Fraser, S. P., & Bosanquet, A. M. (2006). The curriculum? That's just a unit outline, isn't it? *Studies in Higher Education*, 31(3), 269–284. https://doi.org/10.1080/03075070600680521

Gadanidis, G., & Hoogland, C. (2003). The aesthetic in mathematics as story. *Canadian Journal of Science, Mathematics, and Technology Education*, 3(4), 487–498. https://doi.org/10.1080/14926150309556584

Gowers, T., Barrow-Green, J., & Leader, I. (2008). Introduction. In T. Gowers, J. Barrow-Green, & I. Leader (Eds.), *The Princeton companion to mathematics* (pp. 1–76). Princeton University Press.

Harel, G. (2021). The learning and teaching of multivariable calculus: A DNR perspective. *ZDM*, 53, 709–721. https://doi.org/10.1007/s11858-021-01223-8

Harel, G., & Dubinsky (Eds.). (1992). The concept of function: Aspects of epistemology and pedagogy. Mathematical Association of America.

Leavy, P. (2018). Introduction to arts-based research. In P. Leavy (Ed.), *Handbook of arts-based research* (pp. 3–21). The Guildford Press.

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, 81(3), 365–384. https://doi.org/10.1007/s10649-012-9408-8

Martínez-Planell, R., & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM*, 53, 695–707. https://doi.org/10.1007/s11858-021-01233-6

McGee, D., Moore-Russo, D., & Martinez-Planell, R. (2015). Making implicit multivariable calculus representations explicit: A clinical study. *Problems, Resources, and Issues in*

Mathematics Undergraduate Studies, 25(6), 529–541. https://doi.org/10.1080/10511970.2015.1038858
McNiff, S. (2018). Philosophical and practical foundations of artistic inquiry: Creating paradigms, methods, and presentations based in art. In P. Leavy (Ed.), *Handbook of Arts-Based Research* (pp. 22–36). Guilford Press.
Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior*, 60, 1–16. https://doi.org/10.1016/j.jmathb.2020.100806
Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *CBMS Issues in Mathematics Education*, 16, 235–261. https://doi.org/10.1090/cbmath/016/09
Mkhatshwa, T. P. (2022). Quantitative and covariational reasoning opportunities provided by calculus textbooks: The case of the derivative. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2129497
National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics.
Plut, D., & Pesic, J. (2003). Toward a Vygotskian theory of textbook. *Psihologija*, 36(4), 501–515. https://doi.org/10.2298/PSI0304501P
Stephens, J., & McCallum, R. (1998). *Retelling stories, framing culture: Traditional stories and metanarratives in children's literature*. Garland Publishing.
Stewart, J., Clegg, D., & Saleem, W. (2020). *Calculus* (9th ed.). Cengage Learning.
Tremaine, R., & Hagman, J. E. (2023). Creativity and subjectivity: Two researchers' poetic re-presentations of critical student perspectives of mathematical sciences. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1117–1124).
Weinberg, A., Wiesner, E., Benesh, B., & Boester, T. (2012). Undergraduate students' self-reported use of mathematics textbooks. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 22(2), 152–175. https://doi.org/10.1080/10511970.2010.509336
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, 95(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0
Zorn, P. (2015). 2015 CUPM curriculum guide to majors in the mathematical sciences. Mathematical Association of America. https://www.maa.org/sites/default/files/CUPM%20Guide.pdf

Student Use of Series Expansions as an Approximation Technique in Physics Modeling Tasks

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As part of a larger project to investigate student use of mathematics in upper-division physics courses, we have examined how students use series expansions as a means of approximating and simplifying complicated expressions in theory-oriented physics courses. Student responses were collected on pre- and post-instruction written tasks in which students were prompted to use a series to approximate an expression arising from a problem in electricity and magnetism. Despite prior experience with series in calculus and physics, students struggled to determine appropriate quantities in which to expand and did not attend to units or the convergence of their series. While student success rates improved after targeted instruction, many students expanded in quantities that were neither dimensionless nor small, and thus unproductive for modeling.

Keywords: calculus, series, physics, applications

This work is part of a project to investigate student use of mathematics in upper-division physics courses. A key lens has been the study of how concepts studied in calculus and other mathematics courses are implemented in modeling tasks in physics, and how students perceive and take up these concepts given the different disciplinary context. For this study, the research questions are: To what extent are upper-division physics students successful in using series to model simple physical systems? What aspects of this task are challenging? What implications does this have for instruction, both in physics and in mathematics?

Background and Previous Research

Power series are commonly covered in an introductory calculus sequence. In physics, Taylor and Maclaurin series are used frequently as modeling tools, for simplifying analytical solutions and constructing approximations. As part of a project to study student use of mathematics in upper-division physics and support curricular interventions, we have investigated the reasoning used by students in using series to manipulate physics expressions and use mathematical tools for modeling physical systems, including the use of series as approximation tools.

While often physics students will be asked to perform a full Taylor series expansion, in many cases it will suffice to use a binomial expansion. A typical usage is to generate an exact expression from a physical and mathematical model, then to replace a portion of this expression with a power series, in powers of a small, dimensionless ratio of two physical quantities. For example, for the expression for the electric field of an electric dipole (a pair of charges with equal absolute value but opposite sign), shown in Figure 1, the task directs students to consider points far from the dipole. The statement 'far from the dipole' can be interpreted to mean that the distance y is considerably greater than the distance d between the two charges, and thus mathematized as $y>>d$. Students can perform algebraic manipulation to produce an expression proportional to $(1-d/2y)^{-2}$, which is of the form $(1+x)^p$ with $|x|<1$ as stated in the course text (Boas, 2006). With $|x|<1$ the series will converge, and higher-order terms fall off quickly, allowing the complicated expression to be replaced with a relatively simple polynomial.

Previous research on student learning of series comes from both the mathematics education and physics education research (PER) communities. RUME studies have tended to focus on conceptual understanding and visualization. Alcock and Simpson (2004) focused on use of visualization of series. Martin and Oehrtman (2011) investigated the concept of convergence in terms of metaphors, including a part / whole metaphor and a cutting metaphor. Champney and Kuo (2012) focused on visual images of the Taylor series as an approximation.

PER studies have often focused instead on application to problems, Smith et al. (2013) reported that students lacked fluency in both creating and using series, despite prior exposure in mathematics and physics as well as evidence of conceptual understanding of the graphical meaning of series terms. Wilcox et al. (2013) investigated Taylor series as a modeling tool in upper-division classical mechanics. From this analysis they developed the ACER framework for student use of mathematics in physics, named for its four components: activation of a tool, construction of a model, execution of the relevant mathematics, and reflection on the result. Student difficulties in each of the four phases were described, including those with expansions about a point other than zero and a lack of meaningful reflection. In both PER studies, students had to make choices about the series expansion, but in neither case was it necessary to manipulate an expression to produce a small, dimensionless ratio with which to expand.

Context and Methods

Context for the Research

This work has taken place in the context of a course on mathematical methods for physics ("Math Methods"), a required course in many physics departments that is generally intended to prepare students for the mathematics encountered in upper-division theory courses. Course prerequisites include three semesters of calculus and two semesters of introductory physics; all students would have encountered series and the relevant physics concepts. For this report, we describe data that were collected in twelve in-person sections of the course from different semesters over the period 2009 to 2022, all taught by the same instructor at a comprehensive university serving a diverse student population. Instruction focused on mathematical ideas used as modeling tools in physics and included relevant post-tests on course quizzes. Total enrollment over these semesters is approximately 160 students and the performance was roughly similar across sections while accounting for fluctuations. Approximate demographics for these semesters were: 78% male, 22% female, 45% white, 33% Latino, 18% Asian.

Instruction in Math Methods was interactive and used instructional materials developed in response to both prior research and results from early versions of the ungraded quiz. Students in small groups worked on guided worksheets. Tasks in the worksheets included qualitative questions about the first three terms in a Taylor series for several points on an arbitrary graph (adapted from Smith et al., 2013), followed by procedural tasks including evaluation of terms of series for sine and cosine. Students were then asked to interpret their results by comparing their series for sine and cosine to the original functions in terms of values and behavior (slope, curvature). Student groups then revisited the dipole task and executed the expansion, with guiding questions that directed their attention to the units and magnitude of the expansion variable and what implications this had for series convergence and meaning. The purpose of this report is to examine aspects of student reasoning, but the results also have implications for the instructional materials that are addressed in the discussion section.

Method

For this study, we present the results of written responses to tasks administered as part of course assessments, including ungraded and graded quizzes. Each task involved the use of series to derive an approximation from an exact analytical result. In all cases, the generic form of the

Consider the electric field at points along the axis of an electric dipole, $E = \frac{+kQ}{\left(y - \frac{d}{2}\right)^2} + \frac{-kQ}{\left(y + \frac{d}{2}\right)^2}$ for a point far from the dipole, with y much larger than d.

The first term of the expression is $\frac{+kQ}{\left(y - \frac{d}{2}\right)^2}$. Use the binomial expansion (given) to construct the first three terms of a series for this term given that $y >> d$. Explicitly identify what x and p you are using. What simplification is allowed by the fact that $y >> d$?

Figure 1. Task from initial ungraded quiz. A diagram was included (not shown due to space considerations). The problem statement in the quiz included the expression for the binomial expansion from the course text (Boas 2006), with the form $(1+x)^p$. *Graded quizzes were similar but included different contexts (E field from charged ring, potential from charged disk).*

terms of a binomial series was available to students, either as explicit information included in the problem statement, or as a supplemental sheet of equations (e.g., in given information on a quiz).

The initial task for the ungraded quiz involved a series expansion for an expression for the electric field of a dipole for points along the axis containing the two charges (Fig. 1). This task is often encountered in introductory electricity and magnetism as an example and/or textbook problem (Halliday et al., 2011), and the approximate field proportional to the inverse cube of distance is familiar to experienced physicists. Graded quizzes after instruction included other tasks of similar structure (expression given, perform expansion to generate approximation) that are not shown due to space limitations. These tasks included four examples from electricity and magnetism, involving expressions for the electric field or potential for an extended object with a distributed charge (e.g., a ring or disk) or for a simple model of a crystal lattice.

Written responses were collected on an ungraded quiz (dipole, Figure 1, $N = 156$) on the first day of instruction in the Math Methods course over twelve semesters. Additional responses were collected on the first graded quiz (other tasks, not shown, $N = 161$). Student responses were coded by the lead researcher and student assistants using qualitative inductive analysis (Otero and Harlow, 2009); the initial data were coded without *a priori* categories, based on correctness and the overall approach taken. Initial codes were then refined, informed by an analysis of the steps in the task. Categories are described in results, below.

Results

Categories coded for included: mathematization of relative distances (e.g., $y>>d$), choice of expansion variable, dimensionality and magnitude of expansion variable, identification of exponent, execution of expansion, and interpretation of results (see Table I). For each category, specific choices in the response were recorded, e.g., whether to expand in powers of (y-d/2) or to factor the expression and expand in powers of $d/2y$ or $2y/d$. For example, expanding in (y-d/2) would be coded as neither small nor dimensionless. In contrast, $2y/d$ is coded as dimensionless but is not small compared to one. Nearly all codes of 'small' were also coded 'dimensionless.'

Responses on the initial ungraded quiz, before instruction in the Math Methods course, illustrated the difficulty of the tasks but provided relatively little insight into student reasoning. Very few students produced complete and correct responses to the task. A third of the responses

Table 1. Categories coded for in student responses, and examples of the coding. Examples are drawn from post-test questions involving a ring or disk.

Step	Examples
Mathematize	"far from the ring so z>>R"
Choose expansion variable	
dimensionless	"$x=R^2/z^2$"
small compared to one	"use z/L, <<1 b/c L>>z"
Identify exponent in binomial	"p=-3/2, negative brings to numerator"
Execute expansion	[omitted due to space constraints]
Interpret results physically	"we can cut out [terms] so →kQ/z which is a point charge from far away"

were either blank or nonproductive and uncodable, despite ample time to respond. Under 10% of responses transformed the expression into a form suitable for the expansion, i.e., $(1+x)^p$ with x dimensionless and $0<|x|<1$. Students who identified such a dimensionless quantity frequently neglected to attend to convergence, choosing a series with powers of a quantity large compared to one (e.g., y/d). Around 3% of students expanded in a dimensionless quantity small compared to one, one that would lead to a series that converges.

Student responses to the question about the significance of $y>>d$ further suggest the importance of the interplay between mathematical calculation and more qualitative physical reasoning. A large group of coded responses included answers stating incorrectly that the electric field would be zero. This category included two subcategories. One set of codes (~20% of students) included assertions, without calculation, that the electric field is zero because y is large. The second group of codes (~30%) included an algebraic calculation or symbolic argument, stating explicitly that $y>>d$ allows one to set d=0 or set $y-d/2=y$, in which case the expression for E reduces to zero. It is not clear whether the students in the second group force the mathematics to support their intuition that E=0 or simply perform the calculations and reach that result. The former might be explained with a dual-process model (Kryjevskaia et al., 2021).

Table 2 compares results before and after instruction. The post-test quizzes involved four tasks different from the dipole, all involving electricity and magnetism. The results are shown as a single combined entry because there was very little difference in response patterns. (For example, the fraction of students choosing a dimensionless x with which to expand was 72% in tasks involving electric potential of a disk, 78% in tasks involving electric field of a ring, and 76% in tasks involving electric forces in a crystal lattice.)

Table 2. Responses to series approximation tasks before and after tutorial instruction. In the table, x and 'exponent' refer to the binomial $(1+x)^p$ from the course textbook (Boas, 2006).

Instruction	Appropriate quantity x	x unitless	($\|x\|<1$)	Appropriate exponent
Pre (N = 156)	3%	8%	3%	3%
Post (N = 161)	56%	76%	60%	67%

More students successfully chose a dimensionless quantity with which to expand, but roughly 20% of those who did still chose quantities that were large compared to one, expanding in a series that would not converge and was not productive in modeling.

All post-test examples led to series in which there were fractional and/or negative exponents, which proved to be challenging. For example, given the expression for electric potential of a charged disk, $V = 2\pi k\sigma\left(\sqrt{z^2 + R^2} - \sqrt{z^2}\right)$, the students who answered incorrectly frequently expanded with a positive integer exponent (1 or 2, rather than $(1 + {R^2}/{z^2})^{1/2}$ with exponent ½).

While the algebraic manipulation required means that success on these tasks will be linked, there may also be a hierarchy of ideas; most of the students who correctly identified a small argument also chose a dimensionless argument and the correct exponent. The converse was not true; roughly one in three students who identified the correct exponent had incorrect series terms.

Discussion

Prior research may have underestimated the difficulty for students in using this technique by providing examples in which the variable of expansion was already dimensionless and small compared to one. Even after targeted instruction, nearly half of the students expanded these symbol-rich expressions in powers of a quantity that would not be productive for modeling.

While series are a point of shared emphasis in calculus and physics courses, the emphasis differs in important ways. Power series in physics must be dimensionless, but few students on the pretest chose to expand in a dimensionless quantity. Whereas a mathematics course might provide a 'cleaned up' expression to simplify student calculations (and instructor assessment), students in physics will encounter expressions that are symbolically rich and quantities that have units. Despite the strong emphasis in calculus on convergence, physics students need support to understand how this idea plays out in physics contexts. Students are likely to encounter an expression in a form inconvenient for expansion and must make choices consistent with convergent behavior that consider the dimensionality of quantities in physics. In our data set, many students have chosen to expand in powers of a quantity greater than one, for which a series would not converge and thus would not be useful in modeling. Instructors should be cognizant of the disciplinary differences their students will encounter and provide appropriate support.

The case of the dipole illustrates that it is sometimes necessary to perform an expansion rather than simply arguing that $y \gg d$ implies that $d \sim 0$. While ignoring a small quantity might be productive in certain circumstances while modeling, here it eliminates the interesting physics of the dipole configuration, in which the small extra distance to one charge has important physical consequences, thus highlighting disciplinary practices.

We are using these results to modify the instructional materials and develop a set of modular curricular materials that can be adapted to the purposes of different upper-division physics courses. Preliminary materials seem to provide support for students in identifying dimensionless quantities but students seem to need additional scaffolding to recognize when and whether their series are convergent. Additional research and development in ongoing. Current versions of the materials are hosted at PhysPort.org.

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References

Alcock, L. J., & Simpson, A. P. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1), 1-32.

Boas, M. (2006). *Mathematical Methods in the Physical Sciences,* Wiley.

Champney, D. & Kuo, E. (2012). An evolving visual image of approximation with Taylor series: A case study, in Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education, edited by Stacy Brown, Sean Larsen, Karen Marrongelle, and Michael Oehrtman (SIGMAA on RUME, Portland, OR, 2012), Vol. 1, pp. 94–107.

Halliday, E., Resnick, R., & Walker, J. (2011). *Fundamentals of Physics,* Wiley.

Kryjevskaia, M., Heron, P. R., & Heckler, A. F. (2021). Intuitive or rational? Students and experts need to be both. *Physics Today,* 74(8), 28-34.

Martin, J., Oehrtman, M., Roh, K. H., Swinyard, C. & Hart-Weber, C. (2011). Students' reinvention of formal definitions of series and pointwise convergence. In (Eds.) S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman, Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education, Vol. 1, pp 239-254. Portland, Oregon.

Otero, V. K., & Harlow, D. B. Getting Started in Qualitative Physics Education Research. *Rev. PER 2* **no.1**, (2009).

Tutorials in Mathematical Methods in Physics, PhysPort, retrieved from https://www.physport.org/curricula/MathMethodsTutorials/

Smith, T. I., Thompson, J. R., & Mountcastle, D. B. (2013). Student understanding of Taylor series expansions in statistical mechanics. *Physical Review Special Topics-Physics Education Research*, *9*(2), 020110.

Wilcox, B. R., Caballero, M. D., Rehn, D. A., & Pollock, S. J. (2013). Analytic framework for students' use of mathematics in upper-division physics. *Physical Review Special Topics-Physics Education Research*, *9*(2), 020119.

Analysis Across Geometry Textbooks to Link Content, Curriculum and Children's Reasoning

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Research showing that geometry textbooks for teacher candidates primarily focus on expanding the content knowledge in geometry by applying geometry to real-world problems (Litoldo & Amaral-Schio, 2021) and using technology to support problem-solving activities and create multiple representations (Jones et al., 2017). In this study, building on the previous research, we looked for how textbooks used in geometry content courses for elementary school teacher candidates might support their pedagogical content knowledge.

The conceptual framework of this study is Ball et al.'s (2008) practice-based theoretical model that refines pedagogical content knowledge which blends content knowledge, pedagogy and curriculum that highlights teacher's knowledge of standards, grade levels when particular topics are taught, and more. The framework guided the current study's data creation and analysis process to learn how textbooks support prospective teachers' pedagogical content knowledge in geometry teaching.

We collected the commonly used textbooks reported by U.S. mathematics courses instructors for elementary teachers (Max & Newton, 2017; Ozturk et al., 2023). Drawing from relational content analysis methodology (Elo et al., 2014), we first analyze the content of five geometry textbooks and identify the instances of pedagogical content knowledge elements while considering the conceptual framework and examine the differences and similarities across the textbooks. We engaged in a selective reduction process that entailed reading the book and reducing the text to categories, from which we could focus on developing codes for informing our research question. After creating the initial codes, we examined the books individually and then came together to compare and contrast the patterns found in each book. After six meetings (about 14 hours), we finalized five codes including *instancing children's written work*, *instancing children's ideas*, *instancing teacher's role and their activities in K-5 classrooms*, *using of learning standards and connection to K-5 curriculum*, and *using of standardized assessment questions*. Next, we created the final table containing the code frequencies within each textbook and across all codes to draw conclusions.

Analysis revealed that of 780 total codes, the highest percentages of codes focused on children's ideas which provides many examples of how children reason through geometric concepts. The second-highest code percentages focused on the teacher's role and classroom connections, such as discussing teachers' responses to children's ideas to advance the latter's geometric reasoning. The third-highest percentage of codes focused on the use of learning standards, which was often listed next to each relevant book session to show how standards could be linked with instructional activities in progressing through grades K–5. The fourth-highest percentages were related to the code for children's written work which would allow prospective teachers to see concrete examples of children's reasoning through geometric concepts. The code with the least frequency related to the use of assessment questions, providing sample questions from actual standardized tests for K–5 students to familiarize prospective teachers with the content for different grade levels. Depending on the instructor's purposes and practices, the coded instances of pedagogical content knowledge in the textbooks could be used to support prospective teachers' knowledge in geometry content courses (Ozturk et al., 2023).

References

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Chapin, S. H., Gibbons, L. K., Feldman, Z., Callis, L. K., & Salinas, A. (2021). The Elementary Mathematics Project: Supporting preservice teachers' content knowledge for teaching mathematics. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 89–113). Springer International Publishing.

Elo, S., Kääriäinen, M., Kanste, O., Pölkki, T., Utriainen, K., & Kyngäs, H. (2014). Qualitative content analysis: A focus on trustworthiness. *SAGE Open*, *4*(1), Article 2158244014522633.

Jones, D., Hollas, V., & Klespis, M. (2017). The presentation of technology for teaching and learning mathematics in textbooks: Content courses for elementary teachers. *Contemporary Issues in Technology and Teacher Education*, *17*(1), 53–79.

Litoldo, B. F., & Amaral-Schio, R. B. (2021). Mathematics textbooks as subject of study: Producing knowledge on the presence of geometry. *The Mathematics Enthusiast*, *18*(3), 502–534.

Ozturk, A., Schramm, L., Milligan, M., Canning, M., Rapol, M., & Wilson, S. (2023). Examining Textbooks to Support Prospective Teachers' Pedagogical Content Knowledge of Geometry. *Ohio Journal of School Mathematics*, *94*(1), 13–20.

Max, B., & Newton, J. (2017). Mathematics preparation of elementary teachers: Results of a national survey. *AMTE Connections*. http://amte.net/connections/2017/09/mathematics-preparation-elementary-teachers-results-national-survey

Exploring Students' Mathematical Convictions Through a Script-writing Task

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Some empirical studies that report on mathematical convictions include proof comprehension (Stylianides et al., 2007) or evaluation (Inglis & Mejia-Ramos, 2008). Some focus on students' convictions (Lockwood et al., 2020; Zaslavsky, 2005). One underreported method in the literature for this topic of research is s*cript-writing*—the creation of a dialogue, between oneself and an imaginary student or classmate, focused on issues that may be encountered and resolved (Zazkis & Zazkis, 2014). This method is often used with teachers as a form of lesson-play (Koichu & Zazkis, 2018; Zazkis et al., 2013) while others have used it to investigate proof comprehension (Brown, 2018; Zazkis, 2014).

The purpose for this study was to explore this method's utility when investigating students' mathematical convictions. The guiding research question was: how do students' responses to a script-writing task describe their mathematical convictions? Take-home script-writing tasks, initial interview recordings, and one follow-up interview recording were collected. The statement and argument presented to students for their task were:

Statement: For $n \in \mathbb{N}$, the sum of the first n consecutive odd numbers is n^2: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Argument: Let n be a natural number and consider $n = 1$. We have $2(1) - 1 = 2 - 1 = 1$ which is 1^2. Assume that $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$ is true for the natural numbers: $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) + (2n + 1) = [1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1)] + (2n + 1) = n^2 + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$. Therefore, $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$ is true for the natural numbers.

The task had instructions on creating a dialog between them and a classmate named Gamma. These instructions asked to introduce and explain the statement and argument to Gamma and to identify problematic points (see Brown (2018, p. 66) for similar instructions).

Taylor, in his final semester, was completely convinced that the statement was true and the argument proved the statement in the initial interview. For his script, he used an example showing the statement to be true and invited Gamma to try examples. In the follow-up interview, Taylor explained he tried to help Gamma "understand intuitively" because it's easier to do that with examples. When asked how he would resolve the issue if Gamma was still not convinced that the statement was true, he responded with "like if I lay down a whole proof?... I guess they're just not convinced…." This was interpreted that Taylor had an assumption that proofs should be convincing. I asked Taylor if he agreed that if Gamma understood the statement and argument more, then Gamma would be more convinced. He responded with "definitely."

Stewie was a fifth-year student who recently completed a transition-to-proof course. In the initial interview, he was 99.99% convinced by the statement and argument saying, "it's hard to be 100% certain on anything." He cited possible human errors for this judgment. In his script, he identified issues regarding the formatting and clarity of the proof. Stewie appeared to explain the components of the proof, like the purpose of the base case, to help Gamma.

In this study, I hoped that the scripting task would influence participants' convictions. However, their convictions remained the same throughout the study. The script-writing task revealed that they viewed ideas of helping a classmate in: 1) comprehension and clarify of the method of proof and 2) conviction in the statement's truth. Future research may include more complex statements and arguments to explore the script-writing task's influence on convictions.

References

Brown, S. A. (2018). Difficult dialogs about degenerate cases: A proof script study. *Journal of Mathematical Behavior*(52), 61-76.

Inglis, M., & Mejia-Ramos, J. P. (2008). How persuaded are you? A typology of responses. *Research in Mathematics Education*, *10*(2), 119-133.

Koichu, B., & Zazkis, R. (2018). "I Understand" Talk in Script Writing: A Case from Euclid's Elements. In R. Zazkis & P. Herbst (Eds.), *Scripting Approaches in Mathematics Education* (pp. 163-184). Springer International Publishing.

Lockwood, E., Caughman, J., & Weber, K. (2020). An essay on proof, conviction, and explanation: multiple representation system in combinatorics. *Educational Studies in Mathematics*(103), 173-189.

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*(3), 145-166.

Zaslavsky, O. (2005). Seizing the Opportunity to Create Uncertainty in Learning Mathematics. *Educational Studies in Mathematics*, *60*(3), 297-321.

Zazkis, D. (2014). Proof-scripts as a lens for exploring students' understanding of odd/even functions. *The Journal of Mathematical Behavior*(35), 31-43.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development.* Springer.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: imaginary conversations on the structure of numbers. *Research in Mathematics Education*, *16*(1), 54-70.

The Precalculus Concept Assessment (PCA) and Prospective Secondary Teachers

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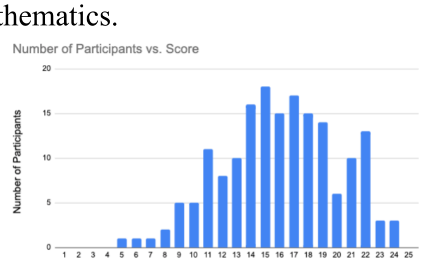
Keywords: Student Cognition, Covariational Reasoning, Pre-Calculus

Carlson et al. (2010) developed the 25-item multiple choice Precalculus Concept Assessment (PCA) to investigate reasoning abilities and meanings researchers (e.g., Carlson et al., 2002; Dubinsky & Harel, 1992; Oehrtman et al., 2008; Thompson & Silverman, 2007) have established as critical for pre-calculus and calculus learning. Since its initial development and validation, the PCA has been administered to thousands of secondary and post-secondary students, providing key insights into their reasoning abilities, as well as their potential success (as measured by grades) in future calculus courses. For instance, using a population of 248 students who were entering a first-semester calculus course taught by six different instructors, Carlson et al. (2010) identified that 77% of the students who scored 13 or higher passed the course (i.e., C or better). Meanwhile, 60% of the students who scored 12 or lower failed (i.e., D, F, or W). The authors illustrated that the correlation between course grades and PCA score were as strong or stronger than other popular educational math tests including the MAA placement test.

Given the insights the PCA provides relative to pre-calculus and calculus students, we grew interested in the extent the PCA can provide useful insights with other populations. Across several semesters, we administered the PCA to 174 undergraduate students upon their entry to a secondary mathematics teacher preparation program. Their program entry typically occurs during their sophomore or junior year of undergraduate studies, and after having taken two math courses beyond a calculus sequence. In this poster, we present on the results of that administration. Specifically, we report on an analysis of the aggregate scores of the population (Figure 1a), as well as their performance on covariational reasoning items (Figure 1b). With respect to the covariational reasoning item performance, we draw on our expertise and research (Moore, 2021; Moore et al., 2022; Moore et al., 2019) to develop hypotheses regarding discrepancies in performance. Specifically, we highlight differences between items based on the level of covariational reasoning targeted by the item. We also explore differences between items based on the items' figurative material and the extent the material afforded enacting quantitative operations versus numerical operations. These differences provide potential insights and implications relative to the participants' grounding to reason about and teach for key concepts of secondary mathematics.



(a)

Covariational Reasoning Items

Question	Correct	Incorrect	% Correct	% Wrong
7	101	73	58.05%	41.95%
8	50	124	28.74%	71.26%
15	70	104	40.23%	59.77%
18	140	33	80.46%	19.54%
19	125	49	71.84%	28.16%
22	74	100	42.53%	57.47%
24	134	40	77.01%	22.99%
25	81	92	46.55%	53.45%

(b)

Figure 1. (a) PCA Participant Scores and (b) Covariational Reasoning Item Performance.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378. https://doi.org/10.2307/4149958

Carlson, M. P., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment (PCA) Instrument: A tool for assessing reasoning patterns, understandings, and knowledge of precalculus level students. *Cognition and Instruction*, *28*(2), 113-145.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Mathematical Association of America.

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of Learning: Progressive Perspectives for Mathematics Education and Related Fields* (pp. 145-171). Springer.

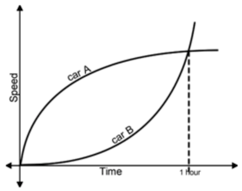
Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education* (pp. 35-69). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_3

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*. https://doi.org/10.1016/j.jmathb.2019.01.008

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Mathematical Association of America.

Thompson, P. W., & Silverman, J. (2007). The concept of accumulations in calculus. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117-131). Mathematical Association of America. https://doi.org/10.5948/upo9780883859759.005

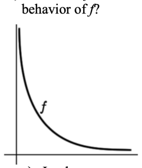
The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph below to answer item 8.



8) What is the relationship between the ***position*** of car A and car B at $t = 1$ hr.?

a) Car A and car B are colliding.
b) Car A is ahead of car B.
c) Car B is ahead of car A.
d) Car B is passing car A.
e) The cars are at the same position.

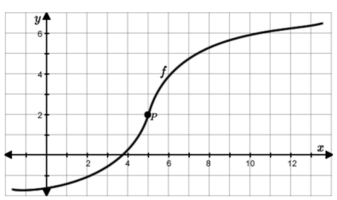
24) A function f is defined by the following graph. Which of the following describes the behavior of f?



I. As the value of x approaches 0, the value of f increases.
II. As the value of x increases, the value of f approaches 0.
III. As the value of x approaches 0, the value of f approaches 0.

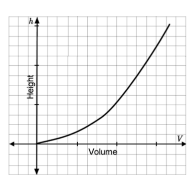
a) I only
b) II only
c) III only
d) I and II
e) II and III

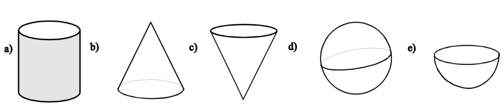
19) Using the graph below, explain the behavior of function f on the interval from $x = 5$ to $x = 12$.



a) Increasing at an increasing rate.
b) Increasing at a decreasing rate.
c) Increasing at a constant rate.
d) Decreasing at a decreasing rate.
e) Decreasing at an increasing rate.

15) The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?

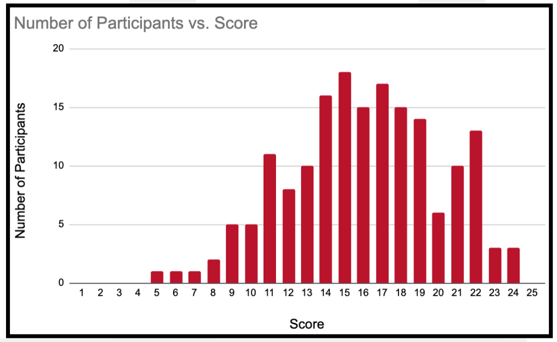




Differences in responses to covariation items provide insights into their covariation meanings.

The prospective teachers have formed indexical associations between graphs and covariation statements, but that does not imply their having abstracted graphs in terms of re-presenting quantitative and covariational operations.

MATCHING GAME



PAPER

Match The Task to the Answer Data Set

The Precalculus Concept Assessment (PCA) and Prospective Secondary Teachers

by Kevin C. Moore, Irma E. Stevens, Anne Waswa, and Sohei Yasuda

SET ONE

Ans.	#	%
a	4	2.3%
b	20	11.49%
c	1	0.57%
d	134	77.01%
e	15	8.62%

SET TWO

Ans.	#	%
a	11	6.32%
b	50	28.74%
c	3	1.72%
d	7	4.02%
e	103	59.20%

SET THREE

Ans.	#	%
a	5	2.87%
b	70	40.23%
c	72	41.38%
d	3	1.72%
e	24	13.79%

SET FOUR

Ans.	#	%
a	44	25.29%
b	125	71.84%
c	5	2.87%
d	0	0%
e	0	0%

An Exploratory Study of Students' Understanding of Graphical Optimization in a Cost Minimization Context

Thembinkosi P. Mkhatshwa
Miami University

This study reports on first-semester calculus students' reasoning about a univariate optimization problem that involves finding the production level at which the cost per yard is minimized when given the graph of a function that represents the relationship between the cost per yard and the number of yards produced by a factory. Analysis of verbal responses and work written by four students when solving the problem revealed that determining the production level at which the cost per yard is minimized was straightforward for all the students. However, explaining how this production level is related to the first derivative of the given function was problematic for most of the students.

Key words: Optimization problems, graphical optimization, problem solving, calculus education

Unlike algebraic optimization that uses algebraic methods (that may sometimes be sophisticated, especially when working with complex objective functions) to solve univariate optimization problems or numerical optimization (that requires some level of technical skills such as proficiency in MATLAB programing) to solve univariate optimization problems, graphical optimization (Bhatti, 2000) is the simplest method for solving univariate optimization problems (UOPs) in that it only requires making sense of graphs of objective functions. A number of studies have reported on several difficulties typically exhibited by students when solving optimization problems algebraically, including formulating the objective function, finding and interpreting critical values or extrema of the objective function, and determining if a critical value(s) results in a minimum/maximum value of the objective function (cf. Borgen & Manu, 2002; Dominguez, 2010; LaRue & Infante, 2015; Mkhatshwa, 2019; Swanagan, 2012). We are not aware of any research that has examined students' thinking in the context of graphical optimization, which is the motivation for this study. To address this knowledge gap, task-based interviews (Goldin 2000) were conducted with four calculus students.

Contrary to findings of studies that have reported on students' thinking about algebraic optimization (cf. Borgen & Manu; Swanagan, 2012), nearly all the students in the present study were successful in finding the critical value or extremum as well as justifying extremum while working with a UOP where the graph of the objective function was provided. To some extent, this may suggest that while students may struggle with solving UOPs algebraically, partly due to the lack of facility with some algebraic techniques such as calculating derivatives of complex objective functions, students have better success with solving UOPs graphically not only because they can visualize the objective function, but also because having access to the graph of the objective function supports their quantitative reasoning (Thompson, 1993; 2011) such as the ease of identifying critical values and extrema. Additionally, three of the four students made remarks that suggested that they had difficulty understanding that the derivative of the objective function ought to be zero at the critical value (i.e., the cost minimizing quantity), something that generally comes easy for students when solving UOPs algebraically.

References

Bhatti M.A. (2000) Graphical optimization. *Practical Optimization Methods, 18*(2), 47-73.

Borgen, K. L., & Manu, S. S. (2002). What do students really understand? *The Journal of Mathematical Behavior*, *21*(2), 151-165.

Dominguez, A. (2010). Single Solution, Multiple Perspectives. In Lesh, R., Galbraith, P. L., Haines, C. R., & Hurford, A. (Eds.), *Modeling Students' Mathematical Modeling Competencies: ICTMA 13* (pp. 223-233). New York: Springer.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Laurence Erlbaum Associates.

LaRue, R., & Infante, N. E. (2015). Optimization in first semester calculus: a look at a classic problem. *International Journal of Mathematical Education in Science and Technology*, *46*(7), 1021-1031.

Mkhatshwa, T. (2019). Students' quantitative reasoning about an absolute extrema optimization problem in a profit maximization context. *International Journal of Mathematical Education in Science and Technology, 50*(8), 1105-1127.

Swanagan, B. S. (2012). *The impact of students' understanding of derivatives on their performance while solving optimization problems* (Doctoral dissertation). University of Georgia, Georgia.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.

Students' Attitudes and Perseverance in Problem-solving in Undergraduate Precalculus

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Keywords: precalculus, curriculum enactment, attitudes, perseverance in problem-solving

This exploratory study investigated the relationships between professors' enactments of a research-based precalculus curriculum and changes observed in students' attitudes towards mathematics and perseverance in problem-solving. While much research focuses on improving student achievement in undergraduate STEM courses, we must also support them in developing the positive dispositions and practices needed to sustain them through years of mathematics-based STEM coursework (Bressoud et al., 2015). When students hold positive attitudes towards mathematics, they are more likely to both succeed in calculus (Sonnert et al., 2020) and persist in completing a STEM degree (Wu et al., 2022). When students persevere in their problem-solving amidst challenges, they make meaning of mathematics and develop productive dispositional factors for STEM majors, including building resilience for overcoming setbacks (Middleton et al., 2015). The Pathways precalculus curriculum (Carlson et al., 2021) leverages research on student thinking to support them in developing problem-solving skills and in making connections across concepts (Moore & Carlson, 2012), and has been tied to improvement in students' covariational reasoning and subsequent success in calculus (McNicholl et al., 2021). However, since curriculum and its enactment can transform students' opportunities to learn in various ways (Stein et al., 2007), including outcomes related to disposition and mathematical practices (DiNapoli & Morales, Jr., 2021; Ruthven, 2011), we chose to focus on the relationships between different enactments of Pathways and students' attitudes and perseverance measures.

Three precalculus professors (Profs. A, B, and C) and 33 of their students participated in this study. Two classroom observations for each professor were video recorded and transcribed. The professors' pedagogical choices were scored on a 0-3 scale along three Pathways-aligned dimensions: support of student problem-solving, understanding and advancing student thinking, and making connections (rubric adapted from Schoenfeld et al., 2014). Student participants completed an Attitudes Towards Mathematics Inventory (ATMI; Tapia & Marsh, 2004) pre- and post-survey, which measured students' mathematical self-confidence, value, enjoyment, and motivation. Student participants also engaged in 12 video-recorded problem-solving sessions. Students' perseverance was measured using the Three-Phase Perseverance framework (3PP; DiNapoli & Miller, 2022), which considered the extent to which students initiated and sustained, and re-initiated and re-sustained upon impasse, productive struggle on a challenging task.

Profs. A, B, and C implemented Pathways in different ways along the dimensions analyzed; over the two observations, Prof. A's scores were almost uniformly low, Prof. B's consistently high, and Prof. C's scores were medium-to-high. Each professor's students exhibited changes in attitudes: Prof A's students shifted towards more negative attitudes across the semester, whereas Profs. B and C's students exhibited positive attitude shifts. Finally, students in all three classes averaged positive 3PP slope scores, indicating some perseverance growth for all, yet Prof. B and C's students experienced about 5 and 4 times the growth, respectively, compared to Prof. A's students. More detailed qualitative and quantitative evidence of our findings will be shared in our presentation. These findings suggest that although the Pathways precalculus curriculum may support the development of positive attitudes toward mathematics and improved perseverance in problem-solving, this potential is influenced by professors' pedagogical choices.

References

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (2015). *Insights and recommendations from the MAA national study of college calculus*. MAA Press.

Carlson, M. P., Oehrtman, M., Moore, K. C. & O'Bryan, A. (2021). *Precalculus pathways to calculus: A problem solving approach*. Hayden-McNeil.

DiNapoli, J., & Miller. E. K. (2022). Recognizing, supporting, and improving student perseverance in mathematical problem-solving: The role of conceptual thinking scaffolds. *Journal of Mathematical Behavior, 66,* 1-20. https://doi.org/10.1016/j.jmathb.2022.100965

DiNapoli, J., & Morales, Jr., H. (2021). Translanguaging to persevere is key for Latinx bilinguals' mathematical success. *Journal of Urban Mathematics Education*, *14*(2), 71-104. https://doi.org/10.21423/jume-v14i2a390

McNicholl, T. H., Frank, K., Hogenson, K., Roat J., & Carlson, M. P. (2021). Improving student success and supporting student meaning-making in large-lecture precalculus classes, *PRIMUS*, *31*(7), 792-810

Middleton, J. A., Tallman, M., Hatfield, N., & Davis, O. (2015). *Taking the severe out of perseverance: Strategies for building mathematical determination*. Spencer.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59.

Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project. (2014). *The TRU Math Scoring Rubric*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. Retrieved from http://ats.berkeley.edu/tools.html.

Sonnert, G., Barnett, M. D., & Sadler, P. M. (2020). The effects of mathematics preparation and mathematics attitudes on college calculus performance. *Journal for Research in Mathematics Education*, *51*(1), 105-125.

Ruthven, K. (2011). Using international study series and meta-analytic research syntheses to scope pedagogical development aimed at improving student attitude and achievement in school mathematics and science. *International Journal of Science and Mathematics Education*, *9*, 419-458.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 319–369). Information Age.

Tapia, M., & Marsh, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly*, *8*(2), 16-22.

Wu, X., Rambo-Hernandez, K., Fuller, E., & Deshler, J. (2022). Predicting STEM persistence from mathematics affect, pedagogical perceptions and calculus I setting, *International Journal of Mathematical Education in Science and Technology, 54*(7), 1-22.

Students' Attitudes and Perseverance in Problem-solving in Undergraduate Precalculus

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This project was supported by a grant from the State of New Jersey

Introduction and Rationale

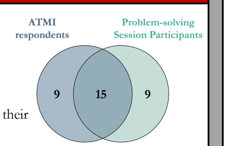
- Student achievement in precalculus is necessary, but not sufficient
  - **Positive attitudes towards mathematics** help students succeed in calculus and persist in completing a STEM degree (Sonnert et al., 2020; Wu et al., 2022).
  - **Persevering in problem-solving** helps students make meaning of mathematics and build resilience for overcoming setbacks (Middleton et al., 2015).
- The **Precalculus: Pathways to Calculus** curriculum has the **potential** to support students in developing positive attitudes and improving their perseverance
  - Content development embedded in **rich, real-world contexts**
    - Reduces **math anxiety** (Karunakaran, 2020)
    - Supports students' value of mathematics (Aikens et al., 2021)
  - Focus on developing students' **problem-solving skills** and **covariational reasoning**
    - Increases **enjoyment** (Schettino, 2016) and **motivation** (Peterson, 2019)
    - Supports **perseverance** development (Barnes, 2019; DiNapoli & Miller, 2022)
  - **Scaffolded questioning** based on research on how students make sense of precalculus concepts (e.g., Frank, 2017; Kuper, 2019; Moore, 2010)
    - Improve students' mathematical **self-efficacy** (Jatisunda, 2020)
    - Helps students initiate and sustain **productive struggle** (Reiser & Tabak, 2014), and thus, their **perseverance** at times of impasse (DiNapoli & Miller, 2022)
- Curriculum enactments can differentially impact
  - Students' **opportunities to learn** (Stein et al., 2007)
  - Students' **attitudes** towards mathematics (Ruthven, 2011)
  - Students' improvement in **perseverance** in problem-solving (DiNapoli & Miller, 2022)

Research Question

What are the relationships between professors' enactments of a research-based precalculus curriculum and changes observed in students' attitudes towards mathematics and perseverance in problem-solving?

Methods

Context and Participants

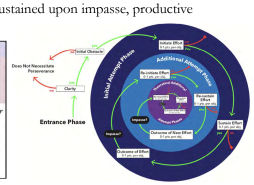


- Public research university in the Northeast U. S.
  - ~23,000 students, HSI, R2: Doctoral University
- Coordinated, entry-level precalculus course
  - ~40 sections and ~1200 students per year, optional PD
- Three precalculus professors (Profs. A, B, and C) and 33 of their students participated.

Data Collection and Analysis

- Two **classroom observations** for each professor were video recorded and transcribed.
  - The professors' pedagogical choices were scored on a 0-3 scale along three Pathways-aligned dimensions: support of student problem-solving, understanding and advancing student thinking, and making connections (Schoenfeld et al., 2014).
- Student participants completed an **Attitudes Towards Mathematics Inventory** (ATMI; Tapia & Marsh, 2004) pre- and post-survey, which measured students' mathematical self-confidence, value, enjoyment, and motivation.
- Student participants also engaged in 12 video-recorded **perseverance in problem-solving sessions**.
  - Students' perseverance was measured using the Three-Phase Perseverance framework (3PP; DiNapoli & Miller, 2022), which considered the extent to which students initiated and sustained, and re-initiated and re-sustained upon impasse, productive struggle on a challenging task.

Pathways Implementation Observation Rubric		
Engaging and Supporting Students in Problem-solving	Understanding and Advancing Students' Thinking	Providing Opportunities for Students to Make Connections
To what extent do classroom activities actively engage students in problem-solving? To what extent does the professor discuss, model, and encourage productive problem-solving practices?	*To what extent does the professor endeavor to understand students' thinking about content and respond in ways that help them to construct more productive understandings?*	*To what extent does the professor support students in making connections: to real-world contexts, across course content, and between functional representations?*



Results

Classroom Observations

		Engaging and Supporting Students in Problem-solving	Understanding and Advancing Students' Thinking	Providing Opportunities for Students to Make Connections
		Pathways Implementation (0 to 3 point scale)		
Prof A	Obs 1	1.00	1.40	2.10
	Obs 2	1.00	1.14	1.70
Prof B	Obs 1	2.25	2.17	2.38
	Obs 2	2.00	2.31	2.40
Prof C	Obs 1	2.20	2.14	2.40
	Obs 2	1.70	1.97	1.40

Classroom Observation Findings

- Prof A had somewhat lower scores across the board compared to Profs B and C, particularly in the Problem-Solving category.
- Prof B's scores were consistent across all categories and were generally highest.
- Prof C's scores were mixed, a bit higher for their first observation than their second.

Note: Maintaining a level 3 in all three categories would be impossible.

Examples of Understanding and Advancing Student Thinking

Prof A's Level 1 Interaction

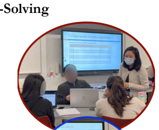
Prof A: As my angle gets bigger, if I measured this, what's this distance relative to the one down here, bigger or smaller?
Student: Smaller.
Prof A: Smaller. … And now as I move further, between pi over two and pi, this distance here is getting bigger because it's moving away. But because it's on the left-hand side, is it positive or negative?
Student: Negative.
Prof A: Negative…

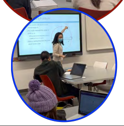
Prof B's Level 3 Interaction

Prof B: So, I know the 10-hour decay factor [is .2722], how can I use [this] to find the five-hour decay factor?
Student 1: You could try and divide it by half…
Student 2: Square root of .2722?
Prof B: Ah. We should explore these options … [Student 1], would you mind telling me your reasoning … for this?
Student 1: Well, since it's a 10-hour decay, and they ask you for like the five-hour growth decay. I just thought if we just divided by two.

Prof B: [Student 2], what's your reasoning for this?
Student 2: Because the five-hour times the five- hour would be the 10 hour.
Prof B: [Takes a class poll, which is split] So, I want to talk about how we can use our intuition to maybe rule this out a little bit. …if [.2722 divided by 2] was the five-hour growth or decay factor, this is … .1372… in five hours, I only have about 14% [of the caffeine] left in my system. But in 10 hours, I have more than that left in my system. That feels weird, right?

Examples of Engaging and Supporting Students in Problem-Solving

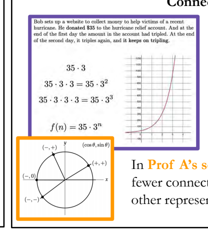




In **Prof C's first observation**, students spent the majority of class working in small groups with Prof B circulating around the room in a supportive role.

In **Prof C's second observation**, they spent more time at the board delivering content.

Examples of Providing Students with Opportunities to Make Connections



In **Prof A's first observation**, they developed the exponential function formula using a real-world context, connected symbolic and graphical representations, and contrasted exponential and linear growth.

In **Prof A's second observation**, they made fewer connections to real-world contexts, other representations, and other content.

Attitudes Towards Mathematics and Perseverance in Problem-Solving

	Average ATMI Changes (Post – Pre, 5-point Likert Scale)					3PP Slope
	Self-Confidence	Value	Enjoyment	Motivation	Overall Shift	(Avg increase per session)
Prof A	0.00	-0.31	-0.17	-0.25	-0.15	0.09
Prof B	0.17	0.37	0.10	0.20	0.21	0.44
Prof C	0.72	0.16	0.30	0.20	0.41	0.34

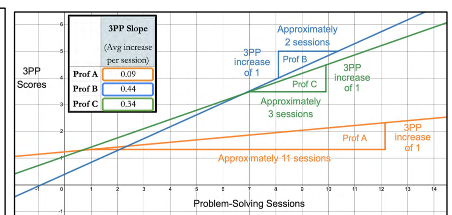
Attitudes Findings

Small shifts in ATMI outcomes:

- Prof A's students: mostly negative shifts in attitudes
- Profs B's students: mostly positive, particularly in value
- Prof C's students: mostly positive, particularly in self-confidence.

Perseverance Findings

- All students showed perseverance growth
  - Prof B & C's 3PP growth ~4-5 times greater than Prof A's
- To show 1 point of 3PP growth:
  - Prof A: ~11 problem-solving sessions
  - Prof B: ~2 problem-solving sessions
  - Prof C: ~3 problem-solving sessions
- Interpreting 1 point of 3PP growth, e.g.:
  - 0→1: none → some effort
  - 2→3: sustaining → productive effort
  - 3→4: first → second effort upon impasse



Discussion

In this early exploratory study, we were able to see some potential connections between different ways of implementing Pathways and students' changes in attitudes towards mathematics and perseverance in problem solving, and many of these perceived connections are consistent with the literature.

- We saw evidence of the **links between actively engaging students in problem-solving and improvements in attitudes and perseverance in problem-solving** (Barnes, 2019; DiNapoli & Miller, 2022; LeSage et al., 2021; Peterson, 2019; Schettino, 2016; Schoenfeld, 1992).

		Problem Solving	ATMI Changes	3PP Slope
Prof A	Obs 1	1.00	-0.15	0.09
	Obs 2	1.00		
Prof B	Obs 1	2.25	0.21	0.44
	Obs 2	2.00		
Prof C	Obs 1	2.20	0.41	0.34
	Obs 2	1.70		

- Prof A's lack of problem-solving pedagogies were due, in part, to COVID concerns.
- Prof B's best problem-solving pedagogies involved leading interactive, whole-class discussions.
- Prof C's best problem-solving pedagogies involved small group work.

- We saw evidence of the **links between understanding and advancing student thinking to improve students' self-confidence, motivation, and perseverance in problem-solving** (DiNapoli & Miller, 2022; Harper et al., 2019; Jatisunda, 2020; Reiser & Tabak, 2014).
  - Valuing student thinking in class can help them develop their mathematical agency.
  - Students seeing how their in-progress thinking can lead to mathematical productivity in class can reinforce the value of productive struggle.

		Student Thinking	Self-Confidence	Motivation	3PP Slope
Prof A	Obs 1	1.40	0.00	-0.25	0.09
	Obs 2	1.14			
Prof B	Obs 1	2.17	0.17	0.20	0.44
	Obs 2	2.31			
Prof C	Obs 1	2.14	0.72	0.20	0.34
	Obs 2	1.97			

- We saw evidence of the **links between providing students opportunities to make connections between content and real-world contexts to their perception of the value of mathematics** (Aikens et al., 2021).

		Making Connections	Value of Math
Prof A	Obs 1	2.10	-0.31
	Obs 2	1.70	
Prof B	Obs 1	2.38	0.37
	Obs 2	2.40	
Prof C	Obs 1	2.40	0.16
	Obs 2	1.40	

- Whether the professor or the students make connections may influence students' value of math.
- Prof A and Prof C scored similarly, but Prof C more often provided students with opportunities to make these connections for themselves.
- Prof B often used real-world scenarios suggested by students to create contextualized problems for in-class discussion.

Main Takeaway of Research #betterposter

This research suggests that although the Pathways precalculus curriculum may support the development of positive attitudes toward mathematics and improved perseverance in problem-solving, this potential is influenced by professors' pedagogical choices.

Scan this QR code to access our paper:

Summarizing the Experiences and Beliefs of Community College Instructors and Students in a Corequisite Mathematics Course

Brianna Bentley
Alamance Community College

Keywords: developmental mathematics corequisite course, community college, mindset, math anxiety, math self-efficacy

At community colleges in North Carolina, the corequisite course model has been adopted for all the entry-level mathematics courses. Research about the corequisite course model has shown that these courses can successfully help students earn their mathematics credit (e.g., Childers et al., 2021; Kashyap & Matthew, 2017; Logue et al., 2019) and can increase students' interest in mathematics (e.g., Campbell, 2015). The purpose of this poster is to summarize my three-article dissertation study that described the experiences, mindset, math anxiety, and math self-efficacy of students taking corequisite courses and instructors teaching corequisite courses, the interactions between student and instructor experiences, and the influence of instructors on students' mindset, math anxiety, and math self-efficacy over the course of the semester.

The first article explored the experiences of students taking a corequisite course and instructors teaching a corequisite course, specifically, what characteristics of corequisite courses students and instructors found to be effective, ineffective, and what changes they would make to corequisite courses. A multiple case study design was used to analyze the student and instructor perspectives of corequisite mathematics classrooms. Semi-structured interviews were completed with 11 students and 13 instructors from four community colleges. Interviews were transcribed and coded, and then codes were grouped into four broader themes: (1) corequisite courses characteristics determined at the college and department level; (2) corequisite courses characteristics determined at the classroom level; (3) changes that have been or that could be made to corequisite courses; and (4) their overall recommendation of corequisite courses.

The second article was a convergent parallel mixed methods study that investigated community college students' change in mindset, math anxiety, and math self-efficacy in a corequisite mathematics course and what characteristics of corequisite courses influence these changes. Student's math self-efficacy significantly increased, while their math anxiety decreased over the course of the semester. Student's mathematical mindset did not significantly change over the course of the semester. Student interview comments could be categorized into two larger themes: (1) how students felt at the beginning of the semester when they signed up for a corequisite course; and (2) how the characteristics of the corequisite course influenced their mindset, math anxiety, and math self-efficacy.

The third article was a convergent parallel mixed methods study that investigated the extent to which community college students' change in mindset, math anxiety, and math self- efficacy in a corequisite mathematics course varied among instructors and how corequisite instructors influenced these changes in students' beliefs. The results from an ANOVA analysis showed that there were significant differences across the seven instructors in students' mean math anxiety gain scores. To investigate how instructors influenced students' mindset, math anxiety, and math self-efficacy, semi-structured interviews were conducted at the end of the semester with 11 students and 13 instructors at four community colleges, and student and instructor comments were grouped into four larger themes: (1) pedagogical decisions; (2) instructor disposition; (3) the classroom environment; and (4) instructor and student interactions.

References

Campbell, P. S. (2015). *Self-efficacy in a co-requisite model of developmental mathematics and college algebra: A qualitative analysis of student perceptions* [Doctoral dissertation, Texas Tech University].

Childers, A. B., Lu, L., Hairston, J., & Squires, T. (2021). Impact and effects of co-requisite mathematics remediation. *PRIMUS, 31*(2), 167-183.

Kashyap, U., & Mathew, S. (2017). Corequisite model: An effective strategy for remediation in freshmen level quantitative reasoning course. Journal of STEM Education: Innovations & Research, 18(2).

Logue, A. W., Douglas, D., & Watanabe-Rose, M. (2019). Corequisite mathematics remediation: Results over time and in different contexts. *Educational Evaluation and Policy Analysis, 41*(3), 294-315.

A Peer Tutor with a Partial Growth Mindset

Katie Bjorkman
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Keywords: Growth mindset, peer tutors, motivation, self-regulation

An undergraduate mathematics peer tutor gave various reasons why not all students were successful in mathematics. I consider his reasons in light of the growth mindset framework and common misconceptions about what it means, reflection on the fact that the beliefs of mathematics educators have been shown to alter student outcomes (Dweck, 2015; Rattan et al., 2012). The undergraduate mathematics peer tutor, Jake (gender-preserving pseudonym), was a case-study participant that participated in an ethnographic study of a mathematics learning center as a figured world (Bjorkman, 2019). The study consisted of ethnographic field notes, video-recording of tutoring interactions used in stimulated recall, and semi-structured interviews of peer tutors which were analyzed using grounded theory methodology (Corbin & Strauss, 1990; Dempsey, 2010; Emerson et al., 1995; Lyle, 2010; Muir, 2010; Strauss & Corbin, 1994).

Jake expressed belief that everyone can learn mathematics, but only if they were willing to "put in the effort." He made statements like "I believe that everyone can do math" and "I think it's important to get things wrong and then you understand." However, Jake also said that "weeding out" through failing a mathematics class, rather than changing learning behavior, is good – then clarified that he didn't think that weeding out was for those who were intellectually unable to do math since, "I believe that everyone can do math… but there's some people who just don't put in the effort." Jake talked elsewhere about this group of unmotivated students as "[W]hen they say they don't go to lecture or don't take notes I almost don't want to help." Jake's model for student success and failure, while on the surface aligned with a growth mindset, is, in fact a limited growth mindset model that does not see self-regulation skills as areas where students can learn and grow and does not consider other factors that may lead students to perform poorly (Dweck, 2015; Granberg et al., 2021; Michaelson, 2017). It is likely that he implicitly communicated his belief about those students' identities as "bad students" to them through his interactions (Holland et al., 1998; Rattan et al., 2012). If Jake as a peer tutor isn't the one to help students who don't yet know "how to college" where should they turn?

The idea of a "growth mindset" has become quite popular in educational circles since the framework was put forward in the 1980's, and more recent research has indicated it can be a useful construct for helping students learn (Denworth, 2019). Yet, in my personal reflection, and experience with other professors, this fixed mindset in terms of student effort and self-regulation seems to exist in more educators than Jake, perhaps even more saliently now as students and educators struggle with the new normal after the pandemic (Bozkurt, 2022). We have long had evidence that teacher beliefs about students are influential in student outcomes (Rosenthal et al., 1966; Timmermans et al., 2021). Explicitly extending a growth mindset to students' self-regulation and motivation and designing courses toward fostering self-motivation and self-regulation as skills may help these "bad students" succeed in mathematics courses, rather than giving up or expending effort on ineffective learning strategies (Granberg et al., 2021).

References

Bjorkman, K. (2019). The Identities of Undergraduate Mathematics Peer Tutors within the Figured World of a Mathematics Learning Center. In *Doctoral Dissertation, University of California, San Digeo*. University of California. https://escholarship.org/uc/item/0xp4r2hb

Bozkurt, A. (2022). Resilience, Adaptability, and Sustainability of Higher Education: A Systematic Mapping Study on the Impact of the Coronavirus (COVID-19) Pandemic and the Transition to the New Normal. *Journal of Learning for Development*, *9*(1), 1–16. https://doi.org/10.56059/jl4d.v9i1.590

Corbin, J., & Strauss, A. (1990). Grounded theory research: Procedures, canons and evaluative criteria. *Qualitative Sociology*, *19*(6), 418–427.

Dempsey, N. P. (2010). Stimulated recall interviews in ethnography. *Qualitative Sociology*, *33*(3), 349–367. https://doi.org/10.1007/s11133-010-9157-x

Denworth, L. (2019). Debate arises over teaching "growth mindsets " to motivate students. *Scientific American*.

Dweck, C. S. (2015). Revisits the Growth Mindset. *Education Week*, *35*(05), 20–24.

Emerson, R. M., Fretz, R. I., & Shaw, L. L. (1995). *Writing Ethnographic Field Notes*. The University of Chicago Press.

Granberg, C., Palm, T., & Palmberg, B. (2021). A case study of a formative assessment practice and the effects on students' self-regulated learning. *Studies in Educational Evaluation*, *68*(August 2020). https://doi.org/10.1016/j.stueduc.2020.100955

Holland, D. C., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and Agency in Cultural Worlds*. Harvard University Press.

Lyle, J. (2010). Stimulated recall: A report on its use in naturalistic research. *British Educational Research Journal*, *29*(6), 861–878. https://doi.org/10.1080/0141192032000137349

Michaelson, L. E. (2017). Causes and Correlates of Apparent Failures to Delay Gratification in Children and Adults. In *PhD Thesis, University of Colorado*. https://doi.org/10.1017/CBO9781107415324.004

Muir, T. (2010). Using Video-Stimulated Recall as a Tool for Reflecting on the Teaching of Mathematics. *Mathematics Education Research Group of Australasia*, 438–445. http://eric.ed.gov/?id=ED520951

Rattan, A., Good, C., & Dweck, C. S. (2012). "It's ok - Not everyone can be good at math": Instructors with an entity theory comfort (and demotivate) students. *Journal of Experimental Social Psychology*, *48*(3), 731–737. https://doi.org/10.1016/j.jesp.2011.12.012

Rosenthal, R., Jacobson, L., & San Francisco, S. (1966). Teachers' Expectancies: Determinants Of Pupils' IQ Gains 1. In *Psychological Reports* (Vol. 19).

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 273–285). Sage Publications.

Timmermans, A. C., Rubie-Davies, C. M., & Wang, S. (2021). Adjusting expectations or maintaining first impressions? The stability of teachers' expectations of students' mathematics achievement. *Learning and Instruction*, *75*, 101483. https://doi.org/10.1016/j.learninstruc.2021.101483

A Peer Tutor with a Partial Growth Mindset

Katie Bjorkman
Richard Bland College of William & Mary

The failure of educators to consider motivation and academic skills as areas of possible growth creates a partial growth mindset that can further disadvantage struggling students.

What is the Context?

The undergraduate mathematics peer tutor, Jake (gender-preserving pseudonym), was a case-study participant in an ethnographic study of a mathematics learning center as a figured world (Bjorkman, 2019).

Data Corpus:
- Ethnographic field notes (Emerson et al., 1995)
- Online survey of tutors
- Video-recording of tutoring interactions used in
- Stimulated recall (Dempsey, 2010; Lyle, 2010; Muir, 2010)
- Semi-structured interviews of peer tutors

which were analyzed using grounded theory methodology (Corbin & Strauss, 1990; Strauss & Corbin, 1994).

[The MLC] gives you a place to socialize with people ... First semester, I wasn't a tutor I found it pretty hard to make friends in engineering… I kind of felt isolated and alone. But once I started working here, and coming here a lot…then you make good friends through that.
Jake, Mathematics Peer Tutor

Who is Jake?

Jake (pseudonym) is a male, white/Caucasian sophomore engineering major who had worked as a tutor in the mathematics learning center (MLC) for less than a year at the time of the study. He expressed strong identification with the MLC and the other peer tutors.

Jake's view of others in the MLC related to their perceived mathematical maturity. The transcripts of Jake's interviews reveal that the likelihood of him using "us/we" pronouns instead of only "I/them" pronouns used to describe tutoring interactions increased as the course level of the student increased. For example, Jake was more likely to say "we worked on a problem" for an interaction with a student in Transition to Higher Mathematics (intro to proofs, 300 level) and more likely to say "I showed them how to do a problem" for a student in Calculus I (100 level).

A common theme for Jake was the role of struggle in attaining mathematical maturity and ability, that is, those 'below' his level could reach it through hard work if they were willing to so.

[The MLC is a place] with a lot of nerds who are struggling just as much as you are… You meet good people. It's a great work environment, a lot of intelligent people which helps you want to be smarter I know that's a big thing I like being here.
Jake, Mathematics Peer Tutor

I believe that everyone can do math… but there's some people who just don't put in the effort.
Jake, Mathematics Peer Tutor

Must it always come back to finding a reason why some children just can't learn, as opposed to finding a way to help them learn?
(Dweck, 2015, p. 2)

PDF of Poster | **Paper (including references)**

Use QR codes to access a PDF version of this poster and the one-page paper with references.

[W]hen they say they don't go to lecture or don't take notes I almost don't want to help…It's when I see students who haven't even tried the problem yet, that I usually get pretty frustrated…. I just kind of interpret that they're here to get the answers to get the grade and leave.
Jake, Mathematics Peer Tutor

Who 'Deserves' Jake's Help?

Jake expressed his beliefs throughout the study of what his role was – and wasn't – as a tutor as well as the ways he felt "really learning" math took place. Students that Jake wanted to help were characterized by having good academic habits like:
- Attending class
- Taking good notes
- Attempting problems before asking for help
- Looking back over their work after checking final answers
- Seeking help from their TA and professors

Jake saw value in mathematics courses "weeding out" students, not on the basis of ability, but of effort they put into learning. This effort was measured in good academic habits and Jake did not express a belief that students could grow to improve these habits/increase their effort or that he had a role in that growth and improvement.

I don't think that [students] should babied because I think they're - I think some sort of weeding out is good, in mathematics especially…. So weeding out I know is a bad term, because it means fail, but, there's some people who just don't put in the effort
Jake, Mathematics Peer Tutor

Discussion

Jake's model for student success and failure, while on the surface aligned with a growth mindset, is, in fact a limited growth mindset model that does not see self-regulation skills as areas where students can learn and grow and does not consider other factors that may lead students to perform poorly (Dweck, 2015; Granberg et al., 2021; Michaelson, 2017). It is likely that he implicitly communicated his belief about those students' identities as "bad students" to them through his interactions (Holland et al., 1998; Rattan et al., 2012). If Jake as a peer tutor isn't the one to help students who don't yet know "how to college" where should they turn?

The idea of a "growth mindset" has become quite popular in educational circles since the framework was put forward in the 1980's, and more recent research has indicated it can be a useful construct for helping students learn (Denworth, 2019). Yet, in my personal reflection, and experience with other professors, this fixed mindset in terms of student effort and self-regulation seems to exist in more educators than Jake, perhaps even more saliently now as students and educators struggle with the new normal after the pandemic (Bozkurt, 2022). We have long had evidence that teacher beliefs about students are influential in student outcomes (Rosenthal et al., 1966; Timmermans et al., 2021). Explicitly extending a growth mindset to students' self-regulation and motivation and designing courses toward fostering self-motivation and self-regulation as skills may help these "bad students" succeed in mathematics courses, rather than giving up or expending effort on ineffective learning strategies (Granberg et al., 2021).

Richard Bland College of WILLIAM & MARY

Learning to Learn Mathematics

Kathleen Guy
Center for the Transformation of Teaching Mathematics
Florida International University

The Workshop

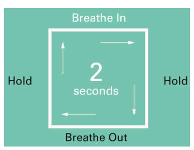
The workshop is being embedded into an extended section of Business Calculus where students meet for 2 hours twice a week vs the standard 75 minutes twice a week.

The workshop has three main educational objectives beyond the goals of the research:
1) Increase the academic success of students in Business Calculus.
2) Expose students to skills that can be used in future classes and their careers.
3) Bring value to the overall academic experience of students.

- Workshop focuses on helping students with nonacademic and mathematics strategy skills specifically tailored for Business Calculus
- Topics include growth mindset, metacognition, time management, mathematics anxiety, mathematics test-taking strategies, calculator skills, procrastination, feedback, goal setting, communication skills, mathematics literacy, and more.
- Workshop includes class discussions, group discussions, personal reflections, group activities, and videos.
- Workshop was piloted in Fall 2023.

Box Breathing

To delineate from regular class time (first 75 minutes) to workshop time (last 45 minutes), I guide the students through the box breathing exercise for around 2 minutes. We started this on the first day of class and do it every class time. This allows students to be exposed to a technique which can reduce stress and anxiety. I also guide them through this on exam day for a longer period.



Examples of Activities

Example 1: Communication Skills
(After watching a short video) On your own, think about how you feel working with others in a school or work setting. Do you enjoy or not enjoy working with others? What do you think makes a good group work? How do you plan on making your calculus group work this semester?

Example 2: Metacognition
(After watching a short video) Have you heard of metacognition before? How would you rate your own metacognition? Using some of the tips in the video, ***how do you plan to study for exam 1***? Be as detailed as possible.

Example 3: Reflecting on an Exam
Let's take some time to dissect the exam. Go through each question and **circle** whether you felt comfortable with the question or not while taking the exam. If the question was partially incorrect or fully incorrect, write the mistake(s) you made or what might have confused you while attempting it.

Learning mathematics in higher education is so much more than learning content. It requires students to have metacognition skills, growth mindset, time management, emotion control, and more! **HOW CAN WE HELP OUR STUDENTS???**

I have created a special workshop which is being facilitated as an educational intervention that helps students learn to learn mathematics by focusing on nonacademic skills and mathematics strategy skills specific to learning Business Calculus.

Conceptual Framework

Scan to **download lesson plans, activities, data collection tools, and more.**

Investigating the effectiveness of the intervention by measuring mathematics anxiety, self-efficacy, and academic achievements throughout the semester. Additionally, investigating the usefulness of the intervention from the student perspective.

FIU FLORIDA INTERNATIONAL UNIVERSITY

The Research

To measure the effectiveness and usefulness of the intervention, I am conducting a quasi-experimental study within an action research paradigm using mixed methods.

Research Questions

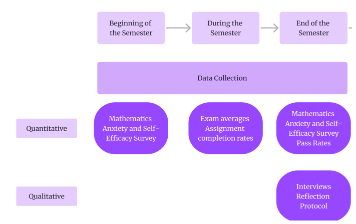
1) What is the effectiveness of an intervention for Business Calculus focused on nonacademic and mathematics strategy skills regarding students' academic achievements, mathematics anxiety, and mathematics self-efficacy?
   a) How do the academic achievements, mathematics anxiety, and mathematics self-efficacy of students enrolled in an intervention for Business Calculus vary, if at all, over the course of a semester with demographics of students as covariates?
   b) How do the academic achievements, mathematics anxiety, and mathematics self-efficacy of students NOT enrolled in an intervention for Business Calculus vary, if at all, over the course of a semester with demographics of students as covariates?
   c) How do the changes, if any, in academic achievements, mathematics anxiety, and mathematics self-efficacy compare between the students enrolled in the intervention and the students not enrolled in the intervention?
2) How do students enrolled in the intervention perceive the usefulness of that intervention?
3) What is the relationship between students' perceived usefulness of the intervention and students' academic achievements, mathematics anxiety, and mathematics self-efficacy?

Data

Two sections of Business Calculus:

Control Group: Standard twice a week 75 min course
Intervention group: Twice a week 2 hours course (where the workshop is embedded in the extra 45 minutes)

Both quantitative and qualitative data are being collected throughout the semester.



The GOAL:

To investigate if providing an innovative, unique, and engaging experience in higher education relating to **learning to learn mathematics** can impact student outcomes in a positive way and provide a resource students find useful

Using Stories to Understand the Attitudes of Students in Entry-Level Mathematics Courses

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Lander University

Chase T. Worley
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Keywords: Mathematical Attitudes; Mathematical Autobiographies

It has long been recognized that attitudes towards mathematics impact how students learn and the degrees to which they achieve in mathematics (Hwang & Son, 2021). Most work on college students' attitudes towards mathematics revolve around students who are pre-service teachers (Watson, 2019; Zazkis, 2015) or in-service teachers (Drake et al., 2001) and its impact on teaching. In this study, we describe the types of mathematical autobiographies that students possess in an entry- or freshman-level mathematics course (e.g., College Algebra). Namely, we add on to Watson's (2019) dissertation findings where she found that pre-service teachers' prompts to humanize mathematics fit into three categories: (1) Friendly - mathematics has been a lifelong friend; (2) Familial - mathematics is like a brother, sister, father, grandmother, etc; and (3) Antagonist - mathematics is a bully or trying to hold a person back.

The goal of our study was to further examine and refine these categories and to assess whether these categories were useful in describing other populations of students, such as those enrolling in a freshman-level, non-majors, mathematics course. To assess this, we asked students participating in a 'Launch into the University' initiative over the summer of 2023 to humanize mathematics and tell their story or relationship with mathematics throughout their schooling. These students were taking the university's freshman-level English and Mathematics courses (e.g., English 101 and College Algebra) with extra support from embedded tutors. Many of these students had their high school learning impacted by COVID-19 and the ensuing pandemic. In our analysis, we found that Watson's (2019) categories remained useful and accurate in describing the types of stories that students wrote in regard to their relationship with mathematics. We also found that an additional category was useful in describing these students' stories. Namely, that mathematics can be a 'Frenemy.' Watson (2019) described Frenemy to be under the category of Antagonist; however, we found that this description was useful as a separate category and more akin to Drake and colleagues' (2001) 'roller-coaster' relationship. We found this distinction to be important as students in the Antagonist category had (mostly) negative leaning stories. This was in contrast to what we saw as a Frenemy or roller-coaster type of relationship, where students expressed how they used to be friends with mathematics (or had a more positive view), and some experience or experiences caused their relationship to change directions. It was also noticeable how impactful the teacher is in impacting a student's story, consistent with previous work on this subject (e.g., Watson, 2019)

In this poster session we hope to prompt the following discussion: *What can we do as instructors to help to shift student's perceptions about mathematics? How can we leverage student's stories to help them succeed in early mathematics courses?*

References

Drake, C., Spillane, J. P., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject-matter context. *Journal of Curriculum Studies*, *33*(1), 1-23.

Hwang, S., & Son, T. (2021). Students' Attitude toward Mathematics and Its Relationship with Mathematics Achievement. *Journal of Education and e-Learning Research*, *8*(3), 272-280.

Watson, L. (2019). *Elementary prospective teachers and the nature of mathematics: An explanatory phenomenological study* [Doctoral dissertation, Middle Tennessee State University]. MTSU JEWL Scholar. https://jewlscholar.mtsu.edu/handle/mtsu/6080

Zazkis, D. (2015). Monsters, lovers and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics*, *35*(1), 33-38.

Professional Mathematicians' Levels of Understanding and Pseudo-Objectification

Kyle Flanagan
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Keywords: Abstraction, Mathematical Understanding, Professional Mathematician

The notion of mathematical understanding has always been central to mathematics education research, including how students understand mathematical concepts (e.g., Wawro et al., 2019; Zandieh, 2000). More recently, researchers have been interested in the mathematical practices of mathematicians, including how they understand mathematical concepts (e.g., Flanagan, 2022; Oehrtman et al., 2019; Shepherd & van de Sande, 2014; Wilkerson-Jerde & Wilensky, 2011). Related to understanding is the notion of abstraction, which includes various process-object theories (e.g., Dubinsky & McDonald, 2001; Sfard, 1991). Likely the most prominent theory of abstraction is Piaget's reflective abstraction and his broader genetic epistemology (Piaget, 1970). In this broader theory, Piaget (1964) distinguished between two different ways to understand a mathematical concept, namely "act on it," and to being able to "understand the process of this transformation, and as a consequence to understand the way the object is constructed" (p. 176).

Using this distinction by Piaget, this study introduces three different theoretical levels of understanding: pseudo-object-level, process-level, and object-level. Pseudo-object-level understanding consists of being able to act on the concept like an object without understanding the underlying processes. Process-level understanding consists of being able to understand the underlying processes of the concept. Object-level understanding consists of being able to both act on the concept and understand the underlying processes. Pseudo-object-level understanding also closely aligns with the notion of a pseudostructural conception (Linchevsky & Sfard, 1991; Sfard, 1992; see also Zandieh, 2000).

Through using these levels of understanding, this study addresses the following two research questions:

1. In what ways do professional mathematicians operate with highly-abstract, advanced mathematical concepts at different levels of understanding?
2. What factors can influence a professional mathematician's level of understanding for a given mathematical concept?

To answer these research questions, three semi-structured Zoom interviews (Hammer & Wildavsky, 1993) were conducted with six professional mathematicians specializing in algebra research. The first interview was a semi-structured task-based interview, where the participants completed two tasks with category theory concepts unfamiliar to the participants. The second interview utilized the participants' own research journal publications to generate discussion on what influenced their understanding of the concepts they used in that journal article. The final interview was more reflective, utilizing stimulated recall to triangulate the other two interviews.

The primary forms of data analysis consisted of conceptual analysis (Thompson, 2008) and thematic analysis (Braun & Clarke, 2006). The results provided evidence that professional mathematicians often function at a pseudo-object-level understanding for various mathematical concepts, and that they tend to operate differently with the concepts depending on their level of understanding. Moreover, numerous factors were shown to influence mathematicians' level of understanding, including the particular way the concept is being utilized in their work, as well as other sociocultural factors like one's field of study or what their research community values.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77-101.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at the university level: An ICMI study* (pp. 275-282). Kluwer Academic Publishers.

