Theoretical Considerations for Designing and Implementing Intellectual Need-Provoking Tasks

Aaron Weinberg — Ithaca College
Michael Tallman — Oklahoma State University
Steven R. Jones — Brigham Young University

The idea of intellectual need (IN) has received much interest from instructors in trying to design tasks that engage students in impasse-driven learning. However, we argue that the literature on IN is currently insufficient for supporting the careful design and implementation of tasks meant to provoke IN. In this paper, we examine two shortcomings: (1) What exactly IN can be created for, and (2) How an instructor might support students in navigating the experience of resolving the confusion and constructing the targeted meanings. For the first of these, we describe the category error of thinking of producing IN for a "topic," and use the idea of conceptual analysis to suggest a way to address this shortcoming. For the second, we bring in control-value theory to explain what an instructor might attend to in order to ensure that the disequilibrium stays productive and does not lead to frustration and disengagement.

Keywords: Intellectual Need, Task Design, Constructivist Pedagogy, Epistemic Emotions

Introduction and Literature Review

Educators and researchers have long described the positive impact of "desirable difficulties" and "failure-driven scaffolding" on student learning (e.g., Dewey, 1938; Kapur, 2015; Sinha et al., 2020). Studies have shown that experiencing cognitive incongruities (e.g., Graesser et al., 2005), working on problems prior to instruction (e.g., Kapur & Rummel, 2012), and experiencing impasses (e.g., VanLehn, 1988; Kapur, 2014) are associated with improved learning and constructive reasoning. Graesser and D'Mello (2012) proposed that these results align with Piaget's (1971, 1985) work on cognitive development—specifically, the idea of disequilibrium, in which the impasses students encounter prompt them to revise their cognitive schemes or construct new ones to accommodate for novel stimuli and experiences.

In the context of learning mathematics, Harel (e.g., 2013), described the role played by cognitive disequilibrium in the learning process in terms of the construct of the *intellectual need*, which is the perceived need to resolve "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (p. 122). Since its introduction, intellectual need has been a widely-cited principle of instructional and curriculum design (e.g., Burger & Markin 2016; Caglayan, 2015; Foster & de Villers 2015; Koichu, 2012; Leatham et al., 2015; Rabin et al., 2013; Zazkis & Kontorovich, 2016). The idea of intellectual need has been enthusiastically endorsed by some instructors. In their planning, these instructors begin with a target mathematical concept ("[x]") and then begin their lesson planning by asking something like: "If [x] is aspirin, then how do I create the headache [i.e., the intellectual need]?" (Meyer, 2015).

Despite its widespread use in the research literature, there is little guidance about how to design and implement effective intellectual need-provoking (IN-P) tasks. Weinberg and Jones (2020, 2022) proposed a theoretically-grounded framework for IN-P task design, but identified issues in need of further clarification, elaboration, or theoretical justification in both their own framework and the research literature on intellectual need generally. First, Weinberg and Jones (2020) noted that the research literature has not clarified what a student can experience intellectual need *for*, an issue that we believe is at the origin of subtle but significant epistemological contradictions in IN scholarship. Second, Weinberg and Jones (2022) highlighted the importance of attending to the role played by students' affective states as they engaged in mathematical activity, but their framework did not clarify how to account for these states in IN task design or implementation.

On its face, it is possible for one to interpret the idea of IN as suggesting that an instructor need only design a task in which their students are required to confront a problem they do not yet possess the intellectual tools to solve. However, we believe this view orients instructors to develop tasks that might stimulate interest and motivation, but fall short of promoting students' constructive mathematical reasoning. In this theoretical paper, we address these gaps in the research literature by presenting an example of a task designed to provoke intellectual need, highlighting some aspects that, upon further inspection, potentially fall short of engendering this cognitive state, and synthesizing additional theoretical perspectives to ameliorate these gaps.

