"When are we ever going to need abstract algebra?"

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Instructors who prepare prospective teachers should provide opportunities for them to make connections between Abstract Algebra and secondary mathematics. This study focuses on an instructor who guided prospective teachers to connect properties of algebraic structures with equation solving procedures by evoking their intellectual needs (Harel, 2013) for computation, structure, causality, and communication. The students resolved such intellectual needs by using ring properties as tools to justify their solutions to equations. We characterize the intellectual needs the instructor evoked, along with her pedagogical moves used to evoke them during class discussion. We discuss how we will extend this analysis and provide implications for teaching.

Keywords: Abstract Algebra, Equation Solving, Prospective Teachers, Intellectual Need

Understanding abstract algebra has the potential to support mathematics teachers in their various teaching practices (e.g., Baldinger, 2018; Murray & Baldinger, 2018; Murray et al., 2017; Serbin, 2021; Wasserman & Stockton, 2013; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). However, this potential is often not realized (e.g., Cofer, 2015; Ticknor, 2012; Wasserman, 2017; Zazkis & Leikin, 2010). For teachers' understanding of advanced mathematics to be useful in their teaching, they need to first make connections between the advanced and secondary mathematics that change their understandings of the content they teach (Wasserman, 2018). One content area in which prospective secondary mathematics teachers (PSMTs) can connect abstract and secondary algebra is equation solving (Wasserman, 2016). Understanding how the properties of algebraic structures are used while solving equations can help PSMTs develop a deeper structural understanding of the often-rote procedures used to solve equations (e.g., Murray & Baldinger, 2018; Serbin, 2020; 2021). This connection between algebraic structures and equations was the focus of a unit in a Mathematics for Secondary Teachers course, which is the unit of analysis for this study. The course instructor guided students to connect abstract and secondary algebra by evoking various types of intellectual need (Harel, 2013) that required students to find a new "tool" they could use to solve equations in $\mathbb{Z}_{12}$ and $\mathbb{Z}_5$. We investigate how the instructor evoked these intellectual needs, which led students to identify ring properties as "tools" to use to meet those needs. This study addresses these research questions: (a) What types of intellectual need did the instructor evoke as she implemented the equation solving task sequence during class? (b) Which pedagogical moves did the instructor use to make the need intrinsic to students?

Literature Review

Equation solving is a fruitful point of connection between abstract and secondary algebra (Wasserman, 2016). By the time PSMTs take an Abstract Algebra course, they are familiar with procedures used to solve equations, including canceling, simplifying, factoring, and using the zero-product property. Learning about the properties of algebraic structures can help PSMTs develop a deeper understanding of why those equation solving procedures work (e.g., Serbin, 2021; 2022). Wasserman (2014) demonstrated how attending to the assumptions underlying algebraic manipulation used in equation solving helped teachers understand the axioms and properties of algebraic structures. Murray and Baldinger (2018) similarly showed how teachers developed a deeper understanding of the properties used in equation solving and how they were connected to algebraic structures. Equation solving has also been used as a context for students to build formal reasoning about algebraic structures from their intuition. Cook (2015) showed how students could reinvent ring axioms by attending to properties that allow them to solve equations in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$. Cook (2014) also showed how students discerned properties of integral domains and fields by reasoning about zero-divisors and multiplicative inverses while solving linear equations. Overall, equation solving has been documented as a useful context in which students can connect abstract algebra and secondary algebra. Guiding PSMTs to make these connections is a fruitful approach for helping them better understand the content they teach.

