Variation in Students' Thinking about Sameness and Comparisons to Mathematicians

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Sameness is a topic threaded throughout all levels of mathematics and yet it has not received much focus from the mathematics education research community. Our work categorizes the dimensions of variation that students highlighted when asked to describe sameness using a recent framework on sameness in mathematics. We captured students' ideas about sameness through a pair of open response surveys at multiple doctoral-granting institutions across the United States. In this presentation, we report on the variation found in the student surveys and also compare our findings to a prior study of surveyed mathematicians.

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Introduction and Background

The concept of equivalence is fundamental in mathematics (Asghari, 2009; 2019). Students are introduced early in their schooling to a specific type of equivalence, mathematical equality, and continue to use equality throughout their mathematics courses (e.g., National Governors Association Center for Best Practices, 2010). Despite their familiarity with the concept, students do not always develop a robust conceptual understanding of equality, which can hinder their ability to engage with middle school algebra topics like solving equations (e.g., Alibali et al., 2007; Kieran, 1981). In undergraduate mathematics courses, students encounter more complex ideas of equivalence like the notion of isomorphism in abstract algebra. Studies have shown that students, professors, and researchers all associate isomorphism and sameness (e.g., Dubinsky et al., 1994; Leron et al., 1995; Rupnow, 2019; 2021; Weber & Alcock, 2004).

Early work on students' understanding of mathematical sameness focused on alternative interpretations of operations and objects from those of the researchers (Melhuish & Czocher, 2020). Still, little is known about how undergraduate and graduate students understand sameness. Recent studies focused on mathematicians' understandings of sameness have categorized notions of sameness according to salient features and examined norms while interpreting sameness (Rupnow et al., 2022; Rupnow & Sassman, 2022). While mathematicians are able to recognize and distinguish between types of sameness, it remains unknown whether students do so in similar ways or what aspects they attend to.

Furthermore, math education research has benefited from comparing experts' understandings to those of students, such as in the context of example usage (Lockwood et al., 2016; Lynch & Lockwood, 2019) and proof production (Weber & Alcock, 2004). These types of studies provide opportunities to leverage mathematicians' expertise to find potentially productive connections and ways of thinking to teach students. In this paper we examine the ways in which mathematics undergraduate and graduate students describe the notion of sameness and how these compare to mathematicians.

Theoretical Perspective

In this study, we employed the notion of example space (e.g., Watson & Mason, 2005) to characterize how undergraduate and graduate students view mathematical sameness. Example spaces are the collections of examples that are associated with a particular concept and include the ways of making such examples. The structure of an example space can be described in terms of dimensions of possible variation, which refers to the aspects that are allowed to vary, and the range of variation of a particular topic, which involves the different options in the dimensions (Watson & Mason, 2005). In particular, we drew on the community example space created from a survey of mathematicians, largely algebraists (Rupnow et al., 2022), which characterized dimensions of variation of sameness used and range within those dimensions. Rupnow and colleagues (2022) highlighted five dimensions of variation: universes of discourse, concepts, objects, properties, and qualities. We adopted this framework to analyze our student data. In so doing, we provide a community example space for undergraduate and graduate students to highlight dimensions and range of variation of mathematical sameness. We also explore the differences and similarities between students' descriptions of sameness and those of mathematicians. In particular, we address the following research questions:

1. Which dimensions of variation did students highlight when describing mathematical sameness?
2. In what ways were the examples and characterizations of sameness provided by our students similar and different to those provided by algebraists?

Methods

Data Collection

We conducted a pair of open response surveys at doctoral-granting institutions across the United States. Each survey consisted of fifteen questions aimed to capture students' ideas about sameness. The first survey, Survey-A, invited actively registered abstract algebra students at three different institutions. A member of the research team visited each of the three courses to personally invite students to participate. During data collection, each of the courses offered at the selected institutions were cross-listed; that is, the courses were listed in the course catalogs as available to both undergraduate and graduate students. Therefore, the 15 students who provided responses to Survey-A were either undergraduate or master's students currently enrolled in abstract algebra. The follow up survey, Survey-G, was also conducted at three doctoral-granting institutions across the United States and invited any current master's or doctoral student in mathematics. For Survey-G, students were emailed an invitation through departmental lists of graduate students, there were 12 responses in total. Responses to both Survey-A and Survey-G serve as the dataset, consisting of 27 total student responses.

