A Precalculus Student's Use of Slope in Finding Linear Equations and Linear Approximations

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This article reports mathematical understanding of one of the precalculus students who participated in a teaching experiment. The experiment was designed to help students strengthen the understanding of the concept of rate of change and linear equations to prepare them for the calculus concepts: tangent line equation and linear approximation. Before the experiment, the student understood slope mostly as an algebraic ratio and could not use the slope to determine the y-intercept of the line. With the intervention, she strengthened her knowledge of slope as both algebraic and geometric ratios and used her knowledge to construct linear equations. In addition, she was able to use slope to determine linear approximations of function values in various functions. The results suggest that when provided with appropriate tasks and prompts, precalculus students can be better prepared for their upcoming calculus concepts.

Keywords: Slope, Linear equation, Tangent line, Linear approximation

Do precalculus students adequately understand linear equations for their upcoming calculus courses? In my teaching experience, I found that most calculus students struggled to find the equations of tangent lines or linear approximations. I initially thought that their struggles stemmed from their insufficient skills or understandings related to derivatives. However, after realizing that many students with correct derivatives could not construct tangent line equations, I speculated that their struggles were also due to their lack of understanding of linear equations. So I conducted a survey with precalculus students on two conversion problems from graph to equation and confirmed that students indeed had difficulty constructing a line equation unless the *y*-intercept could be easily observed from graph. Thus, in order to help them develop understanding of slope and linear equations, I designed and researched a teaching experiment (Steffe & Thompson, 2000) with a research question: How do precalculus students understand linear equations and in what ways understanding linear equations helps them construct equations of tangent lines or approximate function values? In this report, I share my preliminary findings and seek for suggestions and inputs for the future direction of research.

Theoretical and Concept Background

Slope as Algebraic and Geometric Representations

Representations, especially algebraic and geometric, play a central role in mathematics curriculum and instruction. Yet many students do not understand mathematical concepts by connecting representations or have difficulty connecting representations in mathematical problem solving (Dufour-Janvier, 1987; Even, 1998). Among the concepts requiring the connections of representations, rate of change has attracted a lot of research (such as Carlson et al., 2002; Nagel et al., 2019; and Thompson & Carlson, 2017). I here describe the rate of change in linear equations—slope—, using the framework by Nagle, Martínez-Planell, and Moore-Russo (2019).

Nagel et al. (2019) categorizes the concept of slope within the APOS framework (Arnon et al., 2014). Slope as an algebraic ratio is the formula, $\frac{y_2-y_1}{x_2-x_1}$. At an action stage of slope as an algebraic ratio, an individual knows the formula as a memorized fact and can compute the value with the ordered pairs provided. At the transition level (from action to process) of slope as an algebraic ratio, she understands that slope is independent of any two points and may also imagine obtaining the equation y-y_1 = m(x-x_1) or $y = mx+b$ without explicitly doing it. Yet she may understand slope only in algebraic aspect and has no geometric referents to justify the formula, $\frac{y_2-y_1}{x_2-x_1}$.

In comparison, slope as a geometric ratio is the rise over the run in a geometric figure. At an action stage of slope as a geometric ratio, an individual may see slope as a static image and do a simple counting to find slope. At this stage, she has no understanding that the numerator changes as the denominator changes; as such, she may struggle with the sign of slope. At the transition level (from action to process) of slope as a geometric ratio, the individual knows that slope is independent of the size of the triangle being used. As such, she may use slope to determine relationships of properties or to describe the behaviors of a line, such as increasing or decreasing.

When an individual reaches the transition level of slope as both algebraic and geometric ratio as well as understands slope as a parameter in $y = mx+b$ by freely moving between algebraic and geometric ratios, she reaches a process stage of understanding of slope. With a process conception, she could imagine conversions of slope among arithmetic, algebraic, and geometric representations, without explicitly doing the conversions (Nagle et al., 2019).

