Characterizing a Teacher's Ways of Thinking about Teaching the Idea of Sine Function

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Over the past thirty years, researchers have developed constructs to investigate teachers' knowledge base and how it relates to their teaching. These constructs include Mathematical Knowledge for Teaching (MKT), Mathematical Meanings for Teaching (MMT), and Key Pedagogical Understandings (KPUs). In this report, I describe each of these constructs, their uses, and how they are related. I then describe a recently introduced construct, Ways of thinking about Teaching an Idea (WTTI) (Carlson, Bas-Ader, O'Bryan, & Rocha, in press) and illustrate the usefulness of this construct for investigating a teacher's mathematical understanding of sine function and how this understanding is related to the teacher's instructional goals and actions.

Keywords: Mathematical Knowledge for Teaching, Meanings, and Ways of Thinking

Introduction

Mathematics educators have asked the question, "what do teachers need to know to teach students?" for quite some time. In the late 1800s, debates about teachers' knowledge primarily focused on teachers' experience with the subject matter (Shulman, 1986). A century later, emphasis on teachers' subject matter knowledge lost traction as new policies concerning teachers' evaluation and testing were implemented. During the 1960s and 1970s, research programs shifted their focus to teachers' pedagogy with little regard for teachers' content knowledge. Shulman (1986) described the education community's disregard for teachers' content knowledge as the "missing paradigm problem" and responded by proposing that researchers "blend properly the two aspects of teachers' capacities" (p.5) and begin focusing on teachers' pedagogical content knowledge, a type of knowledge that "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p.6). Following Shulman (1986), teachers' knowledge in the context of teaching became a focus of mathematics education research. This area of research on teachers' knowledge is now known by many as *Mathematical Knowledge for Teaching (MKT)*(Thompson & Thompson, 1996).

Mathematical Knowledge for Teaching

In the last three decades, researchers have turned their attention to teachers' MKT to improve the quality of mathematics teaching and learning in the United States. While the literature on teachers' MKT is vast, various research groups' foci and ontological stances differ significantly.

A Practice-Based Theory of MKT

Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) sought to answer the question, "what do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball, Hill, & Bass, 2005, p. 17). Ball's work on teacher knowledge began in 1990 when she studied 252 pre-service teachers' understanding of division with fractions. In this study, most teacher candidates could correctly complete the procedures of dividing fractions. However, few were able to describe a situation that represented the division of fractions accurately to students (Ball, 1990).

Following Ball's (1990) findings of teachers limited mathematical understandings, Ball and Bass (2003) proposed focusing on the mathematical work of teachers, including how and where teachers use their mathematical knowledge in practice. Ball and Bass' analyses revealed that the mathematical knowledge needed for teaching differs from that of mathematicians and other practitioners of mathematics. Namely, mathematics teaching requires teachers to have knowledge of (1) the work involved in problem-solving, (2) how to "unpack" their mathematical understandings, and (3) how to help students in making connections across mathematical domains (ibid). These empirical analyses lead to Hill, Schilling, and Ball's (2004) proposal that teachers' MKT is multidimensional, and Shulman's (1986) original category of subject matter knowledge be subdivided to include teachers' *common content knowledge (CCK)* and *specialized content knowledge (SCK*).

In the past twenty years, these researchers have explored the multi-dimensional nature of teachers' knowledge when teaching and classified teachers' MKT into five types of specialized knowledge (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005). These researchers have also identified positive links between teacher knowledge and student performance (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004, Hill et al., 2008).

A Piagetian Theory of MKT

While Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2003; Hill, Rowan, & Ball, 2005) primarily focused on identifying how teachers must know mathematics to teach it, Silverman and Thompson (2008) focused on the nature of teachers' MKT and how it develops. Silverman and Thompson (2008) made two propositions, (1) teachers' MKT is grounded in personally powerful understandings of mathematical concepts, and (2) teachers' MKT is created through the transformation of those concepts from an understanding having pedagogical potential to an understanding that has pedagogical power.

