Student Quantitative Understanding of Single Variable Calculus

Jason Samuels
City University of New York

Student understanding of key concepts from Single Variable Calculus and the role of infinitesimals was investigated. Analysis was done using a framework which highlights quantitative reasoning. Data showed robust understanding of instantaneous rate and change across multiple representations. I examined the rich student connections between those concepts expressed discretely in Algebra and expressed instantaneously in Calculus, and how conceptions of infinitesimal quantities were formed in order to support those connections. Implications for instruction are considered.

Keywords: Calculus, quantitative reasoning, infinitesimals, derivative

Introduction & Literature Review

First semester Calculus is a critical course for all STEM majors, a required course containing foundational ideas for future study. Student success has been below desired levels for a long time, and efforts toward improving student success are extensive and ongoing (Bressoud, Mesa & Rasmussen, 2015). There has been much investigation into student understanding of Single Variable Calculus (SVC), exploring conceptions of limit (Oehrtman, 2009; Tall, 1992), derivative (Zandieh, 2000; Park, 2013), integral (Jones, 2015; Sealey, 2014), and the Fundamental Theorem of Calculus (Carlson, Smith & Persson, 2003; Radmehr & Drake, 2017). Some work has been done to connect those conceptions to a student's related prior conceptions (Thompson, 1994; Pustejovsky, 1999; Samuels, 2011). However, none of these studies investigated these questions for students learning Calculus using infinitesimals.

Recently there has been a growing call to revise Calculus instruction away from the Analysis-based approach using limits and toward a more quantitative approach (Augusto-Milner, Jimenez-Rodriguez, 2021). There have been a handful of documented attempts to do so with infinitesimals (Ely, 2021). Some research has theorized the underlying student thinking in such an approach (Ely & Ellis, 2018). None of these studies of infinitesimal-based Calculus instruction has examined how students conceive of important Calculus ideas. These intersecting gaps in the literature suggest the following research questions. (1) What is a characterization of student understanding of key Calculus concepts after one semester of Calculus taught with infinitesimals? (2) How are understandings of Algebra concepts in discrete form used to generate understandings of related Calculus concepts in instantaneous form?

Framework

To analyze student understanding of Calculus, an approach oriented towards quantitative reasoning (Thompson & Carlson, 2017) was used. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017, p425), where a quantity is a measurable attribute combined with a way to measure that attribute. The quantities of SVC and the relationships between them can be described using the ACRA Framework (Samuels, 2022). The framework is summarized here briefly with a delineation of the quantities and some of their relationships. An *amount* is a real value, a *change* is a difference between two amounts, a *rate* is a ratio of two changes, and an *accumulation* is a sum of consecutive changes. One important relationship is that the product of two rates is another rate. (The infinitesimal version is the chain rule, see Table 1.) Changes and rates over real intervals are encountered in a typical Algebra course; henceforth this scenario is referred to as *standard* (Keisler, 1976). Changes and rates over arbitrarily small intervals first arise in SVC. A positive infinitesimal is a positive number which is smaller than all positive real numbers; the usual rules of arithmetic apply. Instead of using limits, one can interpret differentials as infinitesimals to describe the arbitrarily small quantities in Calculus. (For further details, see (Samuels, 2022).) This study was targeted specifically at student conceptions after taking Calculus I. Because

Table 1. Part of the ACRA Framework for Quantities in Calculus

	Real	**Infinitesimal**
Change	$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$	$dx,\ dy$
Rate	$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$	$\frac{dy}{dx}$
Rate Equation	$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$	$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

amount has no separate Calculus version, and because accumulation/integration is covered only briefly in the course, those topics were not explored in this study. Thus the part of the framework used, and the key Calculus concepts on which data was collected, included change and rate, and their relationships.

Methodology

In 2022 Spring the author taught a Calculus I course at a northeastern urban college designed to use infinitesimals instead of limits. Two months after the end of the course, a clinical semi-structured interview (Hunting, 1997) was conducted with one of the students, who had volunteered. A script was prepared beforehand with questions on rate and change, and the relationship between them, in both precalculus and calculus contexts, and in multiple representations (verbal, numerical, graphical, symbolic). Flexibility was allowed for follow-up questions which might be suggested in-the-moment by student responses. The interview was video recorded, transcribed, and coded using the framework.

Results

Ari [a pseudonym] was first asked to "talk about rate in a way that does not involve calculus". He explained that rate is "rise over run… [it] could be expressed as ratio" and that a linear graph has a constant rate. Using numbers in an example he created, "the rate, the amount of inches after every 30 seconds is 0.5, it goes up by 0.5". He then illustrated this with the graph in Figure 1, explaining that "it shows the change for x, which is 30 seconds, and the change in y, which is 0.5".

