Students' Inferences Using a Conditional Statement in Probability: The Relationship between Independence and Covariance of Random Variables

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The purpose of this study was to investigate how students in a probability course interpret a theorem and how they apply the theorem when asked to determine whether two random variables are independent given the value of the variables' covariance. We considered how responses of students with prior training in logic compare to other students, as the theorem is a conditional statement. We used a quasi-experimental design to investigate the impact of two instructional treatments, using isomorphic colloquial statements and using Euler diagrams. Participants largely gave conventional truth values for variants of the conditional statement prior to instruction, but many of them used the converse and/or did not use the contrapositive when given covariance values and asked to determine whether two variables were independent. Our results do not consistently show a significant difference in students' choices based on their prior logic training or either treatment.

Keywords: Probability, Conditional Statements, Logic, Covariance, Independence

Students are frequently presented with conditional statements in undergraduate mathematics courses that do not place a large emphasis on proof; consider as examples the Intermediate Value Theorem and the Divergence Test for Series in Calculus and the Existence and Uniqueness Theorems in Differential Equations. Challenges arise as students with no formal training in logic contend with deciding when and how to apply these tests and theorems while instructors sometimes make assumptions that logical inference is more intuitive than it is. While studying students' use of formal mathematical logic has long been of interest for the training of students majoring in mathematics, it has taken more time for researchers to begin exploring how students' interpretations of conditional statements in courses like calculus might impact their success (Case, 2015; Durst & Kaschner, 2020; Sellers et. al, 2021).

In this study we consider how Probability students interpret and use the following theorem about random variables: "If X and Y are independent random variables, then $\mathrm{Cov}(X,Y) = 0$" (Casella & Berger, 2002, p. 171). The statement's converse, "If $\mathrm{Cov}(X,Y) = 0$, then X and Y are independent" is logically false. (Covariance is a measure of a linear relationship, and variables may be clearly related without being linearly related.) In a probability course, students may be asked to use their knowledge about the independence or covariance of two random variables to make inferences about the other, and more importantly, to recognize in which situations they can do so.

Our study differs from the majority of prior work on students' understanding of conditional statements in two ways. First, our goal is not to help students redevelop conventional rules of logic to then use more broadly. Rather, we are providing students with the truth values of a statement and its three variants (subsequently referred to as the contrapositive, converse, and inverse) and we want to know which analogies and connections might be useful in helping students use this particular conditional statement to make accurate inferences about independence. Secondly, because of the nature of these problems, students have the option to entirely bypass the use of logical inference. While calculations may be cumbersome, covariance can be calculated directly and independence determined using the definition of independence. That is, students may choose to avoid using the logic of conditional statements altogether to answer the problems we give them. We therefore pose the following questions:

1. *How do students use covariance values to determine whether two random variables are independent?*
2. *Are their choices influenced by prior formal logic training?*
3. *Are their choices influenced by connections to isomorphic colloquial examples and/or the use of Euler diagrams?*

Review of Literature and Theoretical Perspective

An important and consistent finding in prior studies of students' interpretations of conditional statements is that students very often interpret conditional statements as biconditionals (Hoyles & Küchemann, 2002; O'Brien et. al, 1971; Wason, 1968). This presents as students assuming inverse and converse statements are true even though they are not. Possibly more significant is the finding that even the same student often does not make the same logical inferences consistently; instead, their inferences depend on the context, particularly when it is a familiar one. (Dawkins & Cook, 2017; Durand-Guerrier, 2003).

The results above must be interpreted in light of which "reasoning domain" the researchers used (Sylianides et. al, 2004). Logic can be taught using meaningful everyday (colloquial) statements, meaningless everyday statements, mathematical statements, or formal syntax (Hub & Dawkins, 2018). Claims have been made that students have less difficulty interpreting verbal and mathematical statements than abstract or symbolic ones (Case, 2015). Moreover, it has been hypothesized that everyday statements may be used to help students develop logical principles. Epp (2003) argues "it is helpful to introduce each principle with examples of sentences whose "natural" interpretation agrees with the one used in standard logic" (p. 895). Stylianides et. al (2004) found evidence to support this. Specifically, students who initially identified a contrapositive statement as false changed their minds after seeing an isomorphic statement in the verbal domain.

