“What makes it eigen-esque-ish?”: Eigentheory Development in a Quantum Mechanics Course

Megan Wawro
Virginia Tech

Kaitlyn Stephens Serbin
University of Texas Rio Grande Valley

Eigentheory concepts are central in mathematics and physics; they serve multiple functions, such as symbolizing physical phenomena and facilitating mathematical computations. The words associated with eigentheory develop and vary over time (e.g., eigenvector, eigenstate), as do the associated symbols (e.g., Ax = λx, H|E_n> = E_n|E_n>). The focus of this study is how the concept of "eigen" develops over time for a quantum mechanics classroom community. We analyze the public displays of form-function relations used in the classroom community (Saxe, 1999). In this preliminary report, we summarize the forms and functions identified in the first 19 class sessions and share progress on our analysis of shifts in the uses of forms and functions over time.

Keywords: eigentheory, linear algebra, quantum mechanics, form-function, classroom analysis

Eigentheory concepts are central in mathematics and physics. They serve multiple functions, such as symbolizing associated physical phenomena and facilitating mathematical computations. Physics students reason about eigentheory concepts in Linear Algebra courses where eigen-equations often first look like $A\boldsymbol{x} = \lambda\boldsymbol{x}$, with $n \times n$ matrix A, $\boldsymbol{x}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$, and λ is a scalar; they may also encounter $T(\boldsymbol{v}) = \lambda\boldsymbol{v}$, where T is a linear operator on vector space V. Students further reason about eigentheory in quantum mechanics, where eigenvalues of various Hermitian operators represent possible measured values of corresponding observables. Furthermore, physics students learn Dirac notation, in which eigenequations take on forms, such as $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ and $H|E_n\rangle = E_n|E_n\rangle$, and function to convey information related to spin and energy, respectively. In quantum mechanics, it is “eigen, eigen, eigen all the way” (Shankar, 2012, p. 30).

Several verbal, symbolic, and written forms can be associated with the same eigentheory concept, and many of these forms can serve different functions (Saxe, 1999) in both math and physics. These form-function relations can develop during class as the community’s common ground is negotiated through verbal or written communication. This development for eigentheory concepts in a Quantum Mechanics course is the focus of this study. We pursue the following research question: *How does the concept of "eigen" develop over time for a quantum mechanics classroom community?* To do so, we leverage a form-function analysis (Saxe, 1999), analyzing the public displays of form-function relations used in the classroom community. We identify each form that was used during whole-class discussion and infer the function of the form. So far, we have identified forms and functions from the first 19 class sessions. Our current analysis involves examining how shifts in the uses of these forms and functions occurred over time.

Literature Review

There is a growing body of literature on student understanding of eigentheory and teaching innovations that support student reasoning about eigentheory (e.g., Altieri & Schirmer, 2019; Beltrán-Meneu et al., 2016; Bouhjar et al., 2018; Gol Tabaghi & Sinclair, 2013; Karakok, 2019; Manogue et al., 2001; Plaxco et al., 2018; Pollock et al., n.d.; Salgado & Trigueros, 2015; Serbin et al., 2020; Wawro et al., 2019b). There are many conceptually complex aspects to a deep understanding of eigentheory. For example, interpreting the symbols $A\boldsymbol{x} = \lambda\boldsymbol{x}$ can involve recognizing the matrix-vector product $A\boldsymbol{x}$ as the same mathematical object as the scalar-vector product $\lambda\boldsymbol{x}$ (Thomas & Stewart, 2011). It can also involve conceptualizing eigenvectors as those $\boldsymbol{x}$ which are stretched by matrix A (e.g., Henderson et al., 2010; Sinclair & Gol Tabaghi, 2010). These correspond to a relational meaning of the equals sign (Knuth et al., 2006) and a functional interpretation of $A\boldsymbol{x} = \boldsymbol{b}$ (Larson & Zandieh, 2013), respectively, and have also been documented with quantum mechanics students (Wawro et al., 2019a; Wawro et al., 2020b).