Mathematical understandings are personally powerful if they carry throughout an instructional sequence, are foundational for learning other ideas, and play into a network of ideas that do significant work in students' reasoning (Thompson, 2008). According to Silverman and Thompson (2008), *Key Developmental Understandings (KDUs)* (Simon, 2006) are an example of mathematical understandings that are personally powerful. A KDU is an individual's understanding of an idea that is critical to their learning of other related ideas (Simon, 2006). We say an individual has developed a KDU for a mathematical idea if they (1) make a conceptual advance or have "a change in [their] ability to think about and/or perceive a particular mathematical relationship" (Simon, 2006, p. 362) and (2) develop an understanding for the idea through their own activity and reflection on their activity.

Silverman and Thompson (2008) propose that "developing MKT involves transforming [personal] KDUs of a particular mathematical concept to an understanding of: (1) how this KDU could empower their students' learning of related ideas; (2) actions a teacher might take to support students' development of it and reasons why those actions might work" (p. 502). More concretely, the development of MKT involves transforming teachers' KDUs into *Key Pedagogical Understandings (KPUs).* A KPU is a mini theory that a teacher has regarding how to help students develop the meanings she intends (Byerley & Thompson, 2017). As such, these researchers developed a framework to describe the process by which teachers' MKT-their KDUs and awareness of their KDUs- develop.

Comparing Research on Teachers' MKT

There are two distinguishing characteristics in research programs focused on teachers' MKT. The first involves the focus of each program. Ball and Colleagues' investigations primarily focused on what teachers do in their teaching and the knowledge required to engage in these teaching actions (Ball & Bass, 2003). In contrast, Silverman and Thompson (2008) focus on the cognitive mechanisms that enable teachers' behaviors. These researchers seek to uncover the nature of teachers' knowledge and how it develops. According to Silverman and Thompson (2008), the second distinguishing feature of MKT research involves their ontological stance that an individual's knowledge is constructed and is idiosyncratic. Ball and colleagues (e.g., Ball 1990, Ball & Bass, 2003, Ball, Hill, & Bass, 2005) describe teachers' knowledge in the sense of a truth about a reality external to the knower. In comparison, Silverman and Thompson describe teacher knowledge in the sense of Piaget and von Glasersfeld (1995)- as schemes and ways of coordinating them that explain how a person might act when teaching. Moreover, Silverman and Thompson describe a teacher's knowledge base in terms of her individual cognitive structures and thought patterns. Silverman and Thompson use the phrase *Mathematical Knowledge for Teaching (MKT)* to describe teachers' schemes or meanings for the ideas they teach and hold at a reflected level.

Mathematical Meanings for Teaching

In 2016, Thompson proposed using the construct *Mathematical Meanings for Teaching* (MMT) rather than MKT to make explicit that he was using the word *knowledge* to describe an individual's schemes or meanings for an idea. An individual's meaning is the space of implications resulting from assimilation to a scheme (Thompson, 2016). Thus, to say an individual has a *meaning* for a word, symbol, expression, or statement means that the individual has assimilated that word, symbol, expression, or statement to a scheme. A scheme is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). When defining *Mathematical Meanings for Teaching*, Thompson (2016) extended the construct *mathematical meaning* to account for characterizations of teachers' actions related to teaching. As such, a teacher's mathematical meanings for teaching an idea include (1) the meanings (schemes) the teacher uses while teaching or thinking about teaching, and (2) the teacher's image of the meanings (schemes) they want students to develop for the idea.

The construct *Mathematical Meanings for Teaching* differs from the construct *Mathematical Knowledge for Teaching* in two distinct ways. First, Thompson's use of the word *meaning* connotes something personal (i.e., meanings existing in the mind of the knower) to readers rather than *knowledge*, which seems less personal and disjoint from the knower (Thompson, 2016). Second, and arguably most important, the construct *Mathematical Meanings for Teaching* is used to describe the *status* of an individual's understandings relative to teaching. Although Ball and Thompson defined teacher's MKT differently, both researchers used the construct MKT to describe teacher knowledge as a target for teachers to *achieve*, not a *state* of teacher's knowledge. For instance, Ball, Hill, and Bass (2005) use the example of multiplying two integers to demonstrate that a teacher's sole ability to compute 35 x 25 correctly is insufficient knowledge. These researchers argue instead that teaching involves knowledge of an effective way to represent the meaning of the algorithm to multiply. In this example, Ball, Hill, and Bass (2005) state what teachers must be able to do and know, but they do not make explicit the ways of thinking that enable the teacher to do or know. As a second example, Silverman and Thompson (2008) outline a framework for developing teachers' knowledge that supports conceptual teaching of a particular mathematical topic. This framework frames teachers' MKT as something that must be attained as opposed to a status of teachers' knowledge that may be advanced.

