Engineering Instructional Practices to Better Support Access and Participatory Equity in Authentic Proof Activity

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The Orchestrating Discussion Around Project (ODAP) was conceived of as an investigation into how instructors might support a student-centered classroom with aims of engaging students in authentic mathematical proof activity. The design-based research focused on engineering tasks and specific high leverage teaching practices to promote student engagement with three focal proofs in Abstract Algebra. In this report, we share a series of project learnings related to supporting students in accessing formal proof activity and promoting more equal participation amongst students. We focus specifically on how complex task launch, structuring group work, and working with student ideas in whole class discussion may be shaped to support more students in authentic mathematical proof activity.

Keywords: design-based research, teaching practices, participatory equity, proof

In the United States, there has been a substantial push for undergraduate mathematics classes to become more student-centered. To this end, several researchers have developed curriculum materials that provide resources for instructors to promote active engagement in classes. In the context of upper division proof-based courses, these innovations often exist within the inquiry paradigm (inquiry-oriented to inquiry-based) where students engage actively in constructing and presenting proofs, or in reinventing abstract definitions and theorems. In general, the field is in consensus that more active classes support deeper learning (Freeman et al., 2014). Others have suggested that active classes or in some cases inquiry in particular can lead to more equitable learning environments (Laursen et al., 2014; Tang et al., 2017; Theobald et al., 2020). Yet, others have argued that instruction geared towards inquiry can also open space for more inequitable interactions as students become more actively involved (Brown, 2018; Ernest et al., 2019). Recent large-scale results reflecting a performance disparity between men and women on a group theory concept assessment (Johnson et al., 2019) in inquiry classes has raised questions about how such instruction may alleviate or exacerbate inequity in classroom environments. That is, when classrooms are opened to be more student-centered, we now have students and instructors engaging with each other in ways that would not be found in a traditional lecture class. Thus, the classroom space can become a place where societal inequities can be reproduced and only certain students fully participate. There are several cases (Haider & Andrews-Larson, 2021; Rasmussen & Kwon, 2007) that illustrate the potential for inquiry-oriented teaching to have a very positive impact on student learning. However, as noted in their recent literature review (Melhuish, Fukawa-Connelly, et al., 2022), we know little about the details of instructional practice that may account for when inquiry approaches are successful in supporting students. As noted by Adiredja and Andrews-Larson (2017), “more work is needed to document if, when, and how these instructional approaches are equitable for all students” (p. 459).

In this paper, we share some lessons we learned during our design-based research project that was designed around participatory learning goals in proof classes: engaging students in authentic activities related to comprehending, validating, and constructing proofs. That is, we explored the question, in what ways might HLTPs be incorporated in the proof setting to support our participatory goals? We made the decision to forefront instructional practices, a set we identify as high-leverage teaching practices, as objects of design. These practices stem from the K-12 literature and have been documented to play a key role in structuring student-centered classes and have the potential to increase access and inclusion. Through cycles of design and engineering we found that our practices supported students in engaging in the type of mathematical activity we valued, yet, if we focused on the who and how of engagement, participation was often unbalanced across students. As we transitioned from the lab setting with small groups of students to the classroom setting, we found a need to disentangle our authentic activity goals into two underlying participation heuristics:

- (Access) Providing access to opportunities to participate in authentic mathematical proof activity.
- (Engagement) Promoting participatory equity in authentic mathematical proof activity engagement.

We share a few key insights from our design cycles. We found that just having high-quality tasks and centering student reasoning was insufficient to support all students in engagement, and thus made several modifications to promote more equitable participation. For the scope of this work, we focus specifically on launching tasks, structuring group work, and selecting student ideas for discussion. We note that our focus on participation is a narrow view on equity, and caution that this exploration reflects just one dimension related to promoting equitable learning environments. While the question is beyond addressing fully in a single project like this, our report thus intends to help address the question, "What instructional practices support students' engagement in authentic proof activity and promote participatory equity in such activity?

Background and Perspective

For the scope of this project, we view learning as participating in activity that resembles a broader community of practice (e.g., Sfard, 1996). That is, the proof-based mathematics classroom is a setting where students can be apprenticed into the work of research mathematicians. To adhere to these goals, we focus on learning in terms of student engagement in what we deem *Authentic Mathematical Proof Activity*. These activities include not just constructing formal proofs, but comprehending and validating them. A full treatment of valued activities can be found in Melhuish, Vroom, et al. (2022).

