What Are Instructors Doing to Initiate Refinements of Students' Definitions?

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A key feature of inquiry-based instruction is to develop formal mathematics by directly building on student thinking and contributions, which often requires engaging students in refining their work. In our experiences we have noticed that it is challenging for instructors to support this refining activity and so we see a need to better understand what instructors can do in-the-moment to initiate refinements. In this study, we investigated how three instructors across four courses guided their students through the refinement of one definition. We identified three different moves that these instructors used to support the definition refinements: (1) suggest an edit with implicit mathematical reasoning, (2) suggest an edit with explicit mathematical reasoning, and (3) ask a question to create a problematic situation. The first two moves focused on the refinement itself whereas the third move focused on a problem to motivate a need for a refinement. We discuss future research directions and implications of our work.

Keywords: Defining, intellectual need, teacher moves

A key feature of inquiry-based instruction is to develop formal mathematics by directly building on student contributions (Kuster et al., 2018; Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007). For instance, in many inquiry-based classrooms students are guided to reinvent formal definitions. To work with the students' contributions, the teacher supports students in refining their definition drafts so that their definition is equivalent to a standard definition given in textbooks. This is our intention with the curriculum materials that we are developing as part of a larger research project (ASPIRE in Math NSF IUSE #1916490). Our task sequences engage students in defining (Zandieh & Rasmussen, 2010) by first engaging them in a task that evokes a concept image (Tall & Vinner, 1981), then students and the instructor generate and negotiate examples and non-examples of the concept, and lastly they work together to refine an informal description of the concept towards a formal definition (see Vroom, 2020).

It is documented in the research literature that it is challenging for teachers to build on student thinking in general (Andrews-Larson et al., 2019; Johnson & Larsen, 2012; Speer & Wagner, 2009) and we have noticed this to be especially the case when refining students' definitions. In particular, students sometimes offer definition drafts that are inconsistent with mathematical norms and conventions since students are not only learning about the concept itself but also are in the midst of learning about formal mathematical language (Vroom, 2022). At the same time, formal mathematical language is highly technical and filled with nuances, but functions to articulate precise mathematical meanings (Halliday, 1978; Schleppegrell, 2007). Thus, we see that it is a non-trivial endeavor to support students to refine their definitions in such a way that they more effectively communicate the students' ideas to a broader mathematical audience.

This study focuses on investigating what instructors do to support students in refining their definitions in such a way that supports them to abide by mathematical norms. Such an investigation will directly support our ongoing research project by informing instructor support materials for using our curriculum. This study will also add to the research literature on inquiry-based instruction by providing insights into how instructors can guide students to refine their drafts of definitions and potentially statements and proofs as well.

Theoretical Grounding

Instructors who teach with the ASPIRE curriculum materials are positioned as brokers (Rasmussen et al., 2009; Vroom, 2020; Zandieh et al., 2017) holding membership in the classroom community as well as having relevant knowledge about mathematical norms and concepts. In earlier work, Vroom (2020) described the role of a broker as they engage students in mathematical activity (e.g., defining) in which they aim to construct some product (e.g., a definition). In the case of defining, the broker engages in a cyclic process in which they iteratively aim to (a) make sense of a students' draft of a definition and its intended meaning in relation to how the mathematics community might interpret the draft and (b) support the students to edit the draft (in the case that it is not likely to be interpreted as the students intended). To support the edits, the broker can strategically share mathematical norms or build on the students' thinking, often creating an intellectual need (Harel, 2008; 2013) for refinement.

Harel claims that students should have an intellectual need to learn what we intend to teach them, where an intellectual need refers to a problematic situation that motivates the construction of the piece of knowledge (Harel, 2013). Thus, an instructor who aims to create intellectual needs for definition refinements would focus efforts on creating problematic situations for students. In the case of definition refinements, we see an intellectual need as referring to a problem with the draft that motivates both an edit and an understanding for why such an edit might resolve the problem. For example, the students in Vroom's (2020) study offered Figure 1A as their definition for a sequence that had "no upper bound". While explaining the meaning of the definition, the student described how it fit for the example in Figure 1B. To do so, he first sketched a horizontal line at k and then pointed out that he could find a sequence term that was greater than k (referring to the sixth term in Figure 1B). The teacher-researcher then attempted to create a problematic situation by pointing out that when the student described the definition with the example, he *first* introduced k but that his definition introduced n first and k last. Here, the teacher-researcher attempted to bring light to a problem: the structure of the definition did not match the structure of how he described his intended meaning with the picture. The teacher-research then explained that the student's meaning was clear when they explained it with the picture and suggested editing the definition so that it better fit his description. The student then altered the definition so that the variables were introduced in the same order as he introduced them with the picture, giving some evidence that the student experienced an intellectual need for refinement.