An Example

We begin by presenting an example of a typical task, some variation of which is commonly included in curricula and instruction to motivate the concept of a derivative. For our example we chose to follow the presentations in some widely-used textbooks[1], viewing textbook presentations as potentially reflecting the authors' instructional intentions. This is supported by the authors' descriptions of their texts as "a teaching tool for instructors and … a learning tool for students" (Stewart et al., 2021, p. xi). Suppose a calculus instructor expects their students to learn about the concept of a derivative and considers how to create an intellectual need for the topic. If they follow the exposition of some of the textbooks we examined (e.g., Larson & Edwards, 2018; Hughes-Hallet et al., 2017; Stewart et al., 2021), they will likely begin by posing either "the tangent problem" (i.e., how can we compute the slope of a tangent line?) or "the velocity problem" (i.e., how can we compute instantaneous velocity?). If an instructor adopted the later approach, they might facilitate an informal discussion of velocity and, typically, define "average velocity" as an *arithmetic*—as opposed to *quantitative—operation* (i.e., division of the change in position by corresponding elapsed time) (Thompson, 1990). Then, the instructor might pose a problem or example in which they demonstrate an inability to perform this computation, either due to insufficient information or an indeterminate form (i.e., "0/0"). The instructor using this approach might highlight the conundrum that "we are dealing with a single instant of time, so no time interval is involved" (Stewart et al., 2021, p. 81) and suggest computing the average velocity over successively smaller time intervals. Then, the instructor might represent the situation using a graph with a point representing the time of interest , and add a second point on the graph with arbitrary coordinates. The instructor then might propose that students imagine that the second point approaches the first point (e.g., Stewart et al., 2021; Hass et al., 2019). Alternatively, the instructor might follow slightly different presentations, such as suggesting that students choose points closer together (e.g., Larson & Edwards, 2018; Herman & Strang, 2020), or that they imagine "zooming in" (e.g., Boelkins et al., 2018; Hughes Hallett et al., 2017).

When the instructor first proposes the problem with finding an instantaneous velocity, the students ostensibly do not possess the intellectual tools to solve this problem. Consequently, an instructor might be tempted to view this initial task as having provoked intellectual need (in their students) for the concept of instantaneous rate of change, and view the definition of the derivative at a point as having resolved this intellectual need. However, as we will argue below, there are additional considerations that are essential for understanding if IN was actually

[1] We examined Stewart et al. (2021), Larson & Edwards (2018), Hughes Hallet et al. (2017), Hass et al. (2018), Herman and Strang (2020), and Boelkins et al. (2018).

produced and what, exactly, it was produced for. Furthermore, the instructor might have only made their students experience confusion, without necessarily helping them navigate the experience of resolving this confusion, and constructing a justification for the necessity of derivative at a point.

DNR-Based Instruction in Mathematics

The concept of intellectual need is one of many theoretical constructs organized within an elaborate framework called DNR-based instruction in mathematics (Harel, 2008). The DNR theoretical framework is grounded in several constructs, the most germane of which (for the present paper) we report here. The triad of mental act, way of understanding, and way of thinking is central to the concepts domain of the DNR framework. Mental acts are the basic applications of cognition to our experiential world; they encompass behaviors such as interpreting, proving, conjecturing, inferring, justifying, explaining, generalizing, and predicting. Harel (2008) defined the cognitive constructs way of understanding and way of thinking and described their respective relations to mental acts:

> A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a way of thinking associated with that mental act (p. 490).

We regard an individual's way of understanding a mathematical idea as synonymous with their meaning for the idea (Thompson et al., 2014). We interpret meanings in terms of Piaget's notion of a scheme as an internalized organization of actions, images, and operations constructed and refined through distinct forms of abstraction (Tallman, 2021).

Harel's mathematics premise frames mathematical knowledge as a union of historically institutionalized ways of understanding and ways of thinking. Additionally, Harel (2008) expressed the the reciprocity between these distinct forms of mathematical knowledge in the duality principle, which states, "Students develop ways of thinking through the production of ways of understanding, and, conversely, the ways of understanding they produce are impacted by the ways of thinking they possess" (p. 899). The *knowing premise* asserts,"Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium" (Harel, 2008, p. 894). Relatedly, the *knowing-knowledge linkage premise* states, "Any piece of knowledge humans know is an outcome of their resolution of a problematic situation" (Harel, 2008, p. 894).

