Proof Without Claim: Problem Packaging

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We present a case study of a student engaging in "proof without claim" tasks. In these tasks, students are presented with proofs in which the proposition being proven (its claim) is removed. Students are asked to ascertain what this missing claim is. This brief report serves (1) to illustrate how our novel proof without claim tasks can provide insight into how students engage with proofs and (2) as an existence proof that some students conceive of proofs as packaging for calculational math problems.

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There is a significant body of research on the teaching and learning of proof-based mathematics (Stylianides et al., 2017). A lot of this literature has taken a deficit approach by highlighting various challenges that students face in proof-oriented content. This includes difficulties with constructing proofs (Iannone & Inglis, 2010; Weber, 2001), reading proofs (Dawkins & Zazkis, 2021, Inglis et al., 2012), translating novel arguments into proofs (Zazkis et al., 2015, Zazkis & Mills, 2017), and determining whether a given proof is valid (Selden & Selden, 2003, Alcock & Weber, 2005). Although it is useful to have a thorough list of challenges that students face, it is also important to understand the roots of these challenges.

In undergraduate mathematics, there is a transition between calculation-based courses and proof-based courses (Moore, 1994). A number of mathematics departments implement courses specifically to ease this transition (David & Zazkis, 2020). So, given the differences between proofs and calculations, the transition-to-proof courses also marks a transition in how students interact with mathematics, and by extension, what they might believe mathematics is. Since students have vastly more experience with calculation-based mathematics when they begin proof-based courses, it is reasonable to make sense of some nuances in students' approaches as an extension of their experience in calculation-based coursework (Dawkins & Zazkis, 2021). Importantly, such experiences might color students' engagement with proofs. Studying these conceptions has the potential to lead to meaningful improvements to proof education.

Through utilizing novel tasks, our work addresses the following question: How do students understand the relationship between a proof and what is being proven (its claim)? Addressing this question sheds light on how student thinking about proof might be influenced by past experiences with calculation-based mathematics. In the particular case study discussed here, we show evidence that at least some students view a proof as a container for a calculation problem. That is, students view proofs as essentially the same kind of task they encountered in their previous calculation-based classes, with its component sentences dressing up the calculation. This single case study is insufficient for making causal claims about the relationship between students' conception and their prior experiences with calculation-based course work. However, we do believe that conceptions transferred from those prior experiences are the most straightforward explanation for this phenomenon.

Task Design, Subjects, and Methods

We investigate this topic by creating and implementing the novel idea of *proof without claim* tasks. These tasks involve providing valid proofs while removing any explicit statements (claims) about what is being proven. Immediately prior to the proof is a statement "We will prove that” followed by a blank space, whereas after the proof is a statement “We have proven that” followed by another blank space. Students are prompted to fill in their interpretation of what should be in these blank spaces (Figure 1).

We developed six of these tasks in order to give insight into how students read and interpret proofs. This includes (1) how students determine what text goes in the blank spaces, and the related issue of (2) whether what they write in one blank space differs from what they write in the other. Our tasks included typical logical structures of claims, and they have employed basic proof strategies, such as proofs of universal claims via arbitrary instantiations, proof by counter-example, and constructive existence proofs.