Tangent Line and Linear Approximation

When a function f is differentiable at a point, $x = x_1$, one can determine its tangent line to the curve with the formula, $y - f(x_1) = f'(x_1)(x - x_1)$. Unfortunately, neither the derivative nor the tangent line equation itself involved in the formula is easy for students. Many students do not conceptually understand that $f'(x)$ is the limit of the average rate of change (Asiala et al., 1997; Orton, 1983; Thompson, 1994). Neither do they understand the structure or meaning of the tangent line, making errors such as claiming the slope of the tangent line as $f(x_1)$ (Orton, 1983) or the tangent line as the derivative of $y = f(x)$ (Amit & Vinner, 1990). Further, many individuals may not be much familiar with point slope form in which the formula is presented. According to Nagel and Moore-Russo (2013), preservice secondary teachers rarely hold the point slope form in their concept map of slope. While 11 of 19 teachers included the slope intercept form, only 3 included the point slope form in their concept map.

There are two common ways to find a linear approximation. The first is to find the tangent line equation using the formula $y - f(x_1) = f'(x_1)(x - x_1)$ and to determine $y = f(x_2)$ by substituting $x = x_2$ into the equation. The second is to find the slope of the tangent line at $x = x_1$, $f'(x_1)$, find dy as $f'(x_1)dx$, and add dy to $f(x_1)$, yielding $f(x_1) + dy$ (see Figure 1). The cognitive processes involved in the two ways are quite different, although they may be intrinsically the same—as $y - f(x_1) = f'(x_1)(x - x_1)$ can be converted to $y = f(x_1)+f'(x_1)(x - x_1) = f(x_1) + dy$.

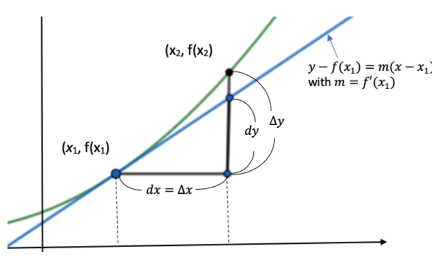


Figure 1. Linear approximation.

Methodology

The participants of this study were three precalculus students at a small state university in the southeast region of the US. I asked precalculus instructors to recommend their students who are active in classroom. I sent an invitation email to all 15 students recommended by their instructors. Three students responded to my email and participated in this study. Student 1 was a local high school student and had an Algebra 2 course at her high school prior to precalculus. Student 2 was a sophomore in biology and had a college algebra course at our university prior to precalculus. Student 3 was a freshman in computer science and had an advanced mathematics course at a high school in southeast Asia where English was the primary language of instruction.

Each of the three students were interviewed for a total of 5-7 sessions, with each session lasting about one and a half hours. The interviews were conducted in two stages. At the first stage—for the first one or two sessions—I used a form of semi-structured clinical interview (Ginsburg, 1997) to assess their level of understanding. The interview questions were designed to investigate their understanding of various forms of linear functions and quadratic functions as well as their ability to use rate of change in constructing function equations.

At the second stage—for the remaining sessions—I used a form of teaching experiment (Steffe & Thompson, 2000) to intervene their understanding, in particular, to strengthen their understanding of rate of change as well as to have them use slope in constructing equations and finding linear approximations. I began interviews with pre-designed tasks. As the interviews progressed, the tasks were modified and more tasks were created based on their responses.

The analysis of data began at the onset of the interviews. After each interview, I viewed the video recordings and prepared a time-coded written record of the content, including rough transcriptions of interesting episodes. At the same time, I developed codes based on the research questions, the framework of the study, and students' understandings observed in the data. During this initial stage of analysis, I came up with hypotheses on students' understanding and difficulties described in this report. In the next stage, I plan to review data by constantly comparing and revising codes in the spirit of constant comparative method by Glaser and Strauss (1967), which will help me confirm, rebut, or revise the hypotheses I made at the first stage of analysis.

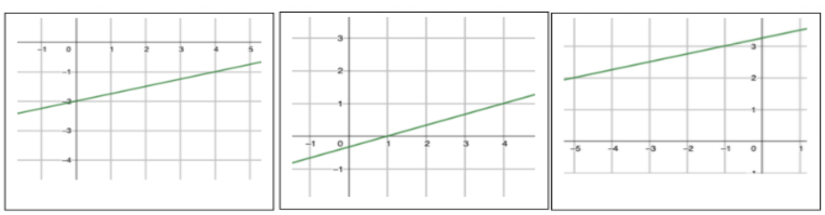


Figure 2. Linear equation task in pre-intervention.