Commensurate with these theoretical premises, *intellectual need* (IN) is a psychological state–with cognitive and affective entailments–resulting from an individual's interpretation of a problematic situation, the resolution of which requires the individual to extend, modify, or create knowledge structures. Unlike confusion, intellectual need results from an individual's conception of a problematic situation in a way that orients them to engage in the constructive activity required to resolve the situation. IN is thus energetic in that it stimulates, rather than impedes, generative mathematical reasoning.

Harel distinguished different categories of IN. A need for *certainty* motivates deductive reasoning; it is a need to prove, or remove doubts. A need for *causality* is a need to identify the cause of some phenomenon. A need for *computation* involves a desire for efficient quantification. A need for *communication* refers to an obligation to persuade others that an assertion is true. Finally, a need for *connection and structure* entails a commitment to identify similarities and analogies and to abstract unifying principles.

Issue #1: Intellectual Need for What?

There is a subtle category error lurking behind a statement like, "I want students to experience an intellectual need for derivative." Replace "derivative" in this sentence with "constant rate of change," "radian measure," "polynomial rings," "Lyapunov functions," and the category error remains. The issue is this: students cannot experience an intellectual need for a mathematical *topic*. A student can, however, experience an intellectual need to engage in the actions that form the experiential basis of a particular *meaning*, or way of understanding, that they expect might enable them to resolve a problem *they* have conceptualized.

We previously described the experience of intellectual need as a stimulant for generative mathematical activity, *not* a state of confusion that paralyzes reasoning and induces an affective state of interest, analogous to the curiosity provoked by having witnessed a compelling magic trick. These affective states result in the student being receptive to *absorbing* (to use a term consistent with Meyer's aspirin-headache metaphor) the mathematical concepts communicated to them, or the skills demonstrated for them, by a knowledgeable expert. This type of affective experience is nearer to how Harel described affective *need*, indisputably useful but certainly not identical in its affordances for supporting students' learning as IN. So where exactly is the category error in aspiring to support students' experience of IN for derivative (or any other mathematical "topic")? Intellectual need cannot, on the one hand, be conceptualized in a manner consistent with Piaget's idea of *equilibration* of cognitive structures–and radical constructivist epistemology generally–while on the other hand promote an exclusively affective state where learning is a product of a *fundamentally empiricist process* (i.e., perception). Theoretical coherence requires that both the inducement of IN and the learning afforded by its experience be a product of the actions, operations, abstractions, and generalizations *of the learner*.

Intellectual need is thus a need to engage in the precise actions that must be internalized and organized to form targeted ways of understanding (i.e., cognitive schemes) through *pseudo-empirical* and *reflecting abstractions* (Piaget, 2001). A meaning, or understanding, exists only in the mind of an individual, and thus consists of an elaborate organization of idiosyncratic images, actions, and operations (diSessa, 1988; Tall & Vinner, 1981; von Glasersfeld, 1995). Successfully engendering IN therefore requires an instructor to be cognizant of the mental actions and conceptual operations entailed in the meanings they want to support or necessitate (Tallman, 2021; Tallman & Frank, 2020). Without such awareness, an instructor cannot design problematic situations that require students to engage in, and abstract from, these actions.

Solution to Issue #1: Conceptual Analysis. Thompson (2008) described conceptual analysis as the process by which an instructor or curriculum designer clarifies the ways of understanding they intend to support, and elucidates how these understandings depend on and contribute to developing particular ways of thinking. In other words, a conceptual analysis is an articulation of "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 43). Conducting a conceptual analysis involves (1) unpacking what it means to conceptualize a mathematical idea in a specific way, and (2) explicating the cognitive activity necessary to construct the targeted understanding. Conducting conceptual analyses is therefore essential to an instructor's development of tasks that effectively necessitate the actions that form the basis of the ways of understanding (i.e., schemes) an instructor or curriculum designer expects students to construct.

