A Calculus Student's Relative Size Reasoning

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The ideas of measurement and measurement comparisons (e.g., fractions, ratios, quotient) are introduced to students in elementary school and understanding these ideas as relative size comparisons (Faulkner, 2013, Joshua et al., 2022) are a part of the Common Core State Standards for Mathematics. Researchers have also reported how conceptual (Thompson et al.,1994) meanings for ideas in calculus such as the difference quotient (Byerley & Thompson, 2014), derivative (Byerley et al., 2012; Zandieh, 2000), and the fundamental theorem of calculus (Thompson,1994) involve a multiplicative comparison of two quantities.

Someone who engages in relative size reasoning would conceptualize comparing two quantities' values in a static or dynamic context as measuring one quantity's magnitude in units of the other quantity's magnitude (Lock, 2021). For instance, suppose a student engaged in relative size reasoning to compare two girls' heights. In this context, both quantities are of the same unit of measurement. Thus, the student, when comparing Lyndsey's height to Darien's height, would imagine Darien's height as the unit of measure used to determine how many times as tall Lyndsey's height is in relation to Darien's height. Therefore, she might say Lyndsey is 5/4 times as tall as Darien.

When the two quantities' values are not measured using the same unit, engaging in relative size reasoning will involve imagining the measure of one quantity's value in multiples of the other quantity's value. For example, speed is a multiplicative relationship between an amount of distance and an amount of time. If someone is traveling at a constant speed and travels 8 feet per 3 seconds, one would imagine that the number of feet one travels will always be 8/3rd times as large as the number of seconds it takes to travel that distance. Relative size reasoning is a way of thinking that a person has developed when conceptualizing the comparison of two quantities' values multiplicatively in static and dynamic contexts (Lock, 2021). The purpose of this poster investigates the questions:

What mental actions do beginning calculus students engage in when responding to tasks designed to assess students' relative size reasoning?

Radical constructivism (Glasersfeld, 1995) is the lens used in this study which takes on the perspective that every individual has their own reality and knowledge is constructed based on previous personal experiences. For this study I conducted 4 clinical interviews (Clements, 2000) where 5 calculus students from a large southwestern university worked on 22 tasks associated with different topics (fractions, rational functions, concavity, trigonometry, etc.) that were designed to elicit students' relative size reasoning. In this poster, I characterize a student's thinking by determining what meanings the student had in the moment while working through two tasks. Future research will aim refine my current framework that describes the mental operations involved in relative size reasoning as well as refine my current multiple-choice assessment based on my analysis of my data.

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Under the advisement of Dr. Marilyn Carlson

Motivation and Research Question

- Common Core State Standards for Mathematics asks that students understand fractions as comparisons of relative size (Faulker, 2013, Joshua et al., 2022).
- This way of thinking supports students in understanding proportional relationships, rates, growth patterns, and derivatives (Byerley 2019; Joshua et al., 2022; Thompson and Thompson 1996, Zandieh 2000).
- Researchers have documented that students in university mathematics courses have difficulties comparing two *quantities* in terms of their *relative size* (e.g. Byerley, 2019; Tallman, 2015).

Research Questions:

- *What mental actions do beginning calculus students engage in when responding to tasks designed to assess students' relative size reasoning?*

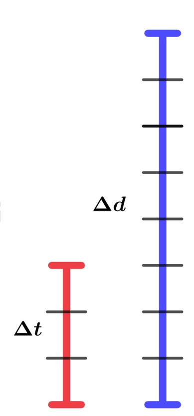
Relative Size Reasoning

- Someone who engages in ***relative size reasoning*** would conceptualize comparing two quantities' values in a static or dynamic context as measuring one quantity’s magnitude in units of the other quantity’s magnitude.
- When the two quantities' values are not of the same type of unit of measure (e.g., speed-relationship between an amount of distance and an amount of time), engaging in relative size reasoning will involve imagining the measure of one quantity’s value in multiples of the other quantity’s value (Lock, 2021).
- Relative size reasoning is a way of thinking (Thompson et al. 2014) that a person has Developed when conceptualizing the comparison of two quantities’ values multiplicatively.

