“The runner jumped back?”: Developing Productive Understandings of Symbolization Through Animations and Applets

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In this report, I discuss one potential role an applet plays in the context of a teaching experiment. Mathematics students are constantly tasked with understanding mathematical notation, which is often context-dependent or sometimes loosely defined. While the purpose of mathematical symbolization is to help mathematicians communicate concisely, researchers have indicated that students have difficulties understanding and utilizing mathematical notation to represent quantities. Therefore, examining the mechanisms by which students can refine their understandings into more conventional ones can be of value to the field of mathematics education. This study provides an example of using applets to help students engage in reflective discourse. The findings of this study indicate that giving students a medium to visualize their mathematics can support them in understanding what their mathematical symbolization actually represents.

Keywords: Symbolization, Applets, Desmos, Teaching Experiment, Average Rate of Change

Symbolization is an integral part in mathematics to represent quantities and operations. As mathematicians, we utilize numbers and symbols to encapsulate information (Pimm, 1991), such as $\frac{dy}{dx}$ to mean “the (instantaneous) rate at which the value of the variable, y, changes with respect to changes in the value of the variable x.” Comprehending and utilizing numbers and symbols is an essential skill in learning mathematics. However, researchers have indicated that students have difficulties interpreting and using mathematical notation to represent quantities (Tall & Vinner, 1981; Torigoe & Gladding, 2007; Chin & Pierce, 2019; Begg & Pierce, 2021). Therefore, investigating how students develop understandings of mathematical notation is an important area of research in mathematics education. Fonger et al.’s (2016) study is an example of the benefits of examining student learning with a focus on students’ representational fluency. Their study found that leveraging students’ ways of thinking supported their ability to reason quantitatively and utilize appropriate symbolization.

Theoretical Background

When a student engages in symbolizing or interpreting mathematical notation, they have an expectation of the calculations they envision and that the parts that compose an expression or formula represent a process of quantifying or representing quantities within a situation. This is what O’Bryan (2018) terms *emergent symbol meaning*. This perspective is important because symbols do not inherently carry meaning; instead, an individual interprets the notation based on their conception of the mathematical problem. For example, consider a situation involving an arithmetic sum of adding up the numbers 1 through 10 (inclusive). One student might utilize the expression $(1+10)*\frac{10}{2}$ [This would be of the form $\frac{n}{2}(a_1+a_{10})$] to represent the process of adding up the first and last numbers and repeating this process $\frac{10}{2}$ times (i.e., 1 + 10, 2 + 9, 3 + 8, …). However, another student might write $\frac{1+10}{2}*10$, to represent their conception that the average of this set of numbers is 5.5 (this holds true due to it being an arithmetic sequence), and they can imagine replacing all the numbers with 5.5 and adding those up 10 times ($5.5 * 10$). Despite using the same numbers and symbols, these students had different ways of conceiving the situation. Another student might incorrectly write $\frac{10}{2} * 10$ because 10 is the highest number of the sequence, and they want to find the middle number of the sequence (and thus write $\frac{10}{2}$) and then add that up 10 times. In this last case, the student can check that their result is incorrect but might not have the means to determine why their symbolization of the situation was incorrect (and how they might make refinements). This last case highlights the importance of determining how we can support students in reflecting on their symbolization and how we might assist students in making adjustments when they have an incorrect response.

One method of helping students reflect on their mathematics is the usage of animations and applets. According to researchers, animations can have an *enabling function* to help reduce the cognitive load (Mayer, 2001) and support reflective discourse by focusing the conversation on students' understanding (Cobb et al., 1997). One way to enable reflective discourse involves having students anticipate what they will see before the animation plays and then explain what they see in the animation (Schnotz & Rasch, 2005; Thompson, 2019).

In this study, I report on a teaching experiment that leveraged Desmos animations and applets. This report's research question is "What role does an animation or applet play in the context of reflective discourse?"

Methodology

This study was conducted by engaging 2 students in individual teaching experiments on the idea of rate of change. The teaching experiment involved six sessions that focused on characterizing and advancing students' ways of thinking about rate of change. In this paper, I discuss one of the sessions and how the students interacted with a Desmos applet.

Teaching experiment methodology (Steffe & Thompson, 2000) involves a sequence of teaching sessions that include a student, a teacher, a witness, and a camera to record each episode. A teaching experiment's purpose is to identify a start and ending point of student progress and how students construct knowledge as they progress through each teaching episode. The goal of a teaching experiment then is to hypothesize and test models of student thinking.