Flanagan, K. (2022). Confronting abstraction: An analysis of mathematicians' concept images and definitions. In Karunakaran, S. S. & Higgins, A (Eds.), *Proceedings of the 24th Annual Conference on Research on Undergraduate Mathematics Education* (pp. 177-184). Boston, MA.

Hammer, D., & Wildavsky, A. (1993). The open-ended, semistructured interview: An (almost) operational guide. In *Craftways: On the organization of scholarly work* (pp. 57-101). Transaction Publishers.

Linchevsky, L., & Sfard, A. (1991). Rules without reasons as process without objects - The case of equations and inequalities. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference of the Psychology of Mathematics Education, 2*, 317-324.

Oehrtman, M., Soto-Johnson, H., & Hancock, B. (2019). Experts' construction of mathematical meaning for derivatives and integrals of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education, 5*, 394-423.

Piaget, J. (1964). Part I: Cognitive development in children: Piaget. Development and learning. *Journal of Research in Science Teaching*, *2*(3), 176–186.

Piaget, J. (1970). *Genetic epistemology*. W.W. Norton & Company Inc.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification- The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA notes, no. 25, pp. 59-84). Mathematical Association of America.

Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding - From novice to expert. *The Journal of Mathematical Behavior, 35*, 74-86.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 31-49).

Wawro, M., Watson, K., & Zandieh, M. (2019). Student understanding of linear combinations of eigenvectors. *ZDM Mathematics Education, 51*, 1111-1123.

Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math?: Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics, 78*(1), 21-43.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CMBS Issues in Mathematics Education, 8*, 103-127.

Is it “fast enough?”: Two undergraduate students’ guided reinvention of the comparison test

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Kristen Vroom
Michigan State University

Keywords: Comparison test, series, sequences, guided reinvention

Series convergence is a key part of the calculus curriculum. Much of the related education research has focused on students’ understandings of sequences and series (Earls, 2017; Eckman & Roh, 2022; Askgun & Duru, 2007; Kung & Speer, 2013; Martínez-Planell et al., 2012), the notation (Larsen et al., 2022), or textbook analysis (González-Martín, 2010). However, there is limited research on ways to support students’ conjecturing about series convergence.

Our data comes from a teaching experiment (Steffe & Thompson, 2000) with two post-Calculus I students, Lara and Stella. Instruction during this experiment was informed by Realistic Mathematics Education’s guided reinvention heuristic (Freudenthal, 2005; Gravemeijer & Doorman, 1999). Here, we focus on the last four (of 11) sessions in which the students were guided to reinvent statements about series convergence. During these sessions, the students played a game in which they imagined moving an object a certain number of feet each day. A sequence $\{x_n\}$ indicated how many feet x_n to move the object on any given day n. Winning the game required predicting the object’s location if the experiment continued indefinitely (i.e., $\sum x_n$ converged and the students reported the value that it converged to). After playing the game with various sequences, the students wrote a “cheat sheet” with tips for which sequences win/lose the game and some warnings for future players. We interpreted much of their cheat sheet as common tests for series convergence (i.e., p-series test, comparison test). For this study, we investigated: *How did two undergraduate students reinvent the comparison test?* We explored how their version of the comparison test emerged from their previous informal ideas by rewatching videos and re-reading transcripts to identify relevant key moments.

Stella and Lara’s reinvention was tied to their understanding of the sequences n^{-1} and n^{-2} and their corresponding series. Using speed as a metaphor, they identified that a winning sequence must converge to zero at a certain “rate” using n^{-2} and n^{-1} as their benchmarks for a sequence which decreased “fast enough” or “too slowly,” respectively. The teacher-researcher then graphed the sequence n^{-1} and a generic sequence that was greater than n^{-1} for all n, asking the students if the generic sequence would win. Lara quickly inferred that it would lose, and Stella explained it converges to zero “at a slower rate” than n^{-1}. They formalized this on their cheat sheet as “if a sequence is always above the sequence n^{-1} it will lose the game” (which we interpret as a specific case of the comparison test). They compared other sequences to n^{-1} and were unsure whether sequences in between n^{-1} and n^{-2} won the game since they decreased “faster” than n^{-1} but “slower” than n^{-2}. To find clarity, they played (and lost) the game with the sequence $(2n)^{-1}$. They generalized this with two warnings: “sequences that are below n^{-1} will not necessarily win the game” and “sequences above n^{-2} might lose,” which we interpret as the comparison test’s possible inconclusiveness. Finally, with the teacher-researcher’s guidance, the two students considered their benchmark sequences (n^{-1}, n^{-2}) as hypothetical losing and winning sequences, and this encouraged them to write the conjecture which we interpret as (part of) the comparison test: “any sequence that is below a known winning sequence but above 0 will win.”

Our study is an existence proof of two students being guided to reinvent the comparison test, showing that students can be supported to conjecture tests for series convergence. In the poster session, we will elaborate on our data and discuss our future work.

References

Akgun, L., & Duru, A. (2007). Misunderstandings and difficulties in learning sequence and series: A case study. *Research in Mathematical Education*, *11*(1), 75-85.

Earls, D. J. (2017). *Students' Misconceptions of Sequences and Series in Second Semester Calculus* (Doctoral dissertation, University of New Hampshire).

Eckman, D., & Roh, K. H. (2022). Students' Intuitive Meanings for Infinite Series Convergence and Corresponding Implications. *North American Chapter of the International Group for the Psychology of Mathematics Education*.

Freudenthal, H. (2005). *Revisiting mathematics education: China lectures* (Vol. 9). Springer Science & Business Media.

González-Martín, A. S. (2010). The concept of series: teachers' conceptions and teaching practices. In *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education (PME34)* (Vol. 3, pp. 33-40).

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational studies in mathematics*, *39*(1-3), 111-129.

Kung, D., & Speer, N. (2013). Do they really get it? Evaluating evidence of student understanding of power series. *PRIMUS*, *23*(5), 419-433.

Larsen, S., Strand, S., & Vroom, K. (2022). How undergraduate students think about summation (sigma) notation. *International Journal of Research in Undergraduate Mathematics Education*, 1-23.

Martínez-Planell, R., Gonzalez, A. C., DiCristina, G., & Acevedo, V. (2012). Students' conception of infinite series. *Educational Studies in Mathematics*, *81*, 235-249.

Steffe, L. P., Thompson, P. W., & von Glasersfeld, E. (2012). Teaching experiment methodology underlying principles and essential elements. In *Handbook of research design in mathematics and science education* (pp. 267-306). Routledge.

Supporting Mathematical Connections Between Abstract Algebra and Secondary Mathematics for Preservice Teachers

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Keywords: Abstract Algebra, Secondary School Mathematics, Preservice Teachers, Connections

The numerous mathematical connections between abstract algebra and secondary school mathematics have led professional organizations and mathematics education researchers to identify abstract algebra as an important course for preservice mathematics teachers (CBMS, 2001; CBMS, 2012; Wasserman et al., 2017). Despite what experts might say are clear mathematical connections between abstract algebra and secondary mathematics, many preservice teachers do not see the relevance of an undergraduate abstract algebra course to their teaching practice. In fact, many of them see no relation at all between abstract algebra and secondary school mathematics (Christy & Sparks, 2015; Ticknor, 2012). To address this need for direct connections between mathematical knowledge and pedagogical practice, this poster presents a portion of the results from a larger qualitative study that followed students with an interest in secondary mathematics teaching through their enrollment in an abstract algebra course (Gray, 2021). In particular, the study investigates the types of tasks and course activities that aid preservice teachers in establishing connections between abstract algebra and secondary school mathematics.

For the larger project, survey data was collected from collegiate mathematics faculty and practicing secondary mathematics teachers regarding mathematical connections between abstract algebra and secondary school mathematics. Mathematical connections that these participants identified as important were subsequently utilized to create short instructional tasks that were situated in secondary school mathematics contexts. The abstract algebra course involved in the larger study included undergraduate and graduate students with various majors meaning that the course was not explicitly designed for preservice teachers. The instructional tasks were implemented as warm-up activities before and after the relevant abstract algebra material was taught. This was an adaptation of Wasserman and his colleague's (2017) model *building up from/stepping down to teaching practice*. While all students in the course completed these activities, this study focuses on seven participants: six undergraduates, all of whom expressed interest in teaching careers, and one graduate student with secondary teaching experience. Data collection for these participants included written work on eight warm-up activities, two written student questionnaires (pre/post course), two individual interviews, one midsemester group interview with the undergraduate participants, and classroom observations of 38 class sessions.

Grounded theory coding (Charmaz, 2006) was used to analyze all data sources. The most common mathematical content connection that the participants made is the connection between algebraic structures and their properties in abstract algebra to various concepts in secondary school mathematics. For example, one task asked participants to solve a system of linear equations and justify/explain each step. Six of the seven participants were able to connect this secondary mathematics problem to abstract algebra concepts like inverse and identity. The seventh participant could identify that connections exist making abstract algebra an important course for preservice teachers but did not display evidence of making any connections. Results suggest that abstract algebra instructors can support preservice teachers by providing them opportunities to explore connections themselves.

References

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. Thousand Oaks, CA: Sage Publications, Inc.

Christy, D., & Sparks, R. (2015). Abstract algebra to secondary school algebra: Building bridges. *Journal of Mathematics Education at Teachers College,* 6(2), 37 – 42.

Conference Board of Mathematical Sciences (2001). The mathematical education of teachers part I. Providence, RI: American Mathematical Society.

Conference Board of Mathematical Sciences (2012). The mathematical education of teachers II (Issues in Mathematics Education, Vol. 17). Providence, RI: American Mathematical Society.

Gray, C. (2021). *Exploring mathematical connections preservice teachers make between Abstract Algebra and secondary mathematics: Implications for learning and teaching* (Doctoral dissertation, University of New Hampshire).

Suominen, A.L. (2015). *Abstract algebra and secondary school mathematics: identifying and classifying mathematical connections* (Doctoral dissertation, University of Georgia).

Ticknor, C.S. (2012). Situated learning in an abstract algebra classroom. *Educational Studies in Mathematics, 81*(3), 307 – 323.

Wasserman, N.H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J.P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS, 27*(6), 559 – 578.

Obstacles of Learning Proof by Cases

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Keywords: Introduction to Proofs, Proof by Cases, Lesson Analysis Manuscript

Epistemological Obstacles (EOs) are defined as challenges that students face when exposed to concepts that contradict their pre-existing mathematical intuition (Norton & Arnold, 2023). The objective of our poster is to address EOs that emerge during the lesson on proving by cases, rather than circumventing them. This approach, as pursued by Norton and Arnold (2023), elicited EOs such as the Principal of Universal Generalization (PUG) and understanding the difference between the role versus the value of a quantified variable (Qrv). PUG entails proving "for all" statements with an arbitrary value. It was mainly encountered during mathematical induction by Norton and Arnold (2022), but its relation to proofs by cases was not studied. When proving statements with PUG, it may not occur to students that they need specific (x=0), or arbitrary (x is even) cases. Qrv teaches students to prioritize maintaining generality; however, loss of generality is allowed with proof by cases as it is preserved in the aggregate. The obstacle comes from students struggling to understand that generality is maintained differently depending on the situation. The two EOs are closely related; Qrv can be thought of as a direct product of PUG.

At the study institution, students generally take introductory proofs their Junior or Sophomore year and are required to have passed the second of a three-part calculus sequence and discrete math. As such, they are expected to have experience with symbolic logic and operations on conditional statements (converse, contrapositive, negation), set theory and proofs on sets, and mathematical induction. Many upper-level students transfer from the local community college, which introduces a deficiency in the information on what students have and have not covered.

Class size ranges from 12-20 students in a room with whiteboards on most walls. An undergraduate learning assistant was present to walk around during group discussions to aid students and note student reactions for post lesson reflections. The instructor has a degree in mathematics, with a specialization in undergraduate mathematics education. He relied on that background to focus the course goals on inculcating social practices, specifically the norms and practices of the mathematics community. He intentionally structured the class to get students to 'play mathematician' by questioning arguments, positing hypotheses, and proving or disproving those hypotheses through routine think-pair-share cycles.

Our course begins with a review of symbolic logic. We then cover translating written propositions into logical forms, performing operations on those propositions, and discovering how universal and existential quantifiers modify them. After covering our first proof method, the direct proof, we move on to proof by cases. This lesson was immediately after direct proof because the instructor was following the order of topics set out in the course textbook suggested by the department. Our poster will detail the lesson tasks, the rational behind the tasks, and student thinking that came up during the lesson. In particular, the instructor was aiming to introduce students to proof by cases, when to use cases, what cases to consider, and how to use the statement 'without loss of generality'. Through reflection on student reactions to lecture content, certain areas of the lesson could be improved. In particular, in-class examples can be chosen in a way that requires students to think more about how the constructs are used. Our intention with this poster is to get ideas on how we can improve the lesson and produce a lesson analysis manuscript (Corey & Jones, 2023) from this material for a future journal submission.

Acknowledgements

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References

Corey, D. L., & Jones, S. R. (2023). Sharing instructional knowledge via lesson analysis. *International Journal of Mathematical Education in Science and Technology, 54*(7), 1320-1327.

Norton, A., & Arnold, R. (2023). Epistemological Obstacles Related to Treating Logical Implications as Actions: The Case of Mary. In Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2). University of Nevada, Reno.

Norton, A., Arnold, R., Vladislav Kokushkin, & Tiraphatna, M. (2022). Addressing the Cognitive Gap in Mathematical Induction. International Journal of Research in Undergraduate Mathematics Education, 9(2), 295–321. https://doi.org/10.1007/s40753-022-00163-2

Obstacles of Learning Proof by Cases

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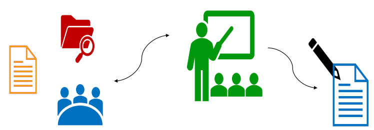
Teaching Materials:

Feedback:

BACKGROUND: Epistemological obstacles (EO's) are challenges students face when exposed to concepts that contradict their pre-existing math intuition. A framework of eliciting and addressing EOs was developed by and practiced at a partnering institution. The particular purpose of this work was to use that EO framework in crafting a proof by cases lesson.

METHODS/CONTEXT

1. Lesson was taught at Eastern US University, class size about 15-20 students.
2. Student Population: had discrete math pre-requisite where a bit of proof was introduced
3. Structured lesson to elicit the EOs that arise in proof by cases- based on suggestions from collaborators



Like the Following:
Task 1) ... For each 'for all' statement, discuss what might make proving the claim tricky. What additional info might you want to know about the arbitrary variables to prove the statement?
a. If n∈"Z", then n^2+n is an even integer
b. Given two integers a, b, if a and b have opposite parity (i.e. characteristic of being even or odd), then a+b is odd
c. ∀x∈R, x≤|x|
d. ∀(x,y)∈R^2 , y=mx for some m∈"R"
e. If x^2−5x+6>0, then x>3 or x<2

RESULTS

- Students found it difficult to determine whether to use cases and which cases to use
- Were some tasks too basic & didn't elicit obstacles? We need insights on how to improve the lesson if so
- Or is proof by cases too easy & there aren't many EOs?
- Or is there only 1 EO (identifying cases thinking is needed and what are the cases) but that itself is a huge challenge?
- Or are things more nuanced because of contextual differences: students at the study institution had experience with baby proofs in discrete mathematics

Students find it difficult to determine **whether** to use cases and **which** cases to use early in the proof writing process.

How might you address such obstacles? What would be productive tasks to move students through this struggle?

Some tasks are too easy:

4) Look over the following proofs. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

b) Prove: $\forall\, a, b \in \mathrm{R}, |ab| = |a||b|$

Assume WLOG, that a>0 and b<0, then

$$|ab| = a(-b) = |a||b|$$

4) Look over the following proofs. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

a) Prove: $\forall\, a, b \in \mathrm{R}, |a+b| \ \leq |a| + |b|$

Case 1: assume a>0, b>0, then $|a+b| = a+b$, and $|a| + |b| = a+b$. Since $a+b = a+b$, then $|a+b| = a+b \leq a+b = |a|+|b|$ as well

Case 2: assume a<0, b<0, then $|a+b| = -(a+b)$, and $|a|+|b| = -a+(-b)$. Since $-(a+b) = -a-b$, then $|a+b| = -(a+b) \leq -a+(-b) = |a|+|b|$ as well

Case 3: assume a and b have opposite sign. Without loss of generality (wlog) assume a>0 and b<0 then

$$|a+b| = a-(-b) = a-|b| \leq |a|+|b|$$

How might **you** adjust these problems?

3b) Given two integers a, b, if a and b have opposite parity (i.e. characteristic of being even or odd), then a+b is odd

2) Find all cases that help prove the following. Time permitting, then prove/disprove the statements

c. $(\forall\, a, b \in \mathrm{R}, ab{=}0)\ (a = 0 \vee b = 0)$

d. $\forall\, n \in \mathrm{Z}$ where n is odd, n divided by 4 leaves a remainder of 1 or 3

This task was added to the second implementation in the current spring semester. Part (d) specifically was added as a challenging problem for students to think about which cases are needed.

4) Look over the following proof. Discuss with your peers whether you think they appropriately prove the statements they address. Note any critiques

Prove: $\forall\, a, b \in \mathrm{R}, |a+b| \ \leq |a| + |b|$

Case 1: assume a>0, b>0, then $|a+b| = a+b$, and $|a| + |b| = a+b$. Since $a+b = a+b$, then $|a+b| = a+b \leq a+b = |a|+|b|$ as well

Case 2: assume a<0, b<0, then $|a+b| = -(a+b)$, and $|a|+|b| = -a+(-b)$. Since $-(a+b) = -a-b$, then $|a+b| = -(a+b) \leq -a+(-b) = |a|+|b|$ as well

Case 3: assume a and b have opposite sign. Without loss of generality (wlog) assume a>0 and b<0 then

$$|a+b| = a-(-b) = a-|b| \leq |a|+|b|$$

Compared to the first implementation, many students did not realize that we need the zero case

6) Determine if the follow statement is true or false. If true, prove your claim. If false, give a counter example: '$\forall\, n \in \mathrm{Z}$ where n is odd, n divided by 4 leaves a remainder of 1 or 3'

This problem was skipped in the first implementation due to time constraints. But in the second implementation, the problem was tackled and asked students to identify cases.

The EO we most frequently noticed was the Principle of Universal Generalization (PUG), which is when students struggle with the concept of maintaining generality in a proof. In our case, they confuse the generality of the proof with the generality of cases. Another is Qrv which is when students mistake the role and the value of a quantified variable.

Acknowledgments
Support for this work was provided by NSF DUE #2141626 and by a 4-VA Collaborative Research Grant.

Online Real Analysis for In-Service Teachers' Meanings for Continuity and Differentiability

Zachary Bettersworth, Western Kentucky University
Nicholas Fortune, Western Kentucky University
Sarah Hartman, Western Kentucky University

In this poster presentation, we present pilot data that emerged from working with secondary math teachers enrolled in a fully online, Introductory Real Analysis course in the Southeastern United States. This pilot data will inform our second attempt at implementing the recommendations of the Project ULTRA (Wasserman et al., 2022) research group in an online context with in-service teachers in a future academic semester.

Keywords: Secondary Teachers, Real Analysis, Continuity, Differentiability, Online

Calculus, and the related mathematical content, is one of the most important, foundational areas of mathematics for people intending to pursue careers in STEM (Bressoud et al., 2016; Rasmussen et al., 2019). Teachers serve a crucial role in the community by providing students with early and continued education about mathematical ideas which build throughout students' K-12 education, up to and including AP Calculus courses (Ball, 2003; Bressoud et al., 2016; Frank & Thompson, 2021). Understanding how in-service teachers develop their own mathematical understanding of ideas related to Real Analysis topics will therefore provide important insights to researchers about the development of teachers' mathematical meanings (Thompson, 2016), or mathematical knowledge, for teaching calculus (Ball, 1990; Hill et al., 2008). Several researchers have reported on students' understanding of mathematical logic in the context of limit and the Intermediate Value Theorem (Roh, 2010; Sellers et al., 2017), and the format of proof-based mathematics courses as well as secondary teachers' views of the utility if proof in their own teaching (Wasserman et al., 2018; Weber, 2004). This work adds to the current RUME literature by demonstrating a burgeoning relationship between secondary teachers' understanding of continuity and differentiability in the context of a fully online Real Analysis course for teachers and their perception of the impact understanding these mathematical ideas has on their teaching.

Our research team collected pilot data during the Fall 2023 academic semester with the goal of better understanding how in-service teachers develop deeper conceptual understandings of important concepts for understanding introductory Real Analysis. The students are currently employed at secondary schools in the surrounding area teaching secondary mathematics. Our research questions are: *1) What meanings for continuity and differentiability do in-service teachers construct in an online analysis course? How are these meanings related? 2) In what ways can the instructional recommendations of Project ULTRA be implemented in the context of an online course for in-service teachers?*

In this poster, we will present preliminary results from our initial implementation of the instructional recommendations from the Project ULTRA research group in their textbook *Understanding Analysis and its Connections to Secondary Mathematics Teaching* (Wasserman et al., 2022). This work represents an initial attempt to extend this research group's instructional recommendations in a fully online Real Analysis course for in-service teachers in the Southeastern United States. We borrowed and added tasks from Sellers et al. (2017) to better understand the teachers' attention to mathematical logic in addition to their conceptual understanding of the concepts of continuity and differentiability when interpreting alternative statements for the Intermediate Value Theorem and the Mean Value Theorem.

References

Ball, D. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449–466.

Ball, D. (2003). What mathematical knowledge is needed for teaching mathematics. *Ecretary's Summit on Mathematics, US Department of Education*, *3*(3), 30–40.

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). Teaching and learning of calculus. *Springer Nature*.

Frank, K., & Thompson, P. W. (2021). School students' preparation for calculus in the United States. *ZDM – Mathematics Education*, *53*(3), Article 3. https://doi.org/10.1007/s11858-021-01231-8

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4), 430–511.

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Brief Report: Characteristics of Precalculus Through Calculus 2 Programs: Insights From a National Census Survey. *Journal for Research in Mathematics Education*, *50*(1), 98–111. https://doi.org/10.5951/jresematheduc.50.1.0098

Roh, K. H. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics*, *73*(3), Article 3.

Sellers, M., Roh, K. H., & David Parr, E. (2017). A comparison of Calculus, Transition-to-Proof, and Advanced Calculus Student Quantifications in complex mathematical statements. *Proceeding of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 285–297.

Thompson, P. W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Researching mathematical meanings for teaching* (pp. 435–461). Taylor & Francis.

Wasserman, N., Fukawa-Connelly, T., Weber, K., Meija-Ramos, J. P., & Abbott, S. (2022). *Understanding Analysis and its Connections to Secondary Mathematics Teaching*. Springer.

Wasserman, N., Weber, K., Villanueva, M., & Meija-Ramos, J. (2018). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis. *The Journal of Mathematical Behavior*, *50*, 74–89.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115–133.

Understanding the IVT and the MVT does not necessarily mean that **secondary teachers view their students' learning** as **metaphorically "continuous."**

Statement 3 = False Not every c value in (a,b) is equal to a corresponding y value, N, between f(a)=2.5 and f(b)=0.
Statement 4 = False. Each y value between f(a)=2.5 and f(b)=0 do not have the same x value in (a,b). This would make the graph not be a function.

-Student E (Graph 1)

Statement 3 – False
There exist real numbers c in $[a, b]$ such that

$f(c) = N$ is not between $f(a)$ and $f(b)$.

Statement 4 – True
Since f is continuous, every N between $f(a)$

and $f(b)$ corresponds to a c in domain (a, b).

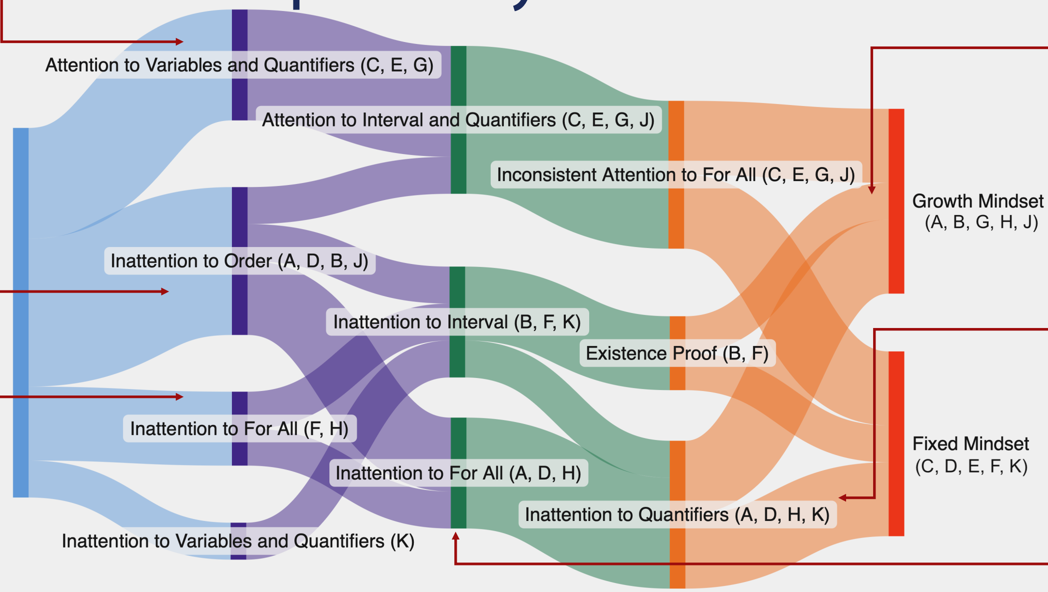
-Student A (Graph 1)

- Statement 3 is true because for all the c values, the N value is always between f(a) and f(b), so we can definitely find at least one case.
- Statement 4 is true because all the N values between f(a) and f(b) correspond to at least one c value, so we can definitely find at least one case.

-Student F (Graph 4)

"I think this statement is a **good preliminary definition** for the graph of a continuous function, but it is **not at the caliber that is necessary** for analysis."

-Student B (Reflection 1)



"**This book is almost antagonistic with its problems** that keep bringing up things to consider, (I know a good textbook should and we are grad students) But **with our kids** we are at the same time trying to **teach then skills to use that work**."

"… it seems to me **such a huge gap** between general Algebra 2 and what the text is saying it is **almost not useful at all**, it has to be **translated several times into a language that they can understand**, and with that translating come the inferences and ambiguity that has caused the problems all along."

-Student D (Reflection 1)

Since Statement 1 says that there exists a point c in [a,b] where $f'(c) = \frac{f(b)-f(a)}{b-a}$. There exists means that there just has to be one c in (a,b) where this is true.

-Student K (Graph 3)

Zac Bettersworth, Nick Fortune, Sarah Hartman
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LITERATURE & PROBLEM STATEMENT

Students' attention to logical quantifiers in the context of theorems related to continuity and differentiability impacts their understanding of theorems in Real Analysis courses (e.g., Roh et al., 2010; Wasserman et al., 2018).

There need to be clear connections between content and teaching, and those connections need to be authentic from a pre-service teacher's perspective (Wasserman et al., 2017).

But what does this look like for in-service teachers?

CONTEXT

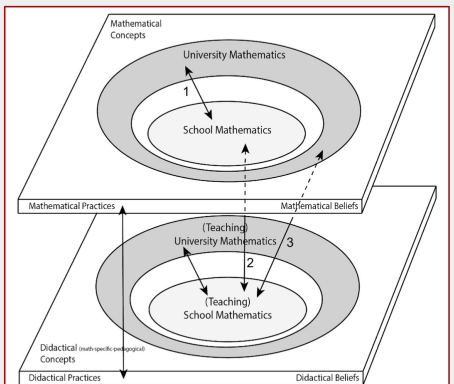
<u>In-service</u> secondary teachers enrolled in a graduate-level, <u>online</u> Real Analysis course at a regional comprehensive University in the Southeastern United States.

2 out of 10 students had completed an undergraduate Real Analysis course prior to

TASKS & REFLECTIONS

Borrowed/modified tasks from Sellers et al. (2017) to better understand the teachers' attention to mathematical logic in addition to their conceptual understanding of the concepts of continuity and differentiability when interpreting alternative statements about the IVT and MVT.

They read Project ULTRA book and reflected on the relationship between their understanding of analysis and the high school mathematics that they teach.



First published by Springer Nature in 2023 in Wasserman, Buchbinder, and Buchholtz's article, "Making university mathematics matter for secondary teacher preparation", in ZDM–Mathematics Education, 55(4), pp. 719-736." *1, 2, and 3 added by authors.*

Proceedings

RESEARCH QUESTIONS

Our work adds to the current RUME literature by demonstrating a burgeoning relationship between <u>**in-service secondary teachers'**</u> understanding of continuity and differentiability in the context of a fully <u>**online**</u> Real Analysis course and their perception of the impact understanding these mathematical ideas has on their teaching.

1) What meanings for continuity and differentiability do in-service teachers construct in an online analysis course? How are these meanings related?

2) In what ways can the instructional recommendations of Project ULTRA be implemented in the context of an online course for in-service teachers?

DATA ANALYSIS

The teachers' responses to the IVT and MVT tasks were coded using Roh et al. (2010) codes in addition to emergent codes.

Teachers' reflection data was coded using open coding with an attention to their meanings for continuity, differentiability, quantifier order in the IVT and MVT statements, and perceptions of teaching or teacher change.

More than half of the pre-service secondary mathematics teachers' interpretations of De Villiers' (1990) roles of proof are often aligned on communication and explanation compared to other roles of proof.

Learner vs. Mathematical Perspectives: Pre-Service Secondary Mathematics Teachers' Interpretations of Roles of Proof

Yi-Yin (Winnie) Ko, Sarah K. Bleiler-Baxter, & Jordan Kirby

Indiana State University, Middle Tennessee State University, & Francis Marion University

Tables, References, & Contact Information

SCAN ME

Background

- Over the past decade, increased attention has been given to teaching and learning proof in school mathematics, as its central roles in mathematics include verification, explanation, discovery, systemization, and communication (e.g., De Villiers, 1990).
- In addition, proving and proof are fundamental to doing mathematics, communicating mathematical ideas, and developing mathematical knowledge (Stylianides, 2007).
- Given that pre-service secondary mathematics teachers (PSMTs) play a critical role in shaping their future students' experiences with proof and proving, we seek to better understand how PSMTs' interpretations of the verification, explanation, discovery, systemization, and communication roles of proof align or misalign with that of the discipline (as represented in De Villiers' (1990) seminal work).

Study

- This study involved 29 PSMTs in the transition-to-proof course titled "Discrete Mathematics" during two semesters (n = 11 in Fall 2020 and n = 18 in Fall 2021) at a Midwestern University in the United States.
- While gaining an appreciation of these five roles that proof can play in mathematics was one of the course goals, the instructor who is the first author did not explicitly name these five roles as a topic of study with students in class until the study week before final exams.
- The primary source of data for this study was PSMTs' written responses to the take-home portion of the final exam, where they were asked to read De Villiers's (1990) article and describe the five given roles of proof using their own words.
- The first author and one research assistant read all PSMTs' responses individually and coded each participant's self-description as not in alignment (N), **in partial alignment (P), or in alignment (Y)** with De Villiers's (1990) explanation.

De Villiers' Roles of Proof & Sample Responses

Verification Proof serves as a means to obtain conviction and establish the truth of a mathematical statement.	Y	*This role deals with the justification behind mathematical proofs. It reasons with the truth of the statements and is used as conviction to proof.*
	P	*To be able to verify a proof can be a difficult task. "Absolute certainty also does not exist in real mathematical research..." (18). This is not the only way verification could be used in fact most are taught to check the answer at the end of a problem. This form of verification can be extremely effective in easing the doubts of an incorrect conclusion.*
	N	*Verification can be used to show that something is incorrect or find mistakes in the problem.*
Explanation Proof serves as a means to promote understanding and illumination of why underlying mathematical concepts are true.	Y	*We must also say why something is true. Many believe the function of explanation is far more important than verification because it gives us an insight into the proof and in return makes us more intelligent.*
	P	*Explanations in proofs play the part of describing why a statement might be true. They are the validation of a proof's truthfulness. The explanation of a proof also helps strengthen the verification of the proof itself.*
	N	*Explanation doesn't necessarily aim to prove something true. It is more about giving reason or explanation as to why it is true.*
Systemization Proof serves as a means to structure unrelated definitions and previously-proven results to gain a global perspective of mathematical concepts.	Y	*Another function of a proof is as a means of systematization. Systematization helps show the logical relationships between statements. These statements are deductive systems such as theorems and definitions. Systematization makes it easier for all of the audience to understand how the proof is built.*
	P	*The systemization of a proof is the organizational process. This role helps simplify a proof and combine similar ideas into a more uniform one. Systemization can help strengthen a proof as well.*
	N	*Systematization is the organization of results. It can help identify errors within the work, connect similar ideas to promote understanding and simplifications, it provides a "global perspective" which I interpreted as a greater understanding within multiple views, and it can provide applications for many things.*
Discovery Proof as a means to expose unexpected results beyond the given mathematical scope or context	Y	*A proof that is written for the sake of discovery follows logical deductions to come to a conclusion that was not necessarily already known or thought to be true. This type of proof can also be used as a way to generalize something—if someone discovers the reasoning behind why a certain hypothesis is true while they are working through examples, that knowledge can then be used to construct a more general proof than before. Proofs of discovery do not always have to be done this way, however. They can also be done just by analyzing the properties of objects that are given.*
	P	*According to de Villiers, there is always a discovery of mathematical results when writing a proof. There are also many instances that new results are found for a proof. Many times, a mathematician has not gone and verified their theorems making their results different.*
	N	*The fourth role of a proof is discovery, in which some concepts and theories are actually discovered on accident through proof. This is not just for new theories to the world, but also new theories to the student. Students can stumble upon untaught theories through class work in proofs.*
Communication Proof serves as a means to expose unexpected results beyond the given mathematical scope or context.	Y	*The final role is communication, where the real value of proofs is seen through debates about the topic. Through debates, we learn new perspectives on topics that we would not have known otherwise and can implement these perspectives into our own work.*
	P	*The article by de Villiers points out that a proof is a way of communicating among people who are working on mathematics. The argument made in the proof is addressed to a human audience. We are trying to convince the audience our results are correct meaning in our explanations we need to use communication so they can understand it.*
	N	*Communication is key to any problem. This means explaining the reasoning good enough to communicate to the reader that it is a proof. It goes hand in hand with explanation. Being able to transcribe the meaning behind the proof to get the reasoning across.*

Preservice Teachers’ Knowledge and Understanding of Fraction Division

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Keywords: Cognitive types, conceptions, content knowledge, content understanding

It is important for preservice teachers (PSTs) to have a deep and connected conceptual understanding of mathematical concepts they will teach in their future classrooms (Ball, 1990; Da Ponte & Chapman, 2015). For this reason, teacher education programs work to provide opportunities for preservice teachers' (PSTs’) to build understanding of the various ways students conceive of significant mathematical ideas and practices (e.g., Ball et al., 2008; Hill, 2010; Li, 2008; Li & Kulm, 2008; Lo & Luo, 2012; Tirosh, 2000). Therefore, it is important to investigate PSTs’ conceptions, connections, notions, and understandings of specific mathematics concepts so that results can be used to inform effective instruction in content courses for future teachers.

The goal of this comparative case study was to investigate PSTs’ *content knowledge* (Ball et al., 2008; Coskun et al., 2023) and *content understanding* (Bair & Rich, 2011; Hill et al., 2004, 2008) as the basis for measuring *cognitive types of teacher content knowledge* (Tchoshanov, 2011). The *cognitive types* include knowledge of facts and procedures (Type 1), knowledge of concepts and connections (Type 2), and knowledge of models and generalizations (Type 3). Our poster presentation will share results from the study for only *Type 2* with a content focus on fraction division. The research question for this portion of the study was: What are PSTs’ mathematical conceptions, connections, and notions of the fraction division concept?

We utilized a linear progression of constructs to guide the analysis of our data to answer the research question. That is, content knowledge □ cognitive types of teacher content knowledge □ specialized content knowledge (SCK) progression □ content understanding □ (conceptions, connections, and notions), with each construct rooted in the literature.

Data was collected from 18 preservice elementary teachers enrolled in mathematics content course at a public university in the Southeastern United States. The study analyzed both quantitative and qualitative data. For the quantitative data, a modified form of the Teacher Content Knowledge Surveys (TCKS) designed by Tchoshanov (2011) was used. For the qualitative data, pre- and post-term exit tickets were used. The SCK progression framework (Bair & Rich, 2011) informed the qualitative data analyzes.

Results also showed that Blair and Rich’s (2011) SCK progression framework was associated with levels of knowledge as measured by the TCKS. We found that 30% of PSTs with less than a numerical average value (< 0.5) of *Type 2* TCKS pre-post scores were those at Level 0 and Level 1 of SCK progression. Also, 40% of participants who were above average in TCKS scores were at Level 3 and Level 4 and the rest were at average. This study informed mathematics teacher educators and instructors of content mathematics courses for PSTs on the characterization of the kind and qualities of aspects of content knowledge and content understanding that PSTs possess and how to support them in specific mathematical concepts.

References

Bair, S. L., & Rich, B. S. (2011). Characterizing the development of specialized mathematical content knowledge for teaching in algebraic reasoning and number theory. *Mathematical Thinking and Learning*, *13*(4), 292-321.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The elementary school journal*, *90*(4), 449-466. https://doi.org/10.1086/461626
Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554
Coskun, S. D., Sitrava, R. T., & Bostan, M. I. (2023). Pre-service elementary teachers' noticing expertise of students' mathematical thinking: The case of fractions. *International Journal of Mathematical Education in Science and Technology*, *54*(6), 982-999.
Da Ponte, J. P., & Chapman, O. (2015). Prospective mathematics teachers' learning and knowledge for teaching. In *Handbook of international research in mathematics education* (pp. 287-308). Routledge.
Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *41*(5), 513-545.
Hill, H. C, Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-Specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400. http://www.jstor.org/stable/40539304
Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, *105*(1), 11–30. https://doi.org/10.1086/428763
Li, Y. (2008). What do students need to learn about division of fractions? *Mathematics teaching in the middle school*, *13*(9), 546-552. https://doi.org/10.5951/MTMS.13.9.0546
Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM*, *40*, 833-843.
Lo, J. J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education*, *15*, 481-500.
Tchoshanov, M. (2011). Relationship between teacher content knowledge, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics, 76*(2), 141–164.
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for research in Mathematics Education*, *31*(1), 5-25. https://doi.org/10.2307/749817

Preservice Teachers' Knowledge and Understanding of Fraction Division

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Introduction

It is important for preservice teachers (PSTs) to have a deep and connected conceptual understanding of mathematical concepts they will teach in their future classrooms (Ball, 1990; Da Ponte & Chapman, 2015). For this reason, teacher education programs work to provide opportunities for preservice teachers' (PSTs') to build understanding of the various ways students conceive of significant mathematical ideas and practices (e.g., Ball et al., 2008; Hill, 2010; Li & Kulm, 2008; Lo & Luo, 2012). The goal of this study was to investigate PSTs' *content knowledge* (Ball et al., 2008; Coskun et al., 2023) and *content understanding* (Bair & Rich, 2011; Hill et al., 2008) as the basis for measuring *cognitive types of teacher content knowledge* (Tchoshanov, 2011).

Theoretical Framework

The study utilized a linear progression of constructs to guide the analysis of our data to answer the research question. That is, content knowledge → cognitive types of teacher content knowledge → specialized content knowledge (SCK) progression → content understanding → (conceptions, connections, and notions), with each construct rooted in the literature.

Methodology

Participants

- ✓ A total of 18 elementary preservice teachers in a mathematics content course were involved.
- ✓ Ten of the participants were selected for further qualitative analysis. Those were the top 5 and lower 5 PSTs with Teacher Content Knowledge Surveys (TCKS) scores.

Data Source and Design

Written Exit Tickets. PSTs were provided with both pre-and post exit tickets modified from the TCKS (Tchoshanov, 2011) as the qualitative data.

Instruments, Tools, Constructs, and Data Coding

Instruments and Tools

Analyzed with the SCK Progression Framework (Blair & Rich, 2011)

Five-level SCK Progression Framework

Indicators: entry (Level 0), emerging (Level 1), developing (Level 2), maturing (Level 3), and deep and connected mathematical knowledge for teaching (Level 4).

Data Coding: Bair and Rich's framework to provide a fine-grained process for coding PSTs' *content understanding* using the five-level progression of indicators.

Research Question

What are PSTs' mathematical conceptions, connections, and notions of the fraction division concept?

Big Takeaways

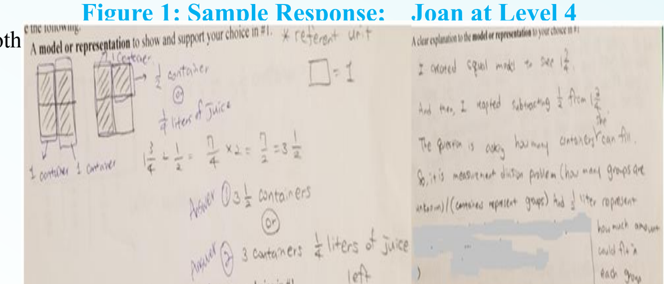
Content Knowledge

Blair and Rich's (2011) SCK Progression Framework **Levels** appear to be associated with the Cognitive Type of Teacher Content Knowledge as measured by the TCKS (Tchoshanov, 2011).

Content Understanding

PSTs with strong and explicit fraction division conceptions, connections, and notions were at Level 4 or Level 3 but getting nearer to Level 4.

Figure 1: Sample Response: Joan at Level 4



Joan: *"the question is asking how many ½ liters containers can be filled? So, it is a measurement division problem".* ***Fraction Division Conception***

Joan: "I created equal model to see $1\frac{3}{4}$ and then I *repeated subtracting* $\frac{1}{2}$ *from* $1\frac{3}{4}$".

Joan's model brings out (i) the problem structure, (ii) the action in the problem, (iii) appropriate representational connections to the problem situation and the context of solution, and (iv) the possible ways the solution can be expressed

Future Research

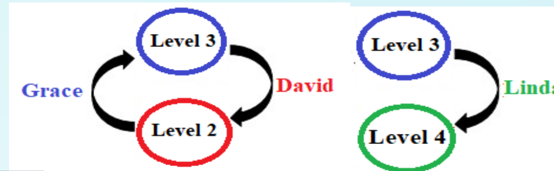
Investigating PSTs cognitive processes that interact with their conceptions, connections, and notions about specific mathematical concepts

Results

Level 0	Level 1	Level 2	Level 3	Level 4
Emily(**b**)	Sarah(**b**)	Kelly(**t**)	Linda(**t**)	Joan(**t**)
	Kate(**b**)	Grace(**b**)	David(**t**)	Lizz(**t**)
		Nina(**b**)		

PSTs with **"b"** against names are the **Bottom 5** for the TCKS scores

Those with **"t"** against them are the **Top 5** for the TCKS scores



Conclusion and Implication

The results of this study will inform instructors content mathematics courses for PSTs the need to integrate pedagogical strategies that will support the development of a deeper conceptual knowledge and understanding of any mathematical concept.

References

Bair, S. L., & Rich, B. S. (2011). Characterizing the development of specialized mathematical content knowledge for teaching in algebraic reasoning and number theory. *Mathematical Thinking and Learning*, *13*(4), 292-321.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The elementary school journal*, *90*(4), 449-466. https://doi.org/10.1086/461626.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Coskun, S. D., Sitrava, R. T., & Bostan, M. I. (2023). Pre-service elementary teachers' noticing expertise of students' mathematical thinking: The case of fractions. *International Journal of Mathematical Education in Science and Technology*, *54*(6), 982-999.

Da Ponte, J. P., & Chapman, O. (2015). Prospective mathematics teachers' learning and knowledge for teaching. In *Handbook of international research in mathematics education* (pp. 287-308). Routledge.

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *41*(5), 513-545.

Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM*, *40*, 833-843.

Lo, J. J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education*, *15*, 481-500.

Tchoshanov, M. (2011). Relationship between teacher content knowledge, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics, 76*(2), 141–164.

Creating Metaphors for Linear Algebra Concepts: Developing Cognitive, Behavioral, and Affective Domains

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Issues with Assessments Practices

Traditional Assessments:

- Privilege symbolism and formalism (Nemirovsky & Bunn, 2022)
- Characterize mathematics as mechanisms (Nemirovsky et al., 2023)
- Viewed as "easy" to implement, non-disruptive, and "real math" (Munter et al., 2014)
- Avoid mathematical aesthetics (Sinclair et al., 2006)

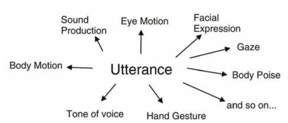
Current recommendations and ways forward:

Assessments should engage students: (Meta)Cognitively, Affectively, and Behaviourally (MAA Instructional Practices Guide).

Targeting Three Domains via Intentional Design of an Embodied Assessment

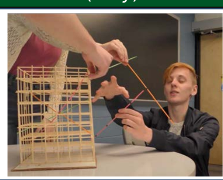
Embodied Cognition

Embodied design applied to "tools whose operatory function is **engineered** specifically so as to . . . **cultivate** . . . the development of particular **sensorimotor schemes** as a condition for masterful control of the environment in accord with **task demands**" and thereby "come to **ground mathematical concepts** we want these students to **learn**" (Abrahamson & Bakker, 2016, p.5, Piaget, 1954)



(Nemirovsky & Ferrara, 2009)

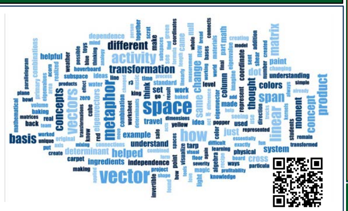
Formative Assessments (Play)



Summative Metaphor Creation

Two-part summative assessment:

1. **Written part:** Create and reflect on your own metaphors about various LA concept pairs (e.g., *Explain how you arrived at this metaphor and how it fits the formal concept.*
1. **Collaborative focus group:** Sharing and co-constructing new improved metaphors with peers.



Research Questions

1. What metaphors do students create for linear algebra concepts? How do they manifest?
2. What cognitive, behavioral, and affective behaviors emerge as students share their metaphors for linear algebra concepts?

Results: Students' Engagement During Summative Assessment

(Meta)Cognitive

Metaphor Resources:

- **Personal Experiences**
  - ***Hobbies***
  - ***Childhood experiences***
  - ***Professional Interests/Major:***
  - ***In-the-Moment***
- **Classroom Experiences**
- **Linguistics**

Evaluating the "Soundness" of Metaphors

A. **Incomplete model of real-world**
B. **Incomplete representation of linear algebra concept**

Extending Metaphor across Multiple LA Concepts: Popcorn Popper

kernels represent the **vectors**. Heat is the **matrix**, it's the one that's changing the vectors. The ones that pop and go into the bowl underneath the popcorn machine, those are in the **column space**, because they get **transformed** into a new location.	And the kernels that pop or just like fly out [throws hand over shoulder] of the machine and land on the floor, those are in the **null space**, because they get sent out of the system entirely.	Kernels that stay in the popcorn machine are like "**eigenvectors** because they didn't' really move" or "pop out".	The degree to which they [popcorn that don't move] got semi-popped is like the **eigenvalue** of the situation because it tells you how much it changed even though it's still like where it was before."

Affective

Behavioral

Slinky Metaphor for Invertible

You can actually ***see it*** *go back … repeats* gesture and then says *"to itself"*

In some of my science classes we used to … how ***sound moves***

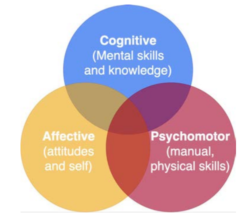
If we could physically see it (grab both ends with eyes focused on her slinky and Billy sweeps his hand)

It always goes back to itself *(brings hands together)*

Findings illustrate the notion that "***Learning – the creation of the new – comes about from situations that were previously unimagined, impossible, unusual, & unexpected … expressed through diagram/gesture [and verbiage] interplay"*** (de Freitas & Sinclair, 2014, p. 109).

Framework for Engagement

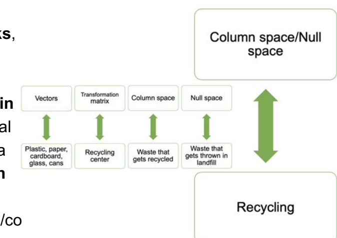
(Anderson & Krathwohl, 2016; Bloom et al., 1956)



The Knowledge Dimension	The Cognitive Dimension					
	Remembering	Understanding	Applying	Analyzing	Evaluating	Creating
Factual knowledge	Listing	Summarizing	Classifying	Ordering	Ranking	Combining
Conceptual Knowledge	Describing	Interpreting	Experimenting	Explaining	Assessing	Planning
Procedural Knowledge	Tabulating	Predicting	Calculating	Differentiating	Concluding	Composing
Meta-cognitive knowledge	Using Appropriately	Executing	Constructing	Achieving	Taking action	Actualizing

Conceptual Metaphors

Appear as **cognitive links**, or mappings, between a **source domain** (abstract/formal concept) and a **target domain** (real-world/physical/concrete).



(Lakoff & Núñez, 1997)

Discussion

1. **Embodiment espouses learning from concrete to abstract** (Tall, 2013)
   - This activity moves students from abstract back to concrete (make full circle)
   - Forced to think deeply about the abstract
2. **Metaphor Assessment Privileges**
   - Blending Creativity & Metacognitive
   - Togetherness
   - Aesthetics of Mathematics (surprise, joy, frustration)
3. **Learning is inventiveness** (de Freitas & Sinclair, 2014)

References

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* Cambridge University Press.

Lakoff, G., & Núñez (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. English *Ed.), *Mathematical reasoning: Analogies, metaphors, and images.* Mahwah, NJ: Lawrence Erlbaum Associates.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*, 159-174.

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How Undergraduates Make Decisions When Co-Validating Mathematical Proofs

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Keywords: Proof, Introduction to Proof, Validation

Mathematical proof is central to the work of mathematicians (e.g., Weber, 2008), and is a center pointe of the undergraduate curriculum for mathematics majors (Stylianides, G., 2007). As undergraduates learn to prove, they need to learn various proving processes, not the least of which are construction and validation (e.g., Ko, & Rose, 2022; de Villiers, 1990; Segal, 2000). Construction and validation of proofs go hand-in-hand, yet they involve unique skills and processes (Kirsten & Greefrath, 2023; Selden & Selden, 2003)). Therefore, it is crucial to understand how undergraduates make decisions about proof construction and proof validation. In recent work, we explored how students in an inquiry-based Introduction-to-Proof (ITP) course make decisions when it comes to *co-constructing* proofs in small groups (Bleiler-Baxter et al., 2023). In this poster, we seek to extend the findings from that research by exploring situations where students are *co-validating* proofs in small groups. We ask: To what aspects of proof do students in an undergraduate inquiry-based ITP course attend when co-validating mathematical proofs within small groups?

This study involves students enrolled in an ITP course at a public university in the southeastern United States in the fall of 2023 taught by the second author. Students in this class were asked, at three separate time points during the semester (early, middle, late), to review sample arguments and to determine if those arguments should "count" as mathematical proof. They were given independent think-time to make their initial validations of the given arguments, and then they were tasked with coming to a consensus in their pre-assigned small-groups, as to whether each argument was or was not a proof and why. We explore the transcripts of audio data of small group discussions across these various points in the semester to investigate how students make decisions about validating proofs.

To analyze the small-group discussions, we use a three-part theoretical coding scheme informed by A. Stylianides' (2007) definition of proof: set of accepted statements, modes of argumentation, and modes of argument representation. We use this definition as a coding scheme to be able to discern what students prioritize when making decisions on what counts as proof. For example, students may focus on the way the definition of divisibility is used in an argument (i.e., set of accepted statements), the way proof by contradiction is employed (i.e., mode of argumentation), or the way a table is used to organize an argument (i.e., mode of argument representation).

Submitting as a poster will allow us to include various student excerpts from transcripts highlighting each of the codes we have used as well as how their frequency changes over the course of the semester. We hope to gain from attendees at the conference what questions they may have about our data and other data analysis methodologies to consider as we begin finalizing our data analysis in the spring.

References

Ko, Y. Y., & Rose, M. K. (2022). Considering proofs: Pre-service secondary mathematics teachers' criteria for self-constructed and student-generated arguments. *Journal of Mathematical Behavior, 68*. https://doi.org/10.1016/j.jmathb.2022.100999

Bleiler-Baxter, S. K., Kirby, Jordan, E., & Reed, S. D. (2023). Understanding authority in small-group co-constructions of mathematical proof. *The Journal of Mathematical Behavior, 70*, 101015. https://doi.org/10.1016/j.jmathb.2022.101015

De Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Kirsten, K., & Greefrath, G. (2023). Proof construction and in-process validation–Validation activities of undergraduates in constructing mathematical proofs. *The Journal of Mathematical Behavior*, *70*, 101064. https://doi.org/10.1016/j.jmathb.2023.101064

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: can undergraduates tell whether an argument proves a theorem. *Journal for Research in Mathematics Education, 34*(1), 4–36. https://doi.org/10.2307/30034698

Segal, J. (2000). Learning about mathematical proof: Conviction and validity. *Journal of Mathematical Behavior, 18(*2), 191-200. https://doi.org/10.1016/S0732-3123(99)00028-0

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 28(*3), 289-321. https://doi.org/10.2307/30034869

Stylianides, G. J. (2007). Investigating the guidance offered to teachers in curriculum materials: The case of proof in mathematics. *International Journal of Science and Mathematics Education, 5*, 191-215. https://doi.org/10.1007/s10763-007-9074-y

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*(4), 431-459. https://doi.org/10.5951/jresematheduc.39.4.0431

Proofs Project Partners: Implementing Research-Based Instruction Across Institutions

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Virginia Tech	Virginia Commonwealth	James Madison
Ahsan Chowdhury	Anderson Norton	Rachel Arnold
George Mason	Virginia Tech	Virginia Tech

Keywords: epistemological obstacles, logical reasoning, proof and proving

The Proofs Project is an NSF-funded research project investigating persistent challenges students experience in Introduction to Proofs courses. These challenges are framed in terms of epistemological obstacles (EOs) students and instructors experience during classroom interaction, even within research-based practice (Brousseau, 2002; Sierpińska, 1987). The project has generated video modules illustrating such obstacles and tasks that instructors can use to elicit and address them. Now, in the second year of the project, instructors at three different universities have begun using project materials in their Introduction to Proofs courses. These three instructors participated in a three-day Proofs Project workshop at Virginia Tech, during the Summer of 2023. Here, we report on their experiences implementing project materials to intentionally elicit and address EOs they have experienced in prior instruction. Data come from surveys conducted at the end of the summer workshop; at the start of the following Fall semester, as they taught their Introduction to Proofs course; and at the end of that semester. In addition to discussion of these findings, Authors 2, 3, and 4 will report on their individual experiences in implementing project materials to elicit and address students' experience of persistent challenges.

At the time of the workshop, one instructor had taught Introduction to Proofs more than five times; one was teaching the course for the fifth time; and one was "terrified" to be teaching the course for the first time. Following the summer workshop, all three instructors were eager to implement project materials and, indeed, had begun making instructional plans using those materials, during the workshop. They anticipated the materials—especially tasks and video clips—would help them address and elicit EOs related to students' treatment of logical implications, quantification, and particular methods for proving, such as proof by cases.

Each instructor elicited EOs that had been identified in prior research. One instructor noted, "For example, even after discussion, students will still take the negation of P implies Q as 'If P, then not Q' or 'If not P, then not Q,' when trying to negate a sentence written in everyday language." (cf., Dawkins & Hub, 2017; Epp, 2003). She found that when students became aware of such obstacles, they were able to begin addressing them on their own, or in discussion with peers. In reflecting on such an approach, another instructor shared the following:

> I think I am gaining an appreciation for the value of eliciting EOs. It's a lot easier to just let some of them fly under the radar, particularly in the classroom. I suppose in the past I had assumed/hoped that students were struggling with some of these concepts while engaging with the homework—that's how I remember experiencing many of them for the first time.

During the presentation, the three instructors will share, compare, and contrast such experiences.

Acknowledgments

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References

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990*, 79–117.

Dawkins, P. C., & Hub, A. (2017). Explicating the concept of contrapositive equivalence. In Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Epp, S. (2003). The role of logic in teaching proofs. *The American Mathematical Monthly, 110*, 886–899.

Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics* 18, 371–397.

Using Trees to See a Forest: Leveraging Machine Learning to Classify Student Thinking

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Keywords: Decision Trees, Qualitative Coding, Machine Learning

Introduction

ChatGPT has caught the world's attention and has researchers considering the ways artificial intelligence can be integrated into human endeavors (Kasneci et al., 2023). This methodological poster will present a systematic method to apply a particular machine learning classification model, Decision Tree (Song & Lu, 2015), to perform and extend the scale of qualitative analysis in mathematics education research.

Implementation of a Decision Tree

There are five general phases to the creation of a decision tree: data preparation, feature analysis, test label creation, decision tree coding, and parameter tuning.

1. Data Preparation: While the original data can take any form (verbal, student work, hand gestures, etc), it will need to be unitized in some way. The size of each unit of data will be affected by the theoretical framework employed and the attributes to be identified from the unit.

2. Feature Analysis: Each unit of data will then be coded for important attributes that are referred to as *features*. Features can be categorical or numeric, and do not need to be of just one type for a single decision tree. For example, features tied to a student's free response to a task could include [a] Presence of a response *(yes/no)*, [b] Presence/Absence of key words/phrases *(binary or number of occurrences)*, or [c] Use of a particular representation *(graphical, algebraic, tabular, etc)*. Identifying features is a critical human task in creating a decision tree, much like qualitatively coding according to a theoretical framework.

3. Test Labels Creation: We need to provide labeled data the decision tree algorithm can learn from. This is the second critical human researcher task that again mirrors a typical qualitative coding task where labels are determined by the theoretical framework employed.

4. Decision Tree Coding: Open source libraires such as Python's pandas library (Reschke, et al., 2020) contain the majority of the technical aspects of employing a machine learning model. After preparing the data, completing the feature analysis, and labeling the data, creating the decision tree is as easy as calling RandomForestClassifier().fit.

5. Parameter Tuning: Numerous parameters can be defined by the researcher to improve the accuracy of predictions as well as the amount of data that is used to train and test the data.

Discussion

We believe that incorporating machine learning can significantly benefit the qualitative coding process and is feasible for many mathematics education researchers. One major benefit to this implementation is the scale and speed at which data can be analyzed. Another major benefit was seen in our codebook. By carefully considering what features might be present and what labels they may correlate to, we have improved the replicability of our qualitative coding based on our extensive codebook. Finally, classification algorithms would provide an excellent engine for automating the analysis of student work based on qualitative coding of student responses. This would allow researchers to bridge the gap and provide practical uses of their esoteric work.

References

Kasneci, E., Seßler, K., Küchemann, S., Bannert, M., Dementieva, D., Fischer, F., Gasser, U., Groh, G., Gunnemann, S., Hullermeier, E., Krusche, S., Kutyniok, G., Michaeli, T., Nerdel, C., Pfeffer, J., Poquet, O., Sailer, M., Schmidt, A., Seidel, T., Stadler, M., & Kasneci, G. (2023). ChatGPT for good? On opportunities and challenges of large language models for education. Learning and individual differences, 103, 102274.

Raschka, S., Patterson, J., & Nolet, C. (2020). Machine learning in python: Main developments and technology trends in data science, machine learning, and artificial intelligence. Information, 11(4), 193.

Song, Y. Y. & Ying, L. U. (2015). Decision tree methods: applications for classification and prediction. Shanghai archives of psychiatry, 27(2), 130.

Analyzing Calculus Textbooks' Introduction of the Dynamic Derivative Using a Framework of Shape-Thinking and (Co)Variational Reasoning

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Keywords: calculus, content analysis, covariational reasoning, shape thinking, textbook analysis

Research over the past few decades demonstrates that calculus students have difficulties making sense of the derivative conceptually, even if they can calculate derivatives fluently (e.g., Orton, 1983; Carlson et al., 2002; Thompson & Carlson, 2017; Epstein, 2013). In a review of literature on ideas foundational to calculus learning, Thompson and Harel (2021) argue that that the underdevelopment of covariational reasoning skills may be the missing link in students' ability to engage deeply with major calculus concepts including the derivative. Covariational reasoning, (i.e., reasoning about how quantities vary simultaneously and in relation to one another) is evident in the MAA's recommended conception of the derivative as dynamic (Bressoud, 2015). Recent work emphasizes the need to promote "derivative as instantaneous rate of change or as a measure of sensitivity of one variable to change in another," rather than the static conception of "derivative as slopes of tangent lines," which does not make explicit the relationship between quantities (Bressoud et al., 2015, p. 18).

A few recent studies suggest that calculus textbooks do not frequently provide opportunities for students to develop covariational reasoning skills with quantities related to the derivative. These studies used Carlson et al.'s (2002) five levels of (co)variational reasoning to compare the opportunities for covariational reasoning across international calculus textbooks (Chen, 2023) and between a regular and applied Calculus textbook (Mkhatshwa, 2022).

This study adds a different perspective on opportunities for reasoning covariationally in the context of the derivative by analyzing introductory material about the derivative in a two widely used calculus textbooks, one traditional (Larson & Edwards, 2018) and one reform (Hughes-Hallet et al., 2013) by applying a novel framework introduced by Tasova et al., 2018. Their framework for analyzing written curriculum, which they used to analyze precalculus content in calculus textbooks, merges Thompson & Carlson's (2017) (co)variational reasoning framework with Moore and Thompson's (2015) framework for static and emergent graphical shape thinking. This multi-layered approach allows for a more nuanced description of the opportunities for students to understand the derivative from a covariational perspective and for exploration of the alignment with a dynamic or static conception of the derivative.

In this poster, I present the results of my directed content analysis (Hsieh et al., 2005) to answer the following questions: *What opportunities does each textbook provide in its narrative and worked example sections for students to reason about quantities and variables related to the derivative as dynamic?* and *What are the similarities/differences between the textbooks in the nature and frequency of these opportunities?* Results are supplemented with an expanded presentation of Tasova et al.'s (2018) framework that includes examples illustrating the range of opportunities textbooks provide. Finally, I present suggestions for both research and practice that focus on supplementing textbook opportunities to foster a dynamic conception and the implications of these results for equity and inclusion in STEM.

Acknowledgments

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References

Bressoud, D. M., Johnston, E., Murphy, C., Rhea, K., Williams, J., & Zorn, P. (2015). The Calculus Sequence. In P. Zorn, C. S. Schumacher, & M. J. Siegel (Eds.), *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* (pp. 17–25). Mathematics Association of America. https://maa.org/sites/default/files/CUPM%20Guide.pdf

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378

Chen, Y. (2023). *An Analysis of Covariational Reasoning Pedagogy for the Introduction of Derivative in Selected Calculus Textbooks*. Teachers College Columbia.

Epstein, J. (2013). The Calculus Concept Inventory–Measurement of the Effect of Teaching Methodology in Mathematics. *Notices of the American Mathematical Society*, *60*(08), 1018. https://doi.org/10.1090/noti1033

Hsieh, H.-F., & Shannon, S. E. (2005). Three Approaches to Qualitative Content Analysis. *Qualitative Health Research*, *15*(9), 1277–1288. https://doi.org/10.1177/1049732305276687

Hughes-Hallett, D., McCallum, W. G., & Marks, E. J. (2013). Calculus: Single and Multivariable (6th ed.). Wiley.

Larson, R., & Edwards, B. H. (2018). Calculus of a single variable (11th ed). Brooks/Cole.

Mkhatshwa, T. P. (2022). Quantitative and covariational reasoning opportunities provided by calculus textbooks: The case of the derivative. *International Journal of Mathematical Education in Science and Technology*, 1–28. https://doi.org/10.1080/0020739X.2022.2129497

Moore, K. C., & Thompson, P. W. (2015). Shape Thinking and Students' Graphing Activity. In T. Fukawa-Connelly, N. Infante, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789).

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14(3), 235–250

Tasova, H. I., Stevens, I. E., & Moore, K. C. (2018). A Framework for Analyzing Written Curriculum from a Shape-Thinking and (Co)Variational Reasoning Perspective. In A. Weinberg, J. Rasmussen, M. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the Twenty-First Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1527–1533). http://sigmaa.maa.org/rume/RUME21.pdf

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), Compendium for research in mathematics education. (pp. 421–456). National Council of Teachers of Mathematics.

Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM – Mathematics Education*, *53*(3), 507–519. https://doi.org/10.1007/s11858-021-01270-1

Static and Dynamic: How Two Textbooks Introduce the Derivative

PRESENTER:
Allison Olshefke

MOTIVATION:

- Even when they can work with derivatives procedurally, students still struggle understanding the derivative conceptually.
- Covariational reasoning may help bridge this gap.
- How do textbooks promote conceptions of the derivative that are productive?

METHODS:

- I compared the intro chapter on derivatives from two commonly used calculus textbooks:
  - Larson's *Calculus*
  - Hughes Hallett et al.'s *Calculus: Single and Multivariable*
- I coded narrative sections of the textbooks using a framework that distinguishes between static and dynamic conceptions of the derivative.

MAIN RESULTS:

Frequency

	Static	Dynamic	Total
Larson	76.4% (n = 84)	23.6% (n = 26)	110
Hughes Hallett	46.5% (n = 33)	53.5% (n = 38)	71

Scan for references & more info!

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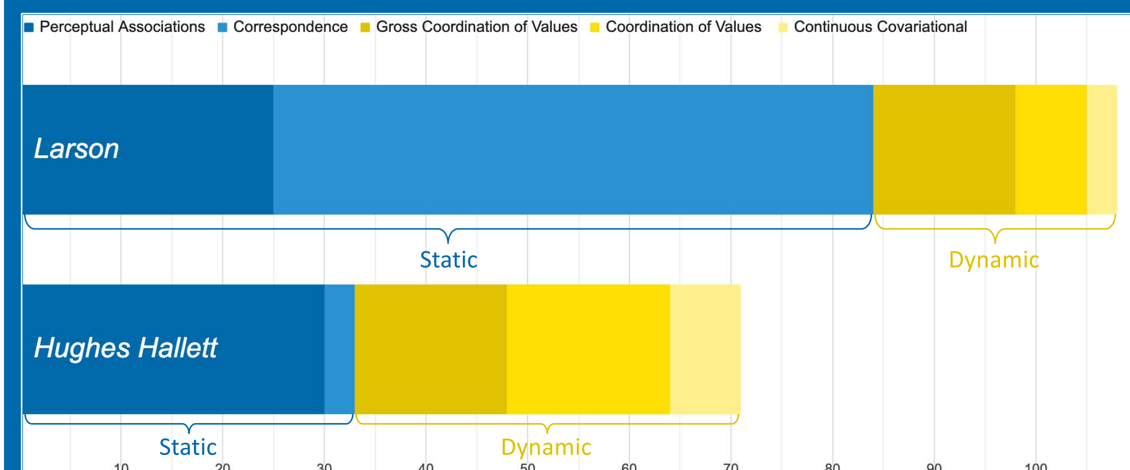
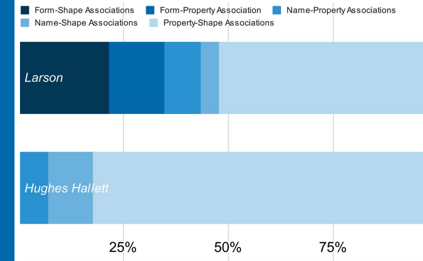
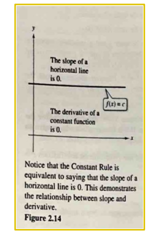
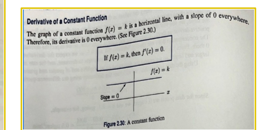
Calculus textbooks differ in the ways that they promote the **derivative** as static and dynamic.

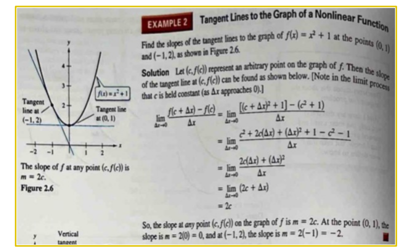








Property-Shape Association worked examples (*Larson* p. 110 & *Hughes Hallett* p. 94)



EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

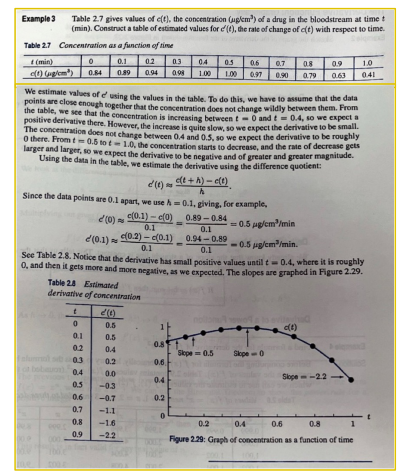
Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure 2.6.

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ can be found as shown below. [Note in the limit process that c is held constant (as Δx approaches 0).]

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(c+\Delta x)^2 + 1] - (c^2 + 1)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2c + \Delta x)$$
$$= 2c$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$.

Larson Correspondence worked example (p. 102)



Example 3 Table 2.7 gives values of $c(t)$, the concentration (μg/cm^3) of a drug in the bloodstream at time t (min). Construct a table of estimated values for $c'(t)$, the rate of change of $c(t)$ with respect to time.

Table 2.7 *Concentration as a function of time*

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c(t)$ (μg/cm^3)	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

We estimate values of c' using the values in the table. To do this, we have to assume that the data points are close enough together that the concentration does not change wildly between them. From the table, we see that the concentration is increasing between $t = 0$ and $t = 0.4$, so we expect a positive derivative there. However, the increase is quite slow, so we expect the derivative to be small. The concentration does not change between 0.4 and 0.5, so we expect the derivative to be roughly 0 there. From $t = 0.5$ to $t = 1.0$, the concentration starts to decrease, and the rate of decrease gets larger and larger, so we expect the derivative to be negative and of greater and greater magnitude.

Using the data in the table, we estimate the derivative using the difference quotient:

$$c'(t) \approx \frac{c(t+h) - c(t)}{h}.$$

Since the data points are 0.1 apart, we use $h = 0.1$, giving, for example,

$$c'(0) \approx \frac{c(0.1) - c(0)}{0.1} = \frac{0.89 - 0.84}{0.1} = 0.5\ \mu\text{g/cm}^3\text{/min}$$
$$c'(0.1) \approx \frac{c(0.2) - c(0.1)}{0.1} = \frac{0.94 - 0.89}{0.1} = 0.5\ \mu\text{g/cm}^3\text{/min}.$$

See Table 2.8. Notice that the derivative has small positive values until $t = 0.4$, where it is roughly 0, and then it gets more and more negative, as we expected. The slopes are graphed in Figure 2.29.

Table 2.8 *Estimated derivative of concentration*

t	$c'(t)$
0	0.5
0.1	0.5
0.2	0.4
0.3	0.2
0.4	0.0
0.5	−0.3
0.6	−0.7
0.7	−1.1
0.8	−1.6
0.9	−2.2

Figure 2.29: Graph of concentration as a function of time

Hughes Hallett Continuous Covariation Worked Example (p. 92-93)

Embodied Noticing in Mathematics: Pre-Service Teachers' Observations and Noticings

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Prior research highlights the critical importance of teachers' professional noticing skills (Jacobs et al., 2010; Sherin et al., 2011). Specifically, the concept of "embodied noticing," or the ability to interpret students' physical gestures and manipulations alongside their verbalizations, is gaining attention (Bondurant et al., 2023; Goldin et al., 2011; Walkoe et al., 2023). In this study, we investigated Pre-Service Teachers' (PTs) abilities to notice and evaluate students' embodied actions in mathematics (Bondurant et al., 2020; Bondurant et al., 2023). The aim of our study was twofold: to investigate PTs' skills in noticing students' embodied actions in math, and to determine how such noticing might contribute to more equitable classroom environments. We employed a sequential exploratory design, incorporating both qualitative and quantitative methods (Creswell & Plano Clark, 2017). Initial qualitative data were gathered from 20 PTs who participated in a 15-week online pedagogical course on elementary math education. PTs were asked to watch a video of a student engaged in a math task and to respond to a set of instructional activity prompts (Musser et al., 2013).

Our findings reveal that most PTs are still at the peripheral or transitional levels of noticing embodiment in all three stages: attending, interpreting, and deciding. In particular, they often overlook the embodied nuances and focus on evaluating the correctness of students' written and verbal responses using a deficit lens. This lack of attention to embodiment may be detrimental as it fails to consider the whole child's mathematical thinking and doing, possibly leading to an inequitable evaluation (Gutiérrez, 2008; 2009).

We recommend several interventions to improve PTs' embodied noticing skills:

1. Implement a mandatory multiple-viewing approach.
2. Utilize on-screen annotation tools for enhanced evaluation.
3. Initially, mute videos to emphasize embodiment, then enable sound for subsequent viewings.

Our study contributes to the field by offering refined instructional prompts, exemplars of embodied noticing, a framework for layering embodiment and noticing, and a rubric for evaluating embodiment in educational contexts. Future research should investigate the long-term impact of these interventions on PTs' noticing, teaching practices, and student outcomes.

Acknowledgments

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References

Bondurant, L., Moss, D., Bertolini-Smith, C., Poling, L., Soto, H., & Troup, J. (2023). Promoting rehumanizing mathematics through embodied-focused noticing. In T. Lamberg, & D. Moss (Eds.). *Proceedings of the forty-fifth annual meeting of the North American*

Chapter of the International Group for the Psychology of Mathematics Education. University of Nevada, Reno, 514-519. https://www.pmena.org/proceedings/

Bondurant, L., Poling, L., & Moss, D. (2020). An analysis of preservice elementary teachers' professional noticing skills in a mathematics education setting. *Journal of Practitioner Research*, *5*(2). https://doi.org/10.5038/2379-9951.5.2.1156

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273 – 285). Thousand Oaks: Sage Publications.

Walkoe, J., Williams-Pierce, C. (., Flood, V. J., & Walton, M. (2023). Toward Professional Development for Multimodal Teacher Noticing. *Journal for Research in Mathematics Education*, *54*(4), 279-285. https://doi.org/10.5951/jresematheduc-2020-0326

EMBODIED NOTICING IN MATHEMATICS: PRESERVICE TEACHERS OBSERVATIONS & NOTICINGS

NSF-EHR-DUE
Grant # 1835409

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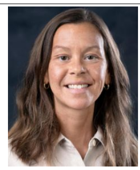

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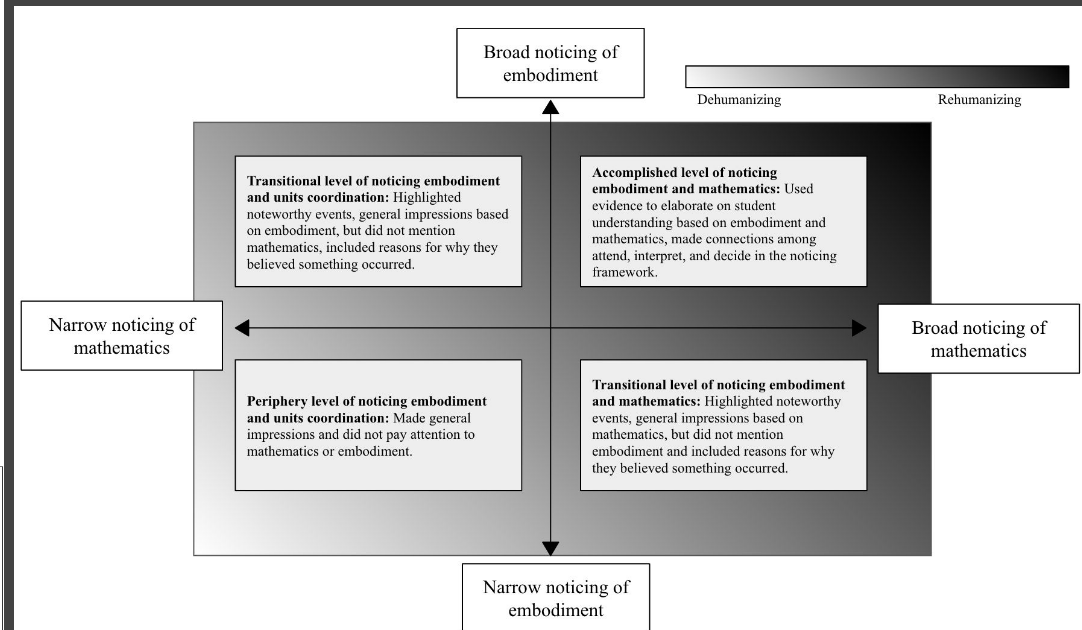
Do preservice teachers notice embodiment and how can noticing embodiment rehumanize maths?



