How a Graduate Mathematics for Teachers Course
"Helped Me See Things from a Different Perspective"

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Secondary teachers and college instructors are required to take several graduate-level mathematics courses to advance in their careers. However, there is a common disconnect between graduate mathematics courses and the teaching of secondary mathematics. We developed a graduate-level mathematics for teachers course that was designed to help teachers connect abstract algebra, real analysis, and statistics to the content and teaching of secondary/early college mathematics. In this study, we examined changes in graduate students' confidence and perceptions of the relevance of advanced mathematics to teaching secondary and college mathematics. Students attributed changes in their perceptions of the connections between advanced and secondary mathematics to the course's implementation of pedagogical scenario tasks. We provide implications for the mathematical preparation of teachers.

Keywords: Graduate Courses, Mathematic Knowledge for Teaching, Perceptions of Relevance of Advanced Mathematics, Confidence

Mathematics teachers enroll in graduate-level mathematics courses for professional growth, graduate degrees, teaching certifications, and mobilities in a future career (e.g., academic specialist, college instructor). They are required to complete a significant number of credit hours, e.g., 18-credit hours to teach dual-enrollment courses. However, teachers are disconnected from the studies of advanced mathematics in graduate courses not only because they took their advanced mathematics courses several years ago, but also because the content and pedagogy of such graduate-level mathematics are not connected to their profession in teaching (Zazkis & Leikin, 2010). These graduate mathematics courses have acted as gatekeepers to professionals in teaching careers, inhibiting them from pursuing graduate studies in mathematics or dual-enrollment teaching certificates. The discontinuity between content courses and classroom teaching has been a long-lasting challenge in teacher preparation. Researchers have been addressing this challenge in their innovative approaches to improve curriculum and instruction for undergraduate-level mathematics courses and in their collaborative efforts to connect content and methods courses (e.g., Wasserman & McGuffey, 2021; Weber et. al., 2020). Graduate-level courses, however, have not been highlighted in the literature as much as undergraduate courses.

One way to address those discontinuities in graduate mathematics courses for teachers is to implement innovative course design and instructional practices that employ and empower teachers' mathematical and pedagogical practices in their learning. This will also provide a positive standpoint of *mathematics for teaching* within the discipline of mathematics by tackling the existing deficit view—inferior to and less rigorous or formal than real mathematics—and by embracing an alternative view—advanced mathematics for the teaching of mathematics.

In Spring 2022, we developed a course for students in the Master's program in mathematics with a teaching concentration at a large public university. The course design emphasized fundamental knowledge and skills in advanced mathematics and their applications to the teaching of secondary school mathematics and lower-level college mathematics including college algebra, calculus, and introductory statistics. The courses adopted student-centered activities in class to facilitate the mathematical development of secondary teachers and their reflection on the pedagogical implications for their teaching practices. In this paper, we present how the participating preservice and in-service teachers in this course perceived (a) changes in their views on connections between advanced mathematics and secondary/college mathematics, (b) changes in their confidence in advanced mathematics and teaching secondary/college mathematics, and (c) effects of the course design on those changes in their views and confidence.

Literature Review and Theoretical Background

Knowledge of advanced mathematics is perceived as an essential component of teachers' content knowledge (e.g., Conference Board of the Mathematical Sciences, 2012), especially when it connects to teachers' understanding of school mathematics. Constructs related to teachers' knowledge of advanced mathematics and its connections to school mathematics can ubiquitously be found in frameworks of mathematical knowledge for teaching (e.g., Ball et al., 2008; Ball & Bass, 2009; McCrory et al., 2012; Wasserman, 2018; Zazkis & Mamolo, 2011). Developing knowledge of advanced mathematics can lead teachers to understand relations among mathematical content and structures, hone mathematical practices, and hold core mathematical values, which have been shown to support teachers in various teaching practices (e.g., Baldinger, 2018; Serbin, 2021; Zazkis & Kontorovich, 2016; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). Taking undergraduate and graduate-level mathematics courses thus has the potential for fostering the development of prospective and in-service teachers' mathematical knowledge, practices, and values. However, these potential outcomes might not come to fruition if the prospective or in-service teachers are not explicitly given opportunities to make connections among the content and practices used in both advanced and secondary mathematics.