The Affordances of Attending to Teachers' MMT

Attending to teachers' mathematical meanings for teaching an idea has many affordances. Investigations into teachers' meanings enable researchers to focus their attention on the mathematical conceptions teachers have and the implications of these understandings on students' learning. In particular, focusing on teachers' MMT enables researchers to shift their focus from the behaviors that a teacher exhibits while teaching to the cognitive mechanisms, meanings, and ways of thinking that enable these behaviors. This shift in focus to teachers' meanings allows mathematics educators to explain why teachers act as they do and how we can improve their teaching (Thompson, 2013; 2016).

As one example, Thompson (2013) described his observation of a ninth-grade teacher's lesson on the point-slope and point-point formulas. In this example, the teacher can write an equation of a line through a point when provided the slope. However, when provided two points and asked to write an equation of a line through the two points, the teacher is unable to do so. Thompson's (2013) efforts to make sense of and characterize the instructor's meaning for constant rate of change led to him uncovering that the instructor's difficulty resided in her schemes for variation, slope, division, and rate of change. As one example, the instructor expressed a meaning for slope as "rise over run" where rise and run represented "two things changing in chunks" (Thompson, 2013, p. 81). With this way of thinking, the instructor could not determine the y-intercept as her "chunk" did not place her at *x=0*.

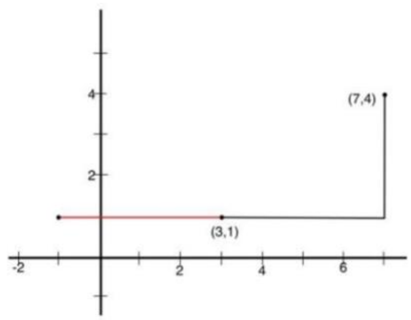

Figure 1. An example provided in Thompson (2013, p. 81)

By investigating instructors' meanings for ideas, we, as mathematics educators, are positioned to enhance our understanding of teachers' practice and develop ways to improve it.

While few mathematics educators have investigated teachers' MMT, the results of the few are alarming. Thompson and colleagues (e.g., Thompson, 2016; Byerly & Thompson, 2017; Yoon & Thompson, 2020; Thompson & Milner, 2018) have repeatedly uncovered the unproductive nature of U.S. mathematics teachers' mathematical meanings for teaching. Yoon and Thompson (2020) have also shown that U.S. teachers' meanings for foundational mathematical ideas are less productive than their South Korean counterparts. The South Korean teachers' conveyance of productive meanings implies that it is reasonable to expect U.S. teachers to be able to construct and convey mathematical meanings in this way (ibid).

One approach for improving US students' mathematical understandings is to support US teachers in advancing their mathematical meanings. Musgrave and Carlson (2017) have demonstrated that teachers' meanings for a foundational mathematical idea can become more productive if these teachers are engaged in interventions designed to support teachers' development of coherent mathematical meanings. "A focus on MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct)" (Thompson, 2013, p. 82).

The Need for a New Construct

Although attention to teachers' mathematical meanings for teaching an idea is needed to support teachers' development of coherent mathematical meanings; researchers who have investigated teachers' MMT have claimed that teachers' development of strong mathematical meanings does not necessarily lead to their conveying these meanings to students (Carlson, O'Bryan, & Rocha, 2022; Rocha, 2021; Tallman & Frank, 2018; Tallman, 2021). Instead, scholars have proposed that teachers' *ways of thinking* are critical to their ability to develop, advance, and convey coherent meanings to students (Carlson, O'Bryan, & Rocha, 2022; Carlson et al., in press; Musgrave & Carlson, 2017; O'Bryan & Carlson, 2016; Rocha & Carlson, 2019; Rocha, 2022; Tallman, 2015; Tallman & Frank, 2018). As one example, these researchers have proposed that quantitative reasoning is a critical *way of thinking* that supports teachers' in (1) developing coherent mathematical meanings, (2) conveying coherent and consistent meanings to students, and (3) designing activities that create opportunities for students to engage in quantitative reasoning (ibid).