We assume that that the classroom is a complex system and that instructional practices, those we consider high-leverage teaching practice, can serve to mitigate complexity in student-centered classrooms and promote space where students can engage in authentic activity. We integrate several scholars' (Ball et al., 2009; Woods & Wilhelm) definitions of high-leverage teaching practices to define these practices as those that:

1. can be implemented routinely,
2. use, shape, or otherwise integrate students' mathematical thinking,
3. have potential to increase equity, access, and/or engagement, and
4. are supported by research connecting the practices to students' learning.

Ultimately, we aim to support participatory equity, defined as "fair distribution of both participation opportunities and participation itself as students engage in the learning process" (Shah & Lewis, 2019, p. 423).

For the scope of this paper, we focus on three such practices: complex task launch, structuring group work, and selecting and working with records of student thinking. These three practices can serve to structure a lesson and have been well studied and developed in the K-12 setting. See Table 1 for descriptions and key citations.

Table 1. Definitions and key citations for focal high leverage teaching practices.

High Leverage Teaching Practice	**Descriptions (found in Melhuish, Dawkins, et al., 2022)**
Complex Task Launch (Jackson et al., 2012; Jackson et al., 2013; Khisty & Chval, 2002; Wood & Wilhelm, 2020)	*The teacher engages students in making sense of tasks via attending to relevant mathematical language, ideas, relationships, task contextual features, and development of a common understanding of the task goals and context. This practice occurs prior to or in parallel with problem-solving, but does not scaffold or directly instruct on solutions to the task*
Structuring Group Work (Cohen, 1994; Esmonde 2009; TeachingWorks, 2018; Webb, 2009)	*The teacher structures and manages partner and group work in order to engage all students meaningfully in mathematical activity. This can include scripts, clear mathematical activity expectations, and/or roles that provide guidance for how students are to interact with each other and the mathematics.*
Selecting and Working with Records of Student Thinking (Durkin et al., 2017; Stein et al., 2008; TeachingWorks, 2018; Wilburne et al., 2018).	*The teacher orchestrates mathematical discussion where (1) students publicly present ideas and (2) students are prompted to meaningfully engage with the ideas through analyzing, critiquing, and/or comparing across student ideas.*

Methods

This report summarizes findings from the ODAP design-based research project. This project consisted of six cycles of design and refinement (two in a lab setting, and four in classroom settings) all occurring at a Hispanic-serving, doctoral-granting research institution in the United States. The focus of instruction was engaging advanced undergraduate mathematics students in authentic proof activity related to three focal theorems in Abstract Algebra: Structural Property Theorem, Lagrange's Theorem, and First Isomorphism Theorem. At a prior RUME conference, we shared some initial work designing and implementing for HLTP from the lab setting where we provided data showing that the K-12 practices could be adapted robustly to this setting (Melhuish et al., 2020). In this paper, we focus primarily on later cycles of development and key aspects of design that emerged during the rather complex transition (see Lamberg & Middleton, 2009) from working with small groups of students in a lab to full classroom implementations.

We focused on key hypotheses about how HLTPs could support students in participation in authentic activity. This analysis occurred in several layers. All implementations were video recorded, transcribed, as well as observed by several members of the research team. Observers took field notes on ways that students did or did not engage in hypothesized activity with attention to issues of access and imbalances in participation. Between lessons, we engaged in debriefing and coming to a consensus related to lesson goals, HLTPs, and student activity. We attended specifically to ways our hypothesized relationships between HLTPs, and student participation were or were not realized. After the completion of each cycle, the team reflected more holistically on prior cycles, revisited important points in the data, and in some cases conducted extended analysis using specific frameworks to analyze elements of our study more systematically. For instance, we analyzed student activity (Melhuish, Vroom, et al., 2022), instructional moves (Melhuish et al., 2020) and distribution of authority in small groups (Hicks et al., 2021). After these debriefs and analyses, we made evidence-based modifications to improve the enactments of our HLTPs and developed new hypotheses of characteristics of HLTP and how they support student activity in proof settings. At the completion of the project, we have engaged in retrospective analyses to better describe elements of the HLTPs leveraging our data corpus. In this report, we share examples that span our project and evidence of what played out in proof contexts and particular instances that led to modifications.

Results

In this section, we focus on several adaptations and refinements we made to the three focal HLTPs for reasons related to either the proof context or participation inequities that emerged during the classroom implementations (see Table 2).

Table 2. Questions related to each focal HLTP and student participation.