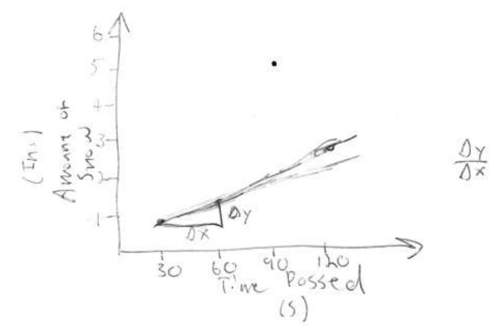

Figure 1. Ari's inscriptions describing rate for a linear graph

In another example, Ari correctly explained how to calculate rate from two points. He used the standard formula, referring to it both as "delta y over delta x" and as "rise over run", and referred to the result as both "rate" and "slope". When asked for contextual examples of rates, Ari gave two: inches per second for snow falling, and miles per hour for a runner. Thus, Ari demonstrated a flexible knowledge of standard change and rate, including how to calculate them and to connect them, in multiple notations and representations.

Next, Ari was asked "can you give me an example where you would need Calculus if you want to talk about a rate?" From the example $y=x^2$, Ari observed that "the rate is always changing...it's not a constant rate. You would have to zoom in infinitely at a certain point on the graph to get, like, a slope. And then it gets straighter... that would represent the instantaneous rate."

The interviewer gave Ari the graph of $y=x^2$ on Desmos with a touch screen, so the graph could be manipulated with fingers. He used a reverse pinch to zoom in at a particular point. He then observed that "when you get really far, it looks like the rise is like – you see how the line is much straighter, so it literally looks like a line – you would get like, the rate on y equals four." Subsequently he explained that this could also be depicted on the original (non-zoomed) graph as the tangent line at the point (2,4) with a slope of 4.

During a discussion of the slope of the magnified section of the graph, Ari wrestled with the magnitudes of the changes. He initially said that delta y was 4 and delta x was 1. After considering the graph at several levels of zoom, he corrected himself and identified the measuring unit of the grid lines of the graph (automatically drawn by the software) as the side of one square. He noted that the length, "when you zoom in, it always changes… it gets smaller". In the following excerpt, Ari starts by saying that infinitely small values do not exist, and by the end, says they do exist and exhibits a multifaceted conception of them. (His inscriptions during this exchange appear in Figure 2.)

Interviewer: We're going to imagine that we did what you said, which is imagine that we zoomed in infinitely far. We're not going to compare it to anything.

Ari: Okay, so if you zoom in it infinitely far, then it's going to keep getting smaller. Okay, so it's like to zero.

I: So once you zoom in infinitely and you say, I've done it, I've zoomed in infinitely, let's imagine that we accomplished that, then what would be the size of that?

Ari: You can't conceive of it. Yeah. Because it's infinite. It's not defined by any value.

…

I: Is there a positive number which is smaller than all positive real numbers?

Ari: No. Well, I mean, zero point 0 0 0 infinite zeros 1…

I: ... So if the length of this [horizontal] bit was point infinite zeroes 1, then what would be the length of this bit, the vertical bit?

Ari: It would be four times that, but it would be infinite still, the zeros would be infinite still, but it's like that same amount times 4. Which creates a change. Which causes a change because it's like 4…

I: Do we have any kind of name for these things? So, like, this horizontal bit of infinite zeros one, or this horizontal bit, which is also infinite zeros, but four times bigger.

Ari: So this [horizontal] bit would be like *dx*… and then the vertical bit is *dy*…

I: And then if you wanted to talk about the rate, how could you write that out? The rate for this specific example.

Ari: So *dy* is like the amount that it goes up an instantaneous rate. So at that point, when you do that a bit, that's how much it's changing at that point in the graph. So that's what *dy* represents. And then *dx* represents, *dx* represents when it changes by one to the right when you zoom out of the original function. So it's like you're relating the linear rate, like, how it would look linearly. This is like just one line, and that was like the rate then. Yeah, that's how it is. Right. But then at a different point, it would be different.

