Planting Seeds through Embodiment to Teach Formal Concepts of Abstract Algebra

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In this descriptive case study, we explored how a mathematics educator integrated embodiment into a first semester abstract algebra course. We found that, in addition to gesture, the instructor encouraged students to interact with physical materials and simulate mathematics using their bodies. Our results offer practical implications by illustrating examples of how embodiment can be incorporated in an abstract algebra classroom.

Keywords: Cosets, Embodied Cognition; Equivalence relation, Factor Group, Homomorphism

Introduction and Literature Review

The purpose of this work is to contribute to the literature on how embodiment, beyond gesture, can be integrated in the teaching of collegiate mathematics. Nathan (2022) defines embodiments as the use of "body-based resources to make meaning and connect new ideas and representations to prior experiences" (p. 4). In this case study, we addressed the research question: In what ways does a mathematics educator, who is knowledgeable about embodiment, integrate embodiment to support students' learning of abstract algebra concepts? We found that instructor-led embodied activities planted seeds for formal definitions, theorems, and proofs. Additionally, our results indicate that embodiment served as a tool to disambiguate referents of students' speech, to position students' contributions as legitimate, to strengthen the classroom community, and to include students who were not yet fluent in formal language.

A majority of the research related to embodiment intersects gesture and K-12 mathematics teaching. Studies show that teachers use gestures to convey mathematical ideas (Alibali & Nathan, 2012; Font et al., 2010), scaffold material (Alibali & Nathan, 2007), or establish common ground (Alibali, et al., 2013). In some of their more recent work, Alibali et al. (2019) found that teachers sometimes repeated students' verbal utterances and added gestures or made gestures, without speaking, in correspondence with students' verbal utterances towards referents. Gestures appeared to highlight students' verbal utterances, correct students' verbiage, or develop shared understanding in the classroom. Research at the intersection of embodiment and collegiate mathematics teaching is sparce and primarily attends to gesture (Lee et al., 2009; Stewart et al., 2019; Weinberg et al., 2015; Wheeler & Champion, 2013). For example, Lee et al. described how a differential equations instructor used pointing gestures to connect a mathematical problem with inscriptions. Wheeler and Champion portrayed an instructor's gesture to represent mapping of a function in abstract algebra. Weinberg et al. highlighted how an abstract algebra instructor's speech, writing, and gestures took on meaning in relation to each other. Beyond gesture, Stewart et al. explored how a linear algebra instructor navigated between embodied, symbolic, and formal instruction. Stewart et al. found that the instructor often used embodied explanations before moving to symbolic or formal instruction. For example, after defining span, the instructor asked the students to "imagine beginning with a new empty set and including a new vector in the set, one at a time" (p. 1257) and then imagine the span of the set. This was an attempt to help students experience linear independence before formally defining it.

Theoretical Framework

For this research study, we drew upon Nathan's (2022) four types of embodiments that he claims are useful for educational settings as the framework. The first type, *body form, movement,* and *perception,* make up our primary experiences, as we exist in "a body that has form and movement as well as specific perceptual capacities" (Nathan, p. 87). The second type, *gesture,* entails "movements of our hands and arms, as well as other body parts, that we make spontaneously as we talk with others and think to ourselves, including pointing and tracing actions" (p. 89). Nathan defines *simulations* as mental events that "enable us to extend the cognitive processes we have for things that are present and familiar so that we can think about possible things and unfamiliar ideas" (p. 90). The last category, *materials*, refers to how "we use physical objects and material in the world to perform epistemic operations" (p. 93). Nathan also describes four ways that embodiment supports student learning. First, *grounding* connects abstract ideas to concrete experiences to facilitate making meaning. *Offloading* leverages "external resources to manage highly constrained cognitive and attentional resources" (p. 99). *Cognitive-sensorimotor transduction* occurs when intellectual processes and sensorimotor processes interact and support learning unconsciously. Finally, *participation* entails communities of practice where newcomers become old timers (Lave & Wenger, 1991). Nathan believes communities of practice support learning "because they develop effective and efficient ways to collaborate, use cultural tools, and application-specific discourse practices" (p. 106).

