How Calculus Students Describe and Use Volume in Calculus and Non-Calculus Tasks

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Existing research indicates that some learning challenges Calculus students face may not be related to the Calculus-specific content, but rather from issues related to prerequisite knowledge. Several Calculus topics such as volume by slicing and volumes of revolution have a prerequisite topic of volume. These Calculus topics tend to be challenging for students in various ways. Better understanding how these students describe and use volume may help inform instructors as well as potential reforms to practices to better support student learning of these Calculus topics.

Keywords: calculus, volume, integration, student description

Existing literature documents the influence of prerequisite knowledge on the learning of limits and differentiation (e.g., Dorko, 2012; Ferrini-Mundy & Graham, 1994; Orton, 1983). It is possible that these issues could also extend to other Calculus topics. In particular, could student understanding of volume influence thinking on volume by slicing or volumes of revolution problems? A 2012 study indicated that, on tasks prompting Calculus students to find the volume of prisms, issues arose with conceptions and calculations, including some students making calculations more suggestive of surface area than volume (Dorko, 2012). Building on findings about understanding of integration (e.g., Jones, 2015; Sealey, 2014), we replicated Dorko's study's design with a focus on volume-related topics found in second-semester Calculus.

Approached from a cognitive perspective, data were collected via written response to surveys distributed to second-semester Calculus students from a large public university. Three questions were prompts to find the volume of a rectangular, a circular, and a triangular prism. Data was also collected in the form of follow up interviews to the survey task. During the interviews, students were asked to solve two volume of revolution tasks, similar to those typically presented in their Calculus class. Students were asked to describe their solutions, as well as why they think their solutions are or are not accurate. Data included what was recorded on paper, as well as video and audio interview recordings. We collected 97 responses to the survey task and have conducted 7 interviews. Of the students who completed the survey, 20% of students used an incorrect formula to calculate the volume of the cylinder and 11% for the triangular prism. These findings are similar to those of Dorko (2012). Analysis of interview data indicates some issues with volume in Calculus tasks as well as uncertainty about the volume ideas. One student stated, "it's the basic geometry that's gonna get me." While this student ultimately correctly completed each task, the student was consistently unsure of the volume formula they were using for both the disk and shell method when setting up each problem. The student was also unable to sufficiently justify the volume formula.

These findings suggest that volume-related ideas are influencing thinking about calculations done in second-semester Calculus. The poster will display student-generated volume sketches and will expand on these findings as well as others from the larger study. This will include findings from additional interviews and discussion of implications.

