In Proving Attempts that Generate Stuck Points, What Characterizes Undergraduate Students' Overall Proving Process?

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Learning to prove mathematical propositions is a cornerstone of mathematics as a discipline (de Villiers, 1990). While the field has generated research that has analyzed the final products of proof (Selden & Selden, 2009) and there are frameworks for analyzing problem-solving processes (e.g., Carlson & Bloom, 2005; Schoenfeld, 1992), much remains to be known about analyzing undergraduate students' proving processes. With a focus on impasses in the proving processes, this study provides a more fine-grained account by characterizing students' overall proving process and navigating actions. The analysis lays the foundation for discussing the relative productivity processes students may take as they are engaging in proving.

Keywords: Proving Processes, Stuck Points, Navigating Actions, Productive Struggles

Since proving is a different mathematical activity as compared to students' prior experience, research has also shown that many undergraduate students struggle to learn to prove, including those who major in mathematics (Moore, 1994; Selden, 2012). Within the proof literature that focuses on undergraduate students, much work has concerned students' final proof products (such as written work or exams). However, such products cannot reveal the sequence of mental and/or physical actions that resulted in the final proof that students present in their written work (Selden & Selden, 2009). In particular, the proving process might entail several places where students get stuck, only some of which ultimately contribute to the final proof.

While the idea of struggle has been examined in problem solving in the middle grades (e.g., Warshauer, 2011), there has been sparse research on this idea in proving at the collegiate level. With a focus on impasses in the proving processes and the research questions "In proving attempts that generate stuck points, what characterizes undergraduate students' overall proving process?", this study provides a more fine-grained account by characterizing both undergraduate students' overall proving process and their navigating actions. Given the difficulty undergraduate students face in higher-level math courses, understanding the ways in which they struggle is important for building more inclusive classroom environments.

Literature Review and Theoretical Framework

View Proving as Problem Solving

A number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Moore, 1994). In fact, Weber (2005) suggested focusing on problem-solving aspects of proof because this "allows insight into some important themes that other perspectives on proving do not address, including the heuristics that mathematicians use to construct proofs" (p. 352). Thus, in this study, I regard proving as problem solving and used Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework to inform my research design. Expanding from Schoenfeld's (1985) framework, Carlson and Bloom (2005) developed four phases for problem-solving processes (Orienting, Planning, Executing, and Checking), each phase with the same four associated problem-solving attributes from Schoenfeld (resources, heuristics, affect, and monitoring). However, the original context of the work of Carlson and Bloom (2005) considered only how experts (like mathematicians) behave to successfully solve problems. Thus, their Multidimensional Problem-

Solving Framework does not characterize where a novice prover may seem to get stuck and how they may behave if they get stuck.

Immersion, Stuck Point and Productive Struggles

Wallas (1926) argued that creative problem-solving begins by first, immersing oneself in the problem to better understand it and exhaust conventional ideas. After a period of *immersion*, one likely reaches an impasse, or a mental block. The immersion phase of creativity shares some commonalities with *productive struggle* (Hiebert & Grouws, 2007), *productive failure* (Kapur, 2014) and other constructivist approaches to teaching in math education in which a student expends great effort during an initial period to explore and try to solve a math problem before receiving direct instruction. Involving in productive struggle in problem-solving is an invaluable mathematical practice which can help learners overcome uncertain obstacles to make sense of mathematics (Middleton et al., 2015). This is essential as students make meaning through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed (Hiebert & Grouws, 2007; Kapur, 2008; Warshauer, 2014). From the student point of view, these *stuck points* have been described as perceived impasses, or moments when a student feels substantially stuck and unsure how to continue (VanLehn et al., 2003).

To date, research on stuck point in proving in the mathematics education literature has been sparse. Identifying a “stuck point” can be hard since it requires finding observable behaviors to serve as indicators of a person’s internal mental state. Savic (2012) conducted a study of how mathematicians recover from proving impasses. In his study, all of the mathematicians were able to recover from the impasses, thus his focus was on their *Incubation Period*: where they temporarily shift their attention away from the problem and do something else. Methodologically, Savic used Livescribe pens that were given to the mathematicians for days to gather real-time data. Although a mathematician may have had a key insight while walking around (for example) that allowed them to get unstuck, from this study, we still know little about the actual cognitive process of getting unstuck. Thus, Savic (2016) had called for future study of proving processes involved several impasses (stuck points) with participants that are less skilled in math (different from mathematicians), since their incubation period might not help them to move forward. In this study, I thus began by characterizing the major actions that were evident in students’ proof construction through “stuck points” in their proving processes.