High Leverage Teaching Practice	Considerations	Key Features
Complex Task Launch	What information do students need to make sense to engage productively in a given task?	• Attention to quantification and referent objects • Discussions of informal ideas that parallel formal tools
Structuring Group Work	What task features are needed to support students in shared engagement amongst group members?	• Eliciting expertise beyond proof construction • Distributing roles and expertise
Selecting and Working with Records of Student Thinking	How can public records represent a multitude of student reasoning? What tools are needed for students to work from each other's records productively?	• Expanding records beyond proof construction • Introducing partial linking ideas

Complex Task Launch

In the proof-based setting, complex task launch involved supporting students in sense making about a focal theorem in ways that may anticipate important conceptual insights or proof structures. Ultimately, complex task launch serves the purpose of supporting access to the major ideas and tools needed to engage in a focal task. Our implementations shared many features of the K-12 versions; however, we found two needs specific to the proof context. First, in addition to questions about meaning, examples, and relationships, we also found a need to ask specific questions about *quantifiers* and *referents.* For example, to engage with the Structural Property Theorem that a group isomorphism preserves commutativity, complex task launch involved unpacking the different definitions involved including isomorphism and abelian. During these conversations, students often provided imprecise explanations, such as articulating "G is abelian" as "*ab*=*ba*." Both quantification (recognizing that a statement applies for all elements in in a group) and referent objects (recognizing that the isomorphism function must be introduced) are essential to the formal proof. Without these tools, many students hit an impasse about how to prove the statement or provided a proof that did not cover all elements in the codomain.

We also found a need to explicitly consider *parallelism* between explorations of theorems and how these link to proofs. For example, in order to prove Lagrange's Theorem, students were positioned to recognize this is a theorem about multiplication, and they were easily able to retrieve a relevant definition of divisibility ($|G| = k|H|$ for some k). However, this formal definition did not evoke the parallel multiplicative structure (G being composed of k disjoint subsets the same size as H) they needed to leverage to create a proof. In our first cycle, the students attempted to create a proof by cases depending on the order of the group, suggesting, "What if we did a couple of cases, like where the order of the group was even, or it was odd?" They introduced ideas of using different remainders to make the argument, soon recognizing the number of cases became unwieldy. We suggest that the formal statement evoked prior knowledge related to number theoretic proofs, rather than drawing on the idea of repeated addition or totaling equal sized sets. We modified this element of task launch in later cycles to incorporate explicit conversation about the *conceptual meaning* of multiplication that better paralleled the proof structure of Lagrange's (e.g., "[L]et's think back to elementary school when we write these things, and we're gonna make a similar type of visual to go with this that's kind of connected to what we mean by multiplication."). In all cycles with these discussions, students built on notions of "repeated addition" or "totalling up" for their proof approaches.

Structuring Group Work

Structuring talk in group work became a major focus of design during the switch between lab setting and whole class setting. In our lab settings, we relied on mechanisms such as think-pair-share and peer review to structure group work. When we switched to whole-class implementations, we found these structures insufficient to disrupt status differentials between students. For example, students were asked to share proofs of the Structural Properties Task written at home and answer "What is one thing that makes sense? What is one thing you have a question about?" in regard to your partner's proof approach. The conversation often devolved into the student with a more complete proof teaching the student with a less complete proof. For example, in one exchange, we noted one of the partner stating they "don't know anything" and the other partner walking them through what was "missing" from their proof. In this case, the introduction of a "function from G to H that is one-to-one and onto."

This led us to modify our focal proof activity. Instead of students peer-reviewing proofs they had written, they were each tasked with making sense of one of two prototypical student proofs provided by the instructor and becoming experts on the proof they read. The student conversations about these proofs were much more balanced both in the quantity of participation and in students' mutual engagement (as opposed to exchanges resembling one-on-one tutoring). This change draws attention to a major theme in our work: *expanding the evoked expertise beyond proof construction.* If the only way to show competence is through constructing formal proofs, we are necessarily amplifying the status of certain students who have become more adept at engaging in this complex and challenging practice.

In other tasks that had less support for equitable small-group discussion, we identified even more foundational issues. For example, when students were tasked with making sense of parts of the First Isomorphism Theorem, some groups of students talked very little and did not appear to engage in meaningful comprehension. In later cycles, we changed the structuring of this task such that each member of the group was responsible for leading a discussion on a specific question, such as identifying the difference between the isomorphism and homomorphism involved in the proof the First Isomorphism Theorem. This structuring served to not only support students' access into the proof, but also *individual and shared accountability* across students.

Selecting and Working with Records of Student Thinking

Finally, when we consider whole class discussion and comparison of student ideas, we found it important to attend carefully to whose ideas made it into the public space. As we anticipated certain approaches to select and sequence, we found the instructor was limited in whose proofs make it to the focus of discussion. In first classroom implementation of the Structural Property Task, the only students with largely complete (though possibly not fully correct) proofs were the more vocal, white men. This informed two modifications: expansion of expertise so that a presenting student would not have to have proven a statement (as discussed in the prior section) and developing *interdependent ways of creating public records*. For example, when students created examples to illustrate the First Isomorphism Theorem, the records were subdivided such that each student had a different role: writing domain and codomain elements, creating the homomorphism map, identifying the kernel, creating the isomorphism map (see Figure 1). In this way, discussions around this student work focused on a truly shared product.

