Proving as cooperative communication: Using Grice's maxims and implicature to understand proof comprehension in undergraduate mathematics education

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In this exploratory theoretical analysis, we apply Grice's maxims for cooperative communication to make sense of the ways that proofs are written and understood. We argue that proving can sometimes be productively viewed as a form of cooperative communication. Grice's maxims place expectations on the prover and provide norms for how proofs should be written. The reader's expectation that the prover is adhering to these norms allow for the possibility of implicature—that is, the prover can imply statements to the reader that are neither explicitly said nor logically necessary. We use this analysis to better understand the practice of proving and to suggest a future line of empirical research.

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A central goal of research in undergraduate mathematics education is to design instruction to enculturate students into the process of proving. We would like students to use proof for the same reasons that mathematicians do (e.g., deVilliers, 1990) and to understand and learn from the proofs that they read (e.g., Conradie & Firth, 2000; Mejía-Ramos et al., 2012). Unfortunately, these goals are often not achieved in undergraduates' advanced mathematics courses, with many students emerging from these courses unable to read or write proofs successfully and, more generally, lacking a sense of what proof is about (e.g., Stylianides et al., 2017).

There has been substantial research into helping students understand what type of reasoning is permissible in a proof. For instance, there has been work into helping students realize that inferences must be based on deductive, rather than empirical, argumentation (e.g., Brown, 2014), to learn new proving techniques such as proof by induction (e.g., Harel, 2001), and about the nature of modus ponens reasoning (e.g., Dawkins & Norton, 2022). This research is important and has substantially increased our understanding of pedagogical issues related to proof. However, we argue that this work alone is not sufficient to enculturate students into the practice of proving. Knowing what types of reasoning is allowable in a proof underdetermines how proofs should be written and read. There are permissible moves in a proof that are almost never carried out in mathematical practice (e.g., introducing variables or making explicit assumptions that are never used again in the proof). There are formal violations of logic and syntax that frequently appear in proofs (e.g., abuses of notation). There are norms of proof writing that are independent of logic (e.g., not using the first-person pronoun "I" and not including information about the wrong turns made in the proving process). Some mathematicians have complained that their students are "tone deaf" with respect to proof and do not have a feel for the game (Weber, 2012). Elsewhere, we have speculated that part of enculturating students into proof is helping them learn the linguistic norms of proving along with the values that they uphold (Dawkins & Weber, 2017). These kinds of rules of the game that are not captured by the grammar or direct meaning of proof might be appropriately categorized as part of the pragmatics of proving. Pragmatics refers to the use and contexts of use. This is an understudied arena of proving practice, which might reflect how proving is often understood as decontextualized and depersonalized (Balacheff, 1988). We argue that proofs still communicate through shared rules.

The goal of this theoretical report is to use Grice's maxims for cooperative communication to better understand how proofs are written and read in mathematical practice (Grice, 1975, 1989). We first document that in mathematics education, proof is often treated as an adversarial form of communication. We argue that it would behoove mathematics educators to also view proof as a form of cooperative communication. Next, we review Grice's maxims for cooperative communication. Critical to this paper is Grice's notion of implicature: when we assume that a speaker is communicating cooperatively with us, we can infer many things that the speaker did not explicitly state. Then we explore the consequences of applying Grice's maxims to proof in mathematical practice and undergraduate mathematics classrooms, showing how proofs invite or even expect implicatures. There are ideas that one can from a proof that are neither explicitly stated nor logically necessary. Finally, we suggest a line of empirical research based on our theoretical explorations.

Proof as a cooperative form of communication

Proof as an adversarial form of communication in mathematics education research

Frequently, proof is framed as a semi-adversarial form of communication: the prover is trying to persuade a skeptic to accept a claim that the skeptic doubts and the skeptic will not (and should not) be swayed easily. Mason (1982) described proving as convincing an enemy and Volminik (1990) defined a proof as an argument that would convince a reasonable skeptic. More recently, Dutilh Novaes (2020) wrote that "a good proof is one that convince a fair but 'tough' opponent" (p. 50). When students are reading proofs, they are encouraged to take a skeptical stance, with Brown (2014) arguing that the capacity to retain doubt is critical to understanding proof. Similarly, Inglis (2022) argued that developing one's inner skeptic when considering proofs may be a critical part to becoming a mature mathematician.

