“Just Get Rid of It:” Students’ Symbolic Forms for Cancellation

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Students use symbolic manipulations when performing cancellation as they solve equations. Rather than performing rote manipulation of symbols, we want students to understand the meanings that these symbols convey. In this paper, we examine three pre-service mathematics teachers’ symbolic forms for additive, multiplicative, and compositional cancellation. Students used symbol templates where they either wrote both the identity and inverses, one but not the other, or neither. We further explore students’ varied conceptual schemas associated with these templates, as well as the relationships that exist among these symbolic forms.

Keywords: Abstract Algebra, Cancellation, Symbolic Forms, Prospective Teachers

Symbols are ubiquitous throughout all mathematics. Instructors typically want their students to use symbols with an understanding of the meanings conveyed by the symbols (e.g., Sherin, 2001). Students’ reasoning about symbols can be analyzed using Sherin’s (2001) symbolic form construct, which is the association of the symbols one writes and their understanding of the meanings those symbols convey. Students’ symbolic forms are present in a variety of contexts, particularly when they reason with symbols while performing cancellation in secondary algebra and abstract algebra contexts. When students cancel terms, they might conceptualize this procedure as “getting rid of terms” or “moving terms to the other side.” This process of cancellation is much more complex than this, as it involves the use of the inverse group axiom, defined as for all elements $a \in G$ where $(G,*)$ is a group, there exists an inverse element $b \in G$, such that $a * b = b * a = e$, where e is the identity element in G. For example, when solving the equation $x + 3 = 1$, one adds the additive inverse of 3 to both sides of the equation to obtain $x + (3 + -3) = 1 + (-3)$, which implies that $x + 0 = -2$. This point of connection that prospective secondary mathematics teachers (PSMTs) can make between abstract algebra and secondary algebra has been the focus of several research studies (e.g., Murray & Baldinger, 2018; Serbin, 2021; Serbin, 2022; Wasserman, 2014). Wasserman (2014) suggested that teachers’ understanding of algebraic structures could support them in “unpacking the interconnected role of inverse and identity elements in the ‘cancellation’ or ‘simplification’ process” (p. 204). We refer to the process of adding an element’s additive inverse to that element to yield the additive identity 0 (e.g., $3 + -3 = 0$) as additive cancellation. We similarly refer to the process of multiplying an element by its multiplicative inverse to yield the multiplicative identity 1 (e.g., $3 \cdot 1/3 = 1$) as multiplicative cancellation. We also refer to the process of composing a function with its inverse to yield the identity function (e.g., $e^{\ln(x)} = i(x) = x$) as compositional cancellation. We hypothesized that students might have similar symbolic forms for these different types of cancellation, as they all involve the explicit or implicit use of inverse and identity elements. We explore the following research question: What symbolic forms do students have for cancellation involving additive cancellation, multiplicative cancellation, and compositional cancellation, and what relationships exist between these symbolic forms?

Literature Review and Theoretical Background

Symbols are used for a variety of purposes in mathematics, some of which include communicating meaning, notating concepts and processes, and illustrating relationships between mathematical objects (Zandieh et al., 2017). Researchers have focused on students' symbol sense (Arcavi, 1994) and their practice of symbolizing, which is broadly defined as reasoning with inscriptions (Zandieh & Andrews-Larson, 2019). Some of this work has addressed students' symbolization in calculus (e.g., Jones, 2013) and linear algebra contexts (e.g., Henderson et al., 2010; Smith et al., 2022; Zandieh & Andrews-Larson, 2019). Reasoning with symbols involves both creating symbols to convey meaning and interpreting the meaning in inscriptions. Sherin (2001) suggested, "The particular arrangement of symbols in an equation expresses a meaning that can be understood" (p. 480). Sherin claimed that students should understand the meanings that the symbols and manipulations convey: "We do not want meaningless symbol manipulation; if students use symbolic expressions, we want them to use the symbols with understanding" (Sherin, 2001, p. 479). However, this is a nontrivial task for students. Furthermore, students often have different interpretations of symbols' meanings (e.g., Jones, 2013). Reasoning with symbols is a central activity in mathematics, so students should have productive understandings of the meanings symbols convey and draw on them as they perform symbolic manipulations.