The nature of a teacher's decentering actions has also been identified as a mechanism for advancing teachers' MMT and their ability to convey coherent meanings to students while teaching (Carlson & Bas Ader, 2021; Carlson et al., in press; Rocha & Carlson, 2019; Teuscher, Moore, & Carlson, 2016). In particular, researchers have argued that decentering or setting aside one's thinking to understand what students understand (Steffe & Thompson, 2000) is useful for explaining and characterizing teachers' responses and interactions with students while teaching (Carlson & Bas Ader, 2021; Carlson et al., in press; Rocha & Carlson, 2019; Tallman & Frank, 2018; Teuscher et al., 2016). As one example, Teuscher et al. (2016) propose that a teachers' ability to decenter is related to the nature of their interactions with students and the development of their mathematical meanings for teaching. Teuscher et al's., (2016) empirical study revealed that teachers' explanations advanced to become more conceptually oriented as the instructors made their speaking and listening to others a specific object of focus and reflection. In 2019, Rocha and Carlson corroborated Teuscher et al.'s (2016) proposal by providing empirical evidence of the reflexive relationship between teachers' meanings and their decentering actions. Rocha and Carlson's' (2019) analysis of an instructor's teaching revealed that the teacher's actions to decenter advanced her MMT by providing the teacher with more refined images of a student's ways of understanding the sine function.

It is relevant then to describe mechanisms for advancing teachers' MMT in ways that support their development of ways of thinking and decentering actions that support the instructor in developing images of students' thinking about an idea and leveraging those images while teaching. Carlson et al. (in press) have classified images of teaching that include ways of thinking about students' ways of learning an idea as a *way of thinking about teaching that idea*. In the next section I illustrate the alignment of these images with an instructor's instructional goals and actions.

Ways of Thinking about Teaching an Idea

The construct *Ways of Thinking about Teaching an Idea (WTTI)* was introduced by Carlson et al. (in press). These researchers leveraged Harel's construct of *way of thinking* to describe a "habitual form of reasoning that governs the application of a variety of specific mathematical schemes" (Thompson et al., 2014, p. 12). A teacher's *Way of Thinking about Teaching an Idea* (WTTI) includes:

1. How the teacher reasons about and understands the idea (the teacher's MMT which includes the ways of thinking the teacher engages in when reasoning about the idea) (Thompson, 2016).
2. The teachers' images (second order models) of students' thinking about and learning that idea (Steffe et al., 1983; Thompson, 2000).
3. The teachers' image of ways of thinking students may engage in to develop and refine their understanding of an idea (ways of learning an idea) (Carlson et al., in press; Silverman & Thompson, 2008).

Carlson et al., (in press) claim that a teacher's WTTI becomes more connected and refined as the teacher engages with students, while attempting to model students' thinking. Repeated efforts to do so typically results in refinements of a teacher's MMT and their images of students' way of learning an idea. The construct WTTI is useful for describing the relationship between teachers' MMT, their ways of thinking about learning an idea, and their actions while teaching. Researchers have demonstrated links between a teacher's MMT and how a teacher interprets and responds to students' questions (decenter) (Carlson et al., in press; Rocha & Carlson, 2019; Tallman, 2015, 2021). Based on these claims of the symbiotic relationship between a teacher's MMT and their decentering actions (Carlson et al., in press; Rocha & Carlson, 2019; Tallman, 2015, 2021), there is a need for research to investigate teachers' decentering actions and the impact of these actions on advancing teachers' MMT and ways of thinking about learning and teaching an idea.

An Example: A Way of Thinking about Teaching Sine Function

A teacher's WTTI is a status of their current meanings and ways of thinking about teaching and learning. As such, a teacher's WTTI can be more or less productive for teaching. Below I present an example of a precalculus teacher's current ways of thinking about teaching sine function. The data presented comes from a clinical interview with an instructor, Uri, who (1) was in his second semester of teaching pre-calculus using a research-based curriculum, and (2) regularly attended a weekly professional development seminar designed to support instructors in engaging in conceptually oriented teaching (Thompson & Thompson, 1996).