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Embodied Utterances:

- Eye Motion
- Facial Expression
- Gaze
- Body Poise
- Sound Production
- Body Motion
- Hand Gesture
- Manipulative Use

Noticing:

- Attending
- Interpreting
- Responding

Findings:

- Written/verbal focus
- Miss embodiment
- Deficit lens

Recommendations

- Expand scope of PT noticing
- Emphasize embodied noticing

Bridging Mathematics: To Care for an Abstraction

Sofía Abreu
Michigan State University

Anthony Dickson
Michigan State University

Keywords: philosophy, ontology of mathematics, stories, isomorphism

This theoretical poster engages with and presents dominant undergraduate mathematics topics as lively forces with rich potentialities for connection beyond disciplinary mathematics. More specifically, this poster offers both (a) a theoretical contribution to engage with undergraduate mathematics as situated, open, and lively (de Freitas & Sinclair, 2013; Mikulan & Sinclair, 2023); and (b) a potential bridge to resurface and generate connections between undergraduate mathematics and philosophical ideas that are often left out of mathematics classrooms. We invite our field to, as Mikulan and Sinclair (2023) write, *care* for an abstraction: "To care for an abstraction is to make sure you haven't extended it too far, forgotten the contingencies on which it depends, applied it flippantly or use it to foreclose thinking" (p. 70).

Doxiadis (2003), when inviting mathematicians to open up to other ways of sharing their work, suggested considering historical, philosophical, aesthetic, and other dimensions of mathematics to understand math differently. They propose to "[e]mbed mathematics in the soul…by embedding it in story" (p. 24). In resonance, this poster understands mathematics as embedded in worlds (rather than fixed, separated/separable from the 'real' world); always emergent, relational, and in transformation, carrying with it histories, desires, and possibilities that are neither predetermined/fixed nor confined within mathematics as a discipline. For example, the concept of *isomorphism* is often fixed to its algebraic definition and approached as the 'endpoint' of some learning trajectory (e.g., Larsen, 2013), narrowing down pedagogical conversations to metaphors of sameness or connections with homomorphisms (Rupnow, 2021). But what if instead we engaged with it as a *starting* point, as a spark? What stories can it awaken and how are they interconnected (or not) with(in) the world? Isomorphism then becomes a kaleidoscope of stories, awakening multiple wonderings: Why does isomorphism seek to preserve *these* specific structures (e.g., group operation), and what possibilities are foreclosed? Connecting isomorphism with notions of preserving a melody without, for example, consideration of an instrument's timbre could invite discussions of what might be left out when prioritizing certain structures at the expense of other dynamics. These stories, moreover, need not be reduced to auxiliary metaphors *towards* the definition of isomorphism and can invite other wonderings: Where does the persistent preoccupation for 'sameness of forms/structures' in dominant mathematics come from? This opens worlds of historical connections to follow, such as ties with coloniality and language (Barton & Frank, 2001; Gutiérrez, 2017) and often-ignored contingencies as well as connections between temporalities and abstractions (Mikulan & Sinclair, 2023), inviting us to trouble the belief that mathematics is 'universal' and 'objective.'

Understanding mathematics as open and entangled with lived/living histories has not only pedagogical implications but also political. The pervasive understanding of mathematics as abstract and thus objective is harmful in multiple ways, concerns that have been raised in K-12 literature (e.g., Fasheh, 2012; Martin, 2019) but are rare in undergraduate mathematics. Thus, we invite our field to open up to engage with mathematics as lively and situated, and question dominant narratives of undergraduate mathematics. As Wertheim (2019) writes, understanding mathematics as living "enables us to recalibrate our relationship with the subject, reframing it as a mode of engagement in which we can highlight its joyous potential for all people" (p. 70).

References

Barton, B. & Frank, R. M. (2001). Mathematical ideas and indigenous languages. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research in mathematics education: An international perspective* (pp. 135–149). Lawrence Erlbaum Associates.

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics, 83*, 453–470.

Doxiadis, A. (2003, January 3–5). *Embedding mathematics in the soul: Narrative as a force in mathematics education* [Opening address]. Proceedings of the Third Mediterranean Conference of Mathematics Education, Athens, Greece.

Fasheh, M. J. (2012). The role of mathematics in the destruction of communities, and what we can do to reverse this process, including using mathematics. In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education*, (pp. 93–106). Sense Publishers.

Gutiérrez, R. (2017, October 5–8). *Living mathematx: Towards a vision for the future* [Paper presentation]. Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Indianapolis, Indiana, United States.

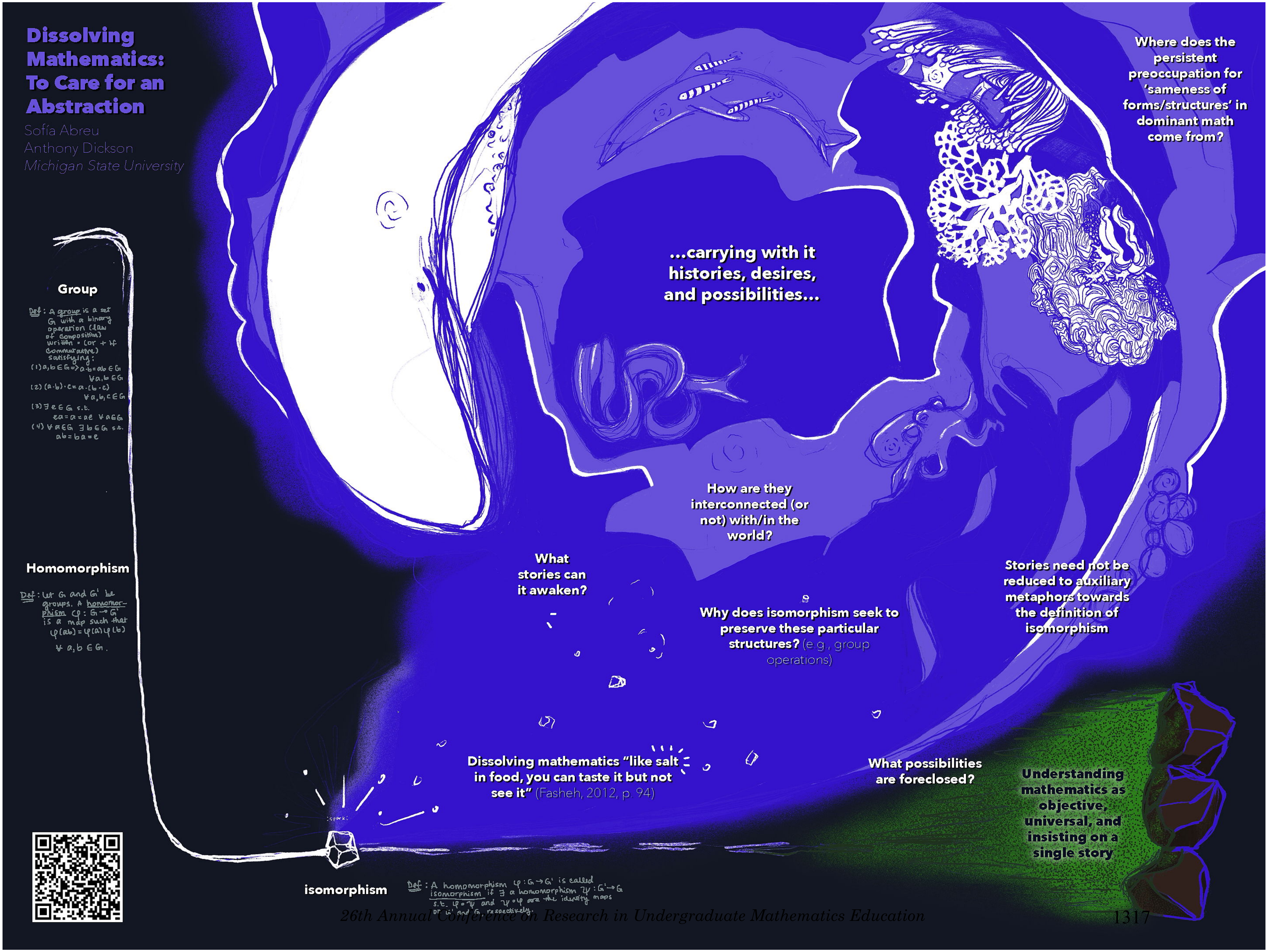
Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behavior, 32*, 712–725.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459–478.

Mikulan, P., & Sinclair, N. (2023). *Time and education: Time pedagogy against oppression.* Bloomsbury Academic.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*, 1–14.

Wertheim, M. (2019). Playing mathematics. *Annals of the New York Academy of Sciences,* 70–74.


Dissolving Mathematics: To Care for an Abstraction
Sofía Abreu
Anthony Dickson
Michigan State University
Group
Def: A group is a set G with a binary operation (law of composition) written · (or + if commutative) satisfying:
(1) $a, b \in G \Rightarrow a \cdot b = ab \in G \quad \forall a, b \in G$
(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$
(3) $\exists e \in G$ s.t. $ea = a = ae \quad \forall a \in G$
(4) $\forall a \in G \ \exists b \in G$ s.t. $ab = ba = e$
Homomorphism
Def: Let G and G' be groups. A homomorphism $\varphi : G \to G'$ is a map such that $\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in G$.
isomorphism
Def: A homomorphism $\varphi : G \to G'$ is called isomorphism if $\exists$ a homomorphism $\psi : G' \to G$ s.t. $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on G' and G respectively.
...carrying with it histories, desires, and possibilities...
Where does the persistent preoccupation for 'sameness of forms/structures' in dominant math come from?
How are they interconnected (or not) with/in the world?
What stories can it awaken?
Why does isomorphism seek to preserve these particular structures? (e.g., group operations)
Stories need not be reduced to auxiliary metaphors towards the definition of isomorphism
Dissolving mathematics "like salt in food, you can taste it but not see it" (Fasheh, 2012, p. 94)
What possibilities are foreclosed?
Understanding mathematics as objective, universal, and insisting on a single story

Student and Instructor Experiences of Equity and Access for
Team-Worthy Tasks in Discrete Mathematics

Marion Campisi
San José State University

Jesus Gonzalez
West Valley College

Shandy Hauk
San Francisco State University

Tim Hsu
San José State University

Mary Rayappan
Hartnell College

Mohammed Yahdi
Hartnell College

Keywords: equity, discrete mathematics, instructor professional growth

For many students, discrete mathematics is a first exposure to formal logical reasoning, proofs, and proving. Undergraduates entering discrete mathematics have much experience with procedural problem solving and many attribute the authority for mathematical sense-making to textbook, instructor, or software (Herbel-Eisenmann, 2007). Like calculus courses, discrete mathematics shares a history of enormous disparities in outcomes correlated to demographic variables. For example at one institution, grade point averages (GPAs) in discrete mathematics, disaggregated by demographic groups privileged in society's majority culture and those marginalized in it were, respectively, 2.28 and 1.72 (Hsu, 2020). Yet subsequent course-taking requires at least a C, 2.0 grade points or higher. The 1.72 is a disqualification for further study.

Research in undergraduate mathematics education, notably in calculus, has indicated that disparities can be addressed by changes in student materials and instruction methods (Bressoud & Rasmussen, 2015). Groupworthy learning opportunities for students, and faculty use of socio-culturally informed instructional approaches, build positive relationships among students, between students and institution, and create active engagement in and outside of class (Archie et al., 2022; Johnson et al., 2020; Laursen et al., 2014; Leyva et al., 2022). Decades of reform in calculus has generated rich curricula, client-discipline informed policy, and most recently, instructor preparation to address the disparity (Yoshinobu et al., 2023). Now, in calculus, the GPA for groups privileged in society is above 2.0 *and* the GPA for historically marginalized groups is near 2.0 (Voigt et al., 2023). For calculus it took several generations of scholarly work to transform the system of structures, values, policies, and people to eke out a less inequitable state. No similar reform effort has occurred for discrete mathematics. Yet.

This poster (Campisi et al., 2024) reports on a state-wide effort in California to transform curriculum and instruction in discrete mathematics. Project research examines student and faculty experience of team-worthy lessons (as substitutes for lecture) and scaffolds for use. Early results indicate increases in students' sense of access to, and engagement in, the intellectual work of discrete mathematics, growth in instructor knowledge of equity-supportive practices, and a reshaping of instructors' views of students' capabilities. Ultimately, the project will:

1. generate a collection of seven team-worthy discrete mathematics lessons,
2. create and refine an asynchronous short-course for faculty who teach with such lessons,
3. gather data from students and faculty to inform lesson and short-course revision,
4. examine implementation in a variety of 2-year and 4-year college settings, and
5. participate in the revision of state policy for the content and processes/practices in discrete mathematics and a similar course in computer science called discrete structures.

Acknowledgements

Based on work supported by the California Education Learning Lab.

References

Archie, T., Hayward, C. N., Yoshinobu, S., & Laursen, S. L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *PLOS One*, *17*(4), e0267097.

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the AMS*, *62*(2), 144-146. https://www.ams.org/journals/notices/201502/rnoti-p144.pdf

Campisi, M., Gonzalez, J., Hauk, S., Hsu, T., Rayappan, M., & Yahdi, M. (2024). Student and instructor experiences of equity and access for team-worthy tasks in discrete mathematics [Poster]. The poster associated with this poster report available at: https://sfsu.box.com/s/3z0ev677cvjgtl2vzsh05wxwd4bdgp2y

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the voice of a mathematics textbook. *Journal for Research in Mathematics Education*, *38*(4), 344-369.

Hsu, T. (2020). An introduction to equity and access in discrete mathematics. *California Educational Learning Lab.*

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504-516.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406-418.

Leyva, L. A., Amman, K., Wolf McMichael, E. A., Igbinosun, J., & Khan, N. (2022). Support for all? Confronting racism and patriarchy to promote equitable learning opportunities through undergraduate calculus instruction. *International Journal of Research in Undergraduate Mathematics Education*, *8*(2), 339-364.

Voigt, M., Hagman, J., Gehrtz, J., Alexander, N., Ratliff, B., & Levy, R. (2023). *Justice through the lens of calculus: Framing new Possibilities for diversity, equity, and inclusion*. Mathematical Association of America. Available online at https://www.researchgate.net/publication/373515596_Justice_through_the_lens_of_calculus_Framing_new_Possibilities_for_diversity_equity_and_inclusion

Yoshinobu, S., Jones, M. G., Hayward, C. N., Schumacher, C., & Laursen , S. L. (2023) A broad doorway to the big tent: A four-strand model for discipline-based faculty development on inquiry-based learning, *PRIMUS, 33*(4), 329-354, DOI: 10.1080/10511970.2022.2072427

Decentering and Interconnecting as Key Practices for Change Agency Leadership in Teaching-focused Professional Development for College Mathematics Instructors

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Keywords: teaching-focused professional development, leadership, decentering, interconnecting

Research in undergraduate mathematics education (RUME) continues to identify features of high quality learning opportunities for students and explore what instructors need to know and do to create those opportunities. RUME also has illustrated features of effective teaching-focused professional development (TPD) for college instructors, characterized the complexity of the system in which TPD occurs, and examined the practices used by providers of TPD in seminars about teaching (Akin et al., 2023; Deshler, et al., 2015; Smith et al., 2021; Yee et al., 2023). Additionally, recent work has explored how development of two skills, decentering and interconnecting, may support professional growth of instructors as future change agents (Hauk & Speer, 2023a,b). *Decentering* is, at its most basic, the act of seeing from someone else's point of view (Teucher et al., 2016). While decentering involves perspective-taking, *interconnecting* is meta-awareness linking across perspectives and contexts and is key in developing and nurturing coalitions, an essential component of local and systemic change (Kotter, 2012; Manville, 2016).

What does it look like if we leverage what is known from the literature to consider the questions: What do providers of TPD need to know and do to create desired types of learning opportunities (for instructors, about teaching) and, more broadly, what are the features of high quality leadership development *for providers to support their growth into facilitators* (who teach about *teaching about teaching*) and *stewards of the discipline* (Bass, 2006)? In addition to supporting faculty in their work to be effective providers of TPD for novice instructors (middle layer of Figure 1) leadership-focused development experiences (outer layers in Figure 1) can scaffold providers to take on roles as stewards and agents for change in the realm of TPD.

In this poster (Hauk & Speer, 2024), we illustrate how stewardship, a particular kind of leadership in the complex system of mathematics instructional development, requires decentering and interconnecting. This model for professional growth of faculty agents for change expands on Figure 1. Earlier work described how instructional practices used by TPD providers (for graduate students) could be beneficial both for learning high-powered teaching approaches for undergraduate mathematics and for building a foundation for future change-agent work. Here we extend the model, with analogous arguments for facilitators and stewards. Ensuring that faculty have capacity as facilitators and as stewards is essential. Such faculty will shape efforts over time, informed by developments in RUME and in response to other developments in the discipline.

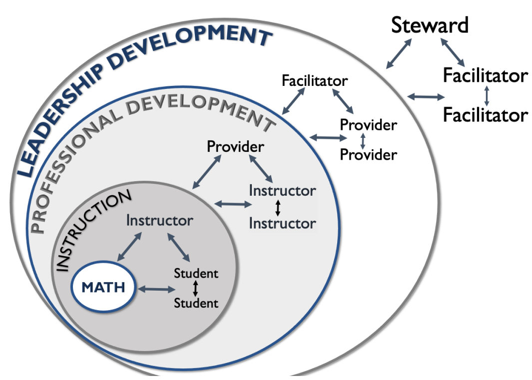

Figure 1. Nested layers of professional growth

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References

Akin, V., Braley, E., & Bookman, J. (2023). Modeling active learning in professional development for teaching. *The Journal of Faculty Development*, *37*(3), 28-39.

Bass, H. (2006). Developing scholars and professionals: The case of mathematics. In Golde, C. M., & Walker, G. E. (Eds.), *Envisioning the future of doctoral education: Preparing stewards of the discipline—Carnegie Essays on the Doctorate* (pp. 101-119). Jossey-Bass.

Deshler, J. M., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. *Notices of the AMS, 62*(6), 638-643.

Hauk, S. & Speer, N. (2023a). Decentering and interconnecting as professional skills in the preparation of new college mathematics instructors. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, Omaha, NE.

Hauk, S., & Speer, N. (2023b). Developing the next generation of change agents in college mathematics instruction. In M. Voigt, J. Hagman, J. Gehrtz, B. Ratliff, N. Alexander, & R. Levy (Eds.), *Justice through the lens of calculus: Framing new possibilities for diversity, equity, and inclusion* (pp. 28–44). https://doi.org/10.48550/arXiv.2111.11486

Hauk, S., & Speer, N. (2024). Decentering and interconnecting as key practices for change agency leadership in teaching-focused professional development for college mathematics instructors [Poster]. The poster associated with this poster report is available at: https://sfsu.box.com/s/zppyqkcf0c3wp4z8e5exr6to4m31nl5y

Kotter, J. P. (2012). *Leading change*. Harvard Business Press.

Manville, B. (2016, May 15). Six leadership practices for 'wicked' problem solving. *Forbes*.

Smith, W. M., Voigt, M., Ström, A., Webb, D. C., & Martin, W. G. (Eds.). (2021). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning* (Vol. 138). American Mathematical Society.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*, 433-456.

Yee, S. P., Wang, J., Hauk, S., & LopezGonzalez, T. (2023). Providers of professional development for novice college mathematics instructors: Perspectives and values about teaching and learning. In S. Cook et al. (Eds.), *Proceedings of the 25th Conference on Research in Undergraduate Mathematics Education*, Omaha, Nebraska.

Decentering and Interconnecting as Key Practices for Change Agency Leadership in Teaching-focused Professional Development for College Mathematics Instructors

MAA CoMInDS

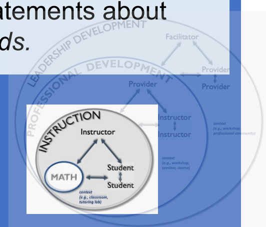
Shandy Hauk, San Francisco State University and Natasha Speer, The University of Maine

Funded by National Science Foundation DUE Awards 1432381, 1654273, & 2021139

Building capacity to steward the development of the next generation of college instructors.

Play Mad Libs™ with the poster

The text to the right describes acts of *decentering* and *interconnecting* done by instructors of mathematics. Use the words on green, blue, and yellow to transform these into statements about *providers, facilitators* and *stewards.*



Glossary

Decentering: the act of seeing from someone else's point of view [1,2].

Interconnecting: meta-awareness linking across perspectives and contexts response to other developments in the discipline [1,3,4].

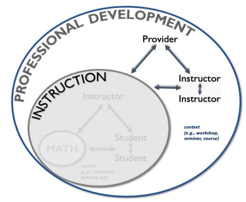
Student: learner of mathematics

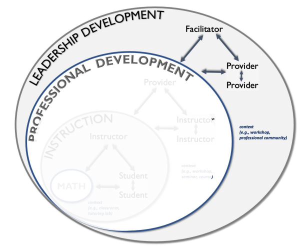
Instructor: person responsible for instruction in mathematics courses

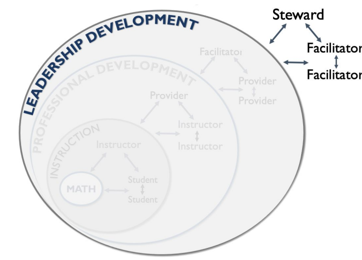
Provider: one who offers teaching-focused professional development to (often novice) instructors

Facilitator: one who offers professional learning for (sometimes novice) Providers.

Steward: one who shapes efforts over time to advance teaching-focused professional learning.







instructor | TPD | instructors | math/pedagogically

provider | facilitation | providers | math/pedagogically

colleague | stewardship | colleagues | professional change

Decentering

- learning to elicit <u>student</u> thinking
- learning how to shape <u>instruction</u> based on <u>student</u> thinking
- scaffolding <u>students</u> to contribute to <u>mathematically dense student</u>-to-<u>student</u> conversations
- being an expert participant rather than sole source of knowledge
- attention to <u>students</u> as (potentially) different from oneself

Interconnecting

- noticing how <u>student</u> conceptions may support or constrain how target <u>learning</u> progresses.
- knowing conceptions as well as the dynamics of communicating about them in a multi-contributor conversation
- selecting formats (e.g., group or pre-meeting activity) for <u>problem-solving about target learning</u>.
- connect across and prioritize <u>instructional / mathematical</u> and contextual factors, to decide what is professionally useful for <u>students</u>.

References

[1] Hauk, S., & Speer, N. (2023). Developing the next generation of change agents in college mathematics instruction. In M. Voigt, J. Hagman, J. Gehrtz, B. Ratliff, N. Alexander, & R. Levy (Eds.), *Justice through the lens of calculus: Framing new possibilities for diversity, equity, and inclusion* (pp. 28–44).

[2] Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*, 433-456.

[3] Kotter, J. P. (2012). *Leading change*. Harvard Business Press.

[4] Manville, B. (2016, May 15). Six leadership practices for 'wicked' problem solving. *Forbes*.

This material is based on work supported by the National Science Foundation (NSF) under Grant Nos. DUE 1432381, 1654273, & 2021139. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. [RUME 2024]

A Comparison of Tutors', Students', and Researchers' Perceptions of College Algebra Mistakes

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San Diego State University

Megan Ryals
University of Virginia

Experiences in first year mathematics classes predict persistence in a STEM major (Seymour & Weston, 2019). Retention becomes increasingly difficult when students need to develop foundational skills in prerequisite courses such as College Algebra. The development of study skills, including metacognitive skills, are often used to improve low success rates. As part of a study on the impact of metacognitive instruction for College Algebra students, we found that when reflecting on the reason for their errors, students often attributed exam errors to "simple mistakes." Researchers identified many of these mistakes as "not simple." Classifications of "simple" or "not simple" mistakes by undergraduate peer tutors, who provide support in campus learning centers, did not consistently align with either the views of students or researchers. We discuss student, tutor, and researcher views of mistakes and how they compare with each other.

Methods

For this IRB approved study, data was obtained from students in a College Algebra support course at a large, Western, public, Hispanic-Serving Institution. In semi-structured interviews, students classified recent errors on an exam as either simple or not simple and provided justification for their classifications. Transcripts of these interviews along with student work on the exams were analyzed to determine students' reasoning for classifying mistakes as "simple or "not simple." Data was later gathered from 11 undergraduate peer tutors at two institutions through an online questionnaire which presented five mistakes from the first college algebra exam. Iterative coding was used to analyze the tutors' reasoning for the mistakes.

Discussion and Conclusion

Researchers defined simple mistakes as ones that "could be made accidentally, would likely not be repeated, or violated a mathematical convention rather than a rule" (Authors, 2020b). Sometimes, tutors' classifications aligned with students' or researchers' classifications of simple and not simple; other times, the tutors were divided. Students sometimes distinguished between simple and not simple by the difficulty level of the problem or by how quickly they could see how to do the problem correctly when reviewing the exam. Student RG stated, "I think that a simple mistake is something that can be fixed without … a large amount of time" Tutors were similar to researchers in that they distinguished between simple and not simple mistakes by noticing if the student had demonstrated understanding previously. Different from both researchers and students, some tutors considered the amount of work it would take to correct the student's misconceptions. Tutor X stated, "A simple mistake is one that can be corrected (and the student would understand where they went wrong), with just a quick clarification. If the mistake requires a more in-depth discussion, then it is not simple." By comparing different views of mistakes between student, tutor, and teacher, we build a deeper understanding of how both tutors and instructors might better address student's misconceptions and misunderstandings.

References

Ryals, M., Hill-Lindsay, S., Burks, L., & Pilgrim, M.E. (2020). Metacognition in college algebra: an analysis of “simple” mistakes. In S. S. Karunakaran, Z. Reed, & A. Higgins (Eds.), Proceedings of the 23rd Conference on Research in Undergraduate Mathematics Education. (pp. 493-501). Boston, MA.

Seymour, E., Hunter, A. B., & Weston, T. J. (2019). Why we are still talking about leaving. *Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education*, 1-53.

Is the mistake simple or not simple?

Researchers say…
Students say…
Tutors say…

What do you say?

Mistake	A	B	C	D	E
Student	S	S	NS	NS	S
Researcher	S	S	NS	NS	NS
Tutors (S/NS)	10/1	3/8	1/10	7/3	7/3

A Comparison of Tutors', Students', and Researchers' Perceptions of College Algebra Mistakes

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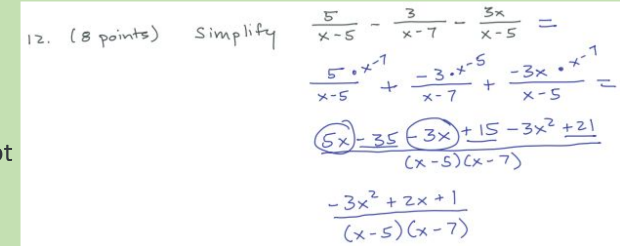
University of Virginia

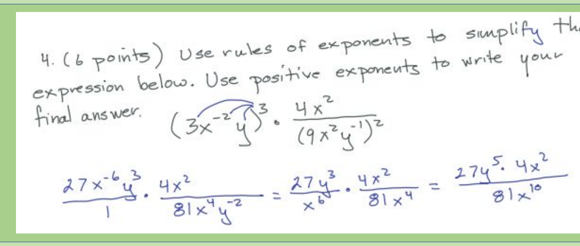
Mistake A

The student correctly began getting a common denominator for each term. The student did not correctly multiply the last numerator, which should have been $-3x^2+21x$.



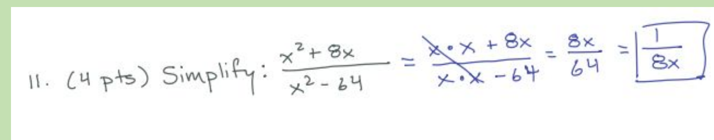
Mistake B

The student initially correctly combined exponents for the y-terms, but did not for the x-terms and the student did not reduce the coefficients.



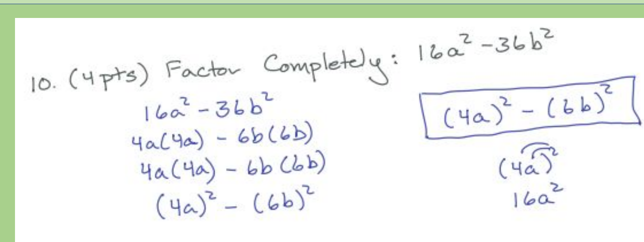
Mistake C

The student incorrectly canceled the x^2 terms in the numerator and denominator, which led to subsequent incorrect steps. Consider the first error of canceling the x^2 terms.



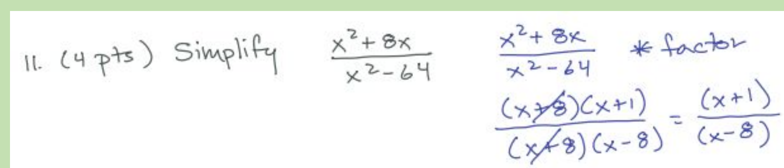
Mistake D

The student recognized the squared terms, but did not factor as a difference of squares to get (4a-6b)(4a+6b)



Mistake E

The student incorrectly factored the numerator as (x+8)(x+1). The student should have factored the numerator as x(x+8).



One Woman's Use of Learning Resources in Conjunction with Her Mathematical Identity

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Keywords: mathematical identity, learning resources, calculus

Calculus courses are critical turning points for university students. Reporting high Drop-Fail-Withdraw rates, retention in these courses is a problem (Voigt et al., 2017). Students, particularly calculus women, navigate decreasing attitudes towards mathematics (Good et al., 2012). For instance, Ellis et al. (2016) found that many women cited lacking confidence as their reasoning for not persisting after their calculus course.

Our study highlights how one woman employed learning resources in conjunction with her mathematical identity, where mathematical identity consists of her dynamic "self-understanding … in the context of doing mathematics" (Martin, 2006, p. 206). We aim to address the following research question: How does one woman calculus student use learning resources and how does her usage of those resources connect with her mathematical identity?

Methods

We conducted three semi-structured interviews with a university freshman, Becca, who was taking a calculus course designed for non-STEM majors at a large southern university. Interviews focused on Becca's perceptions towards her experiences in the course, how she viewed mathematics, and how she viewed herself in mathematical contexts. Analysis consisted of open coding to flag instances in which Becca described learning resources she felt supported her academic success or described a more stable sense of self in reference to mathematics – consistent with the definition of "core mathematical identity" (Cobb & Hodge, 2010).

Results

Becca's learning resources and mathematical identity were interrelated. She viewed herself as an organized mathematics learner focusing on step-by-step approaches, which led her to systematically use learning resources to process the concepts on her own. For example, Becca used apps that provided worked examples of her homework problems. Rather than copying answers, Becca focused on understanding and replicating these processes. We interpret Becca's intentional use of resources as influenced by her identity as an organized mathematics learner.

Becca's use of learning resources also impacted her mathematical identity. In particular, learning resources had a positive impact on her confidence related to mathematics. Reflecting on her instructor's open lab, in which she practiced solving problems, Becca explained, "[…] ***if I can do it now, then I can do it later*** […] when I went to the lab and I was like, so stuck on those problems then I, like, figured it out, it just like, like amazed me […]" *(author's emphasis)*. This quote suggests that the lab gave Becca confidence in her ability to solve problems.

While Becca's use of learning resources may appear to put her in danger of becoming dependent on them, we observe that she has found a way to transfer learning resources to her own knowledge in alignment with how she views mathematics, ultimately leveraging them to perceive herself as capable to persist mathematically.

Acknowledgment

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References

Cobb, P., & Hodge, L. (2010). Culture, identity, and equity in the mathematics classroom. In A. Sfard, K. Gravemeijer, and E. Yackel (Eds), *A Journey in Mathematics Education Research: Insights from the Work of Paul Cobb*, pp. 179– 195.

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PloS one*, *11*(7), e0157447.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, *102*(4), 700–717.

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning*, *8*(3), 197–229.

Voigt, M., Apkarian, N., & Rasmussen, C. (2017). Diverging from the standard fare: Variations in the calculus curriculum. *MAA FOCUS*, February/March, 32-34. Retrieved from http://digitaleditions.walsworthprintgroup.com/publication/?i =392392&ver=html5&p=32

Engaging Preservice Elementary Teachers in Statistical Investigations of Systemic Racism in School Discipline Data

Liza Bondurant, Mississippi State University
Anthony Fernandes, UNC Charlotte
Ksenija Simić-Muller, Pacific Lutheran University
Travis Weiland, University of Houston

In this poster, we share preservice elementary teachers' (PSTs') initial noticings during exploratory data analysis of materials focused on developing content knowledge of statistics as well as normalizing conversations of race in math class. The content covered is typically included in college-level introduction to statistics courses. We summarize PSTs' noticings and wonderings of an interactive data dashboard presenting racial disparities in school discipline.

Racial disparities in school discipline are a significant issue with lasting effects (Huang, 2018; ProPublica, 2023). To illustrate, Black students are suspended at a rate three times higher than their white counterparts (USDEOCR, 2014). These disparities are rooted in prevailing racial narratives that influence teachers' perceptions of Black, Indigenous, and People of Color (BIPOC) students and impact their math identities (Aguirre et al., 2013). Implicit biases may help explain these disciplinary disparities (Gilliam et al., 2016). Since about 80% of U.S. teachers are white, we must raise awareness of these disparities (NCES, 2023).

Inspired by Paulo Freire's critical pedagogy, we focused on critical thinking, empowerment, social justice, and collaborative dialogue to engage PSTs in addressing this issue (Freire, 2004). We developed instructional activities for secondary PSTs, combining statistical investigations with social justice concepts, contributing to the growing body of work on using data analysis to examine social justice issues (Bondurant et al., 2022; Casey et al., 2023). After implementing, we adapted it for elementary PSTs, emphasizing elementary math content and pedagogy, specifically operations with fractions, decimals, and percentages, and utilizing virtual manipulatives to teach math concepts (Suh & Roscioli, 2023).

This poster delves into elementary PSTs' initial observations of an interactive data dashboard focusing on racial disparities in school discipline. We aimed to answer the research question: When presented with such data, what aspects of racial disparities do PSTs notice? Using open coding, we identified themes in PSTs' responses, highlighted representative quotes, and determined frequencies for each theme (Corbin & Strauss, 2014). The findings revealed four key themes in PSTs' responses: "Personal connections" encompassed experiences as students or school employees; "Academics" focused on course offerings, graduation rates, and standardized test scores; and "Discipline" involved comments about behavior, classroom management, and suspension data. Surprisingly, only one PST mentioned discipline, and none discussed racial disparities. We employed nudges in the form of assessing and advancing questions to guide PSTs' attention toward racial disparities (e.g., "What do you notice about the suspension rates of Black and white students and why might this be?"). The impact of these nudges was not within the scope of this poster. Our findings suggest that PSTs may be uncomfortable discussing race and discipline or may not perceive them as relevant to their role as math teachers. These findings point to a need for math teacher educators (MTEs) to explicitly connect mathematics topics with social justice issues (Conway et al., 2022). Future research could explore the impact of different nudging techniques in this context.

References

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). *The impact of Identity in K–8 Mathematics Learning and Teaching: Rethinking Equity-Based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Bondurant, L., Dean, L., & Hudson, R. (2022). Gender pay gap. In Conway IV, B. M., Id-Deen, L., Raygoza, M. C., Ruiz, A., Staley, J. W., & Thanheiser, E. (2022). *Middle school mathematics lessons to explore, understand, and respond to social injustice*. Corwin Press.

Casey, S., Bondurant, L., & Ross, A. (2023). Engaging teachers in the combination of statistical investigation and social justice: Fairness in school funding. *Mathematics Teacher Educator*, *11*(2), 139-142. https://doi.org/10.5951/MTE.2022.0016

Conway IV, B. M., Id-Deen, L., Raygoza, M. C., Ruiz, A., Staley, J. W., & Thanheiser, E. (2022). *Middle school mathematics lessons to explore, understand, and respond to social injustice*. Corwin Press.

Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Thousand Oaks, CA: Sage.

Freire, P. (2004). *Pedagogy of hope: Reliving pedagogy of the oppressed*. Continuum.

Gilliam, W. S., Maupin, A. N., Reyes, C. R., Accavitti, M., & Shic, F. (2016). Do early educators' implicit biases regarding sex and race relate to behavior expectations and recommendations of preschool expulsions and suspensions. *Yale University Child Study Center*, *9*(28), 1-16.

Huang, F. L. (2018). Do Black students misbehave more? Investigating the differential involvement hypothesis and out-of-school suspensions. *The Journal of Educational Research*, *111*(3), 284-294. https://doi.org/10.1080/00220671.2016.1253538

National Center for Education Statistics (NCES) (2023). Characteristics of public school teachers. *Condition of Education*. U.S. Department of Education, Institute of Education Sciences. Retrieved [date], from https://nces.ed.gov/programs/coe/indicator/clr.

ProPublica. (2023, August 28). *Miseducation*. https://projects.propublica.org/miseducation/

Suh, J., & Roscioli, K. (2023). Learning trajectory-based formative assessment & sequenced digital learning activities in math class. In C. L. Webb & A. L. Lindner (Eds.), *Preparing Pre-Service Teachers to Integrate Technology in K-12 Classrooms: Standards and Best Practices*. IGI Global.

U.S. Department of Education Office for Civil Rights (USDEOCR). (2014). *Civil rights data collection, data snapshot: School discipline*. Retrieved from http://www2.ed.gov/about/offices/list/ocr/docs/crdc-discipline-snapshot.pdf

ENGAGING PRESERVICE MATHS TEACHERS in STATS INVESTIGATIONS of SYSTEMIC RACISM in DISCIPLINE DATA




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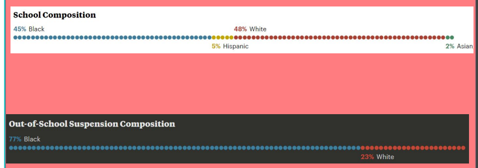
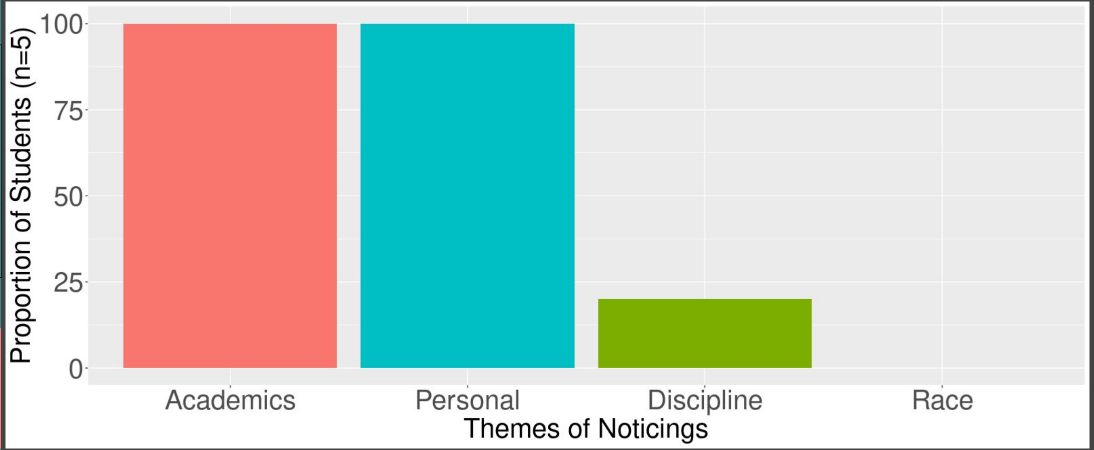
Mississippi State University	*UNC Charlotte*	*Pacific Lutheran University*	*University of Houston*

PROPUBLICA | MISEDUCATION



When presented with an ***interactive data dashboard*** *on* ***racial disparities*** *in school discipline, what do five elementary preservice math teachers notice?*



Not Racial Disparities

Take a picture to **download the full paper**

"I have always wanted to work there because of the environment, salary, and educational opportunities."

"I don't think a test score is appropriate for a child to pass or fail."

Potential reasons for avoidance

- fear
- denial
- acceptance

How to mitigate?

- relate to other marginalized identity markers
- Direct task goals specifically to considering race

First Generation Calculus Students' Beliefs

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Keywords: calculus, first-generation students, beliefs, problem-solving

The retention of college students in the Science, Technology, Engineering, and Mathematics (STEM) fields has been widely studied (e.g.; Rasmussen & Ellis, 2013; Sithole et al., 2017). Less research has focused on students traditionally underserved by the educational system (e.g., Carver et al., 2017). Understanding students' beliefs in mathematics can improve achievement and motivation (Muis, 2004), which has inspired research on the beliefs of several student populations (e.g.; Berkaliev & Kloosterman, 2009; Sangcap, 2010; Sintema & Jita, 2022; Yavus & Erbay, 2015). There is still a need to explore the beliefs of first-generation students, as little confidence in the ability to solve mathematics problems is related to lower academic performance (DeFreitas & Rinn, 2013). Since calculus is key for college STEM retention (Rasmussen & Ellis 2013), our study explores the beliefs that first-generation calculus students hold based on the Indiana Mathematics Beliefs Scale (IBMS) to inform on the retention of first-generation calculus students.

Methods

The study was conducted at a public, Ph.D.-degree granting institution in the midwestern United States implementing a quantitative research approach (Johnson & Christensen 2019). Demographic information such as gender and major was collected via a questionnaire. The IBMS (Fennema & Sherman, 1976; Kloosterman & Stage, 1992) contains 36 Likert-type scale questions measuring six beliefs about mathematical problem-solving, each including six questions that positively or negatively reflect it. The extensively used and validated (e.g.; Ayebo & Mrutu; Berkaliev & Kloosterman 2009) IBMS survey was distributed to 16 calculus classes with a total of 362 students. We collected 224 responses (61.9% response rate) and analyzed responses to identify students who self-identified as first-generation by reporting the highest degree achieved by one of their parents being less than a four-year degree in college. In the analysis, we assigned numerical values to the responses and adjusted for questions that had negative statements toward the beliefs. Then, we used descriptive statistics and an independent sample t-test on the average score of students by belief.

Results & Discussion

From the 94 first-generation students' responses to the demographic survey, we found that this was composed of 40 women, 53 men, and one non-binary student. We also found that within this subset of participants, 14 were black, 41 were Hispanic, and none were indigenous. Respondents most strongly believed that "understanding concepts are important in mathematics" and "mathematics is useful in daily life," but this is not the case for the belief that "there are word problems that cannot be solved with simple, step-by-step procedures." Using an independent samples t-test to compare the average score of each belief for both men and women, there were no significant differences seen ($p>0.05$).

References

Ayebo, A., & Mrutu, A. (2019). An exploration of calculus students' beliefs about mathematics. *International Electronic Journal of Mathematics Education, 14*(2), 385-392.

Berkaliev, Z., & Kloosterman, P. (2009). Undergraduate engineering majors' beliefs about mathematics. *School Science and Mathematics, 109*(3), 175-182.

Carver, S., Van Sickle, J., Holcomb, J. P., Quinn, C., Jackson, D. K., Resnick, A. H., ... & Marquard, A. M. (2017). Operation STEM: increasing success and improving retention among first-generation and underrepresented minority students in STEM. *Journal of STEM Education: Innovations and Research, 18*(3).

DeFreitas, S. C. and Rinn, A. N. (2013). Academic achievement in first generation college students: The role of academic self-concept. *Journal of the Scholarship of Teaching and Learning 13,* 57-67.

Fennema, E., & Sherman, J. A. (1976). Brief reports: Fennema-Sherman Mathematics Attitudes Scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for Research in Mathematics Education, 7*(5), 324-326. https://doi.org/10.5951/jresematheduc.7.5.0324

Kloosterman, P. and Stage, F.K. (1992), Measuring beliefs about mathematical problem solving. *School Science and Mathematics, 92,* 109-115. https://doi.org/10.1111/j.1949-8594.1992.tb12154.x

Johnson, R. B., & Christensen, L. (2019). *Educational research: Quantitative, qualitative, and mixed approaches.* Sage publications.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research, 74*(3), 317-377.

Rasmussen, C., & Ellis, J. (2013). Students who switch out of calculus and the reasons they leave. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Chicago, IL: University of Illinois at Chicago.*

Sintema, E. J., & Jita, T. (2022). Gender differences in high school students' beliefs about mathematical problem solving. *International Journal of Learning, Teaching and Educational Research, 21*(10), 395-417.

Sithole, A., Chiyaka, E. T., McCarthy, P., Mupinga, D. M., Bucklein, B. K., & Kibirige, J. (2017). Student attraction, persistence and retention in STEM programs: Successes and continuing challenges. *Higher Education Studies, 7*(1), 46-59.

Yavuz, G., & Erbay, H. N. (2015). The analysis of pre-service teachers' beliefs about mathematical problem solving. *Procedia-Social and Behavioral Sciences, 174*, 2687-2692.

Intro-to-Proof Students Discuss Logical Implication and Quantification: Themes from the Triangle Group

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Keywords: proofs, logical implication, quantification, epistemological obstacle

Research has shown that students in introduction-to-proof courses encounter epistemological obstacles (EOs, or persistent challenges; cf. Brousseau, 2002) in learning and reasoning about core concepts, even in the face of research-based instruction (e.g., Norton et al., 2022, 2023). Quantification and logical implication are two major course topics that are seen for the first time in the case of most students; consequently, they tend to experience challenges when working with mathematical statements, which are then reflected as specific EOs. In our ongoing research project, we investigate students' experiences in introduction-to-proof courses with particular attention to eliciting and addressing students' EOs. Data from our project include whole-class discussions, one-on-one interviews with students, and small group discussions. In this poster, we present findings from our analyses of the discussions of one small group.

Our methods included analyzing video data from a group of students (the "Triangle Group") working on mathematical tasks during an undergraduate introduction-to-proof course. Tasks were drawn from existing research on the teaching and learning of mathematical proofs (e.g., Hub & Dawkins, 2018; Shipman, 2016). During the completion of these tasks, students in the small group were able to discuss their strategies and any mathematical insights or challenges that they experienced. Consistent with basic qualitative data-analytic methods (Merriam & Tisdell, 2015), we analyzed students' discussions and found patterns in students' reasoning—patterns that constituted the themes shared in this poster.

Three major themes emerged from our analyses. The first major theme involves students establishing constraints on the universal set when reasoning about mathematical statements. For example, the group was asked to negate the statement, "for all nonnegative real numbers x, $\sqrt{x} \leq x$." They responded, "for all nonnegative real numbers less than 1, $\sqrt{x} > x$," constraining the truth set to the interval (0, 1). The second theme involves students providing a counterexample when asked to find the negation of a statement. Using the same statement as above, the Triangle Group said the negation would be given if they "define x and write the exact same thing, [the statement] will be false." Specifically, they said P(x) is false if $x = 0.01$, finding a counterexample to the given statement. Lastly, the third theme explores how small group discussions can help students address EOs. For instance, students in the course were asked to consider how the statement, "the matrix A is invertible," is an existence statement. We inferred that this task elicited hidden quantification (Shipman, 2016), which is the EO caused when quantifiers are not explicitly stated in a mathematical statement. However, the Triangle Group seemed to be able to address this EO through their small-group discussion about the task.

Acknowledgments

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References

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Theory of didactical situations in mathematics: Didactique des mathématiques*, *1970–1990* (pp. 79–117). Kluwer.

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior*, *50*, 90-102.

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.

Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-80). University of Nevada, Reno.

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2023). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295-321.

Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and its Applications*, 35(1), 41–49.

Intro-to-Proof Students Discuss Logical Implication and Quantification: Themes from the Triangle Group

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Motivation

- ***Epistemological obstacle (EO)****:* a persistent challenge in one's learning, even in the fact of best instruction (Brousseau, 2002; Norton et al., 2023)
- Research shows students encounter specific EOs related to core concepts about proof and proving (Arnold et al., in press; Norton et al., 2022, 2023; Shipman, 2016)
- The Proofs Project aims to identify EOs and design instructional strategies for supporting students by eliciting and addressing EOs directly

Methods

- The Proofs Project collected data from whole-class discussions, small-group discussions, and clinical interviews with students
- Tasks designed by The Proofs Project or adapted from published research (e.g., Dawkins & Roh, 2020; Hub & Dawkins, 2018; Vroom, 2020)
- Regular opportunities for discussion provided in class for students to share ideas, strategies, and challenges
- Analyzed video data from small groups of students working on tasks during an intro-to-proofs class
- Focus of this presentation on the "Triangle Group"
- Identified regularities in qualitative findings from our analysis of small-group discussions, constituting the themes shared here (Merriam & Tisdell, 2016)

Theme 1

Students place constraints on the universal set when reasoning about mathematical statements.

Task 1

Determine if the following statement is true or false. "Suppose a, b, and c are integers. If $|a-b|<5$ and $|b-c|<5$, then $|a-c|<5$."

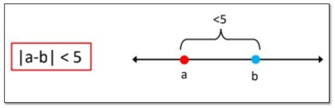

Fig. 1: Visual representation for Task 1

"Since it's an absolute value, we know at least one of them [a or b] has to be <u>over 5</u>. Or... I don't know. There's some sort of <u>constraint</u> about a and b, like <u>over 5 or under 5</u>, or something like that."

Task 2

Negate the statement. "For all nonnegative real numbers x, $\sqrt{x} \leq x$."

"For all nonnegative real numbers <u>less than 1</u>, $P(x)$ is false."

Constrained the universal set to the interval (0,1).

Theme 2

Students provide a counterexamples when asked to find the negation of a statement.

On Task 2, Triangle Group initially found a counterexample to the statement.

"Don't we just need to find a value of x where the square root of that number is greater than? ... $P(.01)$ is false"

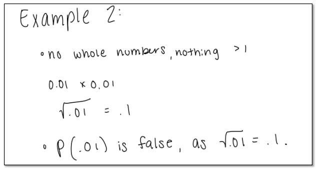

Fig. 2: Group's work on Task 2

Progressing from identifying a counter-example to writing a negation of the statement posed a challenge.

- **S1: So then the whole statement is false. Therefore, $P(x)$ is false.**
- **S2: So, in order to make this statement true—**
- **S3: Well, $P(x)$ is the open statement. So, all of $P(x)$ isn't false. 'Cause, $P(x)$ is true for like 2 or 3. So I think we can just say that, um, just statement P of .01 is false.**

Theme 3

Small group discussions can help students elicit EOs and work together to begin addressing EOs.

Task 3

Explain how the statement "Matrix A is invertible" *is an existence statement.*

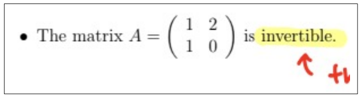

Fig. 3: Statement for Task 3

Task designed to elicit the EO of hidden quantification (Shipman, 2016).

"It's kind of like saying, *there exists* a matrix such that [A] is invertible."

Group member unveils the hidden quantifier, reframing the statement as an existence statement.

Task 4

Prove or disprove the statement.

"If x > 0, then $x + \frac{1}{x} \geq 2$."

Statement contains a hidden universal quantification. Through working together, the group was able to unveil the universal quantifier and work together to negate the statement.

"...the way you would negate it would be 'there exists', right?"

Purpose

To share themes from qualitative analysis of one small group's discussions about research-based tasks during class

References

Arnold, R., Kokushkin, V., Norton, A., & Antonides, J. (in press). Students' treatment of the negation of logical implication. To appear in Proceedings of the 26 annual conference on Research in Undergraduate Mathematics Education.
Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (pp. 79–117). Kluwer.
Dawkins, P.C., & Roh, K.H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *6*(1), 1-22.
Hub, A., & Dawkins, P.C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior*, *50*, 90-102.
Merriam, S.B., & Tisdell, E.J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.
Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-80). University of Nevada, Reno.
Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295-321.
Shipman, B.A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and its Applications*, 35(1), 41–49.
Vroom, K. (2020). Guided reinvention as a context for investigating students' thinking about mathematical language and for supporting students in gaining fluency [Unpublished doctoral dissertation]. Portland State University.

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The Proofs Project Website

Self-reported Student Motivations to Participate in DEI Service Work

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Keywords: Department change, introductory math, student voice, equity

The proposed poster describes information collected from a case study of a DEI initiative in the math department at Kappa University (KU). In Spring 2023, a Networked Improvement Community (NIC) of faculty members and administrators began meeting monthly to discuss DEI in introductory math at KU. After one semester, in recognition of the need to better attend to Gutiérrez' critical axis of equity (2002), the NIC leaders solicited applications from undergraduate and graduate students. Applicants were asked "Why are you interested in joining the KU Networked Improvement Community (NIC)?" We use the 117 application responses as data to answer the question: *What motivates students at KU to get involved in DEI work in the mathematics department? How do students bring their identities into these motivations?* We conceptualize students' desired engagement in the NIC as involving "joint action for a common good," within the "voluntary organizational activity" of NIC participation (Ansala et al., 2016, p. 151). Ansala and colleagues detail five emergent themes which motivated students' engagement with activism: *social* motives of engaging collaboratively, motives of *influence* related to a desire to have a positive impact, motives emphasizing the *benefits* of engaging in activism, motives related to activism as a *lifestyle*, and motives of pure *coincidence.* We utilized the former four themes as first-level *a priori* codes. We also found a need to attend to student identities, framed with a critical sociocultural lens (Esmonde et al., 2009), which articulates both *social identities* and *practice-based* components of a person's identity. The social components are the socially constructed aspects of one's identity, that are both imposed on people by society and that we impose on ourselves. The practice-based identities are developed through our participation in cultural practices. This perspective on identity encourages attention to how social identities interact with and inform practice-based identities. Here, we explore how students evoke their identities in their expressions on wanting to join the DEI work in the mathematics department.

Our preliminary analysis shows that of the 117 applicants, 50 referenced a component of their social identity as salient to their application. The social identities they cite include race, gender, sexual orientation, being a first generation student, being an international student, being low income or rural, and disparate access to calculus before college. Among these students, 27 also reference either having previous positive experience with math, a very strong math identity, or referenced mathematical struggles. Additionally, 104 out of 117 cited a desire to have influence as a component of their motivation for their participation in the NIC, often in the context of making the mathematics program or its associated classes a more positive experience for other students. This often included reference to their own identities; a student remarked, "*as a Latina woman, I usually find myself pretty isolated in the fields of STEM…I would love to be a part of transforming the world of mathematics, especially at KU, into one that is accepting and transformative for people like me.*" In the poster, we highlight student voices behind the motivations and patterns of how students motivate their participation in an equity-oriented NIC.

References

Ansala, L., Uusiautti, S., & Määttä, K. (2016). What are Finnish university students' motives for participating in student activism?. *International journal of adolescence and youth*, *21*(2), 150-163.

Esmonde, I., Brodie, K., Dookie, L., & Takeuchi, M. (2009). Social identities and opportunities to learn: Student perspectives on group work in an urban mathematics classroom. *Journal of Urban Mathematics Education*, *2*(2), 18-45.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. Mathematical Thinking and Learning, 4(2-3), 145-187.

Self-reported Student Motivations to Participate in DEI Service Work

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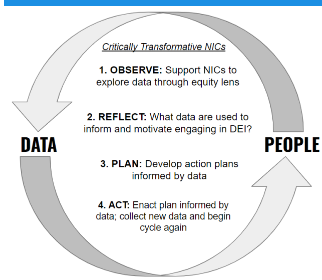
Background

Existing structures in college math departments that don't attend to students' varied identities may compound feelings of exclusion in introductory courses [5]. Thus, there is a need to interrogate and challenge the systems currently in place, particularly at the level of college calculus. Students, as experts in their own experience, play an important role in such DEI initiatives. As D'Ignacio and Klien assert in Data Feminism, leveraging experience and emotion as sources of data has the potential to inspire systemic change [2].

Kappa Networked Improvement Community (NIC)



NIC Improvement Cycle

In 2022, a team of faculty members at Kappa University (Kappa), as part of a multi-institution collaboration, began work toward critically transforming their introductory math courses. They assembled a "Networked Improvement Community" (NIC) of faculty, staff, and administrators who met monthly to discuss potential changes in the Calculus sequence at Kappa. After one semester of work, NIC leaders recognized the need to better attend to Gutiérrez' critical axis of equity [4] in understanding and addressing student experiences in Calculus at Kappa and the intersection of these experiences with student identities and backgrounds. Thus, it became essential to not just gather data on student experiences but to incorporate students into the NIC itself. In Summer 2023, the NIC leaders solicited applications from graduate and undergraduate students to join the NIC. As a case study, we analyze responses to the single application question, "Why are you interested in joining the Kappa NIC?"

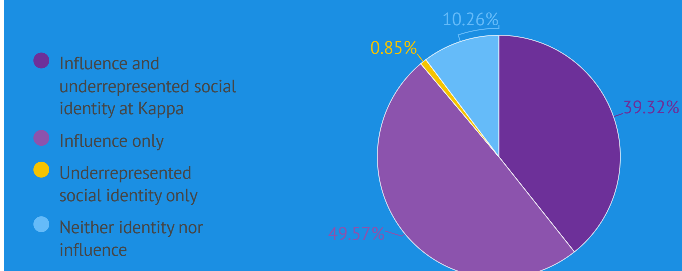
Research Question 1: Motivations

What motivates Kappa students to get involved in math department DEI work?

We conceptualize students' desired engagement in the NIC as involving "joint action for a common good," within the "voluntary organizational activity" of NIC participation [1], and thereby used Ansala et al.'s four themes as *a priori* codes:

- **Influence (89%)** → Desire to participate in decision-making, ensure values are represented, and/or influence change.
- **Benefit (35%)** → Desire to learn more, gain skills and/or apply skills in a way that complements university studies.
- **Social (18%)** → Desire to collaborate and/or make friends. May describe being drawn to DEI through friendship.
- **Lifestyle (16%)** → View activism as part of their identity. May cite prior experiences with activism as motive.

Students cite desire for influence and social identities as motivations to join DEI-focused intro math Networked Improvement Community.



Self-reported applicant motivations, n=117

Student Voices

Influence **Social identity**

As a Latina woman, I usually find myself pretty isolated in the fields of STEM, rarely coming across professors or other students who look like me. I would love to be a part of transforming the world of mathematics, especially at [Kappa], into one that is accepting and transformative for people like me.

Benefit

I am interested because it seems like a good opportunity to help [with] on going research at [Kappa]. I want to get better at interviews which will be beneficial to me.

Social Motivation

What excites me most is the chance to collaborate with a diverse group including faculty, instructors, and fellow students. I am enthusiastic about participating in data collection, analysis, and collaborative efforts within the [Kappa] NIC that accommodate different learning styles and backgrounds.

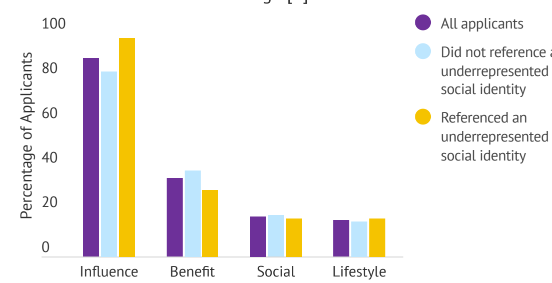
Lifestyle

As a lifelong activist, I have always fought for change and equity as much as is possible in the environments I inhabit. Nowhere is this more relevant than in the courses that I take, where I do not see the diversity of students that I would expect to find at a diverse institution. I want to join the NIC so that I can help make math courses, nationwide, a more welcoming place for all, so that all students are comfortable in the math classroom and all can forget about its looks to focus instead on expanding our knowledge in the field.

Research Question 2: Intersection with identity

How do Kappa students bring their identities into these motivations?

47 of the 117 applicants referenced a component of their social identity [3] that is underrepresented in math at Kappa as salient to their application. Underrepresented identities that were referenced by students related to race (13.7%), gender (8.5%), sexual orientation (3.4%), high school math preparation (4.3%) being a low income or rural student (11.1%), being a first generation student (6%), and being an international student (6%). **Students who referenced an underrepresented identity were significantly more likely than those who did not to also express a desire for influence as a component of their motivation for participation.** This was often in the context of making mathematics courses at Kappa a more positive experience for other students, resonating with Gutiérrez' critical axis of equity and a desire for structural change [4].



Self-reported applicant motivations, n=117

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References

1. Ansala, L., Uusiautti, S., & Määttä, K. (2016). What are Finnish university students' motives for participating in student activism?. International journal of adolescence and youth, 21(2), 150-163.
2. D'Ignazio, C. and Klein, L. (2019). Data Feminism community review site. https://mitpressonpubpub.mitpress.mit.edu/pub/dgv16l22/release/6
3. Esmonde, I., Brodie, K., Dookie, L., & Takeuchi, M. (2009). Social identities and opportunities to learn: Student perspectives on group work in an urban mathematics classroom. Journal of Urban Mathematics Education, 2(2), 18-45.
4. Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. Mathematical Thinking and Learning, 4(2-3), 145-187.
5. Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. The Journal of Higher Education, 92(5), 784-814.

An Integrated Teaching Model for Post-Covid Graduate Teaching Assistants Professional Development

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Keywords: GTA Professional Development, Teacher Preparation, Online Instruction

The roles and expectations for many of the GTAs vary across higher education, from assisting instruction through grading and tutoring on a small scale to teaching entire mathematics courses. Graduate teaching assistants (GTAs) are indispensable assets in many undergraduate learning programs in higher education (Di Bendetti et al., 2022; Lang et al., 2020; Mutambuki & Schwartz, 2018). Higher Education has responded to the growing demand for teaching staff by utilizing GTAs in many gateway courses, including entry-level mathematics courses. It is assumed that if GTAs have a background or a degree in mathematics, they can support and teach mathematics. Unfortunately, many of these GTAs have little to no prior teaching experience (Choate et al., 2021; Di Bendetti et al., 2022; Jonnalagadda et al., 2022; Lang et al., 2020; Mutambuki & Schwartz, 2018). This issue only intensified after many educational programs and clinical practice (student teaching) experiences were altered, shortened, or eliminated outright in March 2020 due to COVID-19 (Choate et al., 2021; Flores & Swennen, 2020). Many students now have a perceived gap (K-12 and post-secondary) in content understanding due to the sudden shift to online learning. This gap is especially impactful on mathematics students and pre-service educators (Choate et al., 2021; Flores & Swennen, 2020; König et al., 2020). Many GTAs entering graduate mathematics programs during this time were expected to support students in the gateway mathematics courses with little to no training or professional development. Many higher education institutions provide some professional development, including one-day workshops, one-credit courses, observation/shadowing opportunities, mentoring, and microinstruction, but few provide any extensive, effective teacher training (Di Bendetti et al., 2022; König et al., 2020; Lang et al., 2020). Even so, many GTAs still are underprepared for future teaching beyond graduation (Choate et al., 2021; Lang et al., 2020). However, with the sudden shift to online learning, there needs to be a new emphasis on GTA professional development and effective teaching training to help engage students affected by the changes in mathematical instruction.

This poster will present the development and implementation of a comprehensive GTA effective teacher training and professional development model created to tackle and address the need for a more conducive and thorough professional development framework for teaching and assisting mathematics GTAs in the post-Covid era and the sudden shift to online instruction. This model is divided into three phases and covers a four-term (two-year) graduate program that incorporates clinical teaching (in-person and online), a mentoring program, micro sessions of professional development, peer and student feedback reflections. This poster will review the model with collected GTA insight from the past three years (Fall 2020- Fall 2023) on the impact of their experiences within this model. Specific highlights will focus on areas GTAs felt most prepared in content understanding, pedagogical preparedness, and beliefs about entering the mathematics classroom. The aim is to provide other mentors insight to better support the educational growth of GTAs and other preservice mathematics teachers.

References

Choate, K., Goldhaber, D., & Theobald, R. (2021). The effects of COVID-19 on teacher preparation. Phi Delta Kappan, 102(7), 52-57.

Di Benedetti, M., Plumb, S., & Beck, S. B. (2022). Effective use of peer teaching and self-reflection for the pedagogical training of graduate teaching assistants in engineering. European Journal of Engineering Education, 1-16.

Flores, M. A., & Swennen, A. (2020). The COVID-19 pandemic and its effects on teacher education. European Journal of Teacher Education, 43(4), 453-456.

Jonnalagadda, R., Singh, P., Gogineni, A., Reddy, R. R., & Reddy, H. B. (2022). Developing, implementing and evaluating training for online graduate teaching assistants based on Addie Model. Asian Journal of Education and Social Studies, 1(10).

König, J., Jäger-Biela, D. J., & Glutsch, N. (2020). Adapting to online teaching during COVID-19 school closure: teacher education and teacher competence effects among early career teachers in Germany. European journal of teacher education, 43(4), 608-622.

Lang, F. K., Randles, C. A., & Jeffery, K. A. (2020). Developing and evaluating a graduate student teaching assistant training course in the chemistry department of a large American university. Journal of Chemical Education, 97(6), 1515–1529.

Mutambuki, J. M., & Schwartz, R. (2018). We don't get any training: The impact of a professional development model on teaching practices of chemistry and biology graduate teaching assistants. Chemistry Education Research and Practice, 19(1), 106-121.

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Stakeholders' Perspectives on a Coordinated Developmental Mathematics Course

Introduction

From my own experience as a graduate student and adjunct instructor of developmental mathematics courses (DMCs), I was interested in contributing to the field of developmental mathematics (DM) research by conducting an exploratory, qualitative study documenting the perspective of educator stakeholders involved with College Algebra, a second-level DMC. The participants, informed by their experiences and values as educators, demonstrated a strong commitment to their students, expressing hope for students' mathematics future and a desire to improve their practice.

Research Questions

1. How do stakeholders describe the purpose, goals, and outcomes of College Algebra?
2. What kind of professional development (PD) do DMC instructors receive and what PD would they like to receive?

Methods

I interviewed five stakeholders involved with College Algebra, the math department's second level DMC.

- John (he/him), adjunct instructor
- Diane (she/her), non-tenure track instructor
- Nancy (she/her), course coordinator
- Sarah (she/her), coordinator of developmental math
- Rae (she/her), leadership in math department

For the past several years, the math department has undertaken efforts to redesign its DMCs in response to the needs of the growing DM student population. These efforts were mentioned by nearly all the participants and meaningfully shaped our conversations.

Big Ideas

Stakeholders unanimously stated the goal of College Algebra as **preparing students for pre-calculus and beyond**.

The stakeholders valued **gaining pedagogical content knowledge** through the **experience of teaching**.

For more details on my study, scan the QR code or email me at katherine.j.burke@temple.edu, I'd love to continue our conversation!

Results

Participants described the development of students' academic maturity as another necessary purpose of this entry level course; they expressed a willingness and desire to help their students make the transition from high school to college. Overall, the participants viewed College Algebra as successfully achieving its preparatory purposes, especially since the DM redesign. Similarly, though PD had not been a requirement for DM instructors in the past, with the concerted attention paid to DMCs going forward, upper-level stakeholders described the creation of PD attuned to the needs of DM instructors; a new grading PD session was well-received and appreciated by a participant of this study.

Discussion and Future Work

The participants of this study provided insight into the systemic operation of developmental mathematics, as well as perceptions of this system and the DM student population. With recent DM redesign efforts, the participants depicted an increasingly cohesive and comprehensive program that works to serve and support both its instructors and students. That said, the means through which the branches of this DM system achieve its aims warrant further study. I am interested in pursuing the following lines of inquiry:

- Building on this study with a larger population of instructional stakeholders, namely adjunct and graduate student instructors
- Repeating this study with first-level DMC educator stakeholders
- Investigating how DMC instructors support their students in practice
- Understanding how DMC instructors' enacted practices are perceived by students

Tracking Graduate Teaching Assistants' Responses through Sustained Professional Development

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Mary E. Pilgrim San Diego State University	Mary Beisiegel Oregon State University	ELITE PD Research Group EHR #2013590, 2013563, and 2013422

Keywords: graduate teaching assistants, professional development, active learning

There is a strong need to support mathematics graduate teaching assistants (MGTAs) in leveraging active learning techniques in ways that are inclusive and equitable (Beisiegel et al., 2019). Mathematics graduate students make up a large section of the workforce involved in teaching introductory mathematics courses at Ph.D. granting universities (Selinski & Milbourne, 2015). Professional development geared specifically towards MGTAs would provide support and guidance (Kuechle, 2022). In this poster we examine: How do MGTAs' responses about teaching with active learning shift through a year of sustained professional development? We track three MGTAs' responses over time and describe their individual trajectories.

This is part of a large multi-site project focused on designing and implementing a multi-year professional development program for MGTAs. We report on data from one of the three sites, a public Southeastern university. Participants are graduate students who were teaching assistants for mathematics courses and took part in a teaching seminar in the fall and then an Introduction to Active Learning course in spring. Each of these professional development structures involved readings and activities in and out of class. These included reading Su (2016)'s *The Secret Mathematical Menu* and sessions on facilitating group work, examining the meaning of equity, considering majority versus marginalized identities, and reflecting on problematic assumptions in teaching. Data consisted of exit tickets and reflections to the aforementioned activities and the overall course from graduate students who consented to the research.

Preliminary analysis of MGTAs' discourse reveals (1) increased sophistication in articulating their thoughts regarding teaching, (2) centering of students, and (3) intellectual need for learning more about issues of diversity, equity, and inclusion in teaching. For example, one participant wrote, "The course has given me more specific examples of what equity and inclusion look like specifically in classroom settings and ideas on how to implement them." Another participant wrote in her exit ticket at the end of the course, "I have learned new ways to be more inclusive" and included what she would still like to learn: "More about changing our own biases." This last finding corroborates MGTAs' readiness for the next phase of the professional development and research: to design and implement a more advanced course around issues of equity, where they can gain deeper awareness and tools to be prepared for teaching. In the poster, we will provide quotes and trajectories for three MGTAs who consistently shared, for how their thoughts shifted through the various professional development activities.

Acknowledgments

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References

Beisiegel, M., Gibbons, C., & Rist, A. (2019). Teacher interrupted: How mathematics graduate teaching assistants (don't) learn about teaching. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97-104). Pretoria, South Africa: PME.

Kuechle, V. A. B. (2022). A Multicase Study of Three Graduate Teaching Assistants Participating in the MDISC Teaching Professional Development. Michigan State University.

Selinski, N. E., & Milbourne, H. (2015). The institutional context. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of College Calculus* (pp. 31-44). Washington, DC: MAA Press.

Su, F. E. (2016). The secret mathematical menu. *MAA FOCUS,* p. 28–30.

The Effect of Small Group Interactions on Opportunities for Student Learning

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Keywords: Group Work, Student Interaction, Calculus

Mathematics instructors frequently incorporate group work as a pedagogical tool to enrich students' comprehension and engagement. This approach promotes social interaction and encourages student autonomy, potentially leading to deeper student learning (e.g., Amit & Fried, 2005; Smith & Confrey, 1991). Our research investigates the intricate relationship between group dynamics and mathematical learning, specifically focusing on students' comprehension of graphs and distances within the Cartesian coordinate system. Understanding the impact of group dynamics on mathematical learning provides insights into how students collaboratively approach mathematical concepts, which can significantly influence their individual learning outcomes. Accordingly, we center our research around the following question: How do social interactions among undergraduate math students completing a graphing activity influence individual opportunities for mathematical learning?

Data for this study was collected from four sections of a Calculus II course at a small, private college in 2022 and 2023. The primary data source includes student work and 23 video recordings of small groups of 2-3 students working on an "Interpreting Graphs for Calculus Activity" during one 50-minute class period. This activity required students to write algebraic expressions to represent distances in one- and two-dimensional graphs. Data was also collected from students completing a pretest item before the activity, two questions on their third exam, and questions on their final exam, all assessing skills relevant to the graphing activity.

We draw on a preliminary analytical framework (Parr et al., 2023) to develop knowledge-based questions (e.g., who is being sought for knowledge?) and five questions related to social dynamics (e.g., who directs the group's activity?) to guide our video data analysis. Throughout video analysis, the first author created content logs, documenting key elements of students' behavior and interactions within the group work. Additionally, the first author documented emerging themes and patterns across group videos. Thus far, analysis has been conducted on ten videos, with plans to continue to analyze additional videos and work from the larger dataset.

Our analysis revealed a diverse spectrum of group interactions among students during the completion of the graphing activity. While some students demonstrated a high level of collaboration, others adopted a more independent approach, primarily cross-checking their answers and communicating only when uncertainties arose. Two prominent themes emerged across groups in the interactions, which impacted their responses to the tasks. First, the role of pattern recognition emerged, in which students validated solutions through pattern identification from previous tasks in the activity, valuing consistency. Second, the extent to which students were willing to correct their peers, reflecting variations in students' perceived comfort levels with correcting their group members, emerged as a significant distinction in varied opportunities for mathematical learning across groups.

The diverse group dynamics observed during students' group completion of the graphing activity highlights the need to comprehend the complex interplay between social interactions and mathematical learning. As our research progresses, we intend to use position codes to analyze selected episodes more closely from the videos that are expected to yield intriguing results.

References

Amit, M., & Fried M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th grade classroom. *Educational Studies in Mathematics, 58*, 145-168. https://doi.org/10.1007/s10649-005-3618-2.

Parr, E. D., Dyer, E. B., Machaka, N., & Krist, C. (2023). Understanding joint exploration: The epistemic positioning underlying collaborative activity in a secondary mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education.* https://doi.org/10.1007/s42330-023-00295-w

Smith, E., & Confrey, J. (1991, April). *Understanding collaborative learning: Small group work on contextual problems using a multi-representational software tool.* A paper presented at the Annual Meeting of the American Educational Research Association. Chicago.

Undergraduate Students' Mathematics Efficacy Change through Collaborative Projects in General Education Mathematics Course

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Keywords: Collaborative learning, General education mathematics, Mathematics efficacy

Some challenges of teaching General Education Mathematics Courses (GEMC) motivated us to investigate how doing collaborative projects affects undergraduate students' self-efficacy in mathematics. The problems we see in the GEMC include students' low self-efficacy in mathematics, low engagement, and negative attitudes toward mathematics. Many students in GEMC described mathematics as a challenging subject area and a mixture of formulas and intense calculation, not recognizing the close connection with real-life. The need to improve the student success rate in GEMC is pervasive (Aycaster, 2001; Harrington et al., 2016; Thiel et al., 2008). These issues led us to questions about changing these low and negatives to higher and positive. In addition to achieving the specific student learning outcomes, three main goals in our teaching include 1) promoting student self-efficacy, 2) boosting student engagement in mathematics, and 3) fostering students' recognition of the relevance of mathematics in the real world. Several research studies have presented that project-based approaches worked positively for students' academic achievement and interest (Han et al., 2015; Jacques, 2017; Larmer & Mergendoller & Boss, 2015; Lou et al., 2012; Remijan, 2017; Rice & Shannon, 2016; Tseng et al., 2013). We added two collaborative projects in Math 1004 (pseudonym), one of the GEMC, to get students more engaged and constructing their meaning in learning.

Five elements were considered in implementing the collaborative projects in Math 1004: 1) own team decision, 2) own topic decision, 3) online shared documentation, 4) presentation together, and 5) peer evaluation. Students could learn technical competency, communication, presentation skills, and mathematical content knowledge throughout this process. Students could develop competency in identifying the problems in the real world and how to present the issues related to the problem and demonstrate techniques.

To explore any change in students' mathematics self-efficacy and recognition of mathematics relevance in real life, the pre-and post-surveys were given in the two sections of the Math 1004 course in the fall of 2021. There were 22 participants in both the pre-survey and the post-survey. In the survey forms, students were asked to mark one of the Likert scales: Strong Disagree, Disagree, Not Sure, Agree, Strongly Agree.

