Magnitude Bars and Covariational Reasoning

Irma E. Stevens
University of Rhode Island

In this report, I examine the recommendations made by researchers to use magnitude bars (or dynagraphs) with students to support their reasoning about quantities. More specifically, I look at implications of an undergraduate pre-service secondary mathematics teacher's reasoning when using tasks with magnitude bars designed to support students' meanings for geometric formulas in dynamic contexts. I illustrate the ways in which a student reasoned, ways that both resonated with researchers' recommendations but also introduced unanticipated difficulties that resulted from the introduction of the magnitude bars. Specifically, I show two different ways of comparing magnitudes a student illustrated in the Painter Problem-using amounts of change reasoning and length comparison reasoning. The results of this study contribute to the literature on using magnitude bars to support students' reasoning by illustrating the different ways in which a student can reason with magnitudes.

Keywords: Pre-Service Teachers, Covariational Reasoning, Dynamic Geometry, Representations

In RUME 2017, Stevens, Paoletti, Moore, Liang, and Hardison (2017) provided principles for designing tasks that promote covariational reasoning—in which students reason about two quantities changing in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Among their considerations to use quantities that are not monotonic, to use shapes strategically, and to use different representational systems, the authors suggest to "use varying segment magnitudes to represent a quantity's magnitude in a situation" (p. 933). In this report, I first (i) review how and why these researchers and others have argued for using dynamic magnitudes to support students' reasoning (ii) describe the methods of a study with pre-service secondary mathematics teachers (hereafter, PSTs) that considered the aforementioned design principles from Stevens et al. (2017), (iii) describe the results of the study by illustrating one PST's reasoning, and (iv) discuss how analysis of the larger study included insights into the affordances and limitations of using varying segment magnitudes that do not have any numerical values or units labeled (hereafter, magnitude bars). Specifically, the results section includes how magnitude bars were able to support students' covariational reasoning but also provided additional sources of confusion about what quantities were being represented. In sum, this report is of a study that aims to answer the question, "What role does the use of magnitude bars play in students' reasoning about relationships between quantities in dynamic geometric contexts?".

The Cases for Varying Magnitudes

This section is a summary of researchers' justifications for using dynamic magnitudes to support students' mathematical reasoning. It includes recommendations from researchers who consider the role of technology in reasoning and are interested in students' cognition.

Varying bars as a way for students to construct productive meanings for graphs

As mentioned in the introduction, Stevens et al. (2017) proposed the following task design principle for designing tasks that promote covariational reasoning—using varying segment magnitudes to represent a quantity's magnitude in a situation. Specifically, the authors stated that using the magnitude bars instead of prompting for a graph immediately offers researchers "insights into students' reasoning while minimizing the influence of the ways of thinking they have developed for graphs (e.g., iconic translations, issues of function/dependency, ways of thinking based on figurative thought)" (p. 933). They also described how the magnitude bars served to help scaffold students into constructing graphs by placing the magnitude bars orthogonally and constructing a multiplicative object (i.e., a point on the graph in this case) and then an emergent trace of all the instantiated pairs (see also a similar use of dynagraphs from Antonini, Baccaglini-Frank, & Lisarelli (2020). This emphasis on using different representations to support productive meanings for graphs relates to the task design principle of Marton's Variation Theory (Kullberg et al., 2017; Marton, 2015) that Johnson (2022) described, though she recommended specifically using different forms of the same type of graph to support students' meanings for rate (building of literature such as Johnson (2012)). Liang & Moore (2021) extend the idea of partitioning magnitudes and its role in students' covariational reasoning. They noted how students' partitioning activities can be rooted in figurative thought by illustrating cases in which students struggled to relate partitions constructed in situations with partitions constructed in graphs intended to represent relationships between quantities from those situations. Thus, collectively, these authors have concluded that magnitude bars can be used to support productive meanings for graphical representations via covariational reasoning but also note the importance of attending to how partitions of magnitudes relate across representation systems.