Applied to the archetypal tasks described above for motivating students' learning of derivative at a point, a conceptual analysis would specify what it means to understand instantaneous rate of change and clarify how this understanding depends on particular conceptions of constant and average rate of change. Moreover, the role of quantitative and covariational reasoning in the structure and development of the targeted understanding would be made explicit. A conceptual analysis would highlight the distinction between two of the meanings that are implied in the example above. One such meaning—a quantitative meaning—involves conceptualizing average rate of change as a proportional relationship between changes in the measures of distance and time and the successive approximation process as involving thinking of these quantities as continuously covarying and measuring this covariation. In contrast, a second meaning—a geometric meaning—involves thinking of the distance-time pairs as represented as points on a graph of the function, thinking of average speeds as the "slantiness" of lines connecting the two points, and then thinking of the "approaching" process as suggesting a pattern in the "slantiness." These two meanings can be coordinated, but this coordination is automatic. Though it is far beyond the scope of this paper to provide an example of such a conceptual analysis we encourage the reader to reflect on the experiences that might be required to support these distinct students' understanding of each of these meanings of instantaneous rate of change.

Issue #2: Managing and Leveraging Students' Experiences

Suppose that the instructor was successful in provoking intellectual need for a particular meaning of the derivative at a point—for example, the quantitative meaning described above (i.e., the limiting value of successive approximations of instantaneous velocity by average velocities over successively smaller amounts of change in time). The research literature on intellectual need is largely silent on what the instructor needs to attend to and how they should act to guide students through the process of constructing the relevant schemes that will enable them to resolve the disequilibrium. In particular, the students are in a state of cognitive disequilibrium and might react negatively to the confusion that is associated with being in such a state. In this section, we adapt theoretical perspectives from the research literature on student affect to highlight these issues and propose aspects of the class' activity that can guide instructors' attention and action.

Control-Value Theory and Epistemic Emotions. Numerous researchers (e.g., Mandler, 1975; 1999; Morton, 2010; Stein & Levine, 1991; Tallman & Uscanga, 2020) have highlighted the critical relationship between the emotions students experience, their engagement with a task, and their subsequent cognitive activity. Pekrun (2000; 2006) proposed a theory of the emotions that students experience in academic settings: the students' emotions are influenced by their perception of the extent of their *control* over their activity and its outcomes, and their emotions are also influenced by their *values*—their perception of the importance of the activity and outcomes. Muis et al. (2015) found that students' experiences of emotions in mathematical settings were closely related to the extent to which they valued mathematics and the degree to which they felt a sense of agency over their engagement with the class' pedagogical practices and of the mathematics they were learning.

Pekrun and Stephens (2012) classified academic emotions into achievement emotions (such as anxiety and contentment); topic emotions (e.g., how the individual feels about the mathematical topics); social emotions (such as pride and shame); and epistemic emotions, which arise when "the object of their focus is on knowledge and learning" (Muis et al., 2015, p. 173). Epistemic emotions arise from "knowledge states involving discrepancy, incongruity, or conflict between cognitive schemas" (Nerantzaki, 2021, p. 2). These emotions include surprise, curiosity, enjoyment, confusion, anxiety, frustration, and boredom (Pekrun et al., 2016) and have been explored by numerous researchers (e.g., Efklides, 2017; Loewenstein, 1994; Mandler, 1975, 1984; Metcalfe et al., 2017; Touroutoglou & Efklides, 2010; Vogl et al., 2020).

We anticipate that curiosity and confusion are the epistemic emotions most closely linked to experiences of intellectual need, though IN is not reduced to students' experience of these emotions, as previously argued. Curiosity is "the complex feeling and cognition accompanying the desire to learn what is unknown" (Kang et al., 2009, p. 963) and it arises when a student experiences a gap in their knowledge (Loewenstein, 1994; Metcalfe et al., 2017); Litman (2008) proposed that epistemic curiosity motivates students to learn new ideas, and, thus, appears to be closely related to the role of intellectual need in the DNR framework. Similarly, confusion occurs when students are unable to assimilate new information into their cognitive schemas and must engage in accommodation (D'Mello et al., 2014); thus, confusion appears to play an important role in students' experiences of intellectual need. However, while curiosity tends to uniformly support students' participation and learning (e.g., Arguel et al., 2019), confusion can either support or hinder student engagement (e.g., Lodge et al., 2018), and D'Mello et al. (2014) distinguished between what they called "productive" and "unproductive" confusion.