To address the commonly documented disconnect between teachers' studies of advanced mathematics and teachers' teaching practices, teacher educators have implemented mathematics for teachers courses with innovative course designs and instructional practices that leverage teachers' mathematical and pedagogical practices in their learning (Goar & Lai, 2021; Wasserman et al., 2019). Researchers have argued for alternative curricula and instruction to explicitly address the mathematical pedagogical knowledge and practices for teachers. The design of innovative mathematics for teachers courses should include an emphasis on content connections and pedagogical mathematical practice connections. Teachers must be given opportunities to make connections among advanced and secondary mathematical content that help them better understand the content they teach. This is the premise of Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching. Wasserman (2018) claimed, "knowledge of nonlocal [advanced] mathematics becomes potentially productive for teaching at the moment that such knowledge alters teachers' perceptions of or ontological understandings about the local content they teach" (p. 121). Thus, advanced mathematics courses for teachers should be designed to guide teachers to make connections that change their understanding of the content they teach and thus have the potential to be useful in their teaching. These courses should also be designed to provide opportunities for teachers to develop *Pedagogical Mathematical Practices* (PMPs), which Wasserman and McGuffey (2021) defined by drawing on the extant literature on mathematical practices (e.g., Rasmussen et. al., 2005) and pedagogical practices (NCTM, 2014). PMPs lie in an intersection of two domains of mathematical practices and pedagogical practices and are the kinds of practices that are common across both mathematicians and mathematics teachers in terms of their actions, habits, and lines of reasoning. Examples of PMPs include attending to assumptions and mathematical constraints or limitations, using special cases to test and illustrate mathematical ideas, exposing logic as underpinning mathematical interpretation, using simpler objects to study more complex objects, and not giving rules without accompanying mathematical explanation. The conceptual framework informing the design of the course under investigation in the current study draws on Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching and Wasserman and McGuffey's (2021) framework of PMPs. These theories provide the rationale for the graduate-level Mathematics for Teachers course design, which is described further below.

Course Design

This graduate-level Mathematics for Teachers course aimed to provide mathematics teachers and college instructors with an opportunity to study key topics in advanced mathematics and to apply the topics to their mathematical knowledge and practice for teaching secondary school mathematics and lower-level college mathematics, such as college algebra, pre-calculus, calculus, and introductory statistics. This course consisted of three modules based on subject areas in mathematics at the graduate level: Abstract Algebra, Analysis, and Probability/ Statistics. Each module included a survey of definitions and basic properties of fundamental mathematical concepts in the corresponding area. The classroom instruction emphasized the role of in-class activities and projects to build students' conceptual understanding and develop their advanced mathematical reasoning and practices.

The Abstract Algebra module was designed using Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching, along with the design heuristics of guided reinvention from Realistic Mathematics Education (Gravemeijer, 1998) and pedagogical mathematical practices (Wasserman & McGuffey, 2021). The module was designed to guide students to develop an intuitive understanding of the structure of groups, rings, integral domains, fields, unique factorization domains, and isomorphisms. It implemented Larsen's (2013) task sequence for the reinvention of groups and group isomorphism, as well as an adaptation of Cook's (2012; 2015) task sequence for reinventing rings. The curriculum then guided students to connect their understanding of the Abstract Algebra content to content covered in secondary and early college mathematics. The course curriculum led students to apply their understanding of the connections between the advanced and secondary content through the implementation of pedagogical scenario tasks. These tasks typically gave samples of hypothetical student work and asked the graduate students to interpret the hypothetical student's understanding and decide how to respond if they were that student's teacher. The pedagogical scenario tasks also were presented in the form of Lesson Play scripting tasks (Zazkis et al., 2013), in which the graduate students were given a script of classroom dialogue between a teacher and students and were prompted to finish a script of the conversation. These tasks were designed to elicit evidence of students' use of mathematical knowledge and PMPs (Wasserman & McGuffey, 2021).