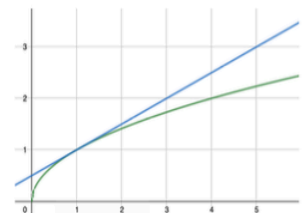
Preliminary Findings

The results presented here are for Student 1 mostly from the interview and written data on the problems in Figures 2 and 3. The first graph of Figure 2 shows an integer valued y-intercept, but the second and third graphs do not. Figure 3 is three linear approximation problems designed at the level of precalculus or algebra. The problem 2 was provided with graphs of the function and the tangent line. Problems 3 and 4 were not provided with any graphs. Before given these problems, students were explained that a tangent line is almost identical to a curve at a given point and that the tangent line can be used to approximate function values.

Line Equations

Slope as an algebraic ratio and the Cartesian Connection. When constructing an equation for line graphs in Figure 2 during the assessment (at the first stage of the interviews), Student 1 chose two lattice points through which the line passes and found the slope of each line by calculating $\frac{y_2-y_1}{x_2-x_1}$ (action as an algebraic ratio). For the first graph, she read the *y*-intercept (0, -2) from the graph and successfully used the slope intercept form to determine the equation, $y = \frac{1}{4}x - 2$. For the second and third graphs, she wrote $y = \frac{1}{3}x + b$ and $y = \frac{1}{4}x + b$, respectively, but incorrectly determined the value of *b* as -0.3 and 3.3, respectively. When asked how she determined *b*, she said she guessed the values based on the locations of the *y*-intercepts.

2. The curve in green is a graph of $y=\sqrt{x}$, and the line in blue is tangent to the curve at $x = 1$. Estimate $\sqrt{1.05}$ using the line.



3. The slope of the tangent line of $f(x) = 2x^2 + 1$ at $x = a$ is $4a$. Estimate $f(2.1)$.

4. Estimate $\sqrt[3]{1.05}$ when the slope of the tangent line of $f(x) = \sqrt[3]{x}$ at $x=$ is $\frac{1}{3\sqrt[3]{a^2}}$.

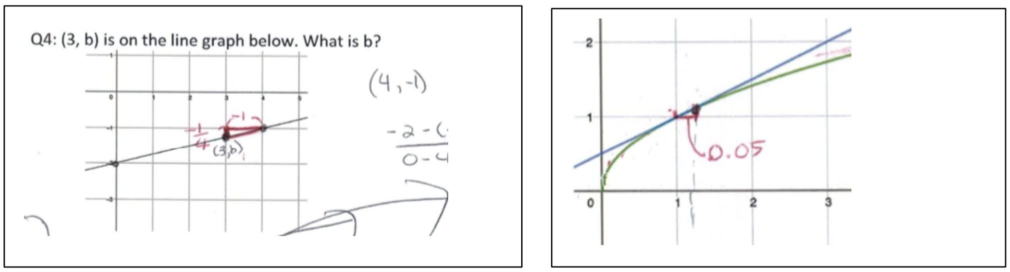
Figure 3. An example of linearization problem in teaching experiment.

During the intervention, I asked her if the $y = 3x+1$ graph passes through the point (2, 7). She answered yes because 3(2) + 1 = 7, showing her understanding of the Cartesian Connection—"a point is on the graph of the line L if and only if its coordinates satisfy the equation of L" (Moschkovich et al., 2013, p. 73). I also asked her what the value of *h* was when the point (4, 1) was on the $y = \frac{1}{3}x + h$ graph. She wrote $1 = \frac{1}{3}(4) + h$ and successfully solved for h, confirming her understanding of the Cartesian Connection in the problem context. I then asked her to determine *b* in the third graph of Figure 2, which she struggled prior to the intervention. She wrote $2 = \frac{1}{4}(-5) + b$ by using the point (-5, 2) from the line graph and claimed that *b* = 13/4. This suggests that students may have difficulty activating their understanding of the Cartesian Connection when the whole graph is presented without a reference of a specific point.