Theoretical Background

This study leverages Harel's (2008) *necessity principle*, which suggests that students learn mathematics when they perceive an intellectual need for it. Harel (2020) defined an *intellectual need* as "A behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (p. 274). When students encounter a problem intrinsic to them, their intellectual need for something is evoked, which can be satisfied by finding a resolution. Harel gave teachers the responsibility of making intellectual needs intrinsic to the students. Harel (2013) found five categories of intellectual needs: (1) *need for certainty*, (2) *need for causality*, (3) *need for computation,* (4) *need for communication*, and (5) *need for structure*. The need for certainty refers to a person's desire to prove whether a conjecture is true or false occurs when students are asked to write proofs. The need for causality is the need to explain the cause that makes an assertion to be true. The students' need for computation can be evoked by the instructor when asking them to solve an equation, as it is described as an individual's need to quantify. A teacher can evoke students' need for communication by formulating spoken language into algebraic expressions and formalizing concepts. The need for structure is a need for organizing knowledge into a logical structure. Harel (2013) proposed the necessity principle can be used in instruction by teachers setting an intellectual need for a particular concept, translating this need into a set of questions, designing a task sequence that allows students to investigate these questions, and helping students identify the concept that is present in their solutions to these tasks. Our study focuses on an instructor's implementation of a task sequence in which she evoked students' various intellectual needs.

Methods

Study Context, Data Collection, and Data Analysis

Data were collected in a senior-level undergraduate Mathematics for Secondary Teachers course at a large research university in the US. This course covered topics from secondary algebra and precalculus that were connected to group and ring axioms and properties. Data were collected in the second unit of the course, which addressed the ring properties used while simplifying expressions and solving linear and quadratic equations in $\mathbb{Z}$, $\mathbb{Z}_5$, and $\mathbb{Z}_{12}$. The data include video and audio recordings of the first three class days of the instructional unit on equation solving. Each class was 75 minutes long. The course was taught by Dr. Grey (pseudonym), a mathematics professor and mathematics education researcher. For each clip of a task implementation, we summarized the task discussion and identified the tools students needed to do the task. We used deductive coding (Miles et al., 2013) to classify the type of intellectual need that was evoked by the instructor, using an a priori list of codes based on Harel's (2013) types of intellectual need: (a) need for structure, (b) need for computation, (c) need for causality, (d) need for communication, and (e) need for certainty. We also used inductive coding (Miles et

Task	Tool Needed for Task	Intellectual Need	Pedagogical Moves Used to Provoke Intellectual Needs
1. What are some sets that have more than one operation?	Awareness of sets with more than one operation	• Need for structure	• Juxtaposing different sets
2. What do you notice about the properties, structure, or rules for $\mathbb{Z}_{12}$?	Awareness of the properties of sets with two operations	• Need for structure • Need for causality • Need for communication	• Asking students to justify their work by using ring axioms • Juxtaposing different sets
3. Find solutions x in $\mathbb{Z}_{12}$ that satisfy the equations and tell which property/ axiom you use on each step. a) $x+3=2$ b) $5x=4$ c) $2x=0$ d) $x+1=2+2x$ e) $3(x+2)+1=4$	a) Additive inverse and additive identity b) Multiplicative inverse and multiplicative identity c) Repeated addition and use of additive inverses d) Additive inverse of cx e) Distributive property	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Challenging student idea to prompt them to revise their work • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Revising students' written work to make it more precise/explicit
4. Find solutions x in $\mathbb{Z}_5$ that satisfy the equations and tell which property/ axiom you use on each step. a) $2x=0$ b) $3(x+2)+1=4$ c) $x+1=2+2x$	a) Multiplicative inverse b) Cancellation rule c) Distributive property and zero-product property	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Challenging student idea to prompt them to revise their work • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets
5. Find solutions x in $\mathbb{Z}$ that satisfy the equations and tell which property/ axiom you use on each step. a) $x+1=2+2x$ b) $x+4=2$ c) $2x=0$ d) $4(x+2)=4$	a) Zero-product property b) Additive identity and additive inverse c) Zero-product property d) Distributive property, zero-product property, and cancellation rule	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets • Revising students' written work to make it more precise/explicit
6. Will even integers behave any different then integers? Does $4(x+2)=4$ have an answer in $2\mathbb{Z}$? What were we not allowed to do and what does that mean?	Compare the properties of $\mathbb{Z}$ *and* $2\mathbb{Z}$ (existence or non-existence of multiplicative identity)	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets
7. a) Complete table comparing rules for $\mathbb{Z}_5$, $\mathbb{Z}_{12}$, $\mathbb{Z}$ *and* $2\mathbb{Z}$. b) Define a ring.	a) Properties for $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ b) Introduce the definition of ring using properties in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ *and* $2\mathbb{Z}$.	• Need for structure • Need for communication	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets

Figure 1. Summary of Intellectual Needs and Pedagogical Moves Used to Evoke These Needs on Tasks

al., 2013) to code the instructor's pedagogical moves used to make the intellectual need intrinsic to students in that video clip. We wrote analytic memos (Maxwell, 2013) that included justifications of our coding and explanations of how Dr. Grey's pedagogical moves evoked an intellectual need and made the need intrinsic to the students. We compared the coded video segments to identify patterns and relationships among the various intellectual needs evoked and the instructor's pedagogical moves used to evoke those intellectual needs.

The Instructional Task Sequence

Dr. Grey used an adaptation of Cook's (2012; 2015) task sequence for the guided reinvention (Freudenthal, 1991) of rings, which leveraged student reasoning about ring properties used to solve equations. Dr. Grey introduced the task sequence (see Figure 1) by comparing sets with one operation to sets that have two operations. Having previously learned about groups, the instructor began by evoking the students' awareness of sets with two operations by asking about the properties of these $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$. She restricted the operations students could use to solve equations in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$ to only addition and multiplication (mod 12 or mod 5), so the tools students needed to use to satisfy their evoked intellectual needs were group/ring axioms or properties. Dr. Grey asked students to justify the algebraic manipulations in their equation solutions by identifying each of the properties used at each step. Students identified properties of additive inverse, multiplicative inverse, and identity. The instructor used these equation tasks to show students that not all properties hold for all sets. As the tasks progressed, students recognized that they needed additional tools to solve equations, such as the distributive property, cancellation rule, and zero-product property. Once they learned the properties in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$, the instructor presented equation solving tasks in $\mathbb{Z}$ and $2\mathbb{Z}$. Students then completed a table where they listed the properties of $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ that they had found while solving equations. This led to the final task where they used the properties in the comparison table to define a ring.

Results

We identified four types of intellectual need evoked through the tasks: the need for structure, computation, causality, and communication. An overview of each task used during class, the tools needed for the task, the intellectual needs evoked by the instructor, and the instructor's pedagogical moves used to evoke those intellectual needs is presented in Figure 1.

How the Instructor Evoked Students' Need for Structure

Dr. Grey evoked students' need for structure at both the beginning and end of the task sequence. She made this need for structure intrinsic to the students by *juxtaposing different sets* together and having students identify similarities and differences in their structure. She posed task 1 (see Figure 1), in which she juxtaposed sets with one operation to sets with two operations to have students recognize a difference in the properties of those two kinds of sets. The students were aware of the structure of groups, so this task led them to see a need to understand the structure of sets with more than one operation. Dr. Grey further used this pedagogical move of juxtaposing sets in task 7, which prompted students to compare the rules used to solve equations in the different sets, $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$. This task allowed students to organize the properties they have learned thus far into a table to visualize which properties held for each of the sets. Dr. Grey's pedagogical move of juxtaposing sets evoked students' intellectual need for structure by organizing/listing the axioms they found for algebraic structures $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ through equation solving. The students' intellectual need for structure prompted them to identify the axioms/properties these sets have in common that they later used to define a ring.