Method

My study of undergraduate students drew participants from the Introduction to Proof course at a large research university in the Midwest. The Introduction to Proof (ITP) course is required for undergraduates majoring in STEM and mathematics education. The course focuses on developing mathematical argumentation techniques and proof strategies. The data for this study consisted of semi-structured task-based interviews with 10 undergraduates enrolled in a transition-to-proof course. Each participant was asked to prove or disprove two mathematical statements. The majority of the interviews took around 70 minutes, which provided enough time for the participants to have at least 30 minutes to engage with each task. In order to capture students’ proving processes with their specific strategies and steps, a think-aloud protocol was used (Schoenfeld, 1985). Interviews were video-recorded, and students’ real-time proof work was captured using a Livescribe™ pen.

As I have discussed in the literature review, in order to account for the proving processes of undergraduate students, the existing problem-solving frameworks need to be expanded. In particular, the role of students’ stuck points in their process is important, as is understanding more about what students do when they get stuck. To do that, I utilize a combination of my own partition of students' process around "stuck point" and a modification of Boero (1999)'s proving phases and Carlson and Bloom (2015)'s Problem Solving framework. My partition was developed from the flowing three steps:

Step 1: Operationalizing and Identifying "Stuck Points". Like I have pointed out, identifying a "stuck point" can be hard since it requires finding observable behaviors to serve as indicators of a person's internal mental state. In this sense, body and facial language become more important in telling whether a person has gotten stuck. For my purposes, stuck meant a period of time during the proving process when a prover felt or recognized that their argument had not been progressing fruitfully and that they had no new ideas. Whether the participant discovered an error was not important; the prover's awareness that the argument had not been progressing, but they were *hesitant about what to do next* was the key focus for me. For the purposes of my study, an impasse (an episode of "stuckness") consisted of interruption in the proving processes that was initiated by a student struggle that was in some way visible, whether voiced, gestured, or written. An impasse ended when the student overcame a stuck point and continued attempting or finishing the task, or the student gave a sign of no resolution and indicated wanting to switch to a different task or end the interview. In total, there were 41 episodes of such "stuck points" and majority of them are in the last four phases according to Boero (1999): Exploring of the Content (identify appropriate argumentations and envisage possible links), Selecting of arguments into deductive chain, Organizing the chain of argument into proof, and finally Write a Formal Proof.

Step 2: Categorizing Major Actions Around Stuck Points. I used the existing Warshauer (2011)'s framework for the four categories of students' stuck point (getting started, carrying out a process, uncertainty in explaining and sense-making, and expressing misconceptions and errors) to enhance my initial coding scheme by developing sub-categories to capture students' actions. Different from Warshauer (2011)'s study that mainly aim to examine the categories of students' struggles in K-12 problem-solving environment with teachers' interaction, my study focuses more on undergraduate students' proving processes and their own navigating actions.

Through this process, several major categories were identified. Each student's proving processes were captured in three major categories initially: their arguments, their stuck point, and their actions. After identifying those bigger categories, smaller categories were then identified inside of each major category. Since I was primarily interested in students' arguments and actions around their stuck points, I refined categories related to proving activity to the following four: initial argument, stuck points, navigating actions, and related outcomes. Once the initial coding scheme with those themes was developed, I completed the first round of data analysis to test and refine those themes. Particularly, I reorganized my coding scheme to highlight students' responses to "stuck points" or impasses. Besides different types of stuck points that were identified by the students, their actions for attempting to resolve those stuck points, monitoring, and persistent actions were also classified and captured.