Transition level as algebraic and geometric ratios. When asked to find the value of *b,* with (3, *b*) on the graph in Figure 4a, she picked two points, (4, -1) and (0, -2), found the algebraic ratio, $\frac{-2-(-1)}{0-4}$, and simplified it to ¼ (action as an algebraic ratio). She then wrote $\frac{b-(-1)}{3-4} = 1/4$ and solved for *b*, showing the understanding that slope between any two points on the line is constant (the transition level of slope as an algebraic ratio). When asked what the change in *y* is if 1 is the change in *x*, she said ¼ without any hesitancy or work, confirming her understanding of slope as an algebraic ratio in the transition level. When I asked her to use the idea on the graph to determine the value *b*, she drew a triangle using points, (4, -1) and (3, b), wrote -1 and -1/4 for the horizontal and vertical segments (Figure 4a), and found *b* as -5/4 after subtracting 1/4 from -1, showing a transition level of slope as a geometric ratio. I then asked her to use her geometric strategy to find the equations of the second and third graphs in Figure 2. She successfully found equations for both. For the third graph, for example, she came up with $y = \frac{1}{4}x + \frac{13}{4}$ by writing *b* as 3+1/4 with no drawing, by increasing Δy of ¼ to the y-coordinate of (-1, 3), hinting her understanding of slope as a process.

Linear Approximation

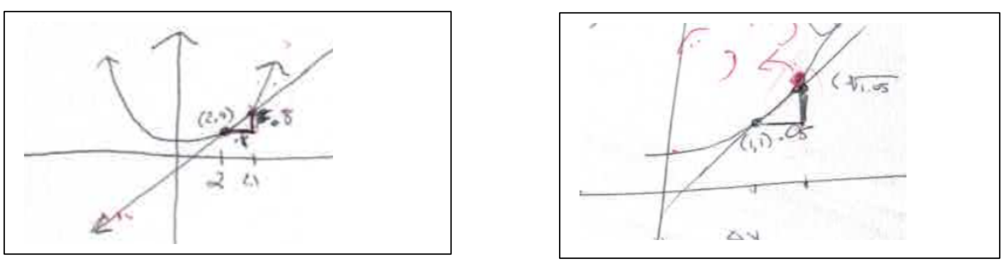
When asked to estimate $\sqrt{1.05}$ using the given tangent line (problem 2, Figure 3), she first determined the slope of 1/2 from the line graph and then the value of *b* based on the point (1, 1), by thinking $\Delta y = -\frac{1}{2}$ when $\Delta x = -1$, without drawing (the transition level of slope as an algebraic ratio). She then substituted x =1.05 into the tangent line equation, $y = \frac{1}{2}x - \frac{1}{2}$, and found the value of y for the approximation of $\sqrt{1.05}$ (the Cartesian Connections). When asked if there was any other way to find the estimate, she drew a triangle (see Figure 4b), found $\Delta y =$ .025 by setting up ½ = $\frac{\Delta y}{0.05}$, and added 0.025 to 1, showing the transition level of slope as a geometric ratio and possibly a process stage of slope.



(a) (b)

Figure 4. Slope in linear equation and its use in approximation.

For both problems 3 and 4 (in Figure 3), she first attempted to represent the functions in graphs. For $y = 2x^2+1$, she correctly drew a graph and its tangent line (Figure 5a) and found an estimate of $f(2.1)$ by performing a similar work to her work in the problem 2 in Figure 3. For problem 4, however, she asked for my help for the $y = \sqrt[3]{x}$ graph as she did not remember the graph. When asked if the shape of a curve mattered in the previous approximation problems, she said no and randomly drew a smooth curve along with its tangent line at x = 1 (Figure 5b). She then successfully estimated $\sqrt[3]{1.05}$ by doing a similar work to the previous problems.



(a) Estimate of $2(2.01)^2+1$ (b) Estimate of $\sqrt[3]{1.05}$

Figure 5. Use of slope in approximation.

Discussion and Future Research

As for the other two students, Student 2 showed at most an action conception of slope as both algebraic and geometric ratios throughout the intervention. Student 3 initially showed the transition level as an algebraic ratio, with little conception of slope as a geometric ratio. With the intervention, he developed a transition level as a geometric ratio. The results suggest that tasks such as those used in this study can strengthen student understanding of slope and linear equations that is necessary for the concepts of the tangent line and linear approximation. However, for students who profoundly lack the understanding of slope and linear equations, such as Student 2 who remained at the action stage despite intervention, more research is needed to investigate their underlying problems.

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