In summary, the results of this pre-survey and post-survey presented a positive change in the participants' self-efficacy in applying mathematical knowledge, problem-solving and explanation quantitatively, mathematical interpretation, and writing ideas quantitively. The findings through these students' positive changes include the following: 1) The participants anticipated the relevance of mathematics in real life through collaborative projects in the GEMC. 2) The participants showed positive change related to efficacy in mathematics after completing the collaborative projects in GEMC.

It is necessary to understand better how collaborative learning approaches work related to students' efficacy in mathematics. Knowing more about what students value in their experiences of learning mathematics through a collaborative learning approach is essential for instructors to develop instructional strategies for positive outcomes in learning.

References

Aycaster, P. W. (2001). Factors impacting success in community college developmental mathematics courses and subsequent courses. *Community College Journal of Research & Practice*, *25*(5–6), 403–416.

Han, S., Capraro, R., & Capraro, M. M. (2015). How science, technology, engineering, and mathematics (STEM) project-based learning (PBL) affects high, middle, and low achievers differently: The impact of student factors on achievement. *International Journal of Science and Mathematics Education*, *13*, 1089–1113.

Harrington, M. A., Lloyd, A., Smolinski, T., & Shahin, M. (2016). Closing the gap: First year success in college mathematics at an HBCU. *Journal of the Scholarship of Teaching and Learning*, *16*(5), 92-106.

Holmes, V. L., & Hwang, Y. (2016). Exploring the effects of project-based learning in secondary mathematics education. *The Journal of Educational Research, 109*(5), 449–463.

Jacques, L. A. (2017). What does project-based learning (PBL) look like in the mathematics classroom? *American Journal of Educational Research*, *5*(4), 428-433.

Larmer, J., Mergendoller, J., & Boss, S. (2015). *Setting the standard for project-based learning*. Association for Supervision and Curriculum Development.

Lou, S. J., Chung, C. C., Dzan, W. Y., & Shih, R. C. (2012). Construction of a creative instructional design model using blended, project-based learning for college students. *Creative Education*, *3*(7), 1281–1290.

Remijan, K. W. (2017). Project-based learning and design-focused projects to motivate secondary mathematics students. *Interdisciplinary Journal of Problem-Based Learning*, *11*(1).

Rice, M., Shannon, L. Y. (2016). Developing project-based learning, integrated courses from two different colleges at an institution of higher education: An overview of the processes, challenges, and lessons learned. *Information Systems Education Journal, 14*(3), 55-62.

Tseng, K. H., Chang, C. C., Lou, S. J., & Chen, W. P. (2013). Attitudes towards science, technology, engineering and mathematics (STEM) in a project-based learning (PBL) environment. *International Journal of Technology and Design Education*, *23*, 87-102.

Thiel, T., Peterman, S., & Brown, M. (2008). Addressing the crisis in college mathematics: Designing courses for student success. *Change: The Magazine of Higher Learning*, *40*(4), 44-49.

Undergraduate Students' Mathematics Efficacy Change through Collaborative Projects in General Education Mathematics Course

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UNIVERSITY OF HARTFORD

Introduction

General Education Mathematics (GEM)

- Quantitative Literacy (Association of American Colleges and Universities, 2005; Bennett et al, 2008; Steen, 2004)
- Students Failing in GEM (Aycaster, 2001; Harrington et al., 2016; Thiel et al., 2008)
- Students are less confident in mathematics (Ellis et al., 2016; Hall & Ponton, 2005; Hill et al., 2010; Sadler & Sonnert, 2017).
- Mathematics is frustrating (Hoyles, 2016; Riegel, 2021).

Motivation

Issues Noticed in My Teaching Experiences

- Low self-efficacy in doing mathematics
- Low engagement
- Low interaction
- Difficulty in developing written communication summaries after/during their verbal discussion.
- Difficulty making the task align with other elements such as context and purposes.

Change in My Teaching

- Provide more meaningful learning opportunities for students to have a more positive orientation towards mathematics.
- Collaborative Projects in GEM

Elements Considered in the Implementation of Collaborative Projects

- Students' own team decision
- Students' own topic decision
- Online shared documentation to progress-check
- Presentation together
- Peer evaluation

Method

Ontological Assumptions

- "Realities are multiple, constructed, and holistic" (Lincoln & Guba, 1985, p. 35).
- Each student's experience constructs their reality, which is related to their learning.
- "Knowledge is constructed by learners as they attempt to make sense of their experiences" (Driscoll, 2000, p. 376).

Research Question

- How do students change in efficacy in mathematics through collaborative projects in the General Education Mathematics course?

Data Collection

- We added two collaborative projects in Math 1004 (pseudonym), one of the GMC.
- Pre-survey and post-survey were given in two sections of the Math 1004 courses in the fall of 2001.
- There were 22 participants in both the pre-survey and the post survey.
- In the Survey forms, students were asked to mark one of the Likert scales: Strongly Disagree, Disagree, Not Sure, Agree, Strongly Agree.

Results

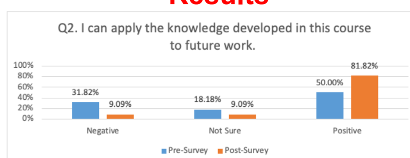


Figure 1: Students' self-efficacy change in applying mathematical knowledge in pre-survey and post-survey.

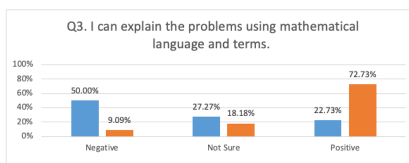


Figure 2: Students' self-efficacy change in problem-solving and explanation mathematically.

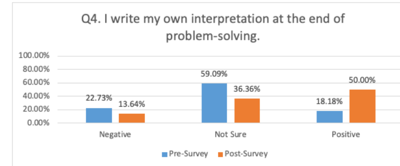


Figure 3: Students' self-efficacy changes in interpretation after mathematical problem-solving.

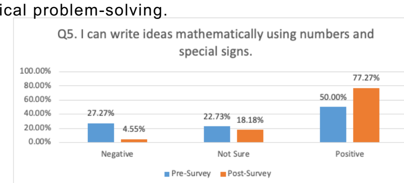
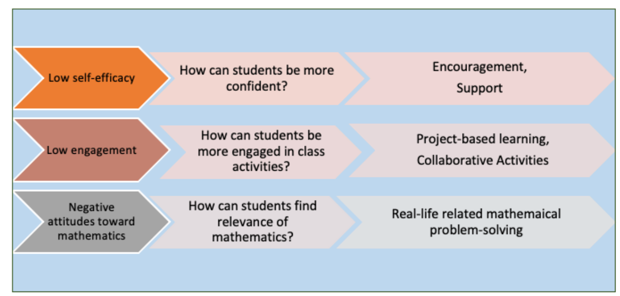


Figure 4: Students' self-efficacy changes in writing ideas mathematically.

The participants showed positive change related to efficacy in mathematics after completing the collaborative projects in GEM.

Discussions

- The participants anticipated the relevance of mathematics in real life through collaborative projects in the GEM.
- The students interacted more actively and constructively during class and outside class.



- Knowing more about what students value in their experiences of learning mathematics through a collaborative learning approach is essential for instructors to develop instructional strategies for positive outcomes in learning.

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Project Adelante: An Anti-deficit Professional Development Program for University Mathematics Instructors

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Keywords: antideficit perspective, professional development, community engagement

In this report, we share the design of a year-long professional development program for university math instructors that we developed and refined as the Anti-deficit Learning and Teaching Project (Adelante). The program is a community learning project wherein minoritized students, STEM peer mentors, and math instructors (graduate students and instructional faculty) build relationships as they share their knowledge and experiences with race, gender, and mathematics. *Culturally relevant pedagogy* (Ladson-Billing, 1995) frames the goals of the community learning in terms of deep mathematical knowledge, cultural knowledge, and sociopolitical consciousness. The program activities are inspired by the *Funds of Knowledge for Teaching* project (Moll et al., 1992) wherein teachers are offered opportunities to build meaningful relationships with students and their communities.

An anti-deficit perspective (Adiredja et al., 2020) guides the learning experience for all participants. Not only are minoritized students assumed to have cultural and intellectual assets for learning, but the project also aims to dismantle *deficit master narratives* (Solórzano & Yosso, 2002) about these students and their capacity to learn. Instructors worked on explicitly challenging deficit narratives about their students as they engaged in the program's activities. The project also takes an anti-deficit approach to instructor development, focusing on their individual growth and agency, joy in teaching, and mental health. We also position ourselves as learners to the experience and wisdom of the staff and students at the university cultural centers.

The core activities for the PD engage teachers to: (a) participate in five PD meetings on anti-deficit teaching and Inquiry Based Learning (IBL) teaching method; (b) lead a five-day math summer bridge workshop in Pre-Calculus, Calculus I, II, Vector Calculus, or Linear Algebra immediately following the meetings; (c) participate in critical conversations about race and gender in STEM with students at the cultural centers; (d) conduct a semi-structure interview with one of their students from the summer workshop about their STEM experience; and (e) participate in group reflection meetings debriefing their experience in the activities.

Preliminary analysis of two of the three cohorts of participants found that most instructors developed a more humanizing approach to their teaching and their students (Gutiérrez, 2018). IBL helped instructors to explicitly challenge deficit narratives about minoritized students in the classroom, wherein most observed their students engaging in deep mathematical reasoning. Interviewing one of their students also shifted deficit narratives that developed in the classroom for some instructors. The workshop served as a space to try out previously learned teaching ideas (student centered teaching) without constraints from curriculum and assessments. Doing so reinvigorated many instructors' passion for teaching, especially those who are more experienced.

Acknowledgments

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References

Adiredja, A. P., Bélanger-Rioux, R., & Zandieh, M. (2020). Everyday examples from students: An anti-deficit approach in the classroom. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 30*(5), 520-538. https://doi.org/10.1080/10511970.2019.1608609

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. M. Goffney & R. Gutiérrez (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx students*. Annual Perspectives in Mathematics Education. Reston, VA: National Council of Teachers of Mathematics, Inc.

Ladson-Billings, G. (1995). Toward a Theory of Culturally Relevant Pedagogy. American Educational Research Journal, 32(3), 465–491. https://doi.org/10.2307/1163320

Moll, L. C., Amanti, C., Neff, D., & González, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice, 21*(2), 132-141. https://doi.org/10.1080/00405849209543534

Solórzano, D.G., & Yosso, T. J. (2002). Critical race methodology: Counter-storytelling as an analytic framework for education research. *Qualitative Inquiry, 8(23),* 23-44. https://doi.org/10.1177/107780040200800103

From Conflict to Unity: Leveraging Incoherence in Student Thinking to Improve Understanding

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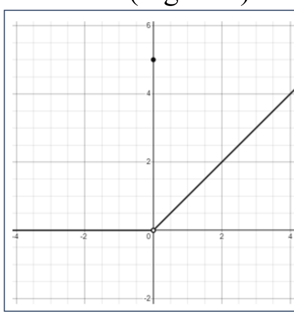
Keywords: schemes, coherence, teaching experiment, proof-texts, continuity, differentiability

Many methodological approaches, such as clinical interviews (Clement, 2000) and teaching experiments (Steffe & Thompson, 2000) are well-equipped for investigating the schemes and constructed knowledge of individuals (Glasersfeld, 1988). Sellers (2020) extended the teaching experiment to the exploratory teaching interview for when the researcher aims to influence student thinking but is not testing a pre-determined hypothesis. Numerous researchers have tasked subjects with reading proofs to probe their thinking (e.g., Dawkins & Zazkis, 2021). I used proof-texts during a teaching experiment which modeled a subject's thinking. This approach was profoundly effective in revealing inconsistencies to the subject in her thinking.

Incoherence as a Basis for Task Design

The lone participant in the study, Rachel, sat for a series of six interviews designed to probe her understanding of the chain rule of differentiation and its underlying ideas, such as continuity, differentiability, and function composition. On several occasions, she evidenced conceptions relevant to the tasks which I conjectured would result in her acknowledging a conflict if presented properly. For example, during the first interview, Rachel stated that a function f was continuous at a if a was in the domain of f. Later in the interview, she said that f was continuous at a if $\lim_{x\to a} f(x)$ existed, regardless of whether a was in the domain of f. Though she did not initially indicate that she found her thinking inconsistent, presenting her with proof-texts which imitated her thinking was effective in helping her recognize conflicts in her thinking and the need to reconcile them (Figure 1).



Claim: f is continuous at $x = 0$.
Proof:
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0^-} f(x) = 0$
- Since the one-sided limits agree, $\lim_{x\to 0} f(x) = 0$.
- Therefore, f is continuous at $x = 0$.

Claim: f is continuous at $x = 0$.
Proof:
- From the graph, we see that $f(0) = 5$.
- Since f is defined at $x = 0$, f is continuous at $x = 0$.

Figure 1: Graph of Function and Accompanying Proof-Texts

Discussion and Future Steps

The interventions in the teaching experiment were designed only to focus on sources of incoherence (Thompson, 2008). As such, my only desired learning outcome for the experiment was to perturb Rachel's schemes (Glasersfeld, 1988) and observe how she assimilated the new stimuli. Resolving conflicts allowed Rachel to cultivate a much deeper understanding of the target concepts. As such, the methodological combination of exploratory teaching interviews (Sellers, 2020) and proof-texts which imitate the subject's thinking warrants further implementation and investigation.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), Handbook of research design in mathematics and science education (pp. 547–589). Lawrence Erlbaum.

Dawkins, P. C., & Zazkis, D. (2021). Using moment-by-moment reading protocols to understand students' processes of reading mathematical proof. Journal for Research in Mathematics Education, 52(5), 510–538. https://doi.org/10.5951/jresematheduc-2020-0151

Glasersfeld, E. (1988). Cognition, construction of knowledge, and teaching. In *Constructivism in Science Education* (pp. 11–30). Springer.

Sellers, M. E. (2020). Students' Quantifications, Interpretations, and Negations of Complex Mathematical Statements from Calculus (Doctoral dissertation, Arizona State University).

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), Research design in mathematics and science education (pp. 267307). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 31-49). Morelia: PME.

Argument-Mirroring Proofs:
A Methodological Approach for Helping Students Recognize Incoherence in Their Thinking

Steven Ruiz, Arizona State University

Theoretical Backing

- Radical Constructivism (Glasersfeld, 1988)
  - *Scheme* – a discrete unit of knowledge possessed by an individual
  - *Perturbation* – when expectation associated with a scheme is different from what occurs
  - *Accommodation* – action taken by an individual to account for a perturbation
    - E.g., adjustment of scope of schemes, rejection of schemes, rejecting perturbation as anomaly
  - *Coherence* – the extent to which a thinker's schemes are consistent with one another (Thompson, 2008).

Glossary of New Constructs

- Extensions to tenets of radical constructivism
  - *Disjoint activated schemes* – schemes evoked by individual in response to specific stimuli
    - "Disjoint" refers to notion that individual's schemes may not be related or that individual may not recognize relationships between them
  - *Schemes targeted for conflict* – disjoint activated schemes which an outside observer conjectures represent lack of coherence in subject's thinking
  - *Perceived conflict* – Explicit recognition by the subject that their thinking was indeed incoherent
- ***Argument-Mirroring Proofs*** – texts which use subject's previously exhibited thinking to form and justify a claim, particularly in effort to perturb their schemes

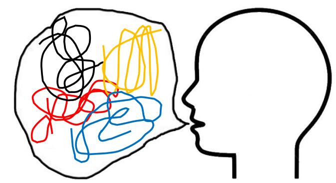
Overview of Case Study

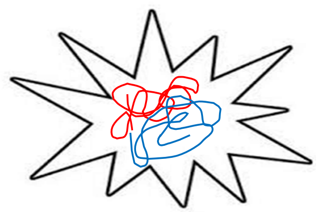
- Participant was an undergraduate student of mathematics education
- Teaching experiment (Steffe & Thompson, 2000)
  - Consisted of clinical interviews (Clement, 2000) and exploratory teaching interviews (Sellers, 2020)
- Task-based interviews focused on continuity, differentiability, and the derivative of composite functions

Argument-Mirroring Proofs, Illustrated

1) Characterize subject's schemes

- Conduct task-based clinical interviews to gather evidence about how student understands mathematical ideas at hand



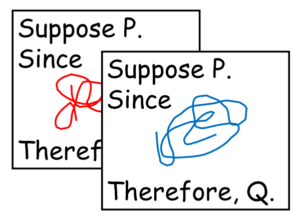


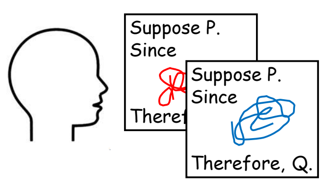
2) Target schemes for conflict

- Code clinical interview data to identify segments of subject's thinking which represent lack of coherence or internal inconsistency

3) Write argument-mirroring proofs

- Construct proof-like texts for use in exploratory teaching interview which model the thinking indicated by the student in the schemes targeted for conflict.



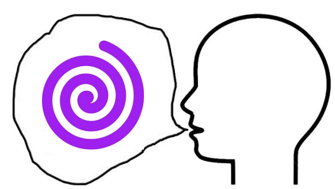


4) Present argument-mirroring proofs

- In an exploratory teaching interview, task the subject with examining the validity of the claims and reasoning in the proof-texts.

5) Characterize subject's schemes

- Code exploratory teaching interview data to gather evidence about how student understanding of mathematical ideas has **changed.**



Interview Tasks

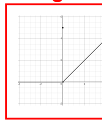
- Clinical interviews
  - Consider where $f(x) = \sqrt{k - 1 - x^2}$ is continuous (resp. differentiable) under various conditions
  - Purpose is to model subject's understanding of the mathematical ideas relevant to the task and target schemes for conflict
- Exploratory teaching interviews
  - Confirm that researcher's model is representative of subject's thinking
  - Present subject with **argument-mirroring proofs** in efforts to perturb their schemes

Illustrative Episode from Study

- Schemes Targeted for Conflict from First Interview

a) If f is defined at a, f is continuous at a.
b) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, then f is continuous at a.
c) Although f is not defined where $x = a$, if $\lim_{x\to a} f(x)$ exists, f is continuous at a.

- **Argument-Mirroring Proofs** from Second Interview



Claim: f is continuous at $x = 0$.
Proof:
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0^-} f(x) = 0$
- Since the one-sided limits agree, $\lim_{x\to 0} f(x) = 0$.
- Therefore, f is continuous at $x = 0$.

Claim: f is continuous at $x = 0$.
Proof:
- From the graph, we see that $f(0) = 5$.
- Since f is defined at $x = 0$, f is continuous at $x = 0$.

- Schemes Targeted for Conflict after Accommodation

a) If f is defined at a, f is continuous at a.
b) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, then f is continuous at a.
c) ~~Although f is not defined where $x = a$, if $\lim_{x\to a} f(x)$ exists, f is continuous at a.~~

→ If f is defined at a, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$ then f is continuous at a.

Discussion and Next Steps

- Key Findings
  - **Argument-mirroring proofs** were unanimously effective in revealing incoherence to subject
  - Subject's thinking became more coherent in response to **argument-mirroring proofs**
- Future Research
  - Conduct similar investigations with students of various mathematical backgrounds
  - Explore other mathematical content areas

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References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), Handbook of research design in mathematics and science education (pp. 547–589). Lawrence Erlbaum.

Glasersfeld, E. (1988). Cognition, construction of knowledge, and teaching. In *Constructivism in Science Education* (pp. 11–30). Springer.

Sellers, M. E. (2020). Students' Quantifications, Interpretations, and Negations of Complex Mathematical Statements from Calculus (Doctoral dissertation, Arizona State University).

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), Research design in mathematics and science education (pp. 267307). Hillsdale, NJ: Erlbaum.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Sage.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 31-49). Morelia: PME.

Student Perceptions on Collaborative Class Projects in Gateway College Calculus Courses

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Keywords: Gateway Calculus, Group work, STEM writing, Real-world applications, peers

Working in small groups can be a daunting activity for students, yet also provide an opportunity to promote peer interactions and build an inclusive community (Tanner, 2013). In this focus group study, we present how students (n = 43) perceive collaborative class projects in Calculus I at Johns Hopkins University, a predominantly-White, “very high research” university in an urban metropolitan area in the Northeastern United States. 51.16% identified as women and 48.84% as men. The most represented majors were Computer Science (23.26%), Mechanical Engineering (16.28%), and Pre-Major (16.28%).

Prior to the focus group, participants filled out a questionnaire that gathered their demographic information (major, gender identity, racial/ethnic identities), first-generation student status, limited-income status, etc.). The questionnaire also asked students to rank these course assessments from most helpful to least helpful to their learning: homework, quizzes, group projects, final portfolio, midterm, and final exam. While homework ranked as the most helpful and tests (final and midterm) as the least helpful, group projects have the highest variability in helpfulness. Hence, in the focus group, Dr. Yap explored this trend by eliciting elaboration on students’ group work experience. Responses were analyzed by qualitative coding of themes (Saldana, 2021) that emerged from the focus groups.

Findings show that participants who highly favored group projects saw these collaborative activities as opportunities: (1) to meet students who can be their friends outside of class, (2) to practice Calculus in a different way, such as being a teacher to a groupmate to explain a concept, and (3) to apply Calculus in practical ways. Meanwhile, other participants disliked group projects because of (1) group dynamics, (2) logistical limitations, and (3) resistance to new ways of learning Math. Examples of group dynamics include dealing with absent or uncooperative groupmates and feeling left out if groupmates are either too far ahead or behind in understanding the class topic. Logistical limitations consist of receiving limited instructions about the group project, including rubrics not being given in advance; however, instructions and rubrics were posted in the syllabus and learning management system. Finally, students complained that the group work was “busy work” forcing them to work with others, was not directly helpful in exams, was too demanding in writing in non-technical ways, was unclear how it was connected to the lessons, and too practical and connected to the real world.

These findings inform the next iteration of Calculus I group projects at Johns Hopkins University. The curriculum team will improve collaborative projects by (1) providing multiple reminders of the already existing pointed instructions and rubrics in multiple modalities, (2) providing best practices in group communication, in regards to norms for participation, work distribution, etc., and (3) providing clear connection to lessons for genuine buy-in from resistant students.

In conclusion, collaborative group projects provide opportunities for improving peer social cohesion and practical knowledge of Calculus but need targeted improvements in implementation and genuine buy-in from potentially resistant students.

References

Tanner, K. (2013). Structure Matters: Twenty-One Teaching Strategies to Promote Student Engagement and Cultivate Classroom Equity. *CBE-LSE, 12*, 322-331.

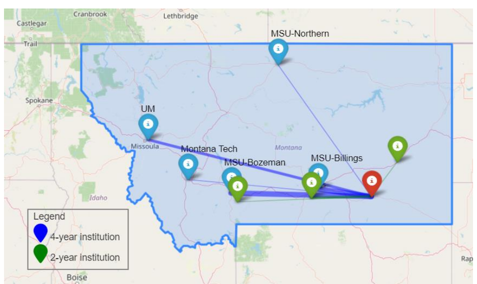
Saldana, J. (2021). *The Coding Manual for Qualitative Researchers*. New York, NY: SAGE.

STEM Course Taking Patterns of Transferred Students from Tribal College

Authors: Shurong Li*, Jeff Hooker**, Gary Ramsey**, Aaron Thomas*, Ke Wu*

* University of Montana, ** Chief Dull Knife College

The purpose of this study is to examine the academic outcomes and performance patterns of students transferring from a tribal college to four-year institutions in the Montana University System(MUS).

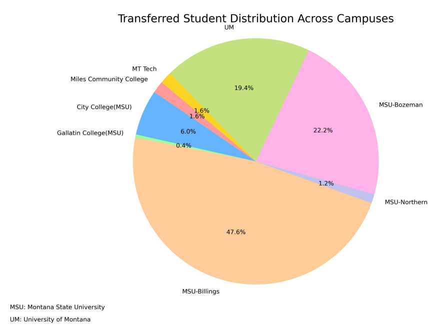


Data source: Merged data between a tribal college and OCHE for 2001 to 2019

Data Variables Overview:

- Transferred student counts by year and term
- Transferred student counts by year and institution
- Student demographic: Ethnicity and Gender
- Transfer destinations within MUS
- Major distribution across institutions
- Enrollment duration in terms of semesters
- Initial status at 4-year institutions
- First-term GPA
- Credit hours completed per term
- Graduation after 4-year MUS enrollment
- STEM courses of identified successful students

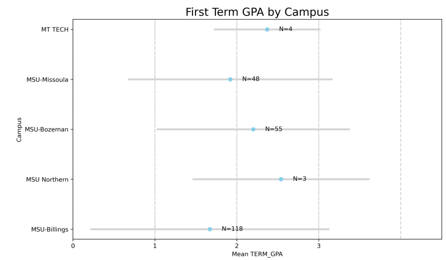
Gillborn, D., Warmington, P., & Demack, S. (2018). QuantCrit: Education, policy,'Big Data'and principles for a critical race theory of statistics. *Race ethnicity and education*, *21*(2), 158-179.



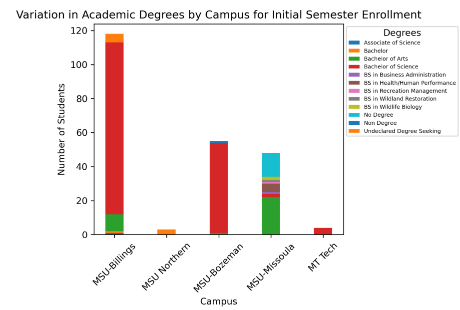
Transferred Students General Demographics

Ethnicity	Frequency	%
Hispanic	4	1.6
Indian	**208**	**83.9**
Missing	4	1.6
Other	1	.4
White	31	12.5
Total	248	100.0

Gender	Frequency	%
Female	165	66.5
Male	83	33.5
Total	248	100



More than 56% of Students Achieved Good Standing in the First Term.



Patterns of Successful Students:

- **"Success" is identified as tribal college students who transferred to a 4-year public institution and earned a STEM degree.**
- **A majority of these successful Native American students are female.**
- **Successful transfer students often return to CDKC for summer science courses.**

Discussion:

- Western success measures, such as degree completion times, may not align with tribal college experiences.
- The role of tribal colleges extends beyond transfer, supporting students through their 4-year institution journey.
- The notable success of Native women in STEM highlights the need to explore the participation of other Native demographics.

Acknowledgement:

- This material is based upon work supported by the National Science Foundation under Grant No.1937225 and No. 1361522.
- The University of Montana acknowledges that we are in the aboriginal territories of the Salish and Kalispell people. Today, we honor the path they have always shown us in caring for this place for the generations to come.
- Chief Dull Knife College acknowledges that we are located on the traditional territory of the Northern Cheyenne people. We honor their deep connection to this land and recognize the importance of their culture, history, and traditions. We are committed to working in partnership with the Northern Cheyenne people to create a future that is respectful of their sovereignty and self-determination.
- This work was supported by the Montana Office of the Commissioner of Higher Education (OCHE) and the University of Montana's Department of Mathematical Sciences Summer Research Graduate Scholarship Program. Our sincere thanks for their essential support and resources.

Postsecondary Mathematics Teaching Methods and Practices: A National Study

Molly C. Bowen
Baylor University

Introduction

Many students find themselves struggling in college-level coursework despite previous successful secondary academic experiences and acceptable college entrance exam scores (Complete College America, 2015; Woods et al., 2018). There have been few studies that specifically address college mathematics faculty knowledge and utilization of best teaching methods and practices and their effectiveness (Benken et al., 2015; Cox, 2007). Recent research has shown that teachers have the greatest impact on student achievement above any other factor of schooling (Chemosit & Rugutt, 2020; Schriver & Harr Kulynych, 2021). Therefore, the researcher explored teaching methods and practices utilized by faculty in the U.S.

Literature Review

Many faculty heavily rely on lecturing and/or direct instruction for most to all of the course and may not know about other teaching methods and/or practices (Blair et al., 2018; Mathematical Association of America, 2017; Mesa et al., 2014; Ngo, 2020; Oleson & Hora, 2014). Unlike PK-12 teachers, faculty are not required to obtain teaching certifications, licenses, or participate in internships or mentorships; they are only required to have an advanced degree (Blair et al., 2018; Chiu & Corrigan, 2019; Gilmore et al., 2014; U.S. Bureau of Labor Statistics, 2016). Inexperience and lack of support may contribute to the heavy use of lecture and/or direct instruction (Fong & Zientek, 2019; Purnomo et al., 2018).

Methodology

This sequential explanatory mixed methods study took place in postsecondary institutions in the U.S. Cluster-random sampling was used to select institutions. Mathematics department chairs were sent a survey containing the TPI (*Teaching Practices Inventory*), ATI (*Approaches to Teaching Inventory*), and demographic questions to forward to faculty (full-time, part-time, tenure-track, and non-tenure track) (Trigwell & Prosser, 2004; Wieman & Gilbert, 2014). It was hypothesized that demographic groups would reveal statistically significant differences in TPI scores, ATI classifications (information-transmission/teacher-focused (ITTF) and conceptual change/student-focused (CCSF)), and between the TPI and ATI. The research questions (RQ): (1) Are there statistically significant differences in the TPI score performance between faculty demographic groups, methods, or practices? (2) What similarities and differences can be found in the ATI classification between faculty demographic groups, methods, or practices?

Results

There were 113 participants from 37 states. For the first RQ, the significant demographic group was membership. The testing revealed that faculty who were members were more likely to utilize more methods and practices that focus on their students. For the second RQ, demographic groups did not reveal significant findings, but individual methods and practices were significant. Faculty who reported utilizing the inquiry, problem-based learning and cooperative learning teaching methods were more likely to be classified by the ATI as CCSF. Also, faculty who utilized more than three methods were more likely to be classified as CCSF. Qualitative and mixed method results are still in progress.

References

Blair, R., Kirkman, E., & Maxwell, J. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2015 CBMS survey*. American Mathematical Society. https://doi.org/10.1090/cbmssurvey/2015

Chemosit, C. C., & Rugutt, J. K. (2020). The impact of professor engagement, student peer interactions, and traditional status on student assessment of quality of teaching and learning. *Educational Research Quarterly*, *44*(1), 64–83.

Chiu, P. H. P., & Corrigan, P. (2019). A study of graduate teaching assistants' self-efficacy in teaching: Fits and starts in the first triennium of teaching. *Cogent Education*, *6*(1), 1579964. https://doi.org/10.1080/2331186X.2019.1579964

Complete College America. (2015). *Data Dashboard*. Complete College America. https://completecollege.org/data-dashboard/

Fong, C. J., & Zientek, L. R. (2019). Instructional practices in developmental mathematics: A multilevel analysis of community college student perceptions. *Journal of College Reading and Learning*, *49*(1), 35–52. https://doi.org/10.1080/10790195.2018.1514283

Gilmore, J., Maher, M. A., Feldon, D. F., & Timmerman, B. (2014). Exploration of factors related to the development of science, technology, engineering, and mathematics graduate teaching assistants' teaching orientations. *Studies in Higher Education*, *39*(10), 1910–1928. https://doi.org/10.1080/03075079.2013.806459

Kilbane, C. R., & Milman, N., B. (2014). *Teaching models: Designing instruction for 21st century learners*. Pearson.

Mathematical Association of America. (2017). *Instructional practices guide*. https://www.maa.org/programs-and-communities/curriculum%20resources/instructional-practices-guide

Mesa, V., Celis, S., & Lande, E. (2014). Teaching approaches of community college mathematics faculty: Do they relate to classroom practices? *American Educational Research Journal*, *51*(1), 117–151. https://doi.org/10.3102/0002831213505759

NCTM. (2014). *Principles to actions: Ensuring mathematical success for all* (6th ed.). NCTM.

Ngo, F. J. (2020). High school all over again: The problem of redundant college mathematics. *The Journal of Higher Education (Columbus)*, *91*(2), 222–248. https://doi.org/10.1080/00221546.2019.1611326

Oleson, A., & Hora, M. T. (2014). Teaching the way they were taught? Revisiting the sources of teaching knowledge and the role of prior experience in shaping faculty teaching practices. *Higher Education; Dordrecht*, *68*(1), 29–45. http://dx.doi.org/10.1007/s10734-013-9678-9

Purnomo, Y. W., Kaur, A., Ismail, S. N. B., Suryadi, D., & Darwis, S. (2018). The consistency between professed teaching practices and assessment practices: A case in mathematics class. *Beta*, *11*(2), 101–113. https://doi.org/10.20414/betajtm.v11i2.223

Schriver, J. L., & Harr Kulynych, R. (2021). Do professor–student rapport and mattering predict college student outcomes? *Teaching of Psychology*, 00986283211037987. https://doi.org/10.1177/00986283211037987

Trigwell, K., & Prosser, M. (2004). Development and use of the approaches to teaching inventory. *Educational Psychology Review*, *16*(4), 409–424. https://doi.org/10.1007/s10648-004-0007-9

U.S. Bureau of Labor Statistics. (2016). *Postsecondary Teachers: Occupational Outlook Handbook: : U.S. Bureau of Labor Statistics*. https://www.bls.gov/ooh/education-training-and-library/postsecondary-teachers.htm

Wieman, C., & Gilbert, S. (2014). The teaching practices inventory: A new tool for characterizing college and university teaching in mathematics and science. *CBE—Life Sciences Education*, *13*(3), 552–569. https://doi.org/10.1187/cbe.14-02-0023

Woods, C. S., Park, T., Hu, S., & Betrand Jones, T. (2018). How high school coursework predicts introductory college-level course success. *Community College Review*, *46*(2), 176–196. https://doi.org/10.1177/0091552118759419

Postsecondary Mathematics Teaching Methods and Practices: A National Study

Molly C. Bowen, Ph.D. Candidate
RUME Conference, Omaha, NE, Feb. 22-24

CURRICULUM & INSTRUCTION GRADUATE PROGRAM

Background

Many students find themselves struggling in college-level mathematics courses (Complete College America, 2017; Woods et al., 2018).

- Teachers have the greatest impact on student achievement (Chemosit & Rugutt, 2020; Schriver & Harr Kulynych, 2021).
- There have been few studies that specifically address college mathematics faculty knowledge (Benken et al., 2015; Cox, 2007).

Therefore, the researcher explored teaching methods and practices utilized by faculty in the U.S.

Quantitative Research Questions

1. Are there statistically significant differences in the TPI score performance between faculty demographic groups, methods, or practices?

2. What similarities and differences exist in the ATI classifications Conceptual Change/Student-Focused (CCSF) or Information Transfer/Teacher-Focused (ITTF) between faculty demographic groups, methods, or practices?

Methods

This study was conducted using sequential explanatory mixed methods.

- **Quantitative Participant Selection Process:**

Institution Selection
Chose institution by type (2-year, 4-year, university) and geographic location

→

Targeted Participants
Department chairs were notified and forwarded the survey to the participants

- **Items in Survey to Participants:**

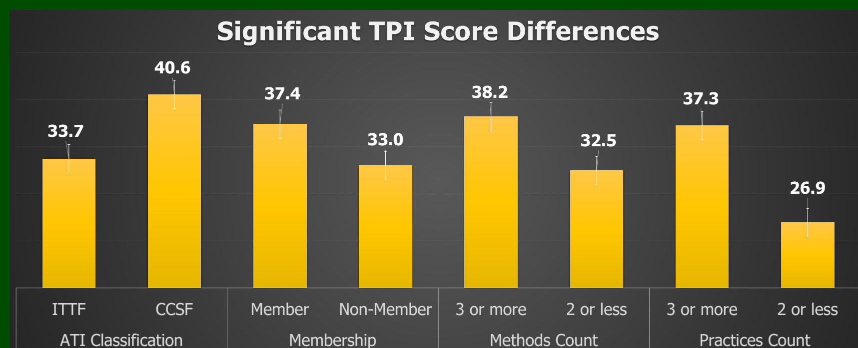
TPI
Results in a score for practices (Wieman & Gilbert, 2014)

ATI
Results in a classification of either conceptual-change/student-focused (CCSF) or information transfer/teacher-focused (ITTF) (Trigwell & Prosser, 2005)

Teaching methods are defined by Kilbane and Milman (2014)

Teaching practices are defined by NCTM (2014).

Faculty who were **student-focused** were **more likely** to **utilize more practices and methods** in their classrooms than those who were teacher-focused. **Membership** in professional organizations also correlated with the use of **more practices**.



CCSF Classification Significant Results

Group	Chi-Squared Value	Probability
Lecture (M)	4.36	0.0368
Inquiry (M)	5.39	0.0203
Problem-Based Learning (M)	5.07	0.0243
Cooperative Learning (M)	5.84	0.0157
Goals (P)	4.06	0.0440
Build Procedural Fluency from Conceptual Understanding (P)	4.59	0.0322
Method Count (3 or more)	9.03	0.0027

Note: M=Method, P=Practice

ITTF Classification Significant Results

Group	Chi-Squared Value	Probability
Lecture (M)	8.27	0.0040
Inquiry (M)	8.55	0.0035
Problem-Based Learning (M)	7.56	0.0060
Cooperative Learning (M)	8.41	0.0037
Goals (P)	4.42	0.0354
Methods Count (3 or more)	11.01	0.0009

Note: M=Method, P=Practice

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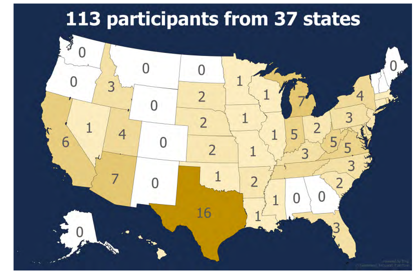
Results



Most Frequent Methods and Practices

- Direct Method: n=85 (75.2%)
- Lecture Method: n=45 (39.8%)
- Problem-Based Learning Method: n=35 (31.0%)
- Implementing Tasks Practice: n=65 (57.5%)
- Mathematical Representations and Connections Practice: n=56 (49.6%)
- Productive Struggle Practice: n=52 (46.0%)

Chi-squared testing revealed that these variables were significant for the ATI classifications. Phi coefficients were calculated to determine an association, which is similar to a regression score.

Positive Association with CCSF (phi > 0.2)	• Inquiry, Problem-based learning, and Cooperative learning methods • Use of 3 or more teaching methods
Positive Association with ITTF (phi > 0.2)	• Lecture • Procedural Knowledge from Conceptual Understanding
Negligible Association (0< phi < 0.2)	• Setting Goals

Discussion

- Faculty who were members of a professional organization had statistically significantly higher TPI scores, which means more practices are being utilized in their classrooms.
- Faculty who reported utilizing the inquiry, problem-based learning, and cooperative learning teaching methods or a combination of multiple methods were more likely to be classified as CCSF (conceptual-change/student-focused) than faculty using other methods.
- Qualitative themes emerged and include faculty experiences as students, professional development, teaching experience, and personal connections.
- Integration of results is in progress.

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Engaging Students as Partners in Critically-Oriented Reform of Postsecondary Mathematics

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Keywords: Critical reforms, Students as Partners, Student voice, Power dynamics

A growing body of research points to the transformative potential of engaging students as partners with faculty to humanize math education (Cook-Sather et al., 2023). Successful partnerships follow a principle of reciprocity, in which students and faculty are both positioned as having expertise which can be leveraged to improve education (Mercer-Mapstone et al., 2017). A larger research study was developed to investigate how the use of Networked Improvement Communities (NIC) composed of key mathematics stakeholders could work to address issues of Diversity, Equity, and Inclusion (DEI) within introductory mathematics courses through critically transformative participatory action research. A unique single case-study emerged from a NIC that recruited students in addition to the faculty as NIC members. We address the following research questions: (a) In what ways was **student voice** prioritized in the NIC? (b) How does the integration of students impact the **power dynamics** within the NIC?

This single case study (Yin, 2009) draws from data collected as part of the ACT UP Math project which is examining the formation of NICs addressing inequities in introductory math courses. Alpha University's NIC presented a unique case study because of their intentional recruitment of students. Out of the eight NIC members, three are students. The NIC met every other week for two hours from January-May 2023 and created two action plans informed by data: (1) dismantling the placement system for introductory mathematics courses and (2) creating programming that connects students to the uses of mathematics.

We conducted a thematic analysis of structured observational field notes of the NIC meetings, semi-structured interviews of NIC members conducted in May 2023, and four reflexive journal entries completed by each NIC member. This process generated cross-cutting themes relevant to the NIC's inclusion of student voice and perceptions of power. These themes are: (1) the intentions of the NIC to prioritize student voice, (2) the inclusion of students as partners (or not) in NIC activities, and (3) reflections on power dynamics by NIC members.

Findings from this study suggest that although the structure of the NIC was intended to uplift student voices and create a space where students and faculty were *equal partners*, outside power structures prevented students from fully viewing themselves as partners with faculty members. For example, NIC student members described how having colleagues in the NIC who were also course instructors outside of the NIC influenced the ways in which they interacted with each other. However, the NIC has made strides mitigating power dynamics over the semester and demonstrating the value of student voice.

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References

Cook-Sather A., Salmeron D., & Smith T. (2023). Humanizing STEM education through student-faculty pedagogical partnerships. *Frontiers in Education, 8*.

Mercer-Mapstone, L., Dvorakova, S. L., Matthews, K. E., Abbot, S., Cheng, B., Felten, P., Knorr, K., Marquis, E., Shammas, R., & Swaim, K. (2017). A Systematic Literature Review of Students as Partners in Higher Education. *International Journal for Students as Partners, 1*(1).

Yin, R. K. (2009). *Case study research: Design and methods* (4th Ed.). Sage.

Exploring Undergraduate Students' Experiences with Standards-Based Grading

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Keywords: Standards-Based Grading, Specifications Grading, Student Perceptions, Grounded Theory

Specifications or Standards-based grading (SBG) is a set of grading approaches that focus on evaluating students' proficiency of learning objectives (Campbell et al., 2020). Over the academic year of 2022-2023, seven Pennsylvania State University faculty from the departments of chemistry, math, physics, and statistics, have implemented SBG in a dozen classes, ranging from introductory to upper division and small (<25 students) to large (~500 students). We present preliminary results of an investigation on students' experience with SBG in these settings. There have been only a few such investigations in higher education (e.g., see Buckmiller et al., 2017).

Our retrospective qualitative study includes student interviews and focus groups. We selected participants from initial survey responses paying attention to diversity among demographics, courses, and expressed opinions about SBG. At the time of this submission, 186 students have completed the survey, and we have conducted interviews with seven students and two focus groups covering seven additional students. While most of the students' comments in the survey describe positive experiences with SBG, we made sure to interview students who described negative or particularly unusual experiences. All authors participate in constant comparative coding, focused coding, and memo writing (Olson et al., 2016).

As we work towards building a grounded theory of students' experiences with SBG, we have identified several preliminary emergent themes. First, many participants feel that SBG leads to deeper learning compared to traditional grading system. Second, students often highlighted the flexibility of SBG systems, particularly the common retake policy on proficiency checks or quizzes. Some students discussed using this flexibility to have more control over their learning, setting up the pace of their study, increasing motivation and reducing stress. One student mentioned difficulties that arose from such flexibility as they had to manage procrastination and work-avoidance behaviors. One focus group discussed the stress they felt related to the slow pace of progress and/or exhaustion from the large number of quizzes. Almost all participants reported some level of initial confusion with how SBG worked. Finally, some participants expressed mistrust towards the attribution of grades in traditional grading systems. These students conveyed that SBG grades felt more reflective of their actual understanding of the material.

Acknowledgments

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References

Campbell, R., Clark, D., & O Shaughnessy, J. (2020). Introduction to the special issue on implementing mastery grading in the undergraduate mathematics classroom. *PRIMUS*, *30*(8-10), 837-848.

Buckmiller, T., Peters, R., & Kruse, J. (2017). Questioning points and percentages: Standards-based grading (SBG) in higher education. *College Teaching*, *65*(4), 151-157.

Olson, J., McAllister, C., Grinnell, L., Gehrke Walters, K., & Appunn, F. (2016). Applying Constant Comparative Method with Multiple Investigators and Inter-Coder Reliability. *The Qualitative Report*. https://doi.org/10.46743/2160-3715/2016.2447

Building a grounded theory of Students' Experiences with Standards-Based Grading.

Authors

Russ deForest, Debbie Gaydos, Neil Hatfield, Joe Houck, Eric Hudson, Louis Leblond, Jennelle Malcos, Michael Steward

Specifications or Standards-based grading (SBG) is a set of grading approaches that focus on evaluating students' proficiency of learning objectives. Often graded on pass/fail or proficient/progressing with multiple opportunities for retakes.

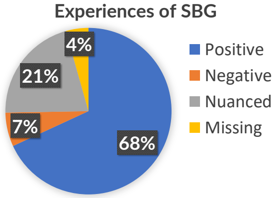
Research Question: What are the experiences of undergraduate students in science and mathematics courses that use Standards-Based/Specifications Grading?

Courses	N	Main Assessment
Intro Calculus	238	Learning Target Quizzes with retakes
Statistics	132	Homework/Projects
Chemistry	739	Learning Target Quizzes with retakes
Physics	90	Learning Target Quizzes with retakes

METHODS

1. Retrospective Survey data (91 respondents).
2. Individual and focus group interviews (24 students to date).
3. Focused coding, memos.
4. Work on the grounded theory in progress.

Experiences of SBG



Students perceive that the control and flexibility of standards-based grading increases motivation and reduces stress but some report procrastination and test-reset behaviors.

[...] and just gather knowledge at your own pace, you know, at your own time, and stress-free.

You were able to just breathe and really understand the work that you were doing

You really had full control over what grade you were going to get.

This Research Is

Proficient	Progressing ✅

Early Findings

SBG = Standards-Based grading
TG = Traditional Grading

Grades as source of motivation
- Trade-offs between minimizing effort and maximizing grade
- Striving to reach mastery in SBG, not just satisfactory
- Low grades are not damning (SBG)
- SBG grades only increase; TG is opposite

Understanding material/content
- Test/reset phenomenon, chunking
  - Felt in both grading systems
  - TG incentivizes short-term learning
  - Some students saw chunking as a positive that helped them learn
- SBG provided deeper learning
  - Better at recall in SBG

Flexibility of SBG
- Both positive and negative
- Student controls learning in SBG
  - Can lead to procrastination
  - Can lower stress

Stress of grading systems (both)
- Many items tested together TG
- Initial unfamiliarity with SBG

The Pennsylvania State University

We thank the Eberly College of Science for a teaching innovation grant.

Exploring Summer Bridge Program Participants' Sense of Belonging in Mathematics

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The University of Alabama

Keywords: sense of belonging; identity; self-efficacy; diversity; bridge programs

Students often leave STEM majors after taking math courses within the first two years of college (e.g., Chen et al., 2013), and historically underrepresented college students are less likely to persist in STEM areas compared to their peers (e.g., Alkhasawneh & Hobson, 2011). A sense of belonging in math may improve students' desire to persist in STEM (Good et al. 2012), so programs that support the development of a math sense of belonging could be a critical component of STEM retention. Summer bridge programs contribute to retention efforts of underrepresented STEM students through diverse representation, community, mentorship, and tutoring. However, little is known about how these programs influence math sense of belonging. This project explores the development of participants' sense of belonging, confidence, and identity in math while involved in a summer bridge program. In particular, I ask: (1) How does the math sense of belonging of bridge program participants change or differ throughout their participation in a bridge program?, and (2) What aspects of bridge programs do participants value and why? How do these valued aspects relate to their sense of belonging in math?

Theoretical Framework

This work is grounded in the academic and social integration components of Tinto's Conceptual Schema for Dropout from College (1975), which identifies academic and social systems as contributors to one's dropout decision. This theoretical framework has been widely adapted for researching retention in STEM.

Methods

Data were collected using surveys and semi-structured interviews. The survey instrument included the 30-item Good et al. (2012) sense of belonging in math scale, the six-item Lubienski et al. (2021) math confidence scale, and short answer questions about students' identity and sense of belonging in math. A coding scheme for belonging explanations, adapted from Rainey et al. (2018), was used for initial content analysis and coding reasons such as math identity, interpersonal relationships, personal interest, or competence. I used qualitative analysis methods and descriptive statistics to produce results.

Results and Conclusions

Participants' feelings of membership and acceptance as measured by the Good et al. (2012) sense of belonging in mathematics scale, declined between the start and end of the summer bridge program. However, this decrease may be attributed to the intensive nature of summer courses. Despite these challenges, students valued the mentoring and program components.

Interviews with one participant revealed confidence in their sense of belonging in math with the reasoning being personal interest and competence, often feeling reassured by peers and mentors in the program. Initial findings indicate a decline in students' overall sense of belonging in math after their first college summer math course. Yet, the positive experiences with the bridge program suggest that continued participation may lead to an improvement in their sense of belonging over time.

References

Alkhasawneh, R & Hargraves, R. (2014). Developing a hybrid model to predict student first-year retention in STEM disciplines using machine learning techniques. *Journal of STEM Education: Innovations and Research,* 15(3), 35-42.

Chen, X. (2013). STEM Attrition: College Students' Paths into and out of STEM Fields. Statistical Analysis Report. NCES 2014-001. *National Center for Education Statistics*.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, *102*(4), 700–717. https://doi.org/10.1037/a0026659

Lubienski, S. T., Ganley, C. M., Makowski, M. B., Miller, E. K., & Timmer, J. D. (2021). "Bold problem solving": A new construct for understanding gender differences in mathematics. *Journal for Research in Mathematics Education*, *52*(1), 12-61.

Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International Journal of STEM Education*, *5*(1). https://doi.org/10.1186/s40594-018-0115-6

Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research*, *45*(1), 89–125. https://doi.org/10.3102/00346543045001089

Exploring Summer Bridge Program Participants' Sense of Belonging in Mathematics

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THE UNIVERSITY OF ALABAMA | College of Arts & Sciences

PURPOSE & MOTIVATION

- Attrition in STEM
- Less than half of American students who enroll in STEM undergraduate programs go on to earn a STEM degree (Wilson et al., 2012).
- Historically underrepresented students are less likely to persist in STEM areas in comparison to their counterparts at the postsecondary level (Alkasawneh & Hobson, 2011; Riegle-Crumb et al., 2019)
- Students often leave STEM majors after taking mathematics courses within the first two years of college (Chen et al., 2013; Seymour & Hunter, 2019)
- Research has revealed that the first year of college is a critical period for attrition in STEM (Chen, 2013), particularly as the transition to college involves students negotiating both academic and personal stressors (Ruble & Seidman, 1996).
- Sense of Belonging (Strayhorn, 2012), Identity (Estrada et al., 2018), and Self-efficacy/Confidence (Rincón, 2018) are some variables that often impact one's ability to successfully academically and socially integrate in STEM.

RESEARCH QUESTIONS

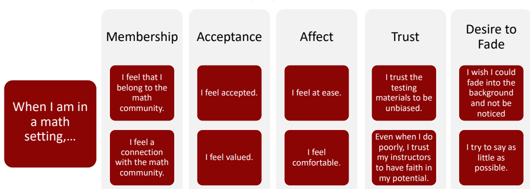
1. How does the math sense of belonging of bridge program participants change or differ throughout their participation in a bridge program?
2. What aspects of bridge programs do participants value and why? How do these valued aspects relate to their sense of belonging in math?

THEORETICAL FRAMEWORK

STEM retention programs are dynamic environments for integrating academic and social systems. This work is grounded in the academic and social integration components of Tinto's Conceptual Schema for Dropout from College (1975), which identifies academic and social systems as contributors to one's dropout decision. This theoretical framework has been widely adapted for researching retention in STEM.

METHODS

Mathematics Sense of Belonging Scale (Good et al., 2012)

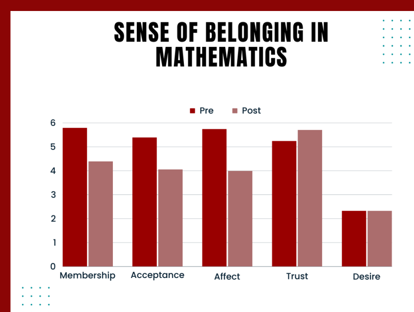


Coding Scheme for Belonging Explanations (Rainey et al., 2018)

Code	Reason for belonging	Reason for not belonging
Interpersonal relationships	Feels socially connected with peers and/or faculty members. May share common interest with peers.	Lacks a social connection with peers. Feels socially different, does not fit in.
Math Identity	Math is a part of their identity as a person.	Lacks a personal connection to the major or material
Personal Interest	Expresses personal interest in course subject or major.	Explicit lack of interest. May find the material boring or unrelated to their reason for choosing their major.
Competence	Feels like they understand major-related material or receives good grades in major-related courses.	Feels like they do not understand major-related material well or receives poor grades in major-related courses.

MAIN FINDINGS

Initial findings indicate a **decline** in **underrepresented STEM bridge** program participants' overall **sense of belonging** in **math** after their first **college summer math course**.

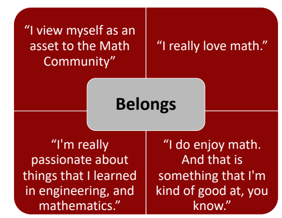


SITE & SAMPLE

- Large, graduate degree-granting, research-intensive PWI, Southeastern part of the United States
  - Louis Stokes Alliance for Minority Participation (LSAMP)
    - 4-week Summer Bridge Program
    - 16 Underrepresented incoming freshmen in STEM

RESULTS

Excerpts of a Student's Responses



"…making sure that I had that foundation…, really contributed to my sense of belonging, because when I would get the problems, it would make me feel like, oh, wow, I can really do the math. You know, this is something I enjoy. It's something I love."

Reasons for Belonging

"…my love for that logical and straightforward process really contributed to my belonging in math."

"And I wanted to continue in something during my undergraduate that has something to do with math. And engineering was the one that appeals most to me because the careers that it will lead to."

"…people would tell me, oh the way that she thought about that and went to explain that, it sounds like you're an engineer, and I'm really passionate about things that I learned in engineering and mathematics. So overall, it was just a combination of things that I'm passionate about, things that I enjoy doing."

Competence | Personal Interest

CONCLUSION

- Participants' feelings of membership and acceptance as measured by the Good et al. (2012) sense of belonging in mathematics scale, declined between the start and end of the summer bridge program.
- Interviews with one participant revealed confidence in their sense of belonging in math with the reasoning being personal interest and competence, often feeling reassured by peers and mentors in the program.
- Initial findings indicate a decline in students' overall sense of belonging in math after their first college summer math course. Yet, positive experiences with the bridge program suggest that continued participation in LSAMP may lead to an improvement in their sense of belonging over time.

FUTURE WORK

- More rounds of coding to draw conclusions from interview data and open-ended survey questions
- Connections among students' sense of belonging in mathematics, mathematics identity, and mathematics confidence
- Longitudinally follow up with participants as they persist through their first year of mathematics courses

REFERENCES

Alkhasawneh, R & Hargraves, R. (2014). Developing a hybrid model to predict student first-year retention in STEM disciplines using machine learning techniques. *Journal of STEM Education: Innovations and Research,* 15(3), 35-42.

Chen, X. (2013). STEM Attrition: College Students' Paths into and out of STEM Fields. Statistical Analysis Report. NCES 2014-001. *National Center for Education Statistics*.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, *102*(4), 700–717. https://doi.org/10.1037/a0026659

Lubienski, S. T., Ganley, C. M., Makowski, M. B., Miller, E. K., & Timmer, J. D. (2021). "Bold problem solving": A new construct for understanding gender differences in mathematics. *Journal for Research in Mathematics Education*, *52*(1), 12-61.

Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International Journal of STEM Education*, *5*(1). https://doi.org/10.1186/s40594-018-0115-6

Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research*, *45*(1), 89–125. https://doi.org/10.3102/00346543045001089

Student's Understanding of the Time Series and State Space Trajectory of the Solution of a Differential Equation

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Keywords: differential equations, mathematical modeling, teaching mathematics

Over the last decades the study of dynamical systems has become an increasingly important area in biology and the life sciences. High profile national documents call for the reform the mathematics instruction for biology and life sciences students in order to focus on modeling and dynamical systems (e.g., National Research Council, 2003). Related to these calls are time series and trajectories in the state space. The representations are central to the solutions of the differential equations describing a dynamical system as they provide important and complementary information about a system's behavior. It is a critical skill to be able to construct the trajectory associated with a given time series, and vice versa.

In this exploratory study, I analyze to what extent students who have taken an undergraduate course focusing on modeling have developed the skill of navigating between these two types of representations. I conducted think-aloud interviews with eight students who had completed the course. Students were given a time series and asked to construct the associated trajectory. They were then asked to do the opposite task. I conducted a thematic analysis (Braun & Clarke, 2006) of the interviews focusing on what level of covariational reasoning (Thompson & Carlson, 2017) the students exhibited when constructing these graphs. Covariational reasoning has been shown to be a core competency to understand functions and engage in mathematical reasoning.

The analysis of the interviews reveals that students showed a variety of levels of covariational reasoning when sketching these graphs. For example, when given a time series, Stan picked points by "going in units of 5 to make it easy". He then plotted the points for his trajectory and connected them with straight lines, exhibiting what Thompson and Carlson (2017) call chunky continuous covariation. When probed about the precision of his trajectory, Stan said that to make it more precise "you would go in units of one". He then indicated that he would pick more points and still connected them with line segments. In other words, he would use the same process but with a smaller time step, further showing chunky continuous covariational reasoning. When asked how confident he was of his answer he said: "Oh, I am definitely very confident on how to do it". In contrast, Carly picked points where the time series "kind of just change[s] direction in general or […] hit[s] a maximum" denoting that she used important features of the functions to pick her points rather than simply choosing points at regular intervals. She then graphed her trajectory using smooth curves, clearly denoting smooth covariational reasoning.

Both students also completed the "opposite task" consisting in starting with a trajectory and sketching an associated time series. While it could be reasonable to expect students to show the same level of covariational reasoning for this new task, the opposite was actually observed. Stan continued to pick points at regular intervals but this time he connected them with smooth curves, which would suggest smooth covariational reasoning. As for Carly, she picked points at regular interval (not based on the shape of the trajectory) and connected them with line segments, showing chunky continuous covariational reasoning. These results suggest that students do not consistently exhibit the same level of covariational reasoning even if the tasks are related. On the poster, I will share additional results and discuss implications for practice.

This work is supported by the National Science Foundation grant DUE-2225258.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

National Research Council. (2003). *BIO2010: Transforming Undergraduate Education for Future Research Biologists*. The National Academies Press. https://doi.org/10.17226/10497

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

An Exploration of Precalculus Students' Reasoning about Exponential Functions

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Keywords: Exponential Functions, Quantitative and Covariational Reasoning, Precalculus

In recent years both the teaching and learning of introductory mathematics courses, such as Precalculus, have made immense strides to improve students' understanding of foundational skills such as covariational reasoning (e.g., Thompson & Carlson, 2017). Despite the emphasis being placed on teaching from a covariational lens, exponential functions continue to be taught from a correspondence lens (Ellis et al., 2016). In other words, the focus is on solving an algebraic expression that defines a static relationship between two variables instead of the underlying covariational relationship between two quantities. This work is a subset of a larger project to support students' covariational reasoning and specifically focuses on exponential functions. We are guided by the following questions: What type of covariational reasoning is present in students' responses on exponential growth/decay questions? How do the students' responses align (or not) with the mathematical goals of each lab?

Methods

This study is situated at a large R1 university in the American Southeast. In Summer 2022, the project began by developing five precalculus lab units based on the findings of small-scale teaching studies and other relevant literature on student thinking (e.g., Cobb et al., 2003; Ellis et al., 2017; Oehrtman et al., 2008). Each lab consists of a brief pre-lab assignment, a lab activity, and an exit ticket. The purpose of this presentation is to detail the ongoing development of two labs which were designed to support students' conceptualization of exponential functions from a covariational reasoning lens. Students' anonymized written work from the two focal labs were used as the primary source of data analyzed for this study. There were approximately 480 students across twenty classes from Fall 2022 and Spring 2023 included in the data set. Data analysis began by identifying which questions from each lab most closely aligned with the learning goals of the corresponding lab. An a priori codebook was created based on the Exponential Growth Learning Trajectory (EGLT) (Ellis et al., 2016). Student responses were coded for correctness and salient reasoning in several iterative rounds (Miles et al., 2020).

Findings and Discussion

One learning goal which we focused on was based on the ninth covariational reasoning ability from Ellis et al.'s (2016) EGLT which states that "any constant change in x results in a proportional multiplicative constant change in y" (p. 160). The main finding was that the majority of students were able to correctly answer questions despite their reasoning not always accurately reflecting the exponential relationships. For example, students agreed that halving the x value does not halve the y value; however, their explanations did not reference proportional multiplicative changes in the y value. Instead, students tended to rely on linear reasoning (i.e., additive changes in both x and y values. The misalignment between students' reasoning and their ability to get a "correct" answer implies that there is a disconnect between how the labs are currently designed and their intended learning goals.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33(5), 352. https://doi.org/10.2307/4149958

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9–13.

Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. Mathematical Thinking and Learning, 18(3), 151–181. https://doi.org/10.1080/10986065.2016.1183090

Miles, M. B., Huberman, A. M., & Saldaña, J. (2020). Qualitative data analysis: A methods sourcebook (Fourth edition). SAGE.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics education (pp. 27–42).

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 421–456). National Council of Teachers of Mathematics.

An Exploration of Precalculus Students' Reasoning about Exponential Functions

Spirit Karcher and Susana Brewer Castano
Florida State University

Background

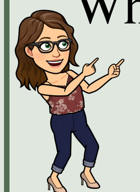
Alice Shrinking is the first of two 50-minute lessons that were designed using the work of **Ellis et al.'s (2016) Exponential Growth Learning trajectory (EGLT)** as a guide. The two labs are designed to support a key shift in student reasoning from **coordinating constant ratios for x-value changes larger than 1 to constant rations for fractional changes** (Ellis et al., 2016).

This poster highlights the findings from an analysis of Desmos responses from **120 students across 5 classes** during a lesson on exponential functions from **Fall 2022** titled **Alice Shrinking**. Student responses were assigned codes deductively based on the salient reasoning in the students' response using the categorizations from **Ellis et al.'s (2016) EGLT** as an a priori codebook.

Research Question

What types of covariational reasoning are salient in undergraduate precalculus students' responses to an exponential decay question with successive changes of an x-value that are greater than 1?

Scan to read our proposal!



When presented with successive additive and multiplicative changes in an exponential decay context, most students reasoned using repeated multiplication or exponentiation of the scale factor.

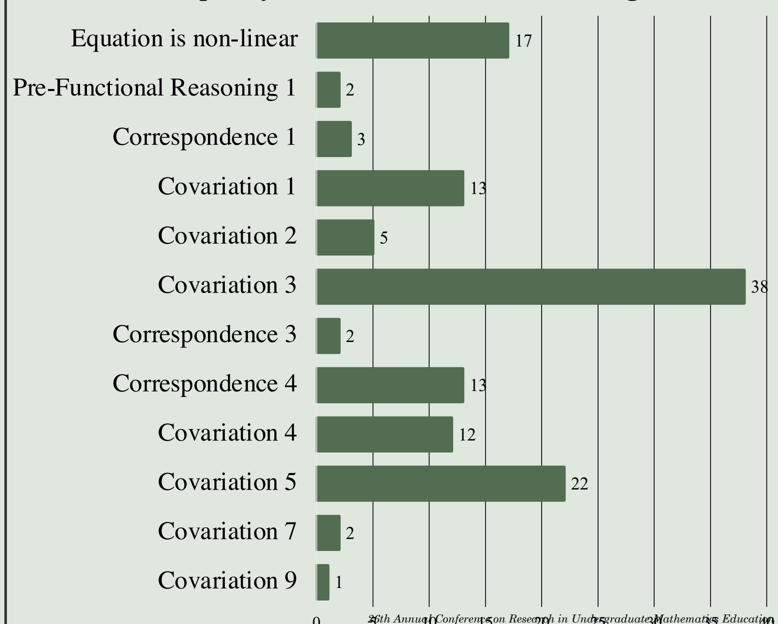
Drink Me: After each sip of potion, the drinker shall shrink to 2/3 their current size. (1 sip = 0.5 oz.)

Your roommate claims that if Alice drinks 1 oz., she would shrink twice as much as if she'd only drank 0.5 oz.

Do you agree or disagree?

Scan for current lesson on Desmos

Frequency of Codes for Student Reasoning



Examples of Most Common Student Reasoning
(Explicit Coordination for Multiple-Unit Changes)

Covariation 3: Repeated Multiplication defined as "coordinating the change in y-values for multiple–unit changes in x-values, but [the student's] mental imagery is grounded in the actions of repeated multiplication" (Ellis et al., 2016, p. 164).	**Covariation 5: Exponentiation Imagery** defined as the "coordination of the ratio of y-values for any Δx > 1; students no longer rely on repeated multiplication imagery" (Ellis et al., 2016, p. 169).
"After drinking 1 ounce of potion Alice would shrink 0.44 times her current height because .5 oz = 2/3 h (height). meaning 1 oz = 2/3 * 2/3 h, equaling 4/9."	"After drinking 1 ounce of potion Alice would shrink to 0.44 times her current height because she would be taking 2 sips; Therefore, she would shrink by (2/3)^2."

Active Assessment for Active Learning

Simone Sisneros-Thiry
California State University, East Bay

Keywords: Active Assessment, Community of Practice, Active Learning

What does it mean for assessment to be aligned with the values of *active learning*? What are the potential benefits and challenges of identifying and defining approaches to assessment consonant with active learning (i.e., *active assessment*)? In the last several years, there has been a surge in interest in developing more flexible assessment structures, with a great deal of momentum for this movement coming from COVID-19 pandemic equity considerations (Kadakia & Bradshaw, 2020; Kim, 2020). Conversations about equity and active learning (Theobald et. al, 2020) and about equity and formative assessment (Kalinec-Craig, 2017) have opened the door to the consolidation of ideas into a framing of "active assessment". This poster focuses on perceived challenges and benefits for defining and designing active assessment. For example, how active assessments may be more or less equitable than timed, independent, written exams. This work takes place at an institution with a variety of supports available, and individual instructors' consideration and implementation of *active assessment* occurs within the ongoing process of professional and pedagogical growth.

This poster presents preliminary results from a qualitative study of exploration of this question in a professional learning community (the Assessment Community of Practice, or ACoP). Pulling ideas from discussions of active learning, formative assessment, and alternative assessment, and from reflections on our own practice, members of the ACoP collaboratively produced two drafts of a definition of *active assessment* in Summer 2023. Both drafts of the definition will be included in the poster. Our goal in developing a shared definition of *active assessment* is to support our implementation of assessment strategies that align with the values and practices of active learning. Throughout the academic year 2023-2024, ACoP members are implementing a variety of active assessment strategies, including variations on standards-based grading, group projects and portfolios, in a broad range of courses, from first-year general education courses through upper division electives for mathematics majors. We continue to engage in discussion about how the strategies do or don't align with our current, working definition. The poster presents preliminary themes that have appeared in these discussions.

Active learning has been defined broadly as instructional activities involving students in doing things and thinking about what they are doing (Bonwell & Eison, 1991). Our consideration of active learning includes Laursen and Rasmussen's four pillars of inquiry-based mathematics education (2019). The ACoP development of a working definition for active assessment also included ideas from Black and Wiliam's (2009) five key aspects of formative assessment and themes presented in the PRIMUS Curated Collection on Assessment (Katz, 2022).

This poster aims to share the motivation, and initial thoughts on the implementation of *active assessment* and present some challenges and questions that have arisen. The conversations generated through the poster session will inform the ongoing research into the development of the definition and practice of active assessment.

Acknowledgement

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References

Black, P. & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability*, *21*(1)*,* 5-31. https://doi.org/10.1007/s11092-008-9068-5

Bonwell, C. C. & Eison, J. A. (1991). *Active learning: Creating excitement in the classroom.* ASHE-ERIC Higher Education Report No. 1. https://eric.ed.gov/?id=ED336049

Kadakia, P. C,, & Bradshaw, A. (2020, May 6). Equitable exams during Covid-19. *Inside Higher Ed*. https://www.insidehighered.com/advice/2020/05/06/changes-instructors-should-consider-administering-and-grading-exams-during-covid

Kalinec-Craig, C. A. (2017). The Rights of the Learner: A Framework for Promoting Equity through Formative Assessment in Mathematics Education. *Democracy and Education*, 25 (2), Article 5. https://democracyeducationjournal.org/home/vol25/iss2/5

Katz, B. P. (2022) Curated Collection: Assessment, *PRIMUS, 32*(5), 636-649.

Kim, J. (2020, April). 5 Reasons to Stop Doing Timed Online Exams During COVID-19. *Inside Higher Ed*. https://www.insidehighered.com/blogs/learning-innovation/5-reasons-stop-doing-timed-online-exams-during-covid-19

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 129-146. https://doi.org/10.1007/s40753-019-00085-6

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G. P., Grummer, J. A., Hennessey, K. M., Hsiao, J., Iranon, N. N., Jones, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., . . . Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences of the United States of America*, *117*(12), 6476–6483. https://doi.org/10.1073/pnas.1916903117

Active Assessment for Active Learning

Simone Sisneros-Thiry
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Institutional Context
Regional, public 4-year university, primarily undergraduate-serving
- Faculty invested in active learning
- STEM Pathway Community of Practice
- Coordination of materials and topics for multi-section courses

Motivation
Create Assessment Community of Practice for connecting across context, focused on **active assessment.**

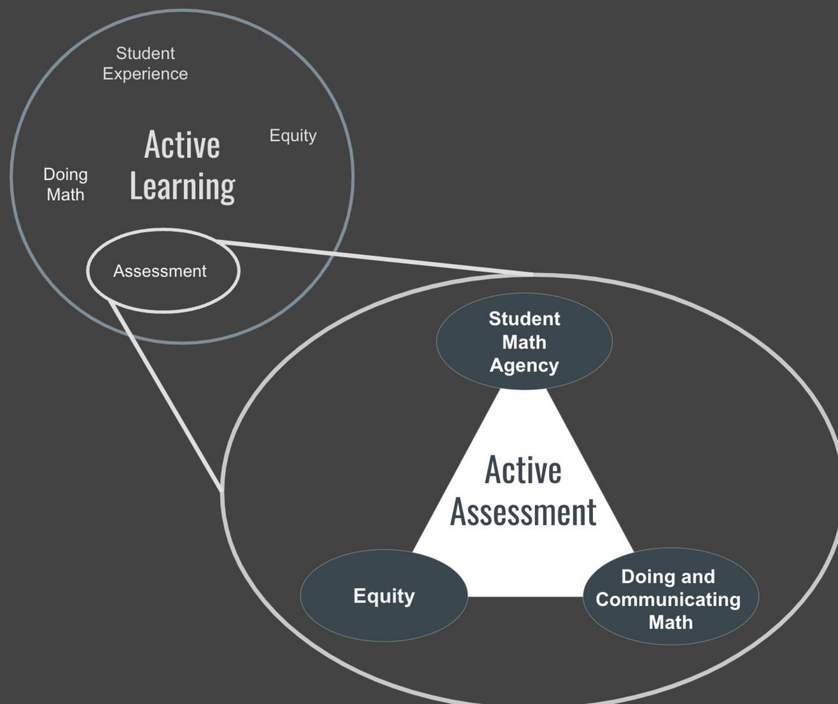
Evolving Definition of Active Assessment

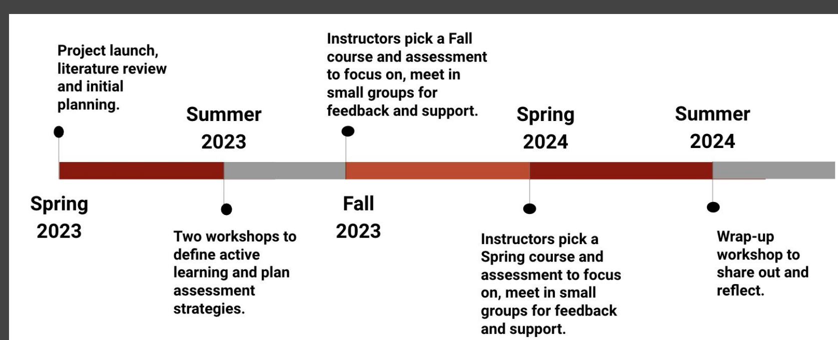
Draft #1 (5/19/2023)
A type of assessment that is rooted in
- *engaging student understanding equitably,*
- *reinforcing communication, reflection, collaboration, and mathematically rich tasks,*
- *valuing diversity in expression while being asset-minded,*
- *and transferring power to students and community*

And captures student progress and facilitates growth in the journey through mathematics.

Draft #2 (7/14/2023)
Active assessment captures student progress and facilitates growth in the journey through mathematics. Active assessment is rooted in
- *engaging student understanding equitably,*
- *reinforcing communication, reflection, collaboration, through mathematically rich tasks,*
- *valuing diversity in expression while being asset-minded, and*
- *giving students the opportunity to share, learn, and expand on ideas as a community.*

Yes, we're starting to understand ***active learning***, so what does ***active assessment*** look like?





2023-2024 Assessment Community of Practice
- Includes lecturer, pre-tenure faculty, and tenured faculty
- Wide range of courses and assessment strategies represented, including:
  - Group Projects and Presentations in upper division Diff. Eq.
  - Final Portfolios in Math for the Arts & Humanities
  - Standards-Based Grading in Precalculus with Algebra

CHALLENGES & NEXT STEPS
- How are current practices opportunities for active assessment development?
  - Standards-based grading.
  - Multiple drafts with collaboration.
  - Summative assessment for the course as formative assessment for students
- How do we operationalize the definition?
  - What does the instructor do?
  - What do the students do?

Related resources at the QR code

This project is supported by a National Science Foundation grant (BCSER #2225295). Any opinions, findings, conclusions or recommendations are those of the authors and do not necessarily reflect the views of the Federal Government.

CAL STATE EAST BAY

RUME 2024

Student Thinking and Challenges when Representing Distance in the Cartesian Plane

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Rhodes College

Samuel Lippe
Rhodes College

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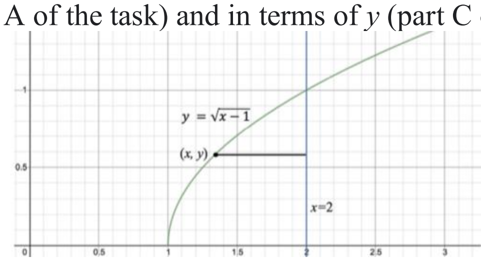
Keywords: Cartesian connection, difference expressions, magnitude interpretation, graphs of functions, survey study

The purpose of this poster presentation is to highlight the variations in student thinking when working with difference expressions in the Cartesian plane — a fundamental skill crucial for grasping numerous Calculus concepts illustrated graphically, such as area, volume, and difference quotients. Previous research has shown students often struggle to conceptualize and express distances in the Cartesian plane (Parr et al., 2021). We view two key understandings, a magnitude interpretation, and the Cartesian connection, as prerequisites to this ability. By magnitude interpretation, we mean interpreting a difference expression as the magnitude of the distance between two corresponding positions in a single dimension (Parr, 2023). The Cartesian connection is the connection between pairs of values of *x* and *y* that satisfy an equation and the ordered pairs of points on the graph of that equation (Moschkovich et al., 1993).

The goal of this study was to investigate the prevalence and nature of challenges that students face when engaging in tasks to represent distances in the Cartesian plane. To investigate this question, we surveyed *n*=169 undergraduate math students at a private liberal arts college located in the southern United States between spring 2022 and fall 2023. On the survey, students were asked to represent distances within the Cartesian plane using expressions in terms of *x* (part A of the task) and in terms of *y* (part C of the task).



a. Represent the horizontal segment's length in terms of *x* (Your expression should have "x" in it).

b. In a sentence or two, explain how you came up with your expression.

c. Represent the horizontal segment's length in terms of *y* (Your expression should have "y" in it).

d. In a sentence or two, explain how you came up with your expression.

Figure 1. Survey item to represent the length of the segment in black in terms of x and in terms of y.

We found that only 16 students (9.47%) correctly responded to both A and C, 22 students (13.02%) correctly answered A but not C, five students (2.96%) correctly answered C but not A, and the remaining 126 students (74.56%) answered both questions incorrectly. We further analyzed the structure of student responses to see if they contained a difference expression, which could indicate the use of a magnitude interpretation. We also analyzed responses for evidence of issues related to the Cartesian connection, including when and how to manipulate the given equation. Our findings highlight a need for instruction on the skills foundational to understanding distances in the Cartesian plane within Calculus education.