Varying bars as having semiotic potential for function

Hollebrands, McCulloch, & Okumus (2021) considered the theory of semiotic mediation to discuss the design idea that a tool (such as the dragging tool in dynamic geometry tasks) has semiotic potential that a teacher can exploit to "guide students to produce *mathematical signs* (definition, proof, mathematical conclusion, generalization, etc.). Here semiotic potential of an artifact relates to the "mathematical meanings related to the artifact and its use" (Bartolini Bussi & Mariotti, 2008, p. 754). For example, they described how dynagraphs elicited students' understandings of the notion of function, referencing, the work of Antonini, Baccaglini-Frank, & Lisarelli (2020). *Dynagraphs* are like magnitude bars where students drag the bar representing the independent variable and can observe the effects of the bar representing the dependent variable, but unlike magnitude bars, they are typically placed on a number line with values labeled. In summary, these authors argued that interacting/dragging the bars in the dynagraphs has the semiotic potential for students to find patterns and generalizations related to ideas of function.

Magnitude bars as a way for students to understand measurement

The idea of motivating ideas about measurement through motion stems from Piaget and colleagues (1960). They and other researchers (e.g., Curry, Mitchelmore, & Outhred, 2006; Kamii, 2006) have reported that students start to conceptualize a length or magnitude via motion (e.g., stepping, sweeping, and pointing motions across straight paths). Moreover, Barrett et al. (2012) proposed a hypothetical learning trajectory that intends to build on this meaning for the quantity by introducing the iteration of single units (eventually without gapping, overlapping, or inconsistent unit size.). These researchers, although not explicitly mentioning magnitude bars, describe an image of movement associated with a length quantity to support students in learning about measurement.

Magnitude bars as a way for students to compare quantities

Lastly, the idea of comparing two magnitude bars to each other at different lengths shows up in the literature as well. These researchers have discussed how comparing dynamic magnitudes can support students at various levels. Yeo (2021) described how "dragging mathematical objects plays a critical role to mediate and formulate fraction understandings" (p. 1) And at the undergraduate level, Beckmann & Izsák (2020) described how a variable-parts perspective (in which the unit measurement varies and is then multiplicatively compared to the output measurement) can be considered when representing measurements graphically to support learning about average rate of change, the Fundamental Theorem of Calculus, and Trigonometry. These researchers highlight how the multiplicative comparison of dynamic magnitudes can support the learning of several different mathematics topics from the elementary to the undergraduate level.

Methods

In part of a semester-long teaching experiment (Steffe & Thompson, 2000), three undergraduate students in a preservice secondary mathematics education program at a large public university in the southeastern U.S individually interviewed with a teacher-researcher and observer(s) to reason about formulas via dynamic geometric objects. Each student participated in 12-15 teaching sessions. Due to space constraints, I discuss only one students' work (Lily) on a single dynamic geometric context (called the Painter Problem), one that she worked on over the course of six total sessions, resulting in about six hours of video data. I video-recorded and screen-captured students' work on a tablet and scans were made of student work.

The overall goal of the teaching experiment was to analyze the mental operations needed to support students in constructing productive meanings for formulas (see Stevens (2019) for full details), but this report focuses specifically on the role of magnitude bars in their reasoning by attending to the ways in which Lily reasoned with magnitude bars in the Painter Problem. To do so, I attended to the descriptions Lily provided for her reasoning when discussing magnitude bars, focusing specifically on (i) what quantities (if any) she was referencing in the situation when describing the magnitude bars (ii) what the goals were for Lily when she was reasoning with the magnitude bars and (iii) what mental operations she was describing when trying to achieve those goals. In particular, I look at students' covariational reasoning—the mental actions involved in reasoning about changing quantities. Of note in this proposal are directional covariational reasoning (i.e., as one quantity increases/decreases another quantity increases/ decreases), and amounts of change reasoning (i.e., as one quantity changes by equal amounts, another quantity changes by increasing/decreasing/equal amounts) (see Carlson et al., 2002).

