What Do Students Do To Develop Understanding of Definitions in Proof-Based Courses?

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Advanced, proof-based mathematics courses typically include learning goals of developing understanding of the relevant content of the class. The content always includes definitions of concept in that they are foundational to theorems and proofs. This study explores what three students believe it means to understand a definition, the actions they take to develop that understanding, and the rationales they use to explain why their actions are productive. The students held divergent beliefs about what it means to understand a definition. While all cited lecture attendance and homework as critical activities in their learning, they diverged on the use of outside resources and the rationales for why they might or might not be useful.

Keywords: proof-based mathematics, student understanding, definitions, learning goals

Mathematical definitions serve an essential purpose in proof-based courses, which require students to attend to these definitions in a distinct way: as stipulated rather than descriptive (Edwards & Ward, 2004). Definitions are often provided during lectures; however, we know little about what students understand from these settings. In one of the few fine-grained studies of what students gain from specific lectures, Lew et al. (2016) described that students may not apprehend the content that professors are attempting to convey. Melhuish et al.'s (2022) recent literature review points to a lack of empirical studies focused on student learning or what students might do outside of class to develop understanding.

As understanding and using definitions constitute a major content goal of proof-based courses, we explore two primary research questions and two that provide context and explanation for the findings of the first two:

- What activities do students engage in to try to develop understanding of a definition?
- Why do they engage in those activities?

To provide context, we also explore:

- What do students believe a definition is in advanced mathematics and why?
- What do students believe it means to understand a definition and why?

Literature Review

In formal mathematics, definitions stipulate the nature of a mathematical concept, and any object that meets the definitional criteria is an example, and any that does not is a non-example. Such definitions are referred to as *stipulative* (Edwards & Ward, 2004). Alcock (2010) suggested that mathematicians desire students to understand the stipulative nature of mathematical definitions. At the same time, extant research suggests that students do not consistently hold similar beliefs about definitions (c.f., Edwards & Ward, 2004; Alcock & Simpson, 2004; Krupnik, et al, 2018), and may operate with the definitions as if *descriptive* (that is they are extracted and created from use) rather than *stipulated.* Some studies have pointed to complexities in students' thinking of definitions attending to issues of mathematical criteria and communicative goals (Zazlavsky & Shir, 2005) or ways students may reinvent stipulative mathematical definitions (Zandieh & Rasmussen, 2010).

In terms of lecture courses, mathematicians have argued that much of the work of learning, such as learning definitions, occurs outside of class (Wu, 1999; Weber & Fukawa-Connelly, in press.). This may involve homework activities (Rupnow, et al, 2020) and other types of at home activities. Yet, we know little about what types of activities students intend to and do engage in at home to meet learning goals. If we focus on definitions, we can find some papers on what mathematicians do to learn new definitions and content (e.g., Parameswaran, 2010; Wilkerson-Jerde & Wilensky, 2011). In the context of definitions, Parameswaran (2010) interviewed and surveyed a set of mathematicians finding they studied examples, used definitions to prove theorems, explored equivalent definitions, and resolved cognitive conflicts while trying to understand a new definition. Others have pointed to how mathematicians' use intuition and diagrams (c.f., Johansen & Misfeldt, 2020; Sinclair& Gol Tabaghi, 2010) in connection with formal definitions. That is, there may be reason to hypothesize that understanding a definition in a formal proof context, may also involve drawing on not just definition, but also concept image (as defined in Tall & Vinner, 1981).

Theoretical Perspective

The practical rationality framework, initially designed to analyze teacher decision making at the K-12 level (Herbst and Chazan, 2003), posits that decision-making is regulated by rationality. More specifically, the knowledge, beliefs, and goals of the teacher mediate the obligations and norms from stakeholders (Chazan, et. al. 2016; Webel 2013) and ultimate decisions. Since its development, it has been extended to investigate the students' decision making. Webel (2013) argued for attention to "both social and individual factors in describing how particular student actions come to take place in mathematics classes" (p. 25). In our study we hold that the beliefs students held and the actions that they took were rational considering the obligations placed on them and their experiences within the mathematics discipline.

Methods

The results discussed here are a part of a pilot study aimed at exploring students' beliefs about definitions. Data collection took place at a large research university in the southern United States. Our first round of data collection was an online questionnaire in which students were asked about what activities they engaged with in between classes that helped them better understand definitions and why they think these activities helped them. This survey was given to students in two undergraduate math classes (modern algebra, real analysis) a few days after their professor had introduced a new definition in class (direct products, continuity). Two students in modern algebra and one student in real analysis completed both the questionnaire and the interview.