Step 3: Capturing Each Students' Proving Process with Map. To capture each student's stuck points for each individual task, I used a flow chart to map out each student's proving processes with their major navigating actions. I split the proving process map into two dimensions. Vertically, the process map captured the four categories as described in Step 2. Horizontally, the map captured the order of each individual argument and action. The arrows in between indicate the directions of the movements and the relationship between the arguments and the actions. Instead of capturing only arguments, I also added a category that captures students' actions to help bridge the gap of the relationship between the argument and the actions, as described above. In *Figure 1* below, I provide an example of one student's proving process map for Task 2. From this process map, one can clearly see the student's arguments, stuck points, navigating actions, and outcomes related to those actions, as well as how one moved from one to the other. This proving process map provides a bird's-eye view of the proving process for each task for each individual student.

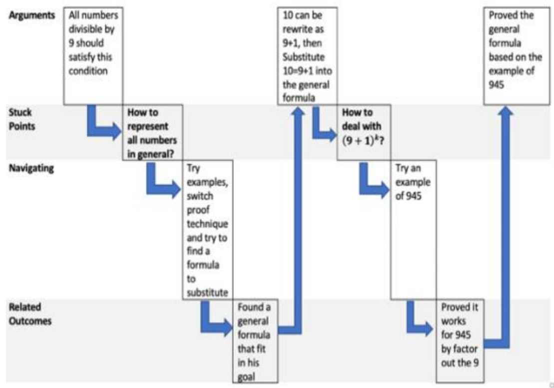

Figure 1. Example of Student's Proving Process Map for Task 2

Result and Discussion

In my subsequent work, the detail analysis of each student's proving process will be provided with map. In this proposal, I turn to focusing on the qualities of navigating actions students used, situated in the context of their entire proving attempts. My analysis resulted in three major types of navigating actions that were generated based on each participant's proving process map and the major categories of my analytical framework. For each task, each participant's proving process was characterized by their process map based on shifts between (1) initial argumentation (Argument), (2) major stuck points (Stuck Points), (3) actions they took to try to navigate out of being stuck (Navigating Actions), and (4) related results (Related Outcome). If they are able to use the related outcomes (i.e., new insight, useful calculation, or new connection) that they generated in a given cycle, then the proving process progresses, either to completion or until they reach a new impasse. In the next sections, I will describe the three major types of proving process around stuck points based on these shifts in the order of increasing complexity. Because of the limited space for this proposal, I will only discuss one student for each type for Task 2 (Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9).

Type 1: No Related Outcome Produced

Trajectory: Arguments⟶ Stuck Points⟷Navigating Actions

For this process type (as in all the types I consider), the student started with some sort of argument and got stuck when trying to construct a proof. The student then took several actions or attempts to try to navigate out of the stuck point but did not produce any related results, thus going back to their navigating actions. In summary, this type can be characterized as going back and forth between stuck points and navigating actions without any related outcome to help one move forward (see *Figure 2*). I will use Nikki's process to explain this type.

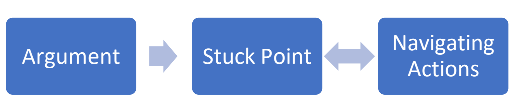

Figure 2. Process Flow for Type 1 Process Around Stuck Point

As soon as Nikki saw the task, she remembered that she had done a similar problem before. She quickly wrote down the problem and figured out what were the “P” and “Q” in this statement to set up for a proof by contrapositive. She then rewrote the statement as “If the sum $x_1 + x_2 + \cdots + x_k$ equals 9, then the number *n* is divisible by 9” (Nikki, Task 2). But Nikki got stuck after she had decided that was the argument that she wanted to prove. “How are they related?” she questioned. At this point she went back to the beginning of the problem and double-checked which statement should be thought of as “P” and which statement as “Q,” then told me that she didn’t know where to go next. She then started trying some examples such as 27, then thought about what happened if the sum was 9 or the sum of a multiple of 9 (like 18). But these navigating actions didn’t help her to go forward. She asked aloud, “How can I represent this example in a more general form?” and she got stuck again with her attempts to try examples. Nikki then tried to look for the general equation for expressing the sum of the digits. She found the expression on the formula sheet and tried to understand the expression again by computing some examples and by rewriting 27 as $2 * 10^1 + 7$. But she was soon stuck again with the k^{th} term: “What does *k* here mean in the expression of 10^k? Is *k* the number of digits or something else?” (Nikki, Task 2). She told me this general form she found (on the provided formula sheet) didn’t really help her, because she didn’t even understand the expression itself. After several attempts, Nikki didn’t produce any related outcome from her navigating actions, and she remained stuck in the end of the episode.

