The Role of Ritual and Personalization in Students' Exploration of Subgroup Generation

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We share preliminary results of a commognitive analysis of the participation of twelve undergraduate students in an exploratory subgroup generation activity in an abstract algebra class. We find that while students' collaborative activity contains markers of both ritualized and explorative participation, their appeals to ritual and personalized formulations of inferences and insights about subgroups serve to broaden access to the conceptual substance of the activity. We discuss a possible asset-oriented interpretation of students' ritualized activity as an aid to exploration and reflective of the disciplinary activity of professional mathematicians.

Keywords: Discourse, Inquiry-Oriented Instruction, Abstract Algebra

Advanced mathematical courses such as abstract algebra provide a space where students are apprenticed into the norms and activity of mathematicians. These classroom spaces involve a complex interaction between student reasoning and activity and norms deriving from the professional mathematician community. Frequently, scholars have analyzed student activity in this setting from a deficit point of view, illustrating ways that students may engage in non-normative activity and fail to meet standards of communication associated with norms of formal proving (see Styliandies et al., 2017 for an overview).

To both illustrate this trend and contextualize our study, we use literature about students and subgroups. Subgroups are one of the fundamental topics in group theory (Melhuish, 2019) and have received considerable attention in literature. Some students have been shown to attend to subset properties but not operation. For example, students may suggest that "odds" are a subgroup of the integers (Brown et al., 1997; Titova, 2013). Similarly, several studies have shown students identify $\{0,1,2\}$ as a subgroup of $\mathbf{Z}_6$ (e.g., Dubinsky et al., 1994; Melhuish, 2018) because they unintentionally switch operation from addition mod 6 to addition mod 3. In contrast, Findell (2002) illustrated how a Cayley table could focus a student on the relevant operation, although generating subgroups from that table proved to be a significant challenge. There have also been some studies that pointed to specific issues with applying subgroup test requirements including production of proofs (Ioannou, 2018) or checking closure by only considering combinations of distinct elements (Hazzan, 1994). In general, we can state that the field has characterized a number of "errors" related to subgroups.

In this report, we attempt to re-examine student activity with subgroups through a more asset-based lens. We share analysis from one open-ended task designed to engage students in generating subgroups for a given, known group. We leverage the framework of Lavie, Steiner, and Sfard (2019) to characterize students' activity as exploratory or ritualistic, noting ways that students engaged productively with the task and identifying dilemmas students needed to resolve. We found that our students negotiated many issues reflective of the literature, but productively drew on personalization and ritual as they engaged in open-ended exploration.

Theoretical Framework

In this study we take a commognitive view of mathematics learning, in which thinking is conceptualized in terms of acts and processes of communication (Sfard, 2008). Central to the commognitive perspective is the notion of *routine*, a repetitive pattern of action that retains some invariant features across different performers and implementations (Lavie et al., 2019). Lavie and colleagues distinguish two types of routines: *rituals*, which are process-oriented and require performance of a rigid sequence of actions; and *explorations*, discursive routines guided by the performer's motivation to communicate an idea or pursue a line of inquiry. According to Lavie et al., students initially can only participate in mathematical activity in ritualized ways, because they must first familiarize themselves with the norms and meta-rules of disciplinary discourse. As they gain familiarity, they may begin to participate in "explorative" ways, generating new truths and pursuing inquiries borne of their own motivation and curiosity. Coles and Sinclair (2019) challenge the notion of ritualized activity as meaningless, arguing that participation in ritual can engender meaning for learners and is not mutually exclusive with exploration.

One specific discursive feature that has been associated with ritualized activity in mathematics is *personalization*, in which operations are presented as human actions on mediators rather than in terms of structural relations among mathematical objects (Ben-Yehuda et al., 2005). Personalization in mathematical talk has been associated with lower achievement in empirical studies of children's mathematical activity (Bills, 2002). Our study aims to challenge narratives of personalization as a marker of less sophisticated or less robust mathematical activity by exploring ways in which personalization can help students access algebraic concepts.

Guided by this theoretical framework, we address the research questions:

1. In what ways might collaborative student work on an open-ended task in an abstract algebra course resemble ritualized or explorative mathematical activity?
2. In what ways might ritual and personalization support students' explorative engagement in such a task?

Context and Method of Study

In this report, we analyze an activity on Lagrange's theorem developed by the Orchestrating Discussion Around Proof project which designed and field-tested abstract algebra lessons that engage students in authentic mathematical proof activity (Melhuish et al., 2022). The first activity in the task sequence involved students generating subgroups in order to anticipate the relationship between the order of a group and its subgroups. The participants in our study are twelve undergraduate mathematics majors who worked on the lesson in four teams (Table 1).