However, formal logic conventions do not align with the way conditional statements are used in everyday language. Epp (2003) provides the following example. A parent may say, "You can go to the movie if you finish your homework" when what she intends is "You can go to the movie if and only if you finish your homework," but the latter rarely gets stated. "If you don't finish your homework, you cannot go" is implied. So while they only state a conditional, the intention is biconditional. It is not surprising that students would interpret conditional statements in mathematics similarly. We interpret the theorem relating independence and covariance as a "generalized conditional statement" (Durand-Guerrier, 2003). There is an implicit universal quantifier within the antecedent, "X and Y are independent random variables," so mathematicians interpret the conditional as "*For all* pairs of random variables X and Y, if X and Y are independent, $\text{Cov}(X, Y) = 0$." Accordingly, mathematicians assign a positive truth value to the statement and its contrapositive and negative truth values to its converse and inverse. Students often, and reasonably so, read conditional statements as "open statements" which have no quantifier and no truth value (Durand-Guerrier, 2003).

Some authors have posited that explicit training in logic and/or exposure to more mathematics should improve students' ability to make correct logical inferences. Durst and Kaschner (2020) for example, show that students often label statements such as "If the limit of a sequence is 0, the corresponding series converges" as "Sometimes True, Sometimes False." They argue that explicit training in logic would assist students in adopting the conventions of formal logic. However, there has been no clear relationship established between current performance and either prior exposure to either mathematics or logic instruction (Attridge et. al, 2016; Cheng et. al, 1986; Inglis & Simpson, 2008).

Our position is that students, in identifying some statements as "sometimes true," have viable reasons for doing so. These students are not conforming to conventional rules of logic but may, in fact, be being more specific. Additionally, while adhering to these conventions may be essential for success in mathematics, we question whether this should be a goal for engineering students. Engineers are trained to become comfortable with uncertainty and this training leads to a reasonable and necessary discomfort with absolutes (Cardella & Atman, 2005; Gainsburg, 2006). Recognizing this, we narrow our focus to identifying analogies that students may use to convince themselves of when independence and covariance may be used to determine the other.

Recently Dawkins and colleagues have argued for developing in students a "subset meaning for conditional truth" (Dawkins & Cook, 2017; Hub & Dawkins, 2017). Their work suggests that given a conditional statement, students initially struggle to determine whether the set of elements that satisfy the *if* are contained in the set of elements that satisfy the *then*, or vice versa. Hub and Dawkins make the case that using Euler diagrams, which provide an illustration of this subset relationship, may facilitate the development of the subset meaning which will then help them move towards consistently making normative logical inferences. For example, a student who places set P within set Q can recognize in the visual representation that elements outside Q are also outside P.

Methods

Data Collection

The data was collected during in-class instruction and testing in the spring of 2022 in a semester-long Applied Probability course for engineering and computer science students at a large public institution in the United States. Prerequisites for the course include multivariable calculus and an introduction to programming. All students from the four sections who were present during this lecture were invited to participate and 72 students provided consent. Of these 72 students, 53 had previously taken or received credit for Discrete Mathematics, which provides an introduction to logical inference and proof techniques.

The second and third authors each taught two sections of the course on the same day when they covered the relationship between covariance and independence of random variables. Each instructor taught one "control" and one "treatment" section, though the treatments differed. After the initial presentation of the theorem in each section, the instructors asked each student to determine whether each of the following statements below (subsequently referred to as the inverse, converse, and contrapositive) were true or false and collected their responses prior to any discussion.

a. If X and Y are not independent, then Cov (X,Y) ≠ 0.

b. If Cov (X,Y) = 0, then X and Y are independent.

c. If Cov (X,Y) ≠ 0, then X and Y are not independent.

The instructors then explained briefly why statement (c) was true but statements (a) and (b) were false. Specifically, they shared an example of two random variables that were dependent but had no linear relationship (thus having a covariance of 0) and briefly reiterated that covariance is specifically a measure of the *linear* relationship between the variables.

At this point in the lesson, instruction in the control and treatment sections diverged. In the Control sections, which closely resembled how instructors of this course have presented this lesson in past semesters, the instructors completed 2 examples where they first determined whether two random variables were independent and then found the covariance of those variables. In the first example, the variables were independent and therefore the statement of the theorem could be applied directly to decide that the covariance must be 0. In the second, the variables were determined to be dependent, so the covariance had to then be calculated.