Methods

The second author served a dual role of instructor (participant) and researcher. She has almost 30 years of teaching experience, has taught abstract algebra numerous times, and her research area centers on undergraduate mathematics education where she adopts an embodied cognition lens (e.g. Soto & Oehrtman, 2022; Soto-Johnson & Hancock, 2019; Soto-Johnson & Troup, 2014). She tends to follow Tall's (2013) three worlds to navigate between embodiment, symbolism, and formalism while teaching. She encourages students to use their bodies, tangible materials, and technology to learn mathematics (e.g. Dittman et al., 2016; Soto, 2021, 2022; Soto-Johnson & Troup, 2014). She values community and requires her students to work with their peers throughout the semester. For the course, the instructor adopted the text *A Book of Abstract Algebra* (Pinter, 1990) because as an undergraduate she used this text to learn abstract algebra and, as a mathematics educator, she appreciates its constructivist format.

The first author attended each lecture, had weekly conversations with the instructor on the role of embodied cognition in her teaching, audio- and video- recorded six weeks of class, and kept field notes. To capture recorded data, the first author set up two cameras. One stationary camera framed the front of the room, and the second hand-held camera was used to focus on the instructor or students. After the course, the first author and the instructor met twice a week to review recorded classroom data, which was divided into seven- and ten-minute segments. The two authors time-stamped instances when the instructor integrated embodiment, took descriptive notes on those instances, and summarized major themes from each segment. Themes included: regesturing students' gestures, delaying formal language until students understood relevant concrete examples, and transforming students' contributions into inscriptions.

After reviewing classroom data, the research team selected video segments that exemplified the instructor's use of embodiment. We fully coded and analyzed four episodes, but for the sake of brevity, we only discuss the episode related to acting out conjugation, showing a subgroup is normal, and the Fundamental Homomorphism Theorem (FHT). The two non-instructor research members re-watched the selected video segments and coded the types of embodiments as well as how each embodiment type appeared to support students' learning per Nathan's (2022) framework. They then used the codes to write descriptive results (using pseudo names for students) on how the instructor integrated embodiment. The instructor reviewed written results and the entire team engaged in round table discussions to ensure the accuracy and validity of the results, which served as member-checking (Merriam & Tisdell, 2015).

Results

In this section, we summarize how the instructor introduced conjugation, normal subgroups, and the FHT by leveraging all four types of embodiments, which appeared to support students' learning in each of the four ways described by Nathan (2022). Table 1 is a summary of these findings.

Table 1. Instances of embodiment in exploring conjugation, normal subgroups and the FHT

Forms of learning→ Types of embodiments	Grounding	Offloading	Cognitive-sensorimotor transduction	Participation
Body form, movement and perception	Vocabulary "to the right of" and "to the left of"		Described conjugation in terms of direction	
Gesture	Held imaginary group elements to gesture the action of conjugation			Gesture to establish common ground Orchestration of conjugation
Simulation	Students acted out conjugation on a normal subgroup to develop embodied meaning			Students searching for other students to help them conjugate
Materialist epistemology		Paddy paper to represent of D_4		