Table 1. Teams and group assignments for the subgroup generation activity.

Team	Group	Students	Team	Group	Students
1	$\mathbf{Z}_{10}$	Carlos, Romiax	**3**	D_4	Brad, Fernando, Mona
2	$\mathbf{Z}_{12}$	Diego, Jake, Paolo, Quinn	**4**	S_3	Alan, Eric, Nathan

Each team worked for 20 minutes to create a list of subgroups of their assigned group. In the previous class session, the instructor (Author 1) had demonstrated a possible approach for generating a subgroup of the group $\mathbf{Z}_8$: choose an element to place in a subset, and then use the closure and inverse properties to identify other elements that must be in the subset if it is a subgroup. Although he modeled how to identify elements that must be included in a subgroup, he did not demonstrate any systematic approach for generating all subgroups of a group.

We audio recorded each team's discussion and photographed all board work, then generated a transcript. To investigate the interplay between ritual and exploration in students' discussions, we analyzed all four team transcripts, noting any talk turns that contained instances of personalization or possible indicators of ritualized or explorative activity (see Table 2). For each instance we further described the problem-solving context, developing analytic descriptions of how personalization, exploration, and ritual supported the group's problem-solving process.

Table 2. Discursive indicators of personalization and ritualized and explorative activity.

	Indicator	Example
Personalization	Talk about human involvement with objects or symbols	"Yeah, we would need 1 and then we would need 11."
Ritualized activity	Description of subgroup generation routine as rigid algorithm with fixed, prescribed steps	"So you do have to add [an element to] itself?"
	Settling conceptual/procedural questions by appeal to authority or analogy with instructor-led example	"Last time over here, [Instructor] adds them, he adds the same element by themselves"
Explorative activity	Exploration of hypotheticals, as evidenced by "if-then" language	"If you compose [a transposition] by itself it just goes back to the identity?"
	Settling conceptual/procedural questions by definition or deduction	"Does 0 always have to be in here?" "Yes. Because it always has to contain the identity."

Preliminary Results

We found that each team's discussion contained some instances of personalization, though different teams used personalization in different ways. Each team's discussion contained some indicators of ritualized mathematical activity, but was largely reflective of explorative activity, with multiple members engaging together in exploration of hypotheticals. For this report, we present analyses of two cases chosen based on the contrast in their discursive features.

Team 1: A Progression from Ritual to Exploration

Team 1 first identified the singleton {0} and the entire group as subgroups of $\mathbf{Z}_{10}$. They then attempted to resolve the question of which operation (addition or multiplication) was the group operation by referring to the instructor's previously worked example. When this failed to clarify the issue, they asked the instructor, who specified that the operation was addition. We interpret this as pointing toward a ritualized form of participation; while it is possible to determine that $\mathbf{Z}_{10}$ is a group under addition but not under multiplication, the group considered the operation to be indeterminate without external specification ("Yeah, well, it could be any binary operation").

As they proceeded with subgroup generation, Team 1 used personalization to generate informal descriptions of subgroup properties, interpreting closure as follows: “So closed means whenever you’re adding, whenever you get another element that’s also in that set.” Throughout the activity, the team frequently referred to a generic “you” operating on elements of subgroups to generate additional required elements.

At one point, Team 1 grappled with whether one should add an element of a subgroup to itself in order to generate additional subgroup elements. Rather than consult the definition of subgroup, Romiax attempted to resolve the issue by referring to the $\mathbf{Z}_8$ example: “Last time over here, he adds them, he adds the same element by themselves.” Carlos appears to accept this as a warrant, then recalls an instance of this himself: “And then he did eight plus eight or something.” It is possible that for Team 1 the ritual of subgroup generation had not yet fully fused with the team’s understanding of the definition of subgroup. This is not to say that the team was inattentive to this definition; on the contrary, the team refers repeatedly to the properties that define a subgroup. We hypothesize that for this team, fluency in the application of the definition of subgroup was a work in progress, and the sense of a subgroup generation ritual or “game” (a descriptor used by the instructor and subsequently by some students) supported their exploratory efforts. After settling these procedural issues, Team 1 went on to fluently explore other subgroup possibilities, determining that the set of even elements of $\mathbf{Z}_{10}$ would form a subgroup, but deducing that the set of odd elements would not satisfy the closure property.

