Linear Algebra Proof Constructions & Tall's Worlds of Mathematics

Kelsey Walters
Purdue University

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Proof is central to mathematics, but proofs are notoriously difficult (Alcock & Weber, 2010; Britton & Henderson, 2009; Caglayan, 2018; Dorier et al., 2000; Epp, 2003; Moore, 1994; Stavrou, 2014; Weber, 2001). Generic example proofs are thought to be effective ways of developing and understanding proofs because they reveal the structure of a proof while providing the prover something concrete to focus on (Leron & Zaslavsky, 2013; Rowland, 2002). When linear algebra students need help writing a proof, they perform better on subsequent tasks if given a generic example proof rather than a formal proof (Malek & Movshovitz-Hadar, 2011). Researchers also recommend emphasizing geometry to develop students' understandings (Dogan, 2018; Ertekin et al., 2010; Hannah et al., 2013; Stewart & Thomas, 2010).

Through this study, I aim to add to the knowledge of linear algebra students' ways of thinking and working with proof constructions by examining the tools and strategies (e.g., generic examples) students use when encouraged to think about linear algebra concepts in different ways (e.g., geometrically). Tall (2013) describes mathematical thinking as occurring in three worlds. The embodied world consists of objects, graphs, diagrams, and their properties. The symbolic world contains symbols, operations, formulas, and calculations. The formal world consists of formal definitions, axioms, and proofs made up of logical deductions from axioms and definitions. My research question was: *What tools and strategies do students choose to use when working on proofs in the embodied, symbolic, or formal world of mathematics?*

To answer this, I conducted a multi-case study with four students enrolled in linear algebra: Rap, Advik, Jun, and Min (pseudonyms). In task-based interviews, I encouraged participants to use a given world of mathematics as they worked on proving or disproving a given statement. I used open coding (Corbin & Strauss, 1990; Strauss & Corbin, 1990) to detail the mathematical tools and strategies students used, and I coded the worlds of mathematics they used. To ensure that my coding was systematic and rigorous, I continually updated a master list of codes, keeping notes on the individual codes and how they were applied. I also performed member checks (Creswell & Miller, 2000; Merriam & Tisdell, 2016).

Comparing tools and strategies across constructions, some were used regardless of the world I encouraged participants to use. For example, Jun used cases and examples. Other strategies and tools were only used when I prompted participants to use a specific world. For example, Rap attempted to work directly from the hypotheses to the conclusion when creating symbolic proofs, but he worked backwards from the conclusion when using formal logic.

All participants used generic examples when asked to use the embodied world and chose other strategies when asked to use the symbolic or formal worlds (with a few exceptions). One interesting task was to prove or disprove the statement: *Let* $S = \{U_1, U_2, \dots, U_p\}$ *be a set of two or more vectors. If* S *is linearly dependent, then there is a subset of* S *that is linearly independent and has the same span as* S*.* In their embodied generic examples, Rap and Advik began with linearly independent vectors and considered other vectors afterwards. However, their symbolic proofs began with a set of dependent vectors, and they removed vectors until they reached an independent set. This shows that one potential strength of the embodied world is to prompt students to create generic examples and to find different ways of structuring an argument.

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Purdue University

Introduction

Proof is central to mathematics, but proofs are notoriously difficult.[1-8] Generic example proofs are thought to be effective ways of developing and understanding proofs because they reveal the structure of a proof while providing the prover something concrete to focus on.[9-10] When linear algebra students need help writing a proof, they perform better on subsequent tasks if given a generic example proof rather than a formal proof.[11] Researchers also recommend emphasizing geometry to develop students' understandings.[12-15] Through this study, I aim to add to the knowledge of linear algebra students' ways of thinking and working with proof constructions by examining the tools and strategies (e.g., generic examples) students use when encouraged to think about linear algebra concepts in different ways (e.g., geometrically).

My research question was: **What tools and strategies do students choose to use when working on proofs in the embodied, symbolic, or formal world of mathematics?**

Methods

I conducted a multi-case study with four students enrolled in linear algebra: Rap, Advik, Jun, and Min. In task-based interviews, I asked participants to use a given world of mathematics as they worked on proving or disproving a given statement.

I used open coding[16-17] to detail the mathematical tools and strategies students used, and I coded the worlds of mathematics they used. To ensure that my coding was systematic and rigorous, I continually updated a master list of codes, keeping notes on the individual codes and how they were applied. I regularly discussed codes with a senior researcher. I also performed member checks.[18-19]

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Tall's Worlds

Tall describes mathematical thinking as occurring in three worlds.[20] The embodied world consists of objects, graphs, diagrams, and their properties. The symbolic world contains symbols, operations, formulas, and calculations. The formal world consists of formal definitions, axioms, and proofs made up of logical deductions from axioms and definitions. Table 1 gives examples of linear algebra concepts in the three worlds.