How the Instructor Evoked Students' Need for Computation

Dr. Grey's goal for the equation solving tasks was to evoke students' need for computation. She made this need intrinsic to them was by *asking students to justify their work by using ring axioms*. In Task 3, she asked students to find solutions to equations in $\mathbb{Z}_{12}$. She asked the students to write every property they had used to solve and why they used it. This pedagogical move allowed students to identify the properties needed to solve the equations. Once students had their solutions and explained their reasoning about them, the instructor evoked students' need for computation by *challenging a student's idea to prompt them to revise their work* by asking them follow-up questions. For example, in Task 3, the instructor prompted students' thinking by asking "What would happen if we could only use addition and multiplication?", when one of the students used division in their solution to an equation in $\mathbb{Z}_{12}$. Dr. Grey also evoked this need for computation by *revising students' written work to make it more precise/explicit*. In tasks 3d and 5d (see Figure 1), she provided a different approach to solving the equations that evoked students' thinking about how to approach these problems and how they could reduce their steps. Lastly, the instructor made this need for computation intrinsic to the students by *directing their attention to where properties can or cannot be used.* She made sure students were aware of how some properties do not hold for all sets. In task 4c (see Figure 1), Dr. Grey explained how the zero-product property holds in $\mathbb{Z}_5$ but not in $\mathbb{Z}_{12}$. Not being able to use this property evoked a need for students to find another way to perform the computation.

How the Instructor Evoked Students' Need for Causality

A need for causality was evoked by the instructor during several tasks. This intellectual need became intrinsic to the students by *asking them to justify their work by using ring axioms.* The task sequence consisted mostly of equation solving tasks, where students needed to show and justify their work in each step. These tasks evoked this intellectual need because students needed to explain why they could go from one line of the equation to the next. During Task 4c (see Fig. 1), students needed to explain why $-1(x+1)=0 \rightarrow x+1=0$ when solving equations in $\mathbb{Z}_5$. This need for causality was met by introducing the name of the property they had implicitly used: the zero-product property. Dr. Grey then evoked students' need for causality by *directing students' attention to where properties can or cannot be used* by asking them if the zero-product property would hold in $\mathbb{Z}_{12}$. This directed students to reason about why or why not this property would hold in another algebraic structure. Altogether, the instructor evoked students' intellectual need for causality, so students could explain why they could use certain ring axioms to solve an equation in a particular set and why some of the properties did not hold for other sets.

How the Instructor Evoked Students' Need for Communication

Dr. Grey evoked a need for communication in students and made this need intrinsic to them by *establishing a shared terminology of mathematical properties*. She asked students to name the properties used to solve equations in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$, which evoked students' need to communicate the names of these properties. This need was satisfied by class's establishment of a shared terminology. For example, Dr. Grey changed a student's expression of "moving 8 to the other side" to use the language of "adding the additive inverse of 8 to each side of equation" in Task 5d (see Figure 1). This shared terminology helped students use precise mathematical language to communicate their justifications of their equation solutions and helped them create new definitions. The need for communication prompted students to give explicit names to the ring axioms they used to solve equations. Aggregating these named properties and comparing whether they held for $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ allowed the class to define a ring on Task 7b.

Discussion and Conclusion

Engaging PSMTs in reasoning about the properties of rings used in equation solving can help them better understand connections between secondary and abstract algebra. A Mathematics for Secondary Teachers course instructor led students to make this connection by evoking students' intellectual needs (Harel, 2013) for structure, computation, communication, and causality on tasks involving equation solving in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$. We identified the pedagogical moves that the instructor used to evoke students' various intellectual needs, which included: (a) asking students to justify their work by using ring axioms, (b) directing students' attention to what properties were involved in cancellation, (c) establishing shared terminology of mathematical properties, (d) juxtaposing different sets, and (e) revising students' written work to make it more precise/explicit. These findings provide a pedagogical implication for mathematics instructors who prepare PSMTs. Instructors can use Dr. Grey's pedagogical moves to make intellectual needs intrinsic to their students. Evoking students' intellectual needs that can be met by using ring axioms seemed to be a beneficial way to guide students to connect abstract and secondary algebra in this study. We plan to continue this analysis by identifying the intellectual needs and pedagogical moves used by this instructor in subsequent class days as the class transitioned from solving linear and quadratic equations to solving equations involving function composition. We pose the following discussion question to the audience: How can we further explore the relationships between the intellectual needs and pedagogical practices used to evoke the needs?

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