Inglis (2022) referred to the dialog between a prover and their skeptical audience as "semi-adversarial" because neither the skeptic nor the prover engages in a "win-at-all-costs" strategy. A reasonable skeptic must grant sensible premises and not repeatedly ask for supporting evidence in bad faith. A reasonable prover should not use rhetoric to mislead the skeptic. Both the prover and the skeptic share the communal goal for identifying the flaw in a proof, should one exist, or otherwise verifying that the proof is airtight (Dutilh Novaes, 2020). Nonetheless, to achieve these shared aims, the reader is asked to take an adversarial role, only granting inferences if it is epistemologically necessary to do so. In short, according to the adversarial account of proving the audience of the proof of a theorem acts as a gatekeeper who checks that the proof passes a high standard of muster, lest a false theorem enter into their collection of mathematical facts (or, in the case of a referee, in the mathematical community's accepted statements).

Proof as cooperative communication

Mathematicians do not always view other mathematicians' proofs adversarially, or even with skepticism. Indeed, when mathematicians read proofs in the published literature, they often do not check proofs for correctness; they presume the proofs are correct (Mejía-Ramos & Weber, 2014; Weber & Mejía-Ramos, 2011). As one mathematician described things, "when I was reading a proof in a journal or a proof that was handed to me by a mathematician friend and they were pretty sure it was true, my assumption would be that the steps are probably correct and that I need to work hard to make sure I understand and can justify each step" (Weber, 2008, p. 448). The point here is that while proofs in the literature and proofs from trusted peers can contain mistakes, mathematicians often do not read the proof scouring for errors. They do not assume unclear statements in a proof are wrong, but rather work to justify these steps. We believe students are in a similar position when they read proofs in a textbook or hear proofs in a lecture.

Grice (1975, 1989) characterized communication as cooperative if the interlocutors are working together to actively achieve a particular goal. Proving can sometimes best be understood as a collaborative communication in which the prover is trying to help their audience see how their theorem can be reached from shared premises by a series of deductive inferences that their audience accepts as clear, indubitable, and not admitting exceptions. Of course, proofs in textbooks and in journals often contain inferences that are neither clear nor obvious. Hamami (2021) explained the situation as follows: The prover hands her audience a proof P. Some steps in P will not be clear to a reader. To address this, a reader will expand P to a longer proof P^*, such that each step in the proof is either an accepted statement or an immediate consequence of a known rule of inference. An accepted statement might be a definition, a result that is established elsewhere in the literature, or a statement that the reader had personally proven in the past. A known rule of inference might be an axiom or the application of a previously proven theorem.

As a brief clarification, while mathematicians sometimes seek out a proof as an end in itself (c.f., Paseau, 2011), they frequently use proof as a means to achieve other epistemic goals (deVilliers, 1990). Possessing the expanded P^* consisting of a series of indubitable inferences can help achieve epistemic aims of proof such as conviction and explanation highlighted by de Villiers (1990) and others (e.g., Hanna, 1990). For instance, if a mathematician harbored significant doubts about the truth of an assertion, seeing how the statement could be reached by a series of indubitable inferences could increase their confidence that the statement was true.

Grice's maxims and implicature

Grice's maxims

When interlocutors are engaged in a cooperative conversation, they are trying to achieve a shared goal. When a speaker offers a contribution to a cooperative conversation, Grice (1975, 1989) argued that the speaker is expected to follow the Cooperative Principle and make contributions that will efficiently facilitate the achievement of these goals. Grice offered four conversational maxims specifying how contributions should be made: The maxim of quality (be truthful); the maxim of quantity (offer the right amount of information); the maxim of relation (be relevant); and the maxim of manner (be clear). These maxims can be regarded as norms or obligations that constrain what contributions a speaker can make in a conversation (in the sense of Herbst et al., 2011 and Yackel & Cobb, 1996). When we apply these maxims of proof, these will yield norms about how proofs should be written (in the sense of Dawkins & Weber, 2017).

Implicature

In a cooperative conversation, a listener presumes that a speaker is following the Cooperative Principle in general and Grice's maxims in particular. Grice's (1975) key insight was that this enables the listener to make inferences that are neither explicitly stated nor deductively implied by the speaker's utterances. To illustrate, consider the following conversational exchange:

Man (speaking to a passerby on the road): My car has run out of gas.

Passerby: There is a gas station around the corner.

The explicit contribution from the passerby is the location of a gas station. Grice (1975) argued that the passerby is also making an implicit contribution, namely that (the passerby believes) the gas station is open. The reason this implication is possible is because the man assumes that the passerby is following the Maxim of Relation (and the passerby assumes this). The man and the passerby are working toward the shared goal of helping the man resolve his problem of finding fuel for his car. If the gas station around the corner were closed, the existence of the gas station would not be useful for achieving this goal. Therefore, by stating the location of the gas station, the passerby is implying that this is open. Grice (1975) referred to *implicature* as the act of making this type of implication and *implicatum* as what is being implied. As we view the situation with proof, Grice's maxims place norms upon the prover. The recipient of the proof makes the presumption that the prover is following Grice's maxims, which allows for the recipient to infer ideas that were neither explicitly stated nor logically necessary consequences of what appears in the proof. These ideas are implicatum. The shared assumption of cooperative communication allows the prover to convey implicata without explicitly stating them.