During the clinical interview, Uri was asked to describe how he wants students to reason about what the values in the table below (see Figure 2) represent. Uri responded by stating that he wanted students to understand that "sine is a function that relates an angle measure with the height [of the terminal point] above the horizontal diameter". Uri further conveyed that he wanted students to view the value of $\sin(\theta)$ as a means of representing the terminal point's height measured in units of the circle's radius. For example (see Figure 2), when prompted to explain the meaning of the point (337° , -0.391) Uri expressed that "the table of values is a snapshot, like if we were to pause an animation, at these instances in angle measure, then this [-0.391] would be the corresponding height above the vertical diameter in radius lengths."

As Uri described the meanings for $\sin(\theta)$ he wanted students to develop, he also conveyed that he wanted students to view sine as representing how the terminal point's height above the horizontal diameter varies with the measure of an angle. When Uri was asked if his prior teaching of sine function has informed his preparation for teaching the same topic during the current semester, Uri conveyed that his previous students' conception of the value of sine as a multiplicative comparison between the terminal point's vertical distance and the circle's radius supported his students' later learning of right triangle trigonometry. In particular, Uri expressed that his former students' conception of the output of the sine function as a multiplicative comparison of two quantities' values supported them in recognizing where the commonly applied trig ratios (SOHCAHTOA) come from.

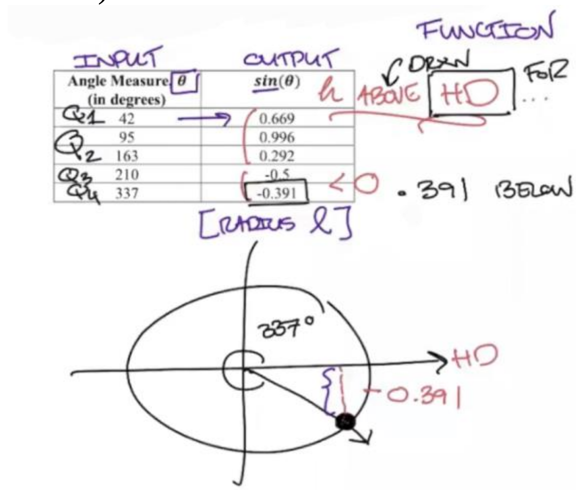

Figure 2. Uri's Description of Sine Function

Uri's description of the meanings he hopes to support students in developing for sine function provides a window into Uri's way of thinking about teaching sine function. In this example, Uri expressed (1) a meaning for sine function that he intended to support students in developing, (2) ways of thinking (quantitative and covariational reasoning) that he anticipated would support students' development of this understanding, and (3) images of student thinking and ways of learning that appeared to influence his instructional goals and actions.

Discussion and Conclusion

In the past thirty years, researchers have used the constructs Mathematical Knowledge for Teaching and Mathematical Meanings for Teaching to describe teachers' knowledge base while teaching. While research on teachers' knowledge base is vast, scholars' use of these constructs varies greatly. The constructs MKT and MMT differ in the nature of the phenomena they describe. Researchers have used the construct MKT to describe a knowledge base as a target for teachers to achieve. In contrast, the construct MMT has been used to describe the current state of teachers' mathematical understandings relative to teaching. While many researchers have advocated for more attention to teachers' MMT, a sole focus on advancing teachers' MMT is insufficient for explaining a teachers' instructional planning and actions. Instead, a focus on a teacher's *ways of thinking about teaching an idea* (that includes teachers' MMT) may provide a more powerful lens for studying a teacher's instructional goals and teaching of a specific idea. This report provides an example of how a teacher's *ways of thinking about teaching* the idea of sine function are related to his instructional goals and actions. In particular, this report highlights the critical role a teacher's mathematical meanings for teaching sine function and ways of thinking about learning and teaching sine function have in the instructor's planning for teaching. The example provided in this report provides empirical support for Carlson et al.(in press) call for the design of professional development seminars focused on developing and advancing teachers' WTTI. Seminars with this focus may support instructors' enactment of teaching practices that focus on understanding and advancing student thinking and learning. As such, further exploration of the role of teachers' WTTI on their enacted teaching actions and mechanisms for developing and advancing teachers' WTTI are needed.

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