Data Collection Methods

The goal of the survey was to document in-the-moment activities that students claimed to engage in to develop understanding of new definitions. During the follow-up interview, students were asked questions about their background, their beliefs about definitions – including what it means to understand a definition – and questions about how they come to understand definitions. For questions relating to how they developed understanding, we encouraged the students to think back to their most recent activity to understand a new definition which often focused on a homework assignment. They were also given the opportunity to respond to other contexts such as preparing for exams or their general practice with new definitions.

Data Analysis Methods

To analyze our data, we began with an open coding scheme in which each interview was first transcribed and then coded for beliefs and activities. All four authors coded Kurt's transcript for both beliefs and actions. For Kris' and Dave's transcripts, each transcript was coded for beliefs by two members of the research team and for actions by three members of the team. We then met to reconcile the coding and any discrepancies were worked out through discussion. Using these codes, we built profiles of each student that addressed the following five areas: what is a mathematical definition, what is the purpose of a definition, what does it mean to understand a definition, what actions are taken to understand a definition, why the student holds each of these beliefs. Note that these are from the student's perspective. The results of this process are addressed in the sections below.

Data and Results

Table 1. Definition Profiles of Participants

Participant	What is a definition?	What is the purpose of definitions?	What it means to understand a definition
Kurt	Rigid criteria categorizing a set of objects ("black and white, either in or out")	The purpose of a definition is to create and prove claims about objects that fit the definition.	• Determine whether an object meets the definition • Developing Intuition
Dave	Axiomatic "Foundation" of mathematics that cannot be changed	The purpose of definitions is to construct and prove mathematical propositions.	• Knowing the definition • Being able to apply the definition in proofs
Kris	Fixed and cannot be argued as incorrect	The purpose of a definition in mathematics is to contribute to the development of propositions and proofs	• Understanding its purpose • Knowing when to use in proofs

Before providing an overview of the at-home activities, we first share a brief overview of the three students' beliefs (Table 1). All the students held stipulated views of mathematical definitions with all three making explicit comments about the definitions being rigid. We also note that all shared a common view that definitions were in service of proofs and propositions. However, the three students reported somewhat distinct views on understanding definitions.

Additionally, we note that none of the students claimed to take any specific activities to learn a definition the day it was presented in class. As we detail below, they all describe some activities that they take to learn definitions, and have well-developed rationales, none are taken immediately after the presentation.

Kurt

Kurt explicitly described four activities that he engages in to try to develop understanding of a definition and we inferred one additional activity, repetition. The first three that he named are attending class, looking at his notes, and doing the homework. He also claimed to look at online videos when those were insufficient or when he thought animations might be helpful.

Kurt explained that he went to lecture diligently, took notes, and asked questions if he was not understanding. In particular, he explained for continuity, I've mostly been relying on lectures for that too, like the first day that it was introduced to us. I was like I don't really get it, so I just I kept like, you know, going to class and like OK like let's try this again and my professor was really good about when we were like getting in deep with continuity being like OK class.

He noted that he asked questions like, "When something isn't continuous, why does it break? Why doesn't it fit that definition? You know? Why does it break? So, I was like, can you show us like a picture of like the epsilon bubble and the delta bubble like show us like how its discontinuous function like breaks that so to speak."

Outside of class he then sought out YouTube animations to further develop his understanding with visuals in the cases he found lecture insufficient elaborating that, "OK, here's that curve, and here's what happens when we do this and this and this, that could be really handy. That's something that's kind of you can't replicate on a chalkboard super easily, you know." He appreciated another perspective but noted sometimes what the professor said was simpler.

Kurt also explained homework as his most valuable tool to develop his understanding of definitions, explaining that he needs to "play with [the definition] himself" and understand how to apply them in proof. This was a prevalent theme throughout the interview.

Reflection. Kurt emphasized several activities that aligned with his view on understanding a definition, including attention to examples (categorizing in or out), and developing intuition from animations (often found online.) He did not see memorizing a definition as part of understanding nor did that influence his activity and uses homework as the primary way to achieve understanding. Although Kurt did not explicitly say using definitions to prove as part of understanding, this was stated as a purpose of definitions and the primary activity for Kurt, likely reflecting the nature of homework prompts.