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Background and Purpose

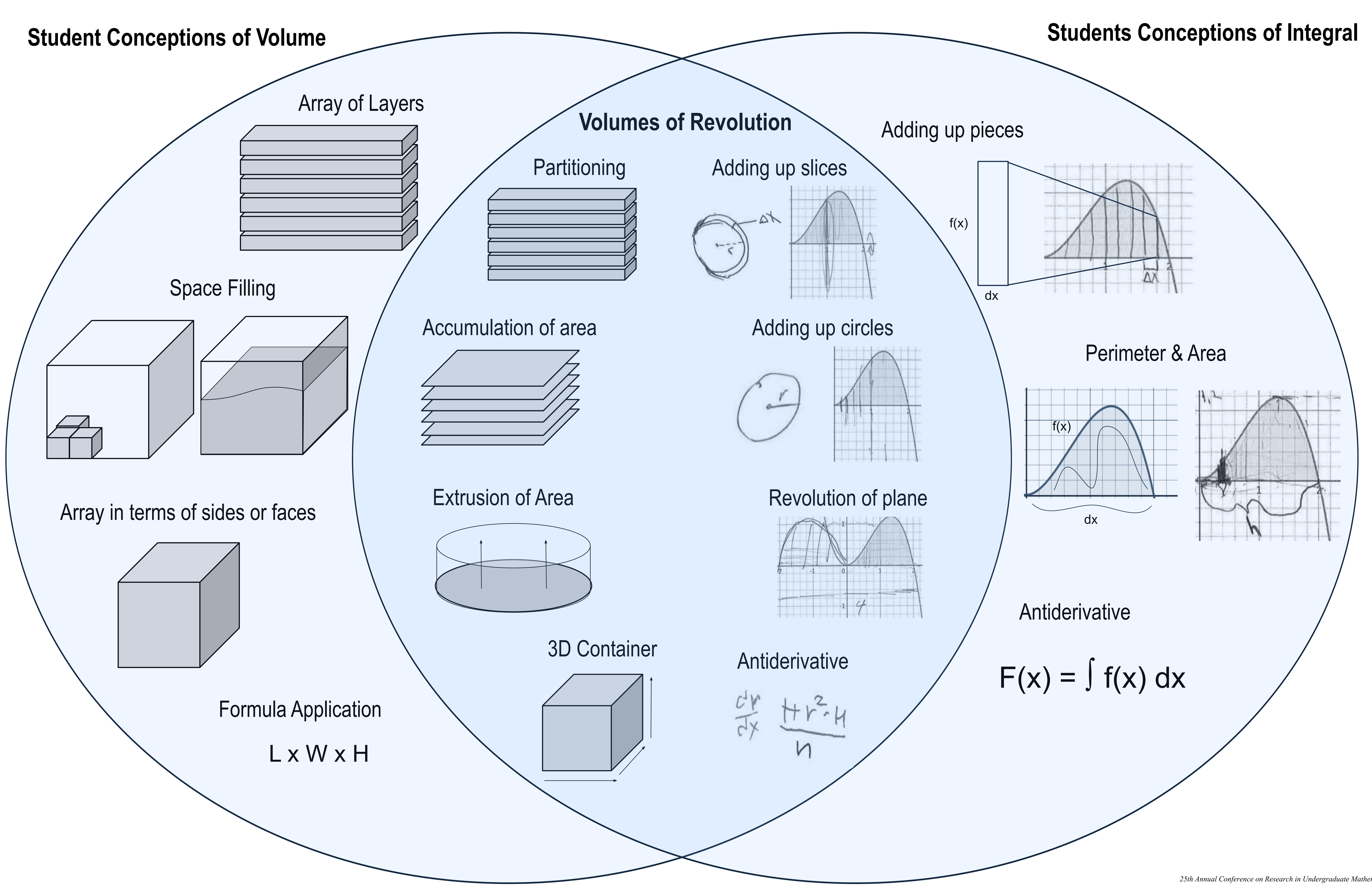
Existing literature documents the influence of prerequisite knowledge on the learning of limits and differentiation (e.g., Dorko, 2012; Ferrini-Mundy & Graham, 1994; Orton, 1983; Stewart & Reeder, 2019). It is possible that this is the case for other Calculus topics. In particular, could students' conceptions of volume influence their thinking on volume by slicing or volumes of revolution problems? Based on the interview data, we contend that Calculus students draw on both their conceptions of the integral and their conceptions of volume when solving these types of problems.

Jones (2013, 2015) investigated how Calculus students conceptualize the integral. Three of the student conceptions of the integral are highlighted in this study.

- **Adding up pieces**
The integral as the area under a curve partitioned into infinitely many rectangular pieces that are summed up. A representative rectangle indicates the multiplication of the function and the differential and applies to all other rectangular pieces. The bounds of integration indicate where the pieces start and stop.
- **Perimeter and Area**
The integral is the area under a curve, but as a singular unit rather than partitioned into pieces. The multiplication inside the integrand is given by the perimeter of the region bounded, with the function being the top curve, the differential being the bottom base, and the bounds of integration being the left and right sides.
- **Antiderivative**
The integral is the inverse process of differentiation. Students use "function matching," where the function inside of the integral must have some original function for which it is the derivative of.

Battista and Clements (1998) investigated how Elementary students conceptualized volume through three-dimensional cube arrays. The categories presented in this study are supported by Dorko (2012) and survey data gathered as part of the larger study of this research.

- **Array of Layers**
Volume is the summation of finitely many three-dimensional unit slices.
- **Space Filling**
Volume is the amount something can hold. This conception can take the form of three-dimensional unit cubes or as the filling of a whole unit.
- **Array in terms of sides or faces**
Volume is bounded by the perimeter or surface area. In this conception, the student measures volume only using the exterior of the shape.
- **Formula Application**
Volume as simply a formula, such as length times width times height, to be plugged in and evaluated.