While both surveys asked how students think about sameness in general and in specific contexts, the wording of the questions was slightly different. The questions we chose as most relevant for analysis are as follows:

- (Q2) What does it mean to be the same in a math context? (both)
- (Q4) How do you know two things are the same in abstract algebra? Is this the same or different from other classes? (Survey A)
- (Q4a) How do you know two things are the same in abstract algebra? (Survey G)
- (Q4b) How is sameness in abstract algebra similar or different from sameness in other branches of mathematics? (Survey G)

Data Analysis

The research team, consisting of abstract algebra education researchers, analyzed the student responses. We used Rupnow and colleagues' (2022) sameness framework describing the various contexts, concepts, objects, properties, and qualities related to sameness. Each member individually coded all student responses to the survey questions provided above. Together, the team compared coding, discussed any discrepancies, and came to group consensus, resulting in total inter-coder agreement. To compare our findings to those of the mathematicians, we considered the differences in prevalence of each dimension.

Results

The results that follow report the universes of discourse, concepts, objects, properties, and qualities used by students when describing sameness. See Rupnow and colleagues (2022) for a detailed definition for each code found in Tables 1-5. Within each table, the frequency ordering of the mathematician paper is maintained to permit comparison of relative frequencies between the mathematicians and students. Following each table, we also provide illustrative examples of common student responses to better characterize the findings.

Table 1. Universe of discourse for sameness

Context[a]	Survey-A	Survey-G	% of total
Linear Algebra	0	4	15%
Topology	2	7	33%
Geometry	0	2	7%
Discrete Math	0	3	11%
Calculus[b]	-	-	-
Arithmetic/pre-university algebra[b]	-	-	-
Analysis	0	5	19%
Category Theory	0	3	11%
Other	0	2	7%
Set Theory	1	2	11%

[a]Some questions prompted consideration of or comparisons to abstract algebra so we do not report frequencies for abstract algebra, but we do discuss ways that discussion of abstract algebra was positioned below.
[b]Not present in graduate student responses

In Table 1, we focus on descriptions of each universe of discourse (context) for discussions about sameness. Among our student participants, topology was the most common context. Some responses broadly noted similarities and differences between abstract algebra and topology: "Another branch of mathematics where I think about sameness is topology. I think sameness in abstract algebra is both similar to and different from sameness in topology." Others linked contexts with particular concepts: "For instance, in topology, we have homeomorphisms that establish the 'sameness' of topological spaces."

Other responses made use of linear algebra and analysis contexts. For instance, a participant focused on co-existence in the same vector space as characterized by the span: "A vector v in a vector space identified with all of its nonzero scalar multiples. In other words, v is the 'same' as w if v = a.w[1] for some nonzero scalar a." Analysis contexts included oblique references to sameness of functions equal almost everywhere (a.e.): "In a more elaborate sense, one can argue that usually in algebra the homomorphisms are bijections whereas in analysis (measure theory) often time ignores sets that are very small."

In general, we note that the prevalence of each universe of discourse roughly mirrors the mathematicians, though our students did not allude to arithmetic or calculus to make points about

[1] We interpret v = a.w as the product **v** = a**w**.

sameness, nor did many use geometry examples. We further note that the preponderance of instances are grounded in areas of math outside abstract algebra and the variety in contexts was driven by the graduate students in the second survey.

Table 2. Concept conveying a type of sameness

Concept	Survey-A	Survey-G	% of total
Isomorphism	9	11	74%
Equality	6	6	44%
Other (e.g., categorical equivalence)	1	4	19%
Equivalence Relation	5	1	22%
Congruence	1	1	7%
Homomorphism	1	4	19%
Homeomorphism	2	6	30%
Diffeomorphism/Homotopy equivalence	0	1	4%
Similar (geometry)	0	1	4%
Linear transformation[a]	-	-	-

[a]Not present in graduate student responses

Table 2 highlights concepts conveying a type of sameness. Among our participants, isomorphism was the most common concept conveying sameness, followed by equality and homeomorphism. Discussions about isomorphism generally were positioned within abstract algebra and emphasized sameness of underlying structures: "We know two things are the same in abstract algebra when we can construct an isomorphism between the two structures. That is, the two structures may not look the same, but they have the same properties." While some references to equality occurred in isolation, many references were used to contrast with isomorphism:

> The meaning of "same" in a math context is a vague one. We can say two groups are the same if there exists an isomorphism between them, but the elements of the respective groups may be different. We may also say things are equal which is a different meaning than isomorphism but also implies a form of sameness.

