A Linear Algebra Instructor's Ways of Thinking of Moving Between the Three Worlds of Mathematical Thinking Within Concepts of Linear Combination, Span, and Subspaces

Avery Madden
University of Oklahoma

Sepideh Stewart
University of Oklahoma

Jeffrey S. Meyer
CSU, San Bernardino

Linear algebra is an important topic for many STEM students and presents unique challenges for teaching and learning. In this study, we analyzed one mathematician's instructional materials and spoken language. Our theoretical framework is based on Tall's (2008) three worlds of mathematical thinking and Harel's (2008) ways of thinking. The goal of the study was to examine the complexities of a mathematician's ways of thinking while moving between the three worlds of mathematical thinking, while orchestrating the important aspects of linear combination, subspaces, and span.

Keywords: subspace, linear combination, span, ways of thinking, three worlds of mathematical thinking

Literature Review

Establishing the current state of linear algebra education research, a survey paper by Stewart, Andrews-Larson, and Zandieh (2019) highlights the students reasoning on the topics of span (e.g., Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013) and subspaces (e.g., Britton & Henderson, 2009). However, most studies focus on these topics and student comprehension, leaving a gap in the investigation of teacher understanding. Furthermore, the survey paper helps indicate extensive research into linear combinations and span, but less so in subspace.

When discussing Subspace, the idea of "Span" and "Spans" is inevitable. These two words are quite similar but mean different things invoking a type of polysemy, which can create challenges for the learning of subjects when it occurs in mathematics (Kontorovich, 2018). This issue overlaps with the understanding of subspace, something that research has documented as a challenge for student understanding (Britton & Henderson, 2009). Specifically concerning subspaces, one study showed that students prefer to use geometric intuition to understand subspaces (Wawro, Sweeny, & Rabin, 2011). Furthermore, students struggled with understanding how many subspaces are in a given space (Fleischmann & Biehler, 2018). Similar struggles exist in learning the concept of linear combination. When researching undergraduates' ability to determine if something is linearly dependent or not, instead of utilizing the characteristics of linear combination, the students preferred to use matrix operations (Aydin, 2014). Navigating these challenges presents important considerations for an instructor's teaching methods.

Looking to how this information is communicated, a teacher's knowledge base is an important aspect of education. A teacher's knowledge base and its relation to student understanding is a deep facet of education that has been studied (Tallman & Frank, 2020). This knowledge base may be accessed through the educator's ways of thinking (Harel, 2008). In their view Tallman and Frank (2020) find for a given topic, "this study provides support for the notion that mathematical ways of thinking comprise a fundamental aspect of teachers' professional knowledge base" (p. 92). A teacher's knowledge base has overlap with how they present information. In the literature, there has been an increasing interest in examining this topic in how an instructor conveys information to students and conducts procedures in the classroom. For example, Stewart, Troup, and Plaxco (2019) discussed an instructor's reflections on his lectures and movements between different mathematical worlds. The study concludes that the instructor was able to give careful attention to the needs of the students after consulting with his own reflections and knowledge base.

Emerging studies have begun examining how digital resources can be used to help students understand linear algebra topics. Previously, introducing drawings into linear algebra education has been a topic of study. Gueudet-Chartier (2004) argues that "the concreteness that seems to lack in linear algebra could be more efficiently provided by the use of drawings" (p. 500). Research into the learning of span has shown that visualization can be helpful in making students more flexible in their reasoning of the topic (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Wawro, Rasmussen, Zandieh, & Larson, 2013b; Cárcamo, Fortuny, & Gómez, 2017). Further, it has been shown that using GeoGebra to assist students in understanding subspace, specifically the zero vector is beneficial (Caglayan, 2019). These studies suggest that linear algebra concepts can be enhanced by visualization tools. Similar studies were conducted on students showing that these students benefited from a more embodied representation (Hannah, Stewart, & Thomas, 2013).

Theoretical Framework

The theoretical framework for this study will utilize Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2010). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, justifying, and problem solving, which are not necessarily unique to mathematics. These several types of mental acts are the foundation of ways of thinking defined as "a cognitive characteristic of a mental act" (p. 269). These ways of thinking have the characteristics of being able to abstract, generalize, structure, visualize, and reason logically. Tall (2010) describes the embodied world as "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (2010, p. 22). In his view, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as a number)" (2010, p. 22). The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (2010, p. 22). Using the above frameworks, the goal of this narrative study is to examine the complexities of a mathematician's ways of thinking while moving between the three worlds of mathematical thinking while orchestrating the important aspects of linear combination, subspaces, and span.

In analogy with Tallman and Frank (2020), we consider moving between the embodied, symbolic, and formal worlds as a characteristic of mental acts and, as such, a way of thinking. It is through Tall's three worlds lens of mathematical thinking that we evaluate the teaching moves of the instructor.