Grice's Maxims, Norms, and Implicature and proof

In this section, we present norms and their associated implicatures for proof presentation for each of Grice's Maxims. In our presentation, we will discuss norms that the maxims impose on the prover and the implicata that a reader may gain.

The Maxim of Relation

The Maxim of Relation asserts that contributions to a conversation should be relevant to the goals of the conversation. Any contribution in a proof P should be relevant to obtaining a chain of indubitable inferences that lead to the conclusion.

> *Norm of proof writing:* Any assumption, deduction, or variable introduced in a proof should be useful in the final chain of argumentation establishing the theorem.
> *Associated implicatum:* If a statement or variable appears in a proof, the reader can expect that it will be built upon at a later stage in the proof.

In other words, proofs do not contain digressions. The prover should not include "logical dead-ends" in a proof and the reader can expect that none will be included. Strictly speaking, from a formalist perspective, adding unused assumptions and including extraneous true statements or valid deductions in a proof does not render it invalid. Nonetheless, this is rarely done. One consequence is that when the prover makes an assumption in the proof, the prover is implying that this assumption will be useful for establishing the theorem. One consequence of the implicatum is that one can understand a proof by seeing why assumptions are necessary or how statements are built upon (e.g., Mejía-Ramos et al., 2012). This consequence only follows based on the expectation that the prover is following the norm above.

The Maxim of Quantity

Grice (1975) summarized the Maxim of Quantity by saying, "make your contribution as informative as required (for the exchange) [and] do not make your contribution more informative than is required" (p. 45). Applying this notion to proof, the prover is aiming to give the reader sufficient information in the proof P so the reader can produce an expanded chain of indubitable inferences P^*, but no more. This suggests some interesting norms and implicata.

> *Norm of proof-writing:* In a conditional statement, "If A, then B" or "Since A, then B", the truth-value of B is tracked by the proof value of A.

Associated implicatum: If "Since A, then B" appears in a proof, there is a general principle for how the truth of A necessitates the truth of B.

Grice (1989) discussed the nature of conditionals more generally. Suppose one asserts, "If John walks to the party, he will be late." Technically speaking, this statement would be true if the speaker knew that John would be late regardless of how he reached the party. However, making such as assertion would be violating the Maxim of Quantity, as the simpler statement "John will be late" would be simpler and more informative. Similarly, if the speaker knew that John would be driving to the party, the statement would be (vacuously) true, but making the assertion would be violating the Maxim of Relation, as John's timing if he were walking would not be relevant in this situation. Hence, when the speaker asserts that "If John walks to the party, he will be late," the implicature is that the speaker is certain neither of how John is getting to the party or whether John will be late, but the speaker believes that the truth of the antecedent implies the proof of the conclusion.

A similar analysis follows from statements such as "if A, then B" or "since A, then B" in a proof. A prover should not proffer such a statement if B could easily be seen in the absence of A, since it would be simpler and more informative to simply state B. Further, to understand the proof, the reader should try to find a chain of reasoning from A to B. There is evidence that this phenomenon is borne out in practice. Some mathematicians will reject statements like "if 13 is prime, then 1013 is prime" in proofs even though they are true because they are not informative. Some will infer that there is a relationship between 13 being prime and 1013 being prime (such as adding 1,000 to a prime number will yield a prime number) and reject the statement not because it is explicitly false, but because the implied warrant is invalid (c.f., Weber & Alcock, 2005). There is further empirical support for this statement. In pedagogical setting, mathematicians do not think it is appropriate to make a statement of the form "If A, B, and C, then D" unless A, B, and C are all relevant to the truth value of D (e.g., Lai et al., 2012). Presumably, this is because, if C was unnecessary, it would be simpler and more informative to state "If A and B, then D."

Norm of proof-writing: The prover will provide enough information in *P* for the reader to expand without extraordinary effort.

Associated implicata: New steps in the proof may require justification, but the justification should not be based on sophisticated ideas outside the proof or be exceptionally difficult.

Recall the passerby who told the stranded motorist that there was a gas station around the corner. The passerby probably would not instruct the man where to look for the gas station after he turned the corner if gas station was clearly visible. The passerby would only provide this information if the location of the gas station was not obvious (e.g., if it were behind a strip mall) and would not be found by a routine search of the area. Similarly, justifications for statements within a proof are only provided if the reader could not find them through ordinary effort. What is interesting here is that the implicature is based on what the passerby did *not* say. If the location of the gas station would not be obvious to the stranded motorist, the Maxim of Quantity would dictate that more direction would be provided. Since the passerby did not provide that information, the location of the gas station should be clear when the motorist walked to it.