While this participant's characterization of isomorphism is more specific than that of equality, we believe it is noteworthy that they used multiple concepts to characterize sameness. Similarly, in keeping with the large number of responses (especially in the second survey) positioning discussion in topology, we observed a number of instances of homeomorphism, especially in contrast with isomorphism. For instance:

> They are the same if there is an isomorphism between them. It may be an isomorphism of groups, rings or modules…The sameness in algebra means that the relationships between elements in one structure is the same as the relationships between the elements of the other structure with respect to the underlying operations. In other situations in math, this may not hold true, but two things might be considered the same if they share some property in common or if there is some special bijective function between them. For example, topological equivalence means there is a homeomorphism between the two sets. They can be considered the same in the sense that one can be molded into the other as opposed to algebraic sameness where the underlying mechanics of two sets are the same but they are labeled differently.

Specifically, while we recognize that questions asking about sameness in abstract algebra likely drove the high number of references to isomorphism, we were encouraged by the number of responses that characterized sameness using more than one concept (6 of 15 in the first survey, 11 of 12 in the second survey). We also observe that while there was a higher percentage of references to homeomorphism and fewer to congruence in this survey than the mathematicians, isomorphism and equality were the most commonly discussed in both settings.

Table 3. Objects to which sameness is applied

Object	Survey-A	Survey-G	% of total
Group	4	6	37%
Ring	2	5	26%
Other (e.g., categories, modules)	3	9	44%
Vector Space	0	4	15%
Set	2	8	37%
Number	1	2	11%
Function	0	2	7%
Field	0	1	4%
Quotients	1	0	4%
Elements	3	3	22%
Triangle	0	1	4%

Table 3 highlights objects that could be 'the same'. In addition to references to groups and rings (similar to the mathematicians), our student participants centered sets and elements as well as a variety of Other objects (e.g., categories, graphs, modules, manifolds). Respondents often listed multiple examples of structures and sometimes used these lists to contrast with sets:

> Most areas of mathematics have a notion of isomorphism between the objects of interest. Exact equality is almost always a much too strong condition to be of any significance when studying mathematical objects. In topology, homeomorphic spaces are looked at as "the same"; in graph theory there is a notion of isomorphic graphs; in analysis one usually studies spaces up to an isometric isomorphism; and even in category theory there is a notion of equivalent categories. This does contrast with set theory where in many cases one does care about exact equality of sets and not just a bijection between them.

Others used lists to contrast whole objects with elements, such as:

> In my experience in linear algebra, the "things" that you'd deal with are structures (groups, rings, fields, modules, ...) and their elements. The elements of the same structure are the same if and only if removing both of them is the same as removing one of them (sameness in the sense of set elements). Two structures are equal if they are isomorphic. This guarantees that they share the same algebraic properties. This will mean that sameness is [an] equivalen[ce] relation with equivalence classes.

We again observe, most of the variety in objects, both in the Other category and generally, were driven by responses to the second survey.

Table 4. Properties conveying sameness or a lack of sameness

Properties	Survey-A	Survey-G	% of total
Cardinality / Bijection	3	8	41%
Representation	1	4	19%
Other (e.g., commutativity, generators)	2	5	26%
Dimension	0	1	4%

Table 4 highlights properties of objects that should be shared for objects to be the same or that could vary and still allow the objects to be the same. Similar choices of properties arose in this survey as in the mathematicians' survey. There were numerous references to bijections, especially as they related to understanding concepts listed above, like isomorphism and homeomorphism. Other properties that students noted included orders of elements within a group, commutativity, and topological invariants. One participant highlighting multiple properties wrote:

> Two isomorphic groups that are essentially the same have similar properties (like the same order of the group or the same orders of its elements) and behave in similar ways with respect to how elements are operated together (e.g., there is a homomorphism between the groups, and the generators of one group correspond to the generators of another group).