Task Design

As mentioned, the tasks used in the teaching experiment as a whole considered the design principles outline in Stevens et al. (2017), and the Painter Problem involving a dynamic rectangle described here is no exception. The context of a dynamic rectangle varying in one dimension to discuss area in relation to width has been used by several other researchers (e.g., Ellis, 2011; Kobiela, Lehrer, & VandeWater, 2010; Matthews & Ellis, 2018; Panorkou, 2020). In this version, the students are given a dynamic geometric sketch in which they can drag the bottom right corner to the right or left. They can also press the "Paint" button to watch the rectangle sweep out to the right (maintaining the constant height of the paintbrush used to create the painted area) and "Reset" to move the bottom right point back to the beginning location (thus removing the rectangle). The novel addition to this varying rectangle in the Painter Problem is the inclusion of magnitude bars. Doing so meets the task design principle of using varying magnitudes from Stevens et al. (2017). In Part II of the problem (Figure 1), the students were told that the orange bar represented the length that a painter had swept out the roller (i.e., the width of the rectangle) and that they were to decide which, if any, of the purple bars represented the area of the rectangle. The top purple magnitude bar was an appropriate bar to represent the area of the rectangle because it has a linear relationship with the orange bar, whereas the bottom purple bar increases at an increasing rate with respect to the orange bar.[1]The students could move around the dots, which they used to mark specific widths of the rectangle and endpoints of the magnitude bars. Other design principles from Stevens et al. (2017) were also met. For example, the choice to allow the student to move the rectangle to the right or left by dragging the points met the design principle of not providing monotonically increasing quantities. The choice to keep the rectangle shape was strategic in that it provided students with a proportional relationship (other tasks considered different relationships between a length dimension and (surface) area). And lastly, throughout the interviews, students were asked to represent the relationship between quantities using different representations (with a formula being the focus of the teaching experiment), meeting that design principle.

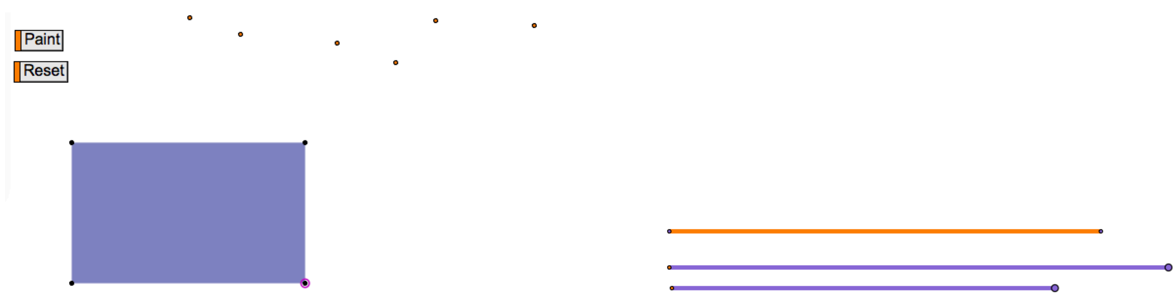

Figure 1. The Painter Problem Part II.

Results

In this section, I describe how Lily reasoned with magnitude bars while considering the relationship between the area and height of a dynamic object, the rectangle in the Painter Problem. Specifically, I attend to her descriptions of (i) the quantities she constructed with the dynamic geometry software (ii) the quantities she constructed with the magnitude bars, and (iii) the quantities she represented in her formulas.

For some context, in the Painter Problem Part I (i.e., prior to the introduction of magnitude bars), Lily had made three conclusions. First, she concluded that the area painted increased as the length that roller swept out increased (a directional covariational relationship as defined by Carlson et. al (2002)). Second, she knew to multiply the length and width to get the area of the rectangle (specifically, her formula was $A=bh$). Lastly, Lily had referenced the idea that "multiplication makes bigger".