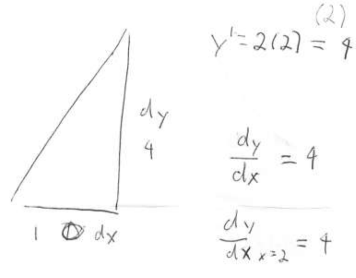

Figure 2. Ari's inscriptions when explaining that the instantaneous rate at a point is 4

In the next excerpt, Ari compared “very small” and “infinitely small” quantities from finite and infinite zooming, and their roles.

Interviewer: You started this by saying you have to zoom in infinitely in order to talk about the [instantaneous] rate. So if you were to zoom in a lot, so you get just keep on going until you get two point 100 zeros and then one, because you zoomed in a lot but not infinitely, is that enough to talk about rate?
Ari: Well, not if it's not infinite.
I: Okay. Why is that not good enough? If I zoom in a lot, why is that not good?
Ari: Because it is an exponential function and at every different point, the rate is always changing because 1.5 to the second power is, say like, okay, two squared is like four, and then three squared is nine, and then four squared is 16. So the difference between those is different. It would be the same the smaller the numbers are. So that's why you can't do that. So that's why it has to be infinite, because then you'll be able to figure out the instantaneous rate.

Ari also described a differential as an instantaneous change, and a derivative as a ratio of two instantaneous changes.

Interviewer: On this graph [in Figure 3], can you indicate for me, can you indicate *dy* and can you indicate *dt* on the graph?
Ari: So when you zoom in infinitely, vertically it would be *dy* [marking *dy*], this would be *dt* [marking *dt*], and then together [drawing the tangent line], that line right there is *dy*/*dt*.
[In response to a question, Ari used an equation he created to calculate *dy/dt* and *dy*.]
I: [What is the difference between] writing *dy* as opposed to writing *dy/dt*.
Ari: Oh, okay, yeah… *dy*/*dt* is like the ratio for the instantaneous change for both variables, for *t* and *y* in relation to each other, like this one [pointing to his earlier calculation for *dy/dt*]. And then when it's just *dy*, it's just the instantaneous change for *y*, which is how much vertically or how much the *y*-value is changing instantaneously at a point.

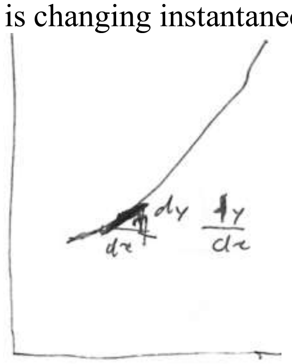

Figure 3. Ari’s graph, used to indicate dy, dt, dy/dt, and the tangent line

Ari discussed derivative in context. When asked for other scenarios in which it would arise, he suggested the context of maximizing profit. He proposed a scenario relating profit in dollars, and quantity in bicycles. When asked for the units of the derivative he noted that it is “dollars per bicycle… Because the output, the value is represented by dollars.”

Ari calculated a derivative using the chain rule (with substitution). For $y=\ln\left(\cos\left(e^{x^2+1}\right)\right)$, he used substitutions $g=x^2+1$, $j=e^g$, $h=\cos(j)$, $y=\ln(h)$, then used $\frac{dy}{dx}=\frac{dy}{dh}\cdot\frac{dh}{dj}\cdot\frac{dj}{dg}\cdot\frac{dg}{dx}$ before expressing the final answer correctly in terms of x. He then described his reasoning about it.

Interviewer: I noticed you made some cancel marks here. Can you talk about that?

Ari: Okay, so pretty much because when you take the derivative of g with respect to x and then j with respect to g, for instance, like, cross out dg because (pause). Well, I just see it as when there's a denominator and numerator, you cross it out because they, like, cancel each other…

I: Are you stating with confidence that's what you did or you stating with uncertainty that's what you did?

Ari: Well, I'm stating with confidence, but I think the problem is that because it's in derivative form. Because I remember when I took Calculus before somewhere, they said that you shouldn't see it as fractions. Like, you shouldn't see it as numbers. Like, it involves infinite number. And it's just kind of like, conceptually, it's not something you should do or see it that way. But it's just kind of like, in your class, I kind of just, like, memorized it this way because I've always hated the chain rule. And I was like, this is a good way to just do it.

When discussing the chain rule, Ari commented on proving derivative properties with limits and how unhelpful that was. "They showed the proof for the chain rule and for the product rule and all the rules, and it was very complicated. It was unlike anything like the calculus class I took, AP Calculus, it was just, like, very different, it was so much more complicated. I was like, what is this?"

Ari talked about which of his understandings improved from studying Calculus with infinitesimals.

Interviewer: Are there things feel that you understand better now after taking this class?