Physical science students face additional complexities in reasoning about eigentheory, given that the mathematical objects and symbols relate to physical referents (Rodriguez et al. (2018). Researchers have suggested that interpreting mathematical symbolic expressions in terms of physical phenomena is a nontrivial endeavor for students (e.g., Caballero et al., 2015; Her & Loverude, 2020; Serbin & Wawro, 2022). Physics students face challenges in interpreting eigentheory symbols because those symbols and their word forms take on additional meanings in quantum mechanics contexts (Dreyfus et al., 2017; Gire & Manogue, 2011; Pina et al., 2022; Wawro et al., 2020a). For example, Wawro et al. (2020a) found that as physics students interpreted the meaning of the symbols in the eigenequations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$, a student was unsure how to resolve the disconnect between their geometric interpretation of the first equation and their quantum mechanical interpretation of the latter equation. Pina et al. (2022) also demonstrated different meanings students had for an eigenequation. They identified three different symbolic forms (Sherin, 2001) that physics students had for a position operator eigenequation that share a symbol template but have different conceptual schemata: a transformation which reproduces the original, an operation taking a measurement of state, and a statement about the potential results of measurement. These studies illustrate the complexity associated with interpreting eigentheory symbols and the varied meanings these symbols convey.

Theoretical Background

Through his anthropological work on the cultural development of mathematical ideas, Saxe (1999) created a framework for investigating the form-function relations created by individuals and a community over time. A *form* is a verbal, symbolic, graphical, or physical representation that takes on mathematical meaning. As individuals engage in activity or communication, they tailor forms to serve certain functions in activity, thereby establishing form-function relations. *Functions* are defined as the “purposes for which forms are used as individuals structure and accomplish practice-linked goals” (Saxe, 1999, p. 20). Forms can be adapted to serve several different functions, and functions can be served by different forms. For example, the form $y = 5x$ can function to conveying a constant rate of change or exhibit covariation of two variables, and the forms *e* and 1 can both serve the function of symbolizing a group identity (Plaxco, 2015).

Saxe and colleagues leverage this framework for analyzing the “reproduction and alteration of a common ground of talk and action over lessons in classroom communities” (Saxe et al., 2015, p. 71). Common ground refers to “shared knowledge of word meanings and norms for communication” that “enables successful communication and coordinated action” (Saxe & Farid, 2021, p. 8). Common ground is created as community members “produce and interpret displays of mathematical thinking, making use of representational forms (linguistic, graphical, gestural) to serve communicative and problem-solving functions” (Saxe et al., 2015, p. 4), is generated in interaction, and is at best taken-as-shared by community members. To identify how forms serve certain functions and how form-function relations shift over time, we perform microgenetic and ontogenetic analysis of the concept of “eigen” over a period of 19 class sessions.

Microgenesis is the process by which individuals construct representations by tailoring forms to serve functions that accomplish goal-directed activity (Saxe et al., 2015). This often occurs in public displays during class and contributes to the alteration of a common ground. Individuals are enabled and constrained by the common ground as they create new form-function relations. They can use familiar forms to serve new functions or recruit new forms to serve existing ones. Saxe et al. (2015) referred to the use of familiar forms or functions as continuity and the use of new forms or functions as discontinuity. *Ontogenesis* is characterized by shifts in microgenetic displays. An ontogenetic analysis involves an "analysis of continuities and discontinuities as individuals reproduce and alter form-function relations" (Saxe et al., 2015, p. 13).

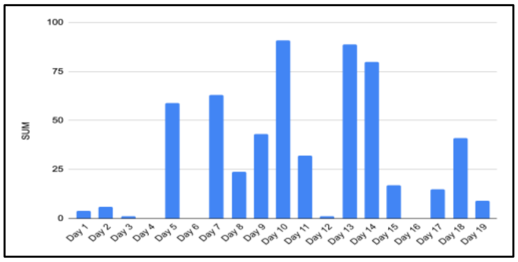
Methods

The data analyzed in this paper come from an in-person, senior-level Quantum Mechanics course taught in a medium, public, research-active university in the northeast US. The semester-long, 3-credit course met three times a week for 50 minutes per class. The first nine weeks (23 days) of class sessions were video recorded, with a focus on capturing the professor and whole-class discussions. The professor was an experienced quantum mechanics instructor and physics education researcher. The course had 17 students and used McIntyre et al. (2012) as its textbook. The data sources were video recordings and associated transcriptions. Only exchanges that occurred with the entire class (as compared to small groups) were analyzed.