Solution to Issue #2: Monitoring and Leveraging Epistemic Emotions. To think about how an instructor can attend to students' epistemic emotions, we can organize the students' engagement with an activity through a model of affective dynamics, such as the one proposed by D'Mello and Graesser (2012) in Figure 1:

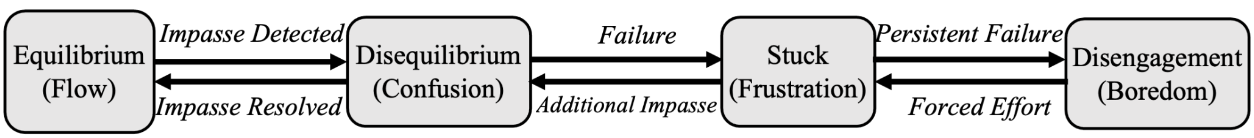

Figure 1. D'Mello and Graesser's (2012) model of affective dynamics

When students begin working on the instantaneous velocity task, the teacher needs to understand the students' perceptions of their roles in mathematics classrooms—a component of their values about mathematical ways of thinking—to help students experience an impasse. When the students recognize their inability to compute the instantaneous speed *and* can articulate the reason for this inability, they shift to a state of disequilibrium; as suggested by D'Mello and Graesser's (2012) model, this shift corresponds with epistemic emotions of either curiosity or confusion (or both).

At this point, as the instructor begins to help students begin the process of articulating and constructing meanings for the targeted schemas, they need to carefully monitor the students' perception of control over their cognitive activity and the targeted outcomes, and also need to remain cognizant of the students' values to help align the activity with those values. In particular, to prevent students from experiencing persistent failure that leads to boredom, the instructor needs the students to experience confusion productively by monitoring their experiences of failure and providing support to move students back into a productively confused state (unfortunately, D'Mello and Graesser (2012) were not able to identify the specific actions that stimulated this transition in affective states).

After working within a state of disequilibrium, the instructor needs to help their students resolve their disequilibrium in a way that leads them to, by Harel's (2008) knowing-knowledge linkage principle, generate the target mathematical meanings for instantaneous velocity. This requires that the teacher helps the students articulate the meanings they need to construct, thereby enabling them to believe that it is possible to resolve the impasses they are experiencing. In the case of instantaneous speed, this could entail helping students articulate the quantities of distance and time involved in the situation, focus on the amounts of change of these quantities, imagine these quantities as varying and co-varying, and then focus on the relationship between the quantities. During this process, the teacher needs to, first, continue to align the class activity with the students' perception of their control over the outcomes of the activity and, second, to help align the act of constructing meaning for instantaneous velocity with the students' values.

Discussion

In this paper, we set out to provide theoretical clarifications on key ideas in designing IN-provoking tasks. In particular, we focused on what IN can be experienced for, what experiencing IN should lead to in terms of student actions, and how to guide students through the disequilibrium toward the resolution of the problem

If the concept of intellectual need is to be consistent with its theoretical premises and epistemological foundations, students' experience of it must induce the precise actions that are abstracted and internalized to form the substrate of cognitive schemes (i.e., *ways of understanding*). Intellectual need is a need for *action*–it sets in motion a constructive process by provoking the experiential basis of knowledge structures, in a manner consistent with Piaget's notion of equilibration and, crucially, the cognitive mechanisms by which equilibrium is established. While affective needs are of value in mathematics education, experiencing them alone does not necessarily place students on a trajectory to constructing targeted ways of understanding. In this paper we distinguished intellectual need from other types of experiences that stimulate interest, provoke curiosity, or foster motivation. We clarified the relation between intellectual need and to constructivist epistemology so that mathematics educators might be better positioned to engender this productive experience through thoughtful task design and implementation. We proposed Thompson's (2008) notion of conceptual analysis as the essential process that positions instructors and curriculum designers to effectively create intellectual need-provoking tasks.