The Desmos Activity (a summary of the activity can be seen here: Desmos Activity)

In the teaching session, students were presented with a situation where a runner traveled 100 meters in 32 seconds at a varying speed (a non-constant speed). Students were tasked with

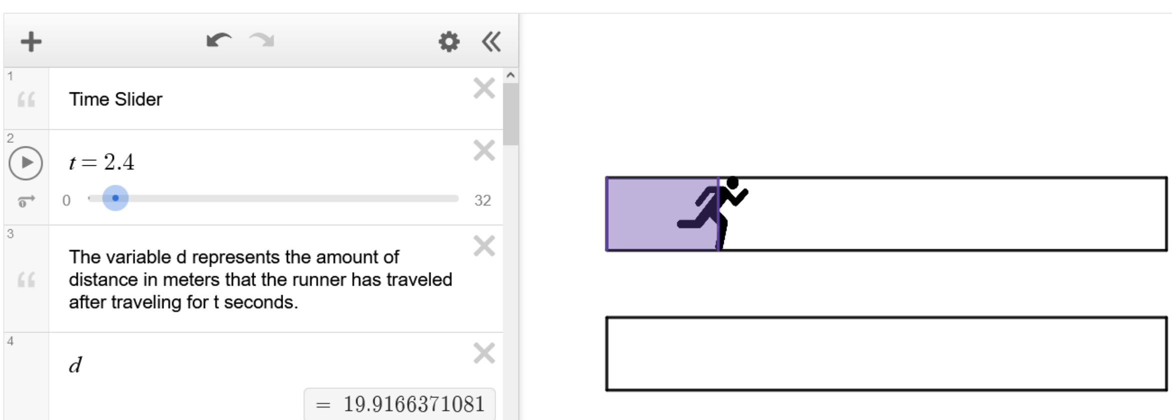

Figure 1: The Desmos Applet for the Runner Task

programming a function on Desmos to model the distance of a hypothetical runner that would run at the average speeds of the first runner in prespecified time intervals. On the Desmos applet (Figure 1) is a slider to vary the amount of time elapsed since the first runner started running (denoted by the variable, t). As students vary the value of t, the applet shows the value of d, the distance in meters that the first runner has traveled after traveling for t seconds.

Students are asked to program a second runner that runs at the average speed of the first runner over pre-defined time intervals. Initially, students are guided to split the 32-second interval (the total time the first runner took to travel 100 meters) into two equally sized time intervals and then utilize the first runner's average speed in each time interval as their programmed runners' constant speed. It is important to note that in the teaching session preceding this one, students developed a meaning for average rate of change as the constant rate of change that would achieve the same net change in the dependent quantity over the same change in the independent quantity. In this session, one goal was to support students in utilizing the value of a rate of change to determine how the related quantities of distance and time would covary. In this task, students were asked to define a function, b, where $b(x)$ represented the distance traveled by the programmed runner after traveling for x seconds. Some of the syntax needed to define a piece-wise function in Desmos was pre-made for the students to allow them to focus on defining a distance function by reducing the cognitive load (Mayer, 2001). As students progress through the activity, they are asked to program the runner using smaller and smaller time intervals with the eventual goal of making their programmed runner run at the same speed as the original runner (the goal is to form a basis for the idea instantaneous rate of change).

Results

This section discusses how the 2 students in the teaching experiment interacted with building the two-interval portion of the task and the role the Desmos applet played in supporting their ability to represent quantities with appropriate mathematical notation.

Scott

Scott was a first-year computer science major and had no trouble determining the appropriate average speeds for each interval. He used the time slider to determine the distance the first runner traveled by 16 seconds (67.098) and used this in his average rate of change expression, $\frac{67.098}{16}x$. However, for the next time interval, he wrote $\frac{100-67.098}{32-16}(x-16)$ (Figure 2). The interviewer then asked Scott what he attempted to represent with his expression. Scott replied that $\frac{100-67.098}{32-16}$ was the first runner's average speed in the second half of the race, and the expression $(x-16)$ represented an amount of time and together that would give you distance. He clarified that he wrote $(x-16)$ because "we have to worry about this interval and so total

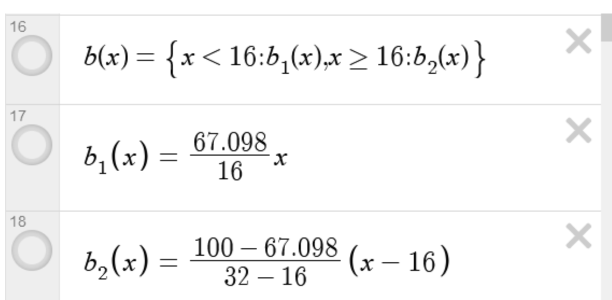

Figure 2: Scott's Initial Attempt at the Runner Task

time, x, that means we are taking away the first interval of time." His explanation suggested that he was using x to represent a number of seconds and that he conceptualized $(x-16)$ to represent a new quantity, the number of seconds elapsed after the first 16 seconds of the race.