Instructor 1 presented the Treatment 1 students with the colloquial statement: "If it is raining, there are clouds" and asked them to determine whether the other variants of this statement, such as "If it is not raining, there are no clouds" were true. This was completed as a whole class discussion and subsequently the instructor made a direct comparison between the truth values of the rain/clouds statements and variants of the theorem.

Instructor 2 presented an Euler diagram illustrating the universal set of all pairs of random variables. (Note that in our conditional statement, the elements of sets P and Q are *pairs* of random variables, which is a potential complication for students in the process of abstraction.) In the diagram, an oval representing the pairs of independent random variables was completely contained within an oval representing the pairs of random variables with a covariance of zero. The class then discussed how a situation could be in the outer oval without being in the inner oval, but it was impossible to be in the inner oval without also being in the outer oval.

At the end of class for all four sections, students individually provided open-ended responses to two problems in which they first computed covariance of two random variables and then were asked to determine whether those variables were independent and justify their answers. In the first example, the covariance was 0 so independence had to be determined using the definition of independence. In the second example, the covariance was not 0 so dependence was guaranteed. We will subsequently refer to these examples as the "In Class Converse" problem and the "In Class Contrapositive" problem, respectively.

We also collected data on a midterm exam and the final exam. The midterm contained a similar converse problem and the final exam a similar contrapositive problem. The midterm and final exam were given 12 days and 6 weeks after the in-class instruction, respectively and there was no significant intervention between assessments.

Analysis

The first and third authors first open coded responses to the two in-class problems. When coding the midterm and final exam problems, we built on and refined the coding schemes from the in-class problems since they were very similar to the in-class problem 1 and 2, respectively. At this stage we were attending to correctness as well as the type of justification provided. The second author then used the coding scheme to code all 4 problems and we refined the coding scheme to reach agreement on approximately 95% of responses across the 4 problems.

For ease of comparing groups by treatment and by exposure to prior logic instruction, we opted to complete a second round of coding to condense the number of codes for each type of problem. For the contrapositive problems, we considered whether the student used the fact that a covariance other than 0 implies variables are dependent (which we labeled "using the contrapositive"). For the converse problems, we asked whether students used the fact that covariance was 0 to (incorrectly) claim that the variables were independent (which we labeled "using the converse"). This resulted in a binary response for each student for each of the 4 problems that we then used in the comparisons discussed below.

Results

Prequiz Truth Values

The truth values that students assigned after being presented with the theorem but prior to instruction were largely in line with conventional logic. Of the 72 participants, 59 of them (82%) stated the inverse was false and (though not the exact same group of students) 59 also claimed the converse was false. 64 students (89%) selected true for the contrapositive statement. 49 students (68%) gave conventional truth values for all three statements. Prior logic training did not appear to impact responses; 68.4% of students who did not have credit for Discrete Math gave conventional truth values for all three statements while 67.9% of those with credit for Discrete Math did the same.

Types of Responses

For the in-class and midterm converse problems, we focused on whether students used the fact that covariance was 0 to (incorrectly) claim that the variables were independent (which we will subsequently refer to as "assuming the converse"). 38% of students used this argument on the in-class converse problem and 31% used it on the midterm. Only 25% used the definition of independence on the in-class problem to correctly conclude that X and Y were dependent. However, the response of 17% of students in class and 10% of students on the final exam was that independence could not be determined since the covariance was 0. These students made a conventional logical inference, although they failed to recognize that independence can be determined directly through use of the definition of independence. This highlights another challenge of introducing and asking students to use this conditional statement; students may focus on the application of the theorem and neglect viable and necessary alternatives. We saw improvement in using the definition of independence on the midterm; 40 of the students (56%) used the definition of independence on the midterm problem to reach the correct conclusion that X and Y were dependent. The remaining students used incomplete or incorrect arguments, some of which referenced the definition of independence and some which did not.