Note this participant commented on cardinality (orders of groups) as well as Other properties of orders of elements and relationships between generators.

Table 5. Qualities of sameness

Qualities	Survey-A	Survey-G	% of total
Levels of Sameness	5	5	37%
Computability / Construction	2	2	15%
Logical equivalence	1	2	11%

Table 5 highlights qualities of sameness, namely ways to compare the types of sameness conveyed by specific concepts. For instance, the following response received a levels of sameness code:

> Two things are exactly the same if, and only if, all of their properties have the same value. We often discuss sameness without meaning, "exactly the same". Here, we reduce our scope to specific properties and check to see if two things have these specific properties in common. Then we call them the same in some sense. Other times, if two things have exceptionally close property values we might call them (knowing that it is strictly untrue) the same. So, in a math context, when one declares two things to be the same, one may mean they are exactly the same. See above. Or, one may mean only some of their properties are the same. This is almost always intuited, or explicitly given, in context.

This segment, which came at the end of an extended response in which five axioms and six definitions were provided to carefully characterize sameness, highlights a distinction between being "exactly the same" and having many properties be the same, which is often sufficient to be called the same in a particular context.

The computability/construction code and logical equivalence code captured ways of demonstrating sameness of object. Computability/construction focused on creating specific mappings (e.g., "We know two things are the same in abstract algebra when we can construct an isomorphism between the two structures."), whereas logical equivalence focused on the use or existence of theorems that can be used to draw the same conclusions, such as the following:

> Often times in mathematics some results appear to be equivalent to some other result although their proof might be completely different. For example, the axiom of choice is equivalent to Zorn's lemma, Tychonoff's theorem, the Well ordering principle, and a whole lot of results in different fields of mathematics.

Unlike the other dimensions, we saw similar frequencies in the descriptions of qualities from both surveys and these frequencies occurred in a similar ordering to that of the mathematicians.

Discussion

In this paper, we have observed many similarities between the students of our data set and previous work with mathematicians (Rupnow et al., 2022). All dimensions of sameness (contexts, concepts, objects, properties, and qualities) in the mathematician study arose in the student data. Many of the particular examples of each dimension, that together comprise the range of variation for each dimension, were also cited. When aggregating across our students, we see similarities to the mathematicians in the ordering of examples cited, including emphasis on isomorphism and equality as concepts conveying sameness, bijections/same cardinality being an important property for sameness, and nuance in qualities of sameness such as the existence of different levels of sameness. Some differences are explainable by the participants' backgrounds, such as frequent emphasis on abstract algebra and algebraic objects (groups, rings) among the mathematicians, who were largely algebraists. Whereas students, some of whom did research in applied math or analysis, gave more emphasis to analysis and other non-algebraic objects.

Examining the data in closer detail, however, it may not be fair to lump together our two student samples. The respondents to Survey-A were students in an abstract algebra course, which included both undergraduates and first-year graduate students in math education. The respondents to Survey-G were exclusively graduate students and included students farther into their programs. We also see some differences in the range of variation discussed. Much of the range in each dimension of variation, which mirrored the mathematicians, was driven by the Survey-G respondents. Except for the concept *equivalence relation* and object *quotient,* the same or more participants in the second sample provided each example within each dimension of variation, despite fewer participants. Thus in this sample, the Survey-G respondents discussed a greater range of topics per person than students responding to Survey-A.

While we recognize this is a small data set, we believe the differences between our two student samples spark questions warranting further research. In particular, it is possible that some formative experiences happen in early graduate school that lead graduate students to write and act more like mathematicians beyond simply taking more courses. Research might examine the impact of studying for preliminary/comprehensive exams on graduate students' perceptions of mathematics and abilities to make connections across their coursework. It is also possible that students who make stronger connections across courses are more likely to want to pursue graduate work in mathematics; thus, some weeding process may occur before graduate school. Other experiences or mechanisms may also be important to graduate students' development of nuanced views of sameness. While RUME is focused on undergraduates' experiences, we nevertheless encourage such work on the transition from undergraduate to graduate students to be included in the RUME community because of the potential impact on understanding what motivates undergraduates for higher study.

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