Although the research literature includes numerous examples of tasks that are designed to provoke intellectual need, there has been little guidance or theoretical framing about how an instructor should engage their students after the task has been presented. The disequilibrium that results from the stimulation of intellectual need is associated with particular affective states, and there is a critical relationship between these states and the students' activity. The perspective of control-value theory allows us to frame this activity to identify and help students manage these affective states by focusing on the students' perceptions of control of their activity and coordinating this with the value students place on their activity and its outcomes. In particular, by focusing on epistemic emotions can help instructors identify particular affective states and their likely antecedents so that the instructor can help students articulate the meanings they need to construct, successfully resolve their disequilibrium, and leverage intellectual need to support student learning.

References

Arguel, A., Lockyer, L., Chai, K., Pachman, M., and Lipp, O. V. (2019). Puzzle- solving activity as an indicator of epistemic confusion. *Frontiers in Psychology, 10*, p. 163. https://doi.org/10.3389/fpsyg.2019.00163

Boelkins, M., Austin, D., & Schlicker, S. (2018). *Active Calculus 2.1*. Grand Valley State University Libraries.
Burger, L., & Markin, M. (2016). A Deeper Look at a Calculus I Activity. In B. Lawler, R. N. Ronau, & M. J. Mohr-Schroeder (Eds.). *Proceedings of the Fifth Annual Mathematics Teacher Education Partnership Conference*. Washington, DC: Association of Public Land-grant Universities
Caglayan, G. (2015). Making sense of eigenvalue–eigenvector relationships: Math majors' linear algebra–Geometry connections in a dynamic environment. *The Journal of Mathematical Behavior, 40*, 131-153. https://doi.org/10.1016/j.jmathb.2015.08.003
D'Mello, S., & Graesser, A. (2012). Dynamics of affective states during complex learning. *Learning and Instruction, 22*(2), 145-157. https://doi.org/10.1016/j.learninstruc.2011.10.001
D'Mello, S., Lehman, B., Pekrun, R., & Graesser, A. (2014). Confusion can be beneficial for learning. *Learning and Instruction 29*, 153–170. https://doi.org/10.1016/j.learninstruc.2012.05.003
Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Collier Books.
diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. B. Pufall (Eds.), *Constructivism in the computer age* (pp. 49–70). Lawrence Erlbaum Associates, Inc.
Efklides, A. (2017). Affect, epistemic emotions, metacognition, and self-regulated learning. *Teachers College Record, 119*(13). http://dx.doi.org/10.1177/016146811711901302
Foster, C. & de Villers, M. (2016). The definition of the scalar product: An analysis and critique of a classroom episode. *International Journal of Mathematical Education in Science and Technology, 47*(5), 750-761. https://doi.org/10.1080/0020739X.2015.1117148
Graesser, A., Chipman, P., Haynes, B., & Olney, A. (2005). Autotutor: an intelligent tutoring system with mixed-initiative dialogue. IEEE Transactions on Education, 48(4), 612e618. https://doi.org/10.1109/TE.2005.856149
Harel, G. (2008). DNR perspective on mathematics curriculum and instruction. Part II: with reference to teachers' knowledge base. *ZDM—The International Journal on Mathematics Education, 40*, 487-500. https://doi.org/10.1007/s11858-008-0146-4
Harel, G. (2013). Intellectual need. In K. R. Leatham (Ed.), *Vital directions for mathematics education research*. New York, NY: Springer.
Hass, J., Heil, C., & Weir, M. D. (2018). *Thomas' Calculus: Single Variable: Based on the Original Work by George B. Thomas, Jr.*, Massachusetts Institute of Technology. Pearson
Strang, G., & Herman, E. (2016). *Calculus Volume 1*. OpenStax. https://openstax.org/books/calculus-volume-1/
Hughes-Hallett, D., Gleason, A. M., & McCallum, W. G. (2017). *Calculus: Single variable*. John Wiley & Sons.
Kang, M. J., Hsu, M., Krajbich, I. M., Loewenstein, G., McClure, S. M., Wang, J. T. Y., & Camerer, C. F. (2009). The wick in the candle of learning: Epistemic curiosity activates reward circuitry and enhances memory. *Psychological Science, 20*(8), 963-973. https://doi.org/10.1111/j.1467-9280.2009.02402.x
Kapur, M. (2014). Comparing learning from productive failure and vicarious failure. *Journal of the Learning Sciences, 23*, 651–677. http://dx.doi.org/10.1080/10508406.2013.819000
Kapur, M. (2015). Learning from productive failure. *Learning: Research and practice, 1*(1), 51-65. http://dx.doi.org/10.1080/23735082.2015.1002195