On the in-class contrapositive problem, 90% of students used the covariance being a value other than 0 to (correctly) claim that X and Y were dependent. In contrast, only 44% of students used this argument on the final exam problem. On the final exam problem, 33% of students "bypassed" the theorem and used the definition of independence correctly. This appears to coincide with the increase in students using the definition of independence correctly on the midterm converse problem. 13% of students provided multiple justifications; specifically, they stated that X and Y must be dependent because the covariance is not 0 and also used the definition (either by stating $P_X(x)P_Y(y) \neq P_{X,Y}(x,y)$ for all x, y or by providing an example.) We grouped these students with those who did not use the contrapositive in our analysis below.

From Covariance to Independence

We might expect students who claimed the converse was true on the prequiz to be more likely to use the covariance of 0 as justification to claim independence on the in-class converse problem. (We highlight the cells in Table 1 below where responses on the prequiz and in-class problem are consistent from our perspective.) However, students who said the converse was false on the prequiz were actually slightly more likely to use covariance being 0 as justification for independence in the example problem than students who had said the converse was true in the prequiz.

Table 1. Converse Prequiz Truth Value versus In Class Problem Approach

Prequiz Response	Used Converse	Did Not Use Converse	TOTAL
Converse: False	22	37	59
Converse: True	5	8	13
TOTAL	27	45	72

A smaller percentage of Treatment 2 students assumed the converse on the In Class Problem than Control 2 students. In contrast, a smaller percentage of Control 1 students assumed the converse than Treatment 1 students. However, these differences were not statistically significant. We built a binary logistic regression model for the response on the In-Class Converse problem for each instructor using Previous GPA, prior logic experience, and treatment as independent variables. For each instructor, p-values for all three variables were each at least 0.1. Trends on the midterm Converse problem closely resembled those on the in-class problem except that on the midterm, treatment for Instructor 1 was a marginally significant predictor ($p = .042$) with Control students less likely to assume the converse.

There was greater consistency between the truth value students selected on the prequiz and the approach to the in-class problem for the contrapositive than for the converse (see Table 2). Specifically, 58 of 72 students claimed the contrapositive was true and used covariance being nonzero to decide X and Y were dependent. However, of the eight students who declared the contrapositive was false on the prequiz, seven of them used covariance being nonzero as their sole justification for the variables being dependent. Furthermore, six students claimed the contrapositive was true but did not use the covariance value in determining independence.

Table 2. Contrapositive Prequiz Truth Value versus In Class Problem Approach

Prequiz Response	Used Contrapositive	Did not Use Contrapositive	TOTAL
Contrapositive: False	7	1	8
Contrapositive: True	58	6	64
TOTAL	65	7	72

Treatment 2 students were more likely than Control 2 students and Control 1 students more likely than Treatment 1 students to use the contrapositive on the In Class Problem. On the final exam problem, Treatment 2 and Control 2 students performed comparably and Control 1 students again used the contrapositive more than Treatment 1 students. These results were not significant when using binary logistic regression and neither GPA nor prior logic training were statistically significant predictors for whether students used the contrapositive.

Discussion and Limitations

Students in this study were more likely to provide conventional truth values for variants of a conditional statement than they were to apply those only when appropriate to determine whether two random variables are independent. Students who provided conventional truth values were not more likely than others to use the contrapositive or avoid using the converse. This may partially explain why prior logic training was not a statistically significant factor impacting problem response. We posit that many students are able to identify contrapositive and converse statements and have learned conventional truth values for the same, but do not hold the same meanings for “true” and “false” as those in the mathematics community. This may help explain why one of the control groups outperformed the treatment group. Each treatment focused on helping students reach agreement with the conventional truth values of the contrapositive and converse statements. Yet, our data suggests that this is not the primary issue. The examples used in instruction with only the control sections accomplished something the treatments did not; they made explicit that the theorem could not be used to solve all problems and that alternative methods to calculating covariance or determining independence are available.

We saw a marked increase in using the definition of independence from in-class to midterm/final exam on both converse and contrapositive problems. It is natural as an instructor to view this shift positively for the converse problems and somewhat negatively for the contrapositive problems. It may be that some students did not have confidence in their ability to apply the theorem correctly during exams and therefore decided to always calculate covariance and determine independence directly. Alternatively, it is possible students simply didn’t think to use the theorem. Students also may have been less likely to use the definition of independence on the in-class problems because of availability bias (Tversky & Kahneman, 1973); they had discussed the theorem more recently than the definition of independence in class, but between class and the midterm and final exam had more time to review. Other explanations are possible of course and this is one question that should be explored in the future with student interviews.

