Successes and Challenges in Supporting Calculus Students to Conceptualize and Express Distance on Graphs

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Conceptualizing and expressing distances on graphs of functions are central to understanding why integrals afford measuring areas and volumes in calculus. In this preliminary report, we share the results of implementing an intervention designed to support these skills with 31 Integral Calculus students. Initial results show that the activity did support some students in learning to accurately expressing the horizontal distance between two functions. However, a closer look at students' work revealed that the tasks may not have supported all of the ways of reasoning we intended. We discuss the instance of two students who successfully completed the activity yet displayed evidence of treating symbols as labels or parameters and following a pattern to complete the tasks rather than conceptualizing distance. We close with a discussion of how to improve the next iteration of the intervention.

Keywords: Graphical Representations, Calculus, Distance, Cartesian Coordinate System

One of the key results of Integral Calculus is the ability to find the area of curved regions and volumes of solids of revolution. In order for students to understand conceptually why the integral affords finding the value of such areas and volumes, students must be able to: 1) conceptualize distances within coordinate systems and 2) describe these distances with algebraic expressions from relevant functions. Prior research suggests that students, even at the undergraduate level, often struggle with one or both of these skills (David, 2019; Knuth, 2000; Parr, 2021), and may benefit from targeted interventions in these areas.

Previously, Parr et al. (2021) described the results of a small teaching experiment (Steffe & Thompson, 2000) that utilized a sequence of tasks designed to support students in conceptualizing and expressing distances algebraically, as part of a hypothetical learning trajectory (Simon & Tzur, 2004). Parr et al. (2021) reported that the tasks showed promise at supporting students and also clarified ways of reasoning that students may use when describing distances of segments. One of these interpretations is what Parr et al. (2021) refer to as a *composed magnitude interpretation* of distances on number lines. This interpretation involves conceptualizing a difference of the form $b–a$ (where $b \geq a$) as representing the distance in one-dimension between a and b. The interpretation of the difference $b–a$ is one of a *magnitude*, or amount of distance, within a graphical representation (Parr, 2021). The interpretation is *composed* as it includes the understanding that b and a themselves are magnitudes of distance. Thus, the difference $b–a$ yields the distance between a and b as a result of finding the difference between the distance between b and the origin, and the distance between a and the origin.

The current study is part of an ongoing effort to learn how to support students in conceptualizing and expressing distances for calculus. In this study, we investigate the effectiveness of implementing a revised set of these tasks in a classroom setting. The research questions this study seeks to answer are:
(1) *To what extent do these tasks support students in learning to conceptualize and express distances on graphs of functions in calculus in terms of the input and output variables?*
(2) *What are the affordances and limitations of these tasks in supporting students to develop a composed magnitude interpretation of distances?*

Methods

The data in this study were collected in Spring 2022 from 31 students enrolled in one of two sections of Integral Calculus taught by the lead author. The Integral Calculus course met three times a week and students participated in the study during two consecutive course meetings, ahead of the unit on Areas and Volumes. At this point in the course, students had been introduced to integrals through an accumulation model, without graphical applications. At the end of the first day of the study, students were given ten minutes to complete the pre-test item (Figure 1) individually. The next class meeting, students were instructed to work in groups of two or three to complete the *Interpreting Graphs for Calculus Activity* over the 50-minute meeting time, which included the same pre-test item as the final task. The instructor circulated groups answering clarification questions about the tasks as needed. Students recorded their group's activity via Zoom. Of the 31 students who consented to have their written work included in the study, 26 also consented to include their group's video recording in the data set.

Pre-Test Item & Interpreting Graphs for Calculus Activity

The pre-test item and the tasks comprising the *Interpreting Graphs for Calculus Activity* used in this study were a revised version of the tasks first described in Parr et al. (2021). The pre-test item (Figure 1) asked students to represent the length of the horizontal segment extending from a point labeled (x, y) on the function $y = \sqrt{x-1}$ to the vertical line $x = 2$ in terms of x, in terms of y, and to explain their reasoning for each response.

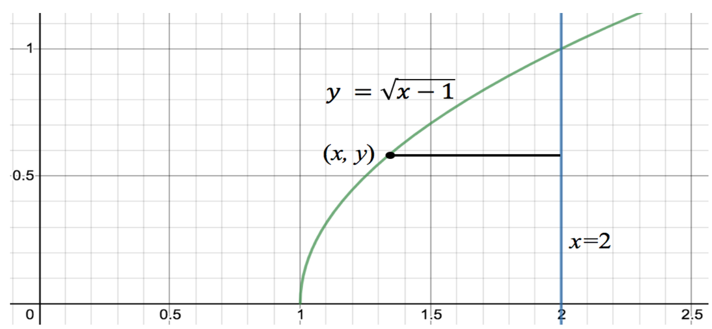
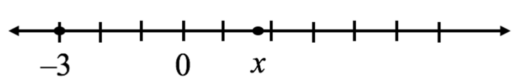


Figure 1. Pre-test item asking students to represent horizontal segment's length (black) in terms of x and y.