The instructor created a simulation in which she assigned students to act as elements of $\langle \mathbb{Z}_8, + \rangle$ and $\langle \mathbb{Z}_4, + \rangle$. She asked students to perform a homomorphism from $\langle \mathbb{Z}_8, + \rangle$ to $\langle \mathbb{Z}_4, + \rangle$; she assigned certain students from $\mathbb{Z}_8$ to map to specific elements of $\mathbb{Z}_4$. Students chose to touch the shoulder of the appropriate student to represent mapping. Together, the class verified that the kernel (K) of the homomorphism formed a subgroup. Then, without using the term conjugation, the instructor explained how to act out conjugation on the subgroup $K = \{Margaret, Cam\}$ of $\mathbb{Z}_8$ by stating, "Margaret [or Cam] is going to be in the middle and is going to be operated on the left-hand side by the inverse and on the right-hand side by the element." While describing conjugation in terms of the simulation, the instructor gestured to "hold" Margaret in the middle, "hold" an element's inverse on the left, and "hold" the element itself on the right, which encouraged her students to imagine conjugation (see Figure 1). The instructor's gesture grounded her explanation of conjugation within the students' performance. Additionally, gesture supported participation because it positioned students as central rather than peripheral participants. Students then created their own sign for conjugation in which Margaret (the identity) stood in the middle, Julie ($5 \in Z_8$) stood on the right, and Jim ($5^{-1} \in Z_8$) stood on the left (see Figure 2).

Figure 1. Gesturing Conjugation of the Kernel

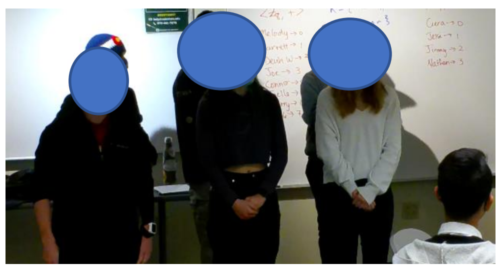

Figure 2. Julie and Jim simulating the action of conjugation on Margaret

The informal language "to the right of" and "to the left of" to describe conjugation is an example of the instructor using body form to support students' learning because spatial relationships *grounded* conjugation in a primary aspect of the human experience – direction. Body form also supported student learning via *cognitive-sensorimotor transduction* because it reversed the complex notion of conjugation in terms of students' physical position in space. Given Margaret was the identity element, the class noticed that the action of conjugation by an element (a student) and its inverse (another student) on Margaret always returned Margaret. Then, by acting out conjugation on Cam, students showed that every element-inverse pair in $\mathbb{Z}_8$ returned Cam under conjugation (see Figure 3).

Overall, simulation served as a tool that helped the instructor guide her students in showing that $K = \{Margaret, Cam\}$ was a normal subgroup. The instructor's use of simulation grounded conjugation and normal subgroup in a meaningful experience before she introduced formal definitions. Moreover, simulation supported student learning via participation. For example, when Patrick conjugated Cam, he looked for his needed classmate (element) to help him perform the operation and students at their desks helped their peers find their inverses, which strengthened the classroom community.

Figure 3. Conjugating Cam by elements of Z_8

After the class (unknowingly) showed that K formed a normal subgroup, the instructor intentionally gestured to recap the simulation. She reminded the class that they began with two groups and "plucked out" a subgroup which generated cosets. Then, via pointing gestures, she emphasized that the two elements in the subgroup they "plucked out" both mapped to the identity in $\mathbb{Z}_4$ (see Figure 4). She also emphasized that the map was a homomorphism and not an isomorphism, which was new to the class. Finally, she emphasized normality of the kernel by stating, "if I take these two elements right **here** [Margaret and Cam] and I operate them by an element from the group and its inverse I somehow end up getting Margaret or Cam." The instructor used gesture to establish common ground because she made sure the class knew that the importance of the simulation was to see an example of a map that was a homomorphism, but not an isomorphism and that the kernel, K, satisfied interesting new properties, i.e., normality.