Integral Conceptions

- **Adding up slices**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation, with the solid partitioned into infinitely many slices, where the function represents the radius of the disk and the differential represents the thickness of the slice.

"Alright, so you could imagine that, if you wanted to revolve it around the axis, you could imagine a bunch of little circles, going around the x-axis. You could like slice it up into a bunch of little circles . . . and if you sum them up using an integral, that will give you a volume. . . . but it's technically gonna be little disks."

- **Adding up circles**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation, with the solid partitioned into infinitely many two-dimensional slices, where the function represents the radius of the circle and the differential represents the infinitesimal difference between each slice.

"We're finding the areas between zero to two, so that's the integral you have. You are forming a bunch of circles with rotating it around the x-axis. . . . they're like slices. . . . it wouldn't be any width."

- **Revolution of plane**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation. The area under the curve is a singular unit of area and the integral "does the work" of revolving it around the axis.

"So we can find the circumference of the circle, which would be the value four, cause you have negative two to two . . . so you have four pi and then we multiply that by, um, the area under a curve."

- **Antiderivative**
The integral is the inverse process of differentiation.

"It would just be, you know, looking at the problem and then applying the differentiation rules, you know, in reverse."

Volume Conceptions

- **Partitioning**
Volume is the summation of infinitely many infinitesimally thin three-dimensional slices.

"You can think about volume of something as first smaller volumes. . . . So if you have a big rectangle, you can first think about it as maybe not like a rectangle, but maybe like a few boxes on top of each other."

- **Accumulation of area**
Volume is the summation of infinitely many two-dimensional regions of area.

"What you could imagine it's volume as turns into it's area as you keep making it's height much much thinner, and if you stack a ton of these essential areas- what are essentially areas- on top of each other, you can effectively get a volume."

- **Extrusion of area**
Volume is a region of area multiplied by some depth.

"A disk being a circle that's kind of raised up."

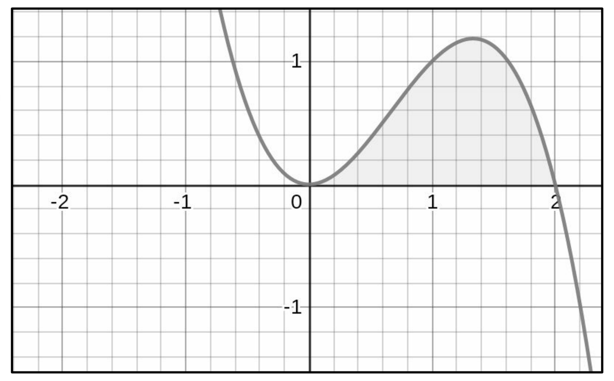
- **3D Container**
Volume is the space inside of a container, bounded by planes.

"to describe volume, I'd say it's like the amount of water a closed container could hold."

Methods

- **Survey tasks (n=97)**
  - distributed to second and third semester Calculus students
  - prompted students to find the volume of three prisms
  - supports literature on student conceptions of volume
- **Follow-up interviews (n=7)**
  - conducted after students completed summative assessment on volumes of revolution
  - included two volumes of revolution tasks
    - revolving a region around the x-axis (disks)
    - revolving a region around the y-axis (shells)
  - students were asked to explain their process

Using a grounded theory inspired approach, segments of the transcribed interviews were coded into emerging categories for both conceptions of the integral and conceptions of volume.



Findings and Implications

- **For instruction**

Our data suggests that students recall their prior conceptions related to volume when solving volumes of revolution or volume by slicing tasks. We contend that **students use both their conceptions of the integral and of volume when solving these tasks.** We suggest that instructors of Calculus should care about their students' understanding of volume. Presenting Calculus students with integration by slicing of simple prism tasks may help reveal students' conceptions of volume and how their conceptions do or do not align with their conceptions of integration.

- **For further research**

The research presented in this poster is part of a larger study investigating student conceptions of the integral and volume. Jones (2015) quantitatively analyzed the productivity of the integral conceptions they identified in both the mathematics and applied contexts. It is beyond the scope of this research to quantitatively analyze the conceptions identified in this poster. We suggest further research **to investigate these conceptions of integral and volume to help determine if the productivity of some concepts over others remains the case in the context of applied Calculus problems.**

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