Ari: I guess the concept of the instantaneous rate of change, I think you showed me, like, there was some Greek letter [epsilon for infinitesimals], and then you multiply it with the number, and that would represent, like, the infinite value, the infinitesimal value, when you showed it to me that way. And that kind of like it helped me… Other classes introduce calculus with limits and that's very complicated, but that's the first thing that they do. Whereas [in this class there was] the delta x delta y thing first with the linear functions and then the rate function or change equation, rate equation. So, yeah, I kind of was able to grasp the concept of instantaneous change better. … When I don't understand something quick enough, sometimes it makes me learn it, because I want to understand it deeply. And then when I don't feel like I can, I just resort to, like with this [chain rule], memorizing and just putting it on the back burner. And other things were much easier. So I was just, like, that's what happened [with limits]. And then that's what kind of just held me back when it came to feeling confident in doing calculus. … I just think [previous Calculus instruction] was really focused on limits… it was your class when I first heard the term infinitesimals.

Discussion

Ari exhibited a rich and connected understanding of key concepts of Calculus. He was able to describe change and rate and the connection between them, both at a standard and infinitesimal level. He exhibited these conceptions in multiple problems across the four mathematical representations: numerical, graphical, symbolic, and verbal. Within the graphical representation, he was able to explain, for both change and rate, the connection between the standard presentation and the infinitesimal presentation. He did this using a verbal explanation, by drawing graphs and by manipulating a graph rendered by a computer grapher. He was very clear about the distinction between "very small" and "infinitesimal" magnitudes, and how the latter was necessary to describe an instantaneous rate, i.e. the derivative.

A significant part of Ari's concept image (Tall & Vinner, 1981) for instantaneous rate relied on the graphical representation, using infinite zooming to generate a straight graph. This conception of local linearity has been shown to be a productive cognitive root (Tall, 1991) in Calculus; several studies have shown that first introducing the derivative in this way leads to robust understanding and performance in Calculus (Tall, 1992; Samuels, 2012), and this study aligns with those results.

Ari described both the connection and the distinction between standard rate and infinitesimal rate. He utilized, first, a conception with finite real quantities, and second, a conscious introduction of the concept of infinity (in this case, infinitely small) to form a new scenario – in this case, quantities on the infinitesimal scale which are the context for infinitesimal change and rate. To characterize these two notions acting in concert, I introduce the term *transfinite thinking*. Thus, it can be said that Ari exhibited transfinite thinking in his conception of the derivative. Learner connections between finite and infinite scenarios have been investigated previously, both in Calculus and non-Calculus contexts (Ely & Samuels, 2019; Oehrtman, 2009; Mamolo & Zazkis, 2008). Here for the first time I provide a rigorous definition of transfinite thinking.

His conception of infinitesimals was given, not immediately, but after a progressions of statements. First, he conceived only of very small real values. Next, he said it would "keep getting smaller", indicating he was considering the *process* rather than the *result*. Then he said that an infinitely small quantity was something "you can't conceive of...because its infinite, its not defined by any value". Next he gave a numerical representation of an infinitesimal as "point zero zero zero infinitely many zeroes, one". Finally, he was able to perform arithmetic operations and make quantitative statements with infinitesimals, as when he referred to one (*dy*) being four times as large as another (*dx*). For Ari, the transfinite reasoning to extend his conception of rate between two points to rate at a single point was very much a multistage process which arose organically during the interview. This could form the basis for a hypothetical learning trajectory (Simon, 1995) for instantaneous rate with infinitesimals.

It is important to note that the idea of representing an infinitesimal as 0.00...1, a decimal with infinitely many zeroes, was never discussed in the Calculus class. This was a construction created by Ari in the moment. (Since he stated that he had never heard of infinitesimals before the Calculus course, I assume he did not hear of this somewhere else.) Research has previously shown that students have both spontaneous and intuitive notions of infinitesimals (Ely, 2010) which sometimes conflict with their classroom instruction. For Ari, it conflicted with his previous Calculus instruction but agreed with his experience in the current course. It is argued, both here and in the literature, that these intuitive notions should be encouraged, since they can form the basis of correct intuitions about Calculus which can be formalized into rigorous mathematics.

In constructing the infinite decimal, Ari gave quantitative meaning to the infinitesimal in the numerical representation. He was able to extend his quantitative reasoning further, making a *quantitative relationship between two infinitesimals*. He stated this relationship both multiplicatively and as a quotient. He described *dy* saying "it would be 4 times that [*dx*], but it would be infinite[simal] still…that same [infinitesimal] amount times 4". Initially he thought that $|dy|=4$, but he realized that $dy = 4$ units, where $dx = 1$ unit. He elaborated on the nature of that unit, describing it as infinitesimal and only apparent on the graph after an infinite zoom. Through unitizing and coordinating infinitesimals, he constructed a coherent quantitative conception of instantaneous rate.