We imported transcripts into MaxQDA, which is a qualitative and mixed methods data analysis software (Kuckartz & Rädiker, 2019). We inductively coded (Miles et al., 2013) of both transcript and screen captures of slides and boardwork. We coded together while watching the videos, discussing our codes, and resolving inconsistencies as needed. We coded the instances in which "eigen" was explicitly used, as well as instances implicitly related to eigentheory. The implicit uses occurred when words such as state, value, or $\hbar/2$ were used in ways compatible with eigen-related interpretations. Our coding system had a nested organization according to form. At the primary level, main forms were separated according to what was characterized by "eigen" (e.g., eigenstate, eigenbasis). The second organization level separated main forms into the specific forms that existed in the data. For instance, the form "eigenvector - verbal" was assigned when "eigenvector" was said, the form "eigenvector - written" when "eigenvector" was written on the board or on a slide, and the form "$|+\rangle$ - symbolic" when the symbol $|+\rangle$ was used to convey eigenvector meaning. The final level of coding corresponds to the function that we interpreted the form to have when the form-function relation was detected. For example, any of the aforementioned eigenvector forms could have associated functions such as being an element of a basis that diagonalizes a spin operator or being part of the computation for determining the probability of measuring a certain spin value. If an instance involved more than one form – for instance, both saying and writing "eigenvector" – the same function was assigned twice; this resulted in two form-function pairs for that instance. We started preliminary ontogenetic analysis by examining how forms, functions, or form-function pairs were used throughout the data.

Preliminary Results

At the time of this preliminary proposal submission, we have coded the whole-class discussions of 19 consecutive class sessions for a total of 576 instances in which a form-function relationship for the concept of "eigen" was communicated. See Figure 1a for the distribution of those instances over the days. We have a small amount of remaining refinement to do (codes from Day 6 and some of Day 9 need to be added, Day 10 needs a final confirmation). We will complete this refinement prior to the conference and will integrate it into our presentation.

(a) 

(b)

Main Form	Number of unique form-function pairs in the secondary form category			Number of unique form-functions pairs in the main form category
	Written	Verbal	Symbolic	
Eigenvalue [188]	13	60	32	105
Eigenvector [73]	15	27	9	51
Eigenstate [229]	23	59	36	118
Eigenequation [65]	5	13	40	58
Eigenbasis [13]	0	8	1	9
Eigen(misc) [8]	0	8	0	8
	Total unique form-function pairs			349

Figure 1. Bar chart indicating the frequency per day of the instances in which a form-function pair for the concept of "eigen" was communicated (a), and summary of the various types of form-function pairs (b).

The form-function pairs are summarized in Figure 1b according to the six main forms of eigenvalue, eigenvector, eigenstate, eigenequation, eigenbasis, and miscellaneous uses (e.g., eigen-esque-ish, eigen math, eigen-ness). The numbers in the square brackets, which sum to 576, indicate the total number of times a form-function pair occurred in the data. The middle three columns organize unique form-function pairs with specificity regarding the form. For example, there were 32 different functions that were associated with a symbolic form accompanying "eigenvalue." For instance, the symbol a_n could be the form used to symbolize the eigenvalue in an eigenequation, the eigenvalues along the diagonal of a diagonalized matrix, or the expectation value of a measurement corresponding to an eigenstate; these are three of the 32 functions for the form "a_n - symbolic" within the Eigenvalue row / Symbolic column. On the other hand, the form "E_n - symbolic" could also pair with these functions in the energy context, such as symbolizing the nonzero entries of a diagonal energy Hamiltonian; that form-function pair is a fourth of the aforementioned 32 form-function pairs. In total, there were 349 unique form-function pairs.

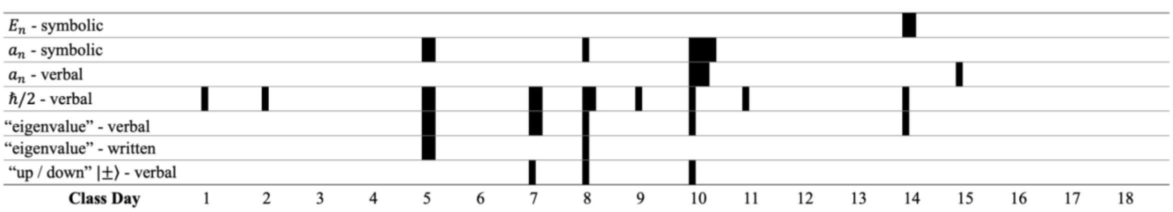


Figure 2. The dispersion of forms with the same function of *being the result of a measurement*.

Functions Associated with Several Different Forms

We performed a preliminary ontogenetic analysis by documenting the instances that different forms were used to serve the same function and determining the dispersion of those forms over time throughout the 19 class days. For example, conveying *the result of a measurement* (Fig. 2) and *a structure inherently tied to an operator* (Fig. 3) were two commonly-used functions, each associated with several forms in our dataset. The dispersion of the forms that served the function of *being the result of a measurement* is shown in Figure 2, and the dispersion of the forms that served the function of *being a structure inherently tied to an operator* is in Figure 3. The width of the black bar conveys the number of times the form occurred on that day (e.g., in Figure 2, the a_n symbolic form occurred twice on day 5, once on day 8, and four times on day 10).