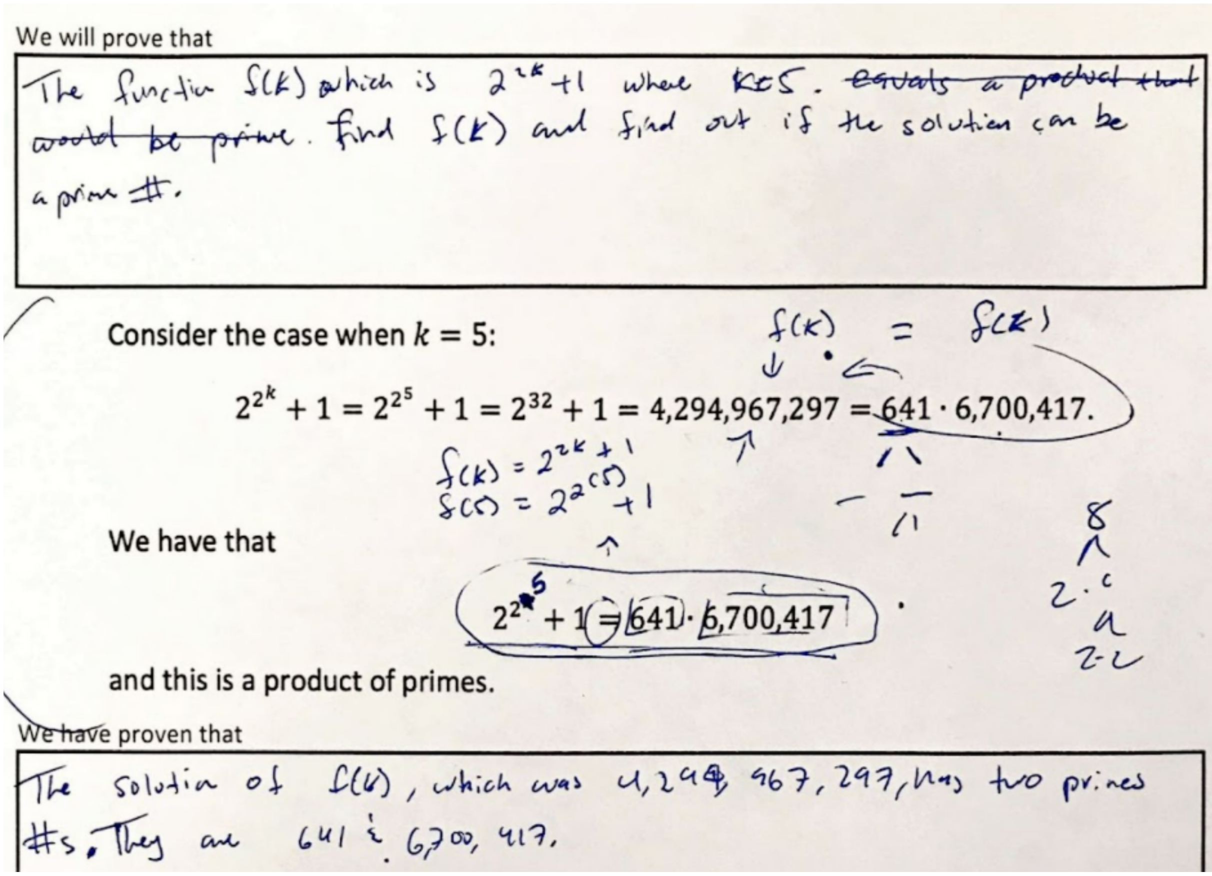
The participant, Maria (pseudonym), was, at the time of the interview, an undergraduate preservice secondary mathematics teacher. She had successfully completed a transition-to-proof course. The interview lasted two and a half hours and can be categorized as an exploratory task-based clinical interview (Clement, 2000).

Results

Our data indicate that Maria views a proof as a package for three different components: the beginning (first blank), the middle (text provided between blanks), and the end (second blank). She brings a calculational orientation (Thompson et al., 1994) toward each individual part; the beginning describes what calculations or math problems are to be performed, the middle carries them out and serves as a holder for the constituent computations and equations, and the end presents the answer or result of these problems or computations. In other words, she appeared to conceptualize proofs as records of answer-getting problems. See Figure 1 for a typical example of Maria’s work. Although there were six tasks involved, we use Figure 1 heavily as a typical example of Maria’s work (due to space limitations).

The First Blank Box

In filling in the blank “claim” boxes, Maria indicated that she viewed the proofs as math problems where the “problem” is the first claim and the “solution” is the second claim. The reason we write quotations around “claims” is that, with the exception of a single task out of six, only once did Maria write in the first box what *we* would consider to be a claim. Her understanding of the first “claim” box as a repository for putting a math problem or prompt is evidenced by her writing of commands such as “find”, “find out”, and “prove”. In other words, rather than writing propositions as her claims, she wrote instructions for what mathematical task to perform. For example, in the task disproving the statement “$2^{2^k}+1$ is prime for all natural numbers k” (see Figure 1), Maria wrote “find f(k)” within the box for the first claim. In one task, she used the command “prove” in the first claim box. While “prove” is not the same as “find”, we note that it shares the aspect of being a command; it is an instruction that might be given to a student to solve a math problem. Interestingly, in two different tasks, she listed “given” information in the first claim box. In one of these tasks, these “given” statements were listed in isolation. We note that, despite not including explicit commands, there is a sense in which writing the “given” information is similar to writing a command; both are pieces of information or symbolic prompts to be operated on or worked with towards a “solution”. In other words, something is “given” in that it is information with which one will perform a task of finding an answer.


We will prove that

The function f(k) which is $2^{2k}+1$ where k≤5. ~~equals a product that would be prime~~. find f(k) and find out if the solution can be a prime #.

Consider the case when $k = 5$:

$$2^{2^k} + 1 = 2^{2^5} + 1 = 2^{32} + 1 = 4{,}294{,}967{,}297 = 641 \cdot 6{,}700{,}417.$$

f(k) = f(k)

$f(k) = 2^{2k}+1$

$f(5) = 2^{2(5)}+1$

We have that

$$2^{2^5} + 1 = 641 \cdot 6{,}700{,}417$$

and this is a product of primes.