References

Moschkovich, J., Schoenfeld, A., and Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T.A. Romberg, E. Fennema, & T.P. Carpenter (Eds.), *Integrating research on the graphical representations of functions* (pp. 69-100). Erlbaum.

Parr, E. D. (2023). Undergraduate students' interpretations of expressions from calculus statements within the graphical register. *Mathematical Thinking and Learning*, *25*(2), 177-207.

Parr, E. D., Ely, R., & Piez, C. (2021). Conceptualizing and representing distance on graphs in Calculus: The case of Todd. *2021 Research in Undergraduate Mathematics Education* Reports. (pp. 214-221).

The manifestation of graph-theoretic methods in mathematics education research: A metasummary of intercontinental conference proceedings

Haile Gilroy
Auburn University

In this poster, I present a metasummary of articles employing graph-theoretic methods (i.e., Graphs, Network Analysis, Maps, etc.) published in proceedings from North American (SIGMAA-RUME) and European (CERME) mathematics education conferences from 2015 to 2022. I use the results of this metasummary to describe international trends in the choice of method and prominent research topic themes in mathematics education research amongst conference proceedings articles employing graph-theoretic methods.

Keywords: meta-analysis, graph theory, research methods, mathematics education research

Leveraging the mathematical definition of a graph (Chartrand et al., 2016), I define a *graph-theoretic method* as a research method that makes use of a set of objects and the relationships between them. In this poster presentation, I answer the following research questions: (RQ1) *What are the graph-theoretic methods present in mathematics education research (i.e., Graph Theory, Social Networks, Mapping, etc.)?* and (RQ2) *What topics in mathematics education research are studied using graph-theoretic methods?*

To answer these questions, I conducted a metasummary, a type of qualitative meta-analysis that relies on descriptions of broad research trends rather than a typical interpretive synthesis (Timulak, 2014), of prominent intercontinental mathematics education conference proceedings published in the past eight years. For an article to be included in this analysis, it must (1) be published in a SIGMAA-RUME or CERME Conference proceeding from 2015 to 2022, a collection of 3,940 articles, and (2) employ a graph-theoretic method, narrowing the analysis to 80 articles (26 from RUME proceedings and 54 from CERME proceedings). The articles analyzed represent a diverse, international perspective of 21 countries spanning 5 continents. A full list of articles analyzed is available at https://aub.ie/1lKIDG.

Open coding (Thornberg & Charmaz, 2014) was used to determine graph-theoretic methods, while a combination of open coding, thematic analysis (Thornberg & Charmaz, 2014), and social network analysis (specifically degree centrality) (yEd – graph editor, 3.23.2) was used to extract prominent research themes. Results of the analysis indicate the existence of 6 graph-theoretic methods: Network Analysis (Hannula & Moreno-Esteva, 2017), Graphs (Johns et. al., 2016), Maps (Andrews-Larson & McCrackin, 2018), Grids (Watson & Jones, 2015), Flowcharts (Anwar & Goedhart, 2020), and Data Visualization (Van Steenbrugge, 2022), which are distributed differently between RUME and CERME articles. The results also indicate that graph-theoretic methods are employed in a wide variety of topics across mathematics education research, whose prominent themes include Teachers (Horsman, 2022), Students (Kanwar & Mesa, 2022), Mathematics Education Research (Hannula & Moreno-Esteva, 2017), Learning (Lyublinskaya & Du, 2022), Setting (Funk et. al., 2022), Pedagogy (Andrews-Larson & McCrackin, 2018), Curriculum (Henriksen, 2022), and Hot Topics (Kim & Andrews-Larson, 2022). The articles cited here exemplify each type of method and research theme.

References

Andrews-Larson, C. & McCrackin, S. (2018). The Next Time Around: Shifts in Argumentation in Initial and Subsequent Implementations of Inquiry-Oriented Instructional Materials. In A.

Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.) *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 932-940). San Diego, CA.

Anwar, L. & Goedhart, M. (2020). Understanding geometric proofs: scaffolding pre-service mathematics teacher students through dynamic geometry system (dgs) and flowchart proof. In U.T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 104-111). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Chartrand, G., Lesniak, L., & Zhang, P. (2016). *Graphs & Digraphs Sixth Edition* (pp. 3). CRC Press.

Funk, R., Moore-Russo, D., Uhing, K., & Williams, M. (2022). Studying the Potential for Pedagogical Change Based on Perceptions and Proposed Initiatives using the Four Frames. In S.S. Karunakaran & A. Higgins (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 970-975). Boston, MA.

Hannula, M.S. & Moreno-Esteva, E.G. (2017). Identifying subgroups of CERME affect research papers. In T. Dooley & G. Gueudet (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10, February 1-5, 2017)* (pp. 1098-1105). Dublin, Ireland: DCU Institute of Education and ERME.

Henriksen, B. (2022). Didactic transposition of natural numbers in the first year of compulsory schooling: a case of comparative curricula analysis. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 2192-2199). Free University of Bozen-Bolzano and ERME.

Horsman, R. (2022). Collaborative knowledge creation of teacher subject knowledge using concept maps. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 3606-3613). Free University of Bozen-Bolzano and ERME.

Johns, G., Nakamura, C., & Grosse, C. (2016). Using the effect sizes of subtasks to compare instructional methods: A network model. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.). *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 901-908). Pittsburgh, PA.

Kanwar, P. & Mesa, V. (2022). Mapping student real-time viewing of dynamic textbooks to their utilization schemes of questioning devices. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 4002-4003). Free University of Bozen-Bolzano and ERME.

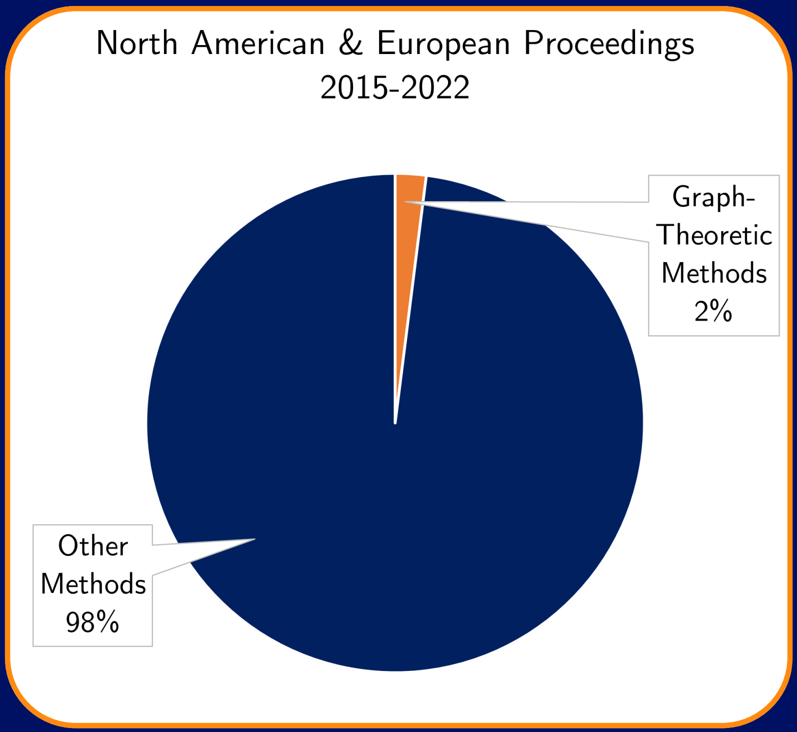
Kim, M. & Andrews-Larson, C. (2022). Can discussion boards disrupt gendered and racialized discussion patterns in math classes? In S.S. Karunakaran & A. Higgins (Eds.). *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1047-1053). Boston, MA.

Lyublinskaya, I. & Du, X. (2022). Preservice mathematics teachers' learning trajectories of Technological Pedagogical Content Knowledge (TPACK) in an online educational technology course. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 2578-2585). Free University of Bozen-Bolzano and ERME.

Thornberg, R. & Charmaz, K. (2014). Grounded Theory and Theoretical Coding. In U. Flick (Ed.), *The SAGE Handbook of Qualitative Data Analysis* (pp. 153–169). SAGE.
Timulak, L. (2014). Qualitative Meta-analysis. In U. Flick (Ed.) *The SAGE Handbook of Qualitative Data Analysis* (pp. 481–495). SAGE.
Van Steenbrugge, H. (2022). Rethinking the notion of textbooks as mediators between the official curriculum and classroom practice. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 4050-4057). Free University of Bozen-Bolzano and ERME.
Watson, K.L. & Jones, S.R. (2015). Adding explanatory power to descriptive power: Combining Zandieh's derivative framework with analogical reasoning. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.). *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1041-1047). Pittsburgh, PA.
yEd—Graph editor (3.23.2). (2023). [Computer software]. yWorks GmbH.

Graph-theoretic methods are uncommon in Mathematics Education research.

However, there is an overwhelming notion that our shared knowledge is a graph.



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NETWORK	1,861
CONNECTION	5,553
RELATIONSHIP	8,935

Mathematics Engagement in College Precalculus: Understanding Students' Experiences Through a Graphing Elicitation Tool

Aida Alibek
University of Georgia

Keywords: mathematics engagement, Precalculus, qualitative methods, phenomenology

In recent years, we have witnessed great efforts to improve college-level mathematics courses by shifting to student-centered instruction and providing better supports for all learners (e.g., Apkarian et al., 2021; Rasmussen et al., 2019). The implementation of pedagogies like active learning, inquiry-based learning, or flipped classroom instruction in gateway mathematics courses highlights the importance of student engagement in improving learning outcomes (Uhing et al., 2021). To increase student engagement, it is critically important that we understand how students experience this phenomenon.

Despite its importance in learning, educational researchers have not come to a clear consensus on a conceptualization of student engagement (e.g., Appleton et al., 2008; Wong & Liem, 2022). Our colleagues in K-12 mathematics education have been inquiring into students' mathematics engagement since the 1990's (Middleton et al., 2017) and a commonly adopted conceptualization comes from Fredricks et al. (2004) of engagement as a multi-dimensional meta-construct capturing students' affect, behavior, and cognition (Middleton et al., 2017).

This poster proposal aims to highlight a part of the methods of a larger study exploring undergraduate students' experiences of mathematics engagement in a college Precalculus course. This study considers students' mathematics engagement holistically, drawing on engagement research in K-12 mathematics education for its conceptual framework and acknowledging that the various dimensions of engagement are interwoven with each other and are extremely dynamic. The overarching research question is: "What is the story of college students' mathematical engagement in Precalculus?" The theoretico-methodological framework of the study is foundationally based in the socio-cultural tradition (Vygotsky, 1978) and built with hermeneutic phenomenology (van Manen, 2014), which means that the experience of the phenomenon of engagement is studied against the backdrop of socio-cultural learning theory.

The focus of the proposed poster is on the use of a graphing elicitation tool as part of the research design, wherein study participants will be asked to sketch a graph of their engagement in their Precalculus course across the semester over the course of 4 interviews. The sketches drawn by the participants will provide the participant and interviewer a common representation to discuss during the conversation and can allow the researcher's questions to be more specific, eliciting rich descriptions of the students' experiences of engagement. In fact, detailed lived experience descriptions are the main goal of phenomenological interviewing (van Manen, 2014).

During the poster presentation, conference attendees will be able to learn about the methodology, see examples of student-produced sketches, and discuss the affordances of the elicitation tool for understanding college students' engagement in the mathematics classroom. Affect graphing is not new in mathematics education research (Riske et al., 2021; Satyam et al., 2022), and pairing the tool with phenomenological interviewing practices can allow us to gain insight into students' experiences, grasp aspects of students' mathematics engagement that are hard to capture through traditional interviewing methods, and study it as the complex dynamic multi-dimensional construct that it is.

References

Apkarian, N., Kirin, D., Gehrtz, J., & Vroom, K. (2021). Connecting the stakeholders: Departments, policy, and research in undergraduate mathematics education. *PRIMUS*, *31*(1), 17–36. https://doi.org/10.1080/10511970.2019.1629135

Appleton, J. J., Christenson, S. L., & Furlong, M. J. (2008). Student engagement with school: Critical conceptual and methodological issues of the construct. *Psychology in the Schools*, *45*(5), 369–386. https://doi.org/10.1002/pits.20303

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, *74*(1), 59–109. https://doi.org/10.3102/00346543074001059

Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In *The compendium for research in mathematics education* (pp. 667–699). The National Council of Teachers of Mathematics.

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., Bressoud, D., & The Progress through Calculus Team. (2019). Brief report: Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, *50*(1), 98–111. https://doi.org/10.5951/jresematheduc.50.1.0098

Riske, A. K., Cullicott, C. E., Mirzaei, A. M., Jansen, A., & Middleton, J. (2021). Student engagement with the “Into Math Graph" tool. *Mathematics Teacher: Learning and Teaching PK-12*, *114*(9), 677–684. https://doi.org/10.5951/MTLT.2020.0322

Satyam, V. R., Bae, Y., Smith, J. P., & Levin, M. (2022). Affect graphing: Leveraging graphical representations in the study of students’ affect in mathematics. *Educational Studies in Mathematics*, *110*(3), 481–501. https://doi.org/10.1007/s10649-021-10131-3

Uhing, K., Hass, M., Voigt, M., Strom, A., & Calleros, E. (2021). Students’ experiences with active learning mathematics. In W. M. Smith, M. Voigt, & A. Ström (Eds.), *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning* (pp. 221–241). American Mathematical Society.

van Manen, M. (2014). *Phenomenology of practice*. Left Coast Press, Inc.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Harvard University Press.

Wong, Z. Y., & Liem, G. A. D. (2022). Student engagement: Current state of the construct, conceptual refinement, and future research directions. *Educational Psychology Review*, *34*(1), 107–138. https://doi.org/10.1007/s10648-021-09628-3

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Aida Alibek, PhD Candidate

Background

Using graphs as elicitation tools to understand students' affect is not new for mathematics education research (Riske et al., 2021; Satyam et al., 2022). This study is using the graphing engagement tool to solicit students' self-described experiences of mathematics engagement in their college Precalculus course.

Engagement is a multi-dimensional meta-construct (Fredricks et al., 2004; Middleton et al., 2017). The key dimensions of engagement in education research are the *affective*, *behavioral*, and *cognitive* dimensions (Appleton et al., 2008; Fredricks et al., 2004; Wong & Liem, 2022).

Stay for a chat if you want to:

- learn how the participants in this study were prompted to share their lived experiences of engagement through graphs,
- view select examples of student-produced graphs with highlighted quotes from interviews,
- discuss the affordances of the elicitation tool for understanding undergraduate students' engagement in the college Precalculus classroom,
- learn about the phenomenological underpinnings of the study design and the graphing prompts.

Methods of the overall study

- 12 participants enrolled in Precalculus
- 8 distinct sections taught by 4 instructors
  - 2 lecture-based sections
  - 6 flipped classroom sections
- partially coordinated Precalculus course

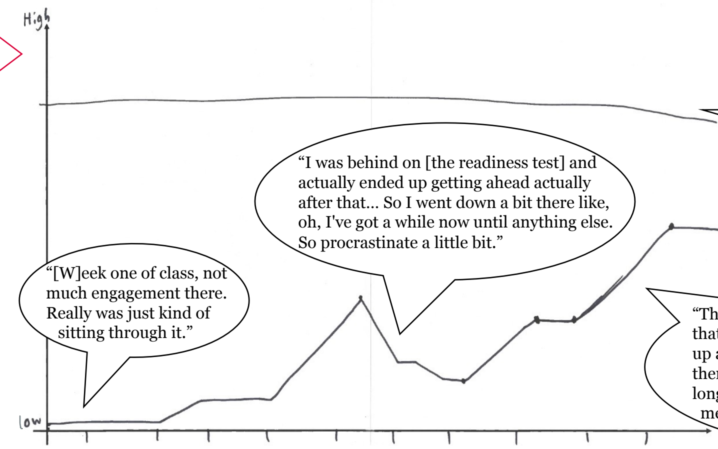
- 4 phenomenological interviews over the course of one semester
- during each interview participants sketch a segment of their engagement graph

For more information please scan the QR code, email me at aida.alibek@uga.edu, or visit: https://bit.ly/AlibekRUME24

Asking students to graph how their engagement in Precalculus shifts throughout the semester is a powerful way to elicit rich qualitative data.

David (he/him)

- Freshman at UGA
- Intended major: Finance
- In a lecture-based classroom

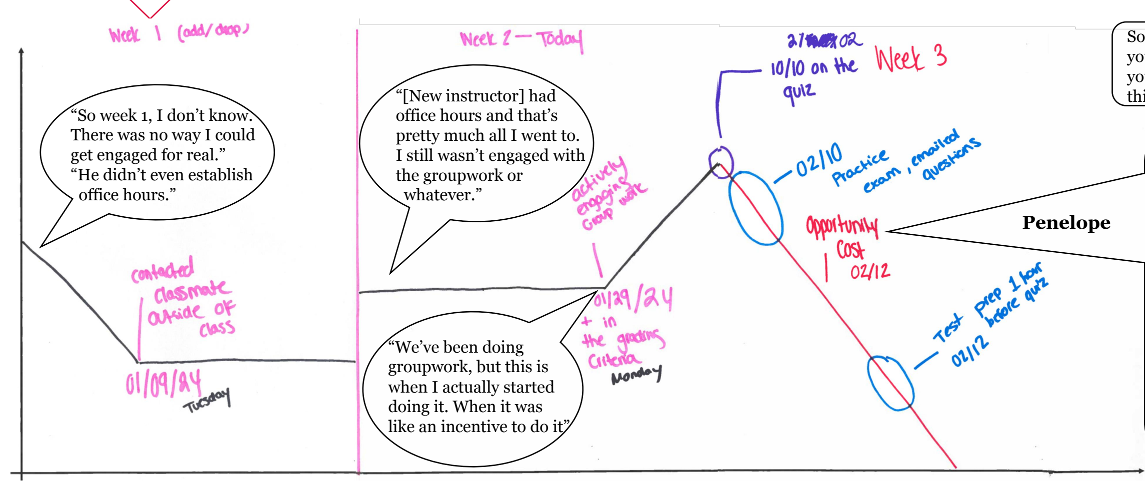


Aida: Is there a limit for you in terms of **engagement** in this class?

David: "Maybe around **here**. Because I got four classes. [M]aybe a B in this class and getting A's in other classes is just better than getting an A in this class and two B's or something like that. ... Two other classes are also [business school] requirements."

Penelope (she/her)

- Freshman at UGA
- Intended major: Accounting
- Switched Precalculus sections after 1 week of classes
- In a flipped classroom



Aida: So what are some ways you have **engaged** in your Precalculus course this past week?

Penelope: "I usually go to office hours, but this week I didn't, because I had a lot of more stuff to do ... [Precalc] is more demanding, so I'm like, okay. It is kinda like an **opportunity cost**. Like I'm giving up time for another class. I made time for this class. It's just, I don't know, I can't find that balance right now."
"I had my RA interview today... I don't have nowhere to live on campus next semester, for real. I'm still a kid. I just turned 19... So it's either I go ahead and prep for my RA interview... or I'm gonna have to sit and study for my test. ... I chose the housing."

Participant and Designer Competing Goals, Resources, and Orientations for Professional Development of Faculty Teaching Mathematics to Future K-8 Teachers

Billy Jackson
University of Illinois, Chicago

Jenq Jong Tsay
University of Texas Rio Grande Valley

Shandy Hauk
San Francisco State

Keywords: faculty professional development, prospective elementary teacher, provider

There is a documented need for professional learning opportunities to support mathematics faculty in working with prospective elementary school teachers (Ping et al., 2018). Recent efforts by the authors and others have generated frameworks for mathematics-faculty-specific, teaching-focused professional development (TPD) (Jackson et al., 2020; Schoenfeld, 2007). In particular, the *Professional Resources and Inquiry into Mathematics Education for K-8 Teacher Education (PRIMED)* project created and implemented a series of online modules for faculty new to teaching mathematics courses for future elementary school teachers. The modules constitute a 15-hour "short-course" for faculty. PRIMED designers had the role of “TPD Providers” (Yee et al., 2023) whose presence was implicit in the asynchronous short-course.

The theory of change informing the work was based on humanizing mathematics instruction for future teachers (MacArthur, 2022; Su, 2020). In the short-course setting, attention to student voices occurred at three levels: children in the future classes of pre-service teachers, the undergraduate pre-service teachers, and the faculty as learners. Providers of the short-course were the implicit instructors in the asynchronous online TPD. The framework for examining designer and faculty views used the **resources** at the disposal of each (knowledge and material resources), their **orientations** (beliefs, preferences, values, etc.), and their **goals** (which exist at multiple levels and change dynamically according to evolving events; Schoenfeld, 2023).

In this poster, we explore short-course design and revision spurred by the distinctions in goals, resources, and orientations of participating novice instructors and Provider-designers. The poster focuses on how the distinctions shaped redesign of the short-course after the initial pilot. The data were designer memos, faculty surveys, and faculty interviews to address the questions:

1. Promise: How does the short-course support faculty learning of target content?
2. Feasibility: Are module materials accessible and useful to faculty?
3. Fidelity: What are the supports needed for participation-as-intended by designers?

Previous findings from research on the use of the short-course included the students of participating faculty demonstrating greater gains in knowledge for teaching mathematics than is common in courses for pre-service K-8 teachers (Hauk et al., 2023). Also, though faculty-participants sought the resource of ready-made materials, research suggested that faculty needed more opportunities to develop the specialized content knowledge used in teaching grades K-8 in addition to knowledge needed for selecting and orchestrating use of course materials).

The work reported in the poster will contribute to the literature because, to date, studies have rarely differentiated among what constitute the professional skills needed as an instructor and professional skills as a teacher-educator, the role of teacher-educator skills in being consumers and selectors of activities (rather than producers or designers of them), and the still understudied forms of motivation behind the work of faculty new to being teacher-educators.

Acknowledgements

First, our thanks to the hundreds who participated in the PRIMED project. This material is based upon work supported by the National Science Foundation under Grant No. DUE-1625215.

References

Hauk, S., Jackson, B., & Tsay, J.J. (2023) Connecting instructor mathematical knowledge for teaching, intercultural competence, and undergraduate learning. In S. Cook, B. Katz, and D. Moore-Russo (Eds.), *Proceedings of the 25th Conference on Research in Undergraduate Mathematics Education*, Omaha, Nebraska.

Jackson, B., Hauk, S., Tsay, J-J., & *Ramirez, A.* (2020). Professional development for mathematics teacher education faculty: Need and design. *The Mathematics Enthusiast 17*(2), Article 8. Available at: https://scholarworks.umt.edu/tme/vol17/iss2/8

Jackson, B., Tsay, J-J., & Hauk, S. (2024). Participant and designer competing goals, resources, and orientations for professional development of faculty teaching mathematics to future K-8 teachers. [Poster]. The poster associated with this poster report: https://sfsu.app.box.com/s/4qx9zymn5sf4usxysb8h1f8s31qocenm

MacArthur, K.A. (2022). *Rehumanizing College Mathematics: Centering the Voices of Latin*, Indigenous, LGTBQ+ and Women STEM Majors* [Doctoral dissertation, Montana State University]. Scholarworks.Montana.edu

Ping, C., Schellings, G., & Beijaard, D. (2018). Teacher educators' professional learning: A literature review. *Teaching and Teacher Education*, *75*, 93-104.

Schoenfeld, A. H. (2007). Method. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 69–110). MacMillan.

Schoenfeld, A. H. (2023). A theory of teaching. In A. Praetorius and C. Y. Charalambous (Eds.), *Theorizing teaching: Current status and open issues* (pp. 159-187). Springer International. Available online at: https://library.oapen.org/bitstream/handle/20.500.12657/61842/1/978-3-031-25613-4.pdf

Su, F. (2020). *Mathematics for human flourishing*. Yale University Press.

Yee, S. P., Hauk, S., Gonzalez, T. L., & Wang, J. (2023) College mathematics instructor professional development providers: Who are they? *School Science and Mathematics, 123*(2), 39-41.

Fostering Set-Based Reasoning for Mathematical Proofs: Student Interaction and Challenges in an Inquiry-Oriented, Transition-to-Proof Course

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Keywords: transition-to-proof course, inquiry-oriented classroom, representations for mathematical logic, set-based reasoning

This poster aims to provide salient insights into an inquiry-oriented transition-to-proof course designed to cultivate undergraduate students' set-based reasoning for mathematical proofs. By set-based reasoning, we refer to a person's reasoning with predicates and relationships between predicates and their truth-sets (Dawkins, 2017). The course emphasized learning fundamental logical principles – such as non-contradiction, contrapositive equivalence, and converse independence – via set-based reasoning. Opportunities were given to students to actively engage with their peers and the instructor team, including the primary instructor for whole-class discussions and research assistants responsible for facilitating small-group discussions. These interactions occurred in the physical classroom and extended to virtual discussion forums.

We examined several aspects of student engagement: peer-to-peer interaction, interaction with the instructor team, assignments and exams, and online discussion forum posts. Our focus was on four key areas: (1) students' participation in building the learning community's acknowledgment of mathematical logic, (2) instructional deliveries in the utilization of set operations and set relationships, (3) formative assessment for students' main takeaways from classroom activities, and (4) student challenges.

Our analysis reveals that (1) students' active participation in sharing their reasoning and reflection on their learning with their peers contributed to developing their collective understanding of logic for mathematical statements and proofs. In addition, (2) the incorporation of mathematical representations (e.g., Euler diagram) and analogies from everyday language (e.g., an empty Hermès Bag for an empty set, fried eggs for proper subset relationships, and Mastercard for non-disjoint relationships) served as a valuable pedagogical tool. These aids fostered student engagement in set-based reasoning for logic. Furthermore, (3) encouraging students to share their main takeaways with their peers in the last five minutes of class and through online discussion forums emerged as an effective formative assessment strategy in gauging students' comprehension and progress. We attended to (4) the challenges faced by students when conveying their reasoning using set-builder notation and Euler diagram, especially those who had prior experience with truth tables for logic and resisted using these new tools introduced in the course for set-based reasoning. Addressing these challenges is an ongoing focus as we refine the course and support student learning in the inquiry-oriented transition-to-proof course.

Acknowledgments

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References

Dawkins, P.C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education,* 3, 496–522. https://doi.org/10.1007/s40753-017-0055-4

Fostering Set-Based Reasoning for Mathematical Proofs: Student Interaction and Challenges in an Inquiry-Oriented, Transition-to-Proof Course

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[1]Arizona State University [2]Texas State University

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Objective

We provide salient insights into an **inquiry-oriented** transition-to-proof (**TTP**) course designed to cultivate undergraduate students' **set-based reasoning** for mathematical proofs.

Aspects of Student Engagement

- ♦Peer-to-peer interaction
- ♦Interaction with the instructor team
- ♦Assignments and exams
- ♦Online discussion forum posts

Curriculum Implementation

Our curriculum was implemented in an in-person TTP course in an inquiry-oriented classroom during the Fall 2023 semester at a large public southwestern university in the United States.

- ♦Unit 1: To cultivate students' set-based reasoning about logic for mathematical proofs (Dawkins, 2017)
- ♦Unit 2: To cultivate proof reading comprehension through apprenticeship
- ♦Unit 3: To humanize mathematics through engaging with proof authors' stories

Analytic Foci

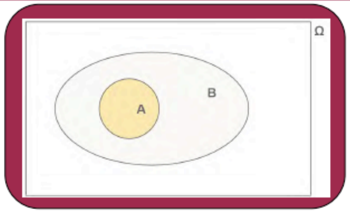
A(1) Students' participation in building the learning community's acknowledgment of mathematical logic

A(2) Instructional deliveries in the utilization of set operations and set relationships

A(3) Formative assessment for students' main takeaways from classroom activities

A(4) Student challenges

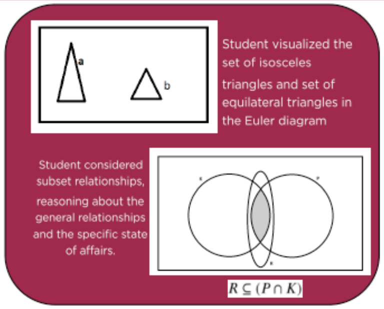
Normative Set Visualizations



Refer to A as the "yolk" of a **fried egg** and B as the "egg white".

Instructor used this description and students adopted these analogies in their verbal and written peer discussions.

Nonnormative Adoption of Set Visualizations



Multimodal **peer interactions, set-based reasoning** tools, and active **reflections** help students make meaning in **transition to proof classroom.**

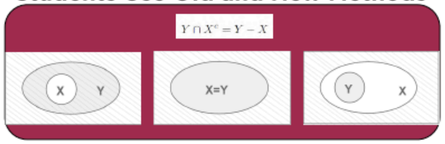
Student Reflections

"Truth sets which equal the empty set have complements that are the universal set and that set can show what proves the original statement!"

"I've found myself using set notation, structure, and different proof approaches in my *linear algebra* class. It's been very helpful using a structured argument rather than winging it!"

"I see sets all the time in my linear algebra class...subsets also come up a lot when talking about vector spaces."

Students Use Old and New Methods



Student used the Euler visualizations, guided by instructional assistant, **to verify a tautology** using set-based reasoning

Some students resisted the use of set-based reasoning, instead using previously acquired tools such as truth tables to evaluate claims about sets throughout the course.

P	Q	P->Q	Q ∩ ¬P
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

All images are researcher generated reproductions of observed student work and are not authentic student generations

Conclusions

C(1) Students' **active participation** in sharing their reasoning and reflection on their learning with their peers.

C(2) Incorporation of **mathematical representations** (e.g., Euler diagram) and **analogies** from everyday language (e.g., fried eggs for proper subset relationships) served as a valuable **pedagogical tool**.

C(3) **Students shared their reflections** in class and in the online discussion forum, **revealing their understanding** of key ideas and application of these tools to other contexts.

C(4) Initial **student challenges** included using new tools in a nonnormative manner or using old tools such as truth tables.

Overall, students exhibited progress and gained ownership in adopting the set-based reasoning tools normatively across various contexts.

Arizona State University

TEXAS STATE UNIVERSITY

This material is based upon work supported by the National Science Foundation under Grants DUE-2141925. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Sense of Belonging of Minoritized Female STEM Students in Introductory Mathematics Classes: A Mixed Methods Study

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Keywords: Sense of Belonging, Minoritized Female Students

Higher education institutions are becoming increasingly diverse in their undergraduate populations, but minoritized female students, especially Black and Latina women, continue to be underrepresented in almost all STEM fields (Hatfield, 2022; Ong et al., 2016). Although minoritized female students often begin college with a strong interest in STEM, they are more likely to leave the STEM major due to an absence of belonging in their introductory courses (Rainey et al., 2016). Researchers argue that sense of belonging at the classroom level, is a key factor in supporting minoritized student persistence, and it may even improve participation and academic performance (Strayhorn, 2019). Understanding female minoritized students' sense of belonging in the introductory mathematics classroom context, therefore, is crucial in better understanding their academic experiences, and by extension, understanding how to better support their persistence and achievement in STEM (Museus et al., 2017).

The following research aim will guide this study: How do their identities as female minoritized students play a role in their sense of belonging in gateway mathematics classrooms at a racially and ethnically diverse open access institution? Specifically, 1. How do Black and Latina female students' sense of belonging in the college algebra and precalculus classrooms compare to students in other racial and gender groups? 2. Does Black and Latina female students' sense of belonging in college algebra and precalculus change from the beginning to the end of the semester? 3. How do Black and Latina female students describe their college algebra and precalculus learning environment, experiences, participation, persistence, support systems and challenges as it relates to their sense of belonging?

Theoretical Perspectives – I frame my study using three conceptual frameworks: (a) sense of belonging (Strayhorn, 2019); (b) intersectionality (Crenshaw, 1991); and (c) authorizing student perspectives (Cook-Sather, 2006).

Research methodology – I use a sequential, explanatory mixed methods design (Creswell, 2015) to conduct quantitative methods using a sense of belonging scale at the beginning and end of the semester. I will then follow with qualitative methods (a math autobiography and semi-structured interviews). Participants in the quantitative phase are students (n=1162) enrolled in college algebra and precalculus during the Fall 2023 semester, at a diverse minority-serving open-access public college in the Southeast with a student population of about 11,000. For the qualitative phase, participants are 12 Black and Latina female students.

Results – At the current date of submission, I have analyzed the beginning semester sense of belonging scale data. These results indicate no statistically significant difference between the mean belonging scores based on students' race or gender, but there is a statistically significant difference in students' mean belonging scores based on their affinity for mathematics. By the date of the poster presentation, I will have also analyzed the end of semester belonging scale data and will present findings on how students' sense of belonging changes throughout the semester based on their racial and gender identities. Understanding how sense of belonging changes over time in introductory mathematics classes is important, as classroom-level belonging has been shown to be a critical element of students' motivation, participation, aspirations, and success.

References

Cook-Sather, A. (2006). Sound, Presence, and Power: 'Student Voice' in Educational Research and Reform. *Curriculum Inquiry 36*(4), 359-390.

Crenshaw, K. (1991). Mapping the margins: Intersectionality, identity politics, and violence against women of color. *Standard Law Review*, *43*(6), 1241-1299.

Creswell, J. (2015). *A concise introduction to mixed methods research*. Thousand Oaks, CA: Sage.

Hatfield, N., Brown, N., & Topaz, C. M. (2022). Do Introductory Courses Disproportionately Drive Minoritized Students Out of STEM Pathways? *PNAS Nexus, 1*(4), 1-10.

Museus, S. D., Yi, V., & Saelua, N. (2017). The Impact of Culturally Engaging Campus Environments on Sense of Belonging. *The Review of Higher Education 40*(2), 187-215.

Ong, M., Ko, L. T., & Hodari, A. K. (2016). Agency of women of color in STEM: Individual and institutional strategies for persistence and success. In E. H. Branch (Ed.), *Pathways, potholes, and the persistence of women in science: Reconsidering the pipeline* (pp. 183–195), Lanham, MD: Lexington Books.

Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2019). A descriptive study of race and gender differences in how instructional style and perceived professor care influence decisions to major in STEM. *International Journal of STEM Education*, *6*(1), 1-13.

Strayhorn, T. L. (2019). *College students' sense of belonging: A key to educational success for all students*. Routledge.

A General Education Data Literacy and Visualization Course Using Service Learning

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Keywords: general education, data literacy, data visualization, service learning

We report on a new course, Data Literacy and Visualization, that fulfills the university general education requirement for quantitative literacy and engages students through service learning. Students learn valuable data skills in an authentic context similar to those they would encounter in the workplace. Specifically, students work with local non-profit organizations who typically have data but lack the in-house expertise to derive value from it.

Motivation

At many institutions of higher education across the country, introductory undergraduate mathematics courses tend to have high failure rates and present a barrier to success for many students (Holm and Saxe 2016). In recent years many institutions have created alternative pathways to prepare students with quantitative literacy, and more recently, data literacy, skills (Dana Center 2020, Heinzman 2022). Data is ubiquitous in all types of organizations today, and individuals who have knowledge and skill in working with data, coupled with discipline-specific expertise, are increasingly valuable assets. Service learning is a method of experiential education that combines classroom instruction with meaningful, community-identified service. This form of engaged teaching and learning emphasizes critical thinking by using reflection to connect course context with real-world experiences (Bringle and Hatcher 1996).

Course Information

The course begins with an introduction to data and visualizations. Then students learn basic statistics including how to compute summary statistics by hand, create histograms, and identify normally-distributed data, then explore the concepts of correlation and regression. They then do these calculations and generate graphs in Excel. The next four weeks are devoted to learning about visualizations in general, including characteristics of good visualizations as well as how they can be misleading, and how to use Tableau, the software used for the projects. The final third of the semester is devoted to with the community partners.

Results

Students leave the course with much more confidence in their mathematical ability than they had at the beginning of the semester. They report enjoying the class and finding it more interesting than other math courses they've taken. In their comments about the course, many students specifically state their appreciation for being able to apply the math they've learned in a real-world context.

Acknowledgments

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References

Bringle, R. G., & Hatcher, J. A. (1996). Implementing service learning in higher education. *The Journal of Higher Education*, *67*(2), 221-239.

Center, C. A. D. (2020). Launch Years: A new vision for the transition from high school to postsecondary mathematics.

Heinzman, E. (2022). "I love math only if it's coding": A case study of student experiences in an introduction to data science. *Statistics Education Research Journal*, *21*(2), 5-5.

Holm, T., & Saxe, K. (2016). A Common Vision for undergraduate mathematics. *Notices of the American. Mathematics Society*.

Digital Task Design to Support Students' Developing Graphing Meanings

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Allison Olshefke, University of Delaware

Keywords: graphs, covariational reasoning, task design

The ability to construct and interpret graphs is critical for students as learners across STEM fields (e.g., Paoletti et al., 2020; Potgieter et al., 2008). However, from algebra through calculus, there are well documented difficulties that students experience with graphs and graphing (e.g., Glazer, 2011; Thompson et al., 2017). A small but growing body of literature addresses ways to support students' development of graphing meanings by utilizing digital tasks (Ellis et al., 2015; Liang & Moore, 2021; Paoletti & Moore, 2017). We describe an emerging digital task design principle: That enabling interaction with theoretically salient aspects of a representation directs students' attention there. We use Paoletti et al.'s (2023) framework to determine which aspects of the representation are theoretically salient and to analyze students' activity.

Building on the research on students' covariational reasoning and graphing meanings (Carlson et al., 2002; Moore, 2021; Moore & Thompson, 2015; Thompson, 2011), Paoletti et al. (2023) argue that providing students with repeated opportunities to bridge their meanings for quantities in situations and the graphical representations of those quantities is productive for developing graphing meanings. This entails constructing quantities from a situation and representing the "amount-ness" (Stevens & Moore, 2017) of the quantity as a magnitude bar. A different part of the process entails conceiving of a coordinate point as a multiplicative object which simultaneously represents the length of magnitude bars along orthogonal axes.

In this poster, we address the question: *How does Paoletti et al.'s (2023) framework inform the iterative design of digital tasks intended to support students' developing graphing meanings?* We use data collected during a two-year design experiment (Cobb et al., 2003) consisting of multiple teaching experiments (Steffe & Thompson, 2000). We describe how Paoletti et al.'s (2023) framework supported iterative cycles of task design, fine-grained analysis of students' activity, and task revision. We draw on examples in which our task design resulted in students' activity that we did not intend. We describe 1) how we use the framework to determine where students' attention could be directed to better elicit the intended activity, and 2) how we applied this emerging digital task design principle to direct students' attention in those ways.

For example, we designed a task to support students' conceiving of a coordinate point as a multiplicative object. We first prompted students to drag dots along the vertical axis to represent increases in one quantity for equal increases in the other. Later in the task, we prompt them to do this again, but also show the related coordinate point in the plane. We intended for students to notice the coordinate points' movement as they manipulated the dots along the vertical axis, but their attention was not drawn to the coordinate point. When revising, we wanted students to attend to the relationship between the coordinate point and the length of the related magnitude bar along the vertical axis. We prompted them to accomplish a familiar goal (arrange dots along the vertical axis to represent a changing situational quantity) through a new type of interaction with the graphical representation (dragging a coordinate point in the plane). We provide implications for teachers and researchers as they adapt or design digital graphing tasks.

Acknowledgments

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References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher*, *32*(1), 9-13.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*, 1–27. https://doi.org/10.1080/10986065.2020.1789930

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145—172). Springer. https://doi.org/10.1007/978-3-030-65632-4_7

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). RUME.

Paoletti, T., Gantt, A. L. & Corven, J. (2023) A local instructional theory for middle school students' emergent reasoning. *Journal for Research in Mathematics Education*, 54(*3*). https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2020). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals. *International Journal of Mathematical Education in Science and Technology*, 1–20. https://doi.org/10.1080/0020739X.2020.1847336

Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior, 48*, 137–151. https://doi.org/10.1016/j.jmathb.2017.08.003

Potgieter, M., Harding, A., & Engelbrecht, J. (2008). Transfer of algebraic and graphical thinking between mathematics and chemistry. *Journal of Research in Science Teaching 45*(2), 197–218. https://doi.org/10.1002/tea.20208

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Erlbaum.

Stevens, I. E., & Moore, K. C. (2017). The Intersection between Quantification and an All-Encompassing Meaning for a Graph. In E. Galindo & J. Newton, (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 709-716). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57).

EMERGING PRINCIPLES OF DIGITAL TASK DESIGN

Digital Task Design to Support Students' Developing Graphing Meanings

EDUCATION
Claudine Margolis
Teo Paoletti
Allison Olshefke

Study Goal: Develop an instructional sequence that supports 6th graders (11-12 years old) in developing emergent graphical shape thinking (Moore & Thompson, 2015).

Methodological Context: Design-based research study (Cobb et al., 2003) with multiple rounds of small-group teaching experiments (Steffe & Thompson, 2000).

Framework: Provide repeated opportunities to develop meanings for situations (MS), develop meanings for graphical representations (MR), and bridge those meanings (MS ←→ MR).

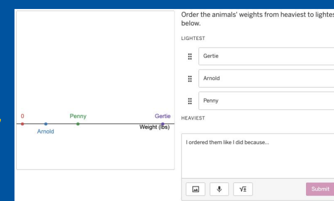
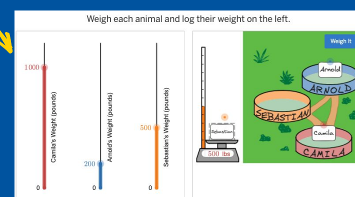
Partial LIT for EGST (Paoletti et al., 2023)

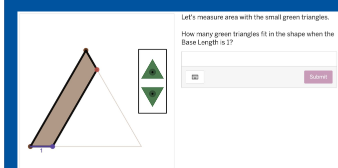
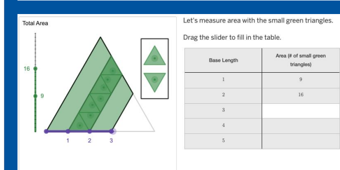
M.S. - Situational quantitative and covariational reasoning	**M.R. - Reasoning with graphical representations of covarying quantities**
M.S.1 - Construct quantities in a contextualized or decontextualized situation.	M.R.1 - Consider a varying segment length as representing a quantity's magnitude.
M.S.2 - Coordinate how two quantities change in relation to each other.	M.R.2 - Consider variations in two orthogonal segment lengths on axes in a coordinate system in relation to two covarying quantities.
M.S.3 - Develop an operative image of covariation that entails a multiplicative object.	M.R.3 - Conceive of or anticipate a point as a multiplicative object in the coordinate system simultaneously representing the two segments' magnitudes.

Direct students' attention to theoretically salient aspects of a representation by making it interactable.

Goal: MS1

New Interactions:
- Measure weight of mystery animals
- Construct magnitude bar representations by dragging

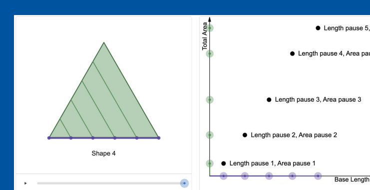
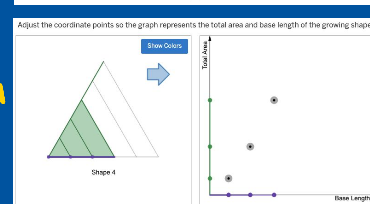



Goal: MS2

New Interactions:
- Drag Base Length
- Construct a table to coordinate values

Goal: MS3←→MR3

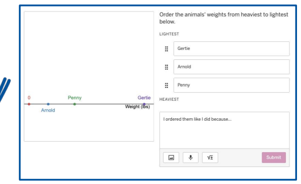
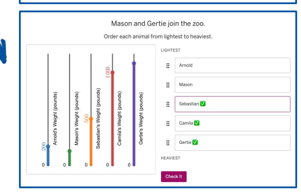
New Interaction:
- Drag coordinate point to arrange orthogonal magnitude bars

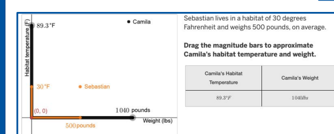
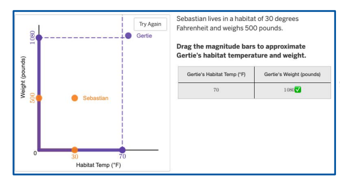



Use visual feedback to represent intended ways of thinking.

Visual Feedback: visual information provided automatically in response to student actions with potential for mathematical meaning (Margolis & Boyce, in press).

Feedback Design: Click "Check It." Magnitude bars reorder to reflect the student's selection.

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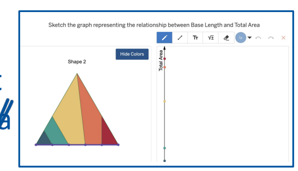
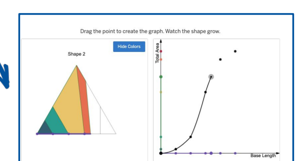

Speaking into the silence: Using multilingual students' own words through poetic transcription to tell their stories within introductory college mathematics education

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Keywords: Multilingualism, student voice, equity, introductory mathematics

As universities continue to become more multicultural, a growing body of literature recognizes the important role that language plays in students' undergraduate mathematics educational experiences (Hwang et al., 2021; Rios, 2023). For example, language mediates how students think, learn, make sense of the world, and socialize (Barwell, 2015). In addition, language, as it intersects with other social identities, positions people within social power hierarchies (Rios, 2023). Driven by these motivations, and within the context of a larger study examining 28 multilingual students' experiences in introductory mathematics, we explore the question: *What are the stories of multilingual students in introductory mathematics?*

We use poetic transcription (Glesne, 1997; Prendergast, 2009; Tremaine, 2022) to showcase the stories of multilingual students of color within these courses. Poetic transcription is an arts-based methodological tool that re-presents interviewees' words in the form of a poem; this allows the interviewee's own words to be used to tell their story while also bringing in space for the researcher's interpretation and analysis, which are used to decide how to present the interviewee's words. In this analysis, our focus is on how experiences of "silence" show up within these experiences, both of being silenced by others and how students use silence as a response for navigating the sociopolitical contexts of the classroom (Mills, 2006). Our results come in the form of poems from each of the six interviews involved in this analysis, chosen because of their discussion of silence. Here, we present an excerpt of one poem to illustrate the power of this methodology. This poem comes from Qamar (she/her), a Syrian refugee living in Saudi Arabia while enrolled as an international student at a large US university, and illustrates her experience of being silenced by multiple calculus instructors at various times.

When I asked that question
She muted me.
She asked, "Does anyone have any other questions?"
Yeah, I have my question,
but you muted me.
You just don't want to speak to me
You're muting me.
I'll stay silent.

On our poster, we will present all six poems in full, and highlight the ways in which silence acts as a navigational response and form of resistance for these multilingual students in undergraduate introductory math courses.

References

Barwell, R. (2015). Tensions in teaching mathematics through a second language. Language Issues: *The ESOL Journal, 26*(1), 52-54.

Glesne, C. (1997). That rare feeling: Re-presenting research through poetic transcription. *Qualitative Inquiry, 3*(2), pp. 202-221.

Hwang, J., Castle, S. D., & Karunakaran, S. S. (2022). One is the Loneliest Number: Groupwork within Linguistically Diverse Classrooms. *PRIMUS, 32*(10), 1140-1152.

Mills, J. (2006). Talking about silence: Gender and the construction of multilingual identities. *International Journal of Bilingualism*, *10*(1), 1-16.

Prendergast, M. (2009). "Poem is What?" Poetic inquiry in qualitative social science research. *International Review of Qualitative Research, 1*(4), pp. 541-568.

Rios J. (2023). Language, Comfort Speaking, and Collaboration: A QuantCrit Analysis of Multilingual Students' Experiences in Introductory College Mathematics Courses. *International Journal of Research in Undergraduate Mathematics Education*.

Tremaine, R. (2022). Opportunities and limitations of poetic transcription as a critical research methodology. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 511-515). Nashville, Tennessee.

Speaking into silence:

Using multilingual students' own words through poetic transcription to tell their stories within introductory college mathematics education

Intro

- As universities become more multicultural, language plays an increasingly important role in students' undergraduate mathematics educational experiences (Hwang et al., 2021; Rios, 2023)
- Language is both a resource that supports learning of mathematics, and is tied to power, how social interactions unfold, and who has access to speak and have their voices heard (Planas & Civil, 2013)

Methods

- Poetic transcription (Glesne, 1997) allowed us to use the students' own words to describe their experiences of silence in college math (Mills, 2006)
- Within the context of a larger study examining 28 multilingual students' experiences in introductory mathematics, we explore: *What are the stories of multilingual students in introductory mathematics?*

Discussion

- The theme of silence and being silenced was a salient part of how multilingual students of color described experiencing their introductory math courses
- Students described using silence in various ways to **navigate** their undergraduate mathematics classrooms, including navigating:
    - feeling uncomfortable speaking, feeling like their contributions were not listened to or valued, and feeling like they were treated differently than peers
- Students used silence as a form of **resistance** to empower their own voices, and discussed their **aspirations** related to using their voices
- These poems highlight the importance, from an instructor standpoint, of recognizing the role silence plays in navigating the classroom, along with the need to make the mathematics classroom more inclusive for multilingual students

Multilingual students of color experienced **being silenced** and **using silence** in ways that are interwoven with their linguistic, navigational, resistance, and aspirational capital (Yosso, 2005)

Scan to read more poems

Jocelyn Rios, Jess Ellis Hagman, Rachel Tremaine, Kaylee Fantin-Hardesty, Sarah Lutz, Kimberly Espinoza, & Hanna Medina

Colorado State University

PODER Promoting Opportunities in Diversity & Equity Research

Nader

(he/him) is an international student from Iran. He is one of few international students at this school who started in Precalculus instead of Calculus.

I'm a silent person
most of the time.
I prefer to work myself.
Nader, do you have any questions?
Ask it,
don't be shy.
No,
I'm okay.
I can work on it–
I can figure it out myself.
It's really hard for me to start talking so much in the group.

I don't feel good when I ask a lot.
The students
-don't like-
when a student asks lots of questions.
When I want to explain complicated things,
people couldn't understand.
My English is not perfect.
I was talking to my classmates
I just said something very weird in Farsi.
No, no I mean, nah, sorry–
I just said nah–
it was no in Farsi–
She was just like,
What?
I said nothing.
It's hard to speak another language.

…Some part of it was bad
Some part of it was good…

When you talk to teacher
and you feel that teachers
-likes-
to communicate with you in class.
She let us to do whatever we want,
how we are comfortable to do things.
I have good communication with my teacher.
I think that makes it much easier.

My friends, they say,
Yeah, we were like you for the first year.
And we get through it.
I was able to get through it.
At the middle of the semester, I feel much better,
I feel more comfortable.
I've been working on it;
I think I might be ready by fall to talk with people
and communicate.

Proportional Reasoning as a Tool to Understand Structural Racism in a Senior Capstone Course

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Keywords: Social Justice, Instructional Activities and Practices

Teaching mathematics in exclusively de-contextualized ways sends students the message that mathematics is not a tool that people can use to make sense of or change the world (Aguirre et al., 2019; Stephan et al., 2021). Lord (2020) found that the perception that mathematics is not useful in making the world better is one reason that some girls who achieved high levels of mathematical success choose not to pursue mathematics in college. Students of Color have shared that the ways mathematics can be used to impact their lives was missing from their secondary mathematics experiences, and that they wanted to use mathematics to promote social justice (Jett, 2019; Rodriguez et al., 2020; Varley Gutierrez, 2009). Adult undergraduate learners have lower levels of math self-efficacy than more "traditional" college students in mathematics content that is perceived as academic, but similar levels when math is seen as useful (Jameson & Fusco, 2014). So, learning about how mathematics can be used in social justice contexts can support students in choosing to continue their mathematics education.

Mathematics is a tool that humans have developed for specific purposes as the needs and wants of their communities have changed over time (Joseph, 2011), including making sense of social justice issues. Proportional reasoning in particular is the foundation of most arguments about equitable treatment and outcomes in politics, media, and education (e.g. Crenshaw, 1989). But students may understand concepts like percentages and ratios procedurally without being able to apply them in social justice contexts (Simic-Muller, 2015). Given the primacy of this content in understanding (in)justice, we wanted to study how a group of college students in a course titled "The Mathematics of Racism" made sense of ratios *and* racism together. We developed a lesson in which students explored claims in newspaper articles about the election of Indigenous Congresspeople through making sense of proportions. Our research question is: To what extent do students see mathematics as connected to social justice issues after the lesson?

Methods and Preliminary Results

The lesson was developed for a synchronous online senior capstone course at a public university in the Pacific Northwest in the Spring of 2023. The data were all 10 participating students' recorded video reflections on the lesson captured in FlipGrid in response to three questions: What did you learn about mathematics? What did you learn about racism? What do you want to know more about? We analyzed the data by identifying which features of Gutstein's (2006) teaching mathematics for social justice framework and Kokka's (2020) critical mathematics consciousness were indicated by students' video reflections.

Students did connect their mathematical learning to social justice by helping them see how mathematics can help them make sense of the world and advocate for change. Every student referred to at least one issue of the representation of Indigenous people in Congress that they thought was unfair or wrong, and all but one student shared that mathematics helped them make sense of the (un)fairness they saw. This is an example of what Gutstein (2006) describes as reading the world with mathematics. Also, 7 students expressed a desire to have things change or to be part of the change themselves, suggesting that students developed their sense of being agents for social change as described by Kokka (2020).

References

Aguirre, J. M., Anhalt, C. O., Cortez, R., Turner, E. E., & Simic-Muller, K. (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice: The Flint water task. *Mathematics Teacher Educator*, *7*(2), 7–26. https://doi.org/10.5951/mathteaceduc.7.2.0007

Crenshaw, K. W. (1989). Demarginalizing the intersection of race and sex: A black feminist critique of antidiscrimination doctrine, feminist theory and antiracist politics. *University of Chicago Legal Forum*, *1989*, 139–168.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Taylor & Francis.

Jameson, M. M., & Fusco, B. R. (2014). Math anxiety, math self-concept, and math self-efficacy in adult learners compared to traditional undergraduate students. *Adult Education Quarterly*, *64*(4), 306–322. https://doi.org/10.1177/0741713614541461

Jett, C. C. (2019). Mathematical persistence among four African American male graduate students: A critical race analysis of their experiences. *Journal for Research in Mathematics Education*, *50*(3), 311–340. https://doi.org/10.5951/jresematheduc.50.3.0311

Kokka, K. (2020). Social justice pedagogy for whom? Developing privileged students' critical mathematics consciousness. *The Urban Review*, *52*(4), 778–803. https://doi.org/10.1007/s11256-020-00578-8

Lord, E. (2020). "I want to help others": Why female A-Level mathematics students reject undergraduate mathematics. In R. Marks (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 40). BSRLM. https://bsrlm.org.uk/wp-content/uploads/2020/05/BSRLM-CP-40-1-09.pdf

Rodriguez, S. L., Bukoski, B. E., Cunningham, K. J., & Jones, A. (2020). Critiquing oppression and desiring social justice: How undergraduate latina students in STEM engage in acts of resistance. *Journal of Women and Gender in Higher Education*, *13*(3), 251–267. https://doi.org/10.1080/26379112.2020.1838297

Simic-Muller, K. (2015). Social justice and proportional reasoning. *Mathematics Teaching in the Middle School*, *21*(3), 162–168.

Stephan, M., Register, J., Reinke, L., Robinson, C., Pugalenthi, P., & Pugalee, D. (2021). People use math as a weapon: Critical mathematics consciousness in the time of COVID-19. *Educational Studies in Mathematics*, *108*(3), 513–532. https://doi.org/10.1007/s10649-021-10062-z

Varley Gutierrez, M. C. (2009). "I thought this U.S. place was supposed to be about freedom": Young Latinas speak to equity in mathematics education and society [Ph.D., The University of Arizona]. In *ProQuest Dissertations and Theses*. https://www.proquest.com/docview/304844995/abstract/1FF6C119F3334EAAPQ/1

Cafecito con Matemática: A bilingual mathematics outreach event that centers family, culture, and community

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Keywords: Outreach, Bilingual mathematics education, Hispanic/Latino communities, Design principles

With the growing emphasis on supporting and celebrating diversity in mathematics, it is important for mathematics departments to build responsible partnerships with local communities and schools. In the context of the Southwest, extant research has documented how outreach initiatives can create bridges between families, school, and universities, helping to facilitate the mathematical participation of students and parents from Hispanic/Latino communities (Civil et al., 2005) and mathematics pre-service teacher professional development (Stoehr et al., 2022). Moreover, research demonstrates the importance of using culturally responsive practices and creating bilingual spaces that affirm Spanish as a valid language for teaching and learning mathematics (Civil, 2007; Civil & Andrade, 2002; Turner et al., 2013).

In this poster, we present *Cafecito con Matemática* (CM), a bilingual mathematics outreach event run by Colorado State University. This event is hosted at a local bilingual elementary school that serves predominantly Hispanic/Latino communities. Occurring monthly during the evenings, CM first provides students and their families with dinner and then engages everyone in fun, culturally relevant mathematics games and activities which are facilitated by school teachers and university members (including mathematics graduate students and pre-service teachers who gain valuable experience interacting with students and their families). All materials for CM are provided in English and Spanish to create accessibility and promote bilingualism. Ultimately, CM creates a space which validates and leverages families' language and mathematical knowledge, while also providing them with resources to help their children develop their learning of, excitement for, and curiosity within mathematics.

Additionally, CM creates a space that celebrates the home languages of our Spanish-speaking mathematics pre-service teachers and graduate students. By intentionally recruiting Spanish speakers to be involved in CM, we are positioning their home language as a valuable resource that strengthens their ability to teach and mentor diverse communities. Through this event these pre-service teachers and graduate students represent the department in a teaching role, serve as role models to the community, and have a space to interact with school students in their home language, and celebrate their culture alongside community members.

In this poster, we further describe the guiding features and implementation of CM in an effort to support others interested in developing similar outreach events within their local communities. We blend a mathematical approach that focuses on hands-on learning experiences with a culturally responsive approach that features mathematics games and activities that draw inspiration from Hispano cultures, communities, and traditions (e.g., Lotería and traditional games from Latin America). We conclude with initial findings from an examination of the benefits of implementing such an event. We focus on multiple perspectives, providing excerpts from students and their families, from the teachers and administration within the schools, and from the mathematics pre-service teachers and graduate students who serve as teaching assistants for CM. We have found that this event is meaningful in many ways for those involved and serves as a way to celebrate culture and diversity in mathematics education.

References

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. *Improving access to mathematics: Diversity and equity in the classroom*, 105–117.

Civil, M. & Andrade, R. (2002). Transitions between home and school mathematics: Rays of hope amidst the passing clouds. In G. de Abreu, A. J. Bishop, & N. C. Presmeg (Eds.), *Transitions between contexts of mathematics practices* (pp.149–169). Kluwer.

Civil, M., Bratton, J., & Quintos, B. (2005). Parents and mathematics education in a Latino community: Redefining parental participation. *Multicultural Education*, *13*(2), 60–64.

Stoehr, K., Salazar, F., & Civil, M. (2022). The power of mothers and teachers engaging in a mathematics bilingual collaboration. *Teachers College Record, 124*(5), 30–48.

Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematics Education, 44*(1), 199–234.

Beginning Teacher's Trajectory in the Figured Worlds of Reform and Traditional Instruction

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NSF Improving Undergraduate STEM Education: Education and Human Resources (IUSE: HER) # 1941720

Rationale

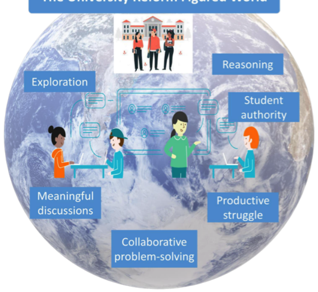
- Beginning teachers transition from the world of reformed teaching of their teacher preparation program to the more traditional world of school teaching (Bjerke & Nolan, 2023; Gainsburg, 2012; NCTM, 2014).
- This transition involves **developing identities** and teaching practices that allow novice teachers to **reconcile the two worlds** (Horn et al., 2008; Ma & Singer-Gabella, 2011).
- Not enough is known about beginning teachers' long-term process of identity formation during the reconciliation of the two worlds.
- We can learn about identity formation by considering beginning teachers' **narratives** about their **current** pedagogical actions and **desired** actions (Heyd-Metzuyanim, 2013; Horn et al., 2008).

We follow the identity formation of one teacher, Olive, by examining her narratives about her current pedagogical actions and her desired ones, as she transitions from being a pre-service teacher (PST) to being an intern (INT) in a mentor teacher classroom, to becoming a new teacher (NT).

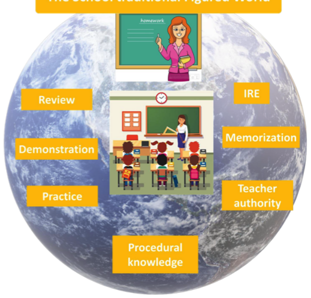
Conceptualization

Drawing on Holland and colleagues (1998), we conceptualize two *figured worlds*, each populated with their own specific actors, pedagogical valued actions, and valued outcomes.

The University Reform Figured World



The School traditional Figured World



Research Question: What was the trajectory of Olive's narratives about her current and desired pedagogical actions in relation to their alignment with the traditional and reform pedagogical actions?

Data collection

Pre-Service Teacher (PST)
- 4 Written reflections on 4 lessons Olive taught during a capstone course on reasoning and proof

Intern (INT)
- 2 interviews on 2 lessons Olive taught as intern in her cooperating teacher's (CT's) classroom.

New Teacher (NT)
- 4 interviews on 4 lessons Olive taught as a new teacher in her own classroom and 2 PLC meetings for new teachers

Findings

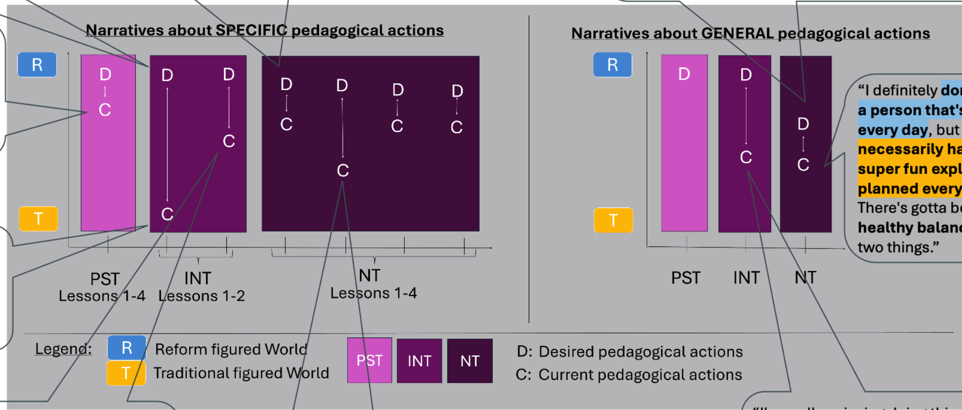
"**I wanna** just go along with what they're saying [...]. [As if saying to them,] "**let's just play with the rules** for the day and" [...], show them [the students] **why** it doesn't make sense."

"I guess my hesitation [...] I'm playing devil's advocate in the situation, if I went and tried to go off on a conceptual tangent with each kid, I think that they would tune it out immediately [...] with so many kids [...] **to try to circulate that room and have a deep conceptual conversation with each of those 28, I don't even think I'd have time** in the block to do that."

"**[In] undergrad and even as an intern** [I] made some lesson plans that were just **absolutely ridiculous** in terms of what I expected the students to understand at a rate or pace that was absolutely **unrealistic** for the children [...] I've definitely become more, I like to consider it, realistic [...] **sometimes I feel bad about it and sometimes I feel guilty about it**. [...] I try to maintain high expectations, but I think I, **I'm a little bit more realistic.**"

"Together [with the students] we defined a **conditional statement** and **discussed how they occur in a variety of settings.** I also asked for students to **provide their own examples.**"

"I would feel accomplished if they [the students] understood **how to combine like terms** on two sides of the equation, **moving chunks with the variable.**"



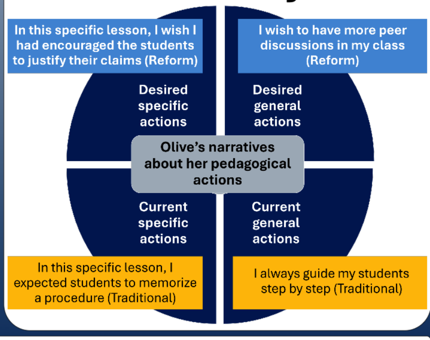
"I definitely **don't wanna be a person that's lecturing every day**, but I also **don't necessarily have some super fun exploration planned every day either.** There's gotta be like a **healthy balance** of those two things."

"I had decided to do that little, what I called an **exploration in the beginning** [of the lesson]. [...] I kind of pushed to do that [...] I wanted them to see it for themselves and to understand, using numbers, **why that was the case.**"

"I liked that some of it [the lesson] felt **exploratory**, but then some of it definitely felt like a **very telling way of teaching**, like just kind of giving them the information as opposed to letting them figure it out."

"I'm really enjoying doing this project because **I have been doing a lot of boring, I feel like, lecture-style things**. And so, **this is like an opportunity for me to stop talking, which is really wonderful.**"

Data Analysis



Discussion

Current Traditional | MIND THE GAP | Desired Reform

- Olive's case revealed two main **critical junctures** along her trajectory of navigating between the university reform figured world and the school traditional figured world:

1. Internship – the opening of a gap between her narratives about her current traditional actions and her desired reform actions; attempting the **close the gap by modifying current actions to align more with desired reform ones.**

2. NT period – "lowering the bar", attempting to close the gap between current and desired actions by modifying the desired actions to be more "realistic" and include more traditional ones.

- Practically, it may be important for teacher educators to anticipate the possibility of a gap opening between current traditional and desired reform pedagogical actions during the internship period.
- It is important to provide continued support to beginning teachers and help them cope with the gap, to address their justifications for the gap, and to find ways to help keep the view of reform teaching "realistic" and desired.

Bjerke, A. H., & Nolan, K. (2023). The return to university after fieldwork: toward disrupting practice-theory challenges identified by mathematics teacher educators. *Frontiers in Education, 8:1129206*.
Gainsburg, J. (2012). Why new mathematics teachers do or don't use practices emphasized in their credential program. *Journal of Mathematics Teacher Education, 15*(5), 359–379.
Heyd-Metzuyanim, E. (2013). The co-construction of learning difficulties in mathematics-teacher-student interactions and their role in the development of a disabled mathematical identity. *Educational Studies in Mathematics, 83*(3), 341–368.
Holland, D., Lachicotte, W. J., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.
Horn, I. S., Nolen, S. B., Ward, C., & Campbell, S. S. (2008). Developing Practices in Multiple Worlds: The Role of Identity in Learning To Teach. *Teacher Education Quarterly, 35*(3), 61–72.
Ma, J. Y., & Singer-Gabella, M. (2011). Learning to Teach in the Figured World of Reform Mathematics: Negotiating New Models of Identity. *Journal of Teacher Education, 62*(1), 8–22.
National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, NCTM.

Statistical Literacy of College Students in a Simulation-Based Introductory Statistics Course

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Keywords: Statistical Literacy, Statistics Education, Simulation-Based Curriculum

The news media is the primary source of information about social and scientific issues for most adults (Bissonnette et al., 2021). Statistical and mathematical products in the news take on many forms, requiring multiple inter-related skills to appropriately comprehend and evaluate (Gal & Geiger, 2022). Additionally, messages in the news are often shaped by political, commercial, or other agendas contributing to inaccurate, misleading, or intentionally manipulative messages (Mehta & Guzmán, 2018). As such, increasing emphasis exists in the literature regarding the need to develop students' critical capabilities in understanding how mathematics and statistics are used (Gal & Geiger, 2022). Hence, this qualitative pilot study investigated the question: what, if any, statistical concepts do students in an introductory statistics course utilizing a simulation-based curriculum apply when critiquing statistics in the news media?

Current trends in statistics education include the use of technology to provide students with dynamic, interactive simulations and the importance of attending to risk, critical literacy, and communication in contexts such as the media (Burrill & Pfannkuch, 2023). Previous studies have utilized media-based tasks to investigate student statistical literacy (e.g., Kaplan & Thorpe, 2010). Other studies found simulation-based curricula improve student outcomes on topics such as confidence intervals, significance testing, and data collection (e.g., Chance et al., 2022). This study extends existing research by investigating how a simulation-based curriculum prepares students to use critical questioning when engaging with news media.

The conceptual framework of this study draws from Watson and Callingham's (2003) six-level construct of statistical literacy. Gal's (2002) characterization of critical questioning, and Gal and Geiger's (2022) nine categories of statistical and mathematical products in the news informed the development of tasks. Five participants, who were enrolled in an introductory statistics course at a mid-sized university, completed task-based interviews during which they commented on three mock news headlines and accompanying articles. Interviews took place after the simulation-based inference topics were covered in class. Participant responses will be coded, after transcription, for the six levels of statistical literacy and how statistical concepts were utilized throughout the task.

Current analysis includes identification of themes in students' responses. Initial findings indicate participants mentioned sampling, study design, and hypothesis tests with additional themes including wanting more information, relying on non-statistical arguments, and struggling to connect statistical ideas to critical questions. During the poster session, further results of the pilot study will be shared and ideas for future directions and adjustments to tasks will be discussed.

References

Bissonnette, M., Chastenay, P., & Francoeur, C. (2021). Exploring adolescents' critical thinking aptitudes when reading about science in the news. *Journal of Media Literacy Education*, *13*(1), 1-13. https://doi.org/10.23860/JMLE-2021-13-1-1

Burrill, G., & Pfannkuch, M. (2023). Emerging trends in statistics education. *ZDM – Mathematics Education*. https://doi.org/10.1007/s11858-023-01501-7

Chance, B., Tintle, N., Reynolds, S., Patel, A., Chan, K., & Leader, S. (2022). Student performance in curricula centered on simulation-based inference. *Statistics Education Research Journal*, *21*(3), 1-24. https://doi.org/10.52041/serj.v21i3.6

Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistical Review*, *70*(1), 1-25. https://doi.org/10.1111/j.1751-5823.2002.tb00336.x

Gal, I., & Geiger, V. (2022). Welcome to the era of vague news: A study of the demands of statistical and mathematical products in the COVID-19 pandemic media. *Educational Studies in Mathematics*, *111*(1), 5-28. https://doi.org/10.1007/s10649-022-10151-7

Kaplan, J. J., & Thorpe, J. (2010). Post secondary and adult statistical literacy: Assessing beyond the classroom. In C. Reading (Ed.), *Proceedings of the 8th International Conference on Teaching Statistics, Ljubljana, Slovenia.* Retrieved from https://iase-web.org/documents/papers/icots8/ICOTS8_5E3_KAPLAN.pdf

Mehta, R., & Guzmán, L. (2018). Fake or visual trickery? Understanding the quantitative visual rhetoric in the news. *Journal of Media Literacy Education*, *10*(2), 104-122. https://doi.org/10.23860/JMLE-2018-10-2-6

Watson, J., & Callingham, R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal*, *2*(2), 3-46. https://doi.org/10.52041/serj.v2i2.553

Conceptualizing and Representing the Cartesian Connection in Calculus

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Keywords: Cartesian Connection, Graphical Representations, Conceptual Analysis, Theoretical Sampling

Calculus concepts require students to express and interpret distances in the Cartesian plane. Yet previous research shows that many students are not easily able to do so (Parr et al., 2021). To effectively represent distances, students must be able to understand and apply the Cartesian connection, that a point is on graph A if and only if its coordinates (x, y) satisfy the equation of A (Moschkovich et al., 1993). This connection requires students to draw on algebraic and graphical concepts to conceive of coordinate points as pairs of distances rather than simply a location on a graph (Parr et al., 2021). Students should also be able to flexibly move between using a graph and its equation to represent distances both in terms of x and y or determine if points lie on a line (Knuth, 2000). We are interested in why some student may not display this flexibility and what underlying issues are preventing students from making the Cartesian Connection.

In this poster, we explore the following research question: What types of reasoning, successes, and challenges arise when undergraduate students engage in tasks meant to develop their ability to make the Cartesian Connection to represent distances? To answer this question, we presented surveys to 169 undergraduate students at a small private college to assess their ability to represent distance in the Cartesian plane both in terms of x and in terms of y. We selected 9 students who did not correctly complete the tasks to conduct exploratory teaching interviews. During the interviews, students worked through a series of tasks to represent distance in the Cartesian plane while thinking out loud and explaining their thought process and reasoning. We analyzed the students' interviews using theoretical sampling (Corbin & Strauss, 2014). Our results highlighted that whether students can make the Cartesian Connection is not a yes or no question, but rather better represented by a spectrum of student thinking. We broke up the Cartesian Connection into 3 components: (1) the definition, (2) the underlying algebraic meanings, (3) the underlying graphical meanings. As part of our results, we offer a conceptual analysis (Thompson, 2008) identifying the mental operations needed to make the Cartesian Connection in this context.

Many different types of student reasoning emerged, yet we found similar obstacles that students encountered when representing distances in the Cartesian plane. Some common obstacles were students having difficulty understanding the phrase "in terms of" as well as students having difficulty associating distance in terms of y for a horizontal segment. We analyzed students' responses according to two dimensions: 1) categories encapsulating all the mental processes needed to make the Cartesian Connection to represent distances and 2) the extent to which students were reliant on either algebraic expressions or graphical representations. This poster will give an overview of the spectrum of student thinking that highlights the complexity and nuance of the Cartesian Connection. We will also describe the implications of our findings for teaching and research.

References

Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory.* Newbury Park, CA: Sage publications.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500-507.

Moschkovich, J., Schoenfeld, A., and Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T.A. Romberg, E. Fennema, & T.P. Carpenter (Eds.), *Integrating research on the graphical representations of functions* (pp. 69-100). Erlbaum.

Parr, E. D., Ely, R., & Piez, C. (2021). Conceptualizing and representing distance on graphs in Calculus: The case of Todd. *2021 Research in Undergraduate Mathematics Education* Reports. (pp. 214-221).

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano and A. Sepulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 31-49). Morelia, Mexico: PME.

Holding on to Reasoning and Proving While Navigating Professional Obligations: Secondary Teachers' Transition from University to Schools

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Keywords: Reasoning and Proof, Beginning Teachers, Teacher Obligations, Teacher Education

Transitioning between university teacher preparation programs and secondary schools is a complex process, fraught with many challenges for beginning mathematics teachers (Klein, 1932/2016). During this transition, many of the ambitious teaching practices experienced in teacher preparation programs may be absent (e.g., Stokking et al., 2003), but some may reemerge later (Grossman et al., 2000). Herbst and Chazan's (2003) theory of *practical rationality* allows for conceptualizing this transitioning process as the socialization of beginning teachers into the profession, as they adopt a decision-making framework rooted in norms and four professional obligations. These obligations are to the *discipline* of mathematics, to students as *individuals* with unique needs, *interpersonal* obligations to the entire class, and obligations to follow the *institutional* policies, curricula, and practices of schools.

To explore how beginning teachers negotiate between their formal education and the realities of classroom practice, specifically, how they integrate reasoning and proving into their teaching, we followed three beginning teachers: Nancy, Olive, and Diane for three years. First, as PSTs in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020), then as interns (INTs), and then as novice teachers (NTs) in their own classrooms. For each teacher, we collected observations, lesson plans, and interviews (3-4 in each setting) and analyzed them in terms of four professional obligations (Chazan et al., 2016) and personal beliefs. Here, we report in aggregate the observed percentages of coded utterances.

As INTs, the participants' discourse was dominated by *institutional* obligation (39%) evident in the expressed need to adhere to the school rules, curriculum, and the cooperating teacher's traditional practices (e.g., "she [the CT] has a system that tends to work for her. So that's what I've been doing"). *Disciplinary* obligation is only expressed in visionary terms – a wish to have students do more explorations. As INTs gained more experience their focus on *institutional* obligation decreased to 17.7% in favor of exposing students to *disciplinary* practices (percent increase of 19) like exploring, conjecturing, and justifying. Having their own classrooms as NTs was characterized by renewed increased *interpersonal* obligation for the classroom (percent increase of 21.5). Despite the added responsibilities, the NTs continued to incorporate reasoning and proof, for example, by including discovery activities. By the end of year 1 the *disciplinary* and *individual* obligations were most prominent, about 30% each. These findings shed light on the process of reconciling various professional obligations while maintaining some focus on reasoning and proof. The findings have implications for those who work to support beginning teachers in their effort to create more reasoning and proof-oriented classrooms.