The Analysis module of this course adopted an instructional model, ULTRA, for teaching in real analysis courses (Wasserman et al., 2017) that emphasizes explicit connections between secondary and advanced mathematics by introducing pedagogical situations in secondary classrooms, stepping up to studies on relevant topics in real analysis, and stepping down to the same situation. This module used selected chapters of *Understanding Analysis and its Connections to Secondary Mathematics Teaching* (Wasserman et al., 2022) written for implementing the ULTRA model. Students participated in group discussions on classroom scenarios given in the textbook and collectively worked on mathematical tasks to review relevant real analysis concepts including epsilon arguments, limits, continuity, and differentiability.

Methods

In this study, we conducted a pre-survey, post-surveys, and semi-structured interviews designed to examine changes in their confidence in high school and advanced mathematics and their perceived content connections and relevance of advanced mathematics for teaching secondary or early college mathematics. Six graduate students enrolled in the course participated in this study and completed a pre-survey at the beginning of the semester and two post-surveys, one in the middle of the semester about the Abstract Algebra and Analysis modules, and the other after the semester about the Probability & Statistics module and the overall course experience. The surveys include 7-point Likert scale items about their confidence and views on connections between secondary and advanced level mathematics. We compared the average scores of students' responses to each of the items to determine changes in their confidence and perceptions about advanced mathematics for teaching. The participants also participated in the interview about their course experiences and changes in their confidence and perceptions after completing the second post-survey. The interviews were audio recorded in Zoom and transcribed for analysis. We used an open coding process (Miles et al., 2013) to examine the participants' perceived changes in their confidence and perceptions of connections between learning advanced mathematics and teaching secondary and early college mathematics. One author independently generated codes in the first round, and the other author confirmed the codes. Then both authors discussed codes, created axial codes, and identified themes explaining how the course experiences affected the ways in which students' confidence and perceptions changed.

Results

Table 1. Mean scores on pre-survey and post-survey items.

Survey Item	Pre-survey Mean Score (N=5)	Post-survey Mean Score (N=6)
1. I am confident with explaining concepts in high school mathematics.	5.80	6.00
2. I am confident with explaining concepts in undergraduate level mathematics.	4.40	5.67
3. I am confident with explaining concepts in graduate level mathematics.	2.20	4.50
4. Understanding advanced mathematics will help me better understand secondary-school level mathematics.	6.60	6.83
5. Understanding advanced mathematics will help me better perform secondary-school level mathematics teaching.	6.60	6.67

Changes in Confidence with Explaining Mathematical Concepts at Different Levels

The comparison between students' responses to the pre and post-surveys indicated that their confidence in explaining mathematical concepts at the high school, undergraduate, and graduate levels increased overall after taking this course. The average scores on the survey items related to their confidence increased from 5.80 to 6.00 at the high school level, 4.40 to 5.67 at the undergraduate level, and 2.20 to 4.50 at the graduate level (See Table 1). Their confidence increased more at the higher levels. The students reported lower confidence in more advanced levels in the beginning but made greater increases at the end (e.g., +0.20 at the graduate level, +1.27 at the undergraduate level, +2.30 at the high school level). Their interview responses indicated that their course experience helped increase their confidence in advanced mathematics and prepared them to take further advanced mathematics courses in their graduate program. Student E said, "I definitely feel more competent in doing certain types of mathematics". Student L said, "I think this class will be perfect for that, for future courses, and I do feel more prepared to go into those [further graduate] classes." However, a student reported that their confidence in advanced mathematics had not changed a lot because the course emphasized less on writing mathematical proofs and a deep investigation of advanced topics in this course than what they had expected from a traditional graduate-level mathematics course. Student R said the course helped increase his confidence not as much as he expected because of less emphasis on proof. He said, "I want to say [my confidence increased] a little bit, not that much since it wasn't really a class where I was taught how to do proofs again because I forgot it, it's been a while. And then I was just expected to do the proofs, not like that. So just a little bit, but not as much as I wanted to." Overall, this course helped students increase their confidence in all levels of mathematics, especially for higher levels in all three subjects in analysis, algebra, and probability & Statistics.