Kapur, M., & Rummel, N. (2012). Productive failure in learning from generation and invention activities. *Instructional Science, 40*(4), 645-650. http://dx.doi.org/10.1007/s11251-012-9235-4
Koichu, B. (2012). Enhancing intellectual need for defining and proving: A case of impossible objects. *For the Learning of Mathematics, 32*(1), 2-7. https://www.jstor.org/stable/23391943
Larson, R., & Edwards, B. H. (2018). *Calculus*. Cengage Learning
Leatham, K., Peterson, B., Stockero, S., & van Zoest, L. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088
Litman, J. A. (2008). Interest and deprivation factors of epistemic curiosity. *Personality and Individual Differences, 44*, 1585–1595. https://doi.org/10.1016/j.paid.2008.01.014
Lodge, J. M., Kennedy, G., Lockyer, L., Arguel, A., & Pachman, M. (2018, June). Understanding difficulties and resulting confusion in learning: an integrative review. *Frontiers in Education, 3*, p. 49. https://doi.org/10.3389/feduc.2018.00049
Loewenstein, G. (1994). The psychology of curiosity: A review and reinterpretation. *Psychological Bulletin 116*, 75–98. https://doi.org/10.1037/0033-2909.116.1.75
Mandler, G. (1975). *Mind and emotion*. Wiley.
Mandler, G. (1984). *Mind and body: Psychology of emotion and stress*. Norton.
Mandler, G. (1999). Emotion. In B. M. Bly, & D. E. Rumelhart (Eds.), *Cognitive science. Handbook of perception and cognition* (2nd ed.). (pp. 367-382) San Diego, CA: Academic Press.
Metcalfe, J., Schwartz, B. L., & Eich, T. S. (2020). Epistemic curiosity and the region of proximal learning. *Current Opinion in Behavioral Sciences, 35*, 40-47. https://doi.org/10.1016/j.cobeha.2020.06.007
Meyer, D. (2015, June 17). If math is the aspirin, then how do you create the headache? [Blog post]. Retrieved from https://blog.mrmeyer.com/2015/if-math-is-the-aspirin-then-how-do-you-create-the-headache/.
Morton, A. (2010). Epistemic emotions. In P. Goldie (Ed.), *The Oxford handbook of philosophy of emotion* (pp. 385–399). New York, NY: Oxford University Press. https://doi.org/10.1093/oxfordhb/9780199235018.001.0001
Muis, K. R., Psaradellis, C., Lajoie, S. P., Di Leo, I., & Chevrier, M. (2015). The role of epistemic emotions in mathematics problem solving. *Contemporary Educational Psychology, 42*, 172-185. https://doi.org/10.1016/j.cedpsych.2015.06.003
Nerantzaki, K., Efklides, A., & Metallidou, P. (2021). Epistemic emotions: Cognitive underpinnings and relations with metacognitive feelings. *New Ideas in Psychology, 63*, 100904. https://doi.org/10.1016/j.newideapsych.2021.100904
Pekrun, R. (2000). A social-cognitive, control-value theory of achievement emotions. In J. Heckhausen (Ed.), *Motivational psychology of human development* (pp. 143–163). Amsterdam: Elsevier.
Pekrun, R. (2006). The control-value theory of achievement emotions: Assumptions, corollaries, and implications for educational research and practice. *Educational Psychology Review, 18*, 315–341. https://doi.org/10.1007/s10648-006-9029-9
Pekrun, R., & Stephens, E. J. (2012). Academic emotions. In K. R. Harris, S. Graham,
T. Urdan, S. Graham, J. M. Royer, & M. Zeidner (Eds.), *APA educational psychology handbook* (pp. 3–31). American Psychological Association. https://doi.org/10.1037/13274-001.