There are a few remarks worth noting. First, the ability for the reader to justify steps within a proof *P* is clearly audience dependent, so how a prover follows this norm depends on the audience. A proof appropriate for a professional number theorist would not be appropriate for a student in a transition-to-proof course, and vice versa. Second, what the readers are capable of doing “without extraordinary effort” is a nebulous concept. At least in pedagogical settings, mathematicians seem to disagree on how much detail a proof should provide (Lai et al., 2012). Research has revealed that students do not appreciate that it is their responsibility to justify some of the steps in a proof (e.g., Weber & Mejía-Ramos, 2014). Nonetheless, when a prover chooses *not* to provide detail for how a particular step can be established, the prover is conveying information about the existence of a type of justification for the step—namely it is a justification that can be formed routinely without extraordinary effort.

In addition to the knowledge of the intended audience, we expect that the application of this maxim is also sensitive to the intended purpose within the proving context. If the proving context is focused on developing an axiomatic system or proving the properties of an axiom system, the prover may adhere to stricter rules regarding what the reader should have to expand on their own. This again portrays how the context of proving influences the rules at play in the cooperation between the prover and reader.

The Maxim of Manner

The Maxim of Manner asserts that speakers try to say things as clearly and orderly as possible. Suppose that you are asked to prove a statement of the form “A if and only if B”. Suppose you have a subproof that A implies B, say $A \to A_1 \to B$, and you have a subproof that B implies A, say $B \to B_1 \to A$. You could, in principle, write a logically valid proof as follows.

1. Assume A
2. Assume B
3. Since A, A_1
4. Since B, B_1.
5. By (3), B
6. By (5), A
7. Since $A \to B$ and $B \to A$, A if and only B.

Of course, no one would write the proof in such a confusing manner.

> *Norm:* Proofs are ordered to make the chain of arguments clear.
> *Implicature:* The justification for how a new line follows in a proof will depend on previous statements in a proof.

Mirin and Dawkins (2022) have illustrated how when a proof involves a string of identities (i.e., showing A=D by showing A=B, B=C, and C=D), mathematicians are sensitive to the order of these identities. They will think norms are being broken and proofs are deficient if the identities are not ordered in the most sensible manner.

> *Norm:* Name objects in proofs according to convention to convey their role or status in the proof.
> *Implicature:* The name of an object will help the reader keep track of its role in the proof or its quantificational status.

The names assigned to mathematical objects is taken to be logically irrelevant. Nevertheless, there are strong norms for how things are usually named. For instance, input variables are frequently named x and bounds on output estimates are usually named ε. Quantification is often encoded using indices and superscripts. A reader who notes that a set indexed by the natural numbers would be justified in drawing the implicature that the set is countable or finite. It would also be strange to denote an object G^* if the variable name G was not used elsewhere in the proof. The superscript is only used to distinguish various objects bearing the label G. As ** stated, one would never give the name "6" to a group. Strictly speaking, these naming conventions bear no logical status and the proofs would remain valid were they not followed. Nevertheless, they are strong conventions of naming because the notation affords implicatures that aide readers in constructing the line of inference intended by the prover.

Discussion and future research

In this paper, we have argued that both in the advanced mathematical classroom and in mathematical practice, it is often productive to frame proof as a cooperative form of communication. We have asserted that Grice's maxims for conversation are useful in two respects. First, Grice's maxims suggest norms for proof writing that place expectations on the prover for how proofs are written. Second, the presumption that the prover is following these norms allow for implicature; the reader of a proof may infer things that are neither explicitly stated nor logically necessitated by the proof text being read.

We believe that this has pedagogical consequences. Much of advanced mathematical instruction (and undergraduate mathematics education research) focuses on the type of reasoning that is admissible in a proof. We agree that this is essential, but argue that this is not enough to fully enculturate a student into the practice of proving. We suggest more effort can be put into instruction as to norms about how proofs are written and how they can be read. There is an intimate relationship between the form of mathematical proofs and the proving practices they express. As Schleppegrell (2004) explained, "It is important for students to develop academic register options in different disciplines because particular grammatical choices are functional for construing the kind of knowledge typical of a discipline" (p. 137).

We also believe that our exploratory theoretical analysis suggests a further line of research. We have made a number of speculative claims about norms for proof writing and opportunities for implicature. Such claims can be empirically tested with breaching experiments by showing both mathematicians and students situations in which those norms were violated, or showing mathematicians and students situations in which our proposed norms were followed and asking them if they believed that the implicata likely held. Mathematicians' responses to such situations would reveal if our speculative claims were accurate and provide us with a richer sense of why these proving norms were operative. Differences between students' and mathematicians' responses could help us unpack one way in which students might not yet have a feel for the game of proving.

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