Students use symbolic manipulations while performing algebraic cancellation as they simplify expressions and solve equations. Instead of performing rote cancellation without attending to the meaning of the symbols or manipulations, students should understand the properties used to manipulate symbols in equations. The properties that allow one to cancel terms while solving equations are the inverse and identity axioms of groups and rings. Wasserman (2014) suggested that understanding these algebraic structures could support teachers in "unpacking the interconnected role of inverse and identity elements in the 'cancellation' or 'simplification' process" (p. 204). Serbin (2021; 2022) found that reasoning about group and ring axioms helped PSMTs understand the shared structure of additive, multiplicative, and compositional identities and inverses. Interpreting the symbols and symbolic manipulations used while canceling terms may involve making sense of how the inverse elements are operated together to yield the identity. In this study, we focus on students' understandings associated with their symbols used in additive, multiplicative, and compositional cancellation. To do so, we leverage Sherin's (2001) construct of symbolic forms.

Symbolic forms (Sherin, 2001) are cognitive resources (Hammer, 2000) that students can activate as they do mathematics. A *symbolic form* is an association of a conceptual schema with a symbol template. A *conceptual schema* is the meaning that symbols convey, and the *symbol template* is the structure or arrangement of written symbols that expresses that meaning. Sherin asserted students develop symbolic forms by associating meaning with the structures symbolized in equations. This association can occur as students write symbolic expressions or interpret them. One symbol template can be associated with several conceptual schemas, and one conceptual schema may be associated with multiple symbol templates (e.g., Jones, 2013). The template-schema pair that students choose to activate while solving a problem depends on the framing or context of the situation (Hammer et al., 2005). We investigate the relationships between the symbolic forms that students activate for the three different types of cancellation.

Methods

Three PSMTs, Amelia, Christina, and Derek, who majored in Mathematics, participated in this study. Each was enrolled in a senior-level Mathematics for Secondary Teachers course at a large university in the US. During the course, the students learned how the properties of groups and rings were used to justify the procedures used to solve equations. Two individual 60-minute task-based clinical interviews (Clement, 2000) were conducted with each of the PSMTs. These two interviews were designed to elicit evidence of the PSMTs' understanding of the content covered in the two course instructional units. The first interview was conducted after the unit on ring properties and equation solving. The second was conducted after the unit on functions and inverses. Some of the interview tasks prompted students to solve equations and describe the properties used and show that two functions are inverses. Given the semi-structured nature of these interviews, the PSMTs were all given the same tasks but were asked different unplanned follow-up questions based on their responses. Both interviews were conducted virtually and transcribed. Copies of the PSMTs' written work from their task responses were also collected.

We analyzed the transcripts and task responses to identify the symbolic forms the PSMTs used to perform cancellation while solving equations. We first classified the type of cancellation according to the operation involved: additive, multiplicative, or compositional. We then inductively coded (Miles et al., 2013) the transcripts by assigning each instance of cancellation present a code that captured the essence of the symbol template and a code that captured the essence of the conceptual schema evident in the data. Each task response contained multiple codes. Furthermore, there were cases in which multiple conceptual schemas were associated with one symbol template, so each of those symbol template-conceptual schema pairs was coded separately. We coded the symbol templates and conceptual schemas together to reach an agreement on the assignment of each code. After, we performed axial coding to group the symbolic forms into broader categories. We identified four axial codes that captured the essence of every symbol template used in these PSMTs' written work of canceling terms: the participant wrote (a) both the identity and inverses, (b) the inverses but not the identity, (c) the identity but not the inverses, and (d) neither the identity nor the inverses. We lastly examined which relationships exist between these symbolic forms. For each participant, we identified the different conceptual schemas that were associated with each of the four symbol template categories. We wrote analytic memos (Maxwell, 2013) to describe patterns and inconsistencies that were present in each participant's use of symbolic forms representing the different types of cancellation and used these to compare the symbolic forms used by each student.