Methods

This qualitative narrative study (Creswell, 2013) took place in the Fall of 2021. The participants in this research consisted of a mathematician and co-author. The mathematician, thereafter, referred to as the instructor, specialized in geometry, linear algebra, and number theory. The teaching material of note is part of a first-year linear algebra course from a university located in the southwest region of the USA taught by the instructor. There were 30 students enrolled in this course. In addition, a mathematics educator, and a research undergraduate assistant from another university in the USA were part of the research team. The data consisted of (a) the instructor's notes (which he authored) on linear combinations, span, and subspaces (in this order) that was available to students (Figure 1), (b) his notes on the blackboard (Figure 2a & b), and a GeoGebra interactive example (Figure 2c), (c) two of his recorded video lectures, and (d) students' weekly reflections and homework. The data analysis for this study consisted of transcribing the second lecture, finding pertinent sections in the notes related to the lecture concepts, and coding the different themes. These different components of analysis were done with the instructor as part of the research team. Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2008) were employed as a framework to analyze the data. We identify and analyze the instructor's one specific way of thinking, namely, the disposition of moving between languages as well as Tall's (2008) three worlds of mathematical thinking (embodied, symbolic, formal). The data from students' work is in the process of being analyzed for future research.

Results

The instructor intentionally ordered his lecture with the goal of building on intuition with more embodied understandings before generalizing to reach into new topics. The instructor gave a variety of connections between topics to aid in understanding. In his lecture, he reminds students of the definitions in the notes and refers to them. Descriptive ways of how to understand and apply these concepts are placed on the board. In the first and second lectures, diagrams of two- and three-dimensional space are drawn on the board to emphasize embodiment (Figures 2a & b).

Linear Combination and Span: the language and movements between the three worlds

We noted the instructor used a variety of informal and geometric language to support the concepts and their definitions. In the notes, the instructor used words like "reach", "line", and "parallelogram" to describe linear combination and span (Figure 1). He repeatedly used the informal language of "mixture" to refer to linear combinations. For example, he asked, "Which vectors? The collection of how many mixtures? Some mixtures? … All mixtures. Right, …all mixtures you can make with these not some, not half, not a few, but all mixtures right." Here the instructor relies on language to give meaning to linear combinations as his way of thinking about the topic. An instance of the instructor's disposition to attend to language is when they anticipated and addressed student confusion stemming from the polysemy "span." Figure 2b breaks the topics of possible linear combinations "span the noun" and spanning sets making subspaces "span the verb" into two categories and highlighting the most crucial pieces of the definition. Harel (2008 considers 'anticipating' as a mental act. This certain mental act motivated the teaching act described above. The instructor further clarifies the distinction between an object and the relationship between two objects: "Right, that's very different than the noun right this is just a set (gestures to W = Span(S) this is like a true false kind of thing (gestures to S spans W) it either does or doesn't span." During the lecture (see Figure 2b), the instructor said: "I would call span the mixture set associated to any collection of vectors, okay and then we could talk about uh span the verb you know if if you know if given a set of vectors right if it's mixture set is the subspace in question, we would say it spans the subspace." Following, the core mixture questions (Figure 1) are set up in such a way as to help internalize what is important about the concept and give these characteristics of linear combinations meaningful language while also pushing towards the concept of span. During the first lecture, some examples with matrix representations are written on the board to assist in symbolic understanding (Figure 2a).

During the second lecture, the word “mixtures” is used. This way of thinking about the concept of linear combinations involves changing the language to give the term linear combination additional “meaning”. The instructor poses the question, “So let’s say what’s the noun associated to this, what is the noun given a collection of vectors, what can we make with it? ... Yeah …so we can make mixtures.” The instructor utilizes the movement between the mathematical worlds of thinking throughout the lecture. In one such example (Figure 2), the instructor moved from the formal to embodied and acknowledged the possibilities of movements to symbolic: “the span of those guys it equals the line, and you can see that visually and if we wanted to, we could verify that symbolically”. For the topic of span, the notes provided a definition of span with emphasis on the word “all” this is done to emphasize any possible linear combination is made up in this set. Meanwhile, the embodiment of the definition invokes the use of visualizing a line that represents all linear combinations of a single vector. This is further extended to the notion of two vectors and a plane. In analyzing the concept of span, in the first lecture, the instructor drew two vectors, proceeded to analyze their linear combinations visually via a parallelogram, and then moved to the symbolical representation (Figure 2a). Next, he showed symbolically how to create an arbitrary vector in R^2 as a linear combination of the original vectors. Performing examples in this order suggest that the instructor comes from a place of visualizing the span first and symbolizes in the matrix form. Visualization continues to be included in the second lecture, with vectors being drawn on the board and displayed in GeoGebra (Figure 2b & c).