However, what was absent from Scott's second expression was the amount of distance traveled by the runner during the first 16 seconds of the race. The interviewer decided to press play on the animation to demonstrate what Scott's expression represented in this situation. When the time elapsed hit the 16-second mark, his runner jumped back to the starting position. Scott was surprised and stated "the runner jumped back?", and he paused to look at his expression, $\frac{100-67.098}{32-16}\,(x-16)$. The interviewer then asked him what he attempted to represent and let the animation rerun. After watching the animation a second time, Scott remarked that he was missing the distance the runner had already traveled and proceeded to add 67.098 to his expression. He stated that he realized that his current expression only represented how his runner's distance would change, instead of his total distance traveled as time elapsed.

Throughout the rest of the teaching session, the Desmos applet played a pivotal role in supporting Scott in connecting his mathematical expressions with how he imagined distance accruing. The Desmos animation provided a visual representation of his mathematical expressions, and when the runner jumped back to the start of the race, Scott was perturbed by what he saw. Since Scott's expectation of what would happen did not align with what he observed, this prompted Scott to reconsider what quantities his symbols represented and then refined his function definition to accurately represent the quantitative relationships he intended. Having the Desmos animation display the runner's movement and distance from the starting line provided Scott with immediate feedback by displaying the quantities being represented by his expression. Seeing his programmed runner make an instantaneous jump back to the starting point prompted Scott to reconsider how to represent the distance he wanted to model (the distance of the programmed runner from the starting line as a function of time elapsed since the start of the race); in particular, he recognized that he needed both an expression to represent how that runner's distance would vary as time varied, $\frac{100-67.098}{32-16}\,(x-16)$, and the initial distance already accrued (67.098 meters) after 16 seconds since the start of the race.

Hans

Hans was a first-year aerospace engineering major and in the previous teaching sessions, he struggled with associating quantities with a mathematical representation. Initially, during the 2-interval portion of the task, Hans only wrote the average speed, $\frac{67.098-0}{16-0}$, and then verbalized that he did not know what he needed to write to represent time passing. The interviewer used the time slider to show the first runner moving as time passed and asked Hans about how long the first runner had traveled for. Hans noted that it was whatever the value of the independent variable and then proceeded to "add time" to his previous expression by multiplying x to his $\frac{67.098-0}{16-0}$. He repeated this for the second interval by typing in $\frac{100-67.098}{32-16}x$, but he did not attend to x as representing the total amount of time elapsed; he also failed to include the distance the runner had traveled during the first 16 seconds (Figure 3a).

For the first issue, the interviewer hypothesized that Hans was thinking "rate multiplied with time equals distance" and was not yet distinguishing between total distance and distance varying within an interval (similar with total time and the amount of time elapsed in the interval). Before playing the animation, the interviewer asked Hans to explain what each portion of his expression represented. Hans identified the average speed for each interval and the amount of time elapsed and then noted that all together, it “represented a change in distance” (Figure 3b). Afterwards, the interviewer asked what the value would represent for specific values of time. Hans then noticed that he needed to change his second expression because “we need $(x - 16)$ to account for the first 16 seconds of our race” and that”$(x - 16)$ is our change in time from 16 seconds.” Even though talking through what he was trying to represent helped Hans in making adjustments, he had not yet noticed his other error.

$$b_1(x) = \frac{(67.098 - 0)}{(16 - 0)}x$$

$$b_2(x) = \frac{100 - 67.098}{32 - 16}x$$

Figure 3a: Hans' mathematical expressions for the average speed runner

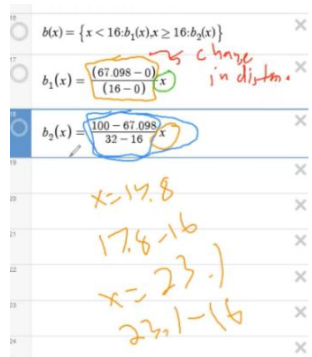

Figure 3b: Hans explaining his symbolization for specific time

To aid Hans in conceptualizing the distance traveled in the first 16 seconds, the interviewer played the animation, and the runner jumped back to the starting line after the runner had traveled for 16 seconds. Unlike Scott, Hans initial reaction was to add the expression $\frac{67.098-0}{16-0}\ x$ to $\frac{100-67.098}{32-16}x$. He appeared to conceptualize the expression $\frac{67.098-0}{16-0}\ x$ as a completed distance traveled that could be determined by multiplying a rate times a time (Table 1). The interviewer then prompted Hans to talk about the quantities he conceptualized and attempted to represent [Lines 4-6] while probing the conception of the starting point for a quantity’s measurement [Lines 7-9]. Eventually, he noted that we still needed “the first distance” and proceeded to add 67.098 to his expression [Line 11]. Hans’ statement that he needed “the first distance” suggested that he shifted to making a distinction between different distances in the situation since he chose to label the distance accrued in the first 16 seconds as the “first distance.”