~~We have~~ proven that

The solution of f(k), which was 4,294,967,297, has two primes #s. They are 641 & 6,700,417.

Figure 1. A typical example of Maria's work

The Second Blank Box

Interestingly, Maria did not write the same content in the second claim boxes as she did in the first claim boxes. Instead, in the second claim boxes, she wrote what we would call propositions. We can compare this with what she tended to do in the first claim box, which was to write an instruction or a calculation problem. Despite ostensibly being a proposition, when viewed in relation to the rest of the task (in particular the content of the first claim box), we understand Maria to be presenting the answer to the "question" or "problem" posed in the first box. Consider, for example, Maria's response to the task shown in Figure 1. While she wrote a command in the first claim box to "find $f(k)$" and to "find if the solution is a prime number", in the second claim box she wrote what her "solution" to $f(k)$ is and noted that it "has" two primes; in other words, the first claim box is for a question or prompt, and the second claim box is for a solution or answer. We see a similar structure in other tasks. For example, in a *proof without claim* task that proves the upper half of the unit circle has area $\frac{\pi}{2}$, Maria reports that the "derivative [sic] for the function $f(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$ is $\frac{\pi}{2}$" in response to her first blank box command to "Given the function $f(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, find the derivative [sic] dx/dy". Another example of this structure is seen in a task that proves the quadratic formula (written with different symbols than the way it usually appears). In the first blank she inscribed the instruction “The equation $x^2 + px + q = 0$ has zeros, Please feel free to prove this how you think is best.” In reply, her second blank box read “The equation $x^2 + px + q = 0$ has two zeroes which are x = [...].” Maria correctly interpreted the last line of the proof, where x is isolated, as a statement of the zeroes of the function. Yet, taken together, the ways she inscribed the first and second blank boxes supports the idea that Maria was thinking of proofs as records of calculation problems.

The Middle

While Maria used the first claim box to write down the “problem” or command and the second claim box the “answer” or result, the middle served as a location for the computations leading to the answer. In the middle of the quadratic proof, Maria spent around five minutes attempting to verify the two algebraic steps—completing the square and rearranging terms. Throughout the study, Maria gave much attention to intermediary steps; her attention was on the particulars of the calculation rather than on the claim.