We identified seven forms that were each associated with the function of *being the result of a measurement* (see Figure 2). This function was conveyed during ten of the class days, spread between days 1 and 15. Furthermore, in one class session (day 10), the class community used five of the seven different forms to serve this same function (see Figure 2). This illustrates how the class community associated the different eigenvalue forms of "a_n - symbolic," "a_n - verbal,"

“$\hbar/2$ - verbal,” and “eigenvalue – verbal,” and the eigenstate form of “up/down $|\pm\rangle$ - verbal” with the same function all in the span of one class session. Furthermore, we identified 15 forms that were each associated with the function of being *a structure inherently tied to an operator* (see Figure 3). By this function, we mean that the speaker was conveying some sort of structural quality imbedded in or inextricable from the operator. For example, a student claimed that certain given states “are eigenstates of the operator.” This conveyed, in the “eigenstate – verbal” form, that eigenstates belong to or are a property operators, an inherent relationship between eigenstate and its operator. This function was conveyed in ten of the class days, spanning from day 5 to day 18, which illustrates a wide dispersion of the use of this function during class. On both day 7 and day 14, there were five different forms used that were all associated with this same function. Overall, these findings from the preliminary ontogenetic analysis illustrate how the class community used several different forms to convey the same function, even within the same class day. The recurrent use of the same functions over time demonstrates the continuity in the class community’s use of those functions. The varied forms associated with the same function gives evidence to discontinuities present in the community’s communicative displays over time. The class members altered existing form-function relations as they tailored different forms to serve the same function. Our remaining ontogenetic analysis will involve additional fine-grained examination of change over time; for instance, how the eigenequation itself changed form across contexts (e.g., abstract, spin, energy, position) but what threads of continuity exist and were leveraged throughout the semester as the classroom engaged with quantum mechanics.

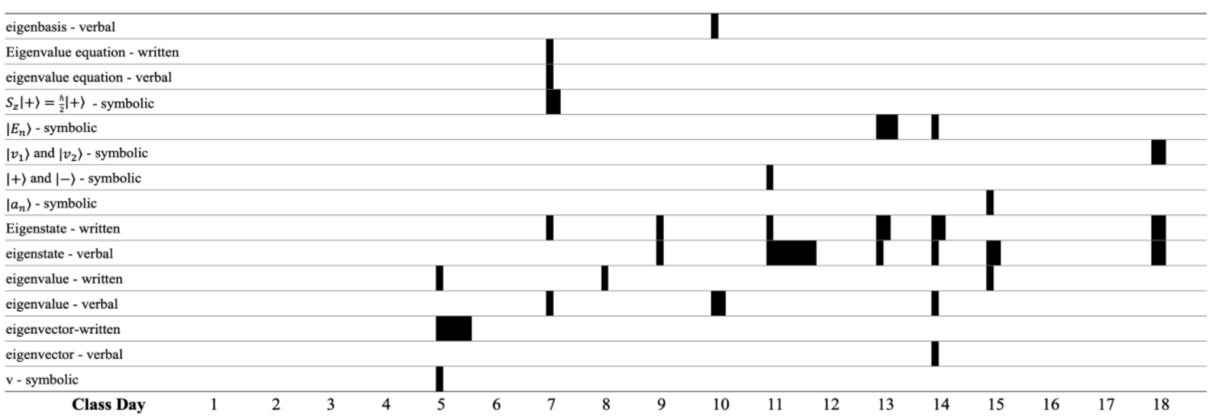


Figure 3. The dispersion of forms with the same function of *being a structure inherently tied to an operator*.

Conclusion

In this study, we are investigating a quantum mechanics class community’s development of eigentheory over a span of 19 class days. We performed microgenetic and preliminary onto-genetic analyses (Saxe, 1999) of form-function pairs that the class used to communicate about eigentheory concepts. We plan to continue this analysis, as well as examine how these form-function relations are reproduced and altered over the remaining class days in our data set that address content related to wave functions. At the conference, we will engage the audience with discussion questions related to the pros and cons for various axial coding choices and the grain size of our codes, as well as solicit feedback specific to working with longitudinal classroom data and communicating about data related to both mathematics and physics.

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