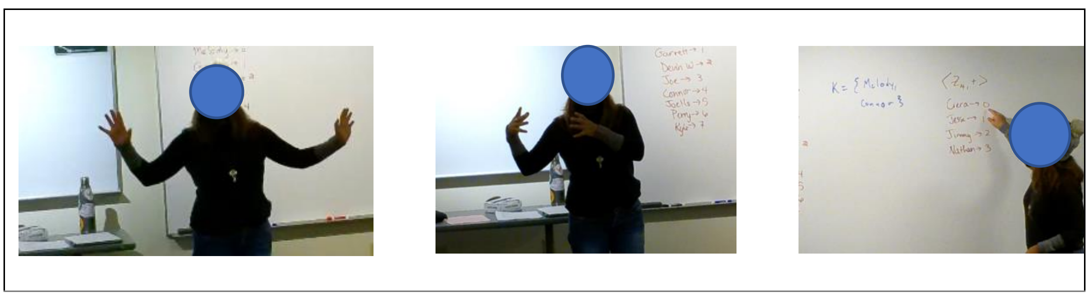

Figure 4. Two groups, pluck out two elements, these elements map to the identity

Following the FHT simulation involving conjugation, the instructor asked the class to complete a problem set related to the group D_4. In conjunction with the problem set, the instructor employed materialist epistemology by holding up a paddy paper and helping the class recall that D_4 represents the symmetries of a square. After distributing paddy paper to each student, she reminded students that all the rotational symmetries keep the square "on the same side." When she noticed Karen flip her square over, the instructor said, "tell me what you just did Karen." Karen replied that she performed a reflection of the square. It seemed that the geometric behavior of the square served to offload information because rather than write down each element of D_4 symbolically, the students could visualize the group D_4 in the material world.

Following this, the instructor encouraged students to look at the Cayley table for D_4 in *Group Explorer* (Carter, 2006), to find patterns in the table, and to reflect on the meaning of the simulation they had just completed. As the students worked in groups, David realized that,

> looking at this [D_4 Cayley table] … you got your warm and your cool colors here. And earlier in class, … we were talking about, … the cool ones are almost subgroups, but they're not because they don't contain the identity. … now we are realizing that if they have the same number of things like they are kind of subsets, but not subgroups [cosets]. Then that's how you construct a homomorphism to your subgroups or subsets [cosets] … and the homomorphism [image] is isomorphic to S_2 (see Figure 5a).

It appeared that David visualized the FHT. Then, Cam also excitedly changed tabs in his browser to visually show a case of the FHT in a map from $\mathbb{Z}_3 \times \mathbb{Z}_3$ to $\mathbb{Z}_3$ because "you can cut it [$\mathbb{Z}_3 \times \mathbb{Z}_3$] into the nine squares and there is a homomorphism to this one [$\mathbb{Z}_3$]" (see Figure 5b).

Figure 5a. D_4 is homomorphic to S_2

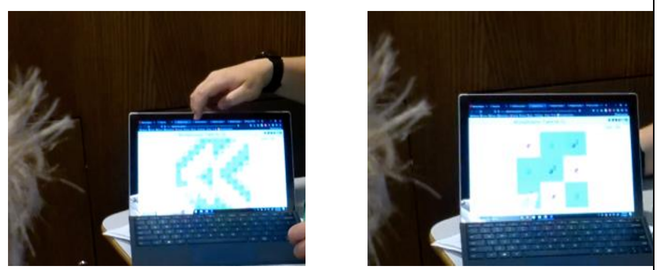

Figure 5b. $Z_3 \times Z_3$ is homomorphic to Z_3

Discussion and Implications

We situate our findings within existing approaches for the teaching of abstract algebra and discuss how gesture, along with other embodiment types, supported shared classroom understandings and helped strengthen the classroom community.

Early instructional design research in abstract algebra was based on actions, processes, objects, and schemas theory (Dubinsky et al., 2001) and the activities, classroom-discussions, and exercises cycle (Asiala et al., 1997). More recently, an inquiry-oriented curriculum was developed by the *Teaching of Abstract Algebra for Understanding* research program, which adheres to the principles of guided reinvention and realistic mathematics education (Gravemeijer & Doorman, 1999; Larsen, et al, 2013; Larsen & Lockwood, 2013). Similar to existing approaches on the teaching of abstract algebra, the instructor in this research study facilitated informal activities before presenting formal definitions, theorems, and proofs. Her approach is novel because she integrated all four types of embodiments described by Nathan's (2022) framework, which also appeared to support student learning in the four ways Nathan described.