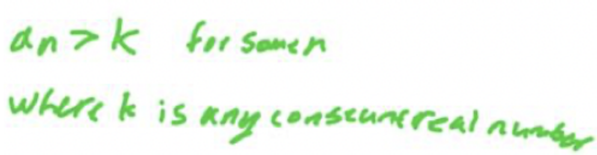
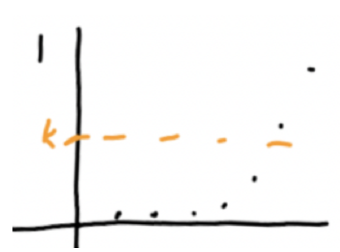
A.	B.
 $a_n > k$ for some n where k is any constant real number	

Figure 1. Students' draft of "no upper bound" definition.

Vroom's (2020) conceptualization of the broker emerged from a successful case study of a broker with a pair of students in a teaching experiment setting. This case was successful in the sense that the teacher-researcher worked with the students to refine the students' mathematical statements so that (1) the statements could be interpreted by the larger mathematical community as the students' intended and (2) the teacher-researcher seemed to create problematic situations that motivated definition refinements. Because the framework emerged in a laboratory setting, we were curious how (if at all) instructors in whole-class settings worked with students in a similar way. Specifically, in this study we aim to answer: what are instructors doing to initiate refinements of students' definitions?

Methods

For this study we investigated how three instructors across four courses guided their students through the refinement of one definition. The three instructors taught Introduction to Proof courses using curricular materials developed as part of the ASPIRE in Math project. One lesson of this curriculum focuses on mathematical language where students are guided to define several concepts and learn about the formal mathematical language that they used to do so. For this study, we focused on the instructors' moves during one of those defining tasks. In this task, students create a formal definition of a sequence they term "Eventually Constant". Students typically describe this sort of sequence as one that at some point gets "stuck" or "becomes constant". While this can be defined formally in several ways, one common formal definition is "A sequence x_n is eventually constant if there exists an $N \in \mathbb{N}$ such that for all $n > N$, $x_n = x_N$." After students develop a shared informal description of the concepts (e.g., a sequence that will get stuck at some point), students work in small groups to draft a definition. Then, students are guided to refine their draft(s) in a whole class setting. We use data from four courses, three of which were university courses and one a community college course. All four courses were conducted remotely with synchronous meetings over Zoom and with students using Google Docs during their group work. Data was collected using screen recordings of the class sessions, and small group work was captured by a researcher entering a breakout room to record the students' work. The focus task occurred over one day of the classes.

We began our analysis by watching the video of each of the defining sessions (four instructional days from the four classes). During this viewing we identified each moment that the instructor made some move to initiate a refinement of the definition. For each move we made note of the definition draft that was being refined and asked ourselves "What did the teacher do/say/ask during this time?" and "Do we see this as an attempt to create a problematic situation? Why or why not?", documenting our answers. We then documented what (if at all) refinement was made as a result of this move. As we documented each move, we constantly compared to previous moves to make note of any similarities or differences. In total we identified 15 teacher moves from the three instructors and through this constant comparison we categorized the 15 moves into three types: (1) Suggest an edit with implicit mathematical reasoning, (2) Suggest an edit with explicit mathematical reasoning, and (3) Ask a question to create a problematic situation.

Results

In what follows, we describe the three categories that we identified for what instructors were doing to motivate a refinement.

Suggest an edit with implicit mathematical reasoning

We identified instances from all three instructors where they suggested an edit with implicit mathematical reasoning (eight total instances). This type of move happened when the instructor highlighted a refinement that seemed (to us) to have mathematical value but the instructor did not explicitly state its value to the students. For instance, in one whole class discussion, an instructor highlighted a group's draft of their definition with the property: "there exists an a such that $x_a = x_{a+b}$ for all $b \in \mathbb{N}$". She read the property out loud and asked "what type of number does a have to be?" and added "we should probably indicate what type of value a is". After a student quickly responded "a natural number", the instructor agreed and edited the property by replacing "there exists an a" with "there exists an $a \in \mathbb{N}$". We see this instance of the instructor suggesting a particular edit that communicated that a is a natural number presumably because it is normative to not only quantify variables but also indicate what set the variable belongs to. This reasoning was left implicit since the instructor only emphasized the edit ("indicate what type of value a is") rather than also highlighting why this edit was advantageous.