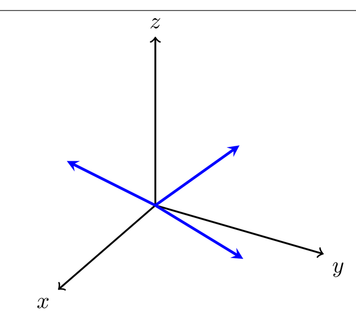
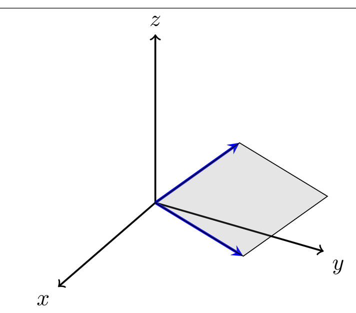
	Embodied World	Symbolic World	Formal World
Linear Independence		$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\1\\0\end{bmatrix}=0$ $a+b=0,\ b=0$ $a=0,\ b=0$	$c_1v_1+c_2v_2+\ldots+c_nv_n=0$ implies $c_1=c_2=\ldots=c_n=0$
Span		$c_1v_1+c_2v_2+\ldots+c_nv_n$ is in span$\{v_1,v_2,\ldots,v_n\}$	span$\{v_1,v_2,\ldots,v_n\}$ is the set of all linear combinations of $v_1,v_2,\ldots,v_n$

Table 1: Linear Algebra Concepts in the Three Worlds of Mathematics

Findings

Comparing tools and strategies across constructions, some were used regardless of the world I asked participants to use. For example, Jun used cases and examples. Other strategies and tools were only used when I prompted participants to use a specific world. For example, Rap attempted to work directly from the hypotheses to the conclusion when creating symbolic proofs, but he worked backwards from the conclusion when using formal logic.

All participants used generic examples when asked to use the embodied world and chose other strategies when asked to use the symbolic or formal worlds (with a few exceptions). In their work on the task in Table 2, Rap and Advik began with linearly independent vectors and considered other vectors afterwards when creating embodied generic examples. However, their symbolic proofs began with a set of dependent vectors, and they removed vectors until they reached an independent set. Table 2 shows parts of Rap's symbolic and embodied work.

Prove or disprove the following statement.
Let $S=\{U_1,\ U_2,\ldots,U_p\}$ be a set of two or more vectors. If S is linearly dependent, then there is a subset of S that is linearly independent and has the same span as S.

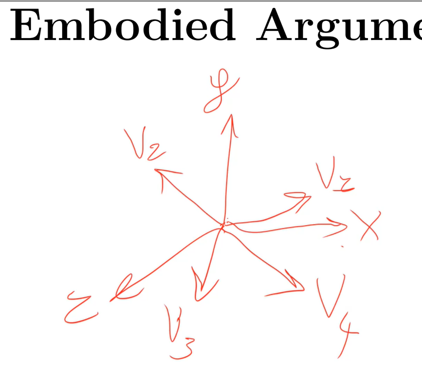
Rap's Symbolic Proof	**Rap's Embodied Argument**
$x_1U_1+x_2U_2\ldots x_pU_p=0$ $x_k\neq 0$ $U_k=\dfrac{x_1U_1\ldots+x_pU_p}{-x_k}$	
"which means that the other $n-1$ vectors is a subset of S, which may or may not be linearly independent. But it has the same span"	"you have V_1 and V_2. V_3 is, in this case, for now, it's linearly independent. Then V_1, V_2, V_3 span the entire space. $\ldots V_4$ is a part of that space."

Table 2: Portions of Rap's Symbolic and Embodied Work

Discussion & Further Research

Participants used generic examples when asked to use the embodied world, whereas other tools and strategies were used regardless of the world I asked participants to use. Further research could determine why students choose particular strategies and how educators can best support their use of various strategies across the mathematical worlds.

One potential strength of using the worlds of mathematics in the classroom is to aid students in thinking about different ways of structuring an argument. Participants also said the three-world framework was helpful for organizing their thoughts.

Through this study, I became interested in how students structure arguments when working in different worlds of mathematics. At times, Rap and Advik structured their arguments differently when reasoning in different worlds. I am interested in studying this further and, if this is a common phenomenon, finding ways to support students' use of various argument structures to develop student understanding of concepts and proofs.

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Contact Information: kelsey.walters.math@gmail.com