Changes in Perceived Content Connections and Relevance of Advanced Mathematics for Teaching Secondary or Early College Mathematics

This graduate-level Mathematics for Teachers course helped graduate students recognize content connections among advanced and secondary mathematics, and it helped change their perceptions regarding the relevance of advanced mathematics for the teaching of secondary mathematics. *Some students reported that they did not have an opportunity to reflect on connections in other graduate mathematics courses.* For example, Student S explained, "I don't think I've thought before about how they related… now I think it would be very helpful in teaching secondary math." This course gave students an opportunity to connect advanced and secondary mathematics, which they said they did not have in other graduate math courses.

Some students reported that after taking this course, they could *better understand advanced mathematical content and secondary mathematical content.* In the survey data, we identified a slight increase in the mean score of the survey item 4 (see Table 1) from 6.60 to 6.83 (on a 7-point scale). The students' responses to open-ended survey items and their interviews gave further insight into how they perceived a change in their understanding of mathematics. For example, one student wrote on a post-survey, "I had always had a bit of trouble understanding groups, rings, and fields, but seeing the connection of this topic with high school content and relating it to higher level thinking made me understand it more." Thus, they acknowledged that relating the advanced mathematics with secondary mathematics during the course helped them better understand the advanced mathematical content. Several students reported a change in their understanding of secondary or early college mathematics. For example, Student L explained,

> "Before, I used to see advanced mathematics as something different from high school…advanced mathematics is really connected rather than just a continuation of one and another. So really seeing the theory behind it helps, like I think I can teach this in a better way, and explain to the students a better way, rather than just saying 'that's just the way it is,' and giving them a little bit of theory behind it."

Student L previously perceived advanced and secondary mathematics as disparate. After the course, she perceived more connections among the content. She explained how understanding the theory from advanced mathematics helped her better understand the content and explain the content to students. Student A similarly acknowledged how studying advanced mathematics helped her make sense of often rote secondary mathematics procedures. She explained,

> "You can do, for example, exponents or logarithms very rotely just by following the procedure, but it doesn't make sense, I think, until you've studied it from a more advanced math point of view, and you can see why it's being done that way."

This illustrates how after taking this course, the students reported perceiving more connections among previously disparate content and better understanding both advanced and secondary math.

The students also acknowledged *a change in their perceptions of the relevance of advanced mathematics for teaching secondary mathematics*. In the survey data, we identified a slight increase in the mean score of the survey item 5 (see table 1) from 6.60 to 6.67 (on a 7-point scale). The students' responses to interview questions provided additional insight into how they perceived advanced mathematics as relevant to and helpful for their teaching. For example, Student E claimed, "It helped me see things from a different perspective. I didn't really ever think that some of the things that we did in the module were relevant to high school teaching or students' thinking at those levels." Students expressed a *wide variety of ways in which they thought advanced mathematics was relevant to and useful for their teaching*. Some of these included using their knowledge of advanced mathematics in their teaching practices of facilitating class discourse, providing visualizations of concepts, attending to precision in the mathematical terms used in definitions, and making sense of students' mathematical thinking. Furthermore, several students acknowledged that advanced mathematics would be helpful in their teaching practices of designing lessons, making content understandable to students, and giving explanations in response to questions. Particularly, Student R explained:

> "There are some kind of students, I call them to be scary… who ask you why…I never really asked myself why. So after having a student like that and taking this class where I see where things come, I can talk like, 'this is a group, this is what is happening.' Yes, this theory helped me out in the way I'm going to be now introducing lectures."