Ari noted that his previous Calculus class began with limits and was "really focused on limits". He tried to understand them, but could not, calling them "complicated" and feeling limits "held me back". He resorted to simply memorizing procedures. Calculus students being forced to abandon understanding for memorization is unfortunate, and also consistent with previous results (Carlson, 1998). Particular difficulty with limits in SVC is also a common finding (Liang, 2016; Cornu, 1991).

An interesting aspect of Ari's situation was that he had previously taken a Calculus course. Students taking Calculus more than once is not only common, it is the majority experience among college Calculus students (Bressoud, Mesa & Rasmussen, 2015). How students negotiate this path is thus highly relevant. He was able to talk about the difference in his understandings in the two courses. Regarding calculating with differentials as in the chain rule, in his first course differentials were not thought of as canceling (i.e. arithmetic operations were not valid) even though those operations resembled the solving procedure, and he "hated the chain rule". In his last course, arithmetic with differentials was allowed, and in the interview he demonstrated the ability to calculate the derivative of a multiply nested function, calling the chain rule "a good way to just do it". Previously, proofs and explanations with limits created confusion, leaving him wondering "what is this?", and not understanding instantaneous rate. The presentation with infinitesimals allowed him to create personal understanding. After first discussing rate with the "delta *x* delta *y* … rate equation" in the standard case and then extending it using infinitesimals to instantaneous rate, he developed the rich understanding as previously described. This is evidence in favor of the calls to redesign Calculus to focus on the quantitative reasoning of infinitesimals and eliminate the Analysis of limits (Augusto-Milner & Jimenez-Rodriguez, 2021). The debate about relying on infinitesimals or limits for the foundation of Calculus has raged for centuries (Ely, 2021), and is likely to continue.

Ari exhibited two types of weakness in his understanding. One, some understandings were temporarily forgotten before being recalled. A likely factor was the researcher's intentional choice to conduct the interview two months after the course ended. Conceptions are likely only to be robust with either recency or reinforcement; without support in subsequent Calculus courses, meaningful and productive Calculus conceptions involving differentials can go away quite quickly and be replaced by unproductive and incorrect ones (Simmons et al., 2022). In the interview, Ari had space to (re)develop his conceptions, and in multiple cases did so with success.

Two, some topics were simply not understood at all. For example, while Ari was able to successfully calculate a derivative using the chain rule, he felt as though he only knew *how* to complete the procedure and he didn't understand *why* it worked. One possible explanation for his conceptual difficulty with this rate relationship was indirectly suggested by Ari. The conflicting instruction in previous Calculus classes may have made him wonder whether or not the cancellation of differentials in the fractions was actually legitimate. If students are taught explicitly that quantitative reasoning and arithmetic manipulation with differentials are incorrect and not allowed (a common practice), it may be challenging to get students subsequently to accept them. That is unfortunate, since those conceptions are mathematically valid and pedagogically productive.

It is important to point out that data in this study was drawn from a single student. I cannot claim that any observations generalize to the entire population. Rather, the analysis sheds light on how a student might possibly form conceptions about Calculus using infinitesimals. Due to the relative novelty of the context, this signifies a contribution to the literature.

Conclusion

In this report, I investigated an understudied area of student understanding of Single Variable Calculus, conceptions of rate and change after a one semester course on Calculus with infinitesimals. The data showed robust student conceptions of instantaneous rate and change across multiple representations. Graphically, change was conceived as a displacement, and a rate was the slope of a straight line, which could be presented as the ratio of two changes. There was an array of calculations with formulas and descriptions of applications. This conception was applied in both the standard and instantaneous contexts, demonstrating a strong link between prior and Calculus knowledge, connected using transfinite thinking. The student was able to articulate the role of infinitesimals in generating a locally constant rate to make meaning out of instantaneous rate. Graphically, this was described with an infinite zoom.

Calculus instruction using infinitesimals is not typical practice. There is a growing body of evidence that it is a propitious approach for students to make quantitative meaning in Calculus, and this study is a contribution in that direction. It also demonstrated how student knowledge of the quantities and relationships within Calculus, and their growth from prior knowledge by transfinite thinking, can be exhibited by students, and productively analyzed using the ACRA Framework.

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