Acknowledgments

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References

Buchbinder, O. & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. Journal of Mathematical Behavior, 58, 100779. https://doi.org/10.1016/j.jmathb.2020.100779

Chazan, D., Herbst, P., & Clark, L. (2016). Research on the teaching of mathematics: A call to theorize the role of society and schooling in mathematics instruction. *Handbook of research on teaching*, *5*, 1039-1098.

Grossman, P. L., Valencia, S. W., Evans, K., Thompson, C., Martin, S., & Place, N. (2000). Transitions into teaching: Learning to teach writing in teacher education and beyond. *Journal of Literacy Research*, *32*(4), 631-662. https://doi.org/10.1080/10862960009548098

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the learning of Mathematics*, *23*(1), 2-14.

Klein, F. (2016). *Elementary mathematics from a higher standpoint: Volume I: Arithmetic, algebra, analysis*. Springer. (Original work published 1932).

Stokking, K., Leenders, F., De Jong, J., & Van Tartwijk, J. (2003). From student to teacher: Reducing practice shock and early dropout in the teaching profession. *European Journal of Teacher Education*, *26*(3), 329-350. https://doi.org/10.1080/0261976032000128175

Holding on to Reasoning and Proving While Navigating Professional Obligations: Secondary Teachers' Transition from University to Schools

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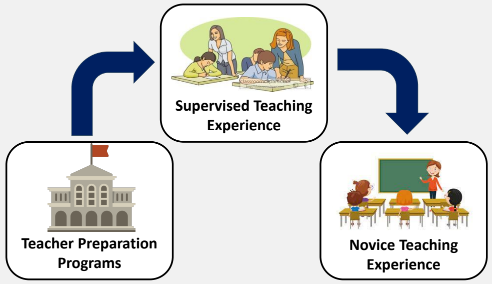
NSF No. # 1941720 The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation

Background



- Transitioning between university teacher preparation programs and secondary schools is a *complex process*, fraught with many challenges for beginning mathematics teachers (Klein, 1932/2016; Smagorinsky et al., 2004).
- Challenges emerge in *balancing commitments* to the university, the cooperation teacher, while developing one's own teaching identity (Bieda et al., 2015).
- The process of socializing in teaching profession involves developing a decision-making framework that takes into account four professional *obligations*: to the discipline of mathematics, to individual students, to students as a class and to the institution of schooling (Herbst & Chazan, 2003).

Goal: To explore how beginning teachers negotiate the four professional obligations as they transition from university to classroom practice; specifically, how they uphold disciplinary obligation by integrating reasoning and proving into their teaching.

The Study

Participants: Three beginning teachers: Olive, Nancy, & Diane

Setting: The study is part of a *longitudinal* project that investigates beginning teachers' expertise to teach mathematics via reasoning and proving.

- Each participant took a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020) in their teacher preparation program.

Data sources for each participant:

- Lesson plans and observations
- Lesson debrief interviews (3-4 for each participant).

Supervised Student Teaching → Solo Student Teaching → First Year of Teaching own class

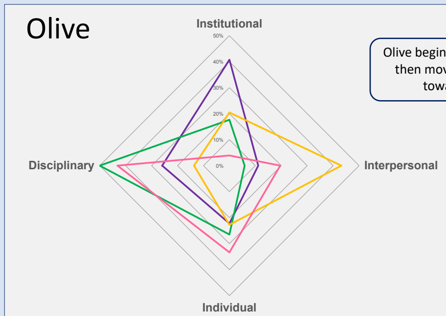
Data analysis:

- Transcribed interviews were coded by three researchers for instances of the four professional obligations.
- For each participant and for each lesson, we aggregated the codes into a single score on each obligation.
- Percentages were calculated to identify trends across lessons and participants.

Theoretical Framework

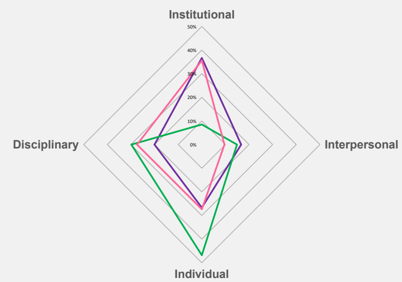
Results

Olive



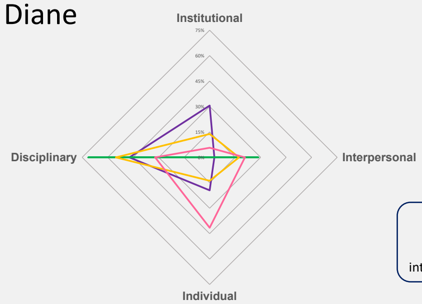
Olive begins by focusing on institutional obligation, then moves toward individual, and finally shifts toward disciplinary and interpersonal

Nancy



Nancy begins by focusing on institutional obligation, then moves toward individual, and then shifts back to institutional followed by disciplinary

Diane



Diane begins by focusing on disciplinary and institutional obligation, then moves toward disciplinary, and in the follow-up interview mostly focuses on individual obligation

Lesson Key: Intern Lesson 1 | Intern Solo Week | Novice Teacher Lesson 1 | Year 1 Follow-up

Conclusions

As interns, the participant's discourse was dominated by ***institutional*** obligation evident in the expressed need to adhere to the school rules, curriculum, and the cooperating teachers' traditional practices. The interns have a desire to implement more exploratory activities.

> In my head, [students] are doing explorations all the time.

> [The CT] has a system that tends to work for her. So that's what I've been doing.

With more experience, the interns started to focus less on *Institutional* obligations and more on ***disciplinary*** obligations (exploring, conjecturing, and justifying).

> I pushed to do an [exploration activity] because in the original lesson plan that [CT] had written like for years prior the idea of flipping the inequality sign is not really explored at all…I wanted [students] to see it for themselves and to understand why that was the case.

As novice teachers, they concentrated more on their ***interpersonal*** obligation to the classroom while still having a strong desire to implement reasoning and proof activities.

> Classroom management pieces were overwhelming in the beginning….But [students] have been really patient with me.

> I want to do more like group activities in the class, and less like direct instruction…and I'd like to do more exploratory stuff

By the end of their first year of teaching, the ***disciplinary*** and ***individual*** obligations were more prominent.

> The advanced group of kids, I like to confuse them because I like for them to unconfuse themselves because I find that's when they really understand something.

> The set of notes I had was this long, dreary like list of properties. So, I'm going to let them do this [exploration] activity instead.

References

- Bieda, K. N., Sela, H., & Chazan, D. (2015). "You are learning well my dear": Shifts in novice teachers' talk about teaching during their internship. *Journal of Teacher Education, 66*(2), 150-169. Buchbinder, O. & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *Journal of Mathematical Behavior, 58*, 100779.
- Grossman, P. L., Valencia, S. W., Evans, K., Thompson, C., Martin, S., & Place, N. (2000). Transitions into teaching: Learning to teach writing in teacher education and beyond. *Journal of Literacy Research, 32*(4), 631-662.
- Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the learning of Mathematics, 23*(1), 2-14.
- Klein, F. (2016). Elementary mathematics from a higher standpoint: Volume I: Arithmetic, algebra, analysis. Springer. (Original work published 1932).
- Smagorinsky, P., Cook, L. S., Moore, C., Jackson, A. Y., & Fry, P. G. (2004). Tensions in learning to teach: Accommodation and the development of a teaching identity. Journal of Teacher Education, 55(1), 8-24.

AI-Facilitated Cross-Cultural Problem-Solving in Calculus

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Keywords: Artificial Intelligence, Cross-Cultural Collaboration, Calculus

Artificial Intelligence agents are used by undergraduates to support their doing of mathematics (D'Mello et al., 2023; Isgandarli, 2022; Khazanchi et al., 2023). For example, ChatGPT is increasingly used by undergraduates and their professors (Davies et al. 2021; Li et al., 2023). How can this reality be leveraged to improve the mathematical learning experiences of students in the U.S. and globally? (Almoubayyed et al., 2023).

The authors report preliminary findings of a research project that explored Large Language Models (LLM) as tools for facilitating cross-cultural discussions about the "big ideas" of calculus. The following question guided the study: (Q1) *What impact do LLMs have on the collaboration and mathematical problem-solving of calculus and precalculus students in an international context?*

In the study, two precalculus students from a state university in the United States (SFSU) and two calculus students from a state university in China (SCUPI) worked collaboratively to solve rich mathematics tasks. Four students participated, two from each university. Across three 90 minute sessions, the students met online (via ZOOM) and engaged in three types of collaborative mathematics tasks---namely, (1) a computational task, (2) an applied problem, and (3) a conceptually-oriented question. At the end of each session, students completed a short exit survey consisting of Likert-style and open-ended prompts that asked about the nature and quality of their interactions with each other and AI as well as their mathematical learning. An AI assistant facilitated the meetings in a student breakout room, with human instructors available as needed (in the main ZOOM room). The online meetings were recorded for further analysis.

In addition to the students' written work on tasks, responses on exit surveys, and ZOOM recordings, the authors report participant responses on a recruitment survey. These four data sources comprise all of the data sources for the study. The research team at SCUPI followed the ethical guidelines and standards established by SFSU.

Data analysis is ongoing at the time of this writing and will include thematic coding of student activities during the meetings, and a discussion analysis using Techno-Mathematical Discourse (TMD) framework, focusing on the AI assistant's role in facilitating discussions including the nature and depth of the mathematical and cultural conversations. The authors compare and contrast interactions of students with each other and AI both within and across cultures as they consider the research question (Q1).

The poster will include illustrations of the AI assistant-in-use, further details of the research protocol, and QR code for accessing the research study AI agent directly. Poster-side conversation will focus on the design of the study, lessons learned about cross-cultural aspects from the first use of the protocol, and advice for revising the protocol.

References

Almoubayyed, H., Bastoni, R., Berman, S. R., Galasso, S., Jensen, M., Lester, L., ... & Ritter, S. (2023, June). Rewriting math word problems to improve learning outcomes for emerging readers: A randomized field trial in carnegie learning's mathia. In *International Conference on Artificial Intelligence in Education* (pp. 200-205). Cham: Springer Nature Switzerland.

Davies, A., Veličković, P., Buesing, L., Blackwell, S., Zheng, D., Tomašev, N., ... & Kohli, P. (2021). Advancing mathematics by guiding human intuition with AI. Nature, 600(7887), 70-74.

D'Mello, S. K., & Graesser, A. (2023). Intelligent tutoring systems: How computers achieve learning gains that rival human tutors. In *Handbook of educational psychology* (pp. 603-629). Routledge.

Isgandarli, E. (2022). *The Relationship Between Student Performance on Aleks Software and Standardized Tests*. University of Missouri-Kansas City.

Khazanchi, R., Di Mitri, D., & Drachsler, H. (2023). Measuring Efficacy of ALEKS as a Supportive Instructional Tool in K-12 Math Classroom with Underachieving Students. *Journal of Computers in Mathematics and Science Teaching*, *42*(2), 155-176.

Li, P. H., Lee, H. Y., Cheng, Y. P., Starčič, A. I., & Huang, Y. M. (2023, August). Solving the Self-regulated Learning Problem: Exploring the Performance of ChatGPT in Mathematics. In *International Conference on Innovative Technologies and Learning* (pp. 77-86). Cham: Springer Nature Switzerland.

AI-Facilitated Cross Cultural Problem Solving in Calculus

PRESENTERS
Tuto LopezGonzalez
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PROJECT OVERVIEW

We engaged in a **case study** with four undergraduates--two from San Francisco State University (USA) and two from Sichuan University (China) who worked (via ZOOM) on collaborative calculus tasks assisted by an AI agent embedded into the sessions. The study explored the impact that the **AI-based agent** had on students' understanding of **foundational ideas of calculus**.

THE TASK

In particular, three types of collaborative mathematics tasks were explored in the sessions: (1) a computational task, (2) an applied problem, and (3) a conceptually-oriented question. At the end of each session, students completed a short exit survey about their interactions and their mathematical learning.

THE AGENT

The Agent has a focus on mathematical guidance and cultural insight. It has three functions: (1) leads math discussions, (2) engages in discussions embodying historical mathematicians, and (3) helps create examples. The adaptive and inclusive approach aims to provide a friendly learning experience.

The agent is programmed using **prompt engineering** techniques such as algorithm of thoughts (AoT) (Bilgehan et al., 2023), program simulation (Scalamogna), and chain-of-thought (CoT).
The result is a single prompt that starts the agent when entered to a large language model (LLM) to initiate the assistant. *Note: An LLM is a mathematical construct employing transformer-based neural networks to model probability distributions over sequences of words.*

DATA COLLECTED

Zoom recordings, Students' work on tasks, Conversation with AI assistant, Responses on surveys

AI has potential as a supportive educational tool. Students used the AI agent to assist in understanding and solving calculus problems.

Open tasks provide better contexts for AI learning agents to assists students.

“We ask the AI to help us understand and see ways to solve open problems"

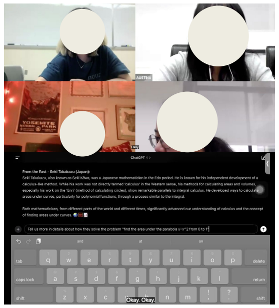
Scan the code to access the agent through GPT-4

MEETING SUMMARIES

First meeting: Meeting each other
The focus was to meet each other, explain the objectives of the study, and provide context on technology setup. Students completed 2 ice-breaker tasks and explored the agent.

Second meeting: Derivative - Optimization
Students explored the notion of derivative and collaboratively solved the Fenced Vegetable Garden task (Find dimensions would maximize the area of the vegetable garden). We used knowledge gained in this session to inform the revision of our AI-based agent for our final meeting.

Third meeting: Integral - Area
Students engaged mathematically and culturally through a study of Eastern and Western contributions to calculus. Students calculated areas under curves, comparing methods used by 16-17th century mathematicians. Students constructed an example "where the value of the area under the graph of a function may not be found" and explain why.



FINDINGS

The Agent Matters: Sometimes the AI leaves out important steps in the calculations. For example, the agent described how Isaac Newton would have calculated the area under f(x) not explaining the evaluation of the anti-derivative at the limits of integration. “if we think, his approach is not very clear. Shall we ask the agent to explain more about that?”

IMPACT

The combination of the AI-agent and tasks designed to engage students' vocational aspirations strengthened student understanding of calculus and culture in ways not always possible with more traditional methods. This highlights the relevance of culturally responsive teaching.

REFERENCES

- D'Mello, S. K., & Graesser, A. (2023). Intelligent tutoring systems: How computers achieve learning gains that rival human tutors. In *Handbook of educational psychology* (pp. 603-629). Routledge.
- Isgandarli, E. (2022). *The Relationship Between Student Performance on Aleks Software and Standardized Tests*. University of Missouri-Kansas City.

Discourse Between Learning Assistants and Calculus Students on Implicit Differentiation

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Keywords: Learning Assistants, Near-Peer Tutors, Calculus, Implicit Differentiation

Learning assistants (LAs) are undergraduate near-peer tutors who, having previously been successful in some course, aid in the instruction of that course, typically by interacting with individual students or groups (Otero et al., 2010). It has been shown that LAs' presence in classrooms is associated with positive student outcomes (Barrasso & Spilios, 2021), but there is little understanding of how LAs contribute to such outcomes. While there are some action-focused categorizations of LA practices, e.g., providing feedback and increasing discussion time (Knight et al., 2015; Thompson et al., 2020) they are not tied to specific content areas. This study contributes a subject-specific perspective on LA classroom practice by examining discourse between LAs and students on the topic of implicit differentiation in a Calculus I course.

LAs were incorporated into the course as one part of a cross-departmental effort to transform introductory STEM courses at a large, public, northeastern university. This Calculus I course consisted of a lecture (~160 students) taught by a faculty member and recitations (~20 students) taught by a graduate teaching assistant. LAs were present only in recitations, and their role was to facilitate small group discussions during work on conceptually oriented activities. This study is focused on one recitation lesson on implicit differentiation. Data consists of 360-video recordings of small groups and students' written work on the activity. Video clips were selected for analysis based upon the presence of the LA with the small group and conversation focus on implicit differentiation. Through this process, 19 clips (~25 minutes of video data) were identified for five LAs across two semesters. LA and student speech were transcribed and analyzed at the turn of talk level for mathematical content and discursive actions. Mathematical content was analyzed using components of a hypothetical learning trajectory (HLT) for implicit differentiation (Buchbinder & Allen, 2024). Discursive actions of LAs were analyzed using Thompson et al.'s (2020) Action Taxonomy for Learning Assistants while those of students were coded using an open process under the constant comparative method (Strauss & Corbin, 1998).

Initial analysis indicates that LAs aid students in learning this topic by coordinating across multiple components of the HLT and integrating previous calculus content knowledge into these components. For example, one LA shared their conceptualization of implicit differentiation as the chain rule, rewriting y as y(x), while another LA described the implicit differentiation procedures as the "normal" derivative then "multiplying by dy/dx". While students tended to ask LAs questions about algebraic procedures, LAs reoriented these questions toward sensemaking across graphical and symbolic representations of implicit equations.

This analysis of LA-student interactions around implicit differentiation sheds light on the nuances of how LAs support student mathematical learning of that content. The study has the potential to bolster our understanding of student learning of the topic of implicit differentiation and gain insight into LAs' instructional moves as they interact with students.

Acknowledgments

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References

Barrasso, A. P., & Spilios, K. E. (2021). A scoping review of literature assessing the impact of the learning assistant model. *International Journal of STEM Education*, *8*(12), 1-18. https://doi.org/10.1186/s40594-020-00267-8

Buchbinder, O. & Allen, M. (2024). *Calculus I students' understanding of implicit differentiation*. To appear in the *Proceeding of the 26th Annual Conference on Research in Undergraduate Mathematics Education*. Omaha, NE, United States.

Knight, J. K., Wise, S. B., Rentsch, J., & Furtak, E. M. (2015). Cues matter: Learning assistants influence introductory biology student interactions during clicker-question discussions. *CBE Life Sciences Education*, *14*(4), 1-14. https://doi.org/10.1187/cbe.15-04-0093

Otero, V., Pollock, S., & Finkelstein, N. (2010). A physics department's role in preparing physics teachers: The Colorado learning assistant model. *American Journal of Physics*, *78*(11), 1218-1224. https://doi.org/10.1119/1.3471291

Thompson, A. N., Talbot, R. M., Doughty, L., Huvard, H., Le, P., Hartley, L., & Boyer, J. (2020). Development and application of the action taxonomy for learning assistants (ATLAs). *International Journal of STEM Education*, *7*(1), 1-14. https://doi.org/10.1186/s40594-019-0200-5

Strauss, A., & Corbin, J. (1998). Basics of qualitative research: Techniques and procedures for developing grounded theory. Sage Publications, Inc.

Discourse Between Learning Assistants and Calculus Students on Implicit Differentiation

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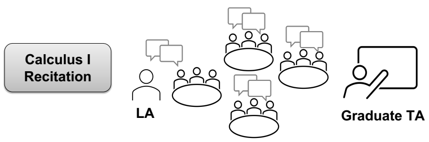
NSF Award No. 2013427. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Background

The Learning Assistant (LA) Model
(Otero et al., 2010)

Learning Assistant: A current undergraduate student who has passed some course and who aids in the instruction of that course.
- **Practice:** LAs help teach students in Calculus I recitations
- **Content:** Weekly meeting with course instructor to refresh course content
- **Pedagogy:** Weekly meetings with a mentor to discuss educational topics (e.g., metacognition)



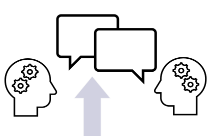
Motivation: Students in LA-supported courses show more *positive content-related outcomes* compared to students without LA support (Barrasso & Spilios, 2021), but little is known about *how* LAs facilitate positive student outcomes. Students *struggle with implicit differentiation* (Martin, 2000; Chu, 2019) offering opportunities for LAs to authentically aid student learning.

Research Question
What mathematical aspects of implicit differentiation do students and LAs discuss during their interactions in Calculus I recitations and ***how*** do they discuss them?

The Study

Setting: Part of a multi-disciplinary project for reforming introductory STEM courses at a large, public, northeastern university.
LAs teach in Calculus I recitations (~20 students each), facilitating small group discussions about conceptually rich activities.
Participants: 5 LAs and 7 different small groups of students; 3-4 students per group.
Data sources: 22 different 360°-video recordings of LA-student conversations on the topic of implicit differentiation
~25 mins of video data across two semesters.

Theoretical Framework

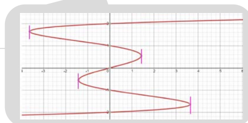


Vygotsky's (1978) Sociocultural Theory
Learning is fundamentally social and mediated by language use

Critical Discourse Analysis (Gee, 2014)		
	Saying (Information)	Utterances coded for *topic* of implicit differentiation discussed (Buchbinder & Allen, 2024)
	Doing (Action)	Utterances coded for *student actions* (open coding) and *LA actions* via ATLA (Thompson et al., 2019)
	Being (Identity)	Speech interpreted within the context of near-peer tutoring in Calculus 1

ID Topics (N=250)

Graphical 37% | Symbolic 26% | Algebra 24% | Calculus 13%



$$x = y^5 - 5y^3 + 4y$$
$$\frac{dy}{dx} = \frac{1}{5y^4 - 15y^2 + 4}$$

"How do you draw the *tangent line*?"
"At each point of *inflection*…"
"So, what is the meaning of *dy/dx*?"

Trend: Conversations focus on prior knowledge and symbolic representations, graphs discussed in connection to computations.

LAs' Actions (N=140)

Directed facilitation 52% | Feedback 33% | Guided facilitation 13% | Advice 2%

"So, if you solve this plugging in one for all the y's, you'll get it."
"You can use the quadratic equation to solve for y squared"

"Yeah, um, I think you just forgot to add the powers at the end, right?"
"Yeah, that's what you did there. Oh, so yes, this is correct."

"Do you see any other points that look like they behave the same way?"
"So, how would you draw the tangent line there?"

Trend: LAs tend to explain concepts and give feedback (e.g., correcting, confirming) on students' work.

Students' Actions (N= 110)

Ask for directions or reassurance 44% | Following LA Instruction 32% | Explaining work or thinking 24%

"Is this right? Like am I on the right track?"
"How do you draw the tangent line?"

"That would make a lot more sense!"
"Oh, my gosh. You're so right."

"So, I said the derivative of y in terms of x the x derivative."
"Okay, I did the (0,-1), but didn't get the same."

Trend: Students seek out LAs to ask questions about next steps. They follow LAs' explanations, but also explain their thinking.

Example of Discourse

ID Topic | Student Actions | LA Actions

Symbolic Evaluation
Ask for directions → Examine student work

Are there other points on the graph, where the slope of the tangent line is the same as you found at (0,1)? Respond using the graph, and then verify algebraically.

Graphic Coordination
Examine student work → Explain next step → Explain thinking → Point out incorrectness → Elicit student thinking → Checking work → Confirm correctness → Explain next step

Significance
The student begins at the symbolic level, but the ***LA engages with graphical representations in complex ways***, before returning to the pragmatic matter of symbols

Symbolic Evaluation
Explain next step → Confirm correctness

Responding to the Research Question

- LA-Student discourse is often *pragmatic*, focusing on correctness of mathematical *procedures* and connections to *previous knowledge*.
- *Implicit differentiation* topics: graphs in connection to computations.
- Students ask LAs questions about next steps, follow LAs' explanations, but also explain their thinking.

Link to paper PDF

References

Barrasso, A. P., & Spilios, K. E. (2021). A scoping review of literature assessing the impact of the learning assistant model. *International Journal of STEM Education*, *8*(12), 1-18. https://doi.org/10.1186/s40594-020-00267-8
Buchbinder, O. & Allen, M. (2024). Calculus I students' understanding of implicit differentiation. To appear in the *Proceeding of the 26th Annual Conference on Research in Undergraduate Mathematics Education*. Omaha, NE, United States.
Chu, C. G. (2019). *Investigation of student understanding of implicit differentiation*. [Doctoral dissertation, The University of Maine]. ProQuest Dissertations and Theses Global.
Gee, J. P. (2014). *An introduction to discourse analysis: Theory and method*. Routledge.
Martin, T. (2000). Calculus students' ability to solve geometric related rates problems. *Mathematics Education Research Journal*, *12*(2), 74-91.
Otero, V., Pollock, S., & Finkelstein, N. (2010). A physics department's role in preparing physics teachers: The Colorado learning assistant model. *American Journal of Physics*, *78*(11), 1218-1224. https://doi.org/10.1119/1.3471291
Thompson, A. N., Talbot, R. M., Doughty, L., Huvard, H., Le, P., Hartley, L., & Boyer, J. (2020). Development and application of the action taxonomy for learning assistants (ATLAs). *International Journal of STEM Education*, *7*(1), 1-14. https://doi.org/10.1186/s40594-019-0200-5
Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* (M. Cole, V. John-Steiner, S. Scribner, E. Souberman, Eds.). Harvard University Press.

Results of Analysis Using the Social Justice Syllabus Design Tool

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Keywords: syllabus, social justice, equity

The re-humanizing of mathematics calls for a variety of classroom-level changes (Su, 2020). Supporting faculty to transform their views of self, students, and authority can begin with the course syllabus (Palmer et al., 2016). This report gives results of the authors' analysis of 39 mathematics class syllabi, from 27 instructors, based on the Social Justice Syllabus Design Tool (Taylor et al., 2019). The tool has 25 questions, each aligned with at least one of three categories: *community, process, relationships*. In using the tool to design a syllabus, one answers reflection questions and makes adjustments to bring the syllabus into greater alignment with recommended practices (i.e., an answer of "Yes"). For this analysis, reviewers answered each of the tool's questions with Yes (1) or No (0) and had the option of offering notes about their selection (e.g., statements from the syllabus that prompted a "Yes" or "No").

On the tool were 10 questions aligned with the community category so 10 opportunities on each syllabus for a "1" in community, 4 opportunities in process, and 11 in the relationship category. Some questions were reverse-coded (i.e., where "No" indicated the *presence* of justice-supportive language). Thus, there were 10+4+11=25 "points" possible for each syllabus.

As an example of poster content, across the 19 syllabi that were for 100-level courses, there were 25*19=475 points possible. The leftmost chart in Figure 1 indicates the proportion of points, out of all 475 possible, aligned with each of the three categories (the gray area indicates the remaining proportion of potential points---social justice foci not present). Similar analysis of nine 200-level course syllabi led to the rightmost chart in Figure 1.

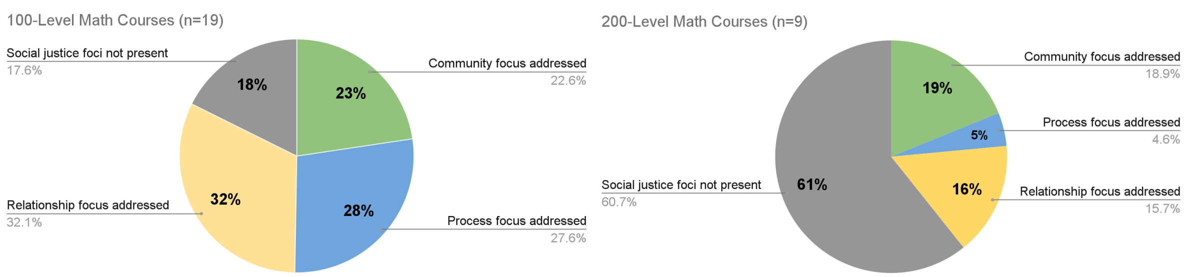


Figure 1. Charts for 100-level and 200-level courses.

In addition to other visuals and comparisons, the poster shares wording. For example, the following illustrate justice-aligned wording (coded with the answer "No" on the reverse-scored question "Does the syllabus read like a legal contract?"):

> This course aims to offer a joyful, meaningful, and empowering experience to every participant.
>
> We (you, your classmates and I) will experience many different class relationships and mathematical ideas this semester. This syllabus documents significant goals and expectations for our class and is a resource for the development of our particular Statistics learning community.

References

Palmer, M. S., Wheeler, L. B., & Aneece, I. (2016). Does the document matter? The evolving role of syllabi in higher education. *Change: The Magazine of Higher Learning*, *48*(4), 36-47.

Su, F. (2020). *Mathematics for human flourishing*. Yale University Press.

Taylor, S. D., Veri, M. J., Eliason, M., Hermoso, J. C. R., Bolter, N. D., & Van Olphen, J. E. (2019). The Social Justice Syllabus Design Tool. *Journal Committed to Social Change on Race and Ethnicity (JCSCORE)*, *5*(2), 133-166.

Results of Analysis Using the Social Justice Syllabus Design Tool

San Francisco State University

Jonathan Balagot, Jordan Guillory, Shandy Hauk, Rebeca Hernandez, Thomas Kunze, Sandra Torres

Background

Poster Purpose: Offer and collect ideas for making sense of (and making sense with) justice as an essential component in syllabus design.

Abstract: Supporting faculty to transform their views of self, students, and authority can begin with the course syllabus. This poster gives results of analysis of 39 mathematics syllabi written by 27 different instructors in a single department. Analysis used Taylor et al. (2019) *Social Justice Syllabus Design Tool.*

Theoretical Perspective

Relationship: *Emphasizes the messages that students receive from a syllabus about the type of relationship that is expected to occur between faculty and student* (p. 147).

- Roles are implicit in "relationships"
- Clear outline for desired student-instructor relationship in their syllabi.

Community: *Highlights the way a syllabus either promotes or inhibits a collaborative and inclusive environment* (p. 150).

- Inherently tied to inclusivity.
- Can establish with collaborative language.

Process: *Draws attention to…how learning can and will occur…how faculty regulate emotions … [and] judgements about student behaviors, and grading outcomes* (p. 153).

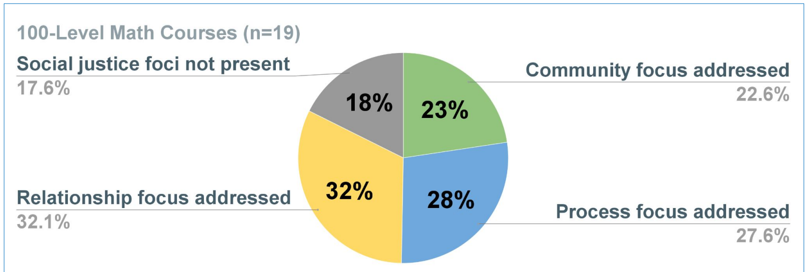
- Grading, practicing, risk-taking
- Success-oriented language (e.g., productive struggle is valuable)

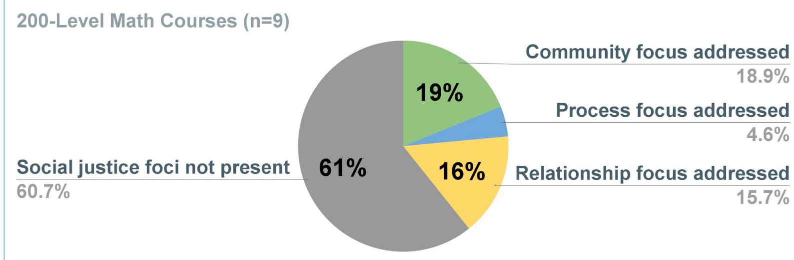
In what ways does a syllabus acknowledge, act, and hold us accountable for increasing justice? What is the evidence of it, for you?

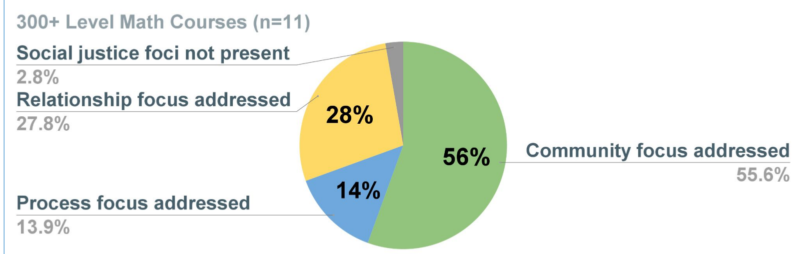
Reference

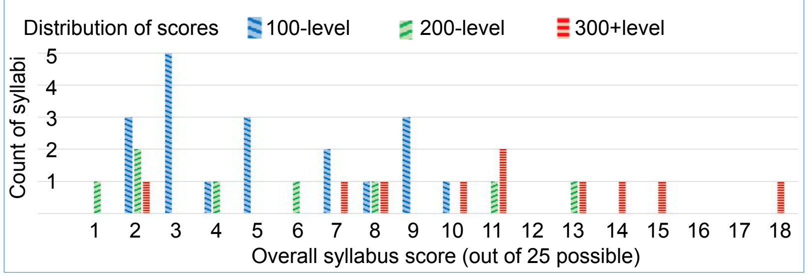
Taylor et al. (2019). The social justice syllabus design tool: A first step in doing social justice pedagogy. *Journal Committed to Social Change on Race and Ethnicity*, *5*(2), 133–166.

Comparisons









Examples

Learning mathematics is all about practice, which means: struggling through confusion, making mistakes, receiving feedback, and having an opportunity for growth. (Process)

We all go through difficult times that can affect our academic life. If you feel this way, regardless of the reason, please do not hesitate to talk to me and ask for extra time or resources. Your well-being is most important. (Relationship)

Please offer and ask (your classmates and me) for help with the material and assignments. Mathematics is often best learned with others, even if that can be a bit uncomfortable sometimes. Start your work early. Come to student hours. Form study groups. (Community)

We (you, your classmates and I) will experience many different class relationships and mathematical ideas this semester. This syllabus documents significant goals and expectations for our class. This syllabus is a resource about the structure, content, and processes for the development of our particular class community. (Community)

What More? What Next?

- Implications for Practice
  - revisit and revise syllabus each term
- Implications for Teaching-focused Professional Development
  - practice with exemplars to support syllabus refinement
- Implications for Research
  - include syllabus score as a covariate
  - explore undergraduate response to syllabi with various score profiles

International and Non-International Students' Experiences in Chavrusa-Style Mathematics Courses

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Keywords: Chavrusa, International Students, Collaborative Learning

A significant group within increasingly diverse U.S. classrooms is international students, whose learning success involves social elements such as ensuring feelings of belonging, respect for one's knowledge, and creative, effective, and confident participation in one's learning environment (Ryan and Viete, 2009). Collaborative learning methods are effective pedagogical techniques to address these needs as they allow learners to interact with each other and build a sense of belonging while exposing each other to varied socio-cultural perspectives (Huijser et al., 2008; Barkley & Cross, 2014). Chavrusa-style learning, modeled after Jewish Talmudic instruction, is a pedagogical technique where students are paired long term and asked to construct understanding from learning materials through debate (Pace, 1992; Kent, 2010). Research by Flint & Mei (2020) indicated that when implemented in mathematics courses, students felt chavrusa-style learning (MathChavrusa) broke down classroom barriers and improved the overall course experience. Due to the heavily social nature of MathChavrusa, it was hypothesized that this method may impact international and non-international students differently. Thus, the purpose of this research was to understand the experiences of international and non-international students in chavrusa-style mathematics courses.

Method

Participants consisted of a total of six international (3) and non-international (3) students who had taken graduate mathematics courses using chavrusa-style learning in the Fall 2021 semester. Data collection consisted of one online semi-structured interview for each participant and eight optional in-person classroom observations. The interview covered participant's impressions of MathChavrusa and perceptions of how it impacted their learning and socialization experiences. Classroom observations focused on partner interactions and were recorded using field notes. Data analysis consisted of coding interview transcripts and field notes using provisional and in vivo coding to develop themes of participants' learning and socialization experiences.

Findings and Future Research

International and non-international students perceived MathChavrusa to be beneficial when developing their mathematical understanding as it provided students with easily accessible sources of support when grappling with new material. Social and schedule compatibility of MathChavrusa partners were issues raised by all participants. The former was influenced by the mode in which MathChavrusa was implemented (i.e., in-person, hybrid, and online learning), with participants noting hybrid learning particularly undermined collaboration and socialization. A benefit noted exclusively by international student participants was that the long-term nature of partnerships allowed them to more easily build relationships that extended beyond the classroom and course. The poster will provide more granular descriptions and discussions of participants' experiences, including the impact of modes of instruction, participant pairings, and individual preferences. In the future, we hope to examine differences in MathChavrusa when used in specific modes of instruction.

References

Barkley, E.F. & Cross, K.P. (2014). Collaborative learning techniques: A handbook for college faculty.

Flint, R. & Mei, B. (2020). Chavrusa-style learning in mathematics classrooms: Instructor and student perspectives. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*. 201-209.

Huijser, H., Kimmins, L., & Evans, P. (2008). Peer assisted learning in fleximode: Developing an online learning community. *Australasian Journal of Peer Learning, 1*(1), 51-60.

Kent, O. A Theory of Havruta Learning. In J.A. Levisohn & S.P. Fendrick (Eds), *Turn it and turn it again*, 286–322. Academic Studies Press.

Pace, A. J. (1992). *Two thousand years of interactive readers: The Jewish tradition of text study and commentary*. Washington, D.C.: ERIC Clearinghouse.

Ryan, J., & Viete, R. (2009). Respectful interactions: Learning with international students in the English-speaking academy. *Teaching in Higher Education, 14*(3), 303-314.

A Growing Need for Asian (American) Perspectives in Math Education

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Undergraduate mathematics features a significant presence of Asian[1] students, whether international or American (NCES). These students naturally bring their views on learning mathematics from their cultural experiences. However, researchers often adopt theories of learning and teaching that may not fully incorporate the students' viewpoints when developing instructional materials and analyzing student reasoning. This proposal aims to foreground Asian scholarship by reviewing existing literature in this regard and intends to open a venue for discussing ways the RUME community could begin addressing this issue.

Keywords: Asian perspectives, Cultural differences, Dispositions, Equity

Despite the prominence of Asian (American) students in STEM classrooms, and Leung's (2001) declaration that an awareness of Asian scholarship "should contribute in a more meaningful way to a sharing of best practices with other cultural traditions," studies rarely reflect the nuanced views they bring when learning mathematics. To address this issue, we focus on two aspects: 1) dispositions - exploring instances where Asian (American) students may uniquely experience mathematics, and 2) the Asian Classroom Paradox - examining the perspectives of math education researchers that render their common frameworks inadequate.

Dispositions.

We explore instances that may be interpreted differently among students. When teaching undergraduate students in the US classroom, we often encounter concerns expressed by those who struggle in mathematics, such as 'math is not my thing.' (Nisbett, 2003; Whang, 1994) The researchers observed that Asian students facing challenges in mathematics tend to perceive it as requiring additional effort, while others may view it as a lack of inherent mathematical ability. Such differences also extend to classroom behavior. (Ing & Victorino, 2016; Kim, 2002) We found literature that may help locate the roots of these dispositions among Asian (American) students, emphasizing memorization, practice, and a perception of education rooted in Confucian dialectics instead of Socratic dialectics. (Fan et al., 2021; Kitayama et al., 2003; Marton et. al, 2005; Wang & Cai, 2007)

Asian Classroom Paradox.

Next, we draw particular attention to studies concerning student-teacher dynamics in Asian mathematics classrooms. This paradox involves a disconnect between assumptions regarding Asian classroom environments and their associated mathematical performance. Western researchers have viewed the Asian classroom as less autonomous than the US classroom pointing out the prevalence of directive lectures from teachers as the primary mode of learning. However, recent studies (Zhou et al., 2012) claim that such a finding is only paradoxical because cultural differences are not accounted for. These studies found that a Confucian framework describes such lectures as student-centered and better discerns the relationship that Asian students have with authority figures like teachers. (Jiang et al., 2021; Li, 2016; Chan et al., 2017)

[1] Asian students include students from Southeast Asia (Philippines, Singapore, Malaysia etc…), South Asia (India, Pakistan, Nepal, etc…), and East Asia (Korea, Japan, China etc…). Due to the positionality of the authors, this proposal is mainly focused on East Asian scholarship.

References

Chan, T., Jiang, B., & Xu, M. M. (2017). US Teachers' Perception of Confucian Teaching Philosophies and Methodologies. New Waves-Educational Research and Development Journal, 20(2), 20-34.

Fan, G., Carlson, K. D., & Thomas, R. D. (2021). Individual Differences in Cognitive Constructs: A Comparison Between American and Chinese Culture Groups. Frontiers in Psychology, 12, 614280. https://doi.org/10.3389/fpsyg.2021.614280

Ing, M., & Victorino, C. (2016). Differences in classroom engagement of Asian American engineering students. Journal of Engineering Education, 105(3), 431-451.

Jiang, Y., Lee, C. K. J., Wan, Z. H., & Chen, J. (2021). Stricter teacher, more motivated students? Comparing the associations between teacher behaviors and motivational beliefs of Western and East Asian learners. Frontiers in Psychology, 11, 564327.

Kitayama, S., Duffy, S., Kawamura, T., and Larsen, J. T. (2003). Perceiving an object and its context in different cultures: a cultural look at new look. Psychol. Sci. 14, 201–206. doi: 10.1111/1467-9280.02432

Kim, H. S. (2002). We talk, therefore we think? A cultural analysis of the effect of talking on thinking. *Journal of Personality and Social Psychology*, *83*(4), 828–842. https://doi.org/10.1037/0022-3514.83.4.828

Li, J. (2016). Humility in learning: A Confucian perspective. *Journal of Moral Education*, *45*(2), 147-165.

Leung, F. K. (2001). In search of an East Asian identity in mathematics education. *Educational studies in mathematics*, *47*, 35-51.

Marton, F., Wen, Q., & Wong, K. C. (2005). 'Read a hundred times and the meaning will appear...'Changes in Chinese University students' views of the temporal structure of learning. Higher Education, 49, 291-318.

National Center for Educational Statistics. (n.d.). Status and Trends in the Education of Racial and Ethnic Groups, Indicator 26: STEM Degrees. Indicator 26: STEM degrees. https://nces.ed.gov/programs/raceindicators/indicator_reg.asp

Nisbett, R. E. (2011). The geography of thought : how Asians and Westerners think differently-- and why. Nicholas Brealey.

Wang, T., & Cai, J. (2007). Chinese (Mainland) teachers' views of effective mathematics teaching and learning. ZDM, 39, 287-300.

Whang, P. A., & Hancock, G. R. (1994). Motivation and mathematics achievement: Comparisons between Asian-American and non-Asian students. *Contemporary Educational Psychology*, *19*(3), 302-322.

Zhou, N., Lam, S.-F., & Chan, K. C. (2012). The Chinese classroom paradox: A cross-cultural comparison of teacher controlling behaviors. Journal of Educational Psychology, 104(4), 1162–1174. doi:10.1037/a002760

The Need for Asian Perspectives in RUME

Matt Furuta Park
Inyoung Lee

Introduction

"There exists an East Asian identity in mathematics education, and an awareness of that identity should contribute in a more meaningful way to a sharing of best practices with other cultural traditions" -Leung (2001)

Examples

These papers are examples of those which touch on their given construct, yet lack solid epistemological foundations by Western standards.

- Mimicry/Imitation (Zhou, J., Guo, W. (2016). Imitation in Undergraduate Teaching and Learning. Journal of Effective Teaching, 16(3), 5-27.)
- Practice (Wang, T., Cai, J. (2007). Chinese (Mainland) teachers' views of effective mathematics teaching and learning. ZDM, 39, 287-300.)
- Memorization (Marton, F., Wen, Q., Wong, K. C. (2005). 'Read a hundred times and the meaning will appear...'Changes in Chinese University students' views of the temporal structure of learning. Higher Education, 49, 291-318.)

Conclusion

The RUME community should incorporate more non-Western philosophy to enrich our learning theories.

Math education research from the West touches upon constructs or phenomena that is not well represented in the literature. Ideas from Asia can help fill in such theoretical gaps. Please join our working group if you are interested in such work.

Full Paper

Survey

Working Group

Additional Readings

- Ausman, M. C., Zhu, Q. (2023). Illuminating the APIDA Experience in Engineering Education: A Scoping Review. 2023 Collaborative Network for Computing and Engineering Diversity (CoNECD).
- Okura, K. (2021). There are no Asians in China: the racialization of Chinese international students in the United States. Identities, 28(2), 147-165.

Additional Information

Take our survey

- What are some things in math education that you feel are important but underrepresented in how math education research is done in the West?
- Briefly describe American cultural practices or attitudes regarding teaching and learning that initially surprised you.

Working Group Goals

- Host a Working Group at the next RUME.
- Get a talk submitted to PMENA and RUME.
- Draft at least two papers integrating the constructs on the left into existing, popular epistemological frameworks.

Argument Mapping for Proof Comprehension

Andrew Baas
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Keywords: argument mapping; proofs; proof comprehension

Proof comprehension has long been recognized as a critical skill for mathematics students (Conradie & Frith, 2000; Cowen, 1991). Cowen specifically states that, while most students will not be proving theorems after graduation, many will need to be able to read and understand mathematical writing. While undergraduate proof comprehension has historically received little attention from researchers (Mejía-Ramos & Inglis, 2009), this has begun to change in the last decade due to theoretical-methodological advancements like the assessment model developed by Mejía-Ramos et al. (2012). These models have overcome challenges inherent to measuring students' proof comprehension, enabling researchers to evaluate efficacy of methods developed to improve students' proof comprehension. For example, two empirical studies have shown that proof presentations which have been assumed to improve proof comprehension (Alcock & Inglis, 2008; Rowland, 2002) actually fail to improve student proof comprehension compared to traditional proof presentations (Lew et al., 2020; Roy et al., 2017).

Borrowing from research in philosophy education, I present argument mapping as a promising instructional technique for improving students' proof comprehension. Argument mapping (also called argument diagramming) is the practice of converting a written argument into a graph-like representation which depicts the logical relationships among statements within the argument. One form of argument mapping, Toulmin diagrams, has been extensively studied both as an analytic tool to investigate proof production (e.g. Aberdein, 2006; Corneli et al., 2019; Pease et al., 2009) and as structured representation to facilitate collaborative argumentation (Wagner et al., 2014; Zambak & Magiera, 2020). Researchers in philosophy education have investigated how argument mapping techniques can improve students' critical thinking (Harrell, 2012; Twardy, 2004). Twardy (2004) demonstrated that first year philosophy students who were taught argument mapping had significantly higher gains on the California Critical Thinking Skills Test (CCTST) than those taught traditionally. Similarly, Harrell (2012) showed that students taking an introduction to philosophy course which taught argument mapping had significantly higher gains on a critical thinking-focused pre/post-test than those who had traditional instruction. While critical thinking in philosophy is not identical to proof comprehension, the skills measured by the CCTST and by Harrell's critical thinking test seem to align with some of the dimensions of proof comprehension laid out by Mejia-Ramos et al. (2012). For example, one goal of Harrell's test was to test whether students were able to "determine how the premises are supposed to support the conclusion" (2012, p. 34), which aligns with the dimension of comprehension "justification of claims" (Mejía-Ramos et al., 2012, p. 9).

Based on this research in the field of philosophy education, argument mapping has the potential to improve student comprehension of proofs. For this poster, I will report a literature review of the current use of argument diagramming in mathematics education and in adjacent fields (like philosophy and logic) and present a study design which seeks to address the question: how does diagramming the argument of a proof impact a student's comprehension of that proof?

References

Aberdein, A. (2006). Managing Informal Mathematical Knowledge: Techniques from Informal Logic. In J. M. Borwein & W. M. Farmer (Eds.), *Mathematical Knowledge Management* (pp. 208–221). Springer. https://doi.org/10.1007/11812289_17

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics*, *69*(2), 111–129. https://doi.org/10.1007/s10649-008-9149-x

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics*, *42*(3), 225–235. https://doi.org/10.1023/A:1017502919000

Corneli, J., Martin, U., Murray-Rust, D., Rino Nesin, G., & Pease, A. (2019). Argumentation Theory for Mathematical Argument. *Argumentation*, *33*(2), 173–214. https://doi.org/10.1007/s10503-018-9474-x

Cowen, C. C. (1991). Teaching and Testing Mathematics Reading. *American Mathematics Monthly*, *98*, 50–53.

Harrell, M. (2012). Assessing the Efficacy of Argument Diagramming to Teach Critical Thinking Skills in Introduction to Philosophy: *Inquiry: Critical Thinking Across the Disciplines*, *27*(2), 31–39. https://doi.org/10.5840/inquiryct201227210

Lew, K., Weber, K., & Mejía Ramos, J. (2020). Do Generic Proofs Improve Proof Comprehension? *Journal of Educational Research in Mathematics*, *30*, 229–248. https://doi.org/10.29275/jerm.2020.08.sp.1.229

Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3–18. https://doi.org/10.1007/s10649-011-9349-7

Mejía-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, *2*, 88–93.

Pease, A., Smaill, A., Colton, S., & Lee, J. (2009). Bridging the Gap Between Argumentation Theory and the Philosophy of Mathematics. *Foundations of Science*, *14*(1–2), 111–135. https://doi.org/10.1007/s10699-008-9150-y

Rowland, T. (2002). Generic proofs in number theory. In *Learning and teaching number theory: Research in cognition and instruction* (pp. 157–183). Ablex Publishing.

Roy, S., Inglis, M., & Alcock, L. (2017). Multimedia resources designed to support learning from written proofs: An eye-movement study. *Educational Studies in Mathematics*, *96*(2), 249–266. JSTOR Journals. https://doi.org/10.2307/45185371

Twardy, C. (2004). Argument Maps Improve Critical Thinking. *Teaching Philosophy*, *27*(2), 95–116. https://doi.org/10.5840/teachphil200427213

Wagner, P. A., Smith, R. C., Conner, A., Singletary, L. M., & Francisco, R. T. (2014). Using Toulmin's Model to Develop Prospective Secondary Mathematics Teachers' Conceptions of Collective Argumentation. *Mathematics Teacher Educator*, *3*(1), 8–26. https://doi.org/10.5951/mathteaceduc.3.1.0008

Zambak, V. S., & Magiera, M. T. (2020). Supporting grades 1–8 pre-service teachers' argumentation skills: Constructing mathematical arguments in situations that facilitate analyzing cases. *International Journal of Mathematical Education in Science and Technology*, *51*(8), 1196–1223. https://doi.org/10.1080/0020739X.2020.1762938

Distributed Leadership in Action: The PAC-Math NIC

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Appalachian State University.

Keywords: Distributed Leadership, Networked Improvement Communities

Distributed leadership (DL) abandons the heroism of an individual leader, and instead, capitalizes on participants' expertise by creating structures that encourage collaboration among an overlapping web of leaders and followers. Our prior work presented a new way of framing DL as 'flipping the spider' of traditional organizational hierarchy (Clark et al., 2022). Our previous research explored how DL-informed workshops can motivate faculty development (Clark et al., 2022) and teacher education (Miller & Braley, 2021). In our current project, we build on this and expand the work to explore how DL works within a Networked Improvement Community (NIC) (Bryk et al., 2015). NICs are described by four key characteristics: (1) the community aims to accomplish a clearly defined outcome, (2) members have a deep understanding of the common challenges of the community and are poised to adapt and create to make progress towards addressing needs, (3) the community action is informed by improvement science and strives for continuous inquiry and learning, and (4) members are coordinated to maximize effort towards addressing complex problems (LaMahieu, 2015). The DL structure provides opportunities for members to step into leadership roles according to their self-identified strengths and curiosities, while a core group of organizers and facilitators provide infrastructure and support.

In October 2023, the Program Assessment Conference for Mathematics (PAC-Math) (NSF award #2306211) brought together mathematics professional development practitioners, evaluation professionals, and higher education administrators to develop a flexible protocol for mathematics departments to use for self-evaluations of their Professional Development for Teaching (PDT) programs for graduate student teaching assistants (GTAs). By replicating the DL model in this NIC context, we aim to provide more nuanced descriptions of the DL anatomy. PAC-Math will advance the frontiers of knowledge about how DL supports NICs.

In this poster, we will review research about DL and describe how DL principles were applied in the planning of the PAC-Math conference by involving all participants. We will highlight how the DL model capitalized on the diversity of voices and shared expertise of participants. We will present some of the outcomes of the conference, conclusions of the external conference evaluation, and how the results of the conference will inform how DL might interplay differently in a NIC than in faculty development. We look forward to hearing from the RUME community about their insights into and experiences with DL and working in NICs.

References

Bryk, A. S., L. Gomez, A. Grunow, & P. LeMahieu. (2015). *Learning to Improve: How America's Schools Can get Better at Getting Better.* Harvard Education Publishing.

Clark, L.J., Braley, E. & Bookman, J. Distributed Leadership: Structure for STEM Faculty Development Workshops. In S.M. Linder, C. Lee & K. High (Eds.) (2022). *Handbook of STEM Faculty Development*. Information Age Publishing.

LaMahieu, P. (2015, August 18). Why a NIC? *Carnegie Foundation for the Advancement of Teaching*. https://www.carnegiefoundation.org/blog/why-a-nic/

Miller, E. R., & Braley, E. A distributed leadership model for online reading groups. In K. Hollebrands, R. Anderson, & K. Oliver (Eds.) (2021). *Online Learning in Mathematics Teacher Education.* Springer International Publishing.

Assessing Graduate Student Professional Development for Teaching:
Needs and Progress With PAC-Math

Jack Bookman, Duke University
Emily Braley, Johns Hopkins University
Jeneva Clark, U. of Tennessee Knoxville

The Program Assessment Conference in Mathematics (PAC-Math, DUE# 2036211) was hosted in October 2023 at the University of Tennessee Knoxville. The conference produced key components to build a protocol for mathematics departments to self-assess their professional development for teaching provided to graduate students in mathematics. We describe previously presented results of a census survey of mathematics PhD programs conducted in Fall 2021 that reinforces the need in higher-ed mathematics for tools to do this assessment work, confirming results from a similar survey conducted in 2015. In this poster we outline the protocol, departmental resources needed to undertake this assessment work and questions of interest for the RUME community.

Keywords: Graduate Student Teaching Assistants, Professional Development, Program Assessment

Most PhD-granting mathematics departments provide Professional Development for Teaching (PDT) to Graduate student Teaching Assistants (GTAs) (Braley et al., 2023; Rasmussen et al., 2019). To sustain and improve these programs, departments and administrators need to be able to assess program effectiveness and establish assessment cycles that inform continuous improvement. From surveys conducted in 2015, in partnership between the CoMInDs project (NSF#1432381) and Progress Through Calculus project (NSF#1430540), and in 2021 by the CoMInDS project, we know that evaluation of GTA PDT has been extremely limited. For example, despite reservations of their validity, student evaluations of teaching are used as a primary evidence source to assess ongoing teacher training efforts (Braley et al., 2023; Speer et al., 2017). Survey respondents also indicated that tools for evaluation would be the one of the most valuable resources to help improve PDT programs.

The PAC-Math (Program Assessment Conference in Mathematics, DUE# 2036211) project aims to improve program assessment practices for PDT programs for GTAs in mathematics. PAC-Math convened a diverse group of experts and oriented them to (1) the current landscape of GTA PDT in PhD granting mathematics departments and (2) the need and desire in the mathematics community to assess PDT. In October 2023 PAC-Math participants met in-person to draft a flexible assessment protocol for self-assessment of PDT programs.

This poster will describe the background and evidence supporting the mathematics community's need for GTA PDT assessment tools or guides. It will provide an outline of the draft protocol developed through the work of the PAC-Math participants with a focus on stakeholders roles and needs, as well as the differing needs at various institutions. We invite the RUME community to scrutinize our results and make suggestions for improving the PAC-Math Protocol. We hope to recruit partners for future work in this area.

References

Braley, E., Bookman, J., Gehrtz, J & Speer, N. A Survey of Programs Preparing Graduate Students to Teach Undergraduate Mathematics. In Cook, S., Katz, B. & Moore-Russo, D. (Eds.) (2023). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. Omaha, NE.

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, *50*(1), 98-111.

Speer, N., Ellis, J., & Deshler, J. Evaluation of Graduate Student Professional Development and Instruction by Mathematics Departments: Results from a National Survey. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.) (2017). *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, California.

PAC MATH

Ready!

ONCE UPON A TIME

- What preparation would Math Departments do before evaluating their GTA PDT programs?
- How would they identify goals?
- What tools, such as logic models or key performance indicators, could help the evaluation planning process?
- Would it help to describe the history of and current state of the program?

THE PLOT

- What data could be collected?
- What are pros and cons for types of data collection?
- Who are the stakeholders and what would each of the stakeholders consider as evidence of success?
- How does data collection reflect the voices and interests of stakeholders?
- What analyses can be done?

PAC MATH

MORAL OF THE STORY

- What form should results take?
- How are results communicated?
- How should stakeholders' responses to results be reflected in reporting?
- How do stakeholders provide feedback to drive continuous improvement and positive change?
- How can the assessment cycle be continued and connected to the broader missions?

Game Not Over!

PAC MATH

SCAN ME

Duke

APPSTATE 125

PILOT DEPARTMENTS

UCONN

CLEMSON UNIVERSITY

h in Undergraduate Mathematics

Mathematics Motivation in Students Enrolled in an Introductory Biology Course

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University of New Hampshire

Melissa Aikens
University of New Hampshire

Keywords: Mathematics Student Motivation, Quantitative Biology, Task Value, Self-Efficacy

Over the last two decades, the field of biology has become increasingly more quantitative leading to more mathematics being integrated into biology education curriculum (Brewer & Smith, 2011). However, students enrolled in biology courses often report having negative feelings towards mathematics, resulting in them being resistant to learning course material (Thompson et al., 2013). To understand these negative feelings, this study focuses on identifying patterns of mathematics motivation among undergraduate students enrolled in a gateway biology course. To do this, the study utilizes Expectancy Value Theory (EVT; Eccles et al., 1983), which proposes that an individual's expectancy for success and their subjective task values are predictors of academic choice, persistence, and performance. Subcomponents of values are (1) utility value, or how useful the task is for future goals, (2) interest value, or the intrinsic interest of a task, (3) attainment value, or the importance of doing well, and (4) cost, or the accumulated negative aspects of engaging in the task. This study looks at two dimensions of cost: emotional cost and effort required to complete the task (Wigfield & Eccles, 2000).

Participants include 538 students in an introductory biology course at a large research-intensive university. At the start of the course, students completed a 7-point Likert Scale survey that examined six variables including self-efficacy, emotional cost, effort cost, intrinsic value, mathematics attainment value, and utility value. Self-efficacy was measured as an indicator of expectancy of success (Eccles & Wigfield, 1995). Each subscale was taken or modified from previously validated surveys (Andrews et al., 2017; Flake et al., 2015; Gaspard et al., 2015; Glynn et al., 2011). The study used Confirmatory Factor Analysis, with "lavaan" R package (Rosseel, 2012) with maximum likelihood robust estimation (Knekta et al., 2019) to verify the factor structure of the variables. All fit indices (CFI, TLI, RMSEA, SRMR) were within suggested cutoff values as suggested by Hu & Bentler (1999) and Steiger (2007).

This study used hierarchical cluster analysis (Wards method and Euclidean distance), to find students with similar motivation characteristics based on task values (mathematics attainment, utility, and intrinsic value), cost (emotional and effort) and self-efficacy (expectancy of success). Using the "NbClust" package (Charrad et al., 2014), we found three different patterns of mathematics motivation within the data: low, moderate and high. The low group includes 45 students that place low value on mathematics (range of task value means: 1.76 - 3.91) and do not feel confident to complete the mathematics required in the course (self-efficacy mean: 2.93). Also, low motivation students felt that mathematics had high costs (emotional and effort cost means > 5.55). In contrast, the high group (164 students) place high value on mathematics and are more confident in the subject (task values and self-efficacy means > 5.60). In addition, students believe mathematics in the course has low costs (means < 2.50). The final group, moderate, is composed of 319 students who are moderately motivated to do mathematics. In this group, mathematics attainment value and utility value had higher means of 5.42 and 5.18, respectively. All other variable means were relatively close to 4. The findings show that students in the introductory biology course can be clustered into three groups (low, moderate, and high), with over half in the moderate cluster. For future analysis, we plan to further examine each cluster, with more of a focus on the moderate cluster.

References

Andrews, S. E., Runyon, C., & Aikens, M. L. (2017). The math–biology values instrument: Development of a tool to measure life science majors' task values of using math in the context of biology. *CBE—Life Sciences Education*, *16*(3), ar45. https://doi.org/10.1187/cbe.17-03-0043

Brewer, C. A., & Smith, D. (2011). Vision and change in undergraduate biology education: a call to action. American Association for the Advancement of Science, Washington, DC, 81. https://doi.org/10.1187/cbe.10-03-0044

Charrad, M., Ghazzali, . N., Boiteau, V., & Niknafs, A. (2014). NbClust: An R Package for Determining the Relevant Number of Clusters in a Data Set. Journal of Statistical Software, 61(6), 1–36. https://doi.org/10.18637/jss.v061.i06

Eccles, J., Adler, T. F., Futterman, R., Goff, S. B., Kaczala, C. M., Meece, J., and Midgley, C. (1983). Expectancies, values and academic behaviors. In Spence, J. T. (ed.), Achievement and Achievement Motives, W. H. Freeman, San Francisco.

Eccles, J. S., & Wigfield, A. (1995). In the mind of the actor: The structure of adolescents' achievement task values and expectancy-related beliefs. Personality and social psychology bulletin, 21(3), 215-225. https://doi.org/10.1177/0146167295213003

Flake, J. K., Barron, K. E., Hulleman, C., McCoach, B. D., & Welsh, M. E. (2015). Measuring cost: The forgotten component of expectancy-value theory. Contemporary educational psychology, 41, 232-244. https://doi.org/10.1016/j.cedpsych.2015.03.002

Gaspard, H., Dicke, A. L., Flunger, B., Schreier, B., Häfner, I., Trautwein, U., & Nagengast, B. (2015). More value through greater differentiation: Gender differences in value beliefs about math. Journal of educational psychology, 107(3), 663. https://psycnet.apa.org/doi/10.1037/edu0000003

Glynn, S. M., Brickman, P., Armstrong, N., & Taasoobshirazi, G. (2011). Science motivation questionnaire II: Validation with science majors and nonscience majors. Journal of research in science teaching, 48(10), 1159-1176. https://doi.org/10.1002/tea.20442

Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. Structural equation modeling: a multidisciplinary journal, 6(1), 1055. https://doi.org/10.1080/10705519909540118

Knekta, E., Runyon, C., & Eddy, S. (2019). One size doesn't fit all: Using factor analysis to gather validity evidence when using surveys in your research. CBE—Life Sciences Education, 18(1), rm1. https://doi.org/10.1187/cbe.18-04-0064

Rosseel, Y. (2012). lavaan: An R Package for Structural Equation Modeling. Journal of Statistical Software, 48(2), 1–36. https://doi.org/10.18637/jss.v048.i02

Steiger, J. H. (2007). Understanding the limitations of global fit assessment in structural equation modeling. Personality and Individual differences, 42(5), 893-898. https://doi.org/10.1016/j.paid.2006.09.017

Thompson, K. V., Nelson, K. C., Marbach-Ad, G., Keller, M., & Fagan, W. F. (2010). Online interactive teaching modules enhance quantitative proficiency of introductory biology students. CBE—Life Sciences Education, 9(3), 277-283. https://doi.org/10.1187/cbe.10-03-0028

Wigfield, A., & Eccles, J. S. (2000). Expectancy–value theory of achievement motivation. Contemporary educational psychology, 25(1), 68-81. https://doi.org/10.1006/ceps.1999.1015

Mathematics Motivation in Students Enrolled in an Introductory Biology Course

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Introduction

- Over the last two decades, the field of biology has become increasingly more quantitative leading to more mathematics being integrated into the biology curriculum (Brewer & Smith, 2011).
- However, students enrolled in biology courses often report having negative feelings towards mathematics, resulting in them not fully engaging in quantitative tasks (Thompson et al., 2013).

Research Questions:

1. What are the mathematics motivational patterns among undergraduate students who are enrolled in a gateway biology course?
2. To what extent is gender and pre-professional status related to these motivational patterns?

Data

Setting and Participants

- Course: Introductory biology course
  - Six sections over 2 years
- Data consists of 473 student survey responses who were life-science majors enrolled in the course.
  - Gender: Male = 121 & Female = 352
  - Pre-Professional Status: Pre-Professional = 211 & Non-Pre-Professional = 262
  - Highest math class taken in High School: Algebra, Geometry or Trigonometry = 70, Pre-Calculus = 202, & Calculus = 201

Instrument & Variables Measured

- The survey is made up of 28 7-point Likert-type items taken or modified from previously published surveys (Andrews et al., 2017; Flake et al., 2015; Gaspard et al., 2015; Glynn et al., 2011).
- The instrument contains items that relate to six constructs:

Construct 1: *Intrinsic value* - interest in using mathematics to understand biology

Construct 2: *Utility value* of mathematics for a career in life science

Construct 3: *Attainment value* – importance of doing the mathematics well

Construct 4: *Emotional Cost* of using mathematics in the biology course

Construct 5: *Effort* required to do mathematics in the biology course

Construct 6: *Expectations for success* in using mathematics in biology courses - measured by self-efficacy (Eccles & Wigfield, 1995)

- Confirmatory Factor Analysis with maximum likelihood robust estimation (Knekta et al., 2019) verified that all fit indices (CFI, TLI, RMSEA, SRMR) were within suggested cutoff values as suggested by Hu & Bentler (1999) and Steiger (2007).

Expectancy Value Theory (Eccles et al., 1983; Wigfield & Eccles, 2000)

Task Values

- Intrinsic Value
- Utility Value
- Attainment Value
- Emotional Cost
- Effort Cost

Expectancy of Success

- Self-Efficacy

Achievement Motivation & Task Completion

Data Analysis

What are the mathematics motivational patterns among undergraduate students who are enrolled in a gateway biology course?	Agglomerative Hierarchical Clustering using Wards method as the algorithm and Euclidean distance as the distance measure. Bootstrapping was used to determine the optimal number of clusters.
To what extent is gender and pre-professional status related to these patterns?	Multinominal Logistic Regression Analysis Outcome Variable: Cluster Independent Variables: Gender (male or female) and Pre-Professional Status (pre-professional or non-pre-professional) Control Variable: Highest Mathematics Course Taken in High School (Algebra/Geometry/Trigonometry, Pre-Calculus, or Calculus)

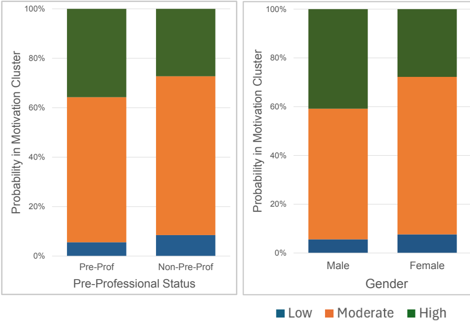
Cluster Analysis Results

Table 1: Means and Standard Deviations of the six affective variables in each motivation cluster.

Variable \ Cluster	Low Motivation N = 45	Moderate Motivation N = 319	High Motivation N = 164
Intrinsic Value	1.76 (± 0.65)	3.56 (± 1.23)	5.62 (± 0.77)
Utility Value	3.76 (± 1.23)	5.42 (± 0.99)	6.20 (± 0.81)
Mathematics Attainment Value	3.91 (± 1.07)	5.18 (± 1.02)	6.13 (± 0.77)
Emotional Cost	6.11 (± 0.60)	4.32 (± 1.05)	2.33 (± 0.91)
Effort Cost	5.55 (± 0.76)	3.74 (± 0.98)	2.10 (± 0.72)
Self-Efficacy	2.93 (± 1.17)	4.72 (± 1.05)	5.98 (± 0.84)

Multinominal Logistic Regression Results

Figure 1: Percent chance each predictor variable category is in low, moderate and high motivation cluster.



Conclusions

- Students in the introductory biology course can be clustered into three groups of motivation (low, moderate, and high), with over half in the moderate cluster.
- Gender and pre-professional status were not statistically significant in predicting motivational patterns.
- Controlling for highest mathematics course taken in High School, the probability that a female will be in the high motivation cluster is 27.8%. In comparison, the probability that a male will be in the high motivation cluster is 40.9%.
- Pre-Professional majors have a 35.7% chance of being in the high motivation cluster while non-pre-professional majors have a 27.3% of being in the high motivation cluster while controlling for highest mathematics course taken in High School.
- Next Steps: Investigate whether students' motivational patterns change as they complete the biology course?

References

- Andrews, S. E., Runyon, C., & Aikens, M. L. (2017). The math–biology values instrument: Development of a tool to measure life science majors' task values of using math in the context of biology. CBE—Life Sciences Education, 16(3), ar45.
- Brewer, C. A., & Smith, D. (2011). Vision and change in undergraduate biology education: a call to action. American Association for the Advancement of Science, Washington, DC, 81. https://doi.org/10.1187/cbe.10-03-0044
- Eccles, J., Adler, T. F., Futterman, R., Goff, S. B., Kaczala, C. M., Meece, J., and Midgley, C. (1983). Expectancies, values and academic behaviors. In Spence, J. T. (ed.), Achievement and Achievement Motives, W. H. Freeman, San Francisco.
- Eccles, J. S., & Wigfield, A. (1995). In the mind of the actor: The structure of adolescents' achievement task values and expectancy-related beliefs. Personality and social psychology bulletin, 21(3), 215-225. https://doi.org/10.1177/0146167295213003
- Flake, J. K., Barron, K. E., Hulleman, C., McCoach, B. D., & Welsh, M. E. (2015). Measuring cost: The forgotten component of expectancy-value theory. Contemporary educational psychology, 41, 232-244. https://doi.org/10.1016/j.cedpsych.2015.03.002
- Gaspard, H., Dicke, A. L., Flunger, B., Schreier, B., Häfner, I., Trautwein, U., & Nagengast, B. (2015). More value through greater differentiation: Gender differences in value beliefs about math. Journal of educational psychology, 107(3), 663. https://psycnet.apa.org/doi/10.1037/edu0000003
- Glynn, S. M., Brickman, P., Armstrong, N., & Taasoobshirazi, G. (2011). Science motivation questionnaire II: Validation with science majors and nonscience majors. Journal of research in science teaching, 48(10), 1159-1176. https://doi.org/10.1002/tea.20442
- Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. Structural equation modeling: a multidisciplinary journal, 6(1), 1055. https://doi.org/10.1080/10705519909540118
- Knekta, E., Runyon, C., & Eddy, S. (2019). One size doesn't fit all: Using factor analysis to gather validity evidence when using surveys in your research. CBE—Life Sciences Education, 18(1), rm1. https://doi.org/10.1187/cbe.18-04-0064
- Steiger, J. H. (2007). Understanding the limitations of global fit assessment in structural equation modeling. Personality and Individual differences, 42(5), 893-898. https://doi.org/10.1016/j.paid.2006.09.017
- Thompson, K. V., Nelson, K. C., Marbach-Ad, G., Keller, M., & Fagan, W. F. (2010). Online interactive teaching modules enhance quantitative proficiency of introductory biology students. CBE—Life Sciences Education, 9(3), 277-283. https://doi.org/10.1187/cbe.10-03- 0028
- Wigfield, A., & Eccles, J. S. (2000). Expectancy–value theory of achievement motivation. Contemporary educational psychology, 25(1), 68-81. https://doi.org/10.1006/ceps.1999.1015

The Journey of Quantitative Literacy Development: Insights from Physics Majors

Qirui Guo, Charlotte Zimmerman, and Suzanne White Brahmia (University of Washington)

INTRODUCTION

What is ……

★**Quantitative Literacy (QL)**: the ability to understand, interpret, and apply numerical and mathematical information across various contexts.

★**Physics Quantitative Literacy (PQL)**: the blending of conceptual and procedural mathematics to generate and apply quantitative models of physical phenomena.

★**Physics Inventory for Quantitative Literacy (PIQL)**: a multiple-choice test designed to assess PQL in everyday and abstract contexts.

METHOD

★Where: University of Washington

★When: between 2020 - 2023

★Who: Matched groups of physics majors (anonymized by code) from 3 levels of courses

✳ Number of people in each matched set:

100 - 200 Level: 198

200 - 300 Level: 60

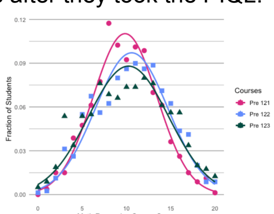
All Three Levels: 42

★How:

✳ Quantitative: We group students by thirds according to their pretest scores and use the same breakdown for posttest.

✳ Qualitative: Free response survey collected from 102 students after they took the PIQL.

What they saw →



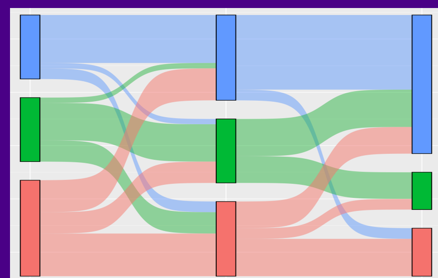
Physics majors develop *algebraic reasoning* **but** they <u>don't</u> perceive it as *"doing physics"*

Students' **transitions** between low, medium, and high scoring groups.

28.6% (15 - 20)

28.6% (11 - 14)

42.9% (2 - 10)



100 → 200 → 300 Level

61.9% (15 - 20)

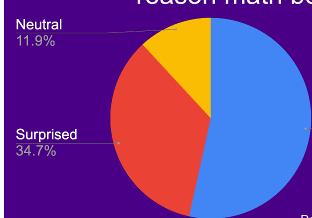
16.7% (11 - 14)

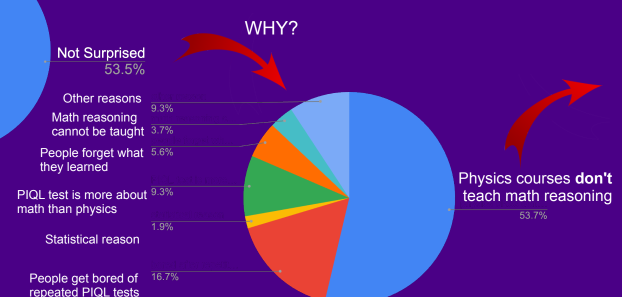
21.4% (2 - 10)

Percentage of students (Score Range)

Max score = 20

Are physics majors <u>surprised</u> that students **don't** typically reason math better after taking introductory physics?





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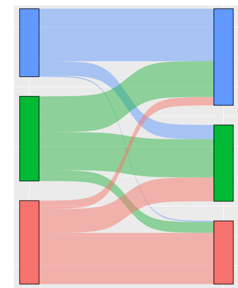
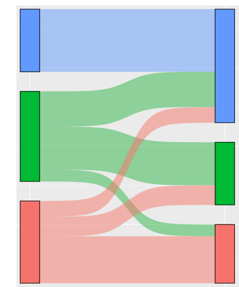
MORE RESULTS



100 → 200 Level



200 → 300 Level

100-200	Number of People	Average Score	Percentage	Number of People	Average Score	Percentage
Low (2 - 10)	73	7.43	**36.1**	57	8.59	**28.2**
Medium (11 - 14)	71	12.5	**35.1**	64	13.9	**31.7**
High (15 - 20)	58	16.7	**20.8**	81	17.9	**40.1**

200-300	Number of People	Average Score	Percentage	Number of People	Average Score	Percentage
Low (4 - 12)	21	8.48	**35**	15	7.93	**25**
Medium (13 - 15)	23	13.8	**38.3**	16	13.9	**26.7**
High (16 - 20)	16	17.1	**26.7**	29	17.6	**48.3**

100-300	Number of People	Average Score	Percentage	Number of People	Average Score	Percentage	Number of People	Average Score	Percentage
Low (3 - 12)	18	9.33	**42.9**	14	9	**33.3**	9	8.44	**21.4**
Medium (13 - 14)	12	13.4	**28.6**	12	13.5	**13.4**	7	13.7	**16.7**
High (15 - 20)	12	16.3	**28.6**	16	16.6	**38.1**	26	17.1	**61.9**

Max score = 20

Why do they think so?