Student R recalled experiences in which he did not know how to respond to students who ask "why." He explained that understanding the theory from advanced mathematics (abstract algebra in this context) helped him understand and explain why certain procedures work in equation solving. Student R illustrated a change in his perception of how advanced mathematics could be useful for him in his teaching practices of responding to students and explaining content.

The *students attributed these changes in their perceptions of the relevance of advanced mathematics for teaching to both the mathematical content covered in the course and the pedagogical scenario tasks used in the course*. First, when asked if anything from the course might have affected a change in their views on the relevance of advanced mathematics for teaching, some acknowledged the mathematical content in the course. Furthermore, several students attributed this change to the course's use of pedagogical scenarios. Student A explained:

> "The scenarios were helpful in that regard, like where we actually were asked to think about that very question of, in a classroom… how to explain something to a student who had some kind of misconception. So how to explain it in a way that would make sense to them… I think that that's what helped, was to actually confront that in a scenario."

These students reported that the pedagogical tasks helped them relate secondary and early college mathematics to advanced mathematics, understand students' mathematical thinking, and learn "how to handle those classroom discussions" (Student L). Student S explained how these tasks helped her become aware of different ways that students can think about mathematics:

> "The scenarios…opened up the panorama about how our students could think outside the box. A lot of kids would think differently about just one topic…. Seeing how that relates to so much more than just basic math…helped me change my point of view."

Overall, these students self-reported changes in their perceptions about the connections between advanced and secondary mathematics and its utility in their teaching of secondary or early college mathematics. The students attributed these changes in perceptions to the mathematical content covered in the course and the pedagogical scenarios used in the course.

Discussion and Conclusion

In this study, we examined how a graduate-level Mathematics for Teachers course supported improving students' confidence in secondary and advanced level mathematics and their perceptions of content connections and relevance of advanced mathematics for teaching secondary and early college mathematics. This course helped students increase their confidence in explaining mathematical concepts at high school, college, and graduate levels, and it was more helpful at the higher levels. More importantly, this course helped them recognize content connections between advanced and secondary mathematics which they had not experienced in the past. Students reported the pedagogical tasks using classroom scenarios helped them to connect the content in advanced mathematics to the teaching of secondary and college mathematics. Students perceived the relevance of learning advanced mathematics for the teaching of secondary mathematics in ways that are meaningful for them, e.g., understanding problematized secondary mathematics content in the given situations from advanced standpoints, and using advanced knowledge of mathematics for reflecting on their teaching practices.

Our course design leveraged Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching, which broadly suggests that for a teacher's knowledge of advanced mathematics to impact their teaching, the advanced mathematics must first change the teacher's ontological understanding of the content they teach. This theory informed our course design heuristics of both (a) guiding students to connect advanced and secondary mathematics to better understand the secondary content they might teach and (b) incorporating pedagogical scenario tasks to connect the advanced mathematics to the *teaching* of secondary mathematics via pedagogical mathematical practices (Wasserman & McGuffey, 2021). Several of the graduate students enrolled in the course attributed the changes in their perceptions of the relevance of advanced mathematics to their teaching to the course's implementation of these pedagogical scenario tasks. Thus, these tasks helped the students recognize how their knowledge of advanced mathematics could be useful in their teaching of secondary or early college mathematics.

The utility of these pedagogical scenario tasks presents fruitful avenues for future research and implications for the preparation of mathematics teachers. We plan to continue our course design efforts by designing curricula that can support prospective and in-service teachers in connecting advanced mathematics to the teaching of secondary mathematics through these pedagogical scenario tasks. We will also further investigate how graduate students use their knowledge of advanced mathematics and their pedagogical mathematical practices (Wasserman & McGuffey, 2021) as they perform these pedagogical scenario tasks. We suggest that both undergraduate and graduate mathematics courses for prospective or in-service teachers should be designed to implement pedagogical scenario tasks, because they provide opportunities for students to identify points of connection between advanced and secondary mathematics.

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