Team 4: Ritual Embedded in Explorative Activity

Team 4 began by identifying the trivial subgroup of the symmetric group S_3. Alan then asked, “Where do we start with the–?” Eric responded, “Yeah, just pick any couple of them and see what they generate.” Eric thus explicitly frames his envisioned approach to the activity as an exploration, where they “pick” group elements and “see” what subgroup they generate. This explorative framing is consistent throughout Eric’s portion of the transcript: “So if you include these ones, you need the whole group, is what we *discovered*.” (emphasis ours).

However, Team 4’s discussion is not without indicators of ritualized activity. At one point, Alan asks, “Dude, just a question, when we’re composing, do we have to do it the other way too?” This question implicitly frames composition as a part of a prescribed routine that must be performed in a specific way. Assuming Alan’s question is about whether closure requires one to be able to compose subgroup elements in different orders, this is a novel dilemma for the team: the instructor’s prior example involved an abelian group. The team avoids the dilemma by discovering that the two permutations they used as starter elements generate the entire group, at which point the question is dropped; this suggests that the team’s successful exploration supersedes the need to clarify procedural aspects of the subgroup generation “game.”

In terms of personalization, we observe that while both Team 1 and Team 4 talk about the subgroup generation routine in terms of human actors performing operations on elements, Team 4 uses personalized language more often to refer to elements they “need”: for example, “if we do this with all of them are we going to end up needing (1 3)? What if we did?” We hypothesize that the language of “needing” elements supports the team’s communication about elements that must be in a subgroup without the use of conditional statements.

Discussion and Implications

In all four discussions, we see moments when team members are uncertain or disagree about the rules of the subgroup “game.” We note two features in how students engage in the game.

Teams often attempted to resolve dilemmas by consulting a prior instantiation of the game, as one might when participating in a ritual whose rules are determined by an external authority. One might consider this antithetical to authentic mathematical activity: one powerful disciplinary norm in mathematics is that we resolve dilemmas by consulting foundational definitions and using deductive reasoning (see Dawkins & Weber, 2017). Second, we observed students engaging in personalization: rather than using abstract mathematical language, they introduce themselves or objects as acting agents. Again, this in some ways defies norms of valued activity in advanced mathematics. Further, literature on how students engage in abstract algebra content often suggests personalization as a deficit or lower level of reasoning (Hazzan, 1999). Others have suggested that using an "I" statement in reference to a quantified statement (in this case related to isomorphism) reflects the lowest level of development related to a quantified statement (Leron et al., 1995). If we were to use the standards of proof communication and disciplinary norms, we might view the student activity as lacking valued disciplinary qualities.

We wish to suggest an alternative, asset-based framing (cf., Adiredja, 2019) that interprets students' personalization and ritualized activity during an exploratory task not as evidence of a deficit with respect to acquisition of formal mathematics, but as resources that facilitate students' fluent participation in the task. We find that the students' personalization of the necessity of an element belonging to a subgroup due to the closure or inverse property (*e.g.*, "*You would need* at least the two [3-cycles] because they are inverses of each other"; emphasis ours) allows students to follow implications of hypotheticals without getting bogged down in the more demanding structure of conditional statements. Similarly, ritualization of the rules of the subgroup generation "game," and the opportunity to discern these rules from a worked example, offers students an alternative pathway to access the task even if, due to uncertainty about a definition or convention, they are unsure whether a certain "move" is permitted or required. This advances one of the goals of [*blinded project*]: to broaden access to mathematical activity beyond those students who quickly master the much more complex "game" of formal mathematics, whose rules are intricate and often difficult to decode. Further, we suggest that personalization and ritualized activity *are* reflective of disciplinary activity in meaningful ways. Weber and Melhuish (2022) argued that personalization is a key part of how mathematicians understand ideas highlighting metaphors and connection to their lives. Additionally, mathematicians also defer to outside sources of authority and have been shown to find arguments more convincing if written by respected authors (Inglis & Mejia-Ramos, 2009) and will defer to authoritative sources to assume validity (Weber & Mejia-Ramos, 2011). One of the main reasons mathematicians read proofs is to find ideas to transfer to their own research (Mejia-Ramos et al., 2012; Weber & Mejia-Ramos, 2011), which is also reflective of much of the students' ritualized activity.

Finally, we note that the instructor's choices of words in introducing the activity (*e.g.*, "game", action verbs such as "generate") likely had some influence on students' participation and discourse. We note that these word choices were reflected in student talk, and envision future work exploring ways in which careful orchestration of a task launch can invite students into exploration of open-ended problems with ritual playing a strategic and productive role.

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