Results

As follows, we describe the four types of symbol templates for cancellation. The symbolic forms for each type of template are presented in Figures 1-4. In the symbol templates, { } refers to an expression with multiple terms, [ ] refers to a monomial, c_i refers to constants, a, b, c refer to set elements, f, f^{-1}, g refer to functions, and $\Longrightarrow$ refers to an implication.

No Attention to Inverses or Identity

All students used symbol templates for multiplicative and additive cancellations that did not contain symbols for the inverses and identity (see Figure 1). Amelia's conceptual schemas were focused on algebraic operations rather than on the structure of operating inverses together to get an identity. She used her conceptual schemas to justify "getting rid of terms" but did not focus on the inverse-identity structure. Christina also seemed to have a computational focus. An exception was when she referred to using multiplicative inverses, however, she only used the terminology to justify removing or "getting rid of" a coefficient. She was the only student who did not write the inverse or identity elements when performing compositional cancellation. Her conceptual schema in this case was still focused on the procedure of doing something to both sides of the equation. Derek demonstrated an understanding of the structure of operating inverse elements together to obtain the identity, even when he did not write out inverse or identity elements. Overall, students tended to ignore inverses and identity when getting the answer or simplifying expressions, moving terms over, getting rid of terms, or operating on both sides of the equation.

Attention to Inverses but Not Identity

Symbol templates containing inverses but not the identity occurred in all cancellations. Students attended to inverses but ignored the identity when doing something to both sides of the equation or getting rid of terms (see Figure 2). Amelia's conceptual schemas focused primarily on the operations that needed to be performed, rather than on the structure of operating inverses together to get an identity. However, when performing compositional cancellation, she used the "operate inverse elements together to get identity" conceptual schema to explain why e and ln cancel. Christina's conceptual schemas were focused on doing something to both sides of the equation to get rid of terms. Derek's conceptual schemas focused on the inverse-identity structure even when his symbol template did not contain the identity. This was how he justified simplifying, getting rid of terms, or doing something to both sides of the equation.

Attention to Identity but Not Inverses

Students used symbol templates that contained the identity but not inverses while performing additive cancellation (see Figure 3) when the additive identity was the only term left on one side of the equation. Amelia used this template when doing something to both sides of the equation.

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	1. $c_1\{\} = c_2\{\} \Rightarrow \{\} = c_3\{\}$	Move over	Amelia
	3. $c_1\{\} = c_2 \Rightarrow \{\} = c_3$	Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
	4. $\frac{[]}{\{\}} = [] \Rightarrow [] = []\{\}$	Do something to both sides	Amelia
		Get rid of something	Amelia
		Get the answer/simplify	Amelia
	8. $\frac{[]}{[]\{\}} + \frac{c_1\{\}}{[]\{\}} = \frac{c_2}{[]\{\}} \Rightarrow [] + c_1\{\} = c_2$	Do something to both sides	Amelia
		Get rid of something	Amelia
		Get the answer/simplify	Amelia
		Use inverses (no attention to identity)	Amelia
	11. $c_1\{\} = c_2 \Rightarrow \{\} = \frac{c_2}{c_1}$	Get rid of something	Christina
		Use inverses (no attention to identity)	Christina
	15. $c_1[] = c_2 \Rightarrow [] = c_3$	Do something to both sides	Derek
		Get the answer/simplify	Amelia, Christina
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Derek
Additive Cancellation	2. $[] + c_1 = [] + c_2 \Rightarrow [] + [] = c_2 - c_1$	Move over	Christina
	4. $[] + c_1 = c_2 \Rightarrow [] = c_3$	Move over	Derek
		Do something to both sides	Derek
		Get the answer/simplify	Amelia, Christina, Derek
		Use inverses (no attention to identity)	Derek
	14. $\{\} + c_1 = c_2 \Rightarrow \{\} = c_3$	Move over	Christina, Derek
		Get the answer/simplify	Amelia
		Uses inverses (no attention to identity)	Derek
Compositional Cancellation	2. $\{\}^k = c_1 \Rightarrow \pm\{\} = c_1$	Do something to both sides	Christina