Reasoning with the idea that multiplication makes bigger

When Lily started the Painter Problem Part II, she immediately chose Bar #1 (the top purple bar) as appropriate because "the value of the area is bigger than the value of the base, and I think,

[1] In Part III of the problem, the students were allowed to determine a number of partitions that would populate along the length of the rectangle. They could also see measurements of the length and area based on their chose unit sizes. The results of this part of the Painter Problem are not discussed in this section. See Stevens (2019) for details.

it can't-I don't think the area can be smaller than the base." She then discussed three different cases of the magnitude bars (which she called "segments") (Figure 2): when the purple segment (area magnitude) was less than, equal to, or greater than the orange segment (length magnitude). Since there were cases in which Bar #2 was not longer than the orange bar, and since multiplication makes larger means that the purple bar should always be longer than the orange bar to represent the multiplicative relationship between length and area, then Bar #2 could not be appropriate. Bar #1 satisfied her condition because its magnitude was always greater than the orange bar. Thus, Bar #1 was Lily's choice.

(a) *(b)* *(c)*

Figure 2. Lily's three cases for Bar #2 in the Painter Problem Part II: (a) area less than length (b) area equal to length and (c) area greater than length.

Lily felt unsatisfied with her justification for Bar #1, worried she was overgeneralizing the rule that "multiplication makes bigger". Thus, she decided to test various numerical values with her formula $A=bh$ to decide if it was a viable generalization. Her results are in Table 1.

Table 1. Lily's list of numerical calculations using A=bh to compare A (which should correspond to the value of the purple bar) and b (which corresponds to the value of the orange bar).

	Painter Problem Part II		
	(Cases) If…	(Conclusions) Then…	Numerical Example
1	$h = 1$	$b = A$	$b=3, h=1 \Rightarrow A=3$
2	(unspecified)	$A < b$	$b=6, h=½ \Rightarrow A=3$ $b=2, h=½ \Rightarrow A=1$
3	$b > h$	$A \leq b$	$b=6, h=½ \Rightarrow A=3$ $b=2, h=½ \Rightarrow A=1$
4	$b < h$	$A \geq b$	$b=6, h=1 \Rightarrow A=6$ $b=.5, h=4 \Rightarrow A=2$ $b=.5, h=1 \Rightarrow A=.5$
5	h is a constant	"If the area starts smaller, it should stay smaller. But if the area starts bigger than the base, then it should stay bigger."	$h=½ \Rightarrow$ "It would always make the area half the base."

As a result of Lily's reasoning with numerical values, she stated that she could no longer rule out Bar #2 as a potential viable candidate based on "my discovery that the area can be smaller than the base, the value of it" (Case 2). But after constructing Case 5, she decided to rule out Bar

#2 again because “[*dragging the corner of the rectangle back and forth*] I don’t think [the area bar] should be less than [the base bar] and then greater than [the base bar].” Beyond saying “I don’t think” and referencing the idea that “multiplication makes bigger”, however, Lily struggled to relate her conclusion to the rectangle in the dynamic situation.

Reasoning with Amounts of Change

In a follow-up session, I asked Lily to consider the case where both purple magnitude bars were always bigger than the orange bar (essentially considering the case where the orange magnitude starts at its location in Figure 1c and extends to its maximum length). After some time, she stated that for equal changes in the base (she specifically stated, “like plus two, plus two, plus two” while dragging the corner of the rectangle in small chunks to the right), the purple magnitude bar should indicate a “constant increase in area.” After some effort, and some additional numerical calculations to support her conclusions (see Figure 3), Lily reasoned using amounts of change; namely, that for the rectangle situation, for equal changes in the base, the amount of area increased by a constant amount. When she made that conclusion, she had no difficulty in deciding that Bar #1 was appropriate. That is, she could identify the magnitude bars as “increasing at a constant amount” (for Bar #1) and “increasing by increasing amounts” (for Bar #2). For example, for Bar #2, she described the situation as adding two “square units” for the first change in length and then four “square units” for the second successive equal change in length. Her discussion of the values was consistent across the situation and the magnitude bars; she associated an increasing number of boxes in the situation with a magnitude bar with increasing changes in magnitude (for the same successive change in the orange magnitude).