Subspace: the language and movements between the three worlds

In the notes (Figure 1), the instructor provided a nonstandard formal definition of subspace relying on the concept of span. His notes continued to discuss a “quick” way to determine if a region is a subspace by checking if it is closed under scaling and addition, but he phrased it in geometric language as “line containment” and “parallelogram containment”, respectively. In the first lecture, the notion of subspace, having a set of “mixtures” and then a spanning set, are all written on the board (Figure 2a). With these components, the same language of the notes is used where a plane is drawn on the board in a projected three-dimensional space. The use of line test and parallelogram test are employed with the inclusion of passing through the origin.

During the second lecture, the instructor used GeoGebra as an activity and demonstration to discuss single vectors, zero vectors, and span vs. spans. With the focus changing to subspace, a GeoGebra three-dimensional model is placed on the top of the board, where the instructor manipulated different vectors and created a variety of subspaces (Figure 2c). After the group activity, an example from the class dialogue continued as follows:

“So, let’s see, A plus B spans the line well that’s a very good point again what is A plus B?... So there’s there’s also there’s all sorts of relationships here so so you could say A plus B spans but A plus B goes by another name D. So, there’s all sorts of algebraic relations between these things.”

Vectors represent quantities we see in the world around us every day. One of the most fundamental ideas associated to quantities is *mixing quantities*. This core idea will be central to everything going forwards.

The idea of mixing vectors is made mathematically precise through two natural operations: scaling and addition. To reason about them verbally, I will discuss them in the context of cereal, where a vector represents one serving size of a given cereal (as in Example 2.2). *Can you reason about them in other contexts?*

Definition 2.8. If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, then a **linear combination** of them is a vector of the form $x\mathbf{v} + y\mathbf{w}$, for real numbers $x, y \in \mathbb{R}$.

More generally, a linear combination of vectors $\mathbf{v}_1 \ldots \mathbf{v}_n \in \mathbb{R}^m$ is an vector

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n, \qquad \text{where } x_1, \ldots x_n \in \mathbb{R}.$$

CORE MIXTURE QUESTIONS

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$ is a collection of vectors, then we seek to understand:

1. *Reach.* What are all the possible vectors that can be made by mixing vectors in S?
2. *Redundancy.* Is one of the vectors in S already a mixture of the other vectors in S?
3. *Realization.* Given a vector $\mathbf{w} \in \mathbb{R}^m$, can $\mathbf{w}$ be made by mixing vectors in S, and if so, how?

To begin, let us start with the question of *reach*. In this situation, we have fixed vectors but can select any and all serving sizes for them. The collection of all possible mixtures which can be made with S is made precise by the notions of *span* and *subspace*.

Definition 2.12. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$, their **span** is the collection of *all* linear combinations of $\mathbf{v}_1 \ldots, \mathbf{v}_n$. Symbolically,

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$

Definition 2.16. Any subset $W \subset \mathbb{R}^m$ which is the span of a collection of vectors is a **subspace**.

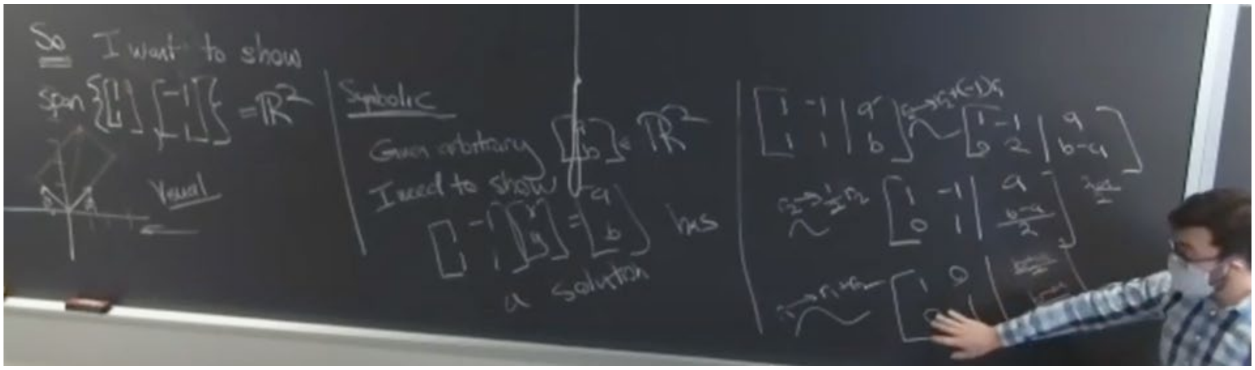
Observe that for our purposes, a span and a subspace are essential the same thing. The reason why we might use the language of one instead of the other is the following: a span means we explicitly know the mixing set S while for a subspace, we know such an S exists, we just do not necessarily know what it is.