Dave

When we consider the actions that Dave takes to understand a definition, we can summarize his activity by saying that he believes that he tends to do better on the coursework when he simply does the homework and try to understand each question conceptually. This approach is believed to lead to remembering definitions and proof strategies. If he does get stuck there is a preference of talking with the professor, but, if necessary, he will use the internet. We noticed that Dave has a nuanced stance on the actions he takes. For instance, he will make statements such as "maybe it's just me. But memorizing does not help. I think understanding the big picture helps a lot more than any amount of memorization." We do not take this statement from Dave to mean that he is against knowing the definition. Instead, we take this to mean that Dave holds the belief that definitions are learned progressively and through doing mathematics as opposed to the use of flash cards or other memorization tools. Also in his internet use, there is a strong preference towards seeking help from the professor before turning to the internet as the internet often lacks the ability to give insight and creates confusion with different notation usage.

In further consideration of Dave's actions, he seems to take actions that he believes will support him in better developing his concept image of the topic at hand. During the interview Dave provided an instance in which talking to the professor was of greater benefit than the internet. In this dialog he stated

And then [Dr. X], I went to his office hours once and I was like I don't understand what this looks like. Help me. So, then he pulls out his Rubik's Cube and he's like see how the face doesn't go anywhere. This is the stabilizer. Like I love stuff like that. I think that's way more helpful to me in the grand scheme of things than like just knowing that definition, then trying to apply it, you know. That I can think about it better if I asked directly for help from the professor versus random Internet strangers online.

In this case, directly talking with the professor allowed Dave to develop a richer concept image of the stabilizer than turning to "random internet strangers."

The rationality for each of these actions was traced back to Dave's intro to proof course. In the following excerpt, Dave describes his experience transition into proof-based mathematics courses

> I can trace it back to intro to advanced math. Umm? There was just kind of like this. I guess wall, right? Because when, when you're like doing computation, computation, computation for years. You hit calculus 3 computation, differential equations computations. And then suddenly, you're just thrown in this theorem land where like everything's completely different, and there's no computation. It's just you need to know how to use definition to prove big idea. It's just a completely different beast, I guess. Like in terms of like math. And I didn't see it coming, so I had to shape up.

In this quote, Dave describes a form of distinction between the computational mathematics that he had been doing and the theoretical mathematics that he was beginning. This distinction between theoretical and application-based mathematics forced the development of new activities for learning.

Reflection. Like Kurt, Dave's activity was tied to his homework. Dave's description of understanding definition aligned with his activity where he explicitly teases apart the ideas of "knowing how" and "knowing what", both of which are a part of understanding. There is more value placed on the "knowing how" category, that is applying definitions to proofs, which we can see in the quote above dealing with the stabilizer.

Kris

Like his peers, Kris uses both lecture and homework as a primary means to engage in definitions. He also attends office hours and seeks out information from the internet. Kris claimed that his two primary activities to develop understanding were to attend lecture and do the homework, "But in my opinion, I think it's, that's the only two ways to help me to understand the definition is you just do the homework or see what Dr. X, like in know the instructor, what they expand for the definition."

Kris also acknowledged the importance of extended time noting that it "takes time to understand the definition and what's the purpose…" and a single lecture is too short, and he may need to revisit ideas on the weekend. He also explains that during lecture it helped when the professor paused and gave "a little bit of time to think about the definition" and how important that time was for him to understand.

While Kris said that those are the only two ways to develop understanding, he specifically describes other activities that he uses to develop understanding including office hours and using the internet as a resource. He would opt for the internet if his instructor was unavailable to meet searching out," Where is this definition come from? And the work, the purpose of this definition for sometimes it helped me to understand more, but some not all the time." He also would seek out different points-of-view on the definitions such as reading online arguments between different people. He also sometimes asked his peers for what the "purpose" of a definition is.

Kris's last observation from his coursework is that while he tries to be able to state theorems in his own words, he does not do this for definitions. He explains that "if I change the words in my, I have a different story than a definition." He elaborated how this led to him developing incorrect quantification in prior classes. That prior experience led to his decision to "force [himself] to memorize" the definition exactly which can also help when they need to be stated on exams.

Reflection. Like other cases, homework and lecture played a substantial role in Kris's activity were using definitions to prove statements was a key activity and addressed the "purpose" of a definition. However, we can see one substantial way that Kris's case differed from the prior. He placed outsized value on memorizing a definition exactly as stated. This may reflect his beliefs on definitions as the "inarguable" parts of mathematics. He also did not go to the internet for animations or intuition, but rather history, which again may reflect how he sees the roles of definitions as foundational.