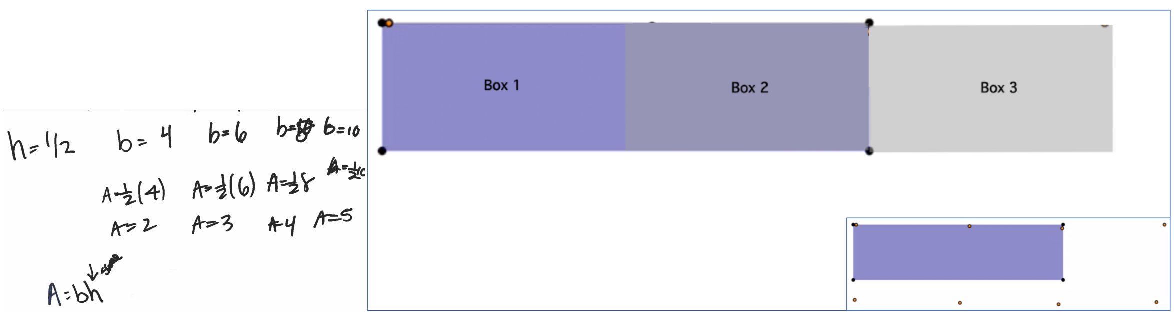

Figure 3. (left) Lily's numerical amounts of change calculations and (right) Lily’s amounts of change in the situation after focusing on magnitudes. Bottom right corner shows Lily’s markers for amounts of change and main figure shows post hoc edits for the reader to indicate the three boxes corresponding to her amounts of change in area.

Discussion and Implications

Researchers have already identified several reasons to include dynamic magnitude bars when introducing students to topics ranging from graphical representations to functions to fractions to trigonometry to the Fundamental Theorem of Calculus. These recommendations stem from ideas that the bars provide a means for comparison (via segmenting, iterating, counting, making multiplicative comparisons, etc.) and that the dynamic nature of the segments supports means for constructing length as a quantity or for observing patterns and generalizing.

In the results of this study, I reported on how the introduction of the magnitude bars supported Lily in covariational reasoning. Lily did not need the magnitude bars to make the conclusion about a directional covariational relationship between quantities. That is, already in Part I, she claimed that as the length the paint roller has swept out increases, the amount of paint increases. However, given the nature of the initial prompt (i.e., to describe the relationship between the two quantities), there was no intellectual need to compare the quantities beyond a directional covariational relationship. Further inquiry was needed but magnitudes are not the only option. If, for instance, she was asked to create a graph, the graph, whether intentional or not, would have been either straight or curved, giving the researcher a natural place to motivate a discussion about rate of change (e.g., Stevens, & Moore, 2016). Or given a formula, the student could be asked to discuss the form of the equation (y=kx, y=mx+b, $y=x^2$, y=sin(x), etc.) to motivate discussions of rates of change. But both of these methods of prompting reasoning about rates of change rely on perceptual features of the representation (i.e., the straightness of the line, the arrangement of the letters) and steps away from the quantities at hand. In doing so, students can turn to relying on conventions they have learned (e.g., shape thinking (Moore & Thompson, 2015)) to justify responses. Instead, by providing students with magnitude bars that represent different relationships between quantities and asking the students to indicate which one is appropriate, students are intellectually motivated to identify differences between the bars that go beyond the directional covariational relationship. For Lily, that prompted her to consider amounts of change covariational reasoning. That is, she considered equal changes in length on both the sketch of the situation and the orange magnitude bar representing that length, observed the resulting equal changes in area, and then identified which of the two magnitude bars also had equal changes. The larger teaching experiment showed similar gains by other students, with more or fewer struggles revolving around the use of numerical values to support their reasoning (see Stevens, 2019). Thus, this study contributes to the knowledge about magnitude bars by illustrating that they can provide a way for students to reason covariationally about amounts of change that do not rely on their understandings of graphs or equations.

Though Lily demonstrated amounts of change reasoning with the magnitude bars, she also demonstrated that it should not be assumed that students will necessarily reason quantitatively. Namely, Lily also compared each purple bar's magnitude to the orange bar's magnitude at specific points. This reasoning was not productive for her because she started relying on generalizations she recalled about mathematical operations (i.e., multiplication makes bigger) rather than attending to how the magnitudes were representing the growing rectangle. Thus, this study also contributes to the knowledge about magnitude bars by illustrating that students' ways of reasoning about the magnitude bars may not result in conclusions that they can relate to the quantities they are being asked to consider in the situation.