Example: Constant speed of 8 feet per 3 seconds

the change in the number of feet one travels will *always* be $\frac{8}{3}$rd times as large as the number of seconds it takes to travel that distance.

$\frac{\Delta d}{\Delta t} = \frac{8}{3}$



constant speed of $\frac{8}{3}$ feet per second

Theoretical Perspective & Methodology

- **Radical constructivism** (Glasersfeld, 1995) is the lens used in this study which takes on the perspective that knowledge is constructed based on one’s previous personal experiences, and knowledge lies in the mind of the individual.
- ***Quantitative reasoning*** is the analysis of a situation into a network of quantities (measurable attributes of objects) and quantitative relationships (Thompson, 1994, 2012).
- ***Covariational reasoning*** refers to the mental actions involved in coordinating the value of two varying quantities and how the values vary in relation to each other (Carlson Jacobs, Coe, Larson, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017)
- This work draws on theories of students' understanding in the moment and communicating mathematics:
  - Students explained their meanings in the moment for each response they had to the questions. Each question posed was used as a point to characterize the students' meanings (Thompson, 2013).
- I report on 2 tasks from conducting five 1.5hr long clinical interviews (Clements, 2000) with 1 calculus student from a large southwestern university worked on 22 tasks from different topics (measurement, exponential growth, rational functions, and concavity) designed to elicit students’ relative size reasoning.

Results

Task 1: Paul is Skiing

Suppose Paul is skiing on a circular trail and he skied 27 miles. If one blip is 14 miles and the entire trail is 9 blips, what angle measure (in degrees) did Paul sweep out as he skied 27 miles on the circular trail?

After reading the problem Gabriel stated, **“we know there are 9 blips so that would be split into 9 pieces and each curvature would be 14 miles”** and then explained that since Paul skied 27 miles, **“that would be like almost two full blips.”** In this excerpt below, Gabriel’s statements and drawings suggest that he coordinated the 3 units of measure: blips, miles, and the circumference of the circular trail.

Excerpt 1

Gabriel: Since the, since the trail is circular and he is going through the, the trail, like in that circle, when you draw out the circle, you could kind of see that the, he is going to like traveling through or by the circumference of the circle.

Interviewer: So would you relate your idea of circumference in this question with what you said earlier about how he skied almost “2 full” <blips>.

Gabriel: Well the almost two full parts of like the split I did of the circle. Since we know **that the trail is made up of 9, equal blips, and each blip is 14 miles, we could kind of split the whole trail into 9 different equal pieces. And since we know that each blip is 14 miles, we know that each piece is also going to equal 14 miles. So when it comes to like something like 27, It's not quite two full pieces since two full pieces is 28**, but it's pretty close.

Interviewer: So how does that relate to what you talked about with circumference?

Gabriel: Since you can also split the circumference into 9 equal pieces and **since he is traveling, on the circumference, then the measure of the circumference is also equal to the 9 blips in total.**

Interviewer: So how would you compare the amount that Paul swept out in comparison to what you just talked about with circumference?

Gabriel: Oh, it was about it was about two ninths of the whole circumference.

Interviewer: How did you think about two and 9 and like, what is that two divided by 9 represent?

Gabriel: **The 9 represents the entirety of the circumference since it is split into 9 equal pieces. And the two is how much of those 9 pieces he was able to actually like get through.**

Gabriel then explains how Paul skiing 2/9 of the circumference is related to the angle measure in degrees that Paul swept out while skiing 27 miles (Excerpt 2)

Excerpt 2

Interviewer: What if we change this question to degrees (instead of measured in radians) how would you imagine going about that?