A primary limitation in this study is the single source of data (student written responses) and our inability to perfectly dichotomize students by their understanding of the conditional statement and its implications. For example, a student who chooses to use the definition of independence when the covariance is 0 does not necessarily believe the converse is false. The student may, in fact, answer this way with little to no knowledge of the theorem or logic principles. As another example, some students provided multiple justifications to one of the problems and we do not know why. They could have been simply checking or confirming their work with a second method. Alternatively, students who use the contrapositive correctly may simply be reiterating a rule they’ve been taught but do not necessarily believe it is always true. Future studies may follow up these types of responses with interviews and slightly modified problems so that students can be more carefully classified. We are particularly interested in the 22 students who claimed the converse was false but then used covariance being 0 as justification for X and Y being independent on the in-class converse problem.

Another limitation is using credit for Discrete Math as a proxy for whether the student had received prior instruction in logic. We cannot guarantee that some of the students who did not have credit for Discrete Math had not taken other courses in logic. Moreover, we did not capture performance in Discrete Math but only whether credit had been achieved.

Finally, the instruction was limited to one 50 minute class period, not by choice but because of the restrictions imposed by the course schedule. This once more highlights the common challenge of how to best prepare students who are not math majors for courses that rely on an understanding of logical implication but do not have time allotted for explicit logic instruction.

References

Attridge, N., Aberdein, A., & Inglis, M. (2016). Does studying logic improve logical reasoning? 2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 27–34. Szeged, Hungary: PME

Austin, H. W. (1984). An assessment of mathematical implication in college students. students. *International Journal of Mathematical Education in Science and Technology*, *15*(3), 327-333.

Cardella, M., & Atman, C. (2005, June). Engineering students' mathematical problem solving strategies in capstone projects. In *2005 Annual Conference* (pp. 10-559).

Case, Joshua, "Calculus Students' Understanding of Logical Implication and its Relationship to their Understanding of Calculus Theorems" (2015). Electronic Theses and Dissertations. Paper 2254.

Casella, G. & Berger, R. L. (2002). *Statistical inference* (2nd ed.). Duxbury.

Cheng, P. W., Holyoak, K. J., Nisbett, R. E. & Oliver, L. M. (1986). Pragmatic versus syntactic approaches to training deductive reasoning. Cognitive Psychology, 18, 293-328.

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 496-522.

Dawkins, P. C., & Cook, J. P. (2017). Guiding reinvention of conventional tools of mathematical logic: students' reasoning about mathematical disjunctions. *Educational Studies in Mathematics*, *94*(3), 241-256.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in mathematics*, *53*(1), 5-34.

Durst, S., & Kaschner, S. R. (2020). Logical Misconceptions with Conditional Statements in Early Undergraduate Mathematics. *Primus*, *30*(4), 367-378.

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886-899.

Gainsburg, J. (2006). The mathematical modeling of structural engineers. *Mathematical Thinking and Learning*, *8*(1), 3-36.

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*(3), 193-223.

Hub, A. & Dawkins, P. C. (2017). A case-study in constructing set-based meanings for conditional truth. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA.

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *The Journal of Mathematical Behavior*, *50*, 90-102.

Inglis, M., & Simpson, A. (2008). Conditional inference and advanced mathematical study. *Educational Studies in Mathematics*, *67*(3), 187-204

O'Brien, T. C., & Shapiro, B. J. (1968). The development of logical thinking in children. *American Educational Research Journal*, *5*(4), 531-542.

O'Brien, T. C., Shapiro, B. J., & Reali, N. C. (1971). Logical thinking-language and context. *Educational studies in mathematics*, 201-219.

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for

quantified variables in complex mathematical statements. *The Journal of Mathematical Behavior*, *61*, 100802.
Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*(1), 133-162.
Tversky, A., & Kahneman, D. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive Psychology*, *5*(2), 207-232.
Wason, P. C. (1968). Reasoning about a rule. Quarterly Journal of Experimental Psychology, 20(3), 273–281.