Figure 1. Symbol Templates Containing No Inverses and No Identity

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	2. $c_1\{\ \} = c_2 \Rightarrow \frac{c_1\{\ \}}{c_1} = \frac{c_2}{c_1} \Rightarrow \{\ \} = c_3$	Move over	Amelia
	6. $\{\ \} \cdot \frac{[\ ]}{\{\ \}} = \{\ \} \cdot [\ ] \Rightarrow [\ ] = \{\ \} \cdot [\ ]$	Do something to both sides	Derek
		Get rid of something	Derek
		Operate inverses together to get identity	Derek
	7. $c_1[\ ] = c_2 \Rightarrow \frac{1}{c_1} \cdot c_1[\ ] = c_2 \cdot \frac{1}{c_1} \Rightarrow [\ ] = c_3$	Use inverses (no attention to identity)	Derek
	9. $\left(\frac{c_1}{\{\ \}} + \frac{c_2}{[\ ]} = \frac{c_3}{[\ ]\{\ \}}\right) \cdot ([\ ]\{\ \}) \Rightarrow [\ ] + c_2\{\ \} = c_3$	Do something to both sides	Christina
		Get rid of something	Christina
	10. $[\ ]\{\ \} \cdot \frac{\{\ \}}{[\ ]\{\ \}} = \frac{c_1}{[\ ]\{\ \}} \cdot [\ ]\{\ \} \Rightarrow \{\ \} = c_1$	Do something to both sides	Derek
		Get rid of something	Derek
	16. $c_1[\ ] = c_2 \Rightarrow \frac{c_1[\ ]}{c_1} = \frac{c_2}{c_1} \Rightarrow [\ ] = c_3$	Do something to both sides	Amelia
Additive Cancellation	1. $[\ ] + c_1 = [\ ] + c_2 \Rightarrow [\ ] + c_1 \pm [\ ] \pm c_1 = [\ ] + c_2 \pm [\ ] \pm c_1$	Move over	Amelia
		Do something to both sides	Amelia
		Get rid of something	Amelia
	3. $\{\ \} = [\ ] + c_1 \pm [\ ] \Rightarrow \{\ \} = c_1$	Move over	Derek
		Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Derek
	6. $[\ ] + c_1 = 0 \Rightarrow [\ ] + c_1 \pm c_1 = 0 \pm c_1$	Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
	9. $[\ ] + c_1 \pm c_1 = c_2 \pm c_1 \Rightarrow [\ ] = c_3$	Move over	Christina
	10. $[\ ] + c_1 \pm c_1 \Rightarrow [\ ]$	Get the answer/simplify	Amelia, Derek
	12. $\{\ \} + c_1 \pm c_1 = c_2 \pm c_1 \Rightarrow \{\ \} = c_3$	Use inverses (no attention to identity)	Christina
	15. $\{\ \} + c_1 \pm c_1 \Rightarrow \{\ \}$	Get the answer/simplify	Amelia
	16. $[\ ] + c_1 = c_2 \Rightarrow [\ ] + c_1 \pm c_1 = c_2 \pm c_1$	Move over	Amelia
		Do something to both sides	Amelia
	17. $\{\ \} + c_1 = c_2 \Rightarrow \{\ \} + c_1 \pm c_1 = c_2 \pm c_1$	Move over	Amelia
		Get rid of something	Christina
Compositional Cancellation	1. $\{\ \}^k = \left(\{\ \}^{\frac{1}{k}}\right)^k \Rightarrow \{\ \} = \{\ \}$	Do something to both sides	Christina
		Get rid of something	Christina
	3. $\{\ \}^k = \left(\sqrt[k]{\{\ \}}\right)^k \Rightarrow \{\ \} \cdots \{\ \}\ (k \text{ times}) = \{\ \}$	Do something to both sides	Derek
	4. $\left(\sqrt[k]{\{\ \}}\right)^k = (c_1)^k \Rightarrow \{\ \} = c_2$	Do something to both sides	Amelia, Derek
		Get rid of something	Derek
		Get the answer/simplify	Derek
		Use inverses (no attention to identity)	Amelia
	5. $\left(\{\ \}^{\frac{1}{k}}\right)^k = (c_1)^k \Rightarrow \{\ \} = c_2$	Get rid of something	Christina
		Get the answer/simplify	Christina
	6. $\ln\{\ \} = c_1 \Rightarrow e^{\ln\{\ \}} = e^{c_1}$	Do something to both sides	Amelia
		Operate inverses together to get identity	Amelia
	7. $\{\ \}^k = c_1 \Rightarrow \sqrt[k]{\{\ \}^k} = \sqrt[k]{c_1}$	Do something to both sides	Amelia
		Get rid of something	Amelia
	8. $e^{\ln(c_1)} = c_1$	Get rid of something	Derek
		Use inverses (no attention to identity)	Amelia
	11. $\left(\sqrt[k]{[\ ]}\right)^k = [\ ]$	Get the answer/simplify	Amelia, Derek
	12. $\sqrt[k]{\{\ \}^k} = \{\ \}$	Get the answer/simplify	Amelia, Derek