As I have noticed from Nikki’s example, this type of process doesn’t involve the production of related outcomes from navigating actions. Students with this process tend to go back and forth between stuck points and navigating actions since no significant related outcome is produced to help them move forward. Thus, in the end of the process, students don’t overcome the stuck point and they are unable to make progress on their argument.

Type 2: Related Outcome Produced but Not Link to the Argument

Trajectory: Arguments⟶Stuck Point 1⟶Navigating Actions⟶Related Outcomes ⟶Stuck Point 2 (⟷Navigating Actions)

The majority of the participants’ proving processes can be characterized as this type. For this process type, like before, a student starts with some sort of argument and gets stuck when trying to prove the argument. The student then takes several actions or attempts to try to navigate out of the stuck point (so far, as in Type 1). However, the student successfully overcomes one stuck point by producing a new idea or connection (a “related outcome”). But they soon run into another stuck point when they try to expand the related outcome into an argument. This process may also include some navigating actions to try to overcome the second stuck point, but the student is unsuccessful in doing so (see *Figure 3*).

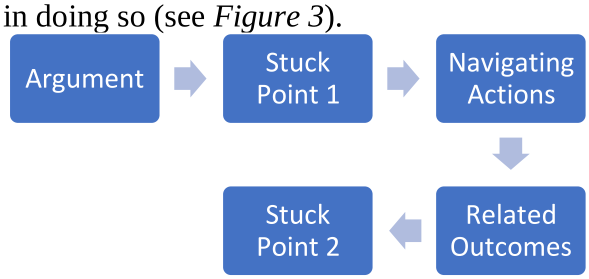

Figure 3. Process Flow for Type 2 Process Around Stuck Point

In Task 2, Devin conjectured the statement works for all multiples of 9 after trying examples like 9, 18, 27. But he then got stuck with how to show the statement in general. He tried to represent a given number in terms of the sum of its digits in a general form and assumed $9/N$ where N is a number such that $ones + tens + \cdots =$ multiple of 9 and $m1, m2, m3$ are the place values of each of the digits. Thus, any number can be represented as $N = 1 * m1 + 10 * m2 + 100 * m3 + \cdots$ But with this generalized formula in hand, how to use this to prove the statement became her new problem. He then thought about whether a number that is a multiple of 9 could be factored differently. Devin took the examples of 36 and 135 to try to factor them out and see whether he can have some insights. He rewrote them as $36 = 4 * 9, 135 = 1 * 100 + 3 * 10 + 5 * 1 = 100 + 30 + 5 = 2 * 5 * 5 + 6 * 5 + 5 = 5 * 27 = 5 * 3 * 9$. But "What do these factorizations tell me?" became his new question, a question he still had in the end.

The examples of Devin illustrate what Type 2 processes look like and how they differ from Type 1 processes in terms of the role of related outcomes. Devin had several related outcomes produced by overcoming some of their stuck points. This helped them to move forward, and, in fact, they were just a few steps away from fully overcoming all the stuck points and successfully proving the statement. Thus, at this point, it will be insightful to compare their processes with at least one related outcome linked with argument.

Type 3: At Least One Related Outcome Linked with Argument

Trajectory: Arguments⟶Stuck Points⟶Navigating Actions⟶Related Outcomes ⟶Arguments

Similar to the previous two types, for this type of process, a student also starts out with an initial argument, but they also get stuck. The student then takes several actions or attempts to try to navigate out of the stuck point (as in Types 1 and 2 process flows). However, the Type 3 process advances to proving the statement using the related outcomes generated in the process and ultimately results in having at least one related outcome linked with the argument. Since there could be multiple stuck points resulting in multiple processes, the student will be categorized in this type as long as at least one productive cycle is complete. The cycle can repeat if a second or third stuck point is encountered. To summarize, this type of process always involves a related outcome each time and the participant is able to leverage these related outcomes to move forward with developing their argument (see *Figure 4*). Note here that students are considered as making productive progress (Type 3) as long as at least one productive cycle is complete, which means they don't need to successfully prove the task in order to be considered as making productive progress.