✳ "…in my experience <u>UW physics does not teach math reasoning skills</u>"

✳ "…the intro physics courses are <u>more about learning the details of the physics rather than broader math reasoning skills</u>"

✳ "…intro physics is too dense with memorizing equations, <u>it does not introduce mathematical creativity very much</u> for better or worse"

✳ "…the intro series of physics courses is also <u>pretty poor at building up mathematical reasoning skills</u>, …, with the math consistently being the most difficult part as well as <u>the most glossed over part</u> of these courses"

DISCUSSION

What are we doing next:

★ Collect more data and look more closely into students who change their groups and who stay in the same groups

★ Compare students' PIQL scores with their course grade and see if there is a connection

Proof Writing and Comprehension in Topology

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The University of Oklahoma

Sepideh Stewart
The University of Oklahoma

Keywords: Topology, mathematics communication, APOS, Tall's worlds of mathematical thinking, proof

The pedagogy of proof has been a topic of research interest in mathematics education literature (e.g., Melhuish et al., 2022). While proof writing is a formal and abstract topic for many students, topology proofs typically demand a level of visual comprehension as well, which places topology in a spot of particular interest. Many argue that learning topology relies on spatial reasoning, yet others have noted that students' shortcomings are typically in the formal, proof-based aspects of problems (e.g., Narli, 2010). Various researchers have sought to understand how we can teach topology by studying how learning takes place (e.g., Cook et al., 2017; Wilkerson-Jerde & Wilensky, 2011), what methods of teaching can support this learning (e.g., Griffiths, 1971; Deogratias, 2022; Estabrooks & McArdle, 2022; Stewart, 2000), and what concepts students typically understand (e.g., Aksu, Gedik, & Konyalioglu, 2021; Cheshire, 2017; Gallager & Infante, 2019). However, there is a lack of research into how students perceive their own experiences in learning topology.

We seek to build a model based on Tall's (2008; 2013) framework of mathematical thinking and APOS theory in order to gain insights into the cognition of topology. In addition, we will examine how students communicate their experiences and think about topology. Tall's three worlds suggests that modes of thinking can be described as three different "worlds" of thought: the embodied, symbolic, and formal worlds (Tall, 2008; 2013). The embodied world refers to aspects of thought informed by perception, the symbolic world contains the thinkable concepts as generalized objects, and the formal, axiomatic world derives meaning from definitions and formal constructions. APOS theory refers to the depth of understanding that learners may demonstrate as they come to understand a topic (Dubinsky & McDonald, 2001). Using the three worlds of mathematical thinking and the action, process, and object levels from APOS, we will create a table synthesizing both ideas inside a grid (APOS on the first column, three worlds on the top row). Networking these theories gives language to what behaviors are exhibited in proof as the writer transitions across various types of thinking and levels of complexity. Changes throughout a proof would be characterized as transitions of thought, demonstrating how various ideas are handled, transformed, or ignored. We will demonstrate this model with a particular example from topology and elaborate on the mathematics behind it in order to develop it.

This three worlds-APOS model can be employed as novice graduate students complete work in topology. Students may understand concepts as actions, then generalized processes, then complete objects, and have a schema to allow them to transition between all three worlds of mathematical thinking. This research study is sought to determine students' struggles and experiences with topology using the model. We seek to answer the following questions: How can this model be used to examine the level of understanding of topology? What improvements can be made to this model? What topics in topology does this model reveal that are typically neglected on the formal level? What topics in topology are transitioned into spatial reasoning? What topics in topology are kept on the formal level of understanding?

References

Aksu, Z., Gedik, S. D., & Konyalioglu, A. C. (2021). Mathematics teacher candidates' approaches to using topology in geometry. *School Science and Mathematics*, *121*(4), 192–200. ERIC. https://doi.org/10.1111/ssm.12464

Cheshire, D. C. (2017). *On the Axiomatization of Mathematical Understanding: Continuous Functions in the Transition to Topology* (ProQuest LLC. 789 East Eisenhower Parkway, P.O. Box 1346, Ann Arbor, MI 48106. Tel: 800-521-0600; ERIC.

Cook, S. A., Hartman, J., Pierce, P. B., & Seaders, N. S. (2017). To each their own: Students asking questions through individualized projects. *PRIMUS*, *27*(2), 235–257. ERIC. https://doi.org/10.1080/10511970.2016.1192076

Deogratias, E. (2022). Developing pre-service mathematics teachers' understanding of metric and topological spaces using reflective questions. *International Online Journal of Education and Teaching*, *9*(1), 1–10. ERIC.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue,U. Krichgraber, J. Hillel, M. Niss & A. Schoenfeld (Eds.), The teaching and Learning of Mathematics at University Level: An ICMI Study (pp. 273-280). Dordrecht: Kluwer Academic Publishers.

Estabrooks, C., & McArdle, D. (2022). The Design and Implementation of a Course in Mathematical Research and Communication. *PRIMUS*, *32*(7), 785–797. ERIC. https://doi.org/10.1080/10511970.2021.1931996

Gallagher, K., & Infante, N. (2019). Undergraduates' Uses of Examples in Introductory Topology: The Structural Example. *Mathematics Education Research Group of Australasia*, 30-J. ERIC.

Griffiths, H. B. (1971). Mathematical Insight and Mathematical Curricula. *Educational Studies in Mathematics*, *4*(2), 153. ERIC.

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, 67, 100986.

Narli, S. (2010). Do students really understand topology in the lesson? A case study. *International Journal of Behavioral, Cognitive, Educational and Psychological Sciences v2* (Issue 3, pp. 121–124). Online Submission; ERIC.

Stewart, I. (2000). A strategy for subsets. *Scientific American*, *282*(3), 96. ERIC.

Tall, D. O. (2008). The transition to formal thinking in mathematics. *MERJ*, *20*(2), 5-24.

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*, Cambridge University Press.

Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math?: Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics*, *78*(1), 21–43. ERIC. https://doi.org/10.1007/s10649-011-9306-5

Proof Writing and Comprehension in Topology

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INTRODUCTION

The pedagogy of proof has been a topic of research interest in mathematics education literature (e.g., Melhuish et al., 2022). While proof writing is a formal and abstract topic for many students, topology proofs typically demand a level of visual comprehension as well, which places topology in a spot of particular interest. Many argue that learning topology relies on spatial reasoning, yet others have noted that students' shortcomings are typically in the formal, proof-based aspects of problems (e.g., Narli, 2010). A portion of the literature can be summarized with the following:

What do students struggle with?	What methods of teaching can help students?	How do students develop understanding?	How can we reinvent how topology is taught?
Countable unions/intersections Non-metrizable spaces Escaping the usual topologies Applying topology to geometry	Topology can be taught motivated from analysis General topologies are abstractions of metric spaces Physical objects can be helpful, though sometimes confusing	Students reinvent and adapt their own axiomatic systems Students reconcile misconceptions Experts learn well from proofs	Classes based around students discovering the concepts do incredibly well Motivations for topology exist in games/art
e.g., Deogratias 2022; Aksu et al. 2021; Cheshire 2017; Gallagher 2019; Narli 2010	e.g., Shipman & Stephenson 2022; Helmstutler et al. 2012; Griffiths 1971	e.g., Cheshire 2017; Wilkerson-Jerde, Michelle, & Wilensky 2011	e.g., Estabrooks & Mcardle 2022; Cook et al. 2017; Stewart 2000; Poggi 1985

Still, there is a lack of research into how students navigate transitions from the spatial concepts to formal logic and how they perceive their own actions in topology.

THEORETICAL BACKGROUND

Tall's Three Worlds of Mathematical Thinking suggests that modes of thinking can be described as three different "worlds" of thought: the embodied, symbolic, and formal worlds (Tall, 2008; 2013). The embodied world refers to aspects of thought informed by perception, the symbolic world contains the thinkable concepts as generalized objects, and the formal, axiomatic world derives meaning from definitions and formal constructions.
APOS theory refers to the depth of understanding that learners may demonstrate as they come to understand a topic (Dubinsky & McDonald, 2001). First, there are actions taken. Processes are actions that are repeated and reflected upon, which then are encapsulated into objects that can be acted on. The schema is the collection of all of these.

DESIGN OF THE MODEL AND RESEARCH QUESTIONS

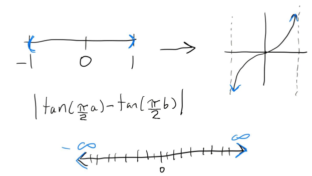
Seeking to gain insight into how students perceive their own actions in their topology proof writing, we network Tall's Three Worlds and APOS Theory together to develop a model. Putting these together gives a holistic perspective of the learner's thoughts and actions. We perform this networking by combining both theories into a single table with APOS columns and Tall's Three Worlds rows. With this model in mind, we seek to answer the following **research questions**:
How can this model be used to examine students' understanding of topology? What topics in topology does this model reveal that are typically neglected on the formal level? What topics in topology are transitioned into spatial reasoning? What topics in topology are kept on the formal level of understanding? What are the limitations of this model and what improvements can be made?

APPLYING THE MODEL: TASK AND SAMPLE SOLUTION

To help identify when transitions are made, the order of the items are numbered with superscripts.

Question 1: Give a metric on the set (-1,1) that induces the usual topology and is complete.

	Action	Process	Object
Embodied	$-1\to-\infty$, $1\to\infty$ [6]	$(-1,1)\to\mathbb{R}$ sends points to infinity [4]	Graph of Tan is infinite 1-manifold [1.5]
Symbolic	Compose distance with Tan [9]	$d(a,b)=\lvert a-b\rvert$ [8]	$\mathbb{R}\cong(a,b)$ [1] Metric has 2 inputs [7]
Formal		$f(x)=\tan(x*\pi/2)$ [5]	Reals are complete [2] $(-1,1)$ doesn't have its limits [3]

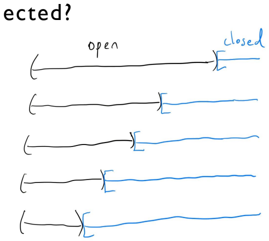


- Why are there no formal actions?
- The world with the most behavior is the symbolic, the stage most referenced is object. Does the performance on this question support this?
- When do transitions from worlds/stages happen?

What is the usual topology on this interval? It's the same as the reals, as is any basic open interval (via some variation on tangent). I must come up with a *metric* that induces the usual topology but is complete as a metric space, so I ask if the space is usually complete. The reals are complete with the usual metric, but clearly, as is the concept of the question, the interval given is not complete with the same metric. So, comparing the two spaces, how are they different from a metric standpoint: While the interval is finite in length, the reals are infinite, so we just need to identify how the interval is sent to the reals via tangent, then combine that with the usual metric on the reals.

Question 2: Is the topology given by $T=\{(0,a) \leq \mathbb{R}^+ \mid 0 < a\}$ path connected?

	Action	Process	Object
Embodied		Path-Connectedness is embedding lines throughout the space [2]	
Symbolic	Compliment of (0,a) is $[a,\infty)$ [5]		Closed sets of R [6]
Formal	Courser topologies are not always subset topologies [1]		Definitions: Topology as a set-family [1.5], path-connected [3], continuous function [4]



- Why was this mostly a symbolic and object oriented proof?
- Does the lack of varying concepts reflect a lack of understanding or completion?

While being a subset of the reals, this is not given the subspace topology, so I need to be mindful of that distinction as I think of this space. My usual intuition for path-connectedness fails since this is not the usual topology. I fall back on the definition of path-connected, and, wanting to make a path between any two points, I realize I'm not sure what a continuous function into this space should look like. The definition of continuous is the preimage of closed sets are closed, so I need to know what sets are closed in this topology. Since these are just compliments of the open sets given to me, they all look the same as $[a,\infty)$. These sets are already closed in the usual topology, so the usual paths are already continuous here, so any straight-line path is a path from any two points of the space, thus the space is path-connected.

REFLECTION

Although the model is adequate at showing how students think about concepts in topology in terms of complexity and form, as the examples above show a lack of formal actions, it needs more research directed towards questions demanding those behaviors.

Please visit: https://sites.google.com/view/proofwritingandcomp

Implementing Active Assessment: Stories from an Instructor Learning Community

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Keywords: Active Assessment, Community of Practice, Active Learning

Recently, there has been a surge in interest in developing more flexible assessment structures, with a great deal of momentum for this movement from COVID-19 pandemic equity considerations (Kadakia & Bradshaw, 2020; Kim, 2020). Active-learning-based instructional practices are now established in undergraduate education (Laursen & Rasmussen, 2019). Yet, how to align assessment with those practices is still developing. The ideas presented are part of a larger project that explores instructor and student perspectives on *active assessment*, assessment strategies grounded in the values and practices of active learning, including attention to equity. The authors are participants in the project. This poster will focus on challenges and benefits that have appeared in the first year for the authors, as a community of practice (CoP), designing and implementing active assessment.

Members of the CoP are mathematics instructors in a variety of career stages and positions, teaching courses from introductory-level courses with co-requisite support, through upper division electives for math majors. This work takes place at an institution with a variety of supports available, and individual instructors' consideration and implementation of *active assessment* occurs within the ongoing process of CoP professional and pedagogical growth.

This poster will share stories about our motivation as well as initial and in-progress thoughts on implementation. It will include details on class sizes and examples of active assessments as well as notes on how they are administered. The stories generated some questions related to equity and the challenges and benefits of *active assessment*:

- We have noticed a challenge in structuring projects as a form of active assessment. Thus, a focal question for the group is: How do we balance promoting group projects and offering individual options for projects in response to the needs and preferences of students?
- Using assessments involving short- and long-answer responses, we noticed students gave short, surface-level responses. This led to another question for the CoP to address: How can we leverage assessment strategies and give qualitative feedback that supports students in engaging with written assignments in a meaningful way?
- Peer feedback on assessments can help students plan their learning and identify strengths and areas for growth. How can we evoke critical peer feedback that encourages students to reflect more deeply on the quality of their work?

In all of these questions, we aim to center equity and access to rich mathematical experiences. A goal of this poster is to generate critical conversations that will inform our ongoing research and practice.

Acknowledgement

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References

Kadakia, P. C,, & Bradshaw, A. (2020, May 6). Equitable exams during Covid-19. *Inside Higher Ed*. https://www.insidehighered.com/advice/2020/05/06/changes-instructors-should-consider-administering-and-grading-exams-during-covid

Kim, J. (2020, April). 5 Reasons to Stop Doing Timed Online Exams During COVID-19. *Inside Higher Ed*. https://www.insidehighered.com/blogs/learning-innovation/5-reasons-stop-doing-timed-online-exams-during-covid-19

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 129-146. https://doi.org/10.1007/s40753-019-00085-6

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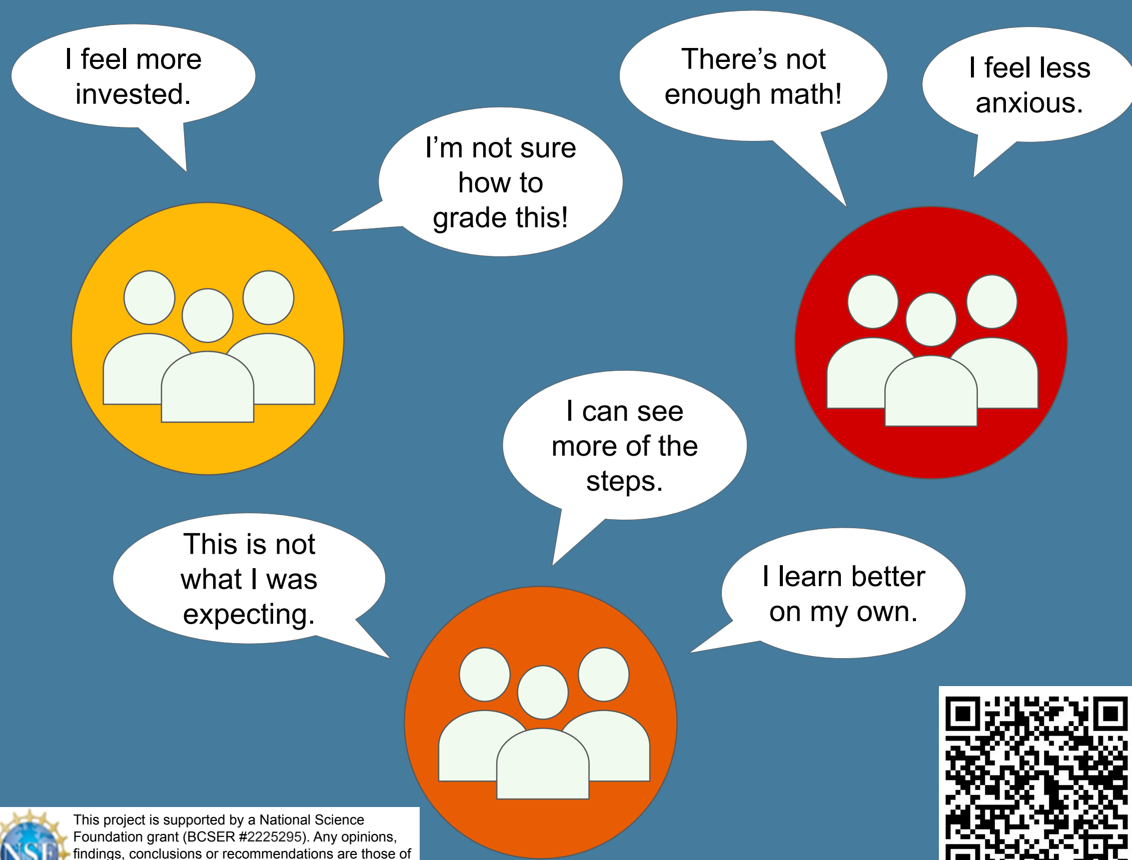
Implementing Active Assessment:
Stories from an Instructor Learning Community

2023-2024 Assessment Community of Practice

- Includes lecturer, pre-tenure, and tenured faculty.
- Wide range of courses and assessment strategies represented.
- Focus on **active assessment** across different contexts.

By using **assessment strategies** aligned with **active learning**, we uncover more about **students' learning process**.



Projects:
Finite Math for Business
Business Majors, 35 students
Students chose to either use real data from a company that produces and sells items or to estimate data from an imaginary future business. Groups built models of business functions and used platforms such as Desmos to make a presentation that explains important points in the model and provide the best consulting solution for the business by mathematics.

- Reducing stress compared to a traditional test.
- Challenging if students cannot work in groups for any reason.
- How flexible we can be if students can't work in project for any reason?

Written Responses:
Math for the Arts and Humanities
Non-STEM majors, 35 students
Students compiled portfolios of their work. As part of the portfolio, students had to write summaries of mathematical content from topics covered in class.

- Writing summaries helps students reinforce content and encourages students to write mathematical ideas in their own words.
- Students typically gave surface level responses and had a difficult time synthesizing material.
- How can we prepare students in a math class to engage in writing at a deeper level?

Peer Feedback:
Intro to differential Equations
STEM Majors, 20 students
Students created mini lessons to teach each other techniques for solving ODEs. They were required to give feedback on their classmates' mini lessons, and use feedback they received to revise their work.

- Student presentations provided insight into their thinking and how they approached each problem.
- Peer feedback did not promote deeper reflection or increase progress toward the learning objectives for either learner.
- How can we evoke critical peer feedback that encourages students to reflect more deeply on the quality of their work?

CAL STATE EAST BAY

This project is supported by a National Science Foundation grant (BCSER #2225295). Any opinions, findings, conclusions or recommendations are those of the authors and do not necessarily reflect the views of the Federal Government.

Is There an Interrelation Between Graduate Students' Teaching of Undergraduate Courses and Their Own Learning of Advanced Mathematics?

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Keywords: mathematics graduate students, teaching, leaning, ROG, PCK

The academic role of mathematics graduate students is complex, as it entails being simultaneously a student, a teacher, and a researcher. The three roles are inseparable from the graduate school experience and shape graduate students' professional identity (Colbeck, 2008). Some studies (e.g., Lai et al., 2016; Speer & Hald, 2008; Schoenfeld, 2015) suggest a deeper consideration of the connection between teaching, research, and learning among graduate students. Furthermore, Beisiegel (2019) believes that some mathematics graduate students perceive teaching and research as contrasting and contradicting roles.

The purpose of this poster is to build and discuss a model of "Interrelation Between Teaching and Learning Mathematics of Graduate Students", based on Schoenfeld's (2015) "Resources, Orientations, and Goals" model along with Ball et al.'s (2008) "Pedagogical Content Knowledge" theoretical framework. As a teacher and researcher, being a student is an inseparable part of graduate students' professional identity (Colbeck, 2008); hence, constant learning is inevitably a part of graduate students' lives, resulting in the accumulation of mathematical knowledge. The framework of "Resources, Orientation, Goals, and Decision-Making" (ROGs) was developed to predict and map one's professional behavior (Schoenfeld, 2015). Schoenfeld's framework made knowledge to be the resource in focus, which, in the presented model, graduate students rely on the most. According to Schoenfeld, "an individual enters into a particular context with a specific body of resources, goals, and orientations" (p. 347), they observe and orient the situation, establish or re-establish goals, and make decisions as a result of this internal analysis (Schoenfeld, 2015). In teaching undergraduate courses, graduate students' decisions and choices are impacted by their previously acquired knowledge. The choice of lecture style, problems solved, examples, and explanations given are based on graduate students' mathematical knowledge acquired through their prior learning. For example, upon completing Real Analysis, graduate students' teaching of limits and integrals may be enhanced by their theory encountered in the class. In addition to mathematical knowledge, teachers have a pedagogical content knowledge acquired through teaching (Ball et al., 2008). As teachers, graduate students have a developed intuition about problems and concepts that are challenging for students and are "able to hear and interpret students' emerging and incomplete thinking" (p. 401) (Ball et al., 2008). The knowledge of the origin of common mistakes and misconceptions among students allows teachers to reliably identify students' thought processes behind solving math problems. The awareness of multiple ways to approach solving a problem, the knowledge behind made choices, and recognition of different strategies applied in the solution are a product of pedagogical content knowledge (PCK). Such gained skill set is expected to help graduate students with answering challenging homework problems in their classes by revealing alternative approaches. In light of the proposed model, our research questions to guide this study are: Does the pedagogical content knowledge gained from teaching undergraduate courses affect graduate students' learning of advanced mathematics? Does the knowledge acquired while studying for graduate courses impact the graduate students' teaching?

References

Ball, L. D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.

Beisiegel, M., Gibbons, C., & Rist, A. (2019). Examining Obstacles to Mathematics Graduate Students Development as Teachers. *North American Chapter of the International Group for the Psychology of Mathematics Education*.Colbeck, C. L. (2008). Professional identity development theory and doctoral education. *New directions for teaching and learning*, *2008*(113), 9-16.

Colbeck, C. L. (2008). Professional identity development theory and doctoral education. *New directions for teaching and learning*, *2008*(113), 9-16.

Lai, Y., Smith, W. M., Wakefield, N. P., Miller, E. R., Goar, J. S., Groothuis, C. M., & Wells, K. M. (2016). Characterizing mathematics graduate student teaching assistants' opportunities to learn from teaching. *Mathematics Education: A Spectrum of Work in Mathematical Sciences Departments*, 73-88.

Schoenfeld, A. H. (2015). How we think: A theory of human decision-making, with a focus on teaching. In *The Proceedings of the 12th International Congress on Mathematical Education: Intellectual and Attitudinal Challenges* (pp. 229-243). Springer International Publishing.

Speer, N., & Hald, O. (2008). How do mathematicians learn to teach? Implications from research on teachers and teaching for graduate student professional development. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and practice in undergraduate mathematics education* (pp. 305–218). Washington, DC: Mathematical Association of America.

Interrelation Between Teaching and Learning Mathematics of Graduate Students

Anna Mikulo & Sepideh Stewart

INTRODUCTION

The academic role of mathematics graduate students is complex, as it entails being simultaneously a student, a teacher, and a researcher. The three roles are inseparable from the graduate school experience and shape graduate students' professional identity (Colbeck, 2008). A study by Beisiegel (2019) reveal mathematics graduate students' view of the relation between teaching and research, where studied participants perceived two roles being "totally different" (p. 494). This perception could lead to research and teaching requiring separately invested time and energy, leaving graduate students with no choice but under-commit to one of the roles. The purpose of this poster is to build and discuss a model of "Interrelation Between Teaching and Learning Mathematics of Graduate Students", based on Schoenfeld's (2015) ROGs model along with Ball et al.'s (2008) PCK theoretical framework.

RESEARCH QUESTIONS

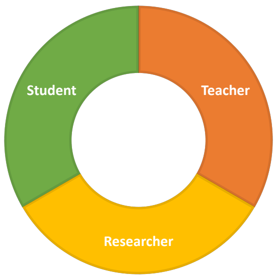
- Does the pedagogical content knowledge gained from teaching undergraduate courses affect graduate students' learning of advanced mathematics?
- Does the knowledge acquired while studying for graduate courses impact the graduate students' teaching?

DESCRIPTION OF THE MODEL

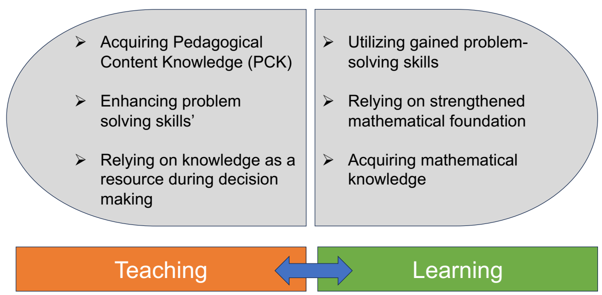
This model relies on two theoretical frameworks: PCK (Ball et al., 2008) and ROGs (Schoenfeld, 2015). During teaching, graduate students acquire pedagogical content knowledge, which enhances their understanding of the material taught and introduces them to different solution approaches undergraduate students take. With wider exposure to problem-solving techniques, we conjecture that producing proofs may become clearer for graduate students. On the other hand, while learning advanced mathematical content, graduate students expand their knowledge, which is the primary resource of their reliance. With knowledge as a resource, according to ROGs, graduate students' decisions in teaching change correspondingly to the resource. Hence, graduate courses students take influence their choices of material presentation, solution approaches, and variety of examples provided.

The model aims to track and establish the existence of a relation between teaching and learning, which would help to understand graduate students' experience as teachers and students and help them witness the influence of this interrelation.

PROFESSIONAL IDENTITY



THE MODEL: INTERRELATION BETWEEN TEACHING AND LEARNING MATHEMATICS OF GRADUATE STUDENTS



THEORETICAL BACKGROUND

The framework of ROGs was developed to predict and map one's professional behavior (Schoenfeld, 2015). Schoenfeld's framework made knowledge to be the resource in focus, which, in the presented model, graduate students rely on the most. According to Schoenfeld, "an individual enters into a particular context" (p. 347), they observe the situation, establish goals, and make decisions as a result of this internal analysis (Schoenfeld, 2015). In teaching undergraduate courses, graduate students' decisions and choices are impacted by their previously acquired knowledge.

As teachers, graduate students have a developed intuition about problems and concepts that are challenging for students and are "able to hear and interpret students' emerging and incomplete thinking" (p. 401) (Ball et al., 2008). The awareness of multiple ways to approach solving a problem, the knowledge behind made choices, and recognition of different strategies applied in the solution are a product of PCK. Such gained skill set is expected to help graduate students with answering challenging homework problems in their classes by revealing alternative approaches.

Graduate students observe norms of social interactions in their professional environment, interpret them based on their personal experience, and adapt such norms according to how they see themselves fit in that professional environment. Adapting Colbeck's definition (2008), the identity is "'what it means to be who one is'" (p. 10) and "individuals' identities are often associated with labels for social positions or roles" (p. 10) (Colbeck, 2008). So, having a professional identity as a *graduate student* entails having teacher, student, researcher, mathematician, tutor, grader, and TA as part of their professional identities.

FUTURE STUDY

To test the developing model, the future research may be a case study using a qualitative research approach and it may be carried out in two phases. The fist phase will investigate the relation between teaching and graduate students' learning where the participants will be University of Oklahoma mathematics graduate students two have taken Real Analysis sequence, have no prior experience teaching Calculus, and are assigned to teach Calculus I or Calculus II. Before the semester of teaching, graduate students will be asked to answer a list of Real Analysis questions that relate to Calculus topics other advanced Real Analysis questions. At the end of the semester, graduate students will be asked a set of similar questions to gage any enhancement of understanding of Real Analysis concepts.

REAL ANALYSIS TEST

1. Find an increasing function f such that $f' = 0$ a.e. but f is not constant on any open interval (p. 129) (Bass, 2013).
2. If f is real-valued and differentiable at each point of $[0, 1]$, is f necessarily absolutely continuous on $[0, 1]$? If not, find a counterexample (p. 129) (Bass, 2013).
3. Prove or disprove: If $F \subseteq \mathbb{R}$ is such that every bounded continuous function from F to $\mathbb{R}$ can be extended to a continuous function from $\mathbb{R}$ to $\mathbb{R}$, then F is a closed subset of $\mathbb{R}$ (p. 72) (Axler, 2020).
4. Prove or disprove: If $A \subseteq \mathbb{R}$ a d $f_1, f_2, \ldots$ is a sequence of uniformly continuous functions from A to $\mathbb{R}$ that converges uniformly to a function $f : A \to \mathbb{R}$, then f is uniformly continuous on A. (p. 71) (Axler, 2020).
5. Suppose $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable. Prove that $f + g$ is Riemann integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ (p. 7) (Axler, 2020).
6. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that f is a Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P on $[a, b]$ such that $U(f, P, [a, b]) - \mathcal{L}(f, P, [a, b]) < \epsilon$ (p. 7) (Axler, 2020).
7. Suppose μ is a signed measure on $(\mathcal{X}, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then $\mu^+ = sup\{\mu(B) \mid B \in B, B \subset A\}$ and $\mu^- = -inf\{\mu(B) \mid B \in B, B \subset A\}$. (p. 98) (Bass, 2013).

The second phase will consist of exploring ways studying advanced mathematics may influence teaching undergraduate courses. The participants will be mathematics graduate students who have not taken Real Analysis sequence and they will be interviewed before and after the sequence. The interview questions will target graduate students' approach to teaching Calculus I or II related topics. The Interviews will be conducted and after before taking Real Analysis sequence.

INTERVIEW QUESTIONS RELATED TO TEACHING

1. Describe your teaching approach of derivatives.
2. Give three examples of continuous functions and justify your choice.
3. Please explain the relation between continuity and differentiability.

REFERENCES AND CONTACT

Creating Statistically Equivalent Versions of a Test of Quantitative Literacy in Physics Contexts

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Keywords: quantitative literacy, assessment, physics

The Physics Inventory of Quantitative Literacy (PIQL) is a 20-item multiple-choice test designed to measure the development of students' physics quantitative literacy (PQL) across multiple physics courses (Olsho et al., 2023; Smith et al., 2020; White Brahmia et al., 2021). Repeated testing, coupled with requiring up to 40 minutes for students to complete the test, could lead to testing fatigue and unreliable results. We seek to create two shorter versions of the PIQL (a.k.a. piqlets) that are statistically equivalent to each other in terms of student performance on three facets of PQL (ratios and proportions, covariation, and signs and negativity).

Han et al. (2015) used a large data set of student responses to a 30-item conceptual physics test to identify combinations of items that produced testlets with the most similar average student scores. They demonstrated the equivalence of the testlets using item response theory (IRT) and correlating individual students' scores across versions. We follow their example by creating 12-item piqlets that each contain four overlapping anchor items (to facilitate comparisons between versions) and eight distinct items. Our work was guided by these research questions:

1) Which combination of items produces piqlets with the smallest score differences?
2) How does the reliability of these piqlets compare to each other and to the PIQL?
3) How similar are the psychometric parameters for the anchor items across piqlets?

Data were collected using the full PIQL in three introductory physics courses at a large public university in the western US (2100–3200 students in each data set). We considered 240 combinations of items for the piqlet versions, subject to constraints that both the content and the format of items were equivalent across the two versions. We calculated a total test score for each piqlet based on the average percentage of items answered correctly, as well as subscores for the three facets of PQL. For each of our three data sets we determined the average score difference between the piqlets, calculated Cronbach's α for each piqlet, and applied IRT analyses to each.

The combinations of items that we identified as being the most similar had overall average score differences from 0.6–1.3%. Cronbach's α values ranged from 0.67 to 0.75, with differences less than 0.01 between versions. We see strong correlations between individual student scores on the two piqlets, with $0.79 \leq r \leq 0.85$. The IRT parameters support the statistical equivalence of the piqlets with parameters of overlapping items agreeing to within 0.1. These preliminary results suggest a strong potential for identifying piqlets that are statistically equivalent for the broader population of mathematics and physics students based on a larger, more diverse, data set.

Acknowledgments

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References

Han, J., Bao, L., Chen, L., Cai, T., Pi, Y., Zhou, S., Tu, Y., & Koenig, K. (2015). Dividing the Force Concept Inventory into two equivalent half-length tests. *Physical Review Special Topics – Physics Education Research, 11*(1), 010112. https://doi.org/10.1103/PhysRevSTPER.11.010112

Olsho, A., Smith, T. I., Eaton, P., Zimmerman, C., Boudreaux, A., & White Brahmia, S. (2023). Online test administration results in students selecting more responses to multiple-choice-multiple-response items. *Physical Review Physics Education Research*, *19*(1), 013101. https://doi.org/10.1103/PhysRevPhysEducRes.19.013101

Smith, T. I., White Brahmia, S., Olsho, A., & Boudreaux, A. (2020). Developing a reasoning inventory for measuring physics quantitative literacy. In S. Smith Karunakaran, Z. Reed, & A. Higgins (Eds.), *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education.* (pp. 949–955). http://sigmaa.maa.org/rume/RUME23.pdf

White Brahmia, S., Olsho, A., Smith, T. I., Boudreaux, A., Eaton, P., & Zimmerman, C. (2021). Physics Inventory of Quantitative Literacy: A tool for assessing mathematical reasoning in introductory physics. *Physical Review Physics Education Research*, *17*(2), 020129. https://doi.org/10.1103/PhysRevPhysEducRes.17.020129

Creating Statistically Equivalent Versions of a Test of Quantitative Literacy in Physics Contexts

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DUE-1832470, DUE-2214283, DUE-2214765; NRC Research Associateship

Introduction

- Improving students' physics quantitative literacy (PQL) is a goal for many physics instructors.
- PQL is the blended relationship between students' general mathematical reasoning and conceptual understanding of physics.
- The **P**hysics **I**nventory of **Q**uantitative **L**iteracy (PIQL) is a 20-item multiple-choice test designed to measure PQL across three facets (Olsho et al., 2023; Smith et al., 2020; White Brahmia et al., 2021).
- PIQL used in multiple physics courses to track student growth
- Some items are multiple-choice-multiple-response (MCMR)
- **Known Problems:**
  - The PIQL takes a long time to complete (40 minutes on average)
  - Repeated testing risks memorization and reduced efficacy
- **GOAL: Create two shorter versions of the PIQL (a.k.a. PIQLets) that are statistically and psychometrically equivalent**

Research Questions

1. Which combination of items produces PIQLets with the smallest score differences? (Figure 1, central Venn diagram)
2. How does the reliability of these PIQLets compare to each other and to the PIQL? (Table 1)
3. How similar are the psychometric parameters for the anchor items across PIQLets? (Table 3)

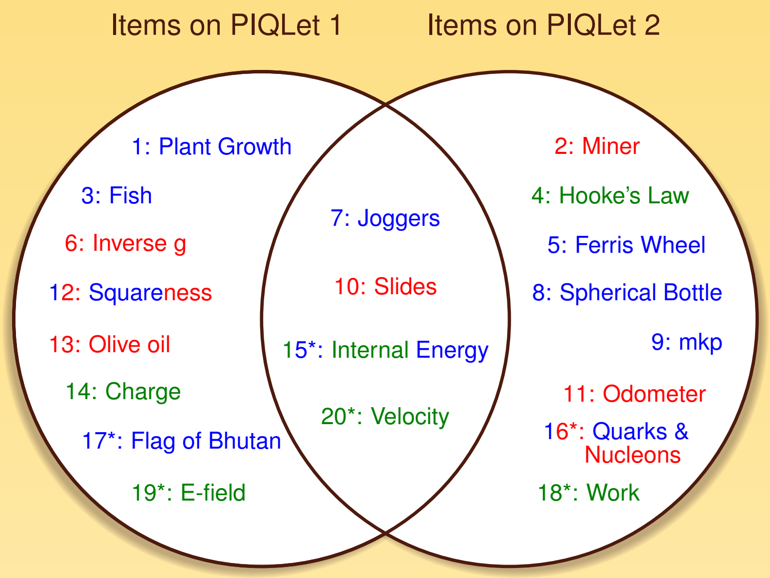
Constraints and Criteria

- Constraints: each pair of PIQLets must have:
  - 12 items on each PIQLet: 4 anchor items in common, 8 unique items
  - the same number of items (anchor and unique) testing each PQL facet
  - the same number of items in each format (single response or MCMR)
- Criteria: the best pair of PIQLets should have:
  - similar total average scores for all three data sets
  - similar average subscores for all three facets
  - good test reliability (Cronbach's α)
  - similar item response theory (IRT) parameters for anchor items

Methods

- Followed example of Han et al. (2015)
- Full 20-item PIQL administered 3 times during introductory physics course sequence at a public university in the Western US
  - Before mechanics (PreMech, $N = 3206$)
  - After mechanics, before electricity & magnetism (PostMech, $N = 2580$)
  - After electricity & magnetism (PostEM, $N = 2136$)
- Each item categorized based on content and format
- Identified 5 candidates for anchor items based on prior analyses
- Identified all combinations of items based on constraints (240 total)
- For each combination of items:
  - Split the full data set into two parts
  - Calculated the differences in the mean total score and mean subscores for each facet for each data set (total of 12 average score differences)
  - Calculate the mean $\langle\Delta\rangle$ and the root-mean-squared Δ_{rms} of the 12 score differences as measures of the total difference between the PIQLets
- Identified a handful of combinations that had the lowest Δ_{rms} and $\langle\Delta\rangle$ close to zero.
  - Calculated Cronbach's α for each PIQLet for each data set
  - Identified the combination with the most similar α values
- For the single most similar PIQLet pair:
  - Applied a three-parameter logistic (3PL) model to calculate IRT item parameters
  - Compared parameters for anchor items between the two PIQLets
- All analyses performed using the `r` computing environment (R Core Team, 2019), specifically using the `mirt` and `tidyverse` packages

Certain combinations of PIQL test items create statistically equivalent testlets



Ratios and Proportions
Covariation
Signed Quantities and Negativity
*MCMR Items

Scan for poster

PIQL on PhysPort
https://www.physport.org

Scores

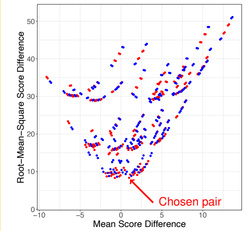


Figure 1. Mean percentage score difference $\langle\Delta\rangle$ and root-mean-square score difference Δ_{rms} for all PIQLet item combinations. Red points have anchor items 7, 10, 15, and 20; blue points have anchor items 1, 10, 15, and 20.

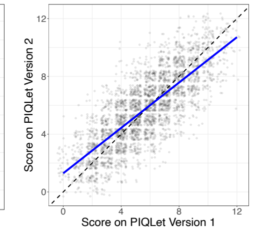


Figure 2. Comparison of PreMech students' scores on the PIQLets. The blue line shows the relationship between the scores on the two versions. The black dashed line represents equal scores on the two versions.

For the chosen pair of PIQLets: the mean total score difference for PreMech data is 1.6%, the mean difference for PostMech data is 1.1%, and the mean difference for PostEM data is 1.3%. In all three data sets PIQLet 2 has the higher mean score.

Statistical Comparisons

Test	PreMech	PostMech	PostEM
PIQL	0.76	0.80	0.82
PIQLet 1	0.68	0.73	0.75
PIQLet 2	0.67	0.72	0.75

Table 1. Cronbach's α for the full PIQL and the closest combination of PIQLets.

Comparison	PreMech	PostMech	PostEM
PIQLet 1 v. P2	0.79	0.82	0.84
PIQL v. P1	0.94	0.94	0.95
PIQL v. P2	0.93	0.94	0.95

Table 2. Pearson's r correlation between scores on the two PIQLet versions, as well as between each PIQLet and the full PIQL.

IRT Parameters

$$\text{3PL Model:} \qquad P_{i,j}(1|\theta_i) = c_j + \frac{1 - c_j}{1 + e^{-1.7a_j(\theta_i - b_j)}}$$

The probability that student i answers item j correctly, given their overall ability and understanding θ_i (i.e., PQL). Each item has three estimated parameters: discrimination a_j, difficulty b_j, and guessing c_j.

	Discrimination a_j			Difficulty b_j			Guessing c_j		
Item	PIQL	P1	P2	PIQL	P1	P2	PIQL	P1	P2
7	0.72	0.73	0.75	-0.27	-0.28	-0.22	0.01	0.00	0.03
10	1.10	1.14	1.09	-0.78	-0.77	-0.79	0.00	0.00	0.00
15	0.93	0.99	0.89	0.40	0.43	0.40	0.01	0.03	0.01
20	1.02	1.08	1.06	0.01	0.03	0.00	0.00	0.01	0.00

Table 3. 3PL IRT parameters for anchor items for the full PIQL and each PIQLet version for our chosen combination.

Summary

- Categorizing testlet similarity by mean scores yielded a pair of PIQLets with very small differences in mean scores, but these differences vary greatly across item combinations (Figure 1).
- Scores are less similar for students at the high and low ends (Figure 2).
- Cronbach's α is very similar across the chosen pair of PIQLets, but slightly smaller than the total PIQL (Table 1). This is expected, given the smaller number of items (Streiner, 2003).
- Student scores across PIQLet versions are strongly correlated (Table 2).
- IRTparameter values for the anchor items are very similar across the two chosen PIQLet versions: a_j differs by 0.1 or less, b_j by 0.06 or less, and c_j by 0.03 or less (Table 3).

Next Steps

- Define difference between PIQLets based on more than mean scores
- Use IRT analyses to compare other PIQLet items based on IRT parameters for anchors items and overall distribution of parameters
- Apply the Stocking-Lord method to define transformations between PIQLets, looking for combinations that require the smallest adjustments

References

Han, J., Bao, L., Chen, L., Cai, T., Pi, Y., Zhou, S., Tu, Y., and Koenig, K. (2015). Dividing the force concept inventory into two equivalent half-length tests. *Phys. Rev. ST Phys. Educ. Res.*, 11:010112.

Olsho, A., Smith, T. I., Eaton, P., Zimmerman, C., Boudreaux, A., and White Brahmia, S. (2023). Online test administration results in students selecting more responses to multiple-choice-multiple-response items. *Phys. Rev. Phys. Educ. Res.*, 19:013101.

R Core Team (2019). R: A Language and Environment for Statistical Computing.

Smith, T. I., White Brahmia, S., Olsho, A., and Boudreaux, A. (2020). Physics Students' Implicit Connections Between Mathematical Ideas. In Smith Karunakaran, S., Reed, Z., and Higgins, A., editors, *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*, pages 940–946, Boston, MA.

Streiner, D. L. (2003). Starting at the beginning: An introduction to coefficient alpha and internal consistency. *Journal of Personality Assessment*, 80(1):99–103. PMID: 12584072.

White Brahmia, S., Olsho, A., Smith, T. I., Boudreaux, A., Eaton, P., and Zimmerman, C. (2021). Physics Inventory of Quantitative Literacy: A tool for assessing mathematical reasoning in introductory physics. *Phys. Rev. Phys. Educ. Res.*, 17(2):020129.

Opening access to professional development: Outcomes of teaching-focused online workshops

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Keywords: Professional development, online workshops, teaching

Introduction

Professional development (PD) workshops on teaching have been shown to effectively increase undergraduate math instructors' use of research-based instructional strategies (Archie et al., 2022). However, face-to-face PD is not accessible for all math instructors, including those with young families, those at under-resourced institutions, and those concerned about the environmental impacts of travel. In response to these constraints, the Mathematical Association of America (MAA) sponsored a series of nine online PD workshops in summer 2022. The MAA solicited proposals and selected teams to lead workshops on a variety of mathematics teaching topics. These teams participated in planning sessions in winter 2022 to prepare their workshops and implement best practices in workshop design (Daly et al., 2021). The workshops served 25 participants each over 24-27 contact hours, conducted via Zoom in two formats: a one-week intensive model and a mini-course format spread over three weeks. The purpose of this preliminary research was to explore the outcomes of the 2022 workshops.

Methods

Participants completed surveys about one month before their workshop (pre), immediately after (post), and 12 months later (follow-up). Survey measures probed instructors' current capacities (knowledge, skill, attitude, and motivation) related to the workshop topic (pre, post), perceptions of workshop quality (post), planned implementation (post) and implementation at one year, and what workshop features helped them learn (post). 119 instructors responded to all three surveys, out of 203 instructors who completed workshops.

Results and Discussion

Overall, participants rated the workshops favorably and reported strong gains in knowledge and skills related to the workshop topic. Most (83%) indicated they would likely implement what they learned in the workshop. Indeed, at the one-year follow-up, 84% of respondents reported some degree of implementation, which is comparable to implementation after face-to-face workshops (Archie et al., 2022). When survey non-responders are included in the total, the self-reported implementation rate drops to 50%. As a group, participants identified the atmosphere, their interactions with others, practical examples, facilitator modeling, and working and connecting with other participants as workshop features that helped them learn the most. Together, these findings suggest that well-designed online professional development can help instructors implement what they learned at the workshops.

Acknowledgments

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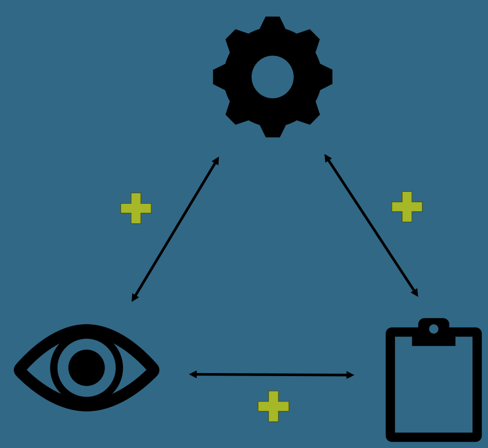
References

Archie, T., Hayward, C. N., Yoshinobu, S., & Laursen, S. L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *Plos One*, 17(4), e0267097.

Daly, D., Ethnography & Evaluation Research, and the Academy of Inquiry Based Learning (2021, June). *AIBL Handbook for Online Professional Development: Lessons Learned from PRODUCT Workshops*. Boulder, CO, and San Luis Obispo, CA: University of Colorado Boulder, Ethnography & Evaluation Research; and Academy of Inquiry Based Learning. https://tinyurl.com/AIBLHandbook

Participants implement what they learned from online professional development

Jenny Valadez Fraire, Tim Archie, Sandra Laursen & Kyra Gallion



Observations and surveys show that online workshops support implementation

Need

Professional development (PD) workshops have been found to effectively increase undergraduate math instructors use of research-based instructional strategies[1].

Benefits of online PD
- Under-resourced institutions
- Instructors with young families
- Environmental impact of travel

Workshops

During the Summer of 2022, the Mathematical Association of America (MAA) sponsored a set of eight virtual PD workshops to address these needs.

Workshop Features
- Facilitators participated in Winter trainings based on PD best practices http://tinyurl.com/2p83ayry
- Conducted via Zoom in two formats: one week intensive model & mini course spread over three weeks
- 25 participants / workshops through 24-27 contact hours

Method

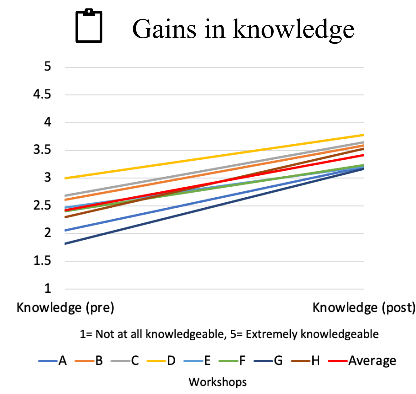
Pre survey and Post Survey
Instructors' current capacities (knowledge, skill, attitude and motivation) related to the workshop topic

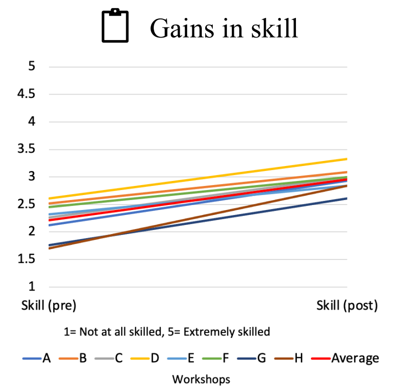
Online Professional Development Observational Checklist (OPDOC)
19 item rubric that assesses workshop features and provides a quantitative score (https://tinyurl.com/27tcrhe6)

119 out of 203 instructors who completed workshops responded to all three surveys

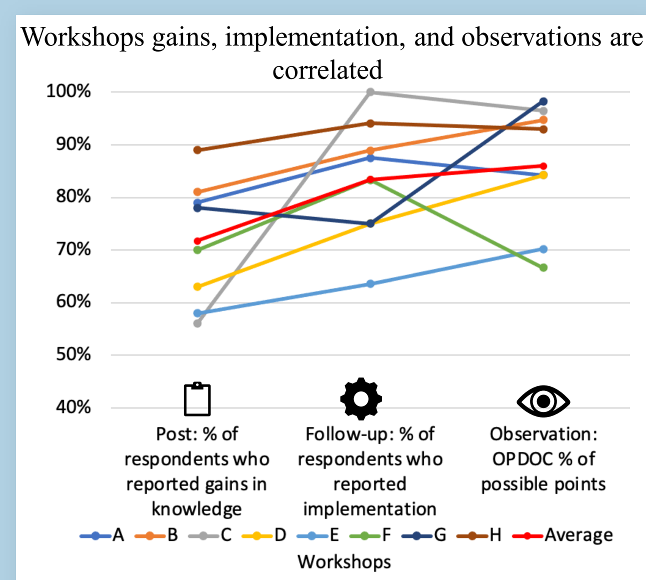
Follow Up Survey
Implementation at one year

Workshop Participant Gains





Results & Discussion



Takeaways

Overall, participants rated the workshops favorably and reported strong gains in knowledge and skills related to the workshop topic.
- Most helpful workshop features included: supportive atmosphere, interactions with others, practical examples, facilitator modeling and collaborating (working & connecting) with other participants
- 83% indicated they would likely implement what they learned in the workshop.
- 84% of respondents reported some degree of implementation at the one-year follow up.
- The high degree of implementation reported by participants is comparable to implementation after face-to-face workshops (Archie et al., 2022), helping to indicate that high quality online PD can be a more accessible form of PD.

References

Archie, T., Hayward, C. N., Yoshinobu, S., & Laursen, S. L. (2022). Investigating the linkage between professional development and mathematics instructors' use of teaching practices using the theory of planned behavior. *Plos One*, 17(4), e0267097.

Daly, D., Ethnography & Evaluation Research, and the Academy of Inquiry Based Learning (2021, June). *AIBL Handbook for Online Professional Development: Lessons Learned from PRODUCT Workshops*. Boulder, CO, and San Luis Obispo, CA: University of Colorado Boulder, Ethnography & Evaluation Research; and Academy of Inquiry Based Learning. https://tinyurl.com/AIBLHandbook

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MAA MATHEMATICAL ASSOCIATION OF AMERICA

NSF

University of Colorado Boulder

Novice Statistics Students' Forms of Reasoning When Reasoning About and With Sampling Distributions

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Keywords: statistics education, sampling distribution, deductive reasoning, inductive reasoning, abductive reasoning

Understanding ideas of sampling, sampling distributions, and statistical inference are important for improving students' statistical literacy and productive citizenship, particularly given calls for students to become "critical consumers of statistically-based results reported in populator media" (GAISE College Report ASA Revision Committee, 2016, p. 8). Despite the importance of understanding statistical inference and sampling distributions, research suggests that students struggle with these ideas (Sotos et al., 2007). One reason for this difficulty stems from the complex and abstract concept of sampling distribution, which requires the coordination of multiple statistical ideas, including distribution, the relationship between sample and population, sampling variability, randomness, and probability (Chance et al., 2004; Saldanha & Thompson, 2002; Noll & Shaughnessy, 2012). I hypothesize that another reason students experience difficulty understanding sampling distributions and statistical inference is because these ideas require students to reason with uncertainty, a kind of reasoning that differs greatly from the forms of reasoning that are expected and highlighted in mathematics, such as deductive and inductive reasoning.

Deduction, induction, and abduction are three classic forms of reasoning that can be modeled with a triadic structure involving a case, rule, and result. (Peirce, 1878; Reid & Knipping, 2010). A case is a specific observation that a condition holds. A condition describes an attribute of something, or a relation between things. A rule is a general proposition that states that if one condition occurs then another one will also occur. A result is a specific observation, similar to a case, but referring to a condition that depends on another one linked to it by a rule. The order in which one links a case, rule, and result determines the kind of inference—deduction, induction, or abduction—needed to gain more knowledge about the situation.

Using this triadic structure, I examined the forms of reasoning—deductive, inductive, and abductive—that novice statistics employed when reasoning about and with sampling distributions. The data come from two clinical interviews, each 60-75 minutes in length, with undergraduate students who recently completed an introductory statistics course. Participants were asked to work through a series of statistical tasks related to sampling distributions. I identified reasoning excerpts—instances in which participants provided justification or an explanation for a claim—from the interview transcripts. Within each reasoning excerpt, I identified case, rule, and result, then examined how the participant linked them, which provided evidence for the form of reasoning they employed.

Preliminary results indicate that participants used all three forms of reasoning when reasoning about and with sampling distributions, though not all three forms were productive ways of reasoning. However, abductive reasoning was powerful for some participants when making inferences from sample data to an unknown population of interest. In this poster presentation, I will provide examples of reasoning excerpts from a small subset of the participants and discuss possible implications for this work.

References

Chance, B., delMas, R., & Garfield, J. B. (2004). Reasoning about sampling distributions In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 295-323). Springer Dordrecht. https://doi.org/10.1007/1-4020-2278-6_13

GAISE College Report ASA Revision Committee. (2016). *Guidelines for assessment and instruction in statistics education (GAISE) college report 2016*. American Statistical Association. https://www.amstat.org/docs/default-source/amstat-documents/gaisecollege_full.pdf

Noll, J., & Shaughnessy, M. (2012). Aspects of students' reasoning about variation in empirical sampling distributions. *Journal for Research in Mathematics Education*, *43*(5), 509-556. https://doi.org/10.5951/jresematheduc.43.5.0509

Peirce, C. S. (1878). Illustrations of the Logic of Science VI: Deduction, Induction, and Hypothesis. *The Popular Science Monthly*, *13*(August), 470-482.

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching*. Sense.

Saldanha, L. A., & Thompson, P. W. (2002). Conceptions of sample and their relationship to statistical inference. *Educational Studies in Mathematics*, *51*(3), 257-270. https://doi.org/10.1023/A:1023692604014

Sotos, A. E. C., Vanhoof, S., Van den Noortgate, W., & Onghena, P. (2007). Students' misconceptions of statistical inference: A review of the empirical evidence from research on statistics education. *Educational Research Review*, *2*(2), 98-113. https://doi.org/10.1016/j.edurev.2007.04.001

Claire Miller
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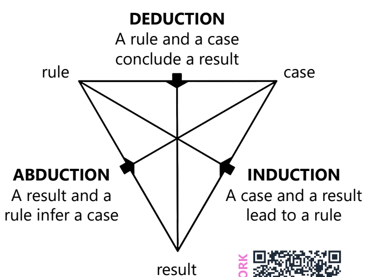
PAPER

Background

Data are everywhere. Being able to make sense of data collected from samples is important for improving students' statistical literacy and their ability to critically evaluate data-based claims.

Forms of Reasoning



FRAMEWORK

Methods

PARTICIPANTS
Undergraduate students, with varying majors, who recently completed an introductory statistics course

→

CLINICAL INTERVIEWS
Two task-based clinical interviews designed to examine participants' existing ways of reasoning about sampling distributions

→

PEIRCE'S (1878) FORMS OF INFERENTIAL REASONING
Identified case, rule, and result in each reasoning excerpt and examined how the participant linked them to infer the form of reasoning (deductive, inductive, or abductive) they employed

Results

Inductive reasoning is useful when estimating sampling variability, and abductive reasoning is powerful when estimating an unknown population parameter.

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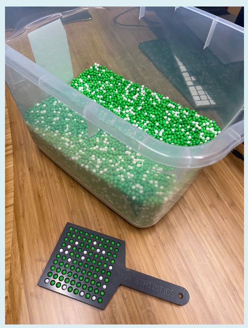
Abductive reasoning is powerful when making inferences from sample data.

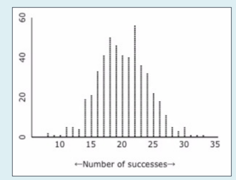
Lyla **induced** a range for a single sample outcome from observing a pattern in multiple sample outcomes.

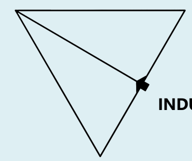


"I would say between 10 and 30. And really just from the samples we've taken so far, we've taken a lot, we haven't seen many go above 30. I don't know if we've seen one. I've seen we've reached 30, but I don't think we've gone above 30 and we haven't really gone below 10 besides these two in 500 [samples]."

I expect that any one single random sample of 100 undergraduates drawn from this population will produce between 10 and 30 business majors

Many random samples drawn from a population of undergraduates with 20% business majors





INDUCTION

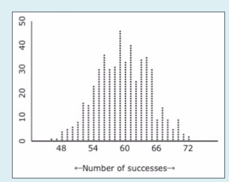
Very few samples produced a number of business majors that was less than 10 or greater than 30

Lorraine **abduced** that her sample could have come from a population with a parameter of 60%.

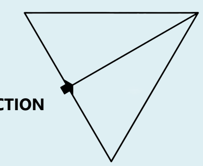
"It looks like [my sample outcome of] 52 would be reasonable. There's a pretty significant number of results that are 52. And then past that is where [the distribution] tends to kind of taper out. So maybe if the true proportion were 60, the lower bound would be like 48 or so."

A population with 60% business majors produced several sample outcomes of 52

The sample I drew could have come from a population with 60% business majors



ABDUCTION



A random sample drawn from a population with an unknown proportion of business majors produced a sample outcome of 52 business majors

Participatory Meta-Research: Opportunities for Engaging Pre-Service Teachers in Educational Research

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Keywords: Meta-Analysis, Meta-Synthesis, Pre-service Teachers, Algebra

As the body of research in the RUME community grows, synthesizing data across these studies becomes more important. This goal can be accomplished through *meta-research,* a method for aggregating studies in a field of research involving both meta-analysis and meta-synthesis. A *meta-analysis* consists of a systematic search, data extraction, and aggregation of quantitative data to explain phenomena in greater detail than individual studies alone (Cooper, 2017). *Meta-synthesis* is the qualitative partner to meta-analysis, and involves a "deliberate process of selecting studies with an emphasis on synthesizing, analyzing, and interpreting across the selected studies" (Thunder & Berry, 2016). Qualitative data is then extracted from the articles, appraised for quality, and then analyzed using any of the traditional qualitative techniques. The results of a meta-synthesis can highlight gaps in a given field, look for themes across studies, and condense swaths of research, making it more useful for practitioners (Erwin et al., 2011).

Participatory synthesis (Wimpenny & Savin-Baden, 2012) takes the perspective that by engaging various participants in the process of meta-research, a study can "ensure all voices are heard", present the "possibility of reciprocal learning", and "progress knowledge within a field" (p.697). For educational research, this presents a great opportunity for pre-service teachers. By participating in meta-research, a pre-service teacher can be exposed to research and interventions that might be relevant to their future classrooms. At the same time, the results of the research can be made richer by incorporating their viewpoints into topics they view as relevant and knowledge they consider useful for the classroom. In addition, if the team is diverse, it presents the opportunity to make the work more equitable by incorporating decisions from a diverse range of viewpoints.

This poster has two purposes. First, we will describe some of the latest techniques of meta-analysis and meta-synthesis through the lens of a current project (Sharpe et al., in prep) looking at algebraic teaching interventions for grades K-12. This includes illustrating the use of cutting-edge software for conducting a meta-synthesis such as Covidence and Raayan and looking at a new technique for extracting qualitative data from an article using artificial intelligence. Second, we will describe some of the challenges and opportunities that have resulted from this process to highlight how pre-service teachers and researchers could benefit. This includes discussions about the definition of an algebraic teaching intervention for screening, a look at some of the interventions that were discovered during search/screening that might be applicable for teaching, and a discussion about what information from an article would be useful for a practicing educator. Our hope is that this poster and the resulting discussion will encourage others to engage in meta-research and consider pre-service teachers as undergraduate researchers for such projects.

References

Cooper, H. (2017). *Research synthesis and meta-analysis: A step-by-step approach (5th ed.)*. Thousand Oaks, CA: SAGE Publications, Inc.

Erwin, E. J., Brotherson, M. J., & Summers, J. A. (2011). Understanding qualitative metasynthesis: Issues and opportunities in early childhood intervention research. *Journal of Early Intervention*, *33*(3), 186-200. https://doi.org/10.1177/1053815111425493

Sharpe, S., Mauntel, M., Choi, B., Aswad, M., Butler, B. (In Prep). PROTOCOL: A systematic review of what type of algebra teaching approach works, for who, and under what conditions: meta-analysis and meta-synthesis.

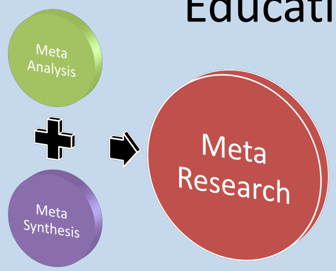
Thunder, K., & Berry, R. Q. (2016). Research commentary: The promise of qualitative metasynthesis for mathematics education. *Journal for Research in Mathematics Education*, *47*(4), 318-337. http://dx.doi.org/10.5951/jresematheduc.47.4.0318

Wimpenny, K., & Savin-Baden, M. (2012). Exploring and implementing participatory action synthesis. *Qualitative Inquiry*, *18*(8), 689-698. https://doi.org/10.1177/1077800412452854

Participatory Meta-Research: Opportunities for Engaging Pre-Service Teachers in Educational Research

What is Meta-Research?

A "**meta-analysis** uses a statistical procedure that aggregates and condenses a body of Quantitative research studies to a common standard metric, such as a mean effect size" (Finlayson & Dixon, 2008; Thunder & Berry, 2016, p.319).

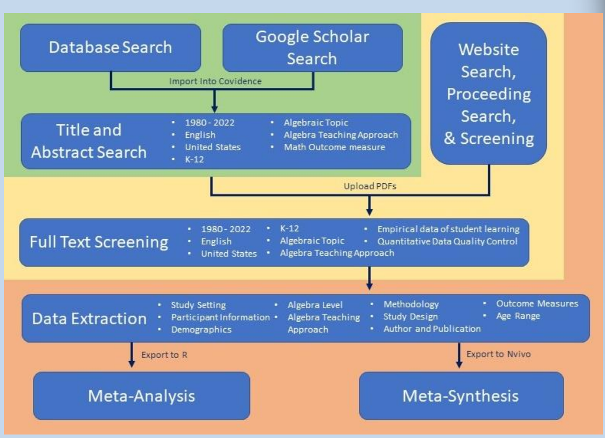


A **qualitative meta-synthesis** uses a "deliberate process of selecting [qualitative research] studies with an emphasis on synthesizing, analyzing, and interpreting findings across the selected studies" (Thunder & Berry, 2016, p.319).

Matthew Mauntel
Sheree Sharpe
Rebecca Butler
Maryam Aswad & Jerome Amedu
Bitna Choi

Participatory Meta-Research can provide a **rich research experience** for undergraduate pre-service teachers/mathematics students while incorporating valuable perspectives into the research synthesis process.

Meta-Research Process



Participatory Meta-Research

Participatory synthesis (Wimpenny & Savin-Baden, 2012) takes the perspective that by engaging various participants in the process of meta-research (p.697), a study can

- "ensure all voices are heard."
- present the "possibility of reciprocal learning."
- "progress knowledge within a field."

For **Pre-Service teachers** this might mean:

- Exposure to research and interventions for the classroom.
- The ability to influence the direction of research.
- Engage with under-represented perspectives and provide their own perspective.

Tools for Meta-Research

Searching:

- Citationchaser.com
- AI assisted Searching/ChatGPT

Screening

- Raayan.ai
- Covidence
- AI Screening

Extraction

- Meta-Reviewer
- AI Supported Memo'ing
- Covidence

Analysis

- R
- Nvivo

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Tasks for Exploring Fraction Understandings of Undergraduate Developmental Mathematics Students

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Texas State University

Keywords: fraction understandings, student thinking, undergraduate developmental mathematics

Many students experience difficulties with fractions (Mesa et al., 2014; Ngo, 2019). Thus, it is unsurprising that fractions act as a gatekeeper for college-level mathematics courses (Ngo, 2019). Research on how undergraduate developmental mathematics students understand fractions is scarce (Alexander, 2013; Mesa et al., 2014). Moreover, existing studies in this area seem to take a deficit perspective, focusing more on what students in this demographic do not understand about fractions rather than reporting what is productive in their thinking (Alexander, 2013; Baker et al., 2012; Doyle et al., 2015).

In this poster, I will report on tasks that I developed to examine the fraction thinking that undergraduate developmental mathematics students can utilize. I implemented these tasks during clinical interviews (Ginsberg, 1997), which come from a larger dissertation study that was designed to (a) examine how undergraduate developmental mathematics students understand fractions and (b) explore pathways to support fraction understandings that will help the student to be successful in future mathematics courses. In designing the tasks, I drew from existing K–12 fraction literature, including Lamon's (2020) fraction understandings and Steffe & Olive's (2010) fraction schemes, which I will explain in the poster. In short, Lamon's fraction understandings include part-whole, operators, quotients, measures, and ratio understandings. Fraction schemes I attend to include part-whole, partitive, and iterative fraction schemes (Steffe & Olive, 2010).

Twelve fraction tasks are spread over two 45-minute sessions for this clinical interview (Ginsberg, 1997). The interview tasks were designed to help me answer the following research question: How do undergraduate developmental mathematics students understand fractions? Specifically, (a) which of the five fraction understandings outlined by Lamon (2020) do the students operate with, and how do they support their fraction understandings; and (b) which of the fraction schemes outlined by Steffe & Olive (2010) do the students operate with and how do they support their fraction understandings?

In this poster, first, I will present an overview of Lamon's (2020) five fraction understandings, three of Steffe & Olive's (2010) fraction schemes, and how I correlated these views of fractions. Second, I will present examples of the tasks that I developed. I will identify which of Lamon's fraction understandings or Steffe & Olive's fraction schemes were intended to be elicited from each task. I will also provide the goals and design principles, as well as follow-up questions for these tasks. Finally, I will discuss some of the affordances and limitations of each task after implementing them in three clinical interviews (Ginsberg, 1997). For example, I designed a number line task that targets the measure understanding of fractions (Lamon, 2020). This task provides insight into the participants' fraction thinking beyond measures. However, due to the complexity of the measure understanding, the task may not reveal enough information to definitively say that the participant has constructed the measure fraction understanding (Lamon, 2020). I hope to provide examples of productive mathematical thinking that undergraduate developmental mathematics students have about fractions. This study aims to counter the deficit narrative prevalent in developmental mathematics.

References

Alexander, C. M. (2013). *Community college developmental education students' understanding of foundational fraction concepts*. (Publication No. 3614168). [Doctoral dissertation, University of California, Davis]. ProQuest One Academic.

Baker, W. J., Czarnocha, B., Dias, O., Doyle, K., Kennis, J. R. & Prabhu, V. (2012). Procedural and conceptual knowledge: *Adults reviewing fractions. Adults Learning Mathematics: An International Journal*, *7*(2), 39–65.

Doyle, K. M., Dias, O., Kennis, J. R., Czarnocha, B., & Baker, W. (2015). The rational number sub-constructs as a foundation for problem solving. *Adults Learning Mathematics: An International Journal*, *11*(1), 21–42.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (4th ed.). Taylor & Francis.

Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K–12 research. *Journal for Research in Mathematics Education*, *45*(2), 173–192.

Ngo, F. (2019). Fractions in college: How basic math remediation impacts community college students. *Research in Higher Education*, *60*, 485–520. https://doi.org/10.1007/s11162-018-9519-x

Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Mahwah: Lawrence Erlbaum.

Tasks for Fraction Understandings of Undergraduate Developmental Mathematics Students

Lucinda Ford
Department of Mathematics, College of Science and Engineering, Graduate College, Texas State University

Background and Goals

- Many students experience difficulties with fractions (Mesa et al., 2014; Ngo, 2019).
- Fractions act as a gatekeeper for college-level mathematics courses (Ngo, 2019).
- Research on the way that undergraduate developmental mathematics students understand fractions is scarce (Alexander, 2013; Mesa et al., 2014).
- Existing studies in this area focus more on what these students do not understand about fractions (Alexander, 2013; Baker et al., 2012; Doyle et al., 2015).
- Goals: (1) Build models of developmental math students' fraction understandings (Lamon, 2020; Steffe & Olive, 2010). (2) Explore pathways to support fraction understandings that help the student to be successful in future mathematics courses.

Research Questions

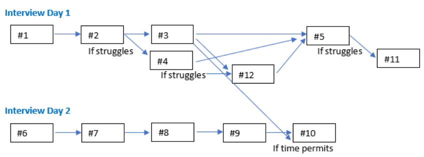
1. How do undergraduate developmental mathematics students understand fractions?
   a. Specifically, which of the five fraction understandings outlined by Lamon (2020) do the students operate with and how do they support their fraction understandings?
   b. Which of the fraction schemes outlined by Steffe & Olive (2010) do the students operate with and how do they support their fraction understandings?
2. What ways of fraction thinking are present (from RQ1) that could specifically help to further the participant's measure fraction understanding?

Task Design Overview

Methods

- Clinical interviews (Ginsberg, 1997)
- 12 questions over two 45-minute sessions
- Video recorded from multiple angles
- 8-12 participants
- Ongoing and retrospective analysis (Steffe & Thompson, 2000)

Potential Paths According to Students' Responses



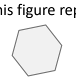
Analysis Framework

Lamon's Five Understandings	Steffe & Olive's Fraction Schemes
Part-Whole: Relating the number of equal parts of a unit to the unit	**Part-Whole Fraction Scheme: Simultaneously partitioning the whole into equal distant, connected, lengths**
Partitioning: Splitting the unit into equal parts **Iterating: Creating copies of the unit fraction**	**Partitive Fraction Scheme: The first true fraction scheme involves comparing the part to the whole. Note: The part-whole fraction scheme is a pre-requisite for the partitive fraction scheme.**
Partitioning and Iterating Beyond the Whole: Creating copies of the unit fraction beyond what is necessary to construct a figure that is equivalent to the whole	**Iterative Fraction Scheme (beyond the whole): Taking a unit fraction or a complex unit and iterating it to create a composite unit. Note 1: The partitive fraction scheme is a pre-requisite for the iterative fraction scheme. Note 2: This generally requires coordinating 3 levels of units.**
Operator: Viewing a fraction as a calculation(s) to be done (e.g., $\frac{a}{b} \cdot k$ means $a \cdot k \div b$)	
Quotient: Not only viewing a fraction as a set of instructions for calculation but assigning meaning to the result	
Measure: This is a geometric meaning of a fraction that involves magnitudes, or lengths, of intervals. **Magnitude recognition: Recognizing that relative size needs to be taken into account when working with the fraction**	Note: the above fraction schemes are part of measurement thinking (Steffe & Olive, 2010, p. 121).
Ratio: This fraction meaning can be comparative of the numerator and denominator as ordered pairs. Note: Not all ratios are fractions.	

This table shows how I correlate Lamon's (2020) fraction understandings with Steffe & Olive's (2010) fraction schemes.

Iterating Beyond the Whole

This figure represents a whole. Draw [   ]



[Goal] To examine whether the student has a partitive unit fraction scheme with a whole greater than one.

a. If student has difficulty, with #2, start with 4/6.
b. **Can you show me 1/6 in your figure?**
c. Round 2: **Let's try another one. Our figure still represents a whole. Draw 15/6**.

- Affordance: Fractions beyond the whole in action
- Limitation: 15/6 may not provide further information if the participant is still constructing an understanding of improper fractions.

Opening Tasks

What does 2/3 mean to you?

[Goal] To examine students' initial immediate response regarding their fraction conception without further prompting that targets specific fraction understandings, as in the following tasks. The student is being asked to read the task prompt to gather more information about how they read/write a fraction.

a. **Can you draw me a picture to show me the 2/3?** (Make sure to follow the student's language.)
b. ***Are there any other ways you can think about this fraction (2/3)?***
c. **Keep going until you exhaust this line of thinking.**

What does 7/3 mean to you?

[Goal] To examine whether they iterate the 1/3 beyond the whole. How they would make sense of an improper fraction in relation to their fraction conception.

a. ***Can you draw a picture of 7/3*** (if not done already-use their language)**?**
b. ***How do you know that's 7/3 and not 7/9?***

- Affordance: Initial reaction before targeted tasks
- Limitation: Not a full story of their understanding

Quotient

Jade shared 5 candies (cherry, chocolate, cinnamon, lime, and peppermint) among 3 friends. What amount of candy did each friend get?

[Goal] To examine whether the student has a quotient understanding of fractions.

a. Have index cards labeled with the candy flavors for the participant to demonstrate the sharing process.
b. If student can do the mental calculations and say that each person gets 5/3 candies each, ask student to use the index cards to show their thinking.
c. If student struggles, ask **Can you show me how to share these candies using the index cards?**

- Affordances:
  - Fair sharing (whole number not divisible by the number of people).
  - Partitioning an object into thirds.
  - Operator vs. quotient understanding
- Limitation: Consideration of number choices

Measure

Directions: Give the participant four index cards marked with 1/2, 2/3, 3/8, and 7/6, respectively.

Without using decimals, put these fractions in order from least to greatest.

[Goal] To examine whether the student has a measurement understanding of fractions.

a. If student indicates they do not know how to do this without fractions, ask the student to give me their best guess.
b. **What were you thinking as you ordered the fractions?**
c. Give the participant a piece of aper with a line across it lengthwise. **If this is a number line, where would you place the fractions as accurately as you can?**
d. *Offer sticky flags for placing numbers (they are moveable).*
e. Pay attention to whether the participant marks 0, 1, etc. and/or benchmarks on the number line.

- Affordances: Operator understanding, commensurate fractions, percentage thinking, and benchmarks
- Limitations: May need further questioning due to the complexities of the measurement understanding

References

Alexander, C. M. (2013). *Community college developmental education students' understanding of foundational fraction concepts.* (Publication No. 3614168). [Doctoral dissertation, University of California, Davis]. ProQuest One Academic.

Baker, W. J., Czarnocha, B., Dias, O., Doyle, K., Kennis, J. R. & Prabhu, V. (2012). Procedural and conceptual knowledge: Adults reviewing fractions. *Adults Learning Mathematics: An International Journal, 7*(2), 39–65.

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Ginsburg, H. (1997). Entering the child's mind: The clinical interview in psychological research and practice. Cambridge University Press.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (4th ed.). Taylor & Francis.

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Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Mahwah: Lawrence Erlbaum.

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Comparing Calculus Students' Use of Problem-Solving Strategies on Related Rates of Change Problems in a Traditional versus Online Homework Format

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Keywords: problem-solving, related rates, calculus, online-homework

With the advent of many online platforms for assigning and grading homework, more and more of the homework in calculus no longer entails paper-and-pencil homework (Dorko, 2020). As such, with the prevalence of associated scaffolding embedded in these platforms, more needs to be understood about whether this tends to over-proceduralize topics, such as related rates of change (RRC), and undermine roles these topics may have in further developing students' capacity in mathematical problem solving or mathematical reasoning. Engelke (2008) maintains that to successfully solve RRC problems, students require well-developed mathematical problem-solving skills. However, Mkhatshwa (2020) asserts that few studies have explored students' reasoning on solving RRC problems. The purpose of this investigation is to explore how problem-solving strategies (PSS) may differ when solving RRC problems presented in a traditional paper-and-pencil format versus RRC problems presented in an online platform which includes typical options for scaffolding help as well as "view an example" features. In this poster, we address the following research questions: (1) How do students' PSS when working online homework on RRC problems compare with their PSS when working paper-and-pencil homework RRC problems? (2) What influence does the 'view an example' feature in online homework have on a student's PSS when working an online RRC homework problem?