In conclusion, the results of this study show that magnitude bars have the potential to support student's covariational reasoning about formulas without relying on numerical values or understandings of different representation systems. I call for the research community to consider how different ways of reasoning with magnitude bars could be productive when learning different mathematical ideas. For example, although Lily's first way of reasoning about the bars did not support her in making a conclusion about the amounts of change relationship between quantities she was asked about, perhaps her comparison of quantities' values would be productive for different mathematical goals.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1350342. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. I thank the entire Advancing Reasoning team for their contributions to this work (see https://sites.google.com/site/advancingreasoning/).

References

Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education*, 6:1–29.

Barrett, J.E., Sarama, J., Clements, D., Cullen, C, McCool, J., Witkowski-Rumsey, C. & Klanderman, D. (2012) Evaluating and Improving a Learning Trajectory for Linear Measurement in Elementary Grades 2 and 3: A Longitudinal Study, Mathematical Thinking and Learning, 14:1, 28–54, DOI: 10.1080/10986065.2012.625075

Bartolini Bussi, M., & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). London, UK: Routledge.

Beckmann, S., & Izsák, A. (2020). A potential foundation for trigonometry and calculus: The variable-parts perspective on proportional relationships and geometric similarity. In Karunakaran, S., Reed, Z., & Higgins, A. (Eds). *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 734–741). Boston, MA.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Curry, M., Mitchelmore, M. and Outhred, L. July 2006. *Development of children's understanding of length, area, and volume measurement principles*, July, Prague, Czech Republic: Paper presented at the Thirtieth Annual Meeting of the International Group for the Psychology of Mathematics Education.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education,* 42(4), 308–345.

Hollebrands, McCulloch, Okumus (2021). High school students' use of technology to make sense of functions within the context of geometric transformations. *Digital Experiences in Mathematics Education*, 7:247–275.

Kamii, C. 2006. Measurement of length: How can we teach it better?. *Teaching children mathematics*, 13(3): 5

Kobiela, M. A., Lehrer, R., & VandeWater, E. (2010, May). Students' developing conceptions of area via partitioning and sweeping *Paper presented at the 2010 annual meeting of the American Educational Research Association*. Denver, CO.

Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM: The International Journal on Mathematics Education*, 49(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4

Johnson, H. (2022) Task design for graphs: rethink multiple representations with variation theory, Mathematical Thinking and Learning, 24:2, 91–98, DOI: 10.1080/10986065.2020.1824056

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, 31(3), 313–330. https://doi.org/10.1016/j.jmathb.2012.01.001

Liang, B & Moore, K. C. (2021). Figurative and operative partitioning activity: students' meanings for amounts of change in covarying quantities, *Mathematical Thinking and Learning*, *23*(4), 291–317, DOI: 10.1080/10986065.2020.1789930

Marton, F. (2015). Necessary conditions of learning. Routledge.

Matthews, P. G., & Ellis, A. B. (2018). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition*, *4*(1), 19–58. https://doi.org/10.5964/jnc.v4i1.97

Moore, K. C., & Thompson, P. W. (2015). *Shape thinking and students' graphing activity*. Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education.

Panorkou, N. (2020). Dynamic Measurement Reasoning for Area and Volume. For the Learning of Mathematics, 40(3), pp. 9–13.

Piaget, J. (1970). *Genetic Epistemology*. New York: W. W. Norton & Company, Inc.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The child's conception of geometry. Edited by: Lunzer, E. A. New York, NY. Norton & Company. 1981 Basic Books, Inc.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Stevens, I. (2019) Pre-service teachers' construction of formulas through covariational reasoning with dynamic objects (Unpublished dissertation). University of Georgia, Athens, GA.

Stevens, I. E., & Moore, K. C. (2016). The Ferris wheel and justifications of curvature. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 644–651). Tucson, AZ: The University of Arizona.

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B. & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 928–936). San Diego, CA.

Yeo, S. (2021). Semiotic mediation through technology: The case of fraction reasoning, *The Mathematics Education 60*(10), 1–19.