Figure 2. Symbol Templates Containing Inverses but Not Identity

However, her conceptual schema did not involve any attention to the identity. Christina and Derek did not attend to the identity when canceling except when they wrote 0 on one side of the equation to not leave it blank. However, even when Derek did not write out inverses, his conceptual schemas referenced the structure of operating inverses together to get the identity.

	Symbol Template	Conceptual Schema	Student who Used Form
Additive Cancellation	7. $\{\} = c_1 \Rightarrow \{\} = 0$	Do something to both sides	Amelia
	11. $[\,] = \{\} \Rightarrow 0 = \{\}$	Operate inverses together to get identity	Derek
	13. $\{\} = \{\} \Rightarrow \{\} = 0$	Get the answer/simplify	Derek
		Use inverses (no attention to identity)	Derek
	18. $\{\} = \{\} \Rightarrow \{\} - \{\} = 0$	Get the answer/simplify	Christina

Figure 3. Symbol Templates Containing Identity but Not Inverses

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	5. $\frac{[\,]}{\{\}}\{\} = [\,]\{\} \Rightarrow [\,] \cdot 1 = [\,]\{\}$	Do something to both sides	Christina
		Get rid of something	Christina
		Operate inverses together to get identity	Christina
	12. $aa^{-1} = \left(\frac{b}{c}\right) \cdot \left(\frac{c}{b}\right) = 1$	Operate inverses together to get identity	Amelia
	13. $(ba)a^{-1} = (ca)a^{-1} \Rightarrow be = ce \rightarrow b = c$	Operate inverses together to get identity	Amelia, Derek
	14. $a \cdot a^{-1} = e$	Operate inverses together to get identity	Derek
Additive Cancellation	5. $\{\} \pm c_1 = c_1 \pm c_1 \Rightarrow \{\} = 0$	Move over	Christina
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Christina
	8. $[\,] \pm [\,] = \{\} \pm [\,] \Rightarrow 0 = \{\}$	Move over	Amelia, Christina
		Operate inverses together to get identity	Christina
Compositional Cancellation	9. $f(g(x)) = \cdots = x$	Operate inverses together to get identity	Amelia, Derek
	10. $f(f^{-1}(x)) = x = f^{-1}(f(x))$	Operate inverses together to get identity	Amelia