Gabriel: **We divide the 360 by 9 in order to get what the measurement of one of those slices is, and then we would be able to kind of find what the angle is of like the, the 27 or at least close to that.**

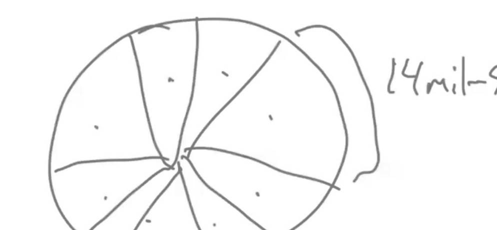
Interviewer: So would you be able to go through that process and kind of walk me through that?

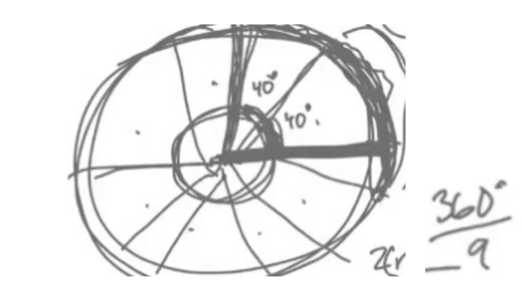
Gabriel: **So 360 divided by 9 is 40, which means that each angle is about 40 degrees. And since it takes up two of those, this angle right here, or 27 takes up about 80 degrees.**

Interviewer: How did you know it was about 80 degrees?

Gabriel: Oh, because of dividing the whole, uh, the whole circle. **Since the circle is going to be 360 degrees, no matter what if I divided that by 9, I got 40 degrees, which meant that every section of this would be also 40. Since 27 takes up about two of those sections, then I'd just add another 40 to that and it would be 80.**

This statement indicates that Gabriel leveraged his understanding of the length of the trail, which is 9 blips, and a full rotation in degree, which he stated is 360 degrees, in order to partition a full rotation into 9 parts and determine the angle measure swept out for each blip. Then he leveraged his thinking about skiing 2 chunks of the circumference to determine Paul swept out 80degrees as he skied 27 miles





Task 2: Gatorade Problem

5gallons of liquid flavoring are poured into a vat. Water will be added to the vat and mixed with the flavoring to produce a drink that will be bottled and sold. The ratio of flavoring to water in the mixture to the total volume of the mixture, x, can be defined as $f(x) = \frac{5}{x-5}$. As the total number of gallons of liquid increases, explain how the value of f varies.

Initially Gabriel read the problem aloud and substituted different values for x as he attempted to make sense of what the rational function conveyed about the context. As an example, Gabriel followed by explaining what the rational function represents when 5 is substituted for 3 (Excerpt 3).

Excerpt 3

Gabriel: So **the number of gallons has to be over 5 in order to produce anything because if it’s 5 or lower, it would just make either zero or a negative number for the denominator, which would mean that there is no like mixture at all.**

Interviewer: What do you mean by that?

Gabriel: There will be less, right? **Because the denominator is the thing that is increasing, and the numerator seems to be constant at a constant state of five.** Let's say we have 6 minus 5 down here. This is the variable. We would have 5 over 1, so that would be equal to 5. If we have like, let's say like 100 down here, minus 5, that would be like 95, which is a number that's a lot smaller.

Interviewer: So, in the first calculation you did where you did 6 minus 5, …what does that 5 represent and what does the 6 represent?

Gabriel: Oh, **The five represents five gallons of liquid flavoring are poured into a vat.** It doesn't really say how many units of the mixture there is. It just kinda says total volume of mixture. **I'm assuming it's gallons so I would assume that this (6) is like how many gallons of the mixture in total there are.** How many gallons of well, like ratio of mixture there is when six gallons of the liquid is put into the mixture.

This suggests Gabriel understood that the function represents the relationship between the total volume of the mixture and the liquid flavoring. In the next excerpt, Gabriel discussed the variation of the function f in the problem context.