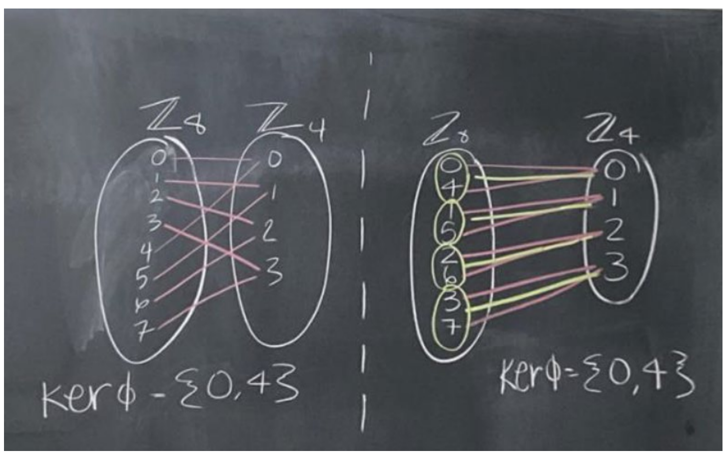

Figure 1. Cycle 5 small group student board work illustrating the FIT

The other major component of comparing and using records of student reasoning was to position students to generalize and see important features of their ideas that may connect to more formal representations. This modification to our task implementation required a large amount of attention and reflection, because, as we compared across implementations, we found that certain discussions could easily devolve into an initiate-respond-evaluate conversation where the instructor was doing majority of the reasoning (and students were left searching for the target insight). To support students in further reasoning and meaningful contribution, we found it useful to introduce *partial information* that linked student ideas and the formal proofs. For example, we provided a partially written isomorphism map for students to complete "β (____) = ____" (First Isomorphism Theorem), a partial function diagram (Structural Property Task), or a series of formal lemma statements that students were asked to pair with their own informal conjectures (Lagrange's Theorem). By providing partial information, the instructor navigated between students' ideas and formal mathematics while providing students explicit tools to increase access and engagement.

Discussion

Working in the formal proof setting, we encountered two substantial hurdles: (1) navigating between formal and informal ideas (Raman, 2003;Weber & Alcock, 2004; Zazkis & Mills, 2017; Zazkis et al, 2016); (2) student status differentials occurring based on students' speed and ease working within the formal proof setting (cf., Weber & Melhuish, 2022). Regarding the first hurdle, we found for complex task launch to provide students access to necessary tools, the conversations had to draw attention to structural ideas (both formal and informal) and help connect imprecise language to quantification or to identifying referent objects involved. Simply unpacking definitions or creating examples was not sufficient to support later proof aims. We also found that orchestrating student discussions to promote comparison and generalization from their proofs and examples required careful attention to ways to connect between formal and informal representations systems (our access heuristic of authentic mathematical proving activity). Often the instructor would introduce an object to focus discussion and promote students in engaging in translating between systems and noticing important ideas.

The second hurdle required us to return to the literature on promoting more equitable group work. As noted by many scholars, active engagement in proof contexts can favor students who are more comfortable with academic language or more quickly engage with the norms of proof (Brown, 2018; Weber & Melhuish, 2022). This can foster divisions of status between students and large inequities in who participates. To better navigate this issue, we incorporated more intentional features related to group-worthy tasks and Complex Instruction (Cohen & Lotan, 1996). By subdividing responsibilities – for example, leading an assigned portion of discussion or taking on a specific role in the construction of an example – and promoting distributed expertise – for example, placing each student in charge of a particular proof approach – we were able to engage more students in authentic activity during group work (our engagement heuristic of authentic mathematical proving activity). This also freed us from constraints we previously experienced in selecting student ideas, as we were no longer tethered to finding certain, anticipated proof approaches or using an example that primarily legitimized only one students' contribution.

We suggest other design researchers consider the dual participation heuristics and engineering of HLTP implementations as they move from small-group lab settings to whole-class instruction. Issues of access and participatory equity become amplified in the latter, where an instructor can no longer interact with each individual student throughout a lesson. HLTPs provide a means to manage some of this transition, but as we found in our project, also require intentional analysis and modification related to our heuristics and the specifics of theorems and proofs involved to meet our participation goals.

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