Alibali, et al. (2019) drew from a compilation of their research to describe three cases illustrating how teacher gestures can create common ground and support students' contributions. In the first case, students sat at their desks so that they were "physically distant from the referents of their utterances," while the middle school teacher was "proximal to those referents" (p. 350). The teacher regestured and revoiced the student's verbal utterances by pointing at symbolic referents to ensure their interpretation was consistent with the student's intentions before moving on. In the second case, Alibali et al. (2019) identified a variation of gesture by a high school geometry teacher who silently gestured over inscriptions located at the board as a student spoke from their desk to "amplify certain voices" (p. 356). In the third case, both the college teacher and student were within arm's reach of the referent (an electrical breadboard). Here, the teacher pointed to connect the students' gestural actions on the electrical breadboard to symbols in a truth table. The instructor from our research study used gesture to establish common ground and support students' contributions in ways that overlap with and extend Alibali et al.'s (2019) three cases. Specifically, the instructor used various types of gesture (besides pointing), she combined gesture with simulation, and her gestures supported student participation.

The instructor's gestures that described conjugation by "holding" group elements (students) to the left, right, and center in relation to each other differed from the three cases described in Alibali et al.'s (2019) work on establishing common ground. Compared to the teachers from Alibali et al., this instructor did not sole use pointing gestures; she also used representational gestures, while describing conjugation. Alibali and Nathan (2012) refer to the merging of iconic and metaphorical gestures as representational gestures. McNeill (2005) categorized metaphoric gestures as those which indicate images of an abstract concept such as holding a function in one's hand and iconic gestures as those which "present images of concrete entities and/or actions" (p. 39). The instructors' gestures along with simulation closed the distance between students and referents because students were the referents of the instructors' gestures. Students responded to her gestural description of conjugation by creating a sign: physically standing to the sides of one another. The instructors' representational gestures, thus, created common ground because students responded with the physical action of standing beside each other and the instructor could ensure that her students understood what she meant by checking that their embodied sign aligned with her knowledge of the conjugation operation.

Nathan (2022) believes communities are helpful for learning because they leverage cultural tools and practices as a means of collaboration. The way that students followed the instructors' descriptive gesture of conjugation by developing their own tool (the sign of standing beside each other to conjugate) is an example of how gesture can strengthen a classroom community and, thus, establish common ground. Through creating and implementing their own sign, students' ideas were positioned as significant and actively used by each other to communicate and work as a team. The entanglement of gesture and simulation illustrates how gesture along with other embodiment types can help establish common ground because the entire class adopted a shared sign, which naturally prompted and reinforced shared understandings.

After the simulation ended, the whole class sat at their desks and the instructor stated, "lets recap what we just did." Unlike the cases described in Alibali et. al.'s (2019) findings on how gesture is implicitly used by teachers to create common ground, this instructor explicitly stated that establishing common ground was her purpose for regesturing. This made the instructors' intentions clear to her class. The instructor used both pointing and representational gestures to review the significance of the simulation. During her recap, students asked why the order of the kernel divides the order of the group. Student questions suggest that when the instructor explicitly stated that her goal in gesturing was to review, she invited students to engage in reflective practices and students felt that they could ask for help establishing common ground.

Our study provides examples of how embodiment could be integrated in abstract algebra classrooms. In response to Johnson et al. (2020), we believe incorporating embodiment can facilitate understanding and has the potential to lead to equitable outcomes where students' contributions are supported. Future research may entail creating an embodied curriculum (Wang & Zheng, 2018) and developing embodied simulation activities for abstract algebra using unrefined human experiences, genuine relationships and collaborations, adjustments tailored to the students' interests and instructors' expertise, and scenarios that embody real-life experiences.

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