Excerpt 4

Interviewer: So here as the number of gallons of liquid varies from 10 to 15, how does the comparison of flavoring to water vary?

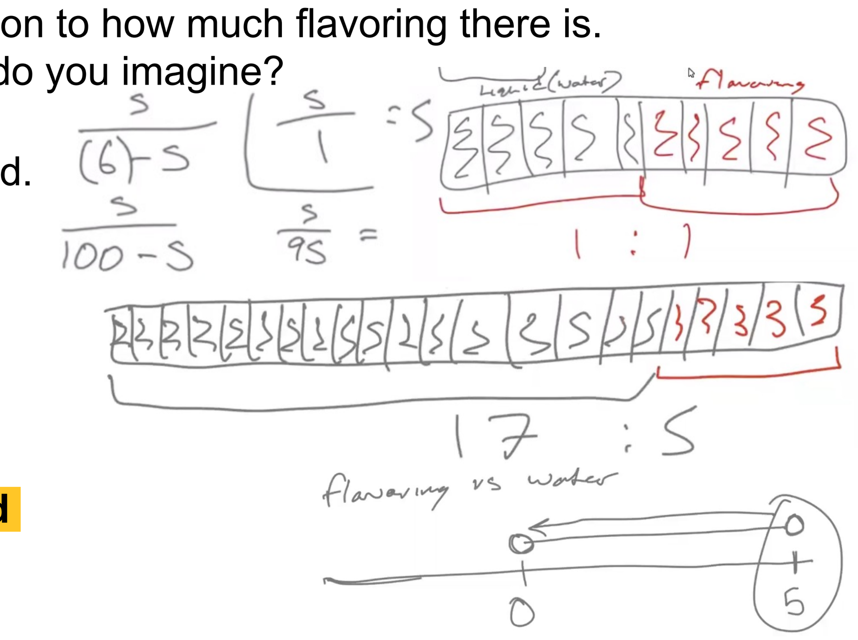
Gabriel: So it's kinda like the same thing like from over here where **the flavoring is overpowered by the amount of liquid that there is. Since the amount of flavoring isn't really changed, it remains at a constant five. If you keep on adding liquid without adding any more of the, the flavoring, then eventually there will be more liquid than the flavoring instead of having like a good, like equal ratio or a constant ratio between the two.**

Interviewer: So, what do you mean by a ratio?

Gabriel: So how much liquid(water) there is in comparison to how much flavoring there is.

Interviewer: When you say comparison and ratio what do you imagine?

Gabriel: So if there's five mixture or, or five parts of the mixture or of the flavoring, then there's five of the liquid. **So as the amount of liquid increases,** the flavoring doesn't also increase meaning that eventually instead of it being like a equal one to one, **the ratio isn't a one-to-one ratio. It's this, 17 to 5, which is what I meant earlier by the liquid overpowering the flavoring. The amount or the proper ratio of the liquid to flavoring decreases… like the more liquid that's added, the more of a smaller fraction the whole equation becomes.**



Conclusion

- This interview suggests that Gabriel’s fraction schemes include to partition a quantity into equal parts, and iterating it repeatedly based on the reference length he chooses.
  - This way of thinking appeared to support him in coordinating several units at the same time in task 1 in order to determine the angle measure Paul swept out as he skied 27miles.
- Gabriel’s language and drawings suggest he reasoned similarly while considering the ratio of flavoring to water as the number of gallons of water increased without bound. Gabriel coordinated the varying number of gallons of total volume, water, and a constant amount of liquid flavoring, while determining the relative size of flavoring to water.
- I argue that Gabriel was engaging in relative size reasoning in task 1 (static context) as well as a in task 2 (dynamic context).

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NOTE: More citations available upon request

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Future Research

- If he continues to reason similarly in other tasks and if other students reason similarly.
- What is the role of relative size reasoning in understanding key ideas of precalculus and calculus.