Suggest an edit with explicit mathematical reasoning

We identified six instances (from two instructors) in which instructors suggested an edit with explicit mathematical reasoning. This type of move happened when the instructor highlighted a refinement and explained why the refinement had mathematical value. For example, one instructor was in a breakout room with two students who wrote the following draft definition in their Google Doc: "The sequence Xn is eventually constant if there exists X_m in X_n such that X_m=X_n for every n>m." The instructor said:

> "so the one thing I would say is usually when you declare a variable with 'there exists' or 'for alls' with sequences a lot of times you'll just do the subscript only. Like when you have X_m in X_n. [...] Here you say X_m in X_n, usually we don't do that, the reason why is that it sort of abuses the notation a little bit. On the one side you're thinking of a specific term sequence and on the other side you're treating it as a set. So that's a little bit of an icky way to do it."

We see this move as suggesting a particular edit ('there exists m in N' instead of 'there exists X_m in X_n') and the instructor explicitly shared the reasoning for that edit. In particular, the students' draft used sequence notation in two different ways: X_m represented a sequence term and X_n represented the sequence itself.

Ask a question to create a problematic situation

We identified only one instance in which an instructor asked a question to create a problematic situation with the students' drafts. In this instance, the students were in breakout rooms tasked with drafting a definition in a shared Google Doc. One group's first attempt read: "The sequence Xn is eventually constant if, eventually, there is a number in the sequence that repeats itself infinitely many times." The instructor added a comment to their definition that said: "Does this allow something like 1, 2, 1, 2, 1, 2, 1, 2… where 1 and 2 are repeated infinitely many times?" We see this as attempting to create a problematic situation because he highlighted a non-example of a sequence for which the property was true. Specifically, the sequence 1, 2, 1, 2, 1, 2, 1, 2… has two numbers (1 and 2) that repeat themselves infinitely many times but this sequence is not eventually constant since it never "gets stuck". This problem could motivate a need to refine the property so that it does exclude all non-examples.

Discussion

We found that the instructors in our study implemented three different moves to support definition refinements for the case of Eventually Constant Sequences. Two of these moves focused on the refinement itself (*suggest an edit with implicit mathematical reasoning* and *suggest an edit with explicit mathematical reasoning*) whereas the third move focused on a problem to motivate a need for a refinement (*ask a question to create a problematic situation*). We do not intend to claim that one move is always preferable. For instance, it might be appropriate for an instructor to suggest an edit with implicit mathematical reasoning when, for instance, the reasoning has been established and taken-as-shared within the classroom community. However, we conjecture creating problematic situations with drafts of students' definitions to be advantageous in many cases. Guided by Harel's (2008; 2013) theoretical perspective, we anticipate that focusing on the problem a refinement can solve rather than the solution itself could better support students in understanding the nuances of the formal mathematical language.

This study is an ongoing project and our results here motivated some additional questions that we wish to further explore. To start, we wonder about the context of the task. In our experience we have seen that students often create an informal definition of the Eventually Constant Sequence concept that structurally parallels a formal definition. Thus it is often the case that any needed refinements are less substantial or more stylistic. This context could explain why we did not see many instances of the last category (*ask a question to create a problematic situation*) especially in comparison to Vroom's (2020) case study. Hence, we wonder whether different defining situations afford more opportunities to create problematic situations. We are also aware that the type of move that an instructor might use to initiate a refinement could be influenced by previous classroom activities. For instance, as we mentioned above, an instructor could decide to keep the mathematical reasoning implicit since it had been established and taken-as-shared within the classroom community. With that in mind, we think it is important to also pay attention to the order the instructors use the moves and what, if any, previous norms had been established. We plan to explore these questions in future research.

Our results also suggest some practical implications. A majority of the instructor moves fell into the *suggest an edit with implicit mathematical reasoning* category. While we have discussed that this move could be appropriate in some cases, this should not be the only move instructors make. As such, when it comes to designing support materials for our curriculum, these results suggest to us that we need to better support instructors to have explicit discussion about the mathematical reasoning for edits. This will be particularly important in the defining tasks that occur early in the task sequence when relevant mathematical norms and conventions have not yet been discussed. We also see this ongoing work as contributing to efforts beyond our own materials, and in particular, we hope that this study will provide insights for inquiry-based instructors who aim to refine students' definitions, conjectures, or proofs.

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