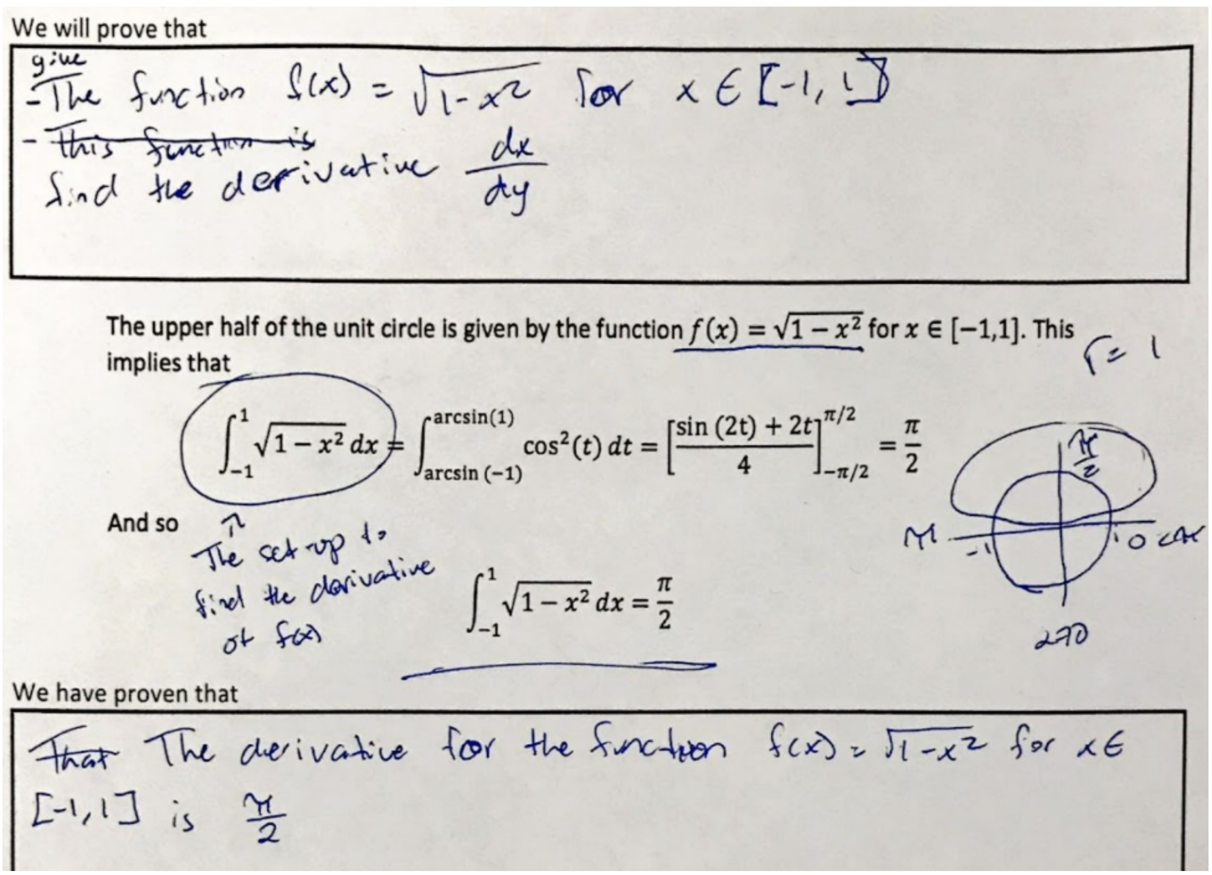


Figure 2. Another typical example of Maria’s work

Figure 2 (above) shows some inscriptions Maria made on the calculus task. This attention given to the intermediary steps is consistent with Maria’s overall view of a proof as packaging a calculation problem; the middle part is what she saw as the work to solve the “problem” being posed. Note that she circles the leftmost integral and inscribes below that it is “The set up to find the derivative [sic] of $f(x)$". In the first blank box she inscribed an instruction to find the derivative, and in the middle she indicates that she demonstrates that the way to start such a calculational-oriented problem is by writing the integral. In other words, her attention to intermediary steps can be understood as her focusing on the "problem" (the computation) instructed by the first box.

Figure 1 also supports our interpretation that the middle serves as a holder for the constituent computations. In drawing three arrows to 4,294,967,297 and labeling it "$f(k)$", Maria indicates that this is the $f(k)$ the proof is having her "find". Below that, she inscribes that "$f(k) = 2^{2^k} + 1$" and that "$f(5) = 2^{2^5} + 1$". Her other command from the first blank box, to "find out if the solution can be a prime number" is explored when she circles 641 and 6,700,417. From the middle of the task she extracts the numerical value of $f(k)$: "$f(k)$ is 4,294,967,297". This then allows her to present her finding as an "answer" in the second blank box and comment on it ("it has two primes"). Yet, Maria was not satisfied with her response and stated that it "has no meaning whatsoever" in relation to her transition-to-proof course. In reflecting on the task in Figure 1, Maria remarked that the mathematics presented in that task was not at all relevant to the mathematics she was learning in proof-based math—"I don't know what this has to do with proofs". For Maria, the task was at the pre-algebra level, because it only involved addition, multiplication, and a few exponents – she was attending only to the calculations when judging the proof's mathematical sophistication. Since this particular proof involved only pre-algebra calculations, she held this in conflict with the notion that proofs are part of higher-level mathematics. This supports our interpretation that what she saw as the mathematics of the proof was the calculational content in the middle, further supporting our interpretation that Maria saw the proofs as records of calculation problems.

Discussion and Future Directions

We reported exploratory findings on an undergraduate preservice teacher who recently completed a transition-to-proof course. This particular student made sense of these tasks by conceptualizing the first claim as a statement of a "problem" to be solved, the last claim as the "answer", and the middle as a location for calculations connecting top to bottom. Hence, ascertaining the claim amounted to determining to which "problem" these calculations correspond. This technique of *proof without claim* gave us insight into how this student conceptualized proofs. We believe prior calculation-based coursework to be a reasonable explanation for the phenomena highlighted in this preliminary work. Hence, future directions can expand the use of these tasks to a larger audience and more systematically explore the connections between such conceptions and students' prior course work.

Furthermore, it may be worth expanding interviews to include comparison tasks, where interviewees are presented with both a container for calculation-type response (e.g. Maria's work) and one for a normative response (e.g. a mathematician's) to a *proof without claim* task. Interviewees' responses to these comparison tasks have the potential to reveal additional nuances of students' conceptions of proof. In particular, since there is no blank claim box that needs filling, there is potential to confirm that some students truly believe the problem-calculation-answer format to be a legitimate proof format and perhaps more desirable than the normative response.

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