Table 1: Hans working through the Runner Task

1	Hans:	So would I… add the top?
2	Int:	So you want to add this? *Highlights $\frac{67.098-0}{16-0}\ x$*
3	Hans:	Yeah
4	Int:	Remember x is going to vary (Hans: oh yeah) so what quantity do you want to add?
5		So think about what you’re trying to represent. This over here *highlights
6		$\frac{100-67.098}{32-16}\ (x - 16)$*, what is this again?
7	Hans:	Um that’s my change in distance.
8	Int:	From what?
9	Hans:	From $x - 16$… like change in distance from… from 16 seconds
10	Int:	Okay so what part are we missing?
11	Hans:	The beginning part of the race… the first distance

The Desmos animation supported Hans in developing fluency in applying an average rate of change in a context and connecting his mathematical expressions with the quantities he attempted to represent. Similar to Scott, Hans was perturbed when the runner instantly jumped back to the starting line, which prompted him to reflect on what quantities his symbols represented. Upon seeing the runner jump back to the starting point, Hans realized that he needed the distance traveled by the runner during the first 16 seconds of the race, and this led to a conversation (Table 1) that supported Hans in making adjustments to accurately represent the quantitative relationships he conceptualized. Additionally, the Desmos applet's slider functionality aided Hans in understanding the role of a variable as representing the value of a varying quantity (Figure 3).

Discussion

Both students made similar errors in failing to distinguish between a change in distance versus total distance traveled. Additionally, before playing the animation, neither student indicated any uncertainty about what their mathematical expression would represent in the situation. In their mind, these students thought that what they symbolized would match what they expected (Figure 4a). However, when what they saw in the animation did not align with what they imagined, they experienced a perturbation that allowed the interviewer to facilitate a conversation to aid the students in reasoning about quantities and how they would represent them. This discussion led students to reconsider what they had symbolized and how to make refinements to represent the quantities they had in mind (Figure 4b).

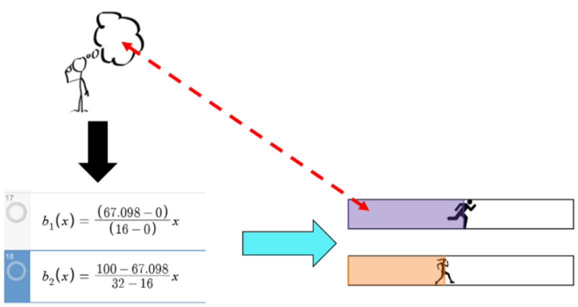

Figure 4a:A student anticipates what they might see

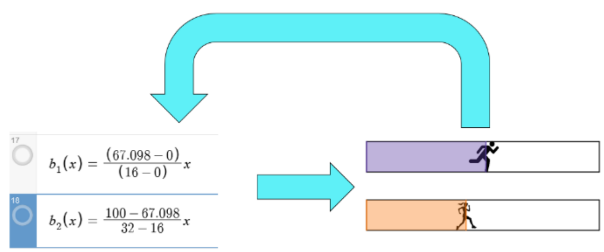

Figure 4b:Students reflect on what they initially symbolized

While the interviewer could have intervened and pointed out the errors in each student's notation, the Desmos animation provided a visualization of what their mathematics would represent and created opportunities for the students to experience perturbations in order to engage in self-reflection. This result suggests that students should have opportunities to explore and engage with what their mathematical symbolization might represent in a particular context.

Limitations and Future Directions

While the results of this study can only be known to be true for these 2 students, the findings of this study provide insight into how we may address the issue of students' difficulties with mathematical notation. Additionally, the tasks in this study utilized kinematic situations, but researchers have indicated that students often struggle to apply their understanding of mathematical ideas outside of kinematic situations (Rasmussen & King, 2000; Marrongelle, 2004).

Future research can further examine the role of applets and animations in a quantitative capacity through classroom studies. Further, more research on the usage of applets and animations should be conducted in non-kinematic situations.

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