This study takes place at a large, urban research university located in the southwestern United States during a 15-week first-semester calculus course which incorporated the use of an online platform for homework. After scoring 318 participants' written work on a free response RRC common midterm exam problem for the extent to which their work exhibited processes for productive problem solving of RRC problems (Engelke, 2008), participants were invited for task-based interviews using a scheme aimed at ensuring that participants from different score ranges would be included in the interviews. Fourteen participated in task-based interviews. Because we were interested in comparing PSS when working paper-and-pencil versus online homework problems, the task-based interviews included four RRC problems, two sets of paired tasks of a paper-and-pencil format problem and a similar problem on the online platform. Qualitative data analysis software was used to code the data using a priori codes from Álvarez et al. (2019) and Carlson and Bloom (2005). Common themes were identified using thematic analysis (Braun and Clark, 2016).

Data analysis reveals more instances of PSS from the exemplary group (top quartile scorers on midterm RRC problem) when compared to the other groups. In addition, participants used more PSS when solving paper-and-pencil RRC problems than when solving online RRC problems. Further, participant's use of PSS decreased when using the 'view an example' feature in the online format. Three categories emerged on participant use of the 'view an example' feature (1) those who mimic the example (ME), (2) those who use the example to learn the process (LP), and (3) those who refuse to use the feature (RF). Participants using this feature for ME used fewer PSS than those whose use was classified as LP or RF. Findings suggest that the format of and resources for online homework RRC problems may need to change to ensure that students engage in using PSS in a manner comparable to traditional paper-and-pencil homework.

References

Álvarez, J.A.M., Rhoads, K., Campbell, R.C. (2019). Toward Designing and Developing Likert Items to Assess Mathematical Problem Solving. In: Liljedahl, P., Santos-Trigo, M. (eds) *Mathematical Problem Solving*. ICME-13 Monographs. Springer, Cham, 231-260.

Braun V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, 3(2), 77-101.

Carlson, M. P., & Bloom, I. (2005). The Cyclic Nature of Problem Solving: An Emergent Multidimensional Problem-Solving Framework. *Educational Studies in Mathematics*, 58(1), 45–75.

Dorko, A. (2020). What do we know about student learning from online mathematics homework?. *In Teaching and learning mathematics online* (pp. 17-42). Chapman and Hall/CRC.

Engelke, N. (2008). Developing the solution process for related rates problems using computer simulations. In Conference on Research in Undergraduate Mathematics Education.

Mkhatshwa, T. P. (2020). Calculus students' quantitative reasoning in the context of solving related rates of change problems. Mathematical Thinking and Learning, 22(2), 139-161.

Comparing Calculus Students’ Use of Problem-Solving Strategies on Related Rates of Change Problems in a Traditional versus Online Homework Format

Tyson C. Bailey
James A. M. Álvarez

THE UNIVERSITY OF TEXAS AT ARLINGTON

Introduction

With the advent of many online platforms for assigning and grading homework, more and more of the homework in calculus no longer entails paper-and-pencil homework (Dorko, 2020). As such, with the prevalence of associated scaffolding embedded in these platforms, more needs to be understood about whether this tends to over-proceduralize topics, such as related rates of change (RRC), and undermine roles these topics may have in further developing students’ capacity in mathematical problem solving or mathematical reasoning. Engelke (2008) maintains that to successfully solve RRC problems, students require well-developed mathematical problem-solving skills. However, Mkhatshwa (2020) asserts that few studies have explored students’ reasoning on solving RRC problems.

The purpose of this investigation is to explore how mathematical problem-solving strategies (MPSS) may differ when solving RRC problems presented in a traditional paper-and-pencil format versus RRC problems presented in an online platform which includes typical options for scaffolding help as well as “view an example” features.

Research Questions

(1) How do students’ mathematical problem-solving strategies when working an online homework related-rates of change problems compare with their mathematical problem-solving strategies when working paper-and-pencil related-rates of change homework problems?

(2) What influence does the ‘view an example’ feature in online homework have on a student’s mathematical problem-solving strategies when working an online related-rates of change homework problem?

Phase 1: Textbook Problem to Online Problem

Frequency of Problem-Solving Strategies Utilized in Phase 1

Student	Q1	Q2
Echo	6	4
Eboy	4	3
Earl	22	0
Ed	5	3
Elsa	12	13
Eve	5	3
Pamela	5	0
Paris	8	4
Pat	7	4
Penny	8	8
Percy	5	0
Peter	8	0
David	4	5
Donald	8	2

23. Time-lagged flights An airliner passes over an airport at noon traveling 500 mi/hr due west. At 1:00 P.M., another airliner passes over the same airport at the same elevation traveling due north at 550 mi/hr. Assuming both airliners maintain their (equal) elevations, how fast is the distance between them changing at 2:30 P.M.?

ates | Question 8, 3.11.23 Part 3 of 3 | HW Score: 0%, 0 of 14 points | Points: 0 of 1 | Save

An airliner passes over an airport at noon traveling 530 mi/hr due west. At 1:00 p.m., another airliner passes over the same airport at the same elevation traveling due north at 580 mi/hr. Assuming both airliners maintain their (equal) elevations, how fast is the distance between them changing at 2:00 p.m.?

The equation relating the horizontal distance between the first airliner and the airport, a, the horizontal distance between the second airliner and the airport, b, and the horizontal distance between the two airliners, c is $a^2+b^2=c^2$.

Differentiate both sides of the equation with respect to t.

$$(2a)\frac{da}{dt}+(2b)\frac{db}{dt}=(2c)\frac{dc}{dt}$$

(Do not simplify.)

At 2:00 p.m., the distance between the airliners is changing at a rate of about 743.4 mi/hr.
(Round to the nearest tenth as needed.)

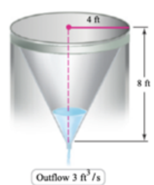
Phase 2: Online Problem to Textbook Problem

Frequency of Problem-Solving Strategies Utilized in Phase 2

Student	Q3	Q4
Echo	8	8
Eboy	3	2
Earl	15	13
Ed	1	6
Elsa	10	11
Eve	3	7
Pamela	1	4
Paris	1	4
Pat	4	5
Penny	8	5
Percy	0	4
Peter	0	1
David	4	4
Donald	0	5

ates | Question 11, 3.11.36 Part 3 of 3 | HW Score: 0%, 0 of 14 points | Points: 0 of 1 | Save

An inverted conical water tank with a height of 8 ft and a radius of 4 ft is drained through a hole in the vertex at a rate of 3 ft^3/s (see figure). What is the rate of change of the water depth when the water depth is 4 ft? (*Hint*: Use similar triangles.)



Let V be the volume of water in the tank and let h be the depth of the water. Write an equation that relates V and h.

$$V=\frac{\pi}{12}h^3$$

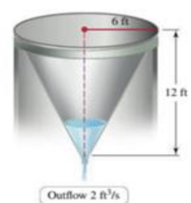
(Type an exact answer, using π as needed.)

Differentiate both sides of the equation with respect to t.

$$\frac{dV}{dt}=\left(\frac{\pi}{4}h^2\right)\frac{dh}{dt}$$

When the water depth is 4 ft, the rate of change of the water depth is about −0.24 ft/s.
(Round to the nearest hundredth as needed.)

36. Draining a tank An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of 2 ft^3/s (see figure). What is the rate of change of the water depth when the water depth is 3 ft? (*Hint:* Use similar triangles.)



http://tinyurl.com/4f9s57cc

Methods

We used student (n=338) departmental midterm performance levels on RRC problems to select participants for task-based interviews. Participants were drawn from three groups, based upon their performance and problem-solving score quartile on the midterm RRC problems.

Of the 14 participants consenting into the study, six came from the fourth quartile, six came from the third quartile, and two came from the second quartile. The task-based interviews entailed solving two pairs of RRC problems (textbook-to-online, online-to-textbook).

Transcripts were coded using thematic analysis techniques (Braun & Clarke, 2006) with a priori codes from Alvarez et al. (2019) and Carlson and Bloom (2005) and also examined for emergent codes.

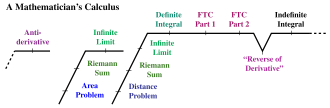
- The problem-solving codes used included orienting/sense-making, planning, representing and connecting, executing, reviewing, justifying, checking.
- Emergent codes included the use of *view an example* for *process*, to *mimic*, or for *sense-making*.

Findings

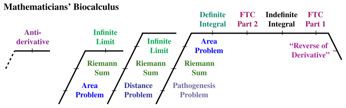
Participants appear to use MPS strategies more frequently when solving the traditional paper and pencil-related rates of change of problems.

The findings suggest that students may be using features of online homework platforms on RRC problems in a manner that circumvents the use of MPSS. That is, ‘view an example’ feature in online homework may be scaffolding the problem-solving process in a manner that leads to students missing opportunities to further develop their MPSS.

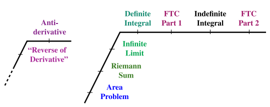
A Mathematician's Calculus



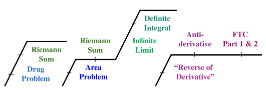
Mathematicians' Biocalculus



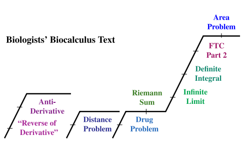
Biomathematicians' Biocalculus Text (Example 1)



Biomathematicians' Biocalculus Text (Example 2)



Biologists' Biocalculus Text



Biologists and Mathematicians tell very different **stories** of the **definite integral**.

On Narratives of the Definite Integral in Biocalculus

Melinda Lanius, Auburn University

Goal: Explore variation in the presentation of the definite integral in calculus texts written by mathematicians and those written by biologists.

Corpus: (biology-oriented texts are classified by authorship teams + preface materials).

- **[Mathematician's Calculus.]** James Stewart. (2016). *Calculus: Early transcendentals* (8th Ed.). Cengage Learning.
- **[Mathematicians' Biocalculus.]** James Stewart & Troy Day. (2015). *Biocalculus: Calculus for Life Sciences.* Cengage Learning.
- **[Biomathematicians' Biocalculus (Example 1).]** Claudia Neuhauser & Marcus L. Roper. (2018). *Calculus for biology and medicine.* Upper Saddle River: Pearson.
- **[Biomathematicians' Biocalculus (Example 2).]** Erin N. Bodine, Suzanne Lenhart, & Louis J. Gross. (2014). *Mathematics for the life sciences.* Princeton University Press.
- **[Biologists' Biocalculus.]** Alan Garfinkel, Jane Shevtsov, & Yina Guo. (2017). *Modeling life: the mathematics of biological systems.* Springer International Publishing AG.

Framework: Analysis via literary tradition of narrative.

- Established use in studying mathematics curriculum (e.g. see Dietiker, 2013; Dietiker, 2015; and Miežys 2023).
- I'm following Netz (2005) - mathematics texts tell a story:
  "*Some passages - descriptive - add detail to the fictional world, constructing its underpinning of reality; other passages - narrative - unfold the plot that takes place in that fictional world*" (2005, p. 262).
- Final product of analysis - a set of chronological diagrams that position the mathematical concepts (descriptions) along a story arc to convey how narrative relates these topics.

How to interpret diagrams:

General features.

- Upward slope (rising action) indicates a concept building on another concept.
- Horizontal lines (a climax or false-climax) are concepts that were separated from the general text as important (e.g. by a box.)
- Downward slope (denouement) indicates a closing thought.

Notable combinations of general features.

- A horizontal line that ends without a decent immediately following it indicates a split narrative, where the reader is to keep this train of ideas in mind, but we start again at height zero to build up another sequence of concepts.
- A horizontal line appearing at a height greater than zero without an upward sloping line preceding it indicates a *deus ex machina*, or the abrupt introduction of a concept in a climactic role without narrative connection to proceeding concepts:

(Surprising) Finding 1:
No narratives used a *deus ex machina* to reach the definite integral.

Note. The *deus ex machina* is a popular narrative device in stories of the derivative, e.g., to introduce the derivative function in Biomathematicians' Biocalculus (Example 1).

(Surprising) Finding 2:
In some narratives, the definite integral was not a climax.

Future Directions.

[1] Explore the scope and rigidity of student's structural expectations and how this impacts learning.

[2] Explore other fruitful constructs from literature, e.g., the analytic tool of implied reader inspired implied student (Ulriksen; 2009).

References

Dietiker, L. (2013). Mathematical texts as narrative: Rethinking curriculum. *For the Learning of Mathematics* 33(3), 14-19.

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, 90, 285-302.

Miežys, V. (2023). Cheap plot tricks and plot holes in mathematical stories. *Educational Studies in Mathematics*, 113, 1-15.

Netz, R. (2005). The aesthetics of mathematics: a study. In Mancosu, P., Jørgensen, K. F. & Pedersen, S. A. (Eds.) *Visualization, Explanation and Reasoning Styles in Mathematics*, pp. 251–293. Springer.

Ulriksen, L. (2009). The implied student, *Studies in Higher Education*, 34(5), 517-532.

Using a Rubric from Pedagogical Partnership to Improve the Accessibility and Usability of Desmos Activities for Students in Calculus 1

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Keywords: Desmos, Pedagogy, Interactive

There was a successful departmental effort to improve the calculus DFWI rates at a midwestern university (Mingus & Koelling, 2021). To maintain this change during the pandemic, instructors developed Desmos-based activities to encourage student engagement and active learning. This platform allows students to interact with tools on activities designed by their instructors. Since the implementation, instructors sought feedback from the students on the digital experience. This study focuses on a partnership between the students and the instructors to develop a rubric to evaluate Desmos activities for the Calculus course.

Our goal is to understand how students interact with digital platforms to further improve students' learning experience. To accomplish this, a pedagogical partnership with undergraduate students was created with the intent to co-develop a rubric to evaluate activities. Pedagogical research with students provides benefits to both the students and instructors. The students gain a greater understanding of course content, learning, teaching, and improved agency. The instructor gains a greater understanding of student thinking and feedback on how to meet student needs. (Bovill et al., 2011, Cook-Sather & Motz-Storey, 2016, Partridge & Sandover, 2010)

An initial rubric was created as a result after the first focus group interview. The rubric consists of three categories: Overall, Between the Slides, and On Each Slide. The Overall category addresses the content in the whole Desmos activity. The Between the Slides category addresses the transitions of the content within the Desmos activity. The On Each Slide category addresses the content within the individual slide.

The poster will consist of the rubric and slides from an activity before and after the application of the rubric. It will be discussed how the team has used the rubric to modify Desmos activities to improve their accessibility and usability for students.

References

Cook-Sather, A. & Motz-Storey, D, (2016) Viewing Teaching and Learning from a New Angle: Student Consultants' Perspectives on Classroom Practice, College Teaching, 64:4, 168-177, DOI: 10.1080/87567555.2015.1126802

Bovill, C., Cook-Sather, A., & Felten, P. (2011). Students as co-creators of teaching approaches, course design and curricula: Implications for academic developers. International Journal for Academic Development, 16(2), 1–5. https://doi.org/10.1080/1360144X.2011.568690

Mingus, T. & Koelling, M. (2021) A Collaborative Approach to Coordinating Calculus 1 to Improve Student Outcomes, PRIMUS, 31:3-5, 393-412, DOI: 10.1080/10511970.2020.1772919

Partridge, L., & Sandover, S. (2010). Beyond 'listening' to the student voice: The undergraduate researcher's contribution to the enhancement of teaching and learning. Journal of UniversityTeaching & Learning Practice, 7(2), 1–19. https://ro.uow.edu.au/jutlp/vol7/iss2/4/

Using a Rubric from Pedagogical Partnership to Improve the Accessibility and Usability of Desmos Activities for Students in Calculus 1

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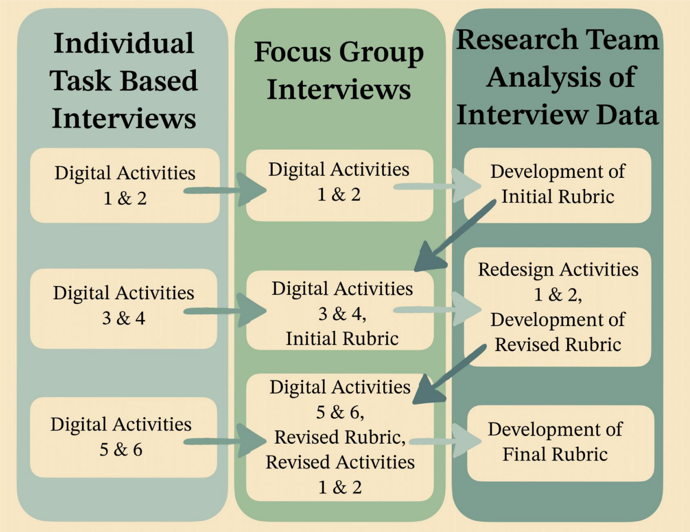
Introduction

- DFWI rates decreased after previous course redesign [3]. Success of the course redesign motivated implementation of Desmos activities.
- Student response to the implementation of activities motivated pedagogical partnership.
- Goal was to create a rubric which instructors could use to evaluate the effectiveness of Desmos activities.

Study Design



- Students were selected from calculus courses taught the semester prior
- Interviews, rubric creation, and slide edits were conducted outside of the classroom

Rubric

Purpose: Evaluate the Desmos Activities to improve students' learning

Breakdown:

- Overall
- Between Slides
- On each Slide

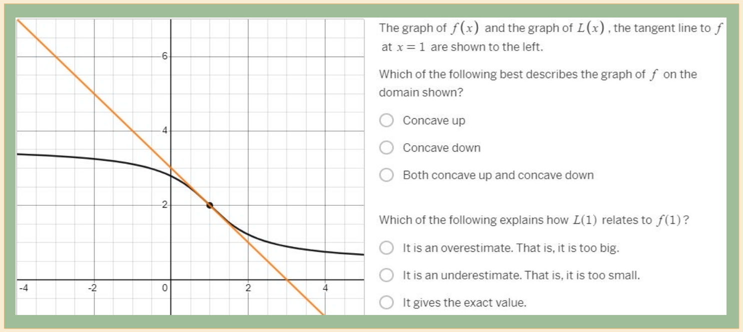
Scan for References & Supplemental Materials

![WMU logo] Western Michigan University

Data Sample

Slide from Desmos Activity 5

Students below are discussing the slide shown during a focus group session after the creation of the initial rubric.



Student A: "I guess if you're able to zoom in, you get more clear understanding of graph. Sometimes, I probably would fall into that route to keep going in and out. If I got bored or distracted in class."

Student B: "Well, I guess it's just a give and take. I feel like it would be a distraction, but it's the kind of distraction that could be learned from…"

Discussion

- Student discussions underscored the necessity for equitable learning material
- Using students as an authority to their experience in the classroom helps us build better curriculum materials.
- The study had to adapt to real-life constraints including availability, student interest, and fluctuations in student turnover.

Transformations in the Plane: Towards Interpretations and Proofs of Linearity

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Florida State University

Christine Andrews-Larson
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Keywords: linear algebra, linear transformations, proof

In recent years, student reasoning in linear algebra has been a rich area of research, but relatively few studies have focused on linearity and proof in this area (Stewart et al., 2019). In this study, our research questions are: (1) What are students' evoked concept images of linear transformations in the context of a Desmos module focused on transformations in the plane? (2) What do students report finding helpful in this module?

Existing literature has examined how students conceive of linear transformations in relation to their interpretations of functions in prior coursework, and how students shift from local to global views of linear transformations (Zandieh et al., 2017; Andrews-Larson et al., 2017), but there is limited work on how students make sense of the properties of linearity, particularly in relation to their proof activity. For our analysis, we draw on the notion of concept image from Tall and Vinner (1981) who described a person's concept image for a particular concept as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152).

The data for this report comes from Desmos activity completed by 54 U.S. undergraduate linear algebra students at the end of a unit on linear transformations. The goals of this activity were to help students identify and prove whether or not given transformations in the plane were linear. We analyzed our data by first developing data-driven codes (deCuir-Gunby et al., 2011) for students' responses to the questions "Explain in your own words what it means for a transformation to be linear and why it matters whether a transformation is linear," and "Which of the examples or slides in this Desmos activity did you find most helpful for your learning and why?" We aim to develop theory-driven codes to relate our findings to those of Zandieh et al. (2017), particularly identifying aspects of students' concept images of linear transformations in terms of: computations, properties, and clusters of metaphorical expressions.

We identified three broad categories of responses regarding the meaning of linear transformations that relate to students' concept images: references to the formal definition, preservation of straightness and parallelism of lines, and references to uniformity in how an image is transformed (e.g. points affected equally, retaining proportional changes in position and magnitude, reversibility, and preservation of orientation). We identified four primary aspects of the Desmos activity that students reported finding helpful: features that helped students develop intuition for linear transformation (e.g. via visualization and distinguishing examples from non-examples), features that helped students relate algebraic to geometric interpretations of linear transformations, and features that helped students make sense of the formal definition and proof methods (e.g., separating out interpretations of the two properties of linear transformations; providing examples of correct proofs and prompts to write a proof). In our poster, we will examine relationships among the meanings students ascribe to linear transformations, what students found helpful, and existing literature.

References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology, 48*(6), 809–829. http://dx.doi.org/10.1080/0020739X.2016.1276225

DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: An example from a professional development research project. *Field Methods*, *23*(2), 1361–55. https://doi.org/10.1177/1525822X10388468

Stewart, S., Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: themes from recent research and evolving research priorities. *ZDM*, *51*, 1017–1030. https://doi.org/10.1007/s11858-019-01104-1

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: the case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0

Transformations in the Plane: Towards Interpretations and Proofs of Linearity

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Christine Andrews-Larson

NSF grant #1914793

MOTIVATION

In recent years, student reasoning in linear algebra has been a rich area of research, but relatively few studies have focused on linearity and proof in this area (Stewart et al., 2019).

RESEARCH QUESTIONS

(1) What are students' evoked concept images of linear transformations in the context of a Desmos module focused on transformations in the plane?

(2) What do students report finding helpful in this module?

PARTICIPANTS

53 undergraduate linear algebra students.

METHODS

We analyzed developed data-driven codes (deCuir-Gunby et al., 2011) for students' responses to prompts in the Desmos module.

Take a picture to access the Desmos unit on Linear Transformations

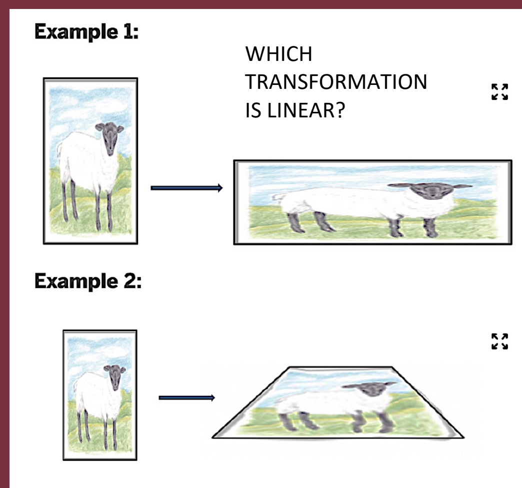
WHAT DOES IT MEAN FOR A TRANSFORMATION TO BE LINEAR?

P005: This means that the order of operations of T(V1) + T(V2), and T(V1 +V2) does not matter. You can transform then add or add then transform. Also applies to scaling with constant c. Scale before the transformation or after, it should be the same in a linear transformation.

P009: For a transformation to be linear, it has to be able to be written linearly in an algebraic form. This is made evident in the formal definition.

P0028. A linear transformation is a transformation that keeps the origin the same, straight lines remain straight, and parallel lines remain parallel. It also follows the properties T(V1+V2) = T(V1)+T(V2) and cT(u)= T(cu). It matter whether a transformation is linear because it helps us understand the nature of transformation and it does not completely change the original thing.

P0018. A transformation is linear when a vector's position relative to other vectors is the same from before the transformation to after, both additively and multiplicatively. It matters because it means the transformation is reversible, and that the relationships between vectors are maintained from before to after the transformation.



Visualization can help students make sense of the formal definition of linear transformations; students take away a variety of interpretations from visualization activities.

RESULTS

We identified three categories of responses that relate to students' concept images based on:

- references to the formal definition,
- preservation of straightness and parallelism of lines, and
- references to uniformity in how an image is transformed (e.g. points affected equally, retaining proportional changes in position and magnitude, reversibility, and preservation of orientation).

Based on student reports, useful features of the module were those that helped students:

- develop intuition for linear transformations (e.g. via visualization; distinguishing examples from non-examples),
- relate algebraic to geometric interpretations of linear transformations, and
- make sense of the formal definition
- model appropriate proof methods (e.g., separating interpretations of two properties of linear transformations; providing examples of correct proofs)

Exploring Mathematical Intuition and its Role in Physics Problem-Solving

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A large body of work in both mathematics and physics education research has explored mathematical reasoning in physics contexts (Redish & Kuo, 2015; Van den Eynde et al., 2020; Serbin & Wawro, 2022; Zimmerman et al. 2023b). Studies show that expert reasoning in physics problem-solving is not purely physical or mathematical, but a blended way of thinking (Redish & Kuo, 2015; Schermerhorn & Thompson, 2023). In this study, we build on the recent work of Zimmerman et al. who observed that one way physics experts reason while modeling is by “mathematical riffing”–manipulating familiar physical models, having a feel for when one might be useful, and quickly rejecting those that are not (Zimmerman et al., 2023a). Just as jazz might seem random and unpredictable to the untrained ear, mathematical riffing is hard to pinpoint and practice for physicists-in-training, even at the graduate level. There is no direct instruction to develop this kind of mathematical intuition on which experts rely to propel their thinking.

While the physics community values mathematical intuition as a mark of expertise, little work has been done to understand what it looks like, or what role it plays in generating quantitative physics models. To better understand mathematical intuition and its role in physics problem-solving, we conducted a pilot study in which we interviewed three physics graduate students after completion of their first year PhD coursework. We asked the interviewees to think aloud while solving unfamiliar graduate-level physics problems that relied on familiar mathematics. The data were coded using a grounded theory approach (Glaser & Strauss, 1967). This poster describes the study and its preliminary findings.

We identified three ways that mathematical ideas drive physics reasoning that, collectively, we call *mathematical impetus*. We see mathematical impetus as part of the set of behaviors associated with riffing. Each feature relies on *interactions with symbols* and is associated with having an *expectation of an outcome* without feeling the need to prove it mathematically. We also developed a description of a few distinct ways that graduate students act on mathematical impetus in tandem with other problem-solving skills. These findings help characterize problem-solving approaches that are typical in graduate classrooms, and can help inform instruction, making expert mathematical intuition more transparent to the learner.

References

Glaser, B. G., & Strauss, A. L. (1967). The Discovery of Grounded Theory. Chicago: Aldine.

Redish, E. F., & Kuo, E. (2015). Language of Physics, Language of Math: Disciplinary Culture and Dynamic Epistemology. Science and Education, 24(5–6). https://doi.org/10.1007/s11191-015-9749-7

Schermerhorn, B.P. & Thompson, J.R. (2023). Making context explicit in equation construction and interpretation: Symbolic blending. Physical Review Physics Education Research, 19(2), 020149(24). https://doi.org/10.1103/PhysRevPhysEducRes.19.020149

Serbin, K.S., Wawro, M. (2022). The Inextricability of Students’ Mathematical and Physical Reasoning in Quantum Mechanics Problems. International Journal of Research in Undergraduate Mathematics Education. https://doi.org/10.1007/s40753-022-00174-z

Van den Eynde, S. & Schermerhorn, B.P. & Deprez, J. & Goedhart, M. & Thompson, J.R. & De Cock, M. (2020). Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics: A case study in the context of the heat equation. Physical Review Physics Education Research, 16(1), 010114(18). https://doi.org/10.1103/PhysRevPhysEducRes.16.010114

Zimmerman, C. & Olsho, A. & Loverude, M. & Brahmia, S. W. (2023). Empirical evidence of the inseparability of mathematics and physics in expert reasoning about novel graphing tasks. https://doi.org/10.48550/arXiv.2308.01465

Zimmerman, C. & Olsho, A. & Loverude, M. & Brahmia, S. W. (2023). Expert covariational reasoning resources in physics graphing tasks. https://doi.org/10.48550/arXiv.2306.00921

Exploring “Mathematical Intuition” and its Role in Physics Problem-Solving

Ella Henry, Charlotte Zimmerman, John Goldak, Suzanne White Brahmia | University of Washington, Department of Physics

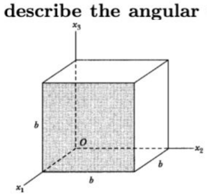
Background

- **This study:** We explore the *“mathematical intuition”* experts rely on to *propel their thinking,* and the role it plays in *generating quantitative physics models.*
- **Analysis framework:** Zimmerman et al. physics expert way of reasoning–*“mathematical riffing”*
  - manipulate familiar physical models, recognize when one might be useful, and quickly reject those that are not [1].
  - analogous to jazz: seems random and unpredictable to physicists-in-training

Methodology

- Interviewed 3 first year graduate students at University of Washington in July 2023.
- Think-aloud interviews that included solving the following classical mechanics problem [2]:

> You wish to study this object’s angular momentum as it rotates with the origin fixed, given its moment of inertia tensor, the equation of motion for rigid body rotations, and the initial conditions
>
> - $\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$
> - $\omega_1(0) = \omega_2(0) = 0\ ,\quad \omega_3(0) = \omega_0,\ \omega_0 > 0$
>
> **What would you do next to describe the angular velocity of this cube?**
>
> $$I_{ij} = mb^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$
>
> $$0 = \frac{d\boldsymbol{L}}{dt} = \frac{dL_i}{dt}\boldsymbol{e}_i + L_i\boldsymbol{\omega} \times \boldsymbol{e}_i$$



- Unfamiliar physics topic that relied on familiar mathematical tools.
- Data coded using a grounded theory; results are emergent from the data.

Some Emergent Ways of Symbolic Reasoning

Defining quantities	providing a working definition or understanding of what a symbol represents
Analogous symbols	using familiar quantities to understand new or unfamiliar ones
Symbols-guided math doing	doing algebraic manipulations, guided by symbols' meanings, interpretations, implications
Symbols-motivated process choosing	choosing a process to move forward in problem solving, cued by symbols' meanings, interpretations, implications
Extracting physical meaning	coming up with a physical interpretation for a mathematical result
Expected outcome	having an expectation for the mathematical/physics implications based on symbols

Mathematical Impetus (bracketing: Symbols-motivated process choosing; Extracting physical meaning; Expected outcome)

Examples of “mathematical impetus” from the transcripts

Symbols-motivated process choosing	Extracting physical meaning	Expected outcome
"But then usually, when I encounter [...] a differential equation like this, [...], where I'm like, fairly certain that there's no closed form solution--- um, we should just linearize? I think?"	"And then we know that DL dt is going to be zero. So this is going to be, um, constant in time."	"yes, that like decouples the set of equations very nicely. Like with this, you get, instead of the nasty i j omega j within the, um, in the differenti---in the differential equations I got, I'll just get a bunch of like [...]decoupled equations."

Preliminary Results

- 3 emergent ways of symbolic reasoning drive physics reasoning; we call this driver ***mathematical impetus***:
  - associated with mathematical riffing.
  - ***interactions with symbols*** with an ***expectation of an outcome*** without feeling the need to prove it mathematically.
- We are embarking on a case study analysis of the individual graduate students to characterize their ***unique ways*** of blending mathematical impetus with other problem-solving skills.

Further Work

- “Mathematical intuition” as a way of knowing is poorly understood and extremely important in physics. There is *no direct instruction to develop it in our classrooms.*
- Toward a characterization of mathematical impetus we will:
  1. continue the analysis of the transcripts to uncover the diverse ways that graduate students use mathematical impetus when solving new physics.
  2. use the codebook developed in this project to study how experts engage with mathematical impetus in generating new physics.

For more materials:

http://tinyurl.com/y8vkvm9x

[1] Zimmerman, C. & Olsho, A. & Loverude, M. & Brahmia, S. W. (2023). Empirical evidence of the inseparability of mathematics and physics in expert reasoning about novel graphing tasks. arXiv:2308.01465v1.
[2] problem modified from chapter 11 of Thornton, S. & Marion, J. (1988). Classical Dynamics of Particles and Systems (3rd edition). Saunders College Publishing.

Pirates, Wobbly Jelly, and Bunnies … Analyzing Applets and Video Games from the Perspective of RME

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Michelle Zandieh, Arizona State University
David Plaxco, Clayton State University

Keywords: Linear Algebra, Technology, Realistic Mathematics Education, Game-based Learning, Dynamic Geometry Software

Realistic Mathematics Education (RME) is a curricular design theory that involves designing materials with realistic starting points and guiding students through four levels of activity: situational, referential, general, and formal (Gravemeijer, 2020, 1999). This theory has been used to develop a multitude of curricular materials including ones that incorporate technology. This poster investigates the design of three different technologies (video games, GeoGebra, and applets) designed to elicit different forms of activity in an RME sequence. Each technology has a 2D and a 3D version and was either could be used or was designed to be used in a linear algebra classroom. This poster will present the design heuristics of each type of technology that lend themselves to a particular type of activity to be evoked with the goal of providing insight into designing technology for use in an RME-type sequence.

The first technology we analyze is a 2D video game called *Vector Unknown* which was designed to convey a realistic starting part for situational activity. *Vector Unknown* was designed to mimic the first task from the Inquiry-Oriented Linear Algebra (IOLA) Magic Carpet Ride Sequence (Wawro et al., 2013). Mauntel et al. (2021) explored the student strategies from the game. *Vector Unknown* gave rise to a new 3D video game called *Vector Unknown: Echelon Seas* (Plaxco et al., 2023). The video game added a new dimension to gameplay and a different design and environment. This new design generated different strategies including the need to adjust the viewpoint in 3D environments.

Mauntel (2023) built upon the foundations of the game Vector Unknown and used an open GeoGebra environment for students to analyze and refer (referential activity) to the game. GeoGebra was intended to be a tool that would help students analyze the game and explore linear combinations more extensively. One issue with this implementation was that students experienced the technology differently as GeoGebra presented linear combinations to students differently depending upon how they entered them into GeoGebra which created some conflicts with how the game presented linear combinations. We investigate this issue, as it is important to consider when technology supplements an already existing phenomenon as opposed to becoming a new phenomenon itself. In this case, for some students, the GeoGebra open environment became a new realistic starting point as opposed to a tool for referring to the video game.

The final context was the use of GeoGebra applets for a sequence on determinants (Wawro et al., 2023) which were used at the end of an RME sequence to promote generalizations. In this case, there were two applets that were explored, 2D and 3D. This allowed students to make generalizations within a context and between contexts. This poster will look at all three technologies and discuss design choices made in each concerning their alignment along an RME sequence and its relation to student activity.

References

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K. (2020) Emergent modeling: an RME design heuristic elaborated in a series of examples. *Educational Designer*, *4*(13). 1–31. https://www.educationaldesigner.org/ed/volume4/issue13/article50/

Mauntel, M., Levine, B., Plaxco, D., & Zandieh, M. (2021). The characterization and evolution of strategies about vector equations in the game Vector Unknown. *Digital Experiences in Mathematics Education (7),* 453–476. https://doi.org/10.1007/s40751-021-00093-9

Mauntel, M. (2023) Structuring and connecting 2D and 3D space using linear combinations. Proceedings for the Twenty-Fifth Annual Conference on Research in Undergraduate Mathematics Education. Omaha, NE. 412-419

Plaxco, D., Mauntel, M., Zandieh, M. (2023) Student strategies playing vector unknown echelon seas, a 3D IOLA videogame. Proceedings for the Twenty-Fifth Annual Conference on Research in Undergraduate Mathematics Education. Omaha, NE. 1111-1116

Wawro, M., Mauntel, M., Plaxco, D. (2023) “The shape will have no volume”: relationships students observed about determinants in a dynamic geometric applet. Proceedings for the Twenty-Fifth Annual Conference on Research in Undergraduate Mathematics Education. Omaha, NE. 403-411

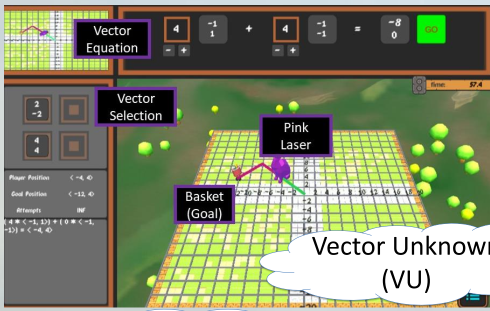
Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-Non Commercial-ShareAlike 4.0 International License.

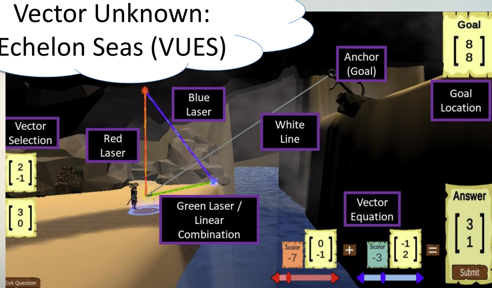
Pirates, Wobbly Jelly, and Bunnies … Analyzing Applets and Video Games from the Perspective of Realistic Mathematics Education (RME)

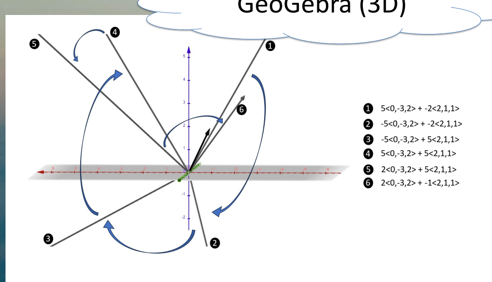
Matthew Mauntel
Michelle Zandieh
David Plaxco

Comparing Contexts

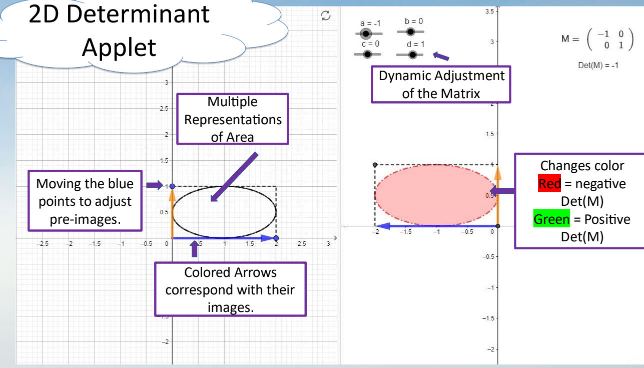
1. Some student strategies between VU and VUES differed because of the presence of the green laser and **white line**. Some students tried to move the green laser onto the **white line**.
2. After playing VU, a student explored linear combinations with GeoGebra 3D. They found they could rotate the result of a linear combination by changing the scalars. This had implications for how they thought about **all possible linear combinations.**







Carefully designed digital environments can serve as realistic starting points for student exploration and promote generalizing activity.

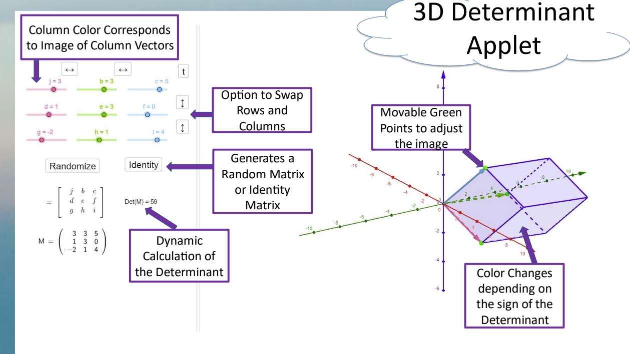


Connections

1. In the 2D and 3D applets we build connections between the matrix, determinant, and associated images using colors and **dynamic color change** associated with the **sign of the determinant**.
2. The **Movable Green Points** were designed to allow students to create biconditional generalizations. We noticed that without the ability to change the images, most generalizations originated with the matrix.
3. Having the option to **switch rows/columns** hints at the importance of **row/column operations**.

Observations Related to RME

1. **Applets and video games** can be great **realistic starting points for RME activities**. If multiple digital environments are used, it is important to consider the connections between design. For example, the student who played VU and then used the GeoGebra applet considered the contexts different and built-up separate strategies.
2. **Randomization** can be a great tool for **generating examples** for generalization and for **designers to scaffold learning**. VU has a difficulty setting (easy) with 0 as an entry because we observed student play and choosing vectors with 0s.



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Exploring The Role of Undergraduate and Graduate Real Analysis Experiences in the Mathematical Trajectories of Women Mathematicians from Historically Disenfranchised Groups: The Cases of Anna and Sasha

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Keywords: real analysis, classroom practices, equity, mathematical identity

As the population of the United States becomes increasingly diverse, more attention to efforts aimed at understanding ways in which colleges and universities can achieve equity in science, technology, engineering, and mathematics (STEM) education is needed to maintain global competitiveness in STEM (Henthorne, 2023; Office of Science and Technology Policy, 2022). As such, since courses in real analysis (undergraduate and graduate or similar courses)—typically required in the undergraduate and graduate mathematics programs in the US—often pose significant challenges to progressing in mathematics (Lew et al., 2016; Weber, 2004), we explore the cases of Anna and Sasha, part of a larger research study exploring the experiences of women mathematicians who come from groups historically disenfranchised in STEM (HDG), with a focus on undergraduate and graduate real analysis.

To investigate the mathematicians' experiences in real analysis, participants completed a 90-minute interview which included questions about their undergraduate and graduate classroom environments, their recollections of important concepts, and the role of their experiences in the development of their mathematical identity. This poster focuses on the following research questions: (1) What belief structures about how participants think of themselves as mathematicians were evoked by their experiences? (2) To what extent did their courses or experiences in real analysis advance their development as mathematicians? Using CRT as a theoretical base, my conceptual framework also incorporates factors that emerge from research literature associated with graduate school experiences of students from HDG (Borum & Walker, 2012; Johnson-Bailey et al., 2009; McAfee & Ferguson, 2006; McGee & Martin, 2011).

For both Anna and Sasha, interview analysis reveals the classroom environment in their real analysis courses negatively affected their identity as a "strong mathematician" even after completing their doctorate. Sasha had the unique experience of having taken real analysis at the undergraduate level in Latin America as well as in the US. She completed her undergraduate and graduate studies in the US and reported that she did well in the courses. She "loved analysis" and incorporates some of the teaching strategies used by her professors in her current teaching. In contrast, Anna describes undergraduate real analysis as not particularly challenging and that she earned "about a B." However, she explained that she failed her graduate course in real analysis because of the amount of information she had to "memorize." Although their performance levels in the courses were distinct, other messages received from their professors and classmates contributed to a similar cost to their identities as mathematicians. Both experienced isolation because of their race or gender throughout their academic career, but Anna experienced it in her analysis class directly. This contributed to Anna's choice to avoid more analysis classes as she "wanted absolutely nothing to do with it," and this continues in her role as a faculty member. The stories of Anna and Sasha compel mathematicians to re-examine the role of real analysis in the curriculum and to integrate anti-deficit strategies that foster a supportive learning environment.

References

Borum, V., & Walker, E. (2012). What makes the difference? Black women's undergraduate and graduate experiences in mathematics. *The Journal of Negro Education*, 81(4), 366-378.

Henthorne, T., Bevevino, D., McCarthy, J., Larkin, A., & Wyner, J. (2023). *A community college guide to excellence and equity in STEM programs.* Aspen Institute. https://highered.aspeninstitute.org/wp-content/uploads/2023/04/Aspen_ExcellenceEquityinSTEM_Spring2023.pdf

Johnson-Bailey, J., Valentine, T., Cervero, R. M., & Bowles, T. A. (2009). Rooted in the soil: The social experiences of Black graduate students at a southern research university. *The Journal of Higher Education*, 80(2), 178-203.

Lew, K., Fukawa-Connelly, T. P., Mejía-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, 47(2), 162-198.

McAfee, L. C., & Ferguson, D. L. (2006, October). Work in Progress: Status and Experiences of Minority Doctoral Students in Science, Technology, Engineering, and Mathematics Disciplines. In *Proceedings. Frontiers in Education. 36th Annual Conference* (pp. 12-13). IEEE.

McGee, E. O., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, 48(6), 1347-1389.

Office of Science and Technology Policy. (2022, December 12). *Equity and Excellence: A Vision to Transform and Enhance the U.S. STEMM Ecosystem.* The White House. https://www.whitehouse.gov/ostp/news-updates/2022/12/12/equity-and-excellence-a-vision-to-transform-and-enhance-the-u-s-stemm-ecosystem/

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior,* 23(2), 115-133.

The MAJORWISE Survey Study: Why Students Leave and Stay in the Mathematics Major

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Keywords: list 3-5 keywords here, Times New Roman 12-point, no italics

An overwhelming 52% of declared mathematics majors nationwide change their major in the course of their college career (Leu, 2017). To better understand retention and attrition in the mathematics major, our research team launched the **Ma**thematics **J**ourneys **o**f **R**etention: **W**hy **I**ndividuals **S**hift **E**ducational Paths (MAJORWISE) project. In this poster, we will address the question: Why do students choose to stay in and leave the mathematics major?

Background

Although several studies have investigated patterns of retention and attrition using quantitative measures (e.g., Rask, 2010; Rasmussen & Ellis, 2013), qualitative studies provide a deeper look into the reasons students provide for leaving mathematics. International qualitative studies have indicated students become disaffected with mathematics at the undergraduate level because they dislike lecture-based instruction and feel their courses place a strong emphasis on memorization (Hall et al., 2022; Rodd & Bartholomew, 2006; Ward-Penny et al., 2011).

Methods

During October 2023, we carried out a nationwide survey in the United States with free-response items of students who have been enrolled in a mathematics major during the years 2013-2023. In addition to demographic information, this survey solicited information such as reasons for enrolling in a mathematics major and timing of and reasons for deciding to either leave or stay in the mathematics major. We recruited participants through professional networks, social media, and snowball recruitment (requesting participants share the survey with others).

We analyzed free-response survey items using a predetermined coding scheme framed by Self-Determination Theory (Ryan & Deci, 2000), with codes for *autonomy*, *relatedness*, and *competency*. We then conducted an inductive analysis of responses in each category to describe specific experiences relevant to choosing to leave or stay in the mathematics major.

Presentation of Findings and Implications

The MAJORWISE online survey received 147 applicable responses. In our poster, we will report detailed findings from our analysis. For example, out of 40 responses to the question, “Why did you consider leaving the mathematics major?,” 22 (55%) were related to autonomy (or a lack thereof), 7 (17.5%) indicated a lack of relatedness, 15 (37.5%) concerned competence, and 13 (32.5%) cited sources of extrinsic motivation (e.g., job opportunities). These findings, along with our in-depth inductive analysis, provide insight into how relevant stakeholders can best promote the motivation and retention of mathematics majors.

Acknowledgements

This project is funded through the MTSU College of Basic and Applied Sciences.

References

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Hall, J., Robinson, T., Flegg, J., & Wilkinson, J. (2022). First-year and final-year undergraduate students' perceptions of university mathematics departments. Mathematics Education Research Journal, 34(2), 189–214. https://doi.org/10.1007/s13394-020-00340-z

Leu, K. (2017). *Beginning college students who change their majors within 3 years of enrollment* (NCES 2018-434; Data Point). National Center for Education Statistics. https://files.eric.ed.gov/fulltext/ED578434.pdf

Rask, K. (2010). Attrition in STEM fields at a liberal arts college: The importance of grades and pre-collegiate preferences. *Economics of Education Review*, *29*(6), 892–900. https://doi.org/10.1016/j.econedurev.2010.06.013

Rasmussen, C., & Ellis, A. B. (2013). Who is switching out of calculus and why. In A. M. Lindmeir & A. Heinze (Eds.), *Proceedings of the 37th Conference for the International Group for the Psychology of Mathematics Education*.

Rodd, M., & Bartholomew, H. (2006). Invisible and special: Young women's experiences as undergraduate mathematics students. Gender and Education, 18(1), 35–50. https://doi.org/10.1080/09540250500195093

Ryan, R. M., & Deci, E. L. (2000). Intrinsic and extrinsic motivations: Classic definitions and new directions. Contemporary Educational Psychology, 25(1), 54-67. https://doi.org/10.1006/ceps.1999.1020

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273 – 285). Thousand Oaks: Sage Publications.

Ward-Penny, R., Johnston-Wilder, S., & Lee, C. (2011). Exit interviews: Undergraduates who leave mathematics behind. For the Learning of Mathematics, 31(2), 21–26.

The MAJORWISE Survey Study: Why Students Leave and Stay in the Mathematics Major

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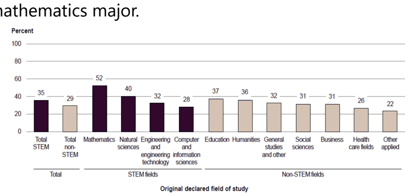
Students leave mathematics for more reasons **than you think.**

The MAJORWISE Project
Mathematics **J**ourneys **o**f **R**etention: **W**hy **I**ndividuals **S**hift **E**ducational Paths

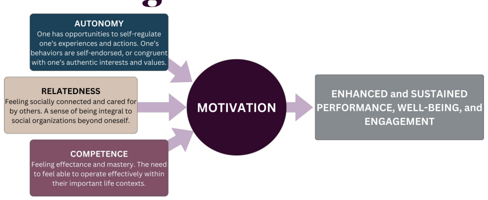
Background

An overwhelming 52% of declared mathematics majors nationwide change their major in the course of their college career (Leu, 2017). In this study, we report results of the MAJORWISE nationwide survey study investigating the reasons students give for choosing to major in mathematics, leave the mathematics major, consider leaving the mathematics major, and ultimately stay in the mathematics major.



Adapted from "Beginning College Students Who Change Their Majors within 3 Years of Enrollment." by Leu, K., 2017, in Data Point. NCES 2018-434. Copyright 2017 by National Center for Education Statistics.

Framing & Methods

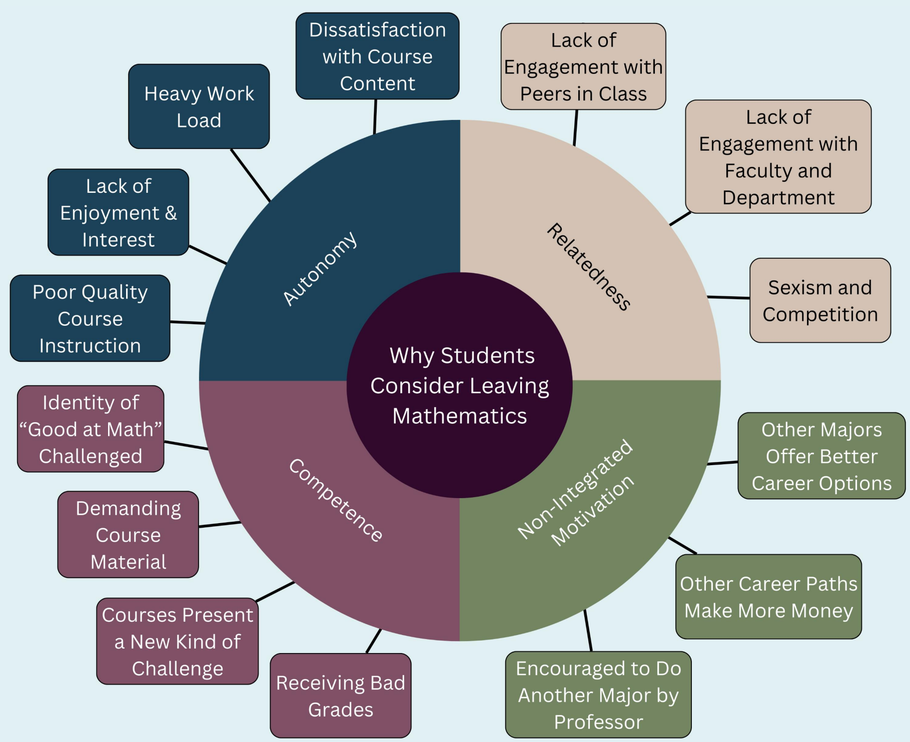


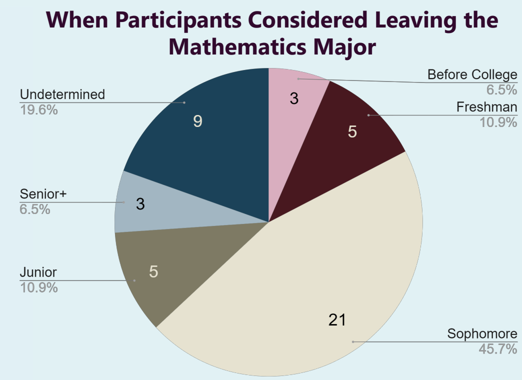
The Survey

- Demographic Information: Race/Ethnicity, Gender, Institution Type
- Basic Major Information: Years enrolled as a mathematics major, mathematics courses taken at the undergraduate level, time of major declaration, title of degree program, status with major (completed, still completing, considered switching, switched), current occupation
- Open-Response Questions for Applicable Participants

Recruitment & Participants

- Recruited participants through professional networks (e.g., RUME, EDGE, Jerry Becker ListServ), social media (e.g., Twitter/X, Facebook), and snowball recruitment.
- The survey received 147 applicable responses out of 176 total responses across 23 states.





Conference Paper, Contact Information, References, and Other Resources

Mathematics and Science Education Doctor of Philosophy Program

Online vs Face-to-Face Instruction: Affordances and Interactions

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Governors State University

Keywords: online instruction, affordances, interactions

The study reported focuses on the affordances of online instruction (Day & Lloyd, 2007), compared with face-to-face instruction. Participants were 51 undergraduate students, STEM majors, enrolled in two Discrete Mathematics sections, taught by the author, at a four-year university in the United States. In spring 2020, the COVID-19 pandemic changed the course instruction from face-to-face to online. The change affected the students' and the instructor's interactions with each other and with the mathematical content (Cohen, Raudenbush, Ball, 2003, p. 122). The following research questions are addressed:

1. How does the instructor use the online affordances to meet the students' learning needs?
2. How do the students use online affordances?
3. How does the online instruction of the course compare with the face-to-face instruction of the course with respect to meeting the students' learning needs, as perceived by instructor and students?
4. How does the online instruction of the course compare with the face-to-face instruction of the course with respect to students' success, as measured by their performance on the final exam?

The following data were collected: Student Questionnaire (SQ), Instructor Questionnaire (IQ), Blackboard Discussion Forum (BB Discussion Forum), and students' answers on the final exam.

SQ included questions related to how students interacted with the mathematical content, the instructor, and their colleagues during the face-to-face and online course instruction, as well as questions about satisfaction with specific instructional elements and preferences for future interactions.

IQ was a self-administered instrument. For both the face-to-face and the online instruction of the course, the questionnaire asked the instructor to rate her satisfaction with the convenience of the interaction with the mathematical content and the students, the quality of the interaction and the time spent, as well as to propose changes to better serve the students' learning needs.

Students posted their homework in Blackboard/Discussion Forum and were required to read their colleagues' postings and comment on one or two of their colleagues' answers. Data collected for this study came from two homework assignments with four tasks related to mathematical induction.

Students' answers on the final exam on two tasks referring to mathematical induction were scored and compared with previous scores from the final exams administered during the previous three semesters, when the course was delivered face-to-face. The scoring rubric for induction tasks was based on the mathematical and pedagogical considerations stated by Ernest (1984).

Data were analyzed using statistics, and social network analysis (Chai, Le, Lee & Lo, 2019; Ye & Pennisi, 2022).

The findings will be discussed and will inform recommendations for practitioners and directions for new research. Among those findings will be those referring to the importance of the BB Discussion Forum as a learning resource for students during the online delivery of the course, and the patterns of interaction between students.

References

Chai, A., Le, J. P., Lee, A. S., & Lo, S. M. (2019). Applying Graph Theory to Examine the Dynamics of Student Discussions in Small-Group Learning. *CBE: Life Sciences Education, 18*, ar29. https://doi.org/https://doi.org/10.1187/cbe.18-11-0222

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, Instruction, and Research. *Educational Evaluation and Policy Analysis*, *25*(2), 119-142. https://doi.org/10.3102/01623737025002119

Day, D., & Lloyd, M. (2007). Affordances of Online Technologies: More than the Properties of the Technology. *Australian Educational Computing, 22*(2), 17-21. https://eprints.qut.edu.au/17291/

Ernest, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics*, *15*, 173-189. https://www.jstor.org/stable/3482247

Ye, D., & Pennisi, S. (2022). Analysing interactions in online discussions through social network analysis. *Journal of Computer Assisted Learning*, *38*(3), 784– 796. https://doi.org/10.1111/jcal.12648

Using Comparative Judgment to Assess the Quality of Explanations in Mathematics

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Matthew Inglis
Loughborough University

Keywords: Comparative Judgment, Explanation, Proof.

We describe evidence of the usefulness of the Comparative Judgment (CJ) approach to assess the quality of explanations in mathematics. In philosophy, some study how certain types of mathematics offer *explanations* of other mathematical phenomena. In the field of education, offering explanations is a central part of teaching mathematics, and understanding those explanations is a vital activity for learners. So, what makes a good explanation in mathematics?

In the philosophy of mathematics, explanation quality has been traditionally studied by investigating the properties of what are deemed to be exemplars of mathematical explanations (e.g., Steiner, 1978), often mathematical proofs. In education, the approach has been to use *general* frameworks (i.e., non-math specific) describing the features that high-quality instructional explanations may have (e.g., Wittwer & Renkl, 2008).

We propose a different approach: using the CJ method to explore the notion of explanation quality *as it exists among mathematicians and undergraduate students*. CJ approaches to understanding human judgment exploit the finding that people are better at comparing two objects against each other than at evaluating one object against specific criteria (Thurstone, 1927). Modern uses of comparative judgment in assessment rely upon the Bradley-Terry model (Bradley & Terry, 1952), which assumes that each stimulus has a numerical parameter which captures its quality on the dimension of interest. By presenting judges with repeated pairs of stimuli and asking them to assess which they would rate higher on the given dimension, empirical estimates of these parameters can be obtained.

Recent empirical studies (Mejía Ramos et al., 2021; Evans et al., 2022) have used the CJ approach to study the extent to which mathematicians and students can reliably judge the quality of explanations in mathematics. In the first study, 38 mathematicians made a total of 760 comparisons between two proofs of the same proposition. For each judgment, two proofs were randomly selected and positioned side-by-side for the participant to think about and select which argument best explained why the proposition holds (without focusing on how the proof might be received by a particular audience), an instruction aimed at investigating the more philosophical sense of explanation described above. Using a similar procedure, in the second study we asked 32 undergraduate students who had taken a Linear Algebra course and 16 mathematicians to judge explanations meant for a hypothetical mathematics undergraduate student who did not understand the concept of an abstract vector space, an instruction aimed at investigating the more pedagogical sense of explanation described above. Using split-half inter-reliability coefficients, we report very high levels of reliability in participants' judgments of explanation quality for both mathematicians and students. Furthermore, in the second study we found a very high correlation between the parameter estimates for mathematicians and those for students. Overall, these studies provide evidence of high level of agreement among participants regarding what makes a good explanation in mathematics. This method opens fascinating avenues for future research on the notion of explanation in mathematics.

References

Bradley, R. A., & Terry, M. E. (1952). Rank analysis of incomplete block designs: The method of paired comparisons. *Biometrika, 39*, 324–345.

Evans, T., Mejía Ramos, J. P., & Inglis, M. (2022) Do mathematicians and undergraduates agree about explanation quality. *Educational Studies in Mathematics, 111*, 445-467.

Mejía Ramos, J. P., Evans, T., Rittberg, C., & Inglis, M. (2021). Mathematicians' assessments of the explanatory value of proofs. *Axiomathes, 31*, 575-599.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies, 34*(2), 135–151.

Thurstone, L.L. (1927). A law of comparative judgement. *Psychological Review, 34*, 273–286.

Wittwer, J., & Renkl, A. (2008). Why instructional explanations often do not work: A framework for understanding the effectiveness of instructional explanations. *Educational Psychologist, 43*(1), 49–64.

A ‘Warm Demander’ in College Mathematics Drives STEM Student Persistence

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Keywords: Community College, STEM, Student Persistence

Within the context of science, technology, engineering, and mathematics (STEM) many studies have shown the reality of the “leaky pipeline” with students dropping out of STEM majors along the way to completing a degree or dropping out of school (Flynn, 2016). Addressing the issues creating the leaky pipeline and students not persisting in STEM degrees is crucial to improving broader issues of diversity in STEM especially for students starting their academic journeys in community colleges.

Theoretical Framework

This work uses the framework of the ‘warm demander’ (Kleinfeld, 1975) from culturally relevant pedagogy to examine the teaching style and pedagogical choices of the instructor. A ‘warm demander’ is an instructor that holds high standards for their students since they know reaching these standards is important for their academic success, but they also create a caring and supportive environment in the classroom.

Research Questions

This work is focused on the case of an effective community college math instructor and considers what instructional moves appear to be driving increased student success. The research question is: What pedagogical choices of an experienced community college math instructor appear to be leading to increased student persistence in STEM trajectories?

Methods and Data

The instructor at the center of this study is an experienced community college math faculty member who has success in increasing student success rates. Success was defined through increased pass rates and persistence of students in follow on courses even if passing courses required multiple attempts. The success of students was defined both from institutional data and through interviews of students and alumni about their experiences and trajectories. Classroom videos were analyzed to understand the pedagogy of the instructor.

Findings and Discussion

The analysis shows that this instructor exhibits many characteristics of a ‘warm demander.’ He brings deliberate language of care into the classroom, not only in communicating his care to students but encouraging them to care about one another. The instructor also creates a warm environment through humor and encouragement. At the same time, he maintains high standards for students, pushing them to come to class on time, engage with the materials and pass demanding quizzes and tests. Powerfully, this combination seems to help students see their own potential even when they struggle. This aspect of instruction should be studied further to see how it may be enacted by other successful instructors and how professional development could be created to teach more instructors how to include warmth and standards in their math classrooms.

References

Flynn, D. T. (2016). STEM field persistence: The impact of engagement on postsecondary STEM persistence for underrepresented minority students. *Journal of Educational Issues*, 2(1), 185-214.

Kleinfeld, J. (1975). Effective teachers of Eskimo and Indian students. *The School Review*, 83(2), 301-344.

An Analysis of Instructor Feedback

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Keywords: Instructor Feedback, Actionable Feedback, Feedback Literacy, Student Success

Carless & Boud (2018) capture the definition of feedback best as "a process through which learners make sense of information… use it to enhance their work or learning strategies." This definition goes beyond feedback solely being the role of teachers and highlights the student component in making sense of the feedback (Carless & Boud, 2018). A more specific type of feedback that will be used in this study involves "actionable feedback" which allows the student more agency to revise their work independently (Hattie & Timperley, 2007).

Research Questions

1. What are the feedback types commonly used by instructors within a student homework sample?
2. Do the feedback types that instructors provide match those that the students prefer/find useful?

Methods, Results, and Benefits

Boise State University instructors were asked to complete an online questionnaire and to grade and leave written feedback on student work samples of the same task. No rubric was provided. A thematic analysis was conducted on their feedback practices and word choice to then classify into groups. The classification of feedback was based on a combination of Wahyuni (2023) and Lyster & Ranta (1997) written feedback clarification. Students' perceptions about feedback were gathered through a questionnaire to gain insight on how they use feedback, how highly they value it, and which feedback type they find most useful.

Results are currently pending, but preliminary results suggest that most instructors use a wide-variety of different feedback types. Most tend to be non-actionable feedback, while there is almost always at least one piece of feedback that provides the student with the agency to help revise their response. Instructors commonly followed a close pattern of feedback across their students.

This study is focused on BSU's Math Department, and as a result, we may be able to identify certain trends within the ways courses are currently taught and provide more awareness to better inform our department's grading practices across our entry level math courses. This information could be especially beneficial to present as training material for new GTAs.

References

Carless, D., Boud, D. (2018) The development of student feedback literacy: enabling uptake of feedback, Assessment & Evaluation in Higher Education, 43:8, 1315-1325. https://doi.org/10.1080/02602938.2018.1463354

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*(1), 81-112. https://doi.org/10.3102/003465430298487

Lyster, R., & Ranta, L. (1997). Corrective feedback and learner uptake: Negotiation of form in communicative classrooms. *Studies in Second Language Acquisition*, *20*, 37-66

Wahyuni, N. (2023). *Investigation on an implementation of mastery grading utilizing revision and oral follow-up assessment in an introduction-to-proof course* (Doctoral dissertation). West Virginia University.

An Analysis of Instructor Feedback

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Introduction

- Instructors across every discipline, especially those in entry-level courses, act as one of the key players in a beginning college student's journey into their course of study and play a pivotal role in students' personal and academic development.
- Their actions and choices in the classroom can highly influence a student's path in college, even including their use of feedback.

Defining Feedback

- Feedback is "a process through which learners make sense of information from various sources and use it to enhance their work or learning strategies" [2]. This goes beyond it solely being the role of teachers and highlights the student component in making sense of the information. We consider feedback to include teachers' comments, questions, and markings (such as check marks, X's, and addition of math content).
- Actionable Feedback allows the student the opportunity to identify an error and strategize about how to correct it themselves [3][4]. This type of feedback promotes student agency and independence.

Research Questions

1) What are the feedback types commonly used by instructors within a student homework sample?
2) Do the feedback types that instructors provide match those that the students prefer/find useful?

Methods

8 Boise State instructors (below Calculus level) completed 2 phases:

- online questionnaire about their grading practices
- a task involving grading and leaving feedback on 7 student work samples

22 Boise State students completed an anonymous questionnaire asking their preferences of feedback.

Classification

This study used a 9-category classification, extended from Lyster & Ranta (1997) and Wahyuni (2023) [5][6].

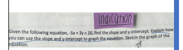
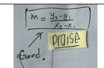
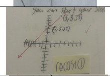
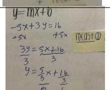
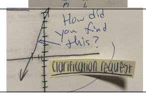
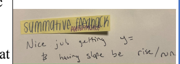
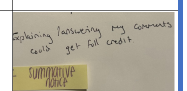
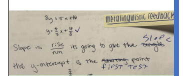
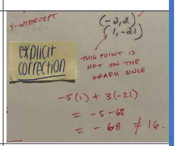
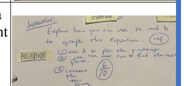
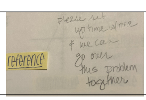
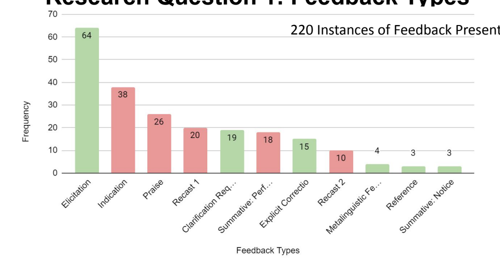
Feedback Type		Actionable?	Justification	Example
Indication		Non-Actionable	No words are included; it can be unclear to students how to proceed.	
Praise		Non-Actionable	Adds no mathematical comments; only offers positive words.	
Recast	1	Non-Actionable	A correction is only provided with no reason as to why there was an error.	
	2	Non-Actionable	All of the work is provided in the feedback, with no reason as to why there was an error.	
Clarification Request		In Between	Comment suggests that the student's work is "ill-informed"; a revision is encouraged.	
Summative	1	Non-Actionable	A comment about a specific performance acts similarly to Praise, as it is only acknowledging the work that was done.	
	2	Actionable	This feedback offers a comment about a certain action to be more aware of in the future.	
Metalinguistic Feedback		Actionable	Feedback is offered about specific word choice used; correction may encourage students to use that terminology in the future.	
Explicit Correction		Actionable	Feedback includes the error detection as well as the reason why it was incorrect.	
Elicitation		Actionable	Feedback is in the form of a question or general comment with instructions on how to complete a given task.	
Reference		Actionable	This directs a student to a certain reference, such as the instructor or a page in the textbook.	

Table 1.1 Nine feedback types are categorized as being actionable or non-actionable.

Results

Research Question 1: Feedback Types



Research Question 2: A Match Between Instructor and Student?

COUNTA of Student		Student Experiences		
Feedback Type	Student Preference	Doesn't Receive	Receives	Grand Total
Actionable	Least	4	3	7
	Top	1	19	20
Actionable Total		5	22	27
Non-Actionable	Least	16	1	17
	Top	0	3	3
Non-Actionable Total		16	4	20
Grand Total		21	26	47

Conclusions

The results indicated that instructor feedback was overall actionable, which students preferred. This study found harmony among these two populations, inconsistent with other findings where teachers believe actionable feedback is less desired by students[1].

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References

1. Amrhein, H.R., & Nassaji, H. (2010). Written Corrective Feedback: What Do Students and Teachers Think is Right and Why? *Canadian Journal of Applied Linguistics/Revue canadienne de linguistique appliquee*, 13(2), 77-101
2. Carless, D., Boud, D. (2018) The development of student feedback literacy: enabling uptake of feedback, Assessment & Evaluation in Higher Education, 43:8, 1315-1325, DOI: 10.1080/02602938.2018.1463354
3. Griffiths, Mutoni., Murdock-Perriera, L., & L Eberhardt, J. (2023). "Can you tell me more about this?": Agentic written feedback, teacher expectations, and student learning. *Contemporary Educational Psychology*, *73*, 102145. https://doi.org/10.1016/j.cedpsych.2022.102145
4. Hattie, J., & Timperley, H. (2007). *The Power of Feedback*. Review of Educational Research, 77(1), 81-112. https://doi.org/10.3102/003465430298487
5. Lyster, R., & Ranta, L. (1997). *Corrective feedback and learner uptake: Negotiation of form in communicative classrooms*. Studies in Second Language Acquisition, 20, 37-66
6. Wahyuni, N. (2023). *Investigation on an Implementation of Mastery Grading Utilizing Revision and Oral Follow-Up Assessment in an Introduction-to-Proof Course* (dissertation).

Opportunities for Reasoning-and-proving with Trigonometric Identities in Precalculus Textbooks

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Keywords: precalculus, trigonometry, proof, reasoning, justification

Outside of geometry, United States high school students have few opportunities to engage with proof (Thompson et al., 2012; Otten et al., 2014). It is therefore likely that for most precalculus students, proving trigonometric identities represents a significant departure from familiar modes of reasoning. As both a thorough understanding of the relationships between trigonometric functions and the ability to reason deductively are critical in undergraduate mathematics, learning trigonometric identities may be an opportune time for students to develop skills at all levels of Stylianides' (2009) taxonomy of reasoning-and-proving (RP) activities.

This poster will summarize a study that examined five precalculus textbooks for the opportunities they afford students to engage in RP activity in the context of trigonometric identities. One textbook was selected from each decade from the 1970s to the 2010s, and each represents a distinct instructional approach (Stylianides, 2008). The research questions were:

1. Across the five textbooks, what opportunities do students have to derive or prove essential trigonometric identities?
2. Across the five textbooks, what types of RP activity, and in what amounts, are expected of students in the context of learning to apply and verify trigonometric identities?

Twenty trigonometric identities were identified as *essential* (defined as necessary in future problems and/or in proofs of other identities). Each textbook's presentation of each identity was assigned one of six author-created codes, capturing the extent to which the text solicited student involvement in deriving it. A modified version of Otten et al.'s (2014) framework for coding RP opportunities in geometry textbooks was used to code justification-based problems in lessons about trigonometric identities. Percentages of items marked with each code were tabulated and trends identified. The poster will report on those trends, of which two were most salient:

1. In nearly all instances of an opportunity to derive an essential identity, the text provided enough guiding detail that a student would not bear the cognitive load of creating a proof.
2. The two 21st century textbooks afforded noticeably more frequent opportunities to make graphical arguments than the three 20th century textbooks.

The second finding is particularly significant in light of Stylianides' (2008) comments on the potential curricular effects of the 2000 NCTM Standards' unified view of reasoning and proof. Because of the small sample of textbooks under analysis, a natural next step would be to repeat the analysis with the precalculus and trigonometry textbooks that have been most popular over time. Significance tests of the differences in proportions of codes could show with greater clarity how responsive trigonometry curricula have been to long-term trends in instructional approaches.

References

Otten, S., Gilbertson, N. J., Males, L. M., & Clark, D. L. (2014). The mathematical nature of reasoning-and-proving opportunities in geometry textbooks. *Mathematical Thinking and Learning, 16*(1), 51–79. https://doi.org/10.1080/10986065.2014.857802

Stylianides, G. J. (2008). Proof in the school mathematics curriculum: A historical perspective. *Mediterranean Journal for Research in Mathematics Education, 7*(1), 23–50.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258–288. https://doi.org/10.1080/10986060903253954

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253

Using Online Resources to Study Mathematics: Students' Perspectives on Effectiveness and Ethics

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Keywords: Information Literacy, Ethics, Technology

Students in undergraduate mathematics classes are using online resources to support their learning with increasing frequency (Erickson, 2019, 2020). The use of such online resources has not proven to be straightforward for students. Undergraduate students have well-documented difficulties locating, evaluating, and making use of information sources regardless of the topic area in question (Scott & O'Sullivan, 2005; Walraven, Brand-Gruwel, & Boshiuzen, 2008). These difficulties may vary depending on a student's familiarity or comfort with the field (Brand-Gruwel, Wopereis, & Vermetten, 2005). Thus, it is important to develop subject-specific models of student information-seeking. This is particularly important for the field of mathematics due to the role that lower-division math classes have often played as a gatekeeper for entry into STEM majors (Martin, Gholson, & Leonard, 2010).

This poster will present findings from a nationwide mixed method research project that has collected surveys from over 300 students across the country and follow-up semi-structured interviews with over 60 students intended to assess how students currently make use of online resources to support their learning in lower-division mathematics classes. These findings specifically address the following questions:

1. From the students' perspective, what does it mean to efficiently use online resources to study mathematics?
2. From the students' perspective, what does it mean to ethically use online resources to study mathematics?

These findings also include a mid-range theory of students' use of online resources based on a grounded theory (Strauss & Corbin, 1994) analysis of the interview transcripts. The poster will present this theoretical model in which we identify triggers for the use of online resources, along with the most common types of resources used and the accompanying goals. For example, students commonly make use of online video lectures if they are having trouble understanding what their instructor is saying in class or look up solutions in an answer engine when they feel that they need to double-check their homework.

We present evidence that students are generally conscientious in their use of such resources, struggling both with how to maximize their efficiency while also expressing concern that their practices are ethical. These two concerns are sometimes balanced against one another as is evidenced by a recurring student suggestion for their peers: start by learning on your own and only make use of online resources when you struggle.

Finally, our poster will present several representative cases in order to better capture the way in which students balance these concerns in different types of mathematics classes. These narratives have the potential to be used in classroom settings in order to help students become more effective and ethical users of online resources.

Acknowledgments

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References

Erickson, A. (2019) Mathematics on the Internet: Charting a Hidden Curriculum. In Weinberg, A., Moore-Russo, D., Soto, H., & Wawro, M. (Eds.). *Proceedings fo the 22nd Annual Conference on Research in Undergraduate Mathematics Education,* 884 - 890. Oklahoma City, OK.

Erickson, A. (2020) Every Mathematics Class is Online: Students' Use of Internet Resources for Self-Directed Learning. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education.* Boston, MA.

Brand-Gruwel, S., Wopereis, I., & Vermetten, Y. (2005). Information problem solving: Analysis of a complex cognitive skill. *Computers in Human Behavior, 21*, 487-508.

Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, *3*(2), 12-24.

Scott, T. J., & O'Sullivan, M. K. (2005). Analyzing student search strategies: Making a case for integrating information literacy skills into the curriculum. *Teacher Librarian*, *33*(1), 21.

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273 – 285). Thousand Oaks: Sage Publications.

Walraven, A., Brand-Gruwel, S., & Boshuizen, H. (2008). Information Problem Solving: A review of problems students encounter and instructional solutions. *Computers in Human Behavior, 24*(3), 623-648.