The three main goals of the *Interpreting Graphs for Calculus Activity* are to support students in (a) conceptualizing horizontal and vertical distances on number lines and the Cartesian plane, (b) representing these distances with difference expressions, and (c) using algebraic relationships among input and output variables to express distances flexibly in terms of either x or y. In this study, we focus on the first two goals and associated tasks. The first task asked students to use a ruler to mark specified distances from 0 on a provided number line, as well as distances between two values. The next set of tasks asked students to represent various difference expressions as segments, involving whole numbers, integers, rational numbers, and variables as distances on both horizontal (Task 2-Figure 2) and vertical number lines (Task 3).

h. Draw a segment to represent x– (–3) on the x-axis below.



i. Place another value of x on the number line above, and draw another segment representing x– (–3).

Figure 2. Portion of Task 2 from Interpreting Graphs for Calculus Activity.

Task 4 reversed this and provided students with horizontal or vertical segments, asking students to express the length of the given segment as a difference. Finally, Task 8, which we refer to as the post-test item, included the same items as the pre-test item to serve as a comparison for student responses before and after completing the activity.

Preliminary Data Analysis

To analyze our results, we first recorded students' responses to the pre-test and post-test items and developed basic categories for these responses, including correct responses, blank/I don't know, or other expressions. Based upon a student's response to the pre-test and post-test items, each research member selected 3-4 students who did not initially answer the pre-test item correctly and developed initial content logs and analytic memos of students' responses to the Interpreting Graphs for Calculus Activity and group videos of the students completing the activity. The goals of this initial analysis of students' work on the activity was to determine if we could identify shifts in the student's thinking about the tasks that may have supported or hindered them in responding to the post-test item correctly, especially related to their development of a composed magnitude interpretation of distances.

Results

The number of students who correctly represented the segment's length in terms of x and y increased significantly from the pre-test to the post-test item on the graphing activity (Table 1).

Table 1. Number of correct responses on the pre-test and post-test item.

Represent segment's length	Pre-Test	Post-Test
in terms of x: $2–x$	12/31 (38.7%)	24/31 (77.4%)
in terms of y: $2–(y^2+1)$	10/31 (32.3%)	20/31 (64.5%)
Both x and y correct	7/31 (22.5%)	20/31 (64.5%)

While the activity appears to be generally effective in its goal, our ongoing data analysis revealed that some students were limited in their use of variables to express lengths of segments, or may have avoided conceptualizing difference expressions as finding distances, even when they gave correct responses on the post-test items. (We note also that the post-test item was completed in groups and may not reflect each individual's understanding). These findings suggest that the activity may not promote some of the central ways of reasoning that we intended for students to develop and use. We illustrate these two issues using the case of two students who were partners for the in-class activity, Kevin and Amir. Kevin and Amir both incorrectly responded to the pre-test item but correctly expressed the horizontal distance in terms of x and y on the post-test item.

Symbols as Labels or Parameters

Both sets of responses from Kevin and Amir indicate that they tended to use symbols (such as x, y, and z) as either labels or parameters, rather than as variables to represent varying values. We found evidence of this tendency beginning with Amir's work on Task 1 in which he labeled segments with symbols x, y, and z that he introduced himself. This tendency was most apparent when it created some hesitation for Amir on Task 2i. Kevin asked Amir about this task and their discussion reveals the limitations of Amir's use of x. We provide their conversation below and Amir's work in Figure 3.

Amir: (*reading 2i*) Oh, so like place a value for x and just draw it, x equals 2…

Kevin: What is it asking, again?
Amir: Place another value of x, like 1, 2, 0, why did I put 0?... It doesn't make sense to place another value of x. I mean even if you put x over here, so x_0 equals x_1, it's the same thing.
Kevin: Ah (*leaning back*)
Amir: I mean, I like the zero thing.
Kevin: So, wait, what are you doing? (*looking at Amir's work*)
Amir: I'm saying x equals 0, I'm setting
Kevin: Okay
Amir: the segment from 0
Kevin: Yeah
Amir: I don't know.