Table 2. Actions taken by Participants

Participant	Actions taken to understand a definition	Beliefs and Rationale for Actions Taken
Kurt	• Attending class • Reviewing notes • Doing homework • Using online resources	• He needs to "play with it [the definition] himself" and understand how to apply them in proof. • Justifies actions using prior experiences in math courses and advice from friends.
Dave	• Doing homework • Talking with professor	• Memorization is not useful to developing the ability to use the definition in writing proofs. • Justified actions based on the goals of understanding/ desired learning outcomes set for himself and prior experience in math courses
Kris	• Attending class • Doing homework • Talking with professor, online resources	• Kris wants to know the historical uses and evolution of a definition, and this can be found via lecture and online

Discussion

The purpose of this study was to address two primary questions:

- What activities do students engage in to try to develop understanding of a definition?
- Why do they engage in those activities?

In addressing these two questions, we also examined two additional questions:

- What do students believe a definition is in advanced mathematics and why?
- What do students believe it means to understand a definition and why?

Regarding these first two questions, through both our survey and interview, we found that there are a range of activities that students engage in when trying to understand definitions and some consistencies. All the students suggested that doing homework and attending lectures were their primary means of developing understanding. Kurt also relied on additional resources, including those found on the internet, as part of their coming to understand. In contrast, Dave preferred not using outside resources like the internet. However, he does acknowledge that he has looked at them in the past. To account for differences in what students saw as useful, we return to the notion of the rationality of students and consider the goals that the students had. We do hold that each of the students utilized resources that were rational from their perspective as they considered their own learning goals. Webel (2013) had stated the importance of understanding the students' goals when trying to examine the motives of their behavior.

The students had relatively similar backgrounds in terms of the mathematics coursework that they had taken. The students always engaged in the activities they believed would best help them achieve their learning goals but had developed very different sets of learning behaviors and rationales to explain them. Kurt, for example, believed that he should develop understanding of the definition through repeated exposure via lecture and practice with homework tasks. In contrast so as to be able to reliably classify sort candidates into the class of example and non-example. In contrast, Kris memorized definitions for exams, but sought to understand the historical reasons that they were created and what theorems they lead to by using lectures, office hours, and resources found on the internet. Dave wanted to know when to use a given definition (akin to Weber's (2001) *strategic knowledge*) and so did the homework, go to office hours, and sometimes use the internet. Dave was leery of the internet due to changes in 'notation' that might confuse him. That is, different goals for understanding appear to support different collections of behaviors.

Through documenting both the activities students engage in when coming to understand a definition and the rationale behind why students engage in activities, we make a minor contribute to the literature on how students engage with definitions as there was little existing literature. We note two limitations that also suggest next directions for study. First, due to our limited sample of three students, we have no reason to believe that they are representative of students in advanced mathematics courses, and so one way to explore that would be to replicate the study with a larger sample. As a second limitation, we found that students provided incongruous responses during the survey and their initial claims during the interviews. We found that students recalled different behavior when prompted with their own survey responses than when asked to describe their behavior without prompting. At the same time, because the survey did not attempt to capture the logic for their behavior, more research is needed in order to explore the in-the moment decision-making processes that students use to seek out outside resources and evaluate them. Finally, in addition to replication, we noticed that preferences and sense of self-efficacy may influence their behavior. Thus, it would be of interest to examine the relationship between students' actions and their self-efficacy.

While we note that this is a small group of students, all three noted that they did not recall their professors describing any practices or behaviors that would support coming to understand definitions outside of class to come to understand a definition. The research on classroom instruction (c.f., Pinto, 2019; Fukawa-Connelly, 2012) suggests that mathematicians engage in modelling behaviors that could support coming to understand the concept in ways that go beyond memorizing the words of the definition. There also is reason to believe that mathematicians want to promote conceptual understanding (Nardi, 2007). Thus, these students do not believe that they have given any instruction about how to develop this understanding or what this understanding might entail, which could explain the divergence in their beliefs about what it means to understand a definition. As a result, we suggest that if mathematics professors have specific actions that they want students to take, they need to more explicitly teach them. At the same time, the literature does not currently allow us to link teaching activities and students' learning very well (c.f., Melhuish, et al, 2022). As a result, explorations of explicit instruction about coming to understand, how that relates to behaviors students actually carry out, and the types of understandings that result would be a valuable addition to the research literature.

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