Figure 4. Symbol Templates Containing Both Inverses and Identity

Attention to Both Inverses and Identity

When the students used cancellation symbol templates that included symbolizations of both inverses and the identity, their associated conceptual schemas focused on the structure of operating inverse elements together to yield the identity element (see Figure 4). Amelia exhibited the "operate inverse elements together to get identity" conceptual schema for multiplicative cancellation only when she was working on abstract algebra tasks. She did not attend to the inverses and identity while solving equations. Furthermore, her symbol template for compositional cancellation contained symbols for the inverses and identity only when verifying that two given functions are inverses of each other. On the other hand, when Christina's symbol templates for additive and multiplicative cancellation contained symbols for the inverses and identity, her conceptual schemas were focused on the inverse-identity structure, i.e., that operating inverse elements together yields the identity. In regard to Derek, when his symbol templates for multiplicative and compositional cancellation included both the inverses and identity, his conceptual schema also referred to the inverse-identity structure of operating inverses together to get the identity. When Derek's symbol template for additive cancellation contained symbols for the inverses and identity, his demonstrated conceptual schema only focused on using inverses, without explicit attention to the identity.

Discussion and Conclusion

We examined three students' symbolic forms for additive, multiplicative, and compositional cancellation. We found that the students used four general types of symbol templates, in which they wrote (a) both the identity and inverses, (b) the inverses but not the identity, (c) the identity but not the inverses, and (d) neither the identity nor the inverses. We then examined the varied conceptual schemas associated with these templates. Amelia only explicitly attended to both multiplicative inverse and identity in the context of abstract algebra tasks and functional inverse and identity only in tasks that required her to determine if two given functions are inverse pairs. However, she did not attend to the structure of both inverse and identity as she performed cancellation while equation-solving. Amelia's evoked conceptual schema seemed to depend on her framing of the task. Ticknor (2012) had a similar finding that teachers' understandings of commutativity, associativity, and inverses were only situated in abstract algebra settings. Christina's symbolic forms were mainly procedural in nature. She attended to the structure of both inverse and identity only for multiplicative and additive cancellation. Her conceptual schemas focused on the equation solving process, as evident in her explanations of moving terms over, doing something to both sides, or getting rid of a term. She did not use any symbol templates that contained symbols for both inverse functions and the identity function when she performed compositional cancellation. She thus did not focus on the structure of the composition of inverse functions yielding the identity function while performing that cancellation. Finally, Derek's symbolic forms always demonstrated an understanding of the structure of operating inverses together to get the identity. Even in cases where he did not write out the identity or inverses, it was clear that he was aware of the inverse-identity structure in the cancellation.

As Amelia's and Christina's conceptual schemas demonstrate, having an understanding of abstract algebra concepts does not necessarily imply students can automatically use it to make sense of secondary mathematics content like equation solving. In Amelia's case, she only attended to both multiplicative inverses and identity in the context of abstract algebra where she is more likely used to that terminology and reverted back to procedural equation-solving, without attention to the inverse or identity, in most other situations. Ideally, teacher educators want PSMTs to understand that operating an element with its inverse yields the identity and be able to use that in other situations. We hypothesize that solving equations while attending to the structure of the inverse, identity, and binary operation will allow students and PSMTs to develop meaningful understandings of the symbolic manipulations involved in equation solving.

Our findings have some implications for teaching and future research. Mathematics instructors who prepare PSMTs should prompt PSMTs to explicitly write out the symbols used to represent inverses and identities while solving equations. Instructors should also guide students to identify which group or ring properties are used while solving equations. This has been found to be a fruitful way for undergraduate students and PSMTs to connect abstract algebra and secondary algebra (e.g., Cook, 2012; Cook, 2015; Murray & Baldinger, 2018; Wasserman, 2014). We intend to extend this analysis of the relationships that exist among students' symbolic forms for different types of cancellation. Future research can explore how mathematics teacher educators can guide PSMTs to make connections across abstract and secondary algebra that are not only situated in abstract algebra contexts.

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