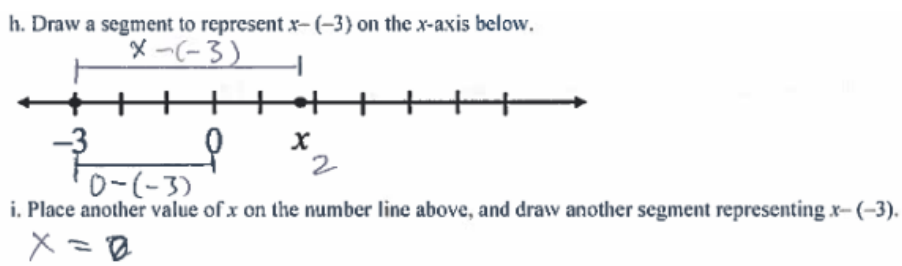


Figure 3. Amir's work, in which he wrote "x= 0," and then labeled "0–(–3)" on a segment below the number line.

Amir explained his interpretation of the directions to "place another value of x" as choosing a value for x, when he listed values "like 1, 2, 0." He goes as far to say that "it doesn't make sense to place another value of x" and continues to then describe different possible x's as "x_0" and "x_1." We infer from these comments that Amir is interpreting x as a parameter (Thompson & Carlson, 2017), in which x may take on different values in different settings, but may not take on different values in the same setting. This interpretation is distinct from interpreting x as a variable in which x may take on multiple varying values in the same context. Amir's interpretation of x as a parameter may help to explain why he thought that it didn't "make sense to place another value of x." Amir's interpretation of x as a parameter is also supported by his written work on this task, in which he sets x to 0, but then he labels the new segment as "0–(–3)" rather than "x–(–3)" as directed, not comfortable with using x to take on two different values in the same number line.

Following a Pattern for Expressing Distances

In addition to treating symbols as labels and parameters, Kevin and Amir also indicate that they may be simply following a pattern in order to express the indicated distances, rather than using a composed magnitude interpretation of distances. Specifically, this pattern involved using a minus sign because previous answers they found were also difference expressions. This issue came to light as Kevin suggested an alternative answer to 4c (Figure 4), a task which asked students to represent the distance between 5 and y on a vertical number line.

Amir: See directions matter. So now like, this above, 5 minus y and this is y minus
Kevin: Yeah. (*pause*) I guess you could say that it's also y plus 5.
Amir: y plus 5 (*slowly, thinking*). Well, yeah, you can say that, too. But because it's differences. Here it doesn't say differences, here it does.
Kevin: I'm just going to use minus for consistency.
Amir: Yeah

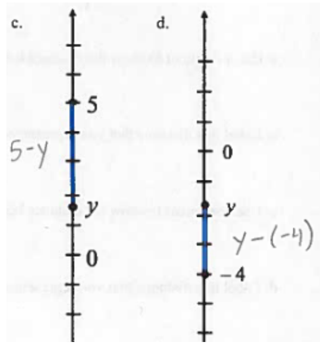


Figure 4. Kevin's work on Task 4c and 4d.

When working on Task 4c, Amir seemed to indicate that the ordering of the symbols mattered. In this case, whether 5 or *y* were on top indicated which was to be first in the expression. At this point, it is unclear whether Amir's distinction about the direction or order of the terms mattering was the result of a composed magnitude interpretation. However, Kevin's suggestion of "y+5" as an alternative expression, and Amir's agreement, indicates that they were not interpreting *y* as a distance from 0 to *y* and 5 as a distance between 0 and 5. Instead, they justify their decision to use the correct answer, 5–*y*, "for consistency," rather than with any reference to distances. This justification suggests that Kevin and Amir may have been learning to use a rule or pattern of "top minus bottom," since they could have just as readily used the expression y+5 to represent the distance between *y* and 5, and may have, if previous expressions had included sums rather than differences. We note that Kevin and Amir provided correct responses to the remaining tasks.

Discussion & Future Plans

At first glance, the *Interpreting Graphs for Calculus Activity* appears promising as an intervention to support students in expressing distances, based on the results of the pre-test and post-test items. However, as designed, they may be limited in the ways of reasoning they can promote. Kevin's and Amir's use of symbols as labels and parameters suggests that the tasks as designed may not effectively support students in conceiving of difference expressions involving symbols as representing varying values of distances. Kevin's and Amir's rule for using a minus sign in their expression to represent distances, to the avoidance of conceptualizing distance measurements given by the expression, may work in many contexts for which a "right minus left" or "top minus bottom" formula yields the correct answer. Yet, we anticipate this rule may not support students in expressing distances in coordinate systems beyond the Cartesian coordinate system. In our future work, we plan to conduct more teaching experiments with students with a next iteration of revised tasks designed to support students in conceptualizing distances more effectively. We hypothesize that it could benefit students to include more reflective prompts throughout the tasks, such as "what does this value represent?" after students calculate the length of each segment and to include some dynamic distances rather than only static distances for students to express with differences. We hope to further discuss with the audience ways to improve the task design to address the two issues we highlight in this study.

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