The containment properties of span can help us determine *which subsets of* $\mathbb{R}^m$ *are subspaces*.

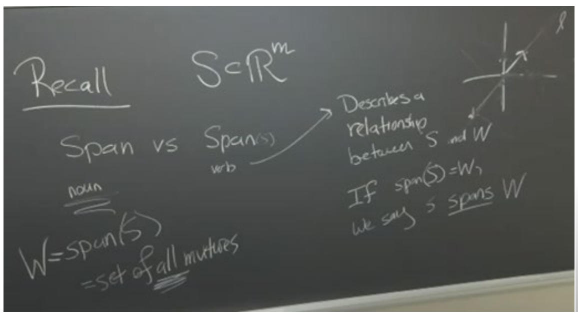
Computational Quick Check 2.17 (Subspaces)**.** If $W \subset \mathbb{R}^m$ is a subset, then W is **not** a subspace if you can find either:

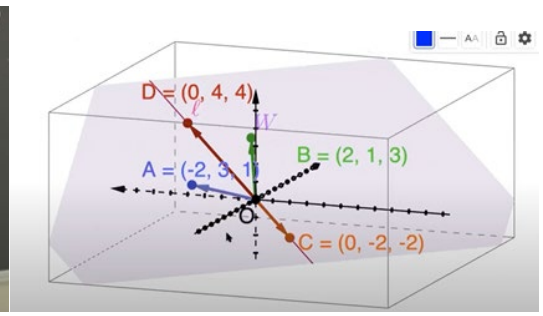
1. **(Line Failure.)** A vector $\mathbf{v} \in W$ for which the line span$(\mathbf{v})$ is *not* contained in W, or
2. **(Parallelogram Failure.)** Vectors $\mathbf{v}, \mathbf{w} \in W$ for which the mixture parallelogram associated to $\mathbf{v} + \mathbf{w}$ is not contained in W.

Figure 1. Linear combinations, Span, and Subspaces from the instructor's notes.



(a)





(b) *(c)*

Figure 2. The instructor is writing on the board during lectures one and two and engaging students via GeoGebra.

During this episode, the instructor focused on how the zero vector and the zero-subspace fit into 3D space. Here is a sample of subsequent discussions:

Instructor: [I]s a single vector a span? No, right, so we have to go back to our precise definition. So, a single vector is not a span right, but a line here is a span. What other things in somehow depicted here are spans of something?

Student 1: Like a point.

Instructor: Uh which point?

Student 2: Any point on the line.

Instructor: Is Anything on the line, Is any point a span? This is a span? (Gestures at a drawing point of a line on board). This is a subspace? No because right a single point cannot be a subspace unless there's one exception.

Here the instructor is connecting the different ideas about what a single vector can generate while also asking them to recall special cases. This response is supported by the picture of a line placed on the board earlier in the lesson (Figure 2b). "Does this guy span the trivial subspace, zero? Some people say no. No! As they take mixtures of this is bigger, it's bigger it's not equal." Here the instructor used geometric language of relative size to support the idea that the span of a non-zero vector cannot be the trivial subspace.

Discussion and Conclusions

Upon analyzing the data in this study, we observed the instructor relied heavily upon reasoning in many representations, which impacted his choices in instruction. We observed how this informed his selection of demonstrations and his attention to language.

We also found that these ways of thinking impacted the choices that he made in instruction. He consistently moved from embodied to symbolic to formal worlds, and back again. His choice of language moved from informal to geometric to formal, and returned to informal. This was observed in both his prepared lecture and his spontaneous responses to questions. The weaving of formal language with informal and familiar language made the topics more accessible. The lecture began using descriptive geometric words such as “sweep out”, “stretch” and “reach” to articulate how these objects look. The instructor used informal terms during the lecture to familiarize informal terms like “mixtures”, “span the noun”, “span the verb”, when talking about these concepts. In their responses to questions, students also used this informal language, indicating that the familiarization may have been effective. The prepared and distributed notes allowed students to focus on active participation within the class. The lecture weaved through multiple understandings of each concept allowing for engagement with familiar concepts, language, and embodied objects, only to then progress to understanding the mathematical machinery of the subjects before being exposed to their deeper meaning. Here the use of language is crucial again as the lecture begins using descriptive words such as “sweep out”, “stretch” and “reach” to articulate how these objects look.

Furthermore, the use of GeoGebra during the lectures supports this meaning by seeing all the different possible combinations of vectors and having students identify what they created. The instructor gives meaning to span and subspace in similar ways by providing a definition and then constructing a visual representation of the topic. The instructor employed the use of language to illustrate how the zero subspace looks, citing that adding anything besides zero would be “bigger” than the zero subspace. He made a clear distinction between the words span and spans when a discrepancy appeared about what a singular vector can make. Following these results, our future research will include an analysis of students’ weekly surveys and some of their homework.

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