Pekrun, R., Vogl, E., Muis, K. R., & Sinatra, G. M. (2016). Measuring emotions during epistemic activities: the Epistemically-Related Emotion Scales. Cognition and Emotion, 31(6), 1268-1276. http://dx.doi.org/10.1080/02699931.2016.1204989

Piaget, J. (1971). Genetic epistemology. New York, NY: W. W. Norton & Company Inc.

Piaget, J. (1985). The equilibration of cognitive structures: The central problem of intellectual development. Chicago: University of Chicago Press.

Piaget, J. (2001). *Studies in Reflecting Abstraction*. New York, NY: Psychology Press.

Rabin, J. M., Fuller, E., & Harel, G. (2013). Double negative: the necessity principle, commognitive conflict, and negative number operations. The Journal of Mathematical Behavior, 32(3), 649-659. https://doi.org/10.1016/j.jmathb.2013.08.001

Sinha, T., Kapur, M., West, R., Catasta, M., Hauswirth, M., & Trninic, D. (2021). Differential benefits of explicit failure-driven and success-driven scaffolding in problem-solving prior to instruction. Journal of Educational Psychology, 113(3), 530–555. https://doi.org/10.1037/edu0000483

Stein, N., & Levine, L. (1991). Making sense out of emotion. In W. Kessen, A. Ortony, & F. Kraik (Eds.), *Memories, thoughts, and emotions: Essays in honor of George Mandler* (pp. 295e322). Hillsdale, NJ: Erlbaum.

Stewart, J., Clegg, D. K., & Watson, S. (2020). *Calculus: early transcendentals*. Cengage Learning.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics, 12*(2), 151-169. https://doi.org/10.1007/BF00305619

Tallman, M. A. (2021). Investigating the transformation of a secondary teacher's knowledge of trigonometric functions. *Journal of Mathematical Behavior, 62*.

Tallman, M. A. & Frank, K. M. (2020). Angle measure, quantitative reasoning, and instructional coherence: An examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education, 23*(1), 69-95. https://doi.org/10.1007/s10857-018-9409-3

Tallman, M. & Uscanga, R. (2020). Managing students' mathematics anxiety in the context of online learning environments. In J.P. Howard & J.F. Beyers (Eds.), *Teaching and learning mathematics online* (pp. 189-216). Boca Raton, FL: CRC Press.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra. Center for Research in Mathematics & Science Education: San Diego State University.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morelia: PME.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing.,* WISDOMe Monographs (pp. 1-24). Laramie, WY: University of Wyoming.

Touroutoglou, A., & Efklides, A. (2010). Cognitive interruption as an object of metacognitive monitoring: Feeling of difficulty and surprise. In A. Efklides, & P. Misailidi (Eds.), *Trends*

and prospects in metacognition research (pp. 171–208). Springer Nature. https://doi.org/10.1007/978-1-4419-6546-2_9
VanLehn, K. (1988). Toward a theory of impasse-driven learning. In H. Mandl & A. Lesgold (Eds.), *Learning issues for intelligent tutoring systems* (pp. 19–41). New York, NY: Springer. http://dx.doi.org/10.1007/978-1-4684-6350-7_2
Vogl, E., Pekrun, R., Murayama, K., & Loderer, K. (2020). Surprised–curious–confused: Epistemic emotions and knowledge exploration. *Emotion, 20*(4), 625–641. https://doi.org/10.1037/emo0000578
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. New York: RoutledgeFalmer.
Weinberg, A., & Jones, S. (2020). A Theorization of Learning Environments to Support the Design of Intellectual Need-Provoking Tasks in Introductory Calculus. In Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). (2020). *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.
Weinberg, A., & Jones, S. (2022). A framework for designing intellectual need-provoking tasks. In Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). (2022). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.
Zazkis, R. & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. *The Journal of Mathematical Behavior. 43*. 98-110. https://doi.org/10.1016/j.jmathb.2016.07.001