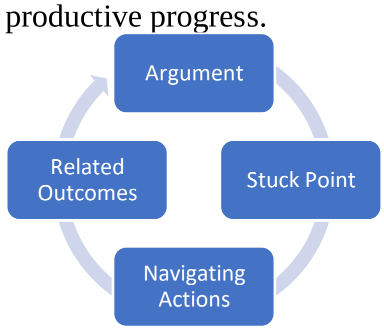

Figure 4. Process Flow for Type 3 Process Around a Stuck Point

Soni successfully overcame several stuck points in both Task 1 and Task 2. For Task 2, Soni started with outlining some of his ideas for finding the differences between counting by 9s and counting by 10s and rewrote 10 as $9 + 1$. Having this general goal in mind about what he would do, he felt the need for a generalized formula to represent any integer with its sum of the digits. Soni looked into the formula sheet and found the formula that any integer can be represented as $a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0$. But then how to use the general formula to prove the statement became his problem. Soni then thought about induction to help him achieve his goal. However, he soon realized that it didn't work here. He decided to go back to his initial thinking about representing 10 as $9 + 1$ and substituting all 10s in the formula as $9 + 1$. However, he got stuck again on how to expand $a_k(9+1)^k + a_{k-1}(9+1)^{k-1} + \cdots + a_0(9+1)^0$. Soni then decided to try a specific example (*Figure 5*) to show it could be factor out by 9, then follow the same way to show the generalized cases. Following this way of thinking, he successfully navigated out of his stuck points and proved the statement in the end.

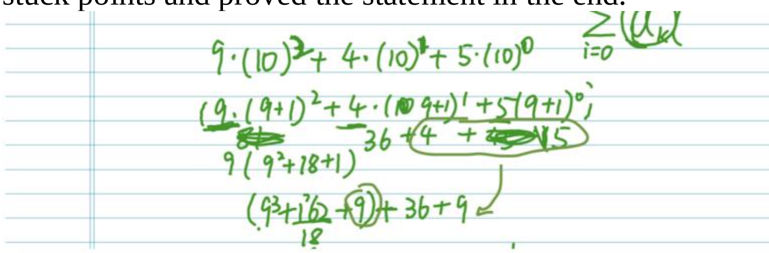

Figure 5. Soni's work of representing 945 using counting by 9s

From Soni, we see what could be described as a more productive struggle engaged in their proving processes since they completed at least one cycle of link back to the argument. Looking at Soni's case, we would see that he was clear about the goals for each of his navigating actions and had a clear understanding about where he got stuck. He also successfully produced several related outcomes from his navigating actions to help him move forward.

Conclusions and Implication

As I discussed in the literature review, although students' and mathematicians' proof practices have been examined in several ways, there are still important aspects that have not been examined in the existing literature. Previous proof construction research, building largely upon the work of Toulmin (1958), focused only on the structure of the argument but not the actual action and the actor (e.g., Pedemonte, 2007). To address those issues and characterize undergraduate students' proving processes, through our analytical process (conceptualizing stuck point, characterizing actions, and capturing general processes using map), I classified three different types of overall processes when undergraduate students face several stuck points and try to navigate out. In general, we can see that even though some of the students' navigating actions might look similar (e.g., look for a generalized formula to represent digits), students in Type 3 seems to make more productive progress compared to the other types. The reason could vary between students, but in general, students in Type 3 are able to clearly identify their stuck points, have a goal for each of their attempts, and able to produce related outcome from their attempt and then link that back to their initial goal/argument.

To conclude, students need to engage in a way that they can make some progress in order to be productive. In general, most of the students interviewed in this study had their process maps and actions categorized as Type 2. If we know where students get stuck in their overall proving process when trying to navigate out stuck points, we will be able to better support their productive struggles and help them to move to Type 3. In the subsequent study, I will describe individual cases in detail to discuss the relative productivity of different possible navigating actions students may take as they are engaging in proving. The different proving processes and navigation actions characterized in this study can help instructors to have a better understanding of students' proving processes when navigating out stuck